APEX Calculus

for University of Lethbridge

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Website: apexcalculus.com[1]

[1] www.apexcalculus.com

Thanks

There are many people who deserve recognition for the important role they have played in the development of this text. First, I thank Michelle for her support and encouragement, even as this “project from work” occupied my time and attention at home. Many thanks to Troy Siemers, whose most important contributions extend far beyond the sections he wrote or the 227 figures he coded in Asymptote for 3D interaction. He provided incredible support, advice and encouragement for which I am very grateful. My thanks to Brian Heinold and Dimplekumar Chalishajar for their contributions and to Jennifer Bowen for reading through so much material and providing great feedback early on. Thanks to Troy, Lee Dewald, Dan Joseph, Meagan Herald, Bill Lowe, John David, Vonda Walsh, Geoff Cox, Jessica Libertini and other faculty of VMI who have given me numerous suggestions and corrections based on their experience with teaching from the text. (Special thanks to Troy, Lee and Dan for their patience in teaching Calc III while I was still writing the Calc III material.) Thanks to Randy Cone for encouraging his tutors of VMI’s Open Math Lab to read through the text and check the solutions, and thanks to the tutors for spending their time doing so. A very special thanks to Kristi Brown and Paul Janiczek who took this opportunity far above and beyond what I expected, meticulously checking every solution and carefully reading every example. Their comments have been extraordinarily helpful. I am also thankful for the support provided by Wane Schneiter, who as my Dean provided me with extra time to work on this project. I am blessed to have so many people give of their time to make this book better.

Preface

A Note on Using this Text. Thank you for reading this short preface. Allow us to share a few key points about the text so that you may better understand what you will find beyond this page.

This text comprises a three—volume series on Calculus. The first part covers material taught in many "Calc 1" courses: limits, derivatives, and the basics of integration, found in Chapters 1 through 6.1. The second text covers material often taught in "Calc 2:" integration and its applications, including an introduction to differential equations, along with an introduction to sequences, series and Taylor Polynomials, found in Chapters 5 through 8. The third text covers topics common in "Calc 3" or "multivariable calc:" parametric equations, polar coordinates, vector-valued functions, and functions of more than one variable, found in Chapters 10 through 15. All three are available separately for free at apexcalculus.com[2], and HTML versions of the book can be found at opentext.uleth.ca[3].

These three texts are intended to work together and make one cohesive text, *APEX Calculus*, which can also be downloaded from the website.

Printing the entire text as one volume makes for a large, heavy, cumbersome book. One can certainly only print the pages they currently need, but some prefer to have a nice, bound copy of the text. Therefore this text has been split into these three manageable parts, each of which can be purchased for about $15 at Amazon.com[4].

For Students: How to Read this Text. Mathematics textbooks have a reputation for being hard to read. High—level mathematical writing often seeks to say much with few words, and this style often seeps into texts of lower—level topics. This book was written with the goal of being easier to read than many other calculus textbooks, without becoming too verbose.

Each chapter and section starts with an introduction of the coming material, hopefully setting the stage for "why you should care," and ends with a look ahead to see how the just—learned material helps address future problems.

- *Please read the text.*

  It is written to explain the concepts of Calculus. There are numerous examples to demonstrate the meaning of definitions, the truth of theorems, and the application of mathematical techniques. When you encounter a sentence you don't understand, read it again. If it still doesn't make sense, read on anyway, as sometimes confusing sentences are explained by later sentences.

[2] apexcalculus.com
[3] opentext.uleth.ca/calculus.html
[4] amazon.com

- *You don't have to read every equation.*

  The examples generally show "all" the steps needed to solve a problem. Sometimes reading through each step is helpful; sometimes it is confusing. When the steps are illustrating a new technique, one probably should follow each step closely to learn the new technique. When the steps are showing the mathematics needed to find a number to be used later, one can usually skip ahead and see how that number is being used, instead of getting bogged down in reading how the number was found.

- *Most proofs have been omitted.*

  In mathematics, *proving* something is always true is extremely important, and entails much more than testing to see if it works twice. However, students often are confused by the details of a proof, or become concerned that they should have been able to construct this proof on their own. To alleviate this potential problem, we do not include the proofs to most theorems in the text. The interested reader is highly encouraged to find proofs online or from their instructor. In most cases, one is very capable of understanding what a theorem *means* and *how to apply it* without knowing fully *why* it is true.

Interactive, 3D Graphics. Versions 3.0 and 4.0 of the textbook include interactive, 3D graphics in the pdf version. Nearly all graphs of objects in space can be rotated, shifted, and zoomed in/out so the reader can better understand the object illustrated. However, the only pdf viewers that support these 3D graphics are Adobe Reader Acrobat (and only the versions for PC/Mac/Unix/Linux computers, not tablets or smartphones).

The latest version of the book, which is authored in PreTeXt, is available in html. In html, the 3D graphics are rendered using WebGL, and should work in any modern web browser.

Interactive graphics are no longer supported within the pdf, but clicking on any 3D graphic within the pdf will take you directly to the interactive version on the web.

APEX – Affordable Print and Electronic teXts. *APEX* is a consortium of authors who collaborate to produce high quality, low cost textbooks. The current textbook—writing paradigm is facing a potential revolution as desktop publishing and electronic formats increase in popularity. However, writing a good textbook is no easy task, as the time requirements alone are substantial. It takes countless hours of work to produce text, write examples and exercises, edit and publish. Through collaboration, however, the cost to any individual can be lessened, allowing us to create texts that we freely distribute electronically and sell in printed form for an incredibly low cost. Having said that, nothing is entirely free; someone always bears some cost. This text "cost" the authors of this book their time, and that was not enough. **APEX Calculus** would not exist had not the Virginia Military Institute, through a generous Jackson—Hope grant, given the lead author significant time away from teaching so he could focus on this text.

Each text is available as a free .pdf, protected by a Creative Commons Attribution - Noncommercial 4.0 copyright. That means you can give the .pdf to anyone you like, print it in any form you like, and even edit the original content and redistribute it. If you do the latter, you must clearly reference this work and you cannot sell your edited work for money.

We encourage others to adapt this work to fit their own needs. One might add sections that are "missing" or remove sections that your students won't need. The source files can be found at github.com/APEXCalculus[5].

You can learn more at www.vmi.edu/APEX[6].

First PreTeXt Edition (Version 5.0). Key changes from Version 4.0 to 5.0:

- The underlying source code has been completely rewritten, to use the PreTeXt[7] language, instead of the original LaTeX.
- Using PreTeXt allows us to produce the books in multiple formats, including html, which is both more accessible and more interactive than the original pdf. html versions of the book can be found at opentext.uleth.ca[8].
- The appendix on differential equations from the "Calculus for Quarters" version of the book has been included as Chapter 8, just after applications of integration. Chapters 8 — 14 are now numbered 9 — 15 as a result.
- In the html version of the book, many of the exercises are now interactive, and powered by WeBWorK.

Key changes from Version 3.0 to 4.0:

- Numerous typographical and "small" mathematical corrections (again, thanks to all my close readers!).
- "Large" mathematical corrections and adjustments. There were a number of places in Version 3.0 where a definition/theorem was not correct as stated. See www.apexcalculus.com[9] for more information.
- More useful numbering of Examples, Theorems, etc. . "Definition 11.4.2" refers to the second definition of Chapter 11, Section 4.
- The addition of Section 13.7: Triple Integration with Cylindrical and Spherical Coordinates
- The addition of Chapter 14: Vector Analysis.

[5] github.com/APEXCalculus
[6] www.vmi.edu/APEX
[7] pretextbook.org
[8] opentext.uleth.ca/calculus.html
[9] apexcalculus.com

A Brief History of Calculus

Calculus means "a method of calculation or reasoning." When one computes the sales tax on a purchase, one employs a simple calculus. When one finds the area of a polygonal shape by breaking it up into a set of triangles, one is using another calculus. Proving a theorem in geometry employs yet another calculus.

Despite the wonderful advances in mathematics that had taken place into the first half of the 17th century, mathematicians and scientists were keenly aware of what they *could not do*. (This is true even today.) In particular, two important concepts eluded mastery by the great thinkers of that time: area and rates of change.

Area seems innocuous enough; areas of circles, rectangles, parallelograms, etc., are standard topics of study for students today just as they were then. However, the areas of *arbitrary* shapes could not be computed, even if the boundary of the shape could be described exactly.

Rates of change were also important. When an object moves at a constant rate of change, then "distance $=$ rate $\times$ time." But what if the rate is not constant—can distance still be computed? Or, if distance is known, can we discover the rate of change?

It turns out that these two concepts were related. Two mathematicians, Sir Isaac Newton and Gottfried Leibniz, are credited with independently formulating a system of computing that solved the above problems and showed how they were connected. Their system of reasoning was "a" calculus. However, as the power and importance of their discovery took hold, it became known to many as "the" calculus. Today, we generally shorten this to discuss "calculus."

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Chapter 1

Limits

The foundation of "the calculus" is the **limit**. It is a tool to describe a particular behavior of a function. This chapter begins our study of the limit by approximating its value graphically and numerically. After a formal definition of the limit, properties are established that make "finding limits" tractable. Once the limit is understood, then the problems of area and rates of change can be approached.

youtu.be/watch?v=37n0cZn6Lyc

Figure 1.0.1 Overview of Calculus

1.1 An Introduction To Limits

We begin our study of *limits* by considering examples that demonstrate key concepts that will be explained as we progress.

Consider the function $y = \frac{\sin(x)}{x}$. When x is near the value 1, what value (if any) is y near?

While our question is not precisely formed (what constitutes "near the value 1"?), the answer does not seem difficult to find. One might think first to look at a graph of this function to approximate the appropriate y values. Consider Figure 1.1.3, where $y = \frac{\sin(x)}{x}$ is graphed. For values of x near 1, it seems that y takes on values near 0.85. In fact, when $x = 1$, then $y = \frac{\sin(1)}{1} \approx 0.84$, so it makes sense that when x is "near" 1, y will be "near" 0.84.

youtu.be/watch?v=rG3XVvFIZPY

Figure 1.1.1 The concept of a limit

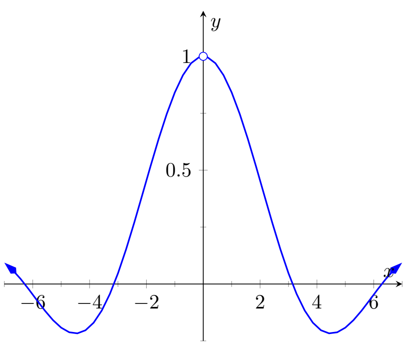


Figure 1.1.2 $\sin(x)/x$

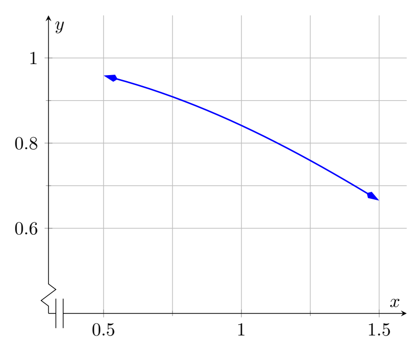


Figure 1.1.3 $\sin(x)/x$ near $x = 1$

Consider this same function again at a different value for x. When x is near 0, what value (if any) is y near? By considering Figure 1.1.4, one can see that it seems that y takes on values near 1. But what happens when $x = 0$? We have

$$y \to \frac{\sin(0)}{0} \to \text{“}\frac{0}{0}\text{”}.$$

The expression $0/0$ has no value; it is **indeterminate**. Such an expression gives no information about what is going on with the function nearby. We cannot find out how y behaves near $x = 0$ for this function simply by letting $x = 0$.

Finding a limit entails understanding how a function behaves near a particular value of x. Before continuing, it will be useful to establish some notation. Let $y = f(x)$; that is, let y be a function of x for some function f. The expression "the limit of y as x approaches 1" describes a number, often referred to as L, that y nears as x nears 1. We write all this as

$$\lim_{x\to 1} y = \lim_{x\to 1} f(x) = L.$$

This is not a complete definition (that will come in the next section); this is a pseudo-definition that will allow us to explore the idea of a limit.

Above, where $f(x) = \sin(x)/x$, we approximated

$$\lim_{x\to 1} \frac{\sin(x)}{x} \approx 0.84 \quad \text{and} \quad \lim_{x\to 0} \frac{\sin(x)}{x} \approx 1.$$

(We *approximated* these limits, hence used the "≈" symbol, since we are working with the pseudo-definition of a limit, not the actual definition.)

Once we have the true definition of a limit, we will find limits *analytically*; that is, exactly using a variety of mathematical tools. For now, we will *approximate* limits both graphically and numerically. Graphing a function can provide a good approximation, though often not very precise. Numerical methods can provide a more accurate approximation. We have already approximated limits graphically, so we now turn our attention to numerical approximations.

Consider again $\lim_{x\to 1} \frac{\sin(x)}{x}$. To approximate this limit numerically, we can create a table of x and $f(x)$ values where x is "near" 1. This is done in Figure 1.1.6.

Notice that for values of x near 1, we have $\sin(x)/x$ near 0.841. The $x = 1$ row is included, but we stress the fact that when considering limits, we are *not* concerned with the value of the function at that particular x value; we are only concerned with the values of the function when x is *near* 1.

Now approximate $\lim_{x\to 0} \frac{\sin(x)}{x}$ numerically. We already approximated the value of this limit as 1 graphically in Figure 1.1.4. Figure 1.1.7 shows the value of $\sin(x)/x$ for values of x near 0. Ten places after the decimal point are shown to highlight how close to 1 the value of $\sin(x)/x$ gets as x takes on values very near 0. We include the $x = 0$ row but again stress that we are not concerned with the value of our function at $x = 0$, only on the behavior of the function *near* 0.

This numerical method gives confidence to say that 1 is a good approximation of $\lim_{x\to 0} \frac{\sin(x)}{x}$; that is,

$$\lim_{x\to 0} \frac{\sin(x)}{x} \approx 1.$$

Later we will be able to prove that the limit is *exactly* 1.

We now consider several examples that allow us explore different aspects of the limit concept.

Example 1.1.8 Approximating the value of a limit.

Use graphical and numerical methods to approximate

$$\lim_{x\to 3} \frac{x^2 - x - 6}{6x^2 - 19x + 3}.$$

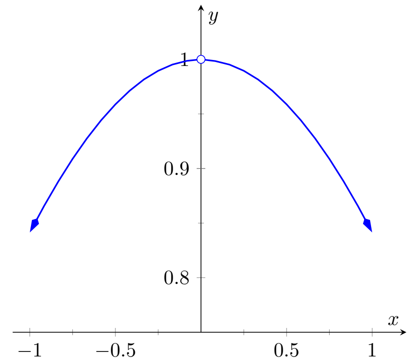


Figure 1.1.4 $\sin(x)/x$ near $x = 0$

youtu.be/watch?v=__qzaSg4y1I

Figure 1.1.5 Investigating $\sin(x)/x$

x	$\sin(x)/x$
0.9	0.870363
0.99	0.844471
0.999	0.841772
1	***0.841471***
1.001	0.841170
1.01	0.838447
1.1	0.810189

Figure 1.1.6 Values of $\sin(x)/x$ with x near 1

x	$\sin(x)/x$
-0.1	0.9983341665
-0.01	0.9999833334
-0.001	0.9999998333
0	***not defined***
0.001	0.9999998333
0.01	0.9999833334
0.1	0.9983341665

Figure 1.1.7 Values of $\sin(x)/x$ with x near 0

Solution. To graphically approximate the limit, graph

$$y = \frac{x^2 - x - 6}{6x^2 - 19x + 3}$$

on a small interval that contains 3. To numerically approximate the limit, create a table of values where the x values are near 3. This is done in Figure 1.1.9 and Figure 1.1.10, respectively.

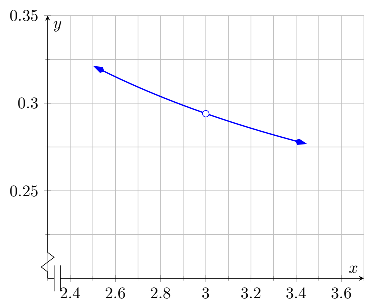


Figure 1.1.9 Graphically approximating a limit in Example 1.1.8

x	$\frac{x^2-x-6}{6x^2-19x+3}$
2.9	0.29878
2.99	0.294569
2.999	0.294163
3	*not defined*
3.001	0.294073
3.01	0.293669
3.1	0.289773

Figure 1.1.10 Numerically approximating a limit in Example 1.1.8

The graph shows that when x is near 3, the value of y is very near 0.3. By considering values of x near 3, we see that $y = 0.294$ is a better approximation. The graph and the table imply that

$$\lim_{x\to 3} \frac{x^2 - x - 6}{6x^2 - 19x + 3} \approx 0.294.$$

Video solution

youtu.be/watch?v=eHx3LmrQZXM

This example may bring up a few questions about approximating limits (and the nature of limits themselves).

1. If a graph does not produce as good an approximation as a table, why bother with it?

2. How many values of x in a table are "enough?" In the previous example, could we have just used $x = 3.001$ and found a fine approximation?

Graphs are useful since they give a visual understanding concerning the behavior of a function. Sometimes a function may act "erratically" near certain x values which is hard to discern numerically but very plain graphically (see Example 1.1.22). Since graphing utilities are very accessible, it makes sense to make proper use of them.

Since tables and graphs are used only to *approximate* the value of a limit, there is not a firm answer to how many data points are "enough." Include enough so that a trend is clear, and use values (when possible) both less than and greater than the value in question. In Example 1.1.8, we used both values less than and greater than 3. Had we used just $x = 3.001$, we might have been tempted to conclude that the limit had a value of 0.3. While this is not far off, we could do better. Using values "on both sides of 3" helps us identify trends.

Example 1.1.11 Approximating the value of a limit.

Graphically and numerically approximate the limit of $f(x)$ as x approaches 0, where

$$f(x) = \begin{cases} x+1 & x<0 \\ -x^2+1 & x>0 \end{cases}.$$

Solution. Again we graph $f(x)$ and create a table of its values near $x = 0$ to approximate the limit. Note that this is a piecewise defined function, so it behaves differently on either side of 0. Figure 1.1.12 shows a graph of $f(x)$, and on either side of 0 it seems the y values approach 1. Note that $f(0)$ is not actually defined, as indicated in the graph with the open circle.

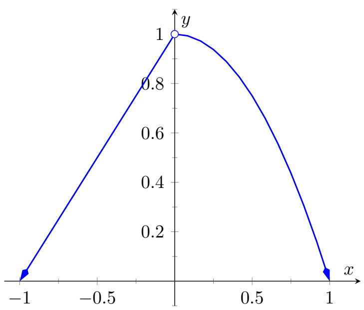


Figure 1.1.12 Graphically approximating a limit in Example 1.1.11

x	$f(x)$
-0.1	0.9
-0.01	0.99
-0.001	0.999
0.001	0.999999
0.01	0.9999
0.1	0.99

Figure 1.1.13 Numerically approximating a limit in Example 1.1.11

Figure 1.1.13 shows values of $f(x)$ for values of x near 0. It is clear that as x takes on values very near 0, $f(x)$ takes on values very near 1. It turns out that if we let $x = 0$ for either "piece" of $f(x)$, 1 is returned; this is significant and we'll return to this idea later.

The graph and table allow us to say that $\lim_{x\to 0} f(x) \approx 1$; in fact, we are probably very sure it *equals* 1.

Video solution

youtu.be/watch?v=7RAiKoLCpgU

1.1.1 Identifying When Limits Do Not Exist

A function may not have a limit for all values of x. That is, we cannot write that $\lim_{x\to c} f(x) = L$ (where L is some real number) for all values of c, for there may not be a number that $f(x)$ is approaching. There are three common ways in which a limit may fail to exist.

1. The function $f(x)$ may approach different values on either side of c.
2. The function may grow without upper or lower bound as x approaches c.
3. The function may oscillate as x approaches c without approaching a specific value.

We'll explore each of these in turn.

youtu.be/watch?v=DSIaDa_ABxo

Figure 1.1.14 Video introduction for Subsection 1.1.1

Example 1.1.15 Different Values Approached From Left and Right.

Explore why $\lim_{x\to 1} f(x)$ does not exist, where

$$f(x) = \begin{cases} x^2-2x+3 & x\le 1 \\ x & x>1 \end{cases}.$$

Solution. A graph of $f(x)$ around $x = 1$ and a table are given in Figures Figure 1.1.16 and Figure 1.1.17, respectively. It is clear that as x approaches 1, $f(x)$ does not seem to approach a single number. Instead, it seems as though $f(x)$ approaches two different numbers. When considering values of x less than 1 (approaching 1 from the left), it seems that $f(x)$ is approaching 2; when considering values of x greater than 1 (approaching 1 from the right), it seems that $f(x)$ is approaching 1. Recognizing this behavior is important; we'll study this in greater depth later. Right now, it suffices to say that the limit does not exist since $f(x)$ is approaching two *different* values as x approaches 1.

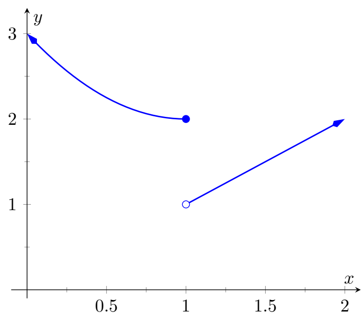


Figure 1.1.16 Observing no limit as $x \to 1$ in Example 1.1.15

x	$f(x)$
0.9	2.01
0.99	2.0001
0.999	2.000001
1.001	1.001
1.01	1.01
1.1	1.1

Figure 1.1.17 Values of $f(x)$ near $x = 1$ in Example 1.1.15

Example 1.1.18 The Function Grows Without Bound.

Explore why $\lim_{x\to 1} \frac{1}{(x-1)^2}$ does not exist.

Solution. A graph and table of $f(x) = \frac{1}{(x-1)^2}$ are given in Figure 1.1.19 and Figure 1.1.20, respectively. Both show that as x approaches 1, $f(x)$ grows larger and larger.

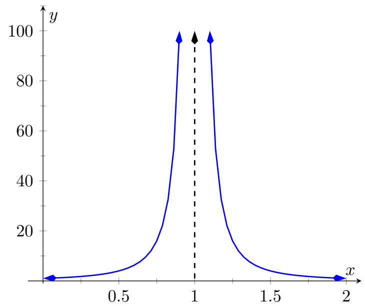


Figure 1.1.19 Observing no limit as $x \to 1$ in Example 1.1.18

x	$f(x)$
0.9	$100.$
0.99	$10000.$
0.999	$1. \times 10^6$
1.001	$1. \times 10^6$
1.01	$10000.$
1.1	$100.$

Figure 1.1.20 Values of $f(x)$ near $x = 1$ in Example 1.1.18

We can deduce this on our own, without the aid of the graph and table. If x is near 1, then $(x-1)^2$ is very small, and:

$$\frac{1}{\text{very small number}} = \text{very large number} .$$

Since $f(x)$ is not approaching a single number, we conclude that

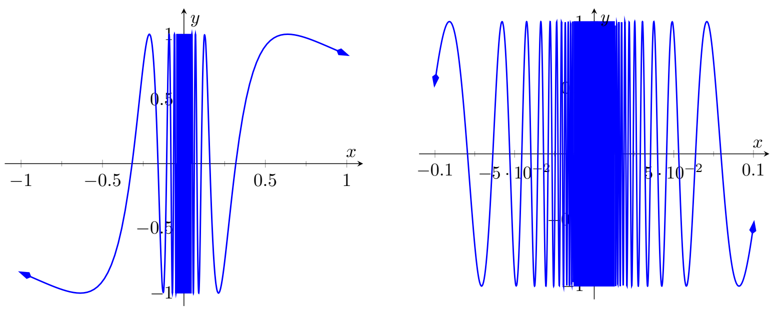
$$\lim_{x\to 1} \frac{1}{(x-1)^2}$$

does not exist.

Example 1.1.22 The Function Oscillates.

Explore why $\lim_{x\to 0} \sin(1/x)$ does not exist.

Solution. Two graphs of $f(x) = \sin(1/x)$ are given in Figure 1.1.23. Figure 1.1.23(a) shows $f(x)$ on the interval $[-1, 1]$; notice how $f(x)$ seems to oscillate near $x = 0$. One might think that despite the oscillation, as x approaches 0, $f(x)$ approaches 0. However, Figure 1.1.23(b) zooms in on $\sin(1/x)$, on the interval $[-0.1, 0.1]$. Here the oscillation is even more pronounced. Finally, in Figure 1.1.24, we see $\sin(1/x)$ evaluated for values of x near 0. As x approaches 0, $f(x)$ does not appear to approach any value.



(a) **(b)**

Figure 1.1.23 Observing that $f(x) = \sin(1/x)$ has no limit as $x \to 0$ in Example 1.1.22

It can be shown that in reality, as x approaches 0, $\sin(1/x)$ takes on all values between -1 and 1 infinitely many times! Because of this oscillation, $\lim_{x\to 0} \sin(1/x)$ does not exist.

1.1.2 Limits of Difference Quotients

We have approximated limits of functions as x approached a particular number. We will consider another important kind of limit after explaining a few key ideas.

Let $f(x)$ represent the position function, in feet, of some particle that is moving in a straight line, where x is measured in seconds. Let's say that when $x = 1$, the particle is at position 10 ft., and when $x = 5$, the particle is at 20 ft. Another way of expressing this is to say

$$f(1) = 10 \text{ and } f(5) = 20.$$

Since the particle traveled 10 feet in 4 seconds, we can say the particle's **average velocity** was 2.5 ft/s. We write this calculation using a "quotient of differences," or, a **difference quotient**:

$$\frac{f(5) - f(1)}{5 - 1} \frac{\text{ft}}{\text{s}} = \frac{10 \text{ ft}}{4 \text{ s}} = 2.5 \text{ ft/s}\,.$$

This difference quotient can be thought of as the familiar "rise over run" used to compute the slopes of lines. In fact, that is essentially what we are doing:

youtu.be/watch?v=NnV1A1KTbg0

Figure 1.1.21 Video presentation for Examples 1.1.15–1.1.18

Video solution

youtu.be/watch?v=gGvvFX5QyjE

x	$\sin(1/x)$
0.1	-0.544021
0.01	-0.506366
0.001	0.82688
0.0001	-0.305614
$1. \times 10^{-5}$	0.0357488
$1. \times 10^{-6}$	-0.349994
$1. \times 10^{-7}$	0.420548

Figure 1.1.24 Observing that $f(x) = \sin(1/x)$ has no limit as $x \to 0$ in Example 1.1.22

youtu.be/watch?v=2NJmd0Jrt4U

Figure 1.1.25 Video introduction to Subsection 1.1.2

given two points on the graph of f, we are finding the slope of the *secant line* through those two points. See Figure 1.1.26.

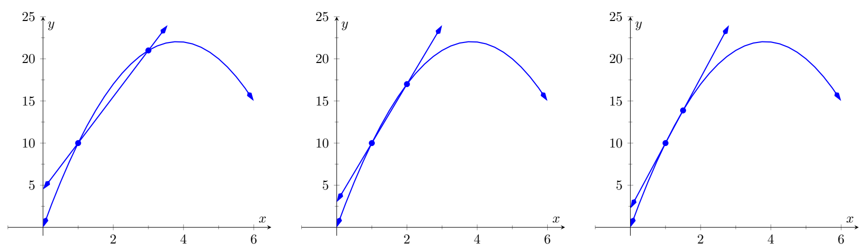
Now consider finding the average speed on another time interval. We again start at $x = 1$, but consider the position of the particle h seconds later. That is, consider the positions of the particle when $x = 1$ and when $x = 1 + h$. The difference quotient (excluding units) is now

$$\frac{f(1+h) - f(1)}{(1+h) - 1} = \frac{f(1+h) - f(1)}{h}.$$

Let $f(x) = -1.5x^2 + 11.5x$; note that $f(1) = 10$ and $f(5) = 20$, as in our discussion. We can compute this difference quotient for all values of h (even negative values!) except $h = 0$, for then we get "0/0," the indeterminate form introduced earlier. For all values $h \neq 0$, the difference quotient computes the average velocity of the particle over an interval of time of length h starting at $x = 1$.

For small values of h, i.e., values of h close to 0, we get average velocities over very short time periods and compute secant lines over small intervals. See Figure 1.1.27. This leads us to wonder what the limit of the difference quotient is as h approaches 0. That is,

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \ ?$$



(a) $h = 2$ **(b)** $h = 1$ **(c)** $h = 0.5$

Figure 1.1.27 Secant lines of $f(x)$ at $x = 1$ and $x = 1 + h$, for shrinking values of h (i.e., $h \to 0$)

As we do not yet have a true definition of a limit nor an exact method for computing it, we settle for approximating the value. While we could graph the difference quotient (where the x-axis would represent h values and the y-axis would represent values of the difference quotient) we settle for making a table. See Figure 1.1.28. The table gives us reason to assume the value of the limit is about 8.5.

Proper understanding of limits is key to understanding calculus. With limits, we can accomplish seemingly impossible mathematical things, like adding up an infinite number of numbers (and not get infinity) and finding the slope of a line between two points, where the "two points" are actually the same point. These are not just mathematical curiosities; they allow us to link position, velocity and acceleration together, connect cross-sectional areas to volume, find the work done by a variable force, and much more.

In the next section we give the formal definition of the limit and begin our study of finding limits analytically. In the following exercises, we continue our introduction and approximate the value of limits.

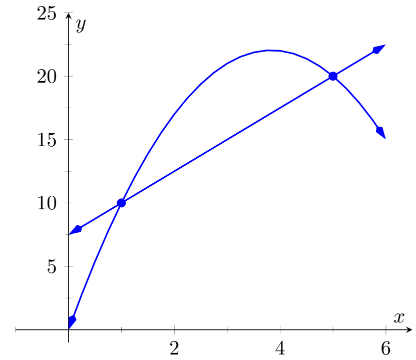


Figure 1.1.26 Interpreting a difference quotient as the slope of a secant line

h	$\frac{f(1+h)-f(1)}{h}$
-0.5	9.25
-0.1	8.65
-0.01	8.515
0.01	8.485
0.1	8.35
0.5	7.75

Figure 1.1.28 The difference quotient evaluated at values of h near 0

youtu.be/watch?v=YplEX5ohJk0

Figure 1.1.29 Video examples for difference quotients: once with direct computation, and then by simplifying first

1.1.3 Exercises

Terms and Concepts

1. In your own words, what does it mean to "find the limit of $f(x)$ as x approaches 3"?

2. An expression of the form $\frac{0}{0}$ is called ______________.

3. (☐ True ☐ False) The limit of $f(x)$ as x approaches 5 is $f(5)$.

4. Describe three situations where $\lim\limits_{x\to c} f(x)$ does not exist.

5. In your own words, what is a difference quotient?

6. When x is near 0, $\dfrac{\sin x}{x}$ is near what value?

Problems

Exercise Group. Approximate the limit numerically and graphically.

7. $\lim\limits_{x\to 1} \left(x^2 + 2x + 2\right)$

8. $\lim\limits_{x\to 1} \left(x^3 + 4x^2 - 4x + 2\right)$

9. $\lim\limits_{x\to 0} \left(\frac{x-5}{x^2-4x}\right)$

10. $\lim\limits_{x\to -4} \left(\frac{x^2+2x-8}{x^2-x-20}\right)$

11. $\lim\limits_{x\to -3} \left(\frac{x^2+10x+21}{x^2+5x+6}\right)$

12. $\lim\limits_{x\to -4} \left(\frac{x^2-13x-32}{x^2+8x+16}\right)$

13. $\lim\limits_{x\to -1} f(x)$, where

$$f(x) = \begin{cases} x+1 & \text{if } x \le -1 \\ -(3x+4) & \text{if } x > -1 \end{cases}$$

14. $\lim\limits_{x\to -2} f(x)$, where

$$f(x) = \begin{cases} x^2 - 2x - 2 & \text{if } x \le -2 \\ 2x+10 & \text{if } x > -2 \end{cases}$$

15. $\lim\limits_{x\to 0} f(x)$, where

$$f(x) = \begin{cases} \cos(x) & \text{if } x \le 0 \\ x^2 + 2x + 1 & \text{if } x > 0 \end{cases}$$

16. $\lim\limits_{x\to \frac{\pi}{6}} f(x)$, where $f(x) = \begin{cases} \sin(x) & x \le \frac{\pi}{6} \\ \cos(x) & x > \frac{\pi}{6} \end{cases}$

17. $\lim\limits_{x\to 0} |x|^x$

18. $\lim\limits_{x\to 0} e^{-e^{1/x}}$

19. $\lim\limits_{x\to -5} \lfloor |x| \rfloor !$, where $|x|$ is the absolute value of x, $\lfloor x \rfloor$ is the floor of x (the greatest integer less than or equal to x), and $x!$ is x factorial.

20. $\lim\limits_{x\to -1} \lfloor |x| \rfloor !$, where $|x|$ is the absolute value of x, $\lfloor x \rfloor$ is the floor of x (the greatest integer less than or equal to x), and $x!$ is x factorial.

Exercise Group. Approximate the limit of the difference quotient, $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$, using $h = \pm 0.1, \pm 0.01$.

21. $f(x) = 2 - 7x$, $a = 3$

22. $f(x) = 9x + 0.06$, $a = -1$

23. $f(x) = x^2 + 3x - 7$, $a = 1$

24. $f(x) = \frac{1}{x+1}$, $a = 2$

25. $f(x) = 5x - 4x^2 - 1$, $a = -3$

26. $f(x) = \ln(x)$, $a = 5$

27. $f(x) = \sin(x)$, $a = \pi$

28. $f(x) = \cos(x)$, $a = \pi$

1.2 Epsilon-Delta Definition of a Limit

This section introduces the formal definition of a limit. Many refer to this as "the epsilon-delta" definition, referring to the letters ε and δ of the Greek alphabet.

Note: the common phrase "the ε-δ definition" is read aloud as "the epsilon delta definition." The hyphen between ϵ and δ is not a minus sign.

Before we give the actual definition, let's consider a few informal ways of describing a limit. Given a function $y = f(x)$ and an x-value, c, we say that "the limit of the function f, as x approaches c, is a value L" if:

Tends	"y tends to L" as "x tends to c."
Approaches	"y approaches L" as "x approaches c."
Near	"y is near L" whenever "x is near c."

The problem with these definitions is that the words "tends," "approach," and especially "near" are not exact. In what way does the variable x tend to, or approach, c? How near do x and y have to be to c and L, respectively?

The definition we describe in this section comes from formalizing "Near". A quick restatement gets us closer to what we want:

youtu.be/watch?v=OGvDIXuWn0g

Figure 1.2.1 An informal definition of the limit

Tolerance Levels
If x is within a certain *tolerance level* of c, then the corresponding value $y = f(x)$ is within a certain *tolerance level* of L.

The traditional notation for the x-tolerance is the lowercase Greek letter delta, or δ, and the y-tolerance is denoted by lowercase epsilon, or ε. One more rephrasing of "Tolerance Levels" nearly gets us to the actual definition:

Named Tolerance Levels
If x is within δ units of c, then the corresponding value of y is within ε units of L.

We can write "x is within δ units of c" mathematically as

$$|x - c| < \delta,$$

which is equivalent to

$$c - \delta < x < c + \delta.$$

Letting the symbol "$\implies$" represent the word "implies," we can rewrite "Named Tolerance Levels" as

$$|x - c| < \delta \implies |y - L| < \varepsilon$$

or

$$c - \delta < x < c + \delta \implies L - \varepsilon < y < L + \varepsilon.$$

The point is that δ and ε, being tolerances, can be any positive (but typically small) values satisfying this implication. Finally, we have the formal definition of the limit with the notation seen in the previous section.

Definition 1.2.2 The Limit of a Function f at a point.

Let I be an open interval containing c, and let f be a function defined on I, except possibly at c. The statement that "the **limit** of $f(x)$, as x approaches c, is L" is denoted by

$$\lim_{x \to c} f(x) = L,$$

and means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all x in

I, where $x \neq c$, if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Mathematicians often enjoy writing ideas without using any words. Here is the wordless definition of the limit:

$$\lim_{x \to c} f(x) = L$$

$$\iff$$

$$\forall\,\varepsilon > 0, \exists\,\delta > 0 \text{ s.t. } 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon.$$

Note the order in which ε and δ are given. In the definition, the y-tolerance ε is given *first* and then the limit will exist *if* we can find an x-tolerance δ that works.

An example will help us understand this definition. Note that the explanation is long, but it will take one through all steps necessary to understand the ideas.

youtu.be/watch?v=npoSY-AFvOY

Figure 1.2.3 Video presentation of Definition 1.2.2

Example 1.2.4 Evaluating a limit using the definition.

Show that $\lim_{x \to 4} \sqrt{x} = 2$.

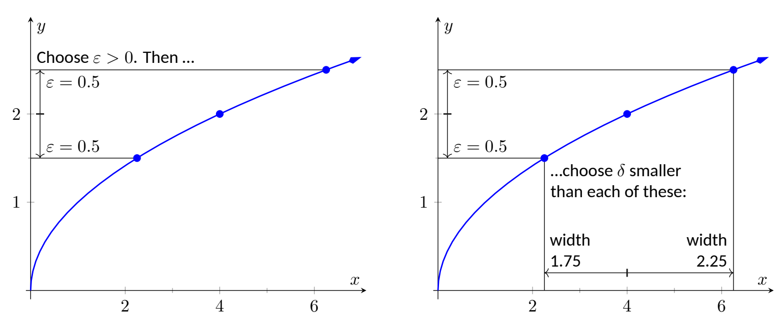
Solution. Before we use the formal definition, let's try some numerical tolerances. What if the y tolerance is 0.5, or in other words $\varepsilon = 0.5$? How close to 4 does x have to be so that y is within 0.5 units of 2? That is, $1.5 < y < 2.5$? In this case, we can proceed as follows:

$$\begin{aligned}
1.5 < y &< 2.5 \\
1.5 < \sqrt{x} &< 2.5 && \text{(Let } y = \sqrt{x}\text{)} \\
1.5^2 < x &< 2.5^2 && \text{(Square the inequality)} \\
2.25 < x &< 6.25 \\
2.25 - 4 < x - 4 &< 6.25 - 4 && \text{(Subtract } 4 \text{ from both sides)} \\
-1.75 < x - 4 &< 2.25
\end{aligned}$$

So, what is the desired x tolerance? Remember, we want to find a δ so that $|x - 4|$ is smaller than δ. Since $1.75 < 2.25$, then if we require $|x - 4| < 1.75$, then we have

$$\begin{aligned}
&|x - 4| < 1.75 \\
\implies\ &-1.75 < x - 4 < 1.75 < 2.25
\end{aligned}$$

Therefore we can have $\delta \leq 1.75$. See Figure 1.2.5.



(a) **(b)**

Figure 1.2.5 Illustrating the $\varepsilon - \delta$ process. With $\varepsilon = 0.5$, we pick any $\delta < 1.75$

Given the y tolerance $\varepsilon = 0.5$, we have found an x tolerance, $\delta < 1.75$, such that whenever x is within δ units of 4, then y is within ε units of 2. That's what we were trying to find.

Let's try another value of ε.

What if the y tolerance is 0.01, i.e. $\varepsilon = 0.01$? How close to 4 does x have to be in order for y to be within 0.01 units of 2? (In other words for $1.99 < y < 2.01$?) Again, we just square these values to get $1.99^2 < x < 2.01^2$, or

$$\begin{aligned} 3.9601 < x < 4.0401 \\ -0.0399 < x - 4 < 0.0401 \end{aligned}$$

What is the desired x tolerance? In this case we must have $\delta < 0.0399$, which is the minimum distance from 4 of the two bounds given above.

What we have so far: if $\varepsilon = 0.5$, then $\delta < 1.75$ leads to $f(x)$ being less than ε from $f(4)$ and if $\varepsilon = 0.01$, then $\delta < 0.0399$ being less than ε from $f(4)$. A pattern is not easy to see, so we switch to general ε try to determine an adequate δ symbolically. We start by assuming $y = \sqrt{x}$ is within ε units of 2:

$$\begin{aligned} |y - 2| &< \varepsilon \\ -\varepsilon < y - 2 &< \varepsilon \\ -\varepsilon < \sqrt{x} - 2 &< \varepsilon && (y = \sqrt{x}) \\ 2 - \varepsilon < \sqrt{x} &< 2 + \varepsilon && \text{(Add 2)} \\ (2-\varepsilon)^2 < x &< (2+\varepsilon)^2 && \text{(Square all)} \\ 4 - 4\varepsilon + \varepsilon^2 < x &< 4 + 4\varepsilon + \varepsilon^2 && \text{(Expand)} \\ -4\varepsilon + \varepsilon^2 < x - 4 &< 4\varepsilon + \varepsilon^2 && \text{(Subtract 4)} \end{aligned}$$

The "desired form" in the last step is "$4 -$ *something* $< x < 4 +$ *something*." Since we want this last interval to describe an x tolerance around 4, we have that either $\delta < 4\varepsilon - \varepsilon^2$ or $\delta < 4\varepsilon + \varepsilon^2$, whichever is smaller:

$$\delta < \min\{4\varepsilon - \varepsilon^2, 4\varepsilon + \varepsilon^2\}.$$

Since $\varepsilon > 0$, we have $4\varepsilon - \varepsilon^2 < 4\varepsilon + \varepsilon^2$, the minimum is $\delta \leq 4\varepsilon - \varepsilon^2$. That's the formula: given an ε, set $\delta \leq 4\varepsilon - \varepsilon^2$.

We can check this for our previous values. If $\varepsilon = 0.5$, the formula gives $\delta < 4(0.5) - (0.5)^2 = 1.75$ and when $\varepsilon = 0.01$, the formula gives $\delta < 4(0.01) - (0.01)^2 = 0.0399$.

So given any $\varepsilon > 0$, set $\delta < 4\varepsilon - \varepsilon^2$. Then if $|x - 4| < \delta$ (and $x \neq 4$), then $|f(x) - 2| < \varepsilon$, satisfying the definition of the limit. We have shown formally (and finally!) that $\lim_{x\to 4} \sqrt{x} = 2$.

Video solution

youtu.be/watch?v=qHWI0eha_rA

The previous example was a little long in that we sampled a few specific cases of ε before handling the general case. Normally this is not done. The previous example is also a bit unsatisfying in that $\sqrt{4} = 2$; why work so hard to prove something so obvious? Many ε-δ proofs are long and difficult to do. In this section, we will focus on examples where the answer is, frankly, obvious, because the non-obvious examples are even harder. In the next section we will learn some theorems that allow us to evaluate limits *analytically*, that is, without using the ε-δ definition.

Example 1.2.6 Evaluating a limit using the definition.

Show that $\lim_{x\to 2} x^2 = 4$.

Solution. Let's do this example symbolically from the start. Let $\varepsilon > 0$ be given; we want $|y-4| < \varepsilon$, i.e., $\left|x^2-4\right| < \varepsilon$. How do we find δ such that when $|x-2| < \delta$, we are guaranteed that $\left|x^2-4\right| < \varepsilon$?
This is a bit trickier than the previous example, but let's start by noticing that $\left|x^2-4\right| = |x-2|\cdot|x+2|$. Consider:

$$\left|x^2-4\right| < \varepsilon \implies |x-2|\cdot|x+2| < \varepsilon \implies |x-2| < \frac{\varepsilon}{|x+2|}.$$

Could we not set $\delta = \frac{\varepsilon}{|x+2|}$?
We are close to an answer, but the catch is that δ must be a *constant* value (so it can't depend on x). There is a way to work around this, but we do have to make an assumption. Remember that ε is supposed to be a small number, which implies that δ will also be a small value. In particular, we can (probably) assume that $\delta < 1$. If this is true, then $|x-2| < \delta$ would imply that $|x-2| < 1$, giving $1 < x < 3$.
Now, back to the fraction $\frac{\varepsilon}{|x+2|}$. If $1 < x < 3$, then $3 < x+2 < 5$ (add 2 to all terms in the inequality). Taking reciprocals, we have

$$\frac{1}{5} < \frac{1}{|x+2|} < \frac{1}{3},$$

which implies

$$\frac{1}{5} < \frac{1}{|x+2|},$$

which implies

$$\frac{\varepsilon}{5} < \frac{\varepsilon}{|x+2|}. \tag{1.2.1}$$

This suggests that we set $\delta < \frac{\varepsilon}{5}$. To see why, let consider what follows when we assume $|x-2| < \delta$:

$$\begin{aligned}
|x-2| &< \delta \\
|x-2| &< \frac{\varepsilon}{5} && \text{(Our choice of } \delta\text{)} \\
|x-2|\cdot|x+2| &< |x+2|\cdot\frac{\varepsilon}{5} && \text{(Multiply by } |x+2|\text{)} \\
\left|x^2-4\right| &< |x+2|\cdot\frac{\varepsilon}{5} && \text{(Simplify left side)} \\
\left|x^2-4\right| &< |x+2|\cdot\frac{\varepsilon}{|x+2|} && \text{(Inequality (1.2.1), } \delta < 1\text{)} \\
\left|x^2-4\right| &< \varepsilon
\end{aligned}$$

We have arrived at $\left|x^2-4\right| < \varepsilon$ as desired. Note again, in order to make this happen we needed δ to first be less than 1. That is a safe assumption; we want ε to be arbitrarily small, forcing δ to also be small.
We have also picked δ to be smaller than "necessary." We could get by with a slightly larger δ, as shown in Figure 1.2.7. The outer lines show the boundaries defined by our choice of ε. The inner lines show the boundaries defined by setting $\delta = \varepsilon/5$. Note how these dotted lines are within the dashed lines. That is perfectly fine; by choosing x within the dotted lines we are guaranteed that $f(x)$ will be within ε of 4.

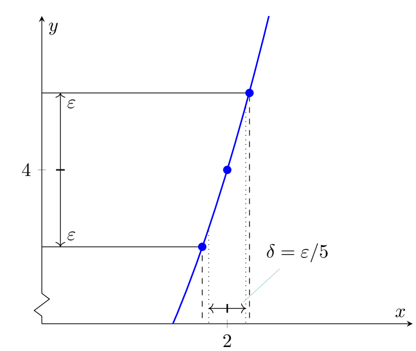


Figure 1.2.7 Choosing $\delta = \varepsilon/5$ in Example 1.2.6

In summary, given $\varepsilon > 0$, set $\delta = \varepsilon/5$. Then $|x-2| < \delta$ implies $\left|x^2-4\right| < \varepsilon$ (i.e. $|y-4| < \varepsilon$) as desired. This shows that $\lim_{x\to 2} x^2 = 4$. Figure 1.2.7 gives a visualization of this; by restricting x to values within $\delta = \varepsilon/5$ of 2, we see that $f(x)$ is within ε of 4.

Video solution

youtu.be/watch?v=QGqoq-xEXyk

Make note of the general pattern exhibited in these last two examples. In some sense, each starts out "backwards." That is, while we want to

1. start with $|x-c| < \delta$ and conclude that
2. $|f(x)-L| < \varepsilon$,

we actually start by doing what is essentially some "scratch-work" first:

1. assume $|f(x)-L| < \varepsilon$, then perform some algebraic manipulations to give an inequality of the form
2. $|x-c| <$ *something*.

When we have properly done this, the *something* on the "greater than" side of the inequality becomes our δ. We can refer to this as the "scratch-work" phase of our proof. Once we have δ, we can formally start the actual proof with $|x-c| < \delta$ and use algebraic manipulations to conclude that $|f(x)-L| < \varepsilon$, usually by using the same steps of our "scratch-work" in reverse order.

We highlight this process in the following example.

Example 1.2.8 Evaluating a limit using the definition.

Prove that $\lim\limits_{x\to 1}(x^3-2x) = -1$.

Solution. We start our scratch-work by considering $|f(x)-(-1)| < \varepsilon$:

$$\begin{aligned} |f(x)-(-1)| &< \varepsilon \\ \left|x^3-2x+1\right| &< \varepsilon && \text{(Now factor)} \\ \left|(x-1)(x^2+x-1)\right| &< \varepsilon \\ |x-1| &< \frac{\varepsilon}{\left|x^2+x-1\right|}. && (1.2.2) \end{aligned}$$

We are at the phase of saying that $|x-1| <$ *something*, where *something* $= \varepsilon/\left|x^2+x-1\right|$. We want to turn that *something* into δ.

Since x is approaching 1, we are safe to assume that x is between 0 and 2. So

$$\begin{aligned} 0 < x &< 2 \\ 0 < x^2 &< 4 && \text{(Squared each term.)} \end{aligned}$$

Since $0 < x < 2$, we can add 0, x and 2, respectively, to each part of the inequality and maintain the inequality.

$$\begin{aligned} 0 < x^2+x &< 6 \\ -1 < x^2+x-1 &< 5 && \text{(Subtracted 1 from each part.)} \end{aligned}$$

In Inequality (1.2.2), we wanted $|x-1| < \varepsilon/\left|x^2+x-1\right|$. The above shows that given any x in $[0,2]$, we know that

$$x^2+x-1 < 5 \qquad \text{which implies that}$$

$$\frac{1}{5} < \frac{1}{x^2+x-1} \qquad \text{which implies that}$$
$$\frac{\varepsilon}{5} < \frac{\varepsilon}{x^2+x-1} \qquad . \qquad (1.2.3)$$

So we set $\delta < \epsilon/5$. This ends our scratch-work, and we begin the formal proof (which also helps us understand why this was a good choice of δ). Given ε, let $\delta < \varepsilon/5$. We want to show that when $|x-1| < \delta$, then $\left|(x^3-2x)-(-1)\right| < \varepsilon$. We start with $|x-1| < \delta$:

$$\begin{aligned} |x-1| &< \delta \\ |x-1| &< \frac{\varepsilon}{5} \\ |x-1| &< \frac{\varepsilon}{|x^2+x-1|} && \text{(Inequality (1.2.3), } x \text{ near 1)} \\ |x-1| \cdot \left|x^2+x-1\right| &< \varepsilon \\ \left|x^3-2x+1\right| &< \varepsilon \\ \left|(x^3-2x)-(-1)\right| &< \varepsilon, \end{aligned}$$

which is what we wanted to show. Thus $\lim_{x\to 1}(x^3-2x) = -1$.

Video solution

youtu.be/watch?v=--_Rq2GX9IY

We illustrate evaluating limits once more.

Example 1.2.9 Evaluating a limit using the definition.

Prove that $\lim\limits_{x\to 0} e^x = 1$.

Solution. Symbolically, we want to take the inequality $|e^x - 1| < \varepsilon$ and unravel it to the form $|x-0| < \delta$. Here is our scratch-work:

$$\begin{aligned} |e^x-1| &< \varepsilon \\ -\varepsilon < e^x - 1 &< \varepsilon && \text{(Definition of absolute value)} \\ 1-\varepsilon < e^x &< 1+\varepsilon && \text{(Add 1)} \\ \ln(1-\varepsilon) < x &< \ln(1+\varepsilon) && \text{(Take natural logs)} \end{aligned}$$

Making the safe assumption that $\varepsilon < 1$ ensures the last inequality is valid (i.e., so that $\ln(1-\varepsilon)$ is defined). We can then set δ to be the minimum of $|\ln(1-\varepsilon)|$ and $\ln(1+\varepsilon)$; i.e.,

$$\delta = \min\{|\ln(1-\varepsilon)|, \ln(1+\varepsilon)\} = \ln(1+\varepsilon).$$

Recall $\ln 1 = 0$ and $\ln x < 0$ when $0 < x < 1$. So $\ln(1-\varepsilon)$ is negative because $1-\varepsilon < 1$; hence we consider its absolute value:

$$\begin{aligned} &|\ln(1-\varepsilon)| \\ &\quad = -\ln(1-\varepsilon) \\ &\quad = \ln\left(\frac{1}{1-\varepsilon}\right). \end{aligned}$$

To determine which is smaller between $|\ln(1-\varepsilon)|$ and $\ln(1+\varepsilon)$ amounts to determining which is smaller between $\frac{1}{1-\varepsilon}$ and $1+\varepsilon$. But

$$\begin{aligned} &(1+\varepsilon)/\left(\frac{1}{1-\varepsilon}\right) \\ &\quad = (1+\varepsilon)(1-\varepsilon) \\ &\quad = 1-\varepsilon^2 < 1, \end{aligned}$$

so $(1+\varepsilon) < \frac{1}{1-\varepsilon}$. And therefore $\ln(1+\varepsilon) < |\ln(1-\varepsilon)|$.

Now, we work through the actual the proof:

$$\begin{aligned} |x-0| &< \delta \\ -\delta < x &< \delta && \text{(Definition of absolute value)} \\ -\ln(1+\varepsilon) < x &< \ln(1+\varepsilon) \\ \ln(1-\varepsilon) < x &< \ln(1+\varepsilon) && \text{(since } \ln(1-\varepsilon) < -\ln(1+\varepsilon)\text{).} \end{aligned}$$

The above line is true by our choice of δ and by the fact that since $|\ln(1-\varepsilon)| > \ln(1+\varepsilon)$ and $\ln(1-\varepsilon) < 0$, we know $\ln(1-\varepsilon) < -\ln(1+\varepsilon)$.

$$1-\varepsilon < e^x < 1+\varepsilon \qquad \text{(Exponentiate)}$$

$$-\varepsilon < e^x - 1 < \varepsilon \qquad \text{(Subtract 1)}$$

In summary, given $\varepsilon > 0$, let $\delta = \ln(1+\varepsilon)$. Then $|x-0| < \delta$ implies $|e^x - 1| < \varepsilon$ as desired. We have shown that $\lim_{x\to 0} e^x = 1$.

We note that we could actually show that $\lim_{x\to c} e^x = e^c$ for any constant c. We do this by factoring out e^c from both sides, leaving us to show $\lim_{x\to c} e^{x-c} = 1$ instead. By using the substitution $u = x - c$, this reduces to showing $\lim_{u\to 0} e^u = 1$ which we just did in the last example. As an added benefit, this shows that in fact the function $f(x) = e^x$ is *continuous* at all values of x, an important concept we will define in Section 1.5.

This formal definition of the limit is not an easy concept grasp. Our examples are actually "easy" examples, using "simple" functions like polynomials, square roots and exponentials. It is very difficult to prove, using the techniques given above, that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, as we approximated in Section 1.1.

There is hope. Section 1.3 shows how one can evaluate complicated limits using certain basic limits as building blocks. While limits are an incredibly important part of calculus (and hence much of higher mathematics), rarely are limits evaluated using the definition. Rather, the techniques of Section 1.3 are employed.

1.2.1 Exercises

Terms and Concepts

1. What is wrong with the following “definition” of a limit?
“The limit of $f(x)$, as x approaches a, is K” means that given any $\delta > 0$ there exists $\varepsilon > 0$ such that whenever $|f(x) - K| < \varepsilon$, we have $|x - a| < \delta$.

2. Which is given first in establishing a limit?
(☐ x-tolerance ☐ y-tolerance)

3. (☐ True ☐ False) ε must always be positive.

4. (☐ True ☐ False) δ must always be positive.

Problems

Exercise Group. Prove the given limit using an ε-δ proof.

5. $\lim\limits_{x \to 4} (2x + 5) = 13$

6. $\lim\limits_{x \to 5} (3 - x) = -2$

7. $\lim\limits_{x \to 3} \left(x^2 - 3\right) = 6$

8. $\lim\limits_{x \to 4} \left(x^2 + x - 5\right) = 15$

9. $\lim\limits_{x \to 1} \left(2x^2 + 3x + 1\right) = 6$

10. $\lim\limits_{x \to 2} \left(x^3 - 1\right) = 7$

11. $\lim\limits_{x \to 2} 5 = 5$

12. $\lim\limits_{x \to 0} \left(e^{2x} - 1\right) = 0$

13. $\lim\limits_{x \to 1} \frac{1}{x} = 1$

14. $\lim\limits_{x \to 0} \sin(x) = 0$

1.3 Finding Limits Analytically

In Section 1.1 we explored the concept of the limit without a strict definition, meaning we could only make approximations. In the previous section we gave the definition of the limit and demonstrated how to use it to verify our approximations were correct. Thus far, our method of finding a limit is

1. make a really good approximation either graphically or numerically, and
2. verify our approximation is correct using a ε-δ proof.

Recognizing that ε-δ proofs are cumbersome, this section gives a series of theorems which allow us to find limits much more quickly and intuitively.

Suppose that $\lim_{x\to 2} f(x) = 2$ and $\lim_{x\to 2} g(x) = 3$. What is $\lim_{x\to 2}(f(x)+g(x))$? Intuition tells us that the limit should be 5, as we expect limits to behave in a nice way. The following theorem states that already established limits do behave nicely.

Theorem 1.3.1 Basic Limit Properties.

Let b, c, L and K be real numbers, let n be a positive integer, and let f and g be functions defined on an open interval I containing c with the following limits:

$$\lim_{x\to c} f(x) = L \qquad \lim_{x\to c} g(x) = K.$$

The following limits hold.

Constants $\lim_{x\to c} b = b$

Identity $\lim_{x\to c} x = c$

Sums/Differences $\lim_{x\to c} (f(x) \pm g(x)) = L \pm K$

Scalar Multiples $\lim_{x\to c} (b \cdot f(x)) = bL$

Products $\lim_{x\to c} (f(x) \cdot g(x)) = LK$

Quotients $\lim_{x\to c} (f(x)/g(x)) = L/K$, *when* $K \neq 0$

Powers $\lim_{x\to c} f(x)^n = L^n$

Roots $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$
(If n is even then require $f(x) \geq 0$ on I.)

Compositions *Adjust the limit requirements to*

$$\lim_{x\to c} f(x) = L \qquad \lim_{x\to L} g(x) = K \qquad g(L) = K.$$

Then $\lim_{x\to c} g(f(x)) = K$.

Many people like to remember the Sum Property as stating that "the limit of the sum is the sum of the limits", and the Product Property as stating that the "limit of a product is the product of the limits."

In practice, the Scalar Multiple Property is often viewed as telling us that we can "take constants out of limits":

$$\lim_{x\to c} (b \cdot f(x)) = b \cdot \lim_{x\to c} f(x).$$

youtu.be/watch?v=da2vdsxd2Fs

Figure 1.3.2 Video presentation of Theorem 1.3.1 (three videos)

We apply the theorem to an example.

Example 1.3.3 Using basic limit properties.

Let

$$\lim_{x\to 2} f(x) = 2 \qquad \lim_{x\to 2} g(x) = 3 \qquad p(x) = 3x^2 - 5x + 7.$$

Find the following limits:

(a) $\lim\limits_{x\to 2}(f(x) + g(x))$ (b) $\lim\limits_{x\to 2}(5f(x) + g(x)^2)$ (c) $\lim\limits_{x\to 2} p(x)$

Solution.

(a) Using the Sums/Differences property, we know that

$$\begin{aligned}\lim_{x\to 2}(f(x)+g(x)) &= \lim_{x\to 2} f(x) + \lim_{x\to 2} g(x)\\ &= 2+3=5.\end{aligned}$$

(b) Using the Scalar Multiples, Sums/Differences, and Powers properties, we find that

$$\begin{aligned}\lim_{x\to 2}(5f(x)+g(x)^2) &= \lim_{x\to 2}(5f(x)) + \lim_{x\to 2}(g(x)^2)\\ &= 5\lim_{x\to 2} f(x) + \left(\lim_{x\to 2} g(x)\right)^2\\ &= 5\cdot 2+3^2=19.\end{aligned}$$

(c) Here we combine the Powers, Scalar Multiples, Sums/Differences and Constants properties. We show quite a few steps, but in general these can be omitted:

$$\begin{aligned}\lim_{x\to 2} p(x) &= \lim_{x\to 2}\left(3x^2-5x+7\right)\\ &= \lim_{x\to 2}\left(3x^2\right) - \lim_{x\to 2}(5x) + \lim_{x\to 2} 7\\ &= 3\left(\lim_{x\to 2} x\right)^2 - 5\lim_{x\to 2}(x) + 7\\ &= 3\cdot 2^2 - 5\cdot 2+7\\ &= 9\end{aligned}$$

Video solution

youtu.be/watch?v=8x42kGfu9ts

Part c of the previous example demonstrates how the limit of a quadratic polynomial can be determined using the properties of Theorem 1.3.1. Not only that, recognize that

$$\lim_{x\to 2} p(x) = 9 = p(2);$$

i.e., the limit at 2 could have been found just by plugging 2 into the function. This holds true for all polynomials, and also for rational functions (which are quotients of polynomials), as stated in the following theorem.

Theorem 1.3.4 Limits of Polynomial and Rational Functions.

Let $p(x)$ and $q(x)$ be polynomials and c a real number. Then:

1. $\lim\limits_{x\to c} p(x) = p(c)$
2. $\lim\limits_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$, *when* $q(c)\neq 0$.

youtu.be/watch?v=NGoUHZESPso

Figure 1.3.5 Video presentation of Theorem 1.3.4 and Theorem 1.3.20

Example 1.3.6 Finding a limit of a rational function.

Using Theorem 1.3.4, find

$$\lim_{x\to -1} \frac{3x^2-5x+1}{x^4-x^2+3}.$$

Solution. Using Theorem 1.3.4, we can quickly state that

$$\begin{aligned}\lim_{x\to -1} \frac{3x^2-5x+1}{x^4-x^2+3} &= \frac{3(-1)^2-5(-1)+1}{(-1)^4-(-1)^2+3}\\ &= \frac{9}{3} = 3.\end{aligned}$$

It was likely frustrating in Section 1.2 to do a lot of work with ε and δ to prove that

$$\lim_{x\to 2} x^2 = 4$$

as it seemed fairly obvious. The previous theorems state that many functions behave in such an "obvious" fashion, as demonstrated by the rational function in Example 1.3.6.

Polynomial and rational functions are not the only functions to behave in such a predictable way. The following theorem gives a list of functions whose behavior is particularly "nice" in terms of limits. In Section 1.5, we will give a formal name to these functions that behave "nicely."

Theorem 1.3.7 Limits of Common Functions.

Let c be a real number in the domain of the given function and let n be a positive integer. The following limits hold:

1. $\lim\limits_{x\to c} \sin(x) = \sin(c)$
2. $\lim\limits_{x\to c} \cos(x) = \cos(c)$
3. $\lim\limits_{x\to c} \tan(x) = \tan(c)$
4. $\lim\limits_{x\to c} \csc(x) = \csc(c)$
5. $\lim\limits_{x\to c} \sec(x) = \sec(c)$
6. $\lim\limits_{x\to c} \cot(x) = \cot(c)$
7. $\lim_{x\to c} a^x = a^c$, if $a > 0$
8. $\lim\limits_{x\to c} \ln(x) = \ln(c)$
9. $\lim\limits_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$

(Item 9 follows from the Identity and Roots rules.)

Example 1.3.8 Evaluating limits analytically.

Evaluate the following limits.

(a) $\lim\limits_{x\to \pi} \cos(x)$

(b) $\lim\limits_{x\to 3} \left(\sec^2(x) - \tan^2(x)\right)$

(c) $\lim\limits_{x\to \pi/2} (\cos(x)\sin(x))$

(d) $\lim\limits_{x\to 1} e^{\ln(x)}$

(e) $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x}$

Solution.

(a) This is a straightforward application of Theorem 1.3.7: $\lim_{x\to\pi} \cos(x) = \cos(\pi) = -1$.

(b) We can approach this in at least two ways. First, by directly applying Theorem 1.3.7, we have:

$$\lim_{x\to 3} \left(\sec^2(x) - \tan^2(x)\right) = \sec^2(3) - \tan^2(3).$$

Using the Pythagorean Theorem, this last expression is 1; therefore

$$\lim_{x\to 3} \left(\sec^2(x) - \tan^2(x)\right) = 1.$$

We can also use the Pythagorean Theorem from the start.

$$\lim_{x\to 3} \left(\sec^2(x) - \tan^2(x)\right) = \lim_{x\to 3} 1 = 1,$$

using the Constants rule. Either way, we find the limit is 1.

(c) Applying the Products rule and Theorem 1.3.7 gives

$$\lim_{x\to\pi/2} \cos(x)\sin(x) = \cos(\pi/2)\sin(\pi/2) = 0\cdot 1 = 0.$$

(d) Again, we can approach this in two ways. First, we can use the exponential/logarithmic identity that $e^{\ln(x)} = x$ and evaluate $\lim_{x\to 1} e^{\ln(x)} = \lim_{x\to 1} x = 1$.

We can also use the Compositions rule. Using Theorem 1.3.7, we have $\lim_{x\to 1} \ln(x) = \ln(1) = 0$ and $\lim_{x\to 0} e^x = e^0 = 1$, satisfying the conditions of the Compositions rule. Applying this rule,

$$\lim_{x\to 1} e^{\ln(x)} = e^{\lim_{x\to 1}\ln(x)} = e^{\ln(1)} = e^0 = 1.$$

Both approaches are valid, giving the same result.

(e) We encountered this limit in Section 1.1. Applying our theorems, we attempt to find the limit as

$$\lim_{x\to 0} \frac{\sin(x)}{x} \to \frac{\sin(0)}{0},$$

which is of the form $\frac{0}{0}$. This, of course, violates a condition of the Quotients rule, as the limit of the denominator is not allowed to be 0. Therefore, we are still unable to evaluate this limit with tools we currently have at hand.

Based on what we've done so far, this section could have been titled "Using Known Limits to Find Unknown Limits." By knowing certain limits of functions, we can find limits involving sums, products, powers, etc., of these functions. We further the development of such comparative tools with the Squeeze Theorem, a clever and intuitive way to find the value of some limits.

Before stating this theorem formally, suppose we have functions f, g, and h where g always takes on values between f and h; that is, for all x in an interval,

$$f(x) \leq g(x) \leq h(x).$$

If f and h have the same limit at c, and g is always "squeezed" between them, then g must have the same limit as well. That is what the Squeeze Theorem states. This is illustrated in Figure 1.3.9.

Theorem 1.3.10 Squeeze Theorem.

Let f, g and h be functions on an open interval I containing c such that for all x in I,

$$f(x) \leq g(x) \leq h(x).$$

If

$$\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x),$$

then

$$\lim_{x\to c} g(x) = L.$$

It can take some work to figure out appropriate functions by which to "squeeze" a given function. However, that is generally the only place where work is necessary; the theorem makes the "evaluating the limit part" very simple.

The Squeeze Theorem can be used to show that limits of $\sin(x)$ can be done by direct substitution, as the videos in Figure 1.3.12 illustrate.

We use the Squeeze Theorem in the following example to finally prove that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

Example 1.3.13 Using the Squeeze Theorem.

Use the Squeeze Theorem to show that

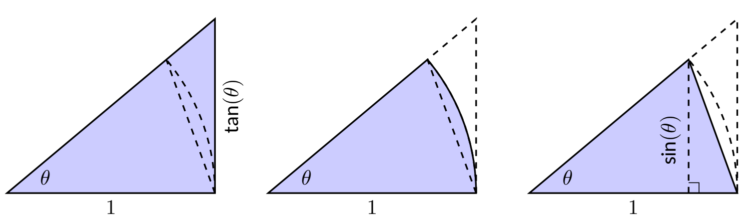
$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

Solution. We begin by considering the unit circle. Each point on the unit circle has coordinates $(\cos(\theta), \sin(\theta))$ for some angle θ as shown in Figure 1.3.14. Using similar triangles, we can extend the line from the origin through the point to the point $(1, \tan(\theta))$, as shown. (Here we are assuming that $0 \leq \theta \leq \pi/2$. Later we will show that we can also consider $\theta \leq 0$.)

Figure 1.3.14 shows three regions have been constructed in the first quadrant, two triangles and a sector of a circle, which are also drawn below. The area of the large triangle is $\frac{1}{2}\tan(\theta)$; the area of the sector is $\theta/2$; the area of the triangle contained inside the sector is $\frac{1}{2}\sin(\theta)$. It is then clear from Figure 1.3.15 that

$$\frac{\tan(\theta)}{2} \geq \frac{\theta}{2} \geq \frac{\sin(\theta)}{2}.$$

(You may need to recall that the area of a sector of a circle is $\frac{1}{2}r^2\theta$ with θ measured in radians.)



(a) **(b)** **(c)**

Figure 1.3.15 Bounding the sector between two triangles

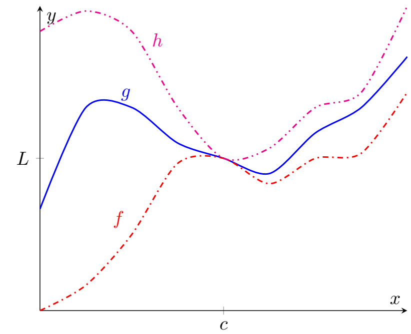


Figure 1.3.9 An illustration of the Squeeze Theorem

youtu.be/watch?v=8Tv-GRQdAVA

Figure 1.3.11 Explaining the Squeeze Theorem

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Figure 1.3.12 Using the Squeeze Theorem to take the limit of $\sin(x)$ at 0

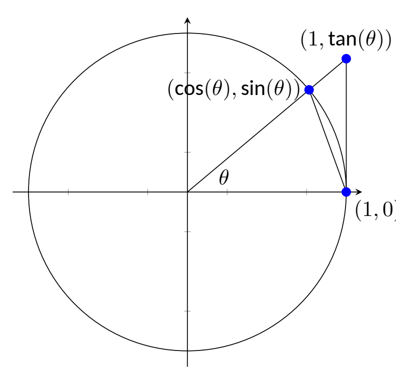


Figure 1.3.14 The unit circle and related triangles

Multiply all terms by $\frac{2}{\sin(\theta)}$, giving

$$\frac{1}{\cos(\theta)} \geq \frac{\theta}{\sin(\theta)} \geq 1.$$

Taking reciprocals reverses the inequalities, giving

$$\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq 1.$$

(These inequalities hold for all values of θ near 0, even negative values, since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$.)
Now take limits.

$$\lim_{\theta\to 0} \cos(\theta) \leq \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} \leq \lim_{\theta\to 0} 1$$
$$\cos(0) \leq \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} \leq 1$$
$$1 \leq \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} \leq 1$$

Clearly this means that $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$.

Video solution

youtu.be/watch?v=pgjv3ojtXh4

With the limit $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$ finally established, we can move on to other limits involving trigonometric functions, as the video in Figure 1.3.16 demonstrates.

Two notes about the Example 1.3.13 are worth mentioning. First, one might be discouraged by this application, thinking "I would *never* have come up with that on my own. This is too hard!" Don't be discouraged; within this text we will guide you in your use of the Squeeze Theorem. As one gains mathematical maturity, clever proofs like this are easier and easier to create.

Second, this limit tells us more than just that as x approaches 0, $\sin(x)/x$ approaches 1. Both x and $\sin(x)$ are approaching 0, but the *ratio* of x and $\sin(x)$ approaches 1, meaning that they are approaching 0 in essentially the same way. Another way of viewing this is: for small x, the functions $y = x$ and $y = \sin(x)$ are essentially indistinguishable.

We include this special limit, along with three others, in the following theorem.

youtu.be/watch?v=Wd464IIls5Y

Figure 1.3.16 Finding limits involving trigonometric functions

Theorem 1.3.17 Special Limits.

1. $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$
2. $\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$
3. $\lim_{x\to 0} (1+x)^{1/x} = e$
4. $\lim_{x\to 0} \frac{e^x-1}{x} = 1$

A short word on how to interpret the latter three limits. We know that as x goes to 0, $\cos(x)$ goes to 1. So, in the second limit, both the numerator and denominator are approaching 0. However, since the limit is 0, we can interpret this as saying that "$\cos(x)$ is approaching 1 faster than x is approaching 0."

In the third limit, inside the parentheses we have an expression that is approaching 1 (though never equaling 1), and we know that 1 raised to any power is still 1. At the same time, the power is growing toward infinity. What happens to a number near 1 raised to a very large power? In this particular case, the result approaches Euler's number, e, approximately 2.718.

In the fourth limit, we see that as $x \to 0$, e^x approaches 1 "just as fast" as $x \to 0$, resulting in a limit of 1.

The special limits stated in Theorem 1.3.17 are called *indeterminate forms*; in this case they are of the form $0/0$, except the third limit, which is of a different form. You'll learn techniques to find these limits exactly using calculus in Section 6.7.

Our final theorem for this section will be motivated by the following example.

Example 1.3.18 Using algebra to evaluate a limit.

Evaluate the following limit:

$$\lim_{x\to 1} \frac{x^2-1}{x-1}.$$

Solution. We begin by attempting to apply Theorem 1.3.4 and substituting 1 for x in the quotient. This gives:

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \frac{1^2-1}{1-1}$$

which is of the form $\frac{0}{0}$, an indeterminate form. We cannot apply the theorem.

By graphing the function, as in Figure 1.3.19, we see that the function seems to be linear, implying that the limit should be easy to evaluate. Recognize that the numerator of our quotient can be factored:

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1}.$$

The function is not defined when $x = 1$, but for all other x,

$$\begin{aligned} \frac{x^2-1}{x-1} &= \frac{(x-1)(x+1)}{x-1} \\ &= \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} \\ &= x+1, \quad \text{if } x \neq 1 \end{aligned}$$

Clearly $\lim\limits_{x\to 1}(x+1) = 2$. Recall that when considering limits, we are not concerned with the value of the function at 1, only the value the function approaches as x approaches 1. Since $(x^2-1)/(x-1)$ and $x+1$ are the same at all points except at $x = 1$, they both approach the same value as x approaches 1. Therefore we can conclude that

$$\begin{aligned} \lim_{x\to 1} \frac{x^2-1}{x-1} &= \lim_{x\to 1}(x+1) \\ &= 2 \end{aligned}$$

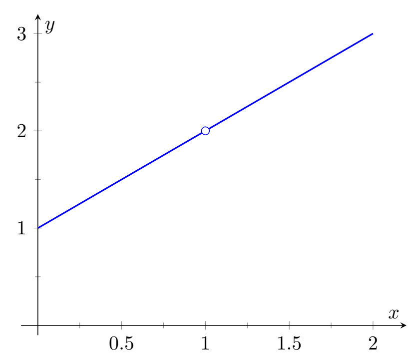


Figure 1.3.19 Graphing f in Example 1.3.18 to understand a limit

The key to Example 1.3.18 is that the functions $y = (x^2-1)/(x-1)$ and $y = x+1$ are identical except at $x = 1$. Since limits describe a value the function is approaching, not the value the function actually attains, the limits of the two functions are always equal.

Theorem 1.3.20 Limits of Functions Equal At All But One Point.

Let $g(x) = f(x)$ for all x in an open interval, except possibly at c, and let $\lim_{x\to c} g(x) = L$ for some real number L. Then

$$\lim_{x\to c} f(x) = L.$$

The Fundamental Theorem of Algebra tells us that when dealing with a rational function of the form $g(x)/f(x)$ and directly evaluating the limit $\lim_{x\to c} \frac{g(x)}{f(x)}$ returns "0/0", then $(x - c)$ is a factor of both $g(x)$ and $f(x)$. One can then use algebra to factor this binomial out, cancel, then apply Theorem 1.3.20. We demonstrate this once more.

Example 1.3.21 Evaluating a limit using Theorem 1.3.20.

Evaluate

$$\lim_{x\to 3} \frac{x^3 - 2x^2 - 5x + 6}{2x^3 + 3x^2 - 32x + 15}.$$

Solution. We attempt to apply Theorem 1.3.4 by substituting 3 for x. This returns the familiar indeterminate form of "0/0". Since the numerator and denominator are each polynomials, we know that $(x - 3)$ is factor of each. Using whatever method is most comfortable to you, factor out $(x - 3)$ from each (using polynomial division, synthetic division, a computer algebra system, etc.). We find that

$$\frac{x^3 - 2x^2 - 5x + 6}{2x^3 + 3x^2 - 32x + 15} = \frac{(x - 3)\left(x^2 + x - 2\right)}{(x - 3)\left(2x^2 + 9x - 5\right)}.$$

We can cancel the $(x-3)$ factors as long as $x \neq 3$. Using Theorem 1.3.20 we conclude:

$$\begin{aligned}\lim_{x\to 3} \frac{x^3 - 2x^2 - 5x + 6}{2x^3 + 3x^2 - 32x + 15} &= \lim_{x\to 3} \frac{(x - 3)\left(x^2 + x - 2\right)}{(x - 3)\left(2x^2 + 9x - 5\right)}\\ &= \lim_{x\to 3} \frac{x^2 + x - 2}{2x^2 + 9x - 5}\\ &= \frac{10}{40}\\ &= \frac{1}{4}.\end{aligned}$$

Video solution

youtu.be/watch?v=MSckoYqdKH4

Example 1.3.22 Evaluating a Limit with a Hole.

Evaluate

$$\lim_{x\to 9} \frac{\sqrt{x} - 3}{x - 9}.$$

Solution. We begin by trying to apply the Quotients limit rule, but the denominator evaluates to zero. In fact, this limit is of the indeterminate form $0/0$. We will do some algebra to resolve the indeterminate form. In this case, we multiply the numerator and denominator by the conjugate of the numerator.

$$\frac{\sqrt{x}-3}{x-9} = \frac{\sqrt{x}-3}{x-9} \cdot \frac{(\sqrt{x}+3)}{(\sqrt{x}+3)}$$
$$= \frac{x-9}{(x-9)(\sqrt{x}+3)}$$

We can cancel the $(x-9)$ factors as long as $x \neq 9$. Using Theorem 1.3.20 we conclude:

$$\begin{aligned}\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} &= \lim_{x\to 9} \frac{x-9}{(x-9)\left(\sqrt{x}+3\right)}\\ &= \lim_{x\to 9} \frac{1}{\sqrt{x}+3}\\ &= \frac{1}{\lim_{x\to 9}\sqrt{x} + \lim_{x\to 9} 3}\\ &= \frac{1}{\sqrt{\lim_{x\to 9} x} + 3}\\ &= \frac{1}{\sqrt{3}+3}\\ &= \frac{1}{6}.\end{aligned}$$

Video solution

youtu.be/watch?v=vOW92eipOu4

We end this section by revisiting a limit first seen in Section 1.1, a limit of a difference quotient. Let $f(x) = -1.5x^2 + 11.5x$; we approximated the limit $\lim\limits_{h\to 0} \frac{f(1+h)-f(1)}{h} \approx 8.5$. We formally evaluate this limit in the following example.

Example 1.3.23 Evaluating the limit of a difference quotient.

Let $f(x) = -1.5x^2 + 11.5x$; find $\lim\limits_{h\to 0} \frac{f(1+h)-f(1)}{h}$.

Solution. Since f is a polynomial, our first attempt should be to employ Theorem 1.3.4 and substitute 0 for h. However, we see that this gives us "$0/0$." Knowing that we have a rational function hints that some algebra will help. Consider the following steps:

$$\begin{aligned}\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0} \frac{-1.5(1+h)^2 + 11.5(1+h) - \left(-1.5(1)^2 + 11.5(1)\right)}{h}\\ &= \lim_{h\to 0} \frac{-1.5(1+2h+h^2) + 11.5 + 11.5h - 10}{h}\\ &= \lim_{h\to 0} \frac{-1.5h^2 + 8.5h}{h}\\ &= \lim_{h\to 0} \frac{h(-1.5h+8.5)}{h}\\ &= \lim_{h\to 0} (-1.5h + 8.5) \quad \text{(using Theorem 1.3.20, as } h \neq 0\text{)}\\ &= 8.5 \quad \text{(using Theorem 1.3.4)}\end{aligned}$$

This matches our previous approximation.

This section contains several valuable tools for evaluating limits. One of the main results of this section is Theorem 1.3.7; it states that many functions that we use regularly behave in a very nice, predictable way. In Section 1.5 we give a name to this nice behavior; we label such functions as **continuous**. Defining that term will require us to look again at what a limit is and what causes limits to not exist.

1.3.1 Exercises

Terms and Concepts

1. Explain in your own words, without using ε-δ formality, why $\lim\limits_{x\to c} b = b$.

2. Explain in your own words, without using ε-δ formality, why $\lim\limits_{x\to c} x = c$.

3. What does the text mean when it says that certain functions' "behavior is 'nice' in terms of limits"? What, in particular, is "nice"?

4. Sketch a graph that visually demonstrates the Squeeze Theorem.

5. You are given the following information:

$$\lim_{x\to 1} f(x) = 0 \quad \lim_{x\to 1} g(x) = 0 \quad \lim_{x\to 1} \frac{f(x)}{g(x)} = 2$$

What can be said about the relative sizes of $f(x)$ and $g(x)$ as x approaches 1?

6. (☐ True ☐ False) $\lim\limits_{x\to 1} \ln x = 0$.

Problems

Exercise Group. Use the following information to evaluate the given limit, when possible.

$$\lim_{x\to 9} f(x) = 6 \qquad \lim_{x\to 6} f(x) = 9 \qquad f(9) = 6$$
$$\lim_{x\to 9} g(x) = 3 \qquad \lim_{x\to 6} g(x) = 3 \qquad g(6) = 3$$

7. $\lim\limits_{x\to 9} (f(x) + g(x))$

8. $\lim\limits_{x\to 9} \left(\frac{3f(x)}{g(x)}\right)$

9. $\lim\limits_{x\to 9} \left(\frac{f(x)-2g(x)}{g(x)}\right)$

10. $\lim\limits_{x\to 6} \left(\frac{f(x)}{3-g(x)}\right)$

11. $\lim\limits_{x\to 9} g(f(x))$

12. $\lim\limits_{x\to 6} f(g(x))$

13. $\lim\limits_{x\to 6} g(f(f(x)))$

14. $\lim\limits_{x\to 6} \left(f(x)g(x) - f(x)^2 + g(x)^2\right)$

Exercise Group. Use the following information to evaluate the given limit, when possible. If it is not possible to determine the limit, state why not.

$$\lim_{x\to 1} f(x) = 2 \qquad \lim_{x\to 10} f(x) = 1 \qquad f(1) = 1/5$$
$$\lim_{x\to 1} g(x) = 0 \qquad \lim_{x\to 10} g(x) = \pi \qquad g(10) = \pi$$

15. $\lim\limits_{x\to 1} (f(x)g(x))$

16. $\lim\limits_{x\to 10} \cos(g(x))$

17. $\lim\limits_{x\to 1} g(5f(x))$

18. $\lim\limits_{x\to 1} 5^{g(x)}$

Exercise Group. Evaluate the given limit.

19. $\lim\limits_{x\to 6} \left(x^2 - 3x + 5\right)$

20. $\lim\limits_{x\to \pi} \left(\frac{x-5}{x-8}\right)^4$

21. $\lim\limits_{x\to \frac{\pi}{6}} \cos(x)\sin(x)$

22. $\lim\limits_{x\to 6} \frac{-(5x+2)}{x+4}$

23. $\lim\limits_{x\to 0} \ln(x)$

24. $\lim\limits_{x\to 2} 4^{x^3-2x}$

25. $\lim\limits_{x\to \frac{\pi}{3}} \csc(x)$

26. $\lim\limits_{x\to 0} \ln(4+x)$

27. $\lim\limits_{x\to\pi} \frac{x^2-4x-2}{2x^2-2x+1}$

28. $\lim\limits_{x\to\pi} \frac{2x-4}{5x-5}$

29. $\lim\limits_{x\to 5} \frac{x^2-11x+30}{x^2-14x+45}$

30. $\lim\limits_{x\to 0} \frac{x^2-7x}{x^2+2x}$

31. $\lim\limits_{x\to 9} \frac{x^2-x-72}{x^2-14x+45}$

32. $\lim\limits_{x\to -8} \frac{x^2+3x-40}{x^2+13x+40}$

33. $\lim\limits_{x\to -6} \frac{x^2+8x+12}{x^2+3x-18}$

34. $\lim\limits_{x\to -4} \frac{x^2+13x+36}{x^2+12x+32}$

Exercise Group. Use the Squeeze Theorem to evaluate the limit.

35. $\lim\limits_{x\to 0} \left(x \sin\left(\frac{1}{x}\right)\right)$

36. $\lim\limits_{x\to 0} \left(\sin(x)\cos\left(\frac{1}{x^2}\right)\right)$

37. $\lim\limits_{x\to 1} f(x)$, where $3x-2 \leq f(x) \leq x^3$

38. $\lim\limits_{x\to 3} f(x)$, where $6x-9 \leq f(x) \leq x^2$

Exercise Group. The following exercises challenge your understanding of limits but can be evaluated using the knowledge gained in Section 1.3.

39. $\lim\limits_{x\to 0} \frac{\sin(8x)}{x}$

40. $\lim\limits_{x\to 0} \frac{\sin(9x)}{8x}$

41. $\lim\limits_{x\to 0} \frac{\ln(1+x)}{x}$

42. $\lim\limits_{x\to 0} \frac{\sin(x)}{x}$, where x is measured in degrees, not radians.

43. Let $f(x) = 0$ and $g(x) = \frac{x}{x}$.

(a) Explain why $\lim\limits_{x\to 2} f(x) = 0$.

(b) Explain why $\lim\limits_{x\to 0} g(x) = 1$.

(c) Explain why $\lim\limits_{x\to 2} g(f(x))$ does not exist.

(d) Explain why the previous statement does not violate the Composition Rule of Theorem 1.3.1.

1.4 One-Sided Limits

We introduced the concept of a limit gently, approximating their values graphically and numerically. Next came the rigorous definition of the limit, along with an admittedly tedious method for evaluating them. Section 1.3 gave us tools (which we call theorems) that allow us to compute limits with greater ease. Chief among the results were the facts that polynomials and rational, trigonometric, exponential and logarithmic functions (and their sums, products, etc.) all behave "nicely." In this section we rigorously define what we mean by "nicely."

In Section 1.1 we saw three ways in which limits of functions can fail to exist:

1. The function approaches different values from the left and right.

2. The function grows without bound.

3. The function oscillates.

In this section we explore in depth the concepts behind Item 1 by introducing the *one-sided limit*. We begin with formal definitions that are very similar to the definition of the limit given in Section 1.2, but the notation is slightly different and "$x \neq c$" is replaced with either "$x < c$" or "$x > c$."

There is a slightly different definition for a left-hand limit, than for a right-hand limit, but both have a lot in common with Definition 1.2.2.

Definition 1.4.1 One Sided Limits: Left- and Right-Hand Limits.

Left-Hand Limit

Let f be a function defined on (a, c) for some $a < c$ and let L be a real number. The statement that the **limit** of $f(x)$, as x approaches c **from the left**, is L, (alternatively, that **the left-hand limit** of f at c is L) is denoted by

$$\lim_{x \to c^-} f(x) = L,$$

and means that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in (a, c)$, if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Right-Hand Limit

Let f be a function defined on (c, b) for some $b > c$ and let L be a real number. The statement that the **limit** of $f(x)$, as x approaches c **from the right**, is L, (alternatively, that **the right-hand limit** of f at c is L) is denoted by

$$\lim_{x \to c^+} f(x) = L,$$

and means that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in (c, b)$, if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Practically speaking, when evaluating a left-hand limit, we consider only values of x "to the left of c," i.e., where $x < c$. The admittedly imperfect notation $x \to c^-$ is used to imply that we look at values of x to the left of c. The notation has nothing to do with positive or negative values of either x or c. It's more like you are adding very small negative values to c to get values for x. A similar statement holds for evaluating right-hand limits; there we consider only values of x to the right of c, i.e., $x > c$. We can use the theorems from previous sections to help us evaluate these limits; we just restrict our view to one side of c.

youtu.be/watch?v=VU8lUocFAfE

Figure 1.4.2 Video presentation of Definition 1.4.1

We practice evaluating left- and right-hand limits through a series of examples.

Example 1.4.3 Evaluating one-sided limits.

Let $f(x) = \begin{cases} x & 0 \le x \le 1 \\ 3 - x & 1 < x < 2 \end{cases}$, as shown in Figure 1.4.4. Find each of the following:

(a) $\lim\limits_{x\to 1^-} f(x)$

(b) $\lim\limits_{x\to 1^+} f(x)$

(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

(e) $\lim\limits_{x\to 0^+} f(x)$

(f) $f(0)$

(g) $\lim\limits_{x\to 2^-} f(x)$

(h) $f(2)$

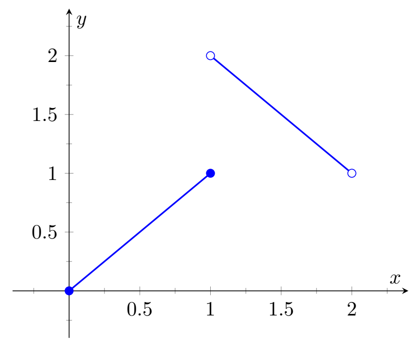


Figure 1.4.4 A graph of f in Example 1.4.3

Solution. For these problems, the visual aid of the graph is likely more effective in evaluating the limits than using f itself. Therefore we will refer often to the graph.

(a) As x goes to 1 *from the left*, we see that $f(x)$ is approaching the value of 1.

Therefore $\lim\limits_{x\to 1^-} f(x) = 1$.

(b) As x goes to 1 *from the right*, we see that $f(x)$ is approaching the value of 2. Recall that it does not matter that there is an "open circle" there; we are evaluating a limit, not the value of the function.

Therefore $\lim\limits_{x\to 1^+} f(x) = 2$.

(c) *The* limit of f as x approaches 1 does not exist, as discussed in Section 1.1. The function does not approach one particular value, but two different values from the left and the right.

(d) Using the definition, and by looking at the graph, we see that $f(1) = 1$.

(e) As x goes to 0 from the right, we see that $f(x)$ is approaching 0. Therefore $\lim_{x\to 0^+} f(x) = 0$. Note we cannot consider a left-hand limit at 0 as f is not defined for values of $x < 0$.

(f) Using the definition and the graph, $f(0) = 0$.

(g) As x goes to 2 from the left, we see that $f(x)$ is approaching the value of 1.

Therefore $\lim\limits_{x\to 2^-} f(x) = 1$.

(h) The graph and the definition of the function show that $f(2)$ is not defined.

Video solution

youtu.be/watch?v=NdBPwaP4Xkk

Note how the left- and right-hand limits were different at $x = 1$. This, of course, causes *the* limit to not exist. The following theorem states what is fairly intuitive: *the* limit exists precisely when the left- and right-hand limits are equal.

Theorem 1.4.5 Limits and One-Sided Limits.

Let f be a function defined on an open interval I containing c, except possibly at c. Then

$$\lim_{x\to c} f(x) = L$$

if, and only if,

$$\lim_{x\to c^-} f(x) = L \text{ and } \lim_{x\to c^+} f(x) = L.$$

The phrase "if, and only if" means the two statements are *equivalent*: they are either both true or both false. If the limit equals L, then the left and right hand limits both equal L. If the limit is not equal to L, then at least one of the left and right-hand limits is not equal to L (it may not even exist).

One thing to consider in Examples 1.4.3–1.4.10 is that the value of the function may/may not be equal to the value(s) of its left/right-hand limits, even when these limits agree.

Example 1.4.6 Evaluating limits of a piecewise-defined function.

Let $f(x) = \begin{cases} 2-x & 0<x<1 \\ (x-2)^2 & 1<x<2 \end{cases}$. Evaluate the following:

(a) $\lim_{x\to 1^-} f(x)$

(b) $\lim_{x\to 1^+} f(x)$

(c) $\lim_{x\to 1} f(x)$

(d) $f(1)$

(e) $\lim_{x\to 0^+} f(x)$

(f) $f(0)$

(g) $\lim_{x\to 2^-} f(x)$

(h) $f(2)$

Solution. In this example, we evaluate each expression using just the definition of f, without using a graph as we did in the previous example.

(a) As x approaches 1 from the left, we consider a limit where all x-values are less than 1. This means we use the "$2-x$" piece of the piecewise-defined function f. As the x-values near 1, $2-x$ approaches 1; that is, $f(x)$ approaches 1.

Therefore $\lim_{x\to 1^-} f(x) = 1$.

A concise mathematical presentation of the above argument could be written as follows:

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} (2-x) \quad (f(x) = x-2 \text{ for } 0<x<1) \\ &= 2-1 = 1 \quad (\text{ properties of limits }) \end{aligned}$$

(b) As x approaches 1 from the right, we consider a limit where all x-values are greater than 1. This means we use the "$(x-2)^2$" piece of f. As the x-values near 1, $(x-2)^2$ approaches 1; that is, we see that again $f(x)$ approaches 1.

Therefore $\lim_{x\to 1+} f(x) = 1$.

Once again, we can present our work computationally as follows:

$$\begin{aligned}\lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+}(x-2)^2 \quad (f(x)=(x-2)^2 \text{ for } 1<x<2)\\ &= (1-2)^2 = 1 \quad (\text{ properties of limits })\end{aligned}$$

(c) *The* limit of f as x approaches 1 exists and is 1, as f approaches 1 from both the right and left.

Therefore $\lim\limits_{x\to 1} f(x) = 1$.

(d) Neither piece of f is defined for the x-value of 1; in other words, 1 is not in the domain of f. Therefore $f(1)$ is not defined.

(e) As x approaches 0 from the right, we consider a limit where all x-values are greater than 0. This means we use the $2-x$ piece of f. As the x-values near 0, $2-x$ approaches 2; that is, $f(x)$ approaches 2.

So $\lim\limits_{x\to 0^+} f(x) = 2$.

(f) $f(0)$ is not defined as 0 is not in the domain of f.

(g) As x approaches 2 from the left, we consider a limit where all x-values are less than 2. This means we use the $(x-2)^2$ piece of f. As the x-values near 2, $(x-2)^2$ nears 0; that is, $f(x)$ approaches 0.

So $\lim\limits_{x\to 2^-} f(x) = 0$.

(h) $f(2)$ is not defined as 2 is not in the domain of f.

We can confirm our analytic result by consulting the graph of f shown in Figure 1.4.7. Note the open circles on the graph at $x = 0$, 1 and 2, where f is not defined.

Video solution

youtu.be/watch?v=vE_7FG2h_LU

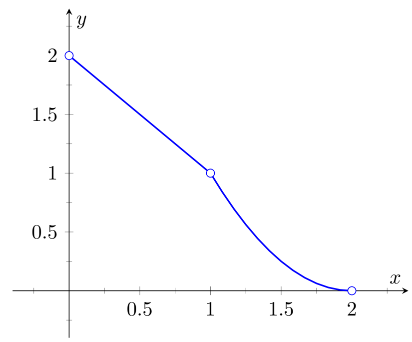


Figure 1.4.7 A graph of f from Example 1.4.6

Example 1.4.8 Evaluating limits of a piecewise-defined function.

Let $f(x) = \begin{cases}(x-1)^2 & 0\le x\le 2, x\ne 1\\ 1 & x=1\end{cases}$ as shown in Figure 1.4.9.

Evaluate the following:

(a) $\lim\limits_{x\to 1^-} f(x)$

(b) $\lim\limits_{x\to 1^+} f(x)$

(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

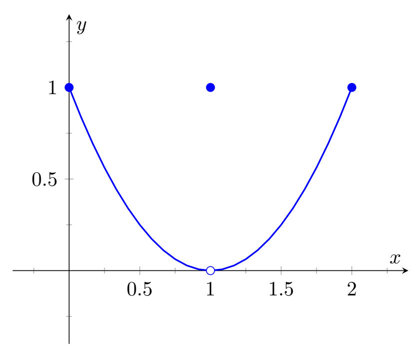


Figure 1.4.9 Graphing f in Example 1.4.8

Solution. It is clear by looking at the graph that both the left- and right-hand limits of f, as x approaches 1, are 0. Thus it is also clear that *the* limit is 0; i.e., $\lim_{x\to 1} f(x) = 0$. It is also clearly stated that $f(1) = 1$.

Video solution

youtu.be/watch?v=HVFazve-Qxc

Example 1.4.10 Evaluating limits of a piecewise-defined function.

Let $f(x) = \begin{cases} x^2 & 0 \leq x \leq 1 \\ 2 - x & 1 < x \leq 2 \end{cases}$ as shown in Figure 1.4.11. Evaluate the following:

(a) $\lim\limits_{x \to 1^-} f(x)$

(b) $\lim\limits_{x \to 1^+} f(x)$

(c) $\lim\limits_{x \to 1} f(x)$

(d) $f(1)$

Solution. It is clear from the definition of the function and its graph that all of the following are equal:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = f(1) = 1.$$

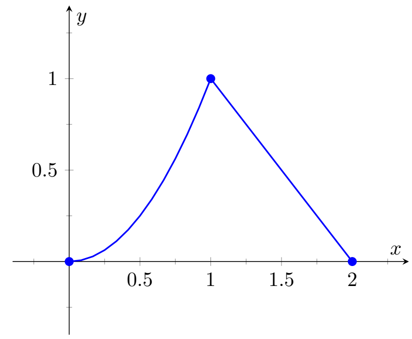


Figure 1.4.11 Graphing f in Example 1.4.10

In Examples 1.4.3–1.4.10 we were asked to find both $\lim_{x \to 1} f(x)$ and $f(1)$. Consider the following table:

	$\lim\limits_{x \to 1} f(x)$	$f(1)$
Example 1.4.3	does not exist	1
Example 1.4.6	1	not defined
Example 1.4.8	0	1
Example 1.4.10	1	1

Only in Example 1.4.10 do both the function and the limit exist and agree. This seems "nice;" in fact, it seems "normal." This is in fact an important situation which we explore in Section 1.5 entitled "Continuity." In short, a **continuous function** is one in which when a function approaches a value as $x \to c$ (i.e., when $\lim_{x \to c} f(x) = L$), it actually *attains* that value at c. Such functions behave nicely as they are very predictable.

Video solution

youtu.be/watch?v=Nn6JoJRK7nk

1.4.1 Exercises

Terms and Concepts

1. What are the three ways in which a limit may fail to exist?

2. (☐ True ☐ False) If $\lim_{x\to 1^-} f(x) = 5$, then $\lim_{x\to 1} f(x) = 5$.

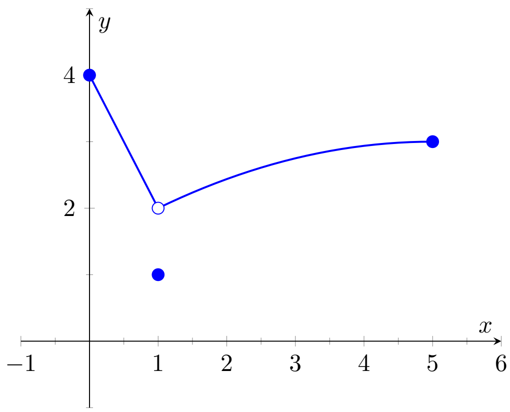
3. (☐ True ☐ False) If $\lim_{x\to 1^-} f(x) = 5$, then $\lim_{x\to 1^+} f(x) = 5$.

4. (☐ True ☐ False) If $\lim_{x\to 1} f(x) = 5$, then $\lim_{x\to 1^-} f(x) = 5$.

Problems

Exercise Group. Evaluate each expression using the given graph of f.

5.



(a) $\lim_{x\to 1^-} f(x)$

(b) $\lim_{x\to 1^+} f(x)$

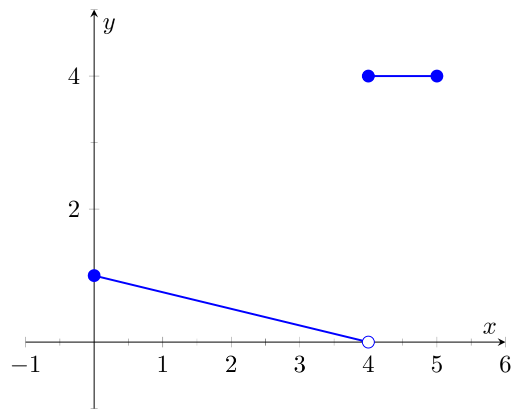
(c) $\lim_{x\to 1} f(x)$

(d) $f(1)$

(e) $\lim_{x\to 0^-} f(x)$

(f) $\lim_{x\to 0^+} f(x)$

6.



(a) $\lim_{x\to 4^-} f(x)$

(b) $\lim_{x\to 4^+} f(x)$

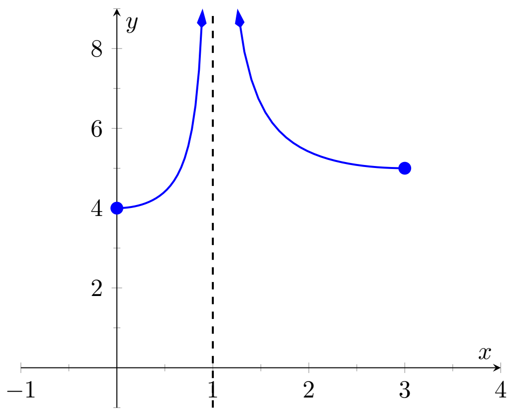
(c) $\lim_{x\to 4} f(x)$

(d) $f(4)$

(e) $\lim_{x\to 0^-} f(x)$

(f) $\lim_{x\to 0^+} f(x)$

7.



(a) $\lim_{x \to 1^-} f(x)$

(b) $\lim_{x \to 1^+} f(x)$

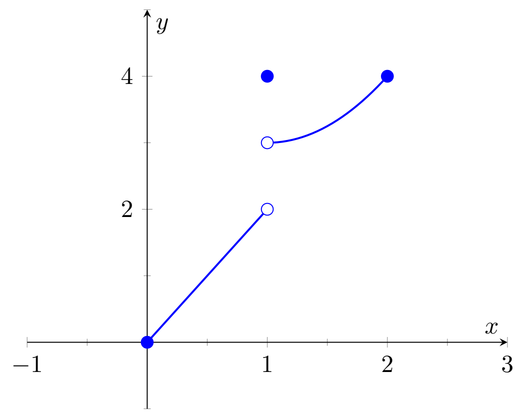
(c) $\lim_{x \to 1} f(x)$

(d) $f(1)$

(e) $\lim_{x \to 3^-} f(x)$

(f) $\lim_{x \to 0^+} f(x)$

8.



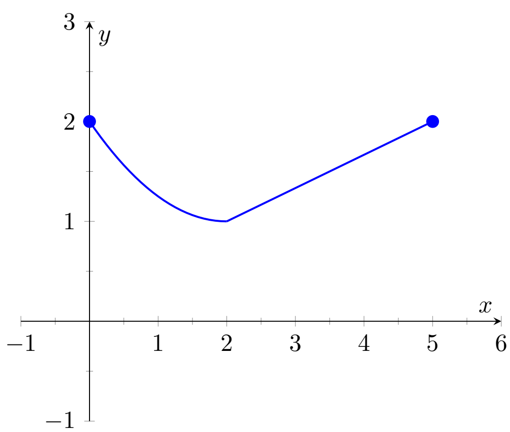
(a) $\lim_{x \to 1^-} f(x)$

(b) $\lim_{x \to 1^+} f(x)$

(c) $\lim_{x \to 1} f(x)$

(d) $f(1)$

9.



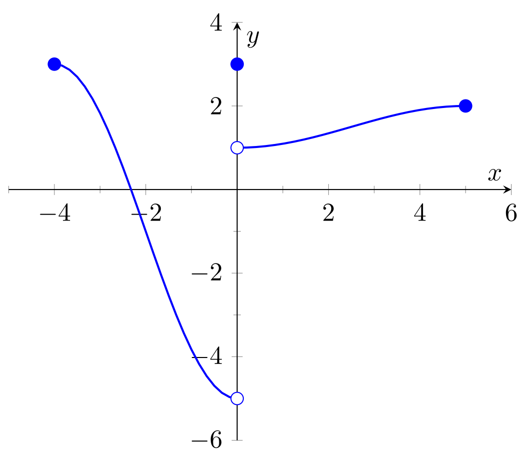
(a) $\lim_{x \to 2^-} f(x)$

(b) $\lim_{x \to 2^+} f(x)$

(c) $\lim_{x \to 2} f(x)$

(d) $f(2)$

10.



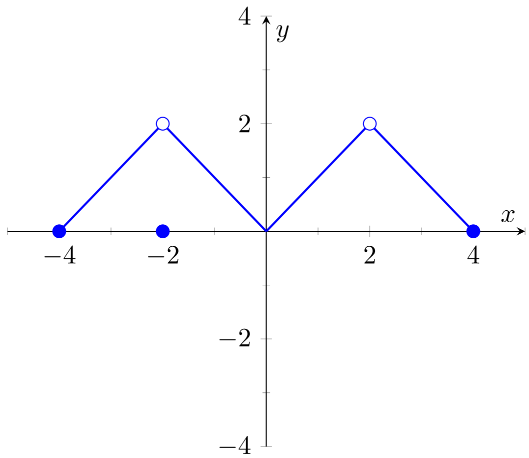
(a) $\lim_{x \to 0^-} f(x)$

(b) $\lim_{x \to 0^+} f(x)$

(c) $\lim_{x \to 0} f(x)$

(d) $f(0)$

11.



(a) $\lim_{x\to -2^-} f(x)$

(b) $\lim_{x\to -2^+} f(x)$

(c) $\lim_{x\to -2} f(x)$

(d) $f(-2)$

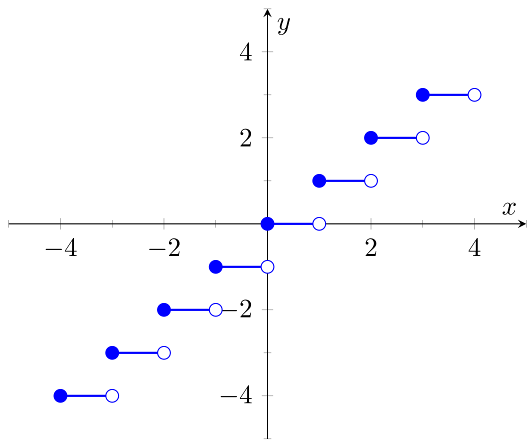
(e) $\lim_{x\to 2^-} f(x)$

(f) $\lim_{x\to 2^+} f(x)$

(g) $\lim_{x\to 2} f(x)$

(h) $f(2)$

12.



Let a be an integer with $-3 \le a \le 3$.

(a) $\lim_{x\to a^-} f(x)$

(b) $\lim_{x\to a^+} f(x)$

(c) $\lim_{x\to a} f(x)$

(d) $f(a)$

Exercise Group. Evaluate the given limits of the piecewise defined function.

13. $f(x) = \begin{cases} x-1 & \text{if } x \le 3 \\ x^2-3 & \text{if } x > 3 \end{cases}$

(a) $\lim_{x\to 3^-} f(x)$

(b) $\lim_{x\to 3^+} f(x)$

(c) $\lim_{x\to 3} f(x)$

(d) $f(3)$

14. $f(x) = \begin{cases} 2x-2x^2-5 & \text{if } x < 3 \\ \sin(x-3) & \text{if } x \ge 3 \end{cases}$

(a) $\lim_{x\to 3^-} f(x)$

(b) $\lim_{x\to 3^+} f(x)$

(c) $\lim_{x\to 3} f(x)$

(d) $f(3)$

15. $f(x) = \begin{cases} x^2+3x-1 & \text{if } x<2 \\ x^3+1 & \text{if } 2\le x\le 5 \\ x^2+4x+81 & \text{if } x>5 \end{cases}$

(a) $\lim_{x\to 2^-} f(x)$

(b) $\lim_{x\to 2^+} f(x)$

(c) $\lim_{x\to 2} f(x)$

(d) $f(2)$

(e) $\lim_{x\to 5^-} f(x)$

(f) $\lim_{x\to 5^+} f(x)$

(g) $\lim_{x\to 5} f(x)$

(h) $f(5)$

16. $f(x) = \begin{cases} \cos(x) & x<\pi \\ \sin(x) & x\ge\pi \end{cases}$

(a) $\lim_{x\to \pi^-} f(x)$

(b) $\lim_{x\to \pi^+} f(x)$

(c) $\lim_{x\to \pi} f(x)$

(d) $f(\pi)$

17. $f(x) = \begin{cases} 1-\cos^2(x) & x<a \\ \sin^2(x) & x\ge a \end{cases}$ where a is a real number.

(a) $\lim_{x\to^-} f(x)$

(b) $\lim_{x\to^+} f(x)$

(c) $\lim_{x\to} f(x)$

(d) $f()$

18. $f(x) = \begin{cases} x+1 & \text{if } x<-1 \\ x-1 & \text{if } x=-1 \\ x+2 & \text{if } x>-1 \end{cases}$

(a) $\lim_{x\to -1^-} f(x)$

(b) $\lim_{x\to -1^+} f(x)$

(c) $\lim_{x\to -1} f(x)$

(d) $f(-1)$

19. $f(x) = \begin{cases} x^2-2x-7 & \text{if } x<-1 \\ x-1 & \text{if } x=-1 \\ -\left(x^2+x+4\right) & \text{if } x>-1 \end{cases}$

(a) $\lim_{x\to -1^-} f(x)$

(b) $\lim_{x\to -1^+} f(x)$

(c) $\lim_{x\to -1} f(x)$

(d) $f(-1)$

20. $f(x) = \begin{cases} a(x-b)^2+c & x<b \\ a(x-b)+c & x\ge b \end{cases}$

(a) $\lim_{x\to b^-} f(x)$

(b) $\lim_{x\to b^+} f(x)$

(c) $\lim_{x\to b} f(x)$

(d) $f(b)$

21. $f(x) = \begin{cases} \frac{|x|}{x} & x\ne 0 \\ 0 & x=0 \end{cases}$

(a) $\lim_{x\to 0^-} f(x)$

(b) $\lim_{x\to 0^+} f(x)$

(c) $\lim_{x\to 0} f(x)$

(d) $f(0)$

1.5 Continuity

As we have studied limits, we have gained the intuition that limits measure "where a function is heading." That is, if $\lim\limits_{x\to 1} f(x) = 3$, then as x is close to 1, $f(x)$ is close to 3. We have seen, though, that this is not necessarily a good indicator of what $f(1)$ actually is. This can be problematic; functions can tend to one value but attain another. This section focuses on functions that *do not* exhibit such behavior.

Definition 1.5.1 Continuous Function.

Let f be a function whose domain contains an open interval I.

1. f is **continuous at a point** c in I if $\lim\limits_{x\to c} f(x) = f(c)$.
2. f is **continuous on the open interval** I if f is continuous at c for all values of c in I. If f is continuous on $(-\infty, \infty)$, we say f is **continuous everywhere** (or **everywhere continuous**).

youtu.be/watch?v=bKi6ReLchfw

Figure 1.5.2 Video presentation of Definition 1.5.1

Note that in Definition 1.5.1, a function f can only be continuous at a point c if c is in the domain of f.

A useful way to establish whether or not a function f is continuous at c is to verify the following three things:

1. $\lim\limits_{x\to c} f(x)$ exists,
2. $f(c)$ is defined, and
3. $\lim\limits_{x\to c} f(x) = f(c)$.

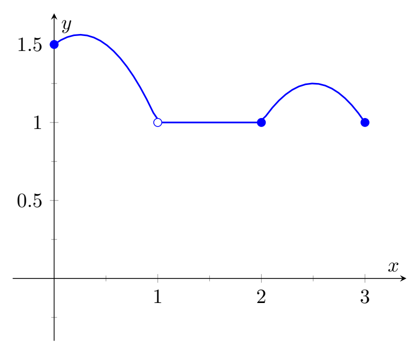


Figure 1.5.4 A graph of f in Example 1.5.3

Example 1.5.3 Finding intervals of continuity.

Let f be defined as shown in Figure 1.5.4. Give the interval(s) on which f is continuous.

Solution. We proceed by examining the three criteria for continuity.

1. The limits $\lim\limits_{x\to c} f(x)$ exists for all c between 0 and 3.
2. $f(c)$ is defined for all c between 0 and 3, *except for* $c = 1$. We know immediately that f cannot be continuous at $x = 1$.
3. The limit $\lim\limits_{x\to c} f(x) = f(c)$ for all c between 0 and 3, except, of course, for $c = 1$.

We conclude that f is continuous at every point of the interval $(0, 3)$ except at $x = 1$. Therefore f is continuous on $(0, 1)$ and $(1, 3)$.

Our definition of continuity (currently) only applies to open intervals. After Definition 1.5.7, we'll be able to say that f is continuous on $[0, 1)$ and $(1, 3]$.

Example 1.5.5 Finding intervals of continuity.

The *floor function*, $f(x) = \lfloor x \rfloor$, returns the largest integer smaller than, or equal to, the input x. (For example, $f(\pi) = \lfloor \pi \rfloor = 3$.) The graph of f in Figure 1.5.6 demonstrates why this is often called a "step function." Give the intervals on which f is continuous.

Solution. We examine the three criteria for continuity.

1. The limits $\lim\limits_{x\to c} f(x)$ do not exist at the jumps from one "step" to the next, which occur at all integer values of c. Therefore the limits

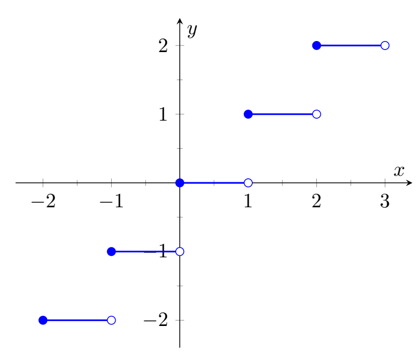


Figure 1.5.6 A graph of the step function in Example 1.5.5

exist for all c except when c is an integer.

2. The function is defined for all values of c.

3. The limit $\lim_{x\to c} f(x) = f(c)$ for all values of c where the limit exist, since each step consists of just a line.

We conclude that f is continuous everywhere except at integer values of c. So the intervals on which f is continuous are

$$\ldots, (-2,-1), (-1,0), (0,1), (1,2), \ldots.$$

We could also say that f is continuous on all intervals of the form $(n, n+1)$ where n is an integer.

Our definition of continuity on an interval specifies the interval is an open interval. We can extend the definition of continuity to closed intervals of the form $[a,b]$ by considering the appropriate one-sided limits at the endpoints.

Definition 1.5.7 Continuity on Closed Intervals.

Let f be defined on the closed interval $[a,b]$ for some real numbers $a < b$.
We say f is **continuous on the closed interval** $[a,b]$ if:

1. f is continuous on (a,b),
2. $\lim_{x\to a^+} f(x) = f(a)$ and
3. $\lim_{x\to b^-} f(x) = f(b)$.

We can make the appropriate adjustments to talk about continuity on half-open intervals such as $[a,b)$ or $(a,b]$ if necessary.

If the domain of f includes values less than a, we say that Item 2 in Definition 1.5.7 indicates that f is **continuous from the right** at a. But if f is undefined for $x < a$, we can say that f is continuous at a without ambiguity.

Similarly, Item 3 indcates that f is **continuous from the left** at b, and if f is not defined for $x > b$, we can simply say that f is continuous at b.

For example, it makes sense to say that the function $f(x) = \sqrt{1-x^2}$ is continuous at 1 and -1, while the floor function in Example 1.5.5 is continuous from the left at 1 and -1, but is not continuous at these points.

Using this new definition, we can adjust our answer in Example 1.5.3 by stating that f is continuous on $[0,1)$ and $(1,3]$, as mentioned in that example. We can also revisit Example 1.5.5 and state that the floor function is continuous on the following half-open intervals

$$\ldots, [-2,-1), [-1,0), [0,1), [1,2), \ldots.$$

This can tempt us to conclude that f is continuous everywhere; after all, if f is continuous on $[0,1)$ and $[1,2)$, isn't f also continuous on $[0,2)$? Of course, the answer is *no*, and the graph of the floor function immediately confirms this.

Continuous functions are important as they behave in a predictable fashion: functions attain the value they approach. Because continuity is so important, most of the functions you have likely seen in the past are continuous on their domains. This is demonstrated in the following example where we examine the intervals of continuity of a variety of common functions.

In this text, when we use the term "closed interval", we mean an interval of the form $[a,b]$, where a and b are real numbers. One may be surprised to learn that intervals of the form $[a,\infty)$, $(-\infty,b]$ and even $(-\infty,\infty)$ are all also considered closed in advanced calculus. While the mathematics supported by this definition of closed is fascinating and important, it is beyond the scope of our purposes here.

Some results, such as The Extreme Value Theorem, are valid for intervals of the form $[a,b]$, but not for intervals such as $[a,\infty)$. The latter interval is closed, but not **bounded**.

A set of real numbers is bounded if there is a number that is greater than every element in the set (an upper bound), and a number that is less than every element in the set (a lower bound). When we do calculus in higher dimensions, we can no longer talk about intervals, but we can still talk about sets being closed and bounded. See Section 14.5 for details.

youtu.be/watch?v=8z07z3yeChY

Figure 1.5.8 Two continuity examples

Example 1.5.9 Determining intervals on which a function is continuous.

For each of the following functions, give the domain of the function and the interval(s) on which it is continuous.

1. $f(x) = 1/x$
2. $f(x) = \sin(x)$
3. $f(x) = \sqrt{x}$
4. $f(x) = \sqrt{1-x^2}$
5. $f(x) = |x|$

Solution. We examine each in turn.

1. The domain of $f(x) = 1/x$ is $(-\infty, 0) \cup (0, \infty)$. As it is a rational function, we apply Theorem 1.3.4 to recognize that f is continuous on all of its domain.

2. The domain of $f(x) = \sin(x)$ is all real numbers, or $(-\infty, \infty)$. Applying Theorem 1.3.7 shows that $\sin(x)$ is continuous everywhere.

3. The domain of $f(x) = \sqrt{x}$ is $[0, \infty)$. Applying Theorem 1.3.7 shows that $f(x) = \sqrt{x}$ is continuous on its domain of $[0, \infty)$.

4. The domain of $f(x) = \sqrt{1-x^2}$ is $[-1, 1]$. Applying Theorems 1.3.1 and 1.3.7 shows that f is continuous on all of its domain, $[-1, 1]$.

5. The domain of $f(x) = |x|$ is $(-\infty, \infty)$. We can define the absolute value function as

$$f(x) = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}.$$

Each "piece" of this piecewise defined function is continuous on all of its domain, giving that f is continuous on $(-\infty, 0)$ and $[0, \infty)$. We cannot assume this implies that f is continuous on $(-\infty, \infty)$; we need to check that $\lim_{x \to 0} f(x) = f(0)$, as $x = 0$ is the point where f transitions from one "piece" of its definition to the other. It is easy to verify that this is indeed true, hence we conclude that $f(x) = |x|$ is continuous everywhere.

Video solution

youtu.be/watch?v=by3ioPN6KRM

Continuity is inherently tied to the properties of limits. Because of this, the properties of limits found in Theorems 1.3.1 and 1.3.4 apply to continuity as well. Further, now knowing the definition of continuity we can re-read Theorem 1.3.7 as giving a list of functions that are continuous on their domains. The following theorem states how continuous functions can be combined to form other continuous functions, followed by a theorem which formally lists functions that we know are continuous on their domains.

Theorem 1.5.10 Properties of Continuous Functions.

Let f and g be continuous functions on an interval I, let c be a real number and let n be a positive integer. The following functions are continu-

ous on I.

Sums/Difference	$f \pm g$
Constant Multiple	$c \cdot f$
Product	$f \cdot g$
Quotient	f/g *(as long as* $g \neq 0$ *on* I*)*
Power	f^n
Root	$\sqrt[n]{f}$ *(If* n *is even then require* $f(x) \geq 0$ *on* I*.)*
Compositions	*Adjust the definitions of* f *and* g *to: Let* f *be continuous on* I*, where the range of* f *on* I *is* J*, and let* g *be continuous on* J*. Then* $g \circ f$*, i.e.,* $g(f(x))$*, is continuous on* I*.*

We have defined what it means for a function to be continuous on an interval, but many functions, such as $f(x) = \tan(x)$, have domains that are the union of more than one interval.

If the domain of a function is a union of intervals, saying that a function is continuous on its domain means that the function is continuous on each of those intervals. But be careful to note that the converse is not true. As we learned in Example 1.5.5, a function can be continuous on a collection of intervals, but not on their union.

Theorem 1.5.12 Continuous Functions.

Let n be a positive integer. The following functions are continuous on their domains.

1. $f(x) = \sin(x)$
2. $f(x) = \tan(x)$
3. $f(x) = \sec(x)$
4. $f(x) = \ln(x)$
5. $f(x) = a^x$ *($a > 0$)*
6. $f(x) = \cos(x)$
7. $f(x) = \cot(x)$
8. $f(x) = \csc(x)$
9. $f(x) = \sqrt[n]{x}$*, where n is a positive integer.*

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Figure 1.5.11 Video presentation of Theorem 1.5.10

As the video example in Figure 1.5.13 illustrates, the above theorems allow us to quickly construct new continuous functions from old ones.

We apply these theorems in the following Example.

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Figure 1.5.13 Continuity of compositions

Example 1.5.14 Determining intervals on which a function is continuous.

State the interval(s) on which each of the following functions is continuous.

1. $f(x) = \sqrt{x-1} + \sqrt{5-x}$
2. $f(x) = x\sin(x)$
3. $f(x) = \tan(x)$
4. $f(x) = \sqrt{\ln(x)}$

Solution. We examine each in turn, applying Theorems 1.5.10 and 1.5.12 as appropriate.

1. The square root terms are continuous on the intervals $[1, \infty)$ and $(-\infty, 5]$, respectively. As f is continuous only where each term is continuous, f is continuous on $[1, 5]$, the intersection of these two intervals. A graph of f is given in Figure 1.5.15.

2. The functions $y = x$ and $y = \sin(x)$ are each continuous everywhere, hence their product is, too.

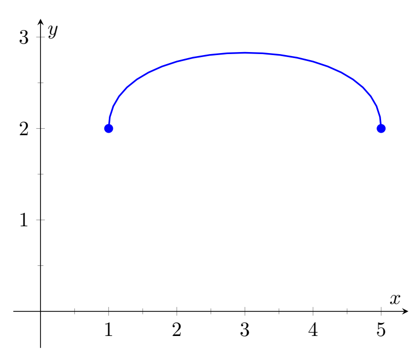


Figure 1.5.15 A graph of $f(x) = \sqrt{x-1} + \sqrt{5-x}$

3. Theorem 1.5.12 states that $f(x) = \tan(x)$ is continuous *on its domain*. Its domain includes all real numbers except odd multiples of $\pi/2$. Thus the intervals on which $f(x) = \tan(x)$ is continuous are
$$\cdots\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots,.$$

4. Here, $f(x)$ is the composition $g(h(x))$, where $g(x) = \sqrt{x}$ and $h(x) = \ln(x)$. The domain of g is $[0, \infty)$, while the range of h is $(-\infty, \infty)$. If we restrict the domain to $[1, \infty)$, then the output from $h(x) = \ln(x)$ is restricted to $[0, \infty)$, on which $g(x) = \sqrt{x}$ is defined. Thus the domain of $f(x) = \sqrt{\ln(x)}$ is $[1, \infty)$.

Video solution

youtu.be/watch?v=6Lm-0eBi-5E

Classification of discontinuities. We now know what it means for a function to be continuous, so of course we can easily say what it means for a function to be *discontinuous*; namely, not continuous. However, to better understand continuity, it is worth our time to discuss the different ways in which a function can fail to be discontinuous. By definition, a function f is continuous at a point a in its domain if $\lim\limits_{x\to a} f(x) = f(a)$. If this equality fails to hold, then f is not continuous. We note, however, that there are a number of different things that can go wrong with this equality.

youtu.be/watch?v=TevrD3qci0Q

Figure 1.5.16 Discussing classification of discontinuities

1. $\lim\limits_{x\to a} f(x) = L$ exists, but $L \neq f(a)$, or $f(a)$ is undefined. Such a discontinuity is called a **removable discontinuity** .

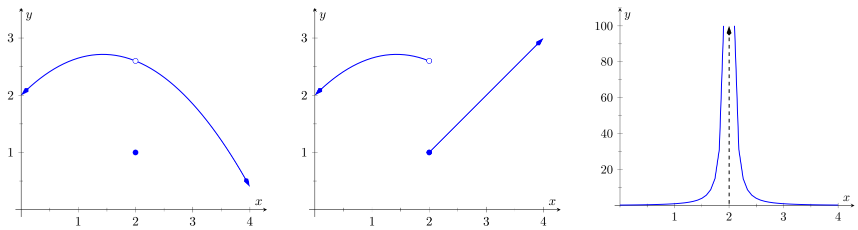
   A removable discontinuity can be pictured as a "hole" in the graph of f. The term "removable" refers to the fact that by simply redefining $f(a)$ to equal L (that is, changing the value of f at a single point), we can create a new function that is continuous at $x = a$, and agrees with f at all $x \neq a$.

2. $\lim\limits_{x\to a^+} f(x) = L$ and $\lim\limits_{x\to a^-} f(x) = M$ exist, but $L \neq M$. In this case the left and right hand limits both exist, but since they are not equal, the limit of f as $x \to a$ does not exist. Such a discontinuity is called a **jump discontinuity**.

   The phrase "jump discontinuity" is meant to represent the fact that visually, the graph of f "jumps" from one value to another as we cross the value $x = a$.

3. The function f is *unbounded* near $x = a$. This means that the value of f becomes arbitrarily large (or large and negative) as x approaches a. Such a discontinuity is called an **infinite discontinuity**.

   Infinite discontinuities are most easily understood in terms of *infinite limits*, which are discussed in Section 1.6.



(a) The graph of a function with a removable discontinuity at $x = 2$

(b) The graph of a function with a jump discontinuity at $x = 2$

(c) The graph of a function with an infinite discontinuity at $x = 2$

Figure 1.5.17 Illustrating three common types of discontinuity

Consequences of continuity. A common way of thinking of a continuous function is that "its graph can be sketched without lifting your pencil." That is, its graph forms a "continuous" curve, without holes, breaks or jumps. This pseudo-definition glosses over some of the finer points of continuity. There are some very strange continuous functions that one would be hard pressed to actually sketch by hand.

However, this intuitive notion of continuity does help us understand another important concept as follows. Suppose f is defined on $[1,2]$, and $f(1) = -10$ and $f(2) = 5$. If f is continuous on $[1,2]$ (i.e., its graph can be sketched as a continuous curve from $(1,-10)$ to $(2,5)$) then we know intuitively that somewhere on the interval $[1,2]$ f must be equal to -9, and -8, and $-7, -6, \ldots, 0, 1/2$, etc. In short, f takes on all *intermediate* values between -10 and 5. It may take on more values; f may actually equal 6 at some time, for instance, but we are guaranteed all values between -10 and 5.

While this notion seems intuitive, it is not trivial to prove and its importance is profound. Therefore the concept is stated in the form of a theorem.

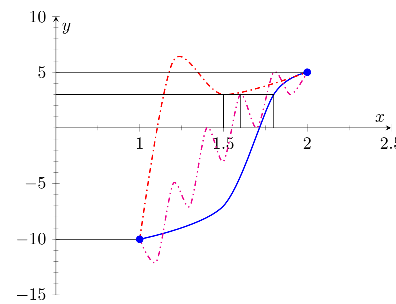


Figure 1.5.18 Illustration of the Intermediate Value Theorem: the output 3 is in between -10 and 5, and therefore any continuous function on $[1,2]$ with $f(1) = -10$ and $f(2) = 5$ will achieve the output 3 somewhere in $[1,2]$

Theorem 1.5.19 Intermediate Value Theorem.

Let f be a continuous function on $[a,b]$ and, without loss of generality, let $f(a) < f(b)$. Then for every value y, where $f(a) < y < f(b)$, there is at least one value c in (a,b) such that $f(c) = y$.

One important application of the Intermediate Value Theorem is root finding. Given a function f, we are often interested in finding values of x where $f(x) = 0$. These roots may be very difficult to find exactly. Good approximations can be found through successive applications of this theorem. Suppose through direct computation we find that $f(a) < 0$ and $f(b) > 0$, where $a < b$. The Intermediate Value Theorem states that there is at least one c in (a,b) such that $f(c) = 0$. The theorem does not give us any clue as to where to find such a value in the interval (a,b), just that at least one such value exists.

There is a technique that produces a good approximation of c. Let d be the midpoint of the interval $[a,b]$, with $f(a) < 0$ and $f(b) > 0$ and consider $f(d)$. There are three possibilities:

1. $f(d) = 0$: We got lucky and stumbled on the actual value. We stop as we found a root.

2. $f(d) < 0$: Then we know there is a root of f on the interval $[d,b]$ — we have halved the size of our interval, hence are closer to a good approximation of the root.

youtu.be/watch?v=Fx7Qu9tZlN4

Figure 1.5.20 Video presentation of Theorem 1.5.19

3. $f(d) > 0$: Then we know there is a root of f on the interval $[a, d]$ — again,we have halved the size of our interval, hence are closer to a good approximation of the root.

Successively applying this technique is called the **Bisection Method** of root finding. We continue until the interval is sufficiently small. We demonstrate this in the following example.

Example 1.5.21 Using the Bisection Method.

Approximate the root of $f(x) = x - \cos(x)$, accurate to three places after the decimal.

Solution. Consider the graph of $f(x) = x - \cos(x)$, shown in Figure 1.5.22. It is clear that the graph crosses the x-axis somewhere near $x = 0.8$. To start the Bisection Method, pick an interval that contains 0.8. We choose $[0.7, 0.9]$. Note that all we care about are signs of $f(x)$, not their actual value, so this is all we display.

Iteration 1: $f(0.7) < 0$, $f(0.9) > 0$, and $f(0.8) > 0$. So replace 0.9 with 0.8 and repeat.

Iteration 2: $f(0.7) < 0$, $f(0.8) > 0$, and at the midpoint, 0.75, we have $f(0.75) > 0$. So replace 0.8 with 0.75 and repeat. Note that we don't need to continue to check the endpoints, just the midpoint. Thus we put the rest of the iterations in Table 1.5.23.

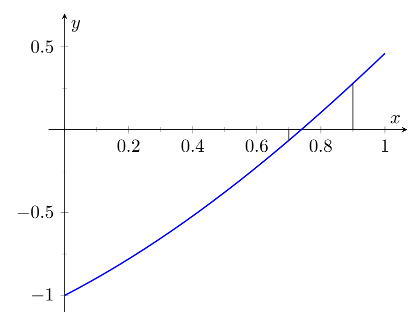


Figure 1.5.22 Graphing a root of $f(x) = x - \cos(x)$

Table 1.5.23 Iterations of the Bisection Method of Root Finding

Iteration #	Interval	Midpoint Sign
1	$[0.7, 0.9]$	$f(0.8) > 0$
2	$[0.7, 0.8]$	$f(0.75) > 0$
3	$[0.7, 0.75]$	$f(0.725) < 0$
4	$[0.725, 0.75]$	$f(0.7375) < 0$
5	$[0.7375, 0.75]$	$f(0.7438) >$
6	$[0.7375, 0.7438]$	$f(0.7407) > 0$
7	$[0.7375, 0.7407]$	$f(0.7391) > 0$
8	$[0.7375, 0.7391]$	$f(0.7383) < 0$
9	$[0.7383, 0.7391]$	$f(0.7387) < 0$
10	$[0.7387, 0.7391]$	$f(0.7389) < 0$
11	$[0.7389, 0.7391]$	$f(0.7390) < 0$
12	$[0.7390, 0.7391]$	

Notice that in the 12th iteration we have the endpoints of the interval each starting with 0.739. Thus we have narrowed the zero down to an accuracy of the first three places after the decimal. Using a computer, we have

$$f(0.7390) = -0.00014, f(0.7391) = 0.000024.$$

Either endpoint of the interval gives a good approximation of where f is 0. The Theorem 1.5.19 states that the actual zero is still within this interval. While we do not know its exact value, we know it starts with 0.739.

This type of exercise is rarely done by hand. Rather, it is simple to program a computer to run such an algorithm and stop when the endpoints differ by a preset small amount. One of the authors did write such a program and found the zero of f to be 0.7390851332, accurate to 10 places after the decimal. While it took a few minutes to write the program, it took less than a thousandth of a second for the program to run the necessary 35 iterations. In less than 8 hundredths of a second, the zero was calculated to 100 decimal places (with less than 200 iterations).

Video solution

youtu.be/watch?v=BH6kUpIgcfg

It is a simple matter to extend the Bisection Method to solve problems similar to "Find x, where $f(x) = 0$." For instance, we can find x, where $f(x) = 1$. It actually works very well to define a new function g where $g(x) = f(x) - 1$. Then use the Bisection Method to solve $g(x) = 0$.

Similarly, given two functions f and g, we can use the Bisection Method to solve $f(x) = g(x)$. Once again, create a new function h where $h(x) = f(x) - g(x)$ and solve $h(x) = 0$.

In Section 4.1 another equation solving method will be introduced, called Newton's Method. In many cases, Newton's Method is much faster. It relies on more advanced mathematics, though, so we will wait before introducing it.

This section formally defined what it means to be a continuous function. "Most" functions that we deal with are continuous, so often it feels odd to have to formally define this concept. Regardless, it is important, and forms the basis of the next chapter.

1.5.1 Exercises

Terms and Concepts

1. In your own words, describe what it means for a function to be continuous.

2. In your own words, describe what the Intermediate Value Theorem states.

3. What is a "root" of a function?

4. Given functions f and g on an interval I, how can the Bisection Method be used to find a value c where $f(c) = g(c)$?

5. (☐ True ☐ False) If f is defined on an open interval containing c, and $\lim_{x\to c} f(x)$ exists, then f is continuous at c.

6. (☐ True ☐ False) If f is defined on an open interval containing c, and f is continuous at c, then $\lim_{x\to c} f(x)$ exists.

7. (☐ True ☐ False) If f is defined on an open interval containing c, and f is continuous at c, then $\lim_{x\to c^+} f(x) = f(c)$.

8. (☐ True ☐ False) If f is continuous on $[a, b]$, then $\lim_{x\to a^-} f(x) = f(a)$.

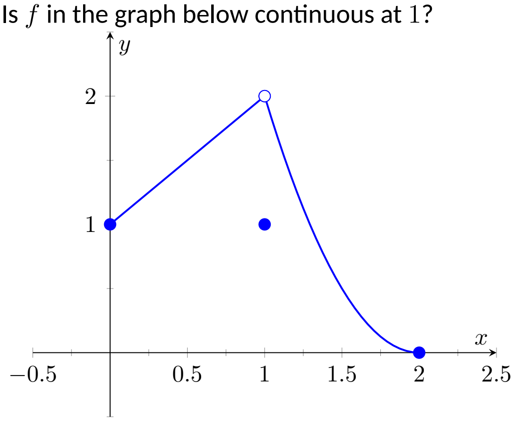
9. (☐ True ☐ False) If f is continuous on $[0, 1)$ and $[1, 2)$, then f is continuous on $[0, 2)$.

10. (☐ True ☐ False) The sum of continuous functions is also continuous.

Problems

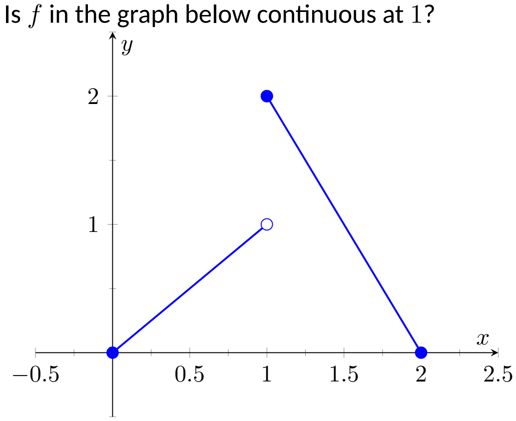
Exercise Group. Use the graph to determine if the function is continuous at the given point.

11. Is f in the graph below continuous at 1?



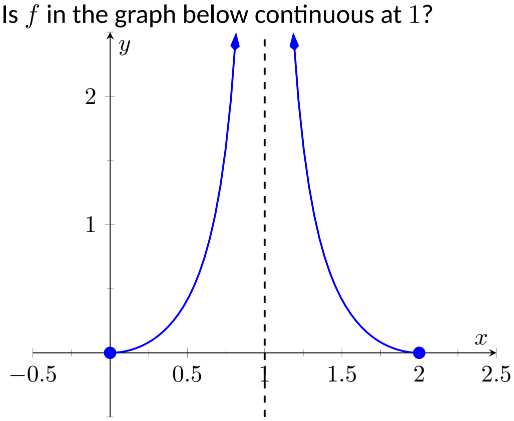
(☐ Yes. ☐ No.)

12. Is f in the graph below continuous at 1?



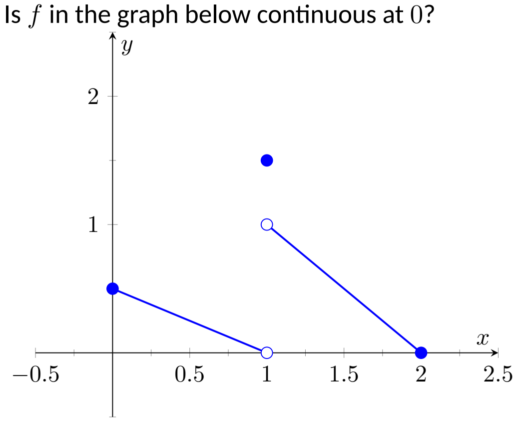
(☐ Yes. ☐ No.)

13. Is f in the graph below continuous at 1?



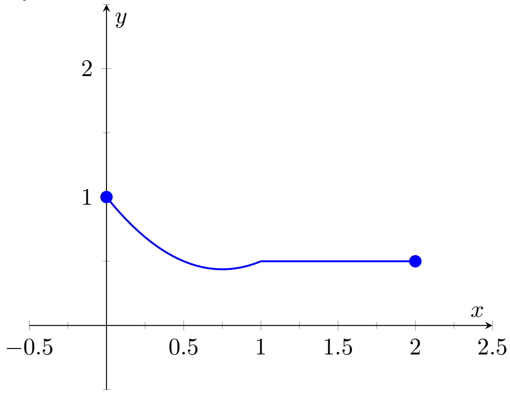
(☐ Yes. ☐ No.)

14. Is f in the graph below continuous at 0?



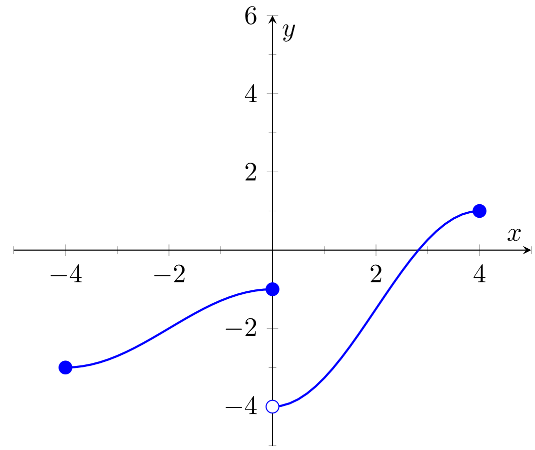
(☐ Yes. ☐ No.)

15. Is f in the graph below continuous at 1?



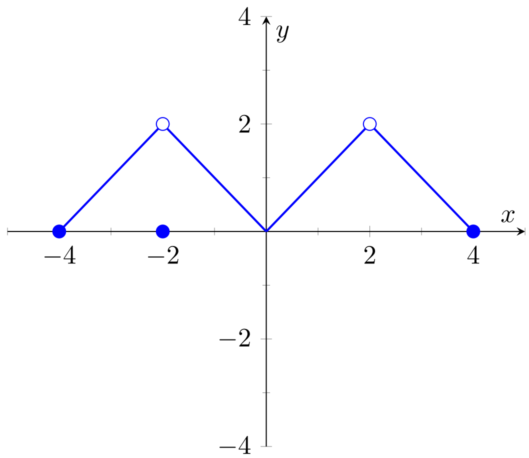
(☐ Yes. ☐ No.)

16. Is f in the graph below continuous at 4?



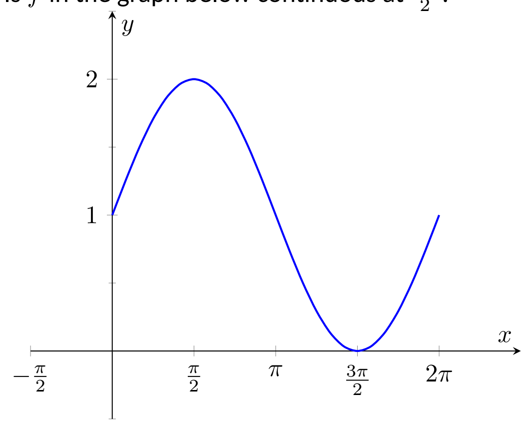
(☐ Yes. ☐ No.)

17. Is f in the graph below continuous at -2, 0, and 2?



At -2: (☐ Yes. ☐ No.)
At 0: (☐ Yes. ☐ No.)
At 2: (☐ Yes. ☐ No.)

18. Is f in the graph below continuous at $\frac{3\pi}{2}$?



(☐ Yes. ☐ No.)

Exercise Group. Determine if f is continuous at the indicated values.

19. $f(x) = \begin{cases} 1 & x = 0 \\ \frac{\sin(x)}{x} & x \neq 0 \end{cases}$

(a) Is f is continuous at 0?

(☐ Yes. ☐ No.)

(b) Is f is continuous at π?

(☐ Yes. ☐ No.)

20. $f(x) = \begin{cases} x^3 - x^2 & \text{if } x < 1 \\ x - 2 & \text{if } x \geq 1 \end{cases}$

(a) Is f is continuous at 0?

(☐ Yes. ☐ No.)

(b) Is f is continuous at 1?

(☐ Yes. ☐ No.)

21. $f(x) = \begin{cases} \dfrac{x^2+5x+4}{x^2+3x+2} & \text{if } x \neq -1 \\ 3 & \text{if } x = -1 \end{cases}$

(a) Is f is continuous at -1?

(☐ Yes. ☐ No.)

(b) Is f is continuous at 10?

(☐ Yes. ☐ No.)

22. $f(x) = \begin{cases} \dfrac{x^2-64}{x^2-11x+24} & \text{if } x \neq 8 \\ 5 & \text{if } x = 8 \end{cases}$

(a) Is f is continuous at 0?

(☐ Yes. ☐ No.)

(b) Is f is continuous at 8?

(☐ Yes. ☐ No.)

Exercise Group. Give the intervals on which the function is continuous.

23. $f(x) = x^2 - 6x + 2$

24. $f(x) = \sqrt{x^2 - 4}$

25. $f(x) = \sqrt{4 - x^2}$

26. $f(x) = \sqrt{3-x} + \sqrt{x+3}$

27. $f(t) = \sqrt{4t^2 - 12}$

28. $g(t) = \frac{1}{\sqrt{49-t^2}}$

29. $g(t) = \frac{1}{8+5t^2}$

30. $f(x) = \pi^x$

31. $g(s) = \log_2(s)$

32. $h(t) = \cos(t)$

33. $f(k) = \sqrt{3 - e^k}$

34. $f(x) = \sin\left(e^x + x^4\right)$

Exercise Group. Test your understanding of the Intermediate Value Theorem.

35. Let f be continuous on $[1, 5]$ where $f(1) = -2$ and $f(5) = -10$. Does a value $1 < c < 5$ exist such that $f(c) = -9$? Why/why not?

36. Let g be continuous on $[-3, 7]$ where $g(0) = 0$ and $g(2) = 25$. Does a value $-3 < c < 7$ exist such that $g(c) = 15$? Why/why not?

37. Let f be continuous on $[-1, 1]$ where $f(-1) = -10$ and $f(1) = 10$. Does a value $-1 < c < 1$ exist such that $f(c) = 11$? Why/why not?

38. Let h be a function on $[-1, 1]$ where $h(-1) = -10$ and $h(1) = 10$. Does a value $-1 < c < 1$ exist such that $h(c) = 0$? Why/why not?

Exercise Group. Use the Bisection Method to approximate, accurate to two decimal places, the value of the root of the given function in the given interval.

39. $f(x) = x^2 + 2x - 4$ on the interval $[1, 1.5]$

40. $f(x) = \sin(x) - \frac{1}{2}$ on the interval $[0.5, 0.55]$

41. $f(x) = e^x - 2$ on the interval $[0.65, 0.7]$

42. $f(x) = \cos(x) - \sin(x)$ on the interval $[0.7, 0.8]$

1.6 Limits Involving Infinity

In Definition 1.2.2 we stated that in the equation $\lim_{x\to c} f(x) = L$, both c and L were numbers. In this section we relax that definition a bit by considering situations when it makes sense to let c and/or L be "infinity."

As a motivating example, consider $f(x) = 1/x^2$, as shown in Figure 1.6.1. Note how, as x approaches 0, $f(x)$ grows very, very large—in fact, it grows without bound. It seems appropriate, and descriptive, to state that

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

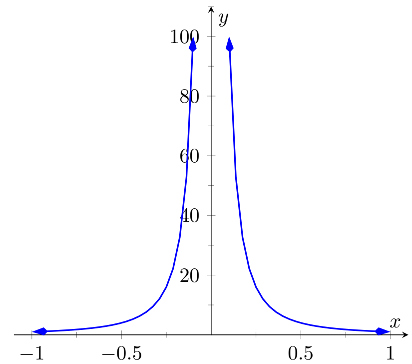


Figure 1.6.1 Graphing $f(x) = 1/x^2$ for values of x near 0

Also note that as x gets very large, $f(x)$ gets very, very small. We could represent this concept with notation such as

$$\lim_{x\to \infty} \frac{1}{x^2} = 0.$$

We explore both types of use of ∞ in turn.

Definition 1.6.2 Limit of Infinity, ∞.

Let I be an open interval containing c, and let f be a function defined on I, except possibly at c.

- The limit of $f(x)$, as x approaches c, is **infinity**, denoted by

  $$\lim_{x\to c} f(x) = \infty,$$

  if given any $N > 0$, there exists $\delta > 0$ such that for all x in I, where $x \neq c$, if $|x - c| < \delta$, then $f(x) > N$.

- The limit of $f(x)$, as x approaches c, is **negative infinity**, denoted by

  $$\lim_{x\to c} f(x) = -\infty,$$

  if given any $N < 0$, there exists $\delta > 0$ such that for all x in I, where $x \neq c$, if $|x - c| < \delta$, then $f(x) < N$.

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Figure 1.6.3 Video presentation of Definition 1.6.2

The first definition is similar to the ε-δ definition in Definition 1.2.2 from Section 1.2. In that definition, given any (small) value ε, if we let x get close enough to c (within δ units of c) then $f(x)$ is guaranteed to be within ε of L. Here, given any (large) value N, if we let x get close enough to c (within δ units of c), then $f(x)$ will be at least as large as N. In other words, if we get close enough to c, then we can make $f(x)$ as large as we want.

It is important to note that by saying $\lim_{x\to c} f(x) = \infty$ we are implicitly stating that *the* limit of $f(x)$, as x approaches c, *does not exist*. A limit only exists when $f(x)$ approaches an actual numeric value. We use the concept of limits that approach infinity because it is helpful and descriptive. It is one *specific way* in which a limit can fail to exist.

We define one-sided limits that approach infinity in a similar way.

Definition 1.6.4 One-Sided Limits of Infinity.

- Let f be a function defined on (a, c) for some $a < c$. We say the limit of $f(x)$, as x approaches c from the left, is infinity, or, the left-hand limit of f at c is infinity, denoted by

$$\lim_{x\to c^-} f(x) = \infty,$$

if given any $N > 0$, there exists $\delta > 0$ such that for all $a < x < c$, if $|x - c| < \delta$, then $f(x) > N$.

- Let f be a function defined on (c, b) for some $b > c$. We say the limit of $f(x)$, as x approaches c from the right, is infinity, or, the right-hand limit of f at c is infinity, denoted by

$$\lim_{x\to c^+} f(x) = \infty,$$

  if given any $N > 0$, there exists $\delta > 0$ such that for all $c < x < b$, if $|x - c| < \delta$, then $f(x) > N$.

- The term *left- (or, right-) hand limit of f at c is negative infinity* is defined in a manner similar to Definition 1.6.2.

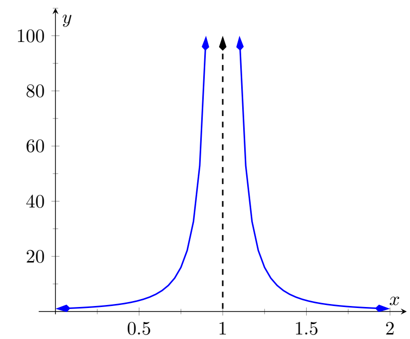

Figure 1.6.6 Observing infinite limit as $x \to 1$ in Example 1.6.5

Example 1.6.5 Evaluating limits involving infinity.

Find $\lim\limits_{x\to 1} \frac{1}{(x-1)^2}$ as shown in Figure 1.6.6.

Solution. In Example 1.1.18 of Section 1.1, by inspecting values of x close to 1 we concluded that this limit does not exist. That is, it cannot equal any real number. But the limit could be infinite. And in fact, we see that the function does appear to be growing larger and larger, as $f(0.99) = 10^4$, $f(0.999) = 10^6$, $f(0.9999) = 10^8$. A similar thing happens on the other side of 1. From the graph and the numeric information, we could state $\lim_{x\to 1} 1/(x-1)^2 = \infty$. We can prove this by using Definition 1.6.2

In general, let a "large" value N be given. Let $\delta = 1/\sqrt{N}$. If x is within δ of 1, i.e., if $|x - 1| < 1/\sqrt{N}$, then:

$$\begin{aligned} |x-1| &< \frac{1}{\sqrt{N}} \\ (x-1)^2 &< \frac{1}{N} \\ \frac{1}{(x-1)^2} &> N, \end{aligned}$$

which is what we wanted to show. So we may say $\lim_{x\to 1} 1/(x-1)^2 = \infty$.

Video solution

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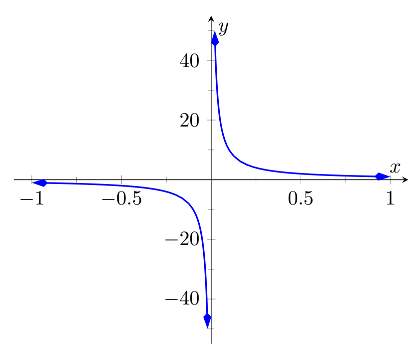

Figure 1.6.8 Evaluating $\lim\limits_{x\to 0} \frac{1}{x}$

Example 1.6.7 Evaluating limits involving infinity.

Find $\lim\limits_{x\to 0} \frac{1}{x}$, as shown in Figure 1.6.8.

Solution. It is easy to see that the function grows without bound near 0, but it does so in different ways on different sides of 0. Since its behavior is not consistent, we cannot say that $\lim_{x\to 0} \frac{1}{x} = \infty$. Instead, we will say $\lim_{x\to 0} \frac{1}{x}$ does not exist. However, we can make a statement about one-sided limits. We can state that $\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.

Video solution

youtu.be/watch?v=JP1k74FZE1I

1.6.1 Vertical asymptotes

The graphs in the two previous examples demonstrate that if a function f has a limit (or, left- or right-hand limit) of infinity at $x = c$, then the graph of f looks similar to a vertical line near $x = c$. This observation leads to a definition.

Definition 1.6.9 Vertical Asymptote.

Let I be an interval that either contains c or has c as an endpoint, and let f be a function defined on I, except possibly at c.
If the limit of $f(x)$ as x approaches c from either the left or right (or both) is ∞ or $-\infty$, then the line $x = c$ is a **vertical asymptote** of f.

youtu.be/watch?v=qIrLL7jbEZw

Figure 1.6.10 Video presentation of Definition 1.6.9

Example 1.6.11 Finding vertical asymptotes.

Find the vertical asymptotes of $f(x) = \frac{3x}{x^2-4}$.

Solution. Vertical asymptotes occur where the function grows without bound; this can occur at values of c where the denominator is 0. When x is near c, the denominator is small, which in turn can make the function take on large values. In the case of the given function, the denominator is 0 at $x = \pm 2$. Substituting in values of x close to 2 and -2 seems to indicate that the function tends toward ∞ or $-\infty$ at those points. We can graphically confirm this by looking at Figure 1.6.12. Thus the vertical asymptotes are at $x = \pm 2$.

Video solution

youtu.be/watch?v=h-1BCF_IsHI

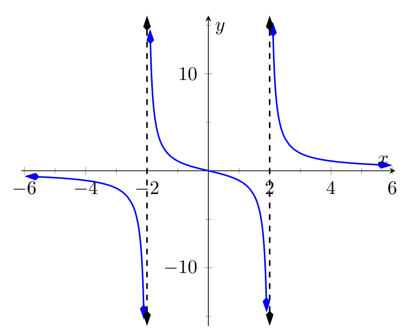

Figure 1.6.12 Graphing $f(x) = \frac{3x}{x^2-4}$

When a rational function has a vertical asymptote at $x = c$, we can conclude that the denominator is 0 at $x = c$. However, just because the denominator is 0 at a certain point does not mean there is a vertical asymptote there. For instance, $f(x) = (x^2 - 1)/(x - 1)$ does not have a vertical asymptote at $x = 1$, as shown in Figure 1.6.13. While the denominator does get small near $x = 1$, the numerator gets small too, matching the denominator step for step. In fact, factoring the numerator, we get

$$f(x) = \frac{(x-1)(x+1)}{x-1}.$$

Canceling the common term, we get that $f(x) = x + 1$ for $x \neq 1$. So there is clearly no asymptote; rather, a hole exists in the graph at $x = 1$.

The above example may seem a little contrived. Another example demonstrating this important concept is $f(x) = (\sin(x))/x$. We have considered this function several times in the previous sections. We found that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$; i.e., there is no vertical asymptote. No simple algebraic cancellation makes this fact obvious; we used the Squeeze Theorem in Section 1.3 to prove this.

If the denominator is 0 at a certain point but the numerator is not, then there will usually be a vertical asymptote at that point. On the other hand, if the numerator and denominator are both zero at that point, then there may or may not be a vertical asymptote at that point. This case where the numerator and denominator are both zero returns us to an important topic.

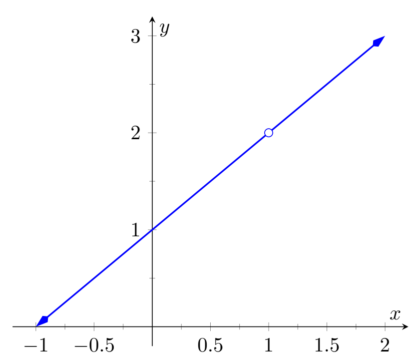

Figure 1.6.13 Graphically showing that $f(x) = \frac{x^2-1}{x-1}$ does not have an asymptote at $x = 1$

1.6.2 Indeterminate Forms

We have seen how the limits $\lim_{x\to 0} \frac{\sin(x)}{x}$ and $\lim_{x\to 1} \frac{x^2-1}{x-1}$ each return the indeterminate form $0/0$ when we blindly plug in $x = 0$ and $x = 1$, respectively. However, $0/0$ is not a valid arithmetical expression. It gives no indication that the respective limits are 1 and 2.

With a little cleverness, one can come up with $0/0$ expressions which have a limit of ∞, 0, or any other real number. That is why this expression is called **indeterminate**.

A key concept to understand is that such limits do not really return $0/0$. Rather, keep in mind that we are taking *limits*. What is really happening is that the numerator is shrinking to 0 while the denominator is also shrinking to 0. The respective rates at which they do this are very important and determine the actual value of the limit.

An indeterminate form indicates that one needs to do more work in order to compute the limit. That work may be algebraic (such as factoring and canceling), it may involve using trigonometric identities or logarithm rules, or it may require a tool such as the Squeeze Theorem. In Section 6.7 we will learn yet another technique called L'Hospital's Rule that provides another way to handle indeterminate forms.

Some other common indeterminate forms are $\infty - \infty$, $\infty \cdot 0$, ∞/∞, 0^0, ∞^0 and 1^∞. Again, keep in mind that these are the "blind" results of directly substituting c into the expression, and each, in and of itself, has no meaning. The expression $\infty - \infty$ does not really mean "subtract infinity from infinity." Rather, it means "One quantity is subtracted from the other, but both are growing without bound." What is the result? It is possible to get every value between $-\infty$ and ∞.

Note that $1/0$ and $\infty/0$ are not indeterminate forms, though they are not exactly valid mathematical expressions, either. In each, the function is growing without bound, indicating that the limit will be ∞, $-\infty$, or simply not exist if the left- and right-hand limits do not match.

1.6.3 Limits at Infinity and Horizontal Asymptotes

At the beginning of this section we briefly considered what happens to $f(x) = 1/x^2$ as x grew very large. Graphically, it concerns the behavior of the function to the "far right" of the graph. We make this notion more explicit in the following definition.

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Figure 1.6.14 Video presentation of Definition 1.6.15

Definition 1.6.15 Limits at Infinity and Horizontal Asymptotes.

Let L be a real number.

1. Let f be a function defined on (a, ∞) for some number a. The limit of f **at infinity** is L, denoted $\lim_{x\to\infty} f(x) = L$, if for every $\epsilon > 0$ there exists $M > a$ such that if $x > M$, then $|f(x) - L| < \epsilon$.
2. Let f be a function defined on $(-\infty, b)$ for some number b. The limit of f **at negative infinity** is L, denoted $\lim_{x\to-\infty} f(x) = L$, if for every $\epsilon > 0$ there exists $M < b$ such that if $x < M$, then $|f(x) - L| < \epsilon$.
3. If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, we say the line $y = L$ is a *horizontal asymptote* of f.

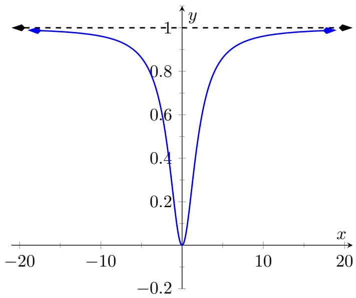
We can also define limits such as $\lim_{x\to\infty} f(x) = \infty$ by combining this definition with Definition 1.6.2.

Example 1.6.16 Approximating horizontal asymptotes.

Approximate the horizontal asymptote(s) of $f(x) = \frac{x^2}{x^2+4}$.

Solution. We will approximate the horizontal asymptotes by approximating the limits $\lim_{x\to-\infty} \frac{x^2}{x^2+4}$ and $\lim_{x\to\infty} \frac{x^2}{x^2+4}$. (A rational function can have at most *one* horizontal asymptote. So we could get away with only taking $x \to \infty$).

Figure 1.6.17(a) shows a sketch of f, and the table in Figure 1.6.17(b) gives values of $f(x)$ for large magnitude values of x. It seems reasonable to conclude from both of these sources that f has a horizontal asymptote at $y = 1$.



x	$f(x)$
10	0.9615
100	0.9996
10000	0.999996
-10	0.9615
-100	0.9996
-10000	0.999996

(a) **(b)**

Figure 1.6.17 Using a graph and a table to approximate a horizontal asymptote in Example 1.6.16

Later, we will show how to determine this analytically.

The video in Figure 1.6.18 shows how to prove the result from Example 1.6.16 using the limit definition.

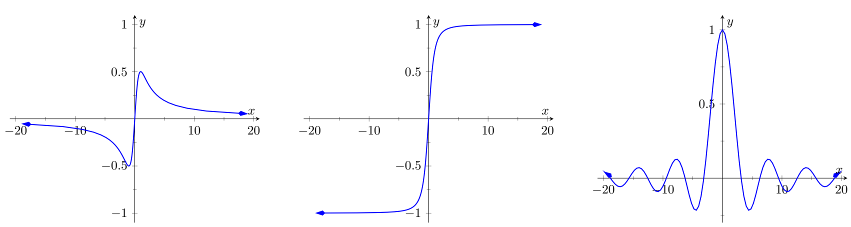
youtu.be/watch?v=kYmfeq-qKiI

Figure 1.6.18 Using an ε-δ proof with Definition 1.6.15 in Example 1.6.16

Horizontal asymptotes can take on a variety of forms. Figure 1.6.19(a) shows that $f(x) = x/(x^2+1)$ has a horizontal asymptote of $y = 0$, where 0 is approached from both above and below.

Figure 1.6.19(b) shows that $f(x) = x/\sqrt{x^2+1}$ has two horizontal asymptotes; one at $y = 1$ and the other at $y = -1$.

Figure 1.6.19(c) shows that $f(x) = \sin(x)/x$ has even more interesting behavior than at just $x = 0$; as x approaches $\pm\infty$, $f(x)$ approaches 0, but oscillates as it does this.



(a) **(b)** **(c)**

Figure 1.6.19 Considering different types of horizontal asymptotes

We can analytically evaluate limits at infinity for rational functions once we understand $\lim_{x\to\infty} \frac{1}{x}$. As x gets larger and larger, $1/x$ gets smaller and smaller, approaching 0. We can, in fact, make $1/x$ as small as we want by choosing a large enough value of x. Given ε, we can make $1/x < \varepsilon$ by choosing $x > 1/\varepsilon$. Thus we have $\lim_{x\to\infty} 1/x = 0$.

It is now not much of a jump to conclude the following:

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \qquad\qquad \lim_{x\to-\infty} \frac{1}{x^n} = 0.$$

Now suppose we need to compute the following limit:

$$\lim_{x\to\infty} \frac{x^3+2x+1}{4x^3-2x^2+9}.$$

A good way of approaching this is to divide through the numerator and denominator by x^3 (hence multiplying by 1), which is the largest power of x to appear in the denominator. Doing this, we get

$$\begin{aligned}\lim_{x\to\infty} \frac{x^3+2x+1}{4x^3-2x^2+9} &= \lim_{x\to\infty} \frac{1/x^3}{1/x^3} \cdot \frac{x^3+2x+1}{4x^3-2x^2+9}\\ &= \lim_{x\to\infty} \frac{x^3/x^3+2x/x^3+1/x^3}{4x^3/x^3-2x^2/x^3+9/x^3}\\ &= \lim_{x\to\infty} \frac{1+2/x^2+1/x^3}{4-2/x+9/x^3}.\end{aligned}$$

Then using the rules for limits (which also hold for limits at infinity), as well as the fact about limits of $1/x^n$, we see that the limit becomes

$$\frac{1+0+0}{4-0+0} = \frac{1}{4}.$$

This procedure works for any rational function. In fact, it gives us the following theorem.

Theorem 1.6.21 Limits of Rational Functions at Infinity.

Let $f(x)$ be a rational function of the following form:

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0},$$

where m, n are positive integers and where any of the coefficients may be 0 except for a_n and b_m. Then:

1. *If $n = m$, then*

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \frac{a_n}{b_m}.$$

2. *If $n < m$, then*

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0.$$

3. *If $n > m$, then $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are both infinite.*

We can see why this is true. If the highest power of x is the same in both the numerator and denominator (i.e. $n = m$), we will be in a situation like the example above, where we will divide by x^n and in the limit all the terms will approach 0 except for $a_n x^n/x^n$ and $b_m x^m/x^n$. Since $n = m$, this will leave us with the limit a_n/b_m. If $n < m$, then after dividing through by x^m, all the terms in the numerator will approach 0 in the limit, leaving us with $0/b_m$ or 0.

youtu.be/watch?v=v5SrtUsdMeU

Figure 1.6.20 Basic examples involving limits at infinity

If $n > m$, and we try dividing through by x^m, we end up with the denominator tending to b_m while the numerator tends to ∞.

Intuitively, as x gets very large, all the terms in the numerator are small in comparison to $a_n x^n$, and likewise all the terms in the denominator are small compared to $b_m x^m$. If $n = m$, looking only at these two important terms, we have $(a_n x^n)/(b_m x^m)$. This reduces to a_n/b_m. If $n < m$, the function behaves like $a_n/(b_m x^{m-n})$, which tends toward 0. If $n > m$, the function behaves like $a_n x^{n-m}/b_m$, which will tend to either ∞ or $-\infty$ depending on the values of n, m, a_n, b_m and whether you are looking for $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$.

Example 1.6.22 Finding a limit of a rational function.

Confirm analytically that $y = 1$ is the horizontal asymptote of $f(x) = \frac{x^2}{x^2+4}$, as approximated in Example 1.6.16.

Solution. Before using Theorem 1.6.21, let's use the technique of evaluating limits at infinity of rational functions that led to that theorem. The largest power of x in f is 2, so divide the numerator and denominator of f by x^2, then take limits.

$$\begin{aligned}\lim_{x\to\infty} \frac{x^2}{x^2+4} &= \lim_{x\to\infty} \frac{x^2/x^2}{x^2/x^2+4/x^2}\\ &= \lim_{x\to\infty} \frac{1}{1+4/x^2}\\ &= \frac{1}{1+0}\\ &= 1.\end{aligned}$$

We can also use Theorem 1.6.21 directly; in this case $n = m$ so the limit is the ratio of the leading coefficients of the numerator and denominator, i.e., $1/1 = 1$.

Video solution

youtu.be/watch?v=cmZ39j1YI-o

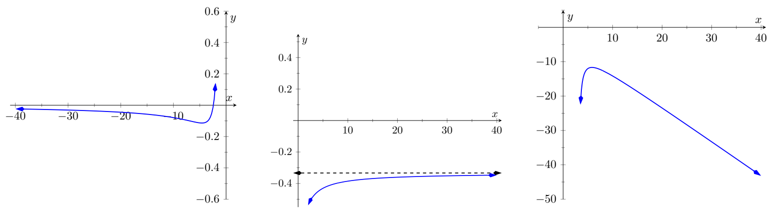
Example 1.6.23 Finding limits of rational functions.

Use Theorem 1.6.21 to evaluate each of the following limits.

1. $\displaystyle\lim_{x\to-\infty} \frac{x^2+2x-1}{x^3+1}$
2. $\displaystyle\lim_{x\to\infty} \frac{x^2+2x-1}{1-x-3x^2}$
3. $\displaystyle\lim_{x\to\infty} \frac{x^2-1}{3-x}$

Solution.

1. The highest power of x is in the denominator. Therefore, the limit is 0; see Figure 1.6.24(a).
2. The highest power of x is x^2, which occurs in both the numerator and denominator. The limit is therefore the ratio of the coefficients of x^2, which is $-1/3$. See Figure 1.6.24(b).
3. The highest power of x is in the numerator so the limit will be ∞ or $-\infty$. To see which, consider only the dominant terms from the numerator and denominator, which are x^2 and $-x$. The expression in the limit will behave like $x^2/(-x) = -x$ for large values of x. Therefore, the limit is $-\infty$. See Figure 1.6.24(c).



(a) **(b)** **(c)**

Figure 1.6.24 Visualizing the functions in Example 1.6.23

With care, we can quickly evaluate limits at infinity for a large number of functions by considering the long run behavior using "dominant terms" of $f(x)$. For instance, consider again $\lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}}$, graphed in Figure 1.6.19(b). The dominant terms are x in the numerator and $\sqrt{x^2}$ in the denominator. When x is very large, $x^2+1 \approx x^2$. Thus

$$\sqrt{x^2+1} \approx \sqrt{x^2} = |x| \qquad\qquad \frac{x}{\sqrt{x^2+1}} \approx \frac{x}{|x|}.$$

This expression is 1 when x is positive and -1 when x is negative. Hence we get asymptotes of $y=1$ and $y=-1$, respectively. We will show this more formally in the next example.

Example 1.6.25 Finding a limit using dominant terms.

Confirm analytically that $y=1$ and $y=-1$ are the horizontal asymptote of $\lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}}$, as graphed in Figure 1.6.19(b).

Solution. The dominating term of f in the denominator is $\sqrt{x^2}=|x|$ so divide the numerator and denominator of f by $\sqrt{x^2}$, then take limits.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} &= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} \cdot \frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}} \\
&= \lim_{x\to\infty} \frac{\frac{x}{|x|}}{\sqrt{\frac{x^2+1}{x^2}}} \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} \quad \text{for } x>0 \\
&= \frac{1}{\sqrt{1+0}} \\
&= 1.
\end{aligned}$$

As $x\to-\infty$, the only thing that changes is the value of $\frac{x}{|x|}$. For $x<0$, we have $\frac{x}{|x|}=-1$, making $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = -1$. Therefore, the horizontal asymptotes are $y=1$ and $y=-1$.

The video in Figure 1.6.26 provides another example similar to Example 1.6.25.

youtu.be/watch?v=vD1-zrRZQTI

Figure 1.6.26 Limits at infinity with a radical function

1.6.4 Exercises

Terms and Concepts

1. (☐ True ☐ False) If $\lim_{x\to 5} f(x) = \infty$, then we are implicitly stating that the limit exists.

2. (☐ True ☐ False) If $\lim_{x\to 5} f(x) = 5$, then we are implicitly stating that the limit exists.

3. (☐ True ☐ False) If $\lim_{x\to 1^-} f(x) = -\infty$, then $\lim_{x\to 1^+} f(x) = \infty$.

4. (☐ True ☐ False) If $\lim_{x\to 5} f(x) = \infty$, then f has a vertical asymptote at $x = 5$.

5. (☐ True ☐ False) $\infty/0$ is not an indeterminate form.

6. List five indeterminate forms.

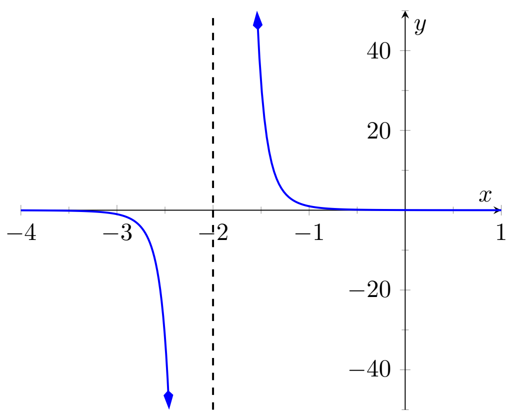
7. Construct a function with a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = 5$.

8. Let $\lim_{x\to 7} f(x) = \infty$. Explain how we know that f is or is not continuous at $x = 7$.

Problems

Exercise Group. Evaluate the given limits using the graph of the function.

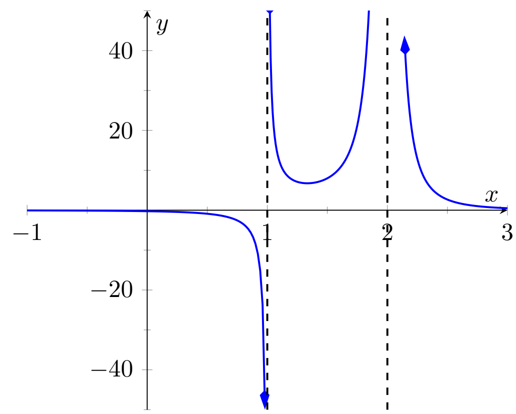
9. $f(x) = \frac{1}{(x+2)^5}$ has the graph:



(a) $\lim_{x\to -2^-} f(x)$

(b) $\lim_{x\to -2^+} f(x)$

10. $f(x) = \frac{1}{(x-1)(x-2)^2}$ has the graph:



(a) $\lim_{x\to 1^-} f(x)$

(b) $\lim_{x\to 1^+} f(x)$

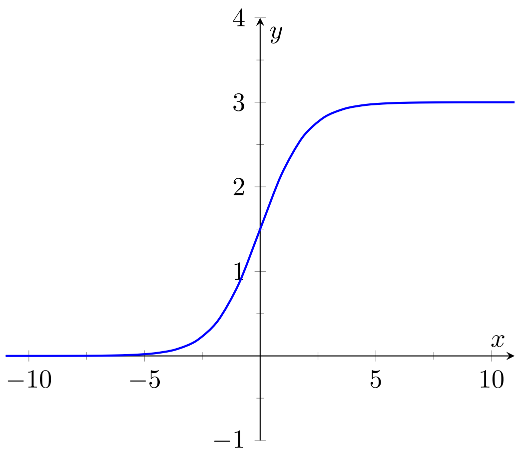
(c) $\lim_{x\to 1} f(x)$

(d) $\lim_{x\to 2^-} f(x)$

(e) $\lim_{x\to 2^+} f(x)$

(f) $\lim_{x\to 2} f(x)$

11. $f(x) = \frac{3}{e^{-x}+1}$ has the graph:



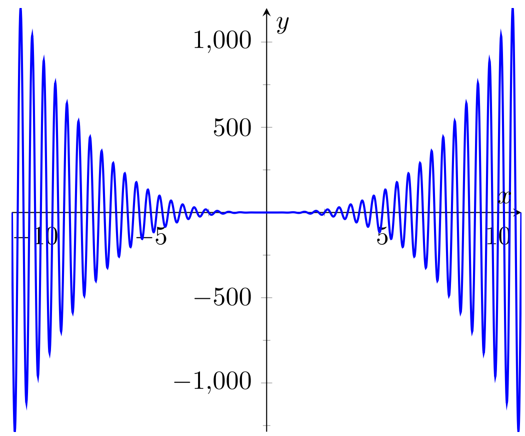
(a) $\lim_{x\to-\infty} f(x)$

(b) $\lim_{x\to\infty} f(x)$

(c) $\lim_{x\to 0^-} f(x)$

(d) $\lim_{x\to 0^+} f(x)$

12. $f(x) = x^3 \sin(4\pi x)$ has the graph:



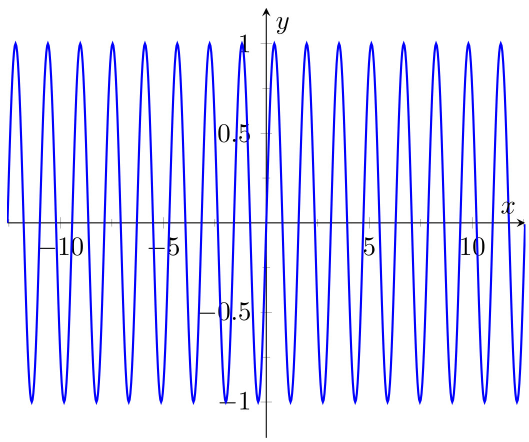
(a) $\lim_{x\to-\infty} f(x)$

(b) $\lim_{x\to\infty} f(x)$

(c) $\lim_{x\to 0^-} f(x)$

(d) $\lim_{x\to 0^+} f(x)$

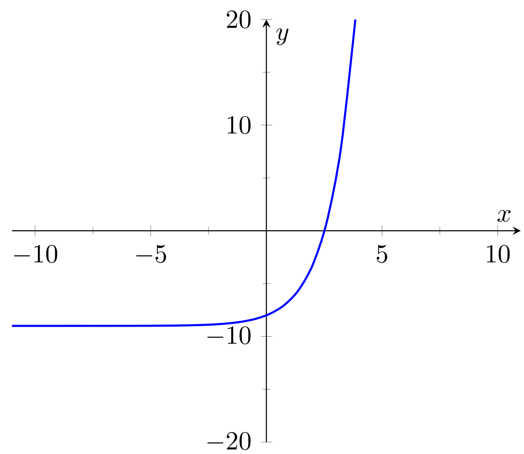
13. $f(x) = \sin(4x)$ has the graph:



(a) $\lim_{x\to-\infty} f(x)$

(b) $\lim_{x\to\infty} f(x)$

14. $f(x) = 2.4^x - 9$ has the graph:



(a) $\lim_{x\to-\infty} f(x)$

(b) $\lim_{x\to\infty} f(x)$

Exercise Group. Numerically approximate the limits.

15. $f(x) = \frac{x^2-x-20}{x^2-3x-40}$

(a) $\lim_{x\to 8^-} f(x)$

(b) $\lim_{x\to 8^+} f(x)$

(c) $\lim_{x\to 8} f(x)$

16. $f(x) = \frac{x^2-4x-5}{x^3+26x^2+225x+648}$

(a) $\lim_{x\to -9^-} f(x)$

(b) $\lim_{x\to -9^+} f(x)$

(c) $\lim_{x\to -9} f(x)$

17. $f(x) = \frac{x^2+13x+40}{x^3+7x^2-24x-180}$

(a) $\lim\limits_{x\to -6^-} f(x)$

(b) $\lim\limits_{x\to -6^+} f(x)$

(c) $\lim\limits_{x\to -6} f(x)$

18. $f(x) = \frac{x^2-x-20}{x^2+3x-4}$

(a) $\lim\limits_{x\to -4^-} f(x)$

(b) $\lim\limits_{x\to -4^+} f(x)$

(c) $\lim\limits_{x\to -4} f(x)$

Exercise Group. Identify the horizontal and vertical asymptotes, if any, of the given function.

19. $f(x) = \frac{2x^2+x-15}{x^2-7x-18}$

20. $f(x) = \frac{5x^2+x-4}{-2x^2-20x-18}$

21. $f(x) = \frac{4x^2-12x+8}{6x^3-36x^2+48x}$

22. $f(x) = \frac{2x^2-12x+16}{-6x-18}$

23. $f(x) = \frac{x^2-10x+24}{3x-18}$

24. $f(x) = \frac{4x^2-44x+96}{-x^2-4x-8}$

Exercise Group. Evaluate the given limit.

25. $\lim\limits_{x\to\infty} \frac{x^3-4x^2-x+2}{3x-3}$

26. $\lim\limits_{x\to\infty} \frac{x^3+9x^2+7x-6}{3x+8}$

27. $\lim\limits_{x\to\infty} \frac{x^3+3x^2-4x+9}{3x^2-3}$

28. $\lim\limits_{x\to\infty} \frac{x^3-5x^2+5x+3}{3x^2+8}$

Chapter Summary. In this chapter we:

- defined the limit,
- found accessible ways to approximate their values numerically and graphically,
- developed a not-so-easy method of proving the value of a limit (ε-δ proofs),
- explored when limits do not exist,
- defined continuity and explored properties of continuous functions, and
- considered limits that involved infinity.

Why? Mathematics is famous for building on itself and calculus proves to be no exception. In the next chapter we will be interested in "dividing by 0." That is, we will want to divide a quantity by a smaller and smaller number and see what value the quotient approaches. In other words, we will want to find a limit. These limits will enable us to, among other things, determine *exactly* how fast something is moving when we are only given position information.

Later, we will want to add up an infinite list of numbers. We will do so by first adding up a finite list of numbers, then take a limit as the number of things we are adding approaches infinity. Surprisingly, this sum often is finite; that is, we can add up an infinite list of numbers and get, for instance, 42.

These are just two quick examples of why we are interested in limits. Many students dislike this topic when they are first introduced to it, but over time an appreciation is often formed based on the scope of its applicability.

Chapter 2

Derivatives

Chapter 1 introduced the most fundamental of calculus topics: the limit. This chapter introduces the second most fundamental of calculus topics: the derivative. Limits describe *where* a function is going; derivatives describe *how fast* the function is going.

2.1 Instantaneous Rates of Change: The Derivative

youtu.be/watch?v=jRW9d25E_ls

Figure 2.1.1 Video introduction to Section 2.1

2.1.1 Introduction

A common amusement park ride lifts riders to a height then allows them to freefall a certain distance before safely stopping them. Suppose such a ride drops riders from a height of 150 feet. Students of physics may recall that the height (in feet) of the riders, t seconds after freefall (and ignoring air resistance, etc.) can be accurately modeled by $f(t) = -16t^2 + 150$.

Using this formula, it is easy to verify that, without intervention, the riders will hit the ground when $f(t) = 0$ so at $t = 2.5\sqrt{1.5} \approx 3.06$ seconds. Suppose the designers of the ride decide to begin slowing the riders' fall after 2 seconds (corresponding to a height of $f(2) = 86$ ft). How fast will the riders be traveling at that time?

We have been given a *position* function, but what we want to compute is a velocity at a specific point in time, i.e., we want an **instantaneous velocity**. We do not currently know how to calculate this.

However, we do know from common experience how to calculate an **average velocity**. (If we travel 60 miles in 2 hours, we know we had an average velocity of 30 mph.) We looked at this concept in Section 1.1 when we introduced the difference quotient. We have

$$\frac{\text{change in distance}}{\text{change in time}} = \frac{\text{“rise”}}{\text{“run”}} = \text{average velocity.}$$

We can approximate the instantaneous velocity at $t = 2$ by considering the average velocity over some time period containing $t = 2$. If we make the time interval small, we will get a good approximation. (This fact is commonly used. For instance, high speed cameras are used to track fast moving objects. Distances are measured over a fixed number of frames to generate an accurate approximation of the velocity.)

Consider the interval from $t = 2$ to $t = 3$ (just before the riders hit the ground). On that interval, the average velocity is

$$\frac{f(3) - f(2)}{3 - 2} = \frac{6 - 86}{1} = -80 \text{ ft/s},$$

where the minus sign indicates that the riders are moving *down*. By narrowing the interval we consider, we will likely get a better approximation of the instantaneous velocity. On $[2, 2.5]$ we have

$$\frac{f(2.5) - f(2)}{2.5 - 2} = \frac{50 - 86}{0.5} = -72 \text{ ft/s}.$$

Units in Calculations. In the above calculations, we left off the units until the end of the problem. You should always be sure that you label your answer with the correct units. For example, if $g(x)$ gave you the cost (in $) of producing x widgets, the units on the difference quotient would be $/widget.

We can do this for smaller and smaller intervals of time. For instance, over a time span of one tenth of a second, i.e., on $[2, 2.1]$, we have

$$\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{79.44 - 86}{0.1} = -65.6 \text{ ft/s}.$$

Over a time span of one hundredth of a second, on $[2, 2.01]$, the average velocity is

$$\frac{f(2.01) - f(2)}{2.01 - 2} = \frac{85.3584 - 86}{0.01} = -64.16 \text{ ft/s}.$$

What we are really computing is the average velocity on the interval $[2, 2+h]$ for small values of h. That is, we are computing

$$\frac{f(2 + h) - f(2)}{h}$$

where h is small.

We really want to use $h = 0$, but this, of course, returns the familiar "$0/0$" indeterminate form. So we employ a limit, as we did in Section 1.1.

We can approximate the value of this limit numerically with small values of h as seen in Figure 2.1.2. It looks as though the velocity is approaching $-64\,\frac{\text{ft}}{\text{s}}$.

Computing the limit directly gives

$$\begin{aligned}
\lim_{h\to 0} \frac{f(2 + h) - f(2)}{h} &= \lim_{h\to 0} \frac{-16(2 + h)^2 + 150 - (-16(2)^2 + 150)}{h} \\
&= \lim_{h\to 0} \frac{-16(4 + 4h + h^2) + 150 - 86}{h} \\
&= \lim_{h\to 0} \frac{-64 - 64h - 16h^2 + 64}{h} \\
&= \lim_{h\to 0} \frac{-64h - 16h^2}{h} \\
&= \lim_{h\to 0} (-64 - 16h) \\
&= -64.
\end{aligned}$$

h	Average Velocity ($\frac{\text{ft}}{\text{s}}$)
1	-80
0.5	-72
0.1	-65.6
0.01	-64.16
0.001	-64.016

Figure 2.1.2 Approximating the instantaneous velocity with average velocities over a small time period h

Graphically, we can view the average velocities we computed numerically as the slopes of secant lines on the graph of f going through the points $(2, f(2))$ and $(2 + h, f(2 + h))$. In Figures 2.1.3–2.1.5, the secant line corresponding to $h = 1$ is shown in three contexts. Figure 2.1.3 shows a "zoomed out" version of f with its secant line. In Figure 2.1.4, we zoom in around the points of intersection between f and the secant line. Notice how well this secant line approximates f between those two points — it is a common practice to approximate functions with straight lines.

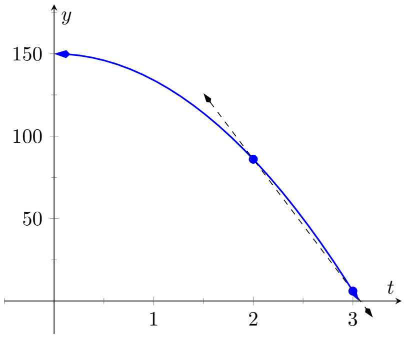


Figure 2.1.3 The function $f(t)$ and its secant line corresponding to $t = 2$ and $t = 3$

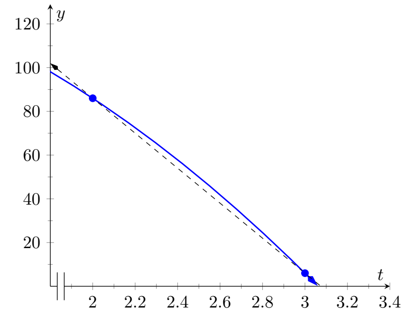


Figure 2.1.4 The function $f(t)$ and a secant line corresponding to $t = 2$ and $t = 3$, zoomed in near $t = 2$

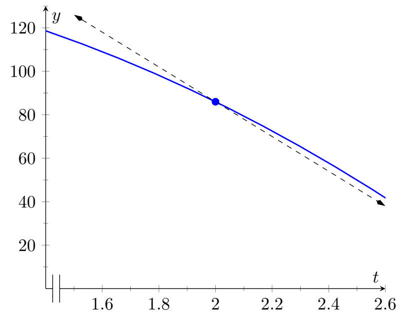


Figure 2.1.5 The function $f(t)$ with the same secant line, zoomed in further

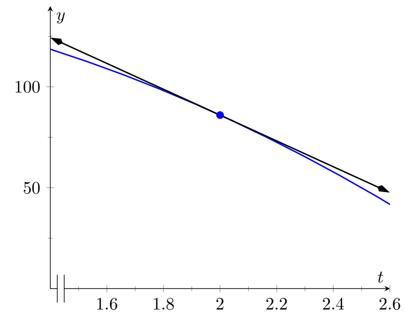


Figure 2.1.6 The function $f(t)$ with its tangent line at $t = 2$

As $h \to 0$, these secant lines approach the **tangent line**, a line that goes through the point $(2, f(2))$ with the special slope of -64. In Figure 2.1.5 and Figure 2.1.6, we zoom in around the point $(2, 86)$. We see the secant line, which approximates f well, but not as well the tangent line shown in Figure 2.1.6.

We have just introduced a number of important concepts that we will flesh out more within this section. First, we formally define two of them.

Definition 2.1.7 Derivative at a Point.

Let f be a continuous function on an open interval I and let c be in I. The **derivative** of f at c, denoted $f'(c)$, is

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h},$$

provided the limit exists. If the limit exists, we say that f is **differentiable** at c; if the limit does not exist, then f is not differentiable at c. If f is differentiable at every point in I, then f is differentiable on I.

Definition 2.1.9 Tangent Line.

Let f be continuous on an open interval I and differentiable at c, for some c in I. The line with equation $\ell(x) = f'(c)(x-c) + f(c)$ is the **tangent line** to the graph of f at c; that is, it is the line through $(c, f(c))$ whose slope is the derivative of f at c.

youtu.be/watch?v=JXFMh21PMx4

Figure 2.1.8 Video presentation of Definition 2.1.7

Some examples will help us understand these definitions.

Example 2.1.10 Finding derivatives and tangent lines.

Let $f(x) = 3x^2 + 5x - 7$. Find:

(a) $f'(1)$

(b) The equation of the tangent line to the graph of f at $x = 1$.

(c) $f'(3)$

(d) The equation of the tangent line to the graph f at $x = 3$.

Solution.

(a) We compute this directly using Definition 2.1.7.

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0} \frac{3(1+h)^2+5(1+h)-7-(3(1)^2+5(1)-7)}{h} \\ &= \lim_{h\to 0} \frac{3(1+2h+h^2)+5+5h-7-1}{h} \\ &= \lim_{h\to 0} \frac{3+6h+3h^2+5+5h-8}{h} \\ &= \lim_{h\to 0} \frac{3h^2+11h}{h} \\ &= \lim_{h\to 0} (3h+11) \\ &= 11. \end{aligned}$$

(b) The tangent line at $x = 1$ has slope $f'(1)$ and goes through the point $(1, f(1)) = (1, 1)$. Thus the tangent line has equation, in point-slope form, $y = 11(x-1)+1$. In slope-intercept form we have $y = 11x - 10$.

(c) Again, using the definition,

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \\ &= \lim_{h\to 0} \frac{3(3+h)^2+5(3+h)-7-(3(3)^2+5(3)-7)}{h} \\ &= \lim_{h\to 0} \frac{3(9+6h+h^2)+15+3h-7-35}{h} \\ &= \lim_{h\to 0} \frac{27+18h+3h^2+15+3h-42}{h} \\ &= \lim_{h\to 0} \frac{3h^2+23h}{h} \\ &= \lim_{h\to 0} 3h+23 \\ &= 23. \end{aligned}$$

(d) The tangent line at $x = 3$ has slope 23 and goes through the point $(3, f(3)) = (3, 35)$. Thus the tangent line has equation $y = 23(x-3)+35 = 23x - 34$.

A graph of f is given in Figure 2.1.11 along with the tangent lines at $x = 1$ and $x = 3$.

Video solution

youtu.be/watch?v=4OMc0gJWcb0

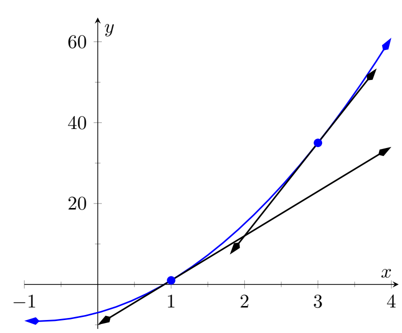


Figure 2.1.11 A graph of $f(x) = 3x^2 + 5x - 7$ and its tangent lines at $x = 1$ and $x = 3$

In Definition 2.1.7, we assumed that the function is continuous, but this is actually not necessary. One can in fact prove that a function has to be continuous at any point where it is differentiable. Or, in other words, a function cannot be differentiable at a point of discontinuity. This is explained in the video in Figure 2.1.12.

youtu.be/watch?v=Ev9hJVbDO1k

Figure 2.1.12 Showing that every differentiable function is continuous

Another important line that can be created using information from the derivative is the *normal line*. It is perpendicular to the tangent line, hence its slope is the negative-reciprocal of the tangent line's slope.

Definition 2.1.13 Normal Line.

Let f be continuous on an open interval I and differentiable at c, for some c in I. The **normal line** to the graph of f at c is the line with equation

$$n(x) = \frac{-1}{f'(c)}(x-c)+f(c),$$

when $f'(c) \neq 0$. (When $f'(c) = 0$, the normal line is the vertical line through $(c, f(c))$; that is, $x = c$.)

Example 2.1.14 Finding equations of normal lines.

Let $f(x) = 3x^2 + 5x - 7$, as in Example 2.1.10. Find the equations of the normal lines to the graph of f at $x = 1$ and $x = 3$.

Solution. In Example 2.1.10, we found that $f'(1) = 11$. Hence at $x = 1$, the normal line will have slope $-1/11$. An equation for the normal line is

$$n(x) = \frac{-1}{11}(x-1)+1.$$

The normal line is plotted with $y = f(x)$ in Figure 2.1.15. Note how the line looks perpendicular to f. (A key word here is "looks." Mathematically, we say that the normal line *is* perpendicular to f at $x = 1$ as the slope of the normal line is the negative-reciprocal of the slope of the tangent line. However, normal lines may not always *look* perpendicular. The aspect ratio of the picture of the graph plays a big role in this. When using graphing software, there is usually an option called `Zoom Square` that keeps the aspect ratio $1:1$

We also found that $f'(3) = 23$, so the normal line to the graph of f at $x = 3$ will have slope $-1/23$. An equation for the normal line is

$$n(x) = \frac{-1}{23}(x-3)+35.$$

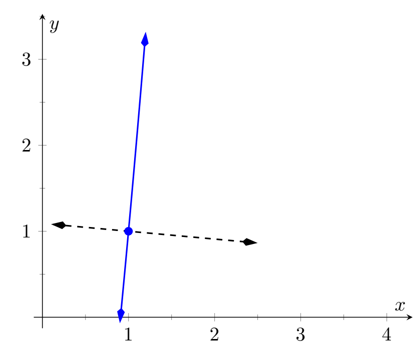


Figure 2.1.15 A graph of $f(x) = 3x^2 + 5x - 7$, along with its normal line at $x = 1$

Linear functions are easy to work with; many functions that arise in the course of solving real problems are not easy to work with. A common practice in mathematical problem solving is to approximate difficult functions with not-so-difficult functions. Lines are a common choice. It turns out that at any given point on the graph of a differentiable function f, the best linear approximation to f is its tangent line. That is one reason we'll spend considerable time finding tangent lines to functions.

One type of function that does not benefit from a tangent line approximation is a line; it is rather simple to recognize that the tangent line to a line is the line itself. We look at this in the following example.

Example 2.1.16 Finding the derivative of a linear function.

Consider $f(x) = 3x + 5$. Find the equation of the tangent line to f at $x = 1$ and $x = 7$.

Solution. We find the slope of the tangent line by using Definition 2.1.7.

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{3(1+h) + 5 - (3+5)}{h} \\ &= \lim_{h\to 0} \frac{3h}{h} \\ &= \lim_{h\to 0} 3 \\ &= 3. \end{aligned}$$

We just found that $f'(1) = 3$. That is, we found the **instantaneous rate of change** of $f(x) = 3x + 5$ is 3. This is not surprising; lines are characterized by being the *only* functions with a *constant rate of change*. That rate of change is called the **slope** of the line. Since their rates of change are constant, their *instantaneous* rates of change are always the same; they are all the slope.

So given a line $f(x) = ax + b$, the derivative at any point x will be a; that is, $f'(x) = a$.

It is now easy to see that the tangent line to the graph of f at $x = 1$ is just f, with the same being true at $x = 7$.

We often desire to find the tangent line to the graph of a function without knowing the actual derivative of the function. While we will eventually be able to find derivatives of many common functions, the algebra and limit calculations on some functions are complex. Until we develop further techniques, the best we may be able to do is approximate the tangent line. We demonstrate this in the next example.

Example 2.1.17 Numerical approximation of the tangent line.

Approximate the equation of the tangent line to the graph of $f(x) = \sin(x)$ at $x = 0$.

Solution. In order to find the equation of the tangent line, we need a slope and a point. The point is given to us: $(0, \sin(0)) = (0, 0)$. To compute the slope, we need the derivative. This is where we will make an approximation. Recall that

$$f'(0) \approx \frac{\sin(0+h) - \sin(0)}{h}$$

for a small value of h. We choose (somewhat arbitrarily) to let $h = 0.1$. Thus

$$f'(0) \approx \frac{\sin(0.1) - \sin(0)}{0.1} \approx 0.9983.$$

Thus our approximation of the equation of the tangent line is $y = 0.9983(x - 0) + 0 = 0.9983x$; it is graphed in Figure 2.1.18. The graph seems to imply the approximation is rather good.

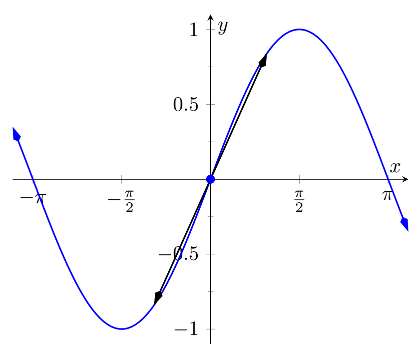


Figure 2.1.18 $f(x) = \sin(x)$ graphed with an approximation to its tangent line at $x = 0$

Recall from Section 1.3 that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, meaning for values of x near

0, $\sin(x) \approx x$. Since the slope of the line $y = x$ is 1 at $x = 0$, it should seem reasonable that "the slope of $f(x) = \sin(x)$" is near 1 at $x = 0$. In fact, since we *approximated* the value of the slope to be 0.9983, we might guess the *actual value* is 1. We'll come back to this later.

Consider again Example 2.1.10. To find the derivative of f at $x = 1$, we needed to evaluate a limit. To find the derivative of f at $x = 3$, we needed to again evaluate a limit. We have this process:

input specific number c	$\longrightarrow$	do something to f and c	$\longrightarrow$	return number $f'(c)$

This process describes a **function**; given one input (the value of c), we return exactly one output (the value of $f'(c)$). The "do something" box is where the tedious work (taking limits) of this function occurs.

Instead of applying this function repeatedly for different values of c, let us apply it just once to the variable x. We then take a limit just once. The process now looks like:

input variable x	$\longrightarrow$	do something to f and x	$\longrightarrow$	return function $f'(x)$

The output is the **derivative function**, $f'(x)$. The $f'(x)$ function will take a number c as input and return the derivative of f at c. This calls for a definition.

Definition 2.1.19 Derivative Function.

Let f be a differentiable function on an open interval I. The function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is the **derivative** of f.

Let $y = f(x)$. The following notations all represent the derivative of f:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}(f) = \frac{d}{dx}(y).$$

Important: The notation $\frac{dy}{dx}$ is one symbol; it is *not* the fraction "dy/dx". The notation, while somewhat confusing at first, was chosen with care. A fraction-looking symbol was chosen because the derivative has many fraction-like properties. Among other places, we see these properties at work when we talk about the units of the derivative, when we discuss the Chain Rule, and when we learn about integration (topics that appear in later sections and chapters).

Examples will help us understand this definition.

youtu.be/watch?v=yPzNYlzA0Js

Figure 2.1.20 Video presentation of Definition 2.1.19

youtu.be/watch?v=TJhJiA_w4mQ

Figure 2.1.21 Explaining derivative notation

Example 2.1.22 Finding the derivative of a function.

Let $f(x) = 3x^2 + 5x - 7$ as in Example 2.1.10. Find $f'(x)$.

Solution. We apply Definition 2.1.19.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{3(x+h)^2 + 5(x+h) - 7 - (3x^2 + 5x - 7)}{h} \\ &= \lim_{h \to 0} \frac{3h^2 + 6xh + 5h}{h} \end{aligned}$$

$$= \lim_{h\to 0}(3h+6x+5)$$
$$= 6x+5$$

So $f'(x) = 6x+5$. Recall earlier we found that $f'(1) = 11$ and $f'(3) = 23$. Note our new computation of $f'(x)$ affirms these facts.

Example 2.1.23 Finding the derivative of a function.

Let $f(x) = \frac{1}{x+1}$. Find $f'(x)$.

Solution. We apply Definition 2.1.19.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\frac{1}{x+h+1}-\frac{1}{x+1}}{h} \end{aligned}$$

Now find common denominator then subtract; pull $1/h$ out front to facilitate reading.

$$= \lim_{h\to 0}\frac{1}{h}\cdot\left(\frac{x+1}{(x+1)(x+h+1)}-\frac{x+h+1}{(x+1)(x+h+1)}\right)$$

Now simplify algebraically.

$$\begin{aligned} &= \lim_{h\to 0}\frac{1}{h}\cdot\left(\frac{x+1-(x+h+1)}{(x+1)(x+h+1)}\right) \\ &= \lim_{h\to 0}\frac{1}{h}\cdot\left(\frac{-h}{(x+1)(x+h+1)}\right) \end{aligned}$$

Finally, apply the limit.

$$\begin{aligned} &= \lim_{h\to 0}\frac{-1}{(x+1)(x+h+1)} \\ &= \frac{-1}{(x+1)(x+1)} \\ &= \frac{-1}{(x+1)^2}. \end{aligned}$$

So $f'(x) = \frac{-1}{(x+1)^2}$. To practice using our notation, we could also state

$$\frac{d}{dx}\left(\frac{1}{x+1}\right) = \frac{-1}{(x+1)^2}.$$

Video solution

youtu.be/watch?v=JKHbXYanjDs

Example 2.1.24 Finding the derivative of a function.

Find the derivative of $f(x) = \sin(x)$.

Solution (a). Before applying Definition 2.1.19, note that once this is found, we can find the actual tangent line to $f(x) = \sin(x)$ at $x = 0$, whereas we settled for an approximation in Example 2.1.17.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} && \text{Derivative definition}\\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} && \text{Angle addition identity}\\
&= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h} && \text{Regrouped and factored}\\
&= \lim_{h\to 0} \left(\frac{\sin(x)(\cos(h)-1)}{h}+\frac{\cos(x)\sin(h)}{h}\right) && \text{Split into two fractions}\\
&= \lim_{h\to 0} \sin(x)\cdot \lim_{h\to 0} \frac{\cos(h)-1}{h}\\
&\quad + \lim_{h\to 0} \cos(x)\cdot \lim_{h\to 0} \frac{\sin(h)}{h} && \text{Product/sum limit rules}\\
&= \sin(x)\cdot 0+\cos(x)\cdot 1 && \text{Applied Theorem 1.3.17}\\
&= \cos(x). && \text{(Are you surprised?)}
\end{aligned}$$

We have found that when $f(x) = \sin(x)$, $f'(x) = \cos(x)$. This should be somewhat amazing; the result of a tedious limit process on the sine function is a nice function. Then again, perhaps this is not entirely surprising. The sine function is periodic — it repeats itself on regular intervals. Therefore its rate of change also repeats itself on the same regular intervals. We should have known the derivative would be periodic; we now know exactly which periodic function it is.

Thinking back to Example 2.1.17, we can find the slope of the tangent line to $f(x) = \sin(x)$ at $x = 0$ using our derivative. We approximated the slope as 0.9983; we now know the slope is *exactly* $\cos(0) = 1$.

Using similar techniques, we can show that the derivative of $\cos(x)$ is $-\sin(x)$. See if you can show this yourself; if you get stuck, you can check out the video in Figure 2.1.25.

Video solution

youtu.be/watch?v=vsnDopbWHXQ

youtu.be/watch?v=-1lOFzhDJAo

Figure 2.1.25 Finding the derivative of $\cos(x)$

Example 2.1.26 Finding the derivative of a piecewise defined function.

Find the derivative of the absolute value function,

$$f(x) = |x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}.$$

See Figure 2.1.27.

Solution. We need to evaluate $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. As f is piecewise-defined, we need to consider separately the limits when $x < 0$ and when $x > 0$.

When $x < 0$:

$$\begin{aligned}
\frac{d}{dx}(-x) &= \lim_{h\to 0} \frac{-(x+h)-(-x)}{h}\\
&= \lim_{h\to 0} \frac{-h}{h}\\
&= \lim_{h\to 0} -1\\
&= -1.
\end{aligned}$$

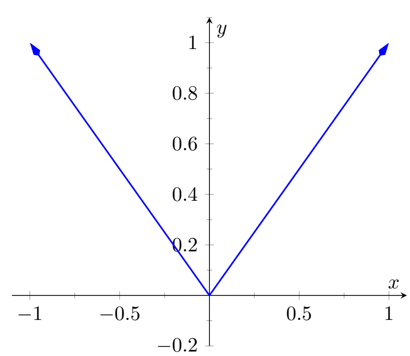


Figure 2.1.27 The absolute value function $f(x) = |x|$. Notice how the slope of the lines (and hence the tangent lines) abruptly changes at $x = 0$.

When $x > 0$, a similar computation shows that $\frac{d}{dx}(x) = 1$.
We need to also find the derivative at $x = 0$. By the definition of the derivative at a point, we have

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}.$$

Since $x = 0$ is the point where our function's definition switches from one piece to the other, we need to consider left and right-hand limits. Consider the following, where we compute the left and right hand limits side by side.

$$\begin{aligned} &\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0^-} \frac{-h-0}{h} \\ &= \lim_{h\to 0^-} -1 \\ &= -1 \end{aligned} \qquad \begin{aligned} &\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0^+} \frac{h-0}{h} \\ &= \lim_{h\to 0^+} 1 \\ &= 1 \end{aligned}$$

The last lines of each column tell the story: the left and right hand limits are not equal. Therefore the limit does not exist at 0, and f is not differentiable at 0. So we have

$$f'(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}.$$

At $x = 0$, $f'(x)$ does not exist; there is a jump discontinuity at 0; see Figure 2.1.28. So $f(x) = |x|$ is differentiable everywhere except at 0.

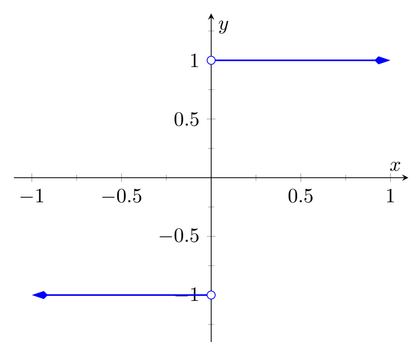


Figure 2.1.28 A graph of the derivative of $f(x) = |x|$

The point of non-differentiability came where the piecewise defined function switched from one piece to the other. Our next example shows that this does not always cause trouble.

Example 2.1.29 Finding the derivative of a piecewise defined function.

Find the derivative of $f(x)$, where

$$f(x) = \begin{cases} \sin(x) & x \le \pi/2 \\ 1 & x > \pi/2 \end{cases}.$$

See Figure 2.1.30.

Solution. Using Example 2.1.24, we know that when $x < \pi/2$, $f'(x) = \cos(x)$. It is easy to verify that when $x > \pi/2$, $f'(x) = 0$; consider:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1-1}{h} = \lim_{h\to 0} 0 = 0.$$

So far we have

$$f'(x) = \begin{cases} \cos(x) & x < \pi/2 \\ 0 & x > \pi/2 \end{cases}.$$

We still need to find $f'(\pi/2)$. Notice at $x = \pi/2$ that both pieces of f' are 0, meaning we can state that $f'(\pi/2) = 0$.

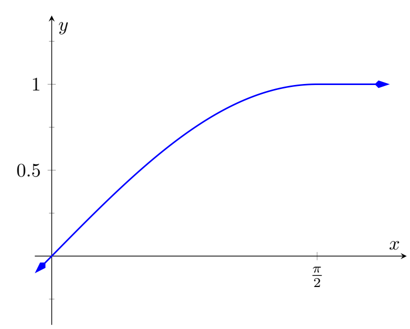


Figure 2.1.30 A graph of $f(x)$ as defined in Example 2.1.29

Being more rigorous, we can again evaluate the difference quotient limit at $x = \pi/2$, utilizing again left- and right-hand limits. We will begin with the left-hand limit:

$$\begin{aligned}
&\lim_{h\to 0^-} \frac{f(\pi/2+h)-f(\pi/2)}{h}\\
&= \lim_{h\to 0^-} \frac{\sin(\pi/2+h)-\sin(\pi/2)}{h}\\
&= \lim_{h\to 0^-} \frac{\sin(\frac{\pi}{2})\cos(h)+\sin(h)\cos(\frac{\pi}{2})-\sin(\frac{\pi}{2})}{h}\\
&= \lim_{h\to 0^-} \frac{1\cdot\cos(h)+\sin(h)\cdot 0-1}{h}\\
&= \lim_{h\to 0^-} \frac{\cos(h)-1}{h}\cdot\lim_{h\to 0^-}\frac{\sin(h)}{h}\\
&= 1\cdot 0\\
&= 0.
\end{aligned}$$

Notice we used Limits of Common Functions to finally evaluate the limit. Now we will find the right-hand limit:

$$\begin{aligned}
&\lim_{h\to 0^+} \frac{f(\pi/2+h)-f(\pi/2)}{h}\\
&= \lim_{h\to 0^+} \frac{1-1}{h}\\
&= \lim_{h\to 0^+} \frac{0}{h}\\
&= 0.
\end{aligned}$$

Since both the left and right hand limits are 0 at $x = \pi/2$, the limit exists and $f'(\pi/2)$ exists (and is 0). Therefore we can fully write f' as

$$f'(x) = \begin{cases} \cos(x) & x \le \pi/2 \\ 0 & x > \pi/2 \end{cases}.$$

See Figure 2.1.31 for a graph of this derivative function.

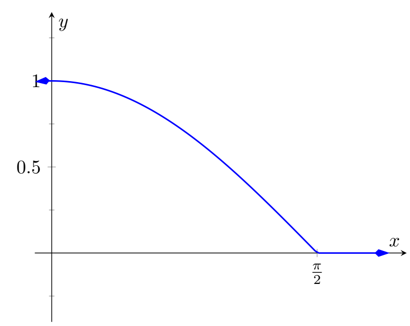


Figure 2.1.31 A graph of $f'(x)$ in Example 2.1.29.

For one more example involving piecewise-defined functions, we turn to the video in Figure 2.1.32.

Recall we pseudo-defined a continuous function as one in which we could sketch its graph without lifting our pencil. We can give a pseudo-definition for differentiability as well: it is a continuous function that does not have any "sharp corners" or a vertical tangent line. One such sharp corner is shown in Figure 2.1.27. Even though the function f in Example 2.1.29 is piecewise-defined, the transition is "smooth" hence it is differentiable. Note how in the graph of f in Figure 2.1.30 it is difficult to tell when f switches from one piece to the other; there is no "corner."

youtu.be/watch?v=q1ZAqRsgVPk

Figure 2.1.32 Determining when a piecewise-defined function is differentiable

2.1.2 Differentiability on Closed Intervals

When we defined the derivative at a point in Definition 2.1.7, we specified that the interval I over which a function f was defined needed to be an open interval. Open intervals are required so that we can take a limit at any point c in I, meaning we want to approach c from both the left and right.

Recall we also required open intervals in Definition 1.5.1 when we defined what it meant for a function to be continuous. Later, we used one-sided limits to extend continuity to closed intervals. We now extend differentiability to closed intervals by again considering one-sided limits.

Our motivation is three-fold. First, we consider "common sense." In Example 2.1.22 we found that when $f(x) = 3x^2 + 5x - 7$, $f'(x) = 6x + 5$, and this derivative is defined for all real numbers, hence f is differentiable everywhere. It seems appropriate to also conclude that f is differentiable on closed intervals, like $[0, 1]$, as well. After all, $f'(x)$ is defined at both $x = 0$ and $x = 1$.

Secondly, consider $f(x) = \sqrt{x}$. The domain of f is $[0, \infty)$. Is f differentiable on its domain — specifically, is f differentiable at 0? (We'll consider this in the next example.)

Finally, in later sections, having the derivative defined on closed intervals will prove useful. One such place is Section 7.4 where the derivative plays a role in measuring the length of a curve.

After a formal definition of differentiability on a closed interval, we explore the concept in an example.

Definition 2.1.33 Differentiability on a Closed Interval.

Let f be continuous on $[a, b]$ and differentiable on (a, b). If the one-sided limits

$$\lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} \qquad\qquad \lim_{h\to 0^-} \frac{f(b+h) - f(b)}{h}$$

exist, then we say f is differentiable on $[a, b]$.

For all the functions f *in this text*, we can determine differentiability on $[a, b]$ by considering the limits $\lim_{x\to a^+} f'(x)$ and $\lim_{x\to b^-} f'(x)$. This is often easier to evaluate than the limit of the difference quotient.

Example 2.1.34 Differentiability at an endpoint.

Consider $f(x) = \sqrt{x} = x^{1/2}$ and $g(x) = \sqrt{x^3} = x^{3/2}$. The domain of each function is $[0, \infty)$. It can be shown that each is differentiable on $(0, \infty)$; determine the differentiability of each at $x = 0$.

Solution. We start by considering f and take the right-hand limit of the difference quotient:

$$\begin{aligned}\lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} &= \lim_{h\to 0^+} \frac{\sqrt{0+h} - \sqrt{0}}{h}\\ &= \lim_{h\to 0^+} \frac{\sqrt{h}}{h}\\ &= \lim_{h\to 0^+} \frac{1}{h^{1/2}} = \infty.\end{aligned}$$

The one-sided limit of the difference quotient does not exist at $x = 0$ for f; therefore f is differentiable on $(0, \infty)$ and not differentiable on $[0, \infty)$.

We state (without proof) that $f'(x) = 1/\big(2\sqrt{x}\big)$. Note that $\lim_{x\to 0^+} f'(x) = \infty$; this limit was easier to evaluate than the limit of the difference quotient, though it required us to already know the derivative of f.

Now consider g:

$$\lim_{h\to 0^+}\frac{g(a+h)-g(a)}{h} = \lim_{h\to 0^+}\frac{\sqrt{(0+h)^3}-\sqrt{0}}{h} = \lim_{h\to 0^+}\frac{h^{3/2}}{h} = \lim_{h\to 0^+} h^{1/2} = 0.$$

As the one-sided limit exists at $x=0$, we conclude g is differentiable on its domain of $[0,\infty)$.

We state (without proof) that $g'(x) = 3\sqrt{x}/2$. Note that $\lim_{x\to 0^+} g'(x) = 0$; again, this limit is easier to evaluate than the limit of the difference quotient.

The two functions are graphed in Figure 2.1.35. Note how $f(x)=\sqrt{x}$ seems to "go vertical" as x approaches 0, implying the slopes of its tangent lines are growing toward infinity. Also note how the slopes of the tangent lines to $g(x)=\sqrt{x^3}$ approach 0 as x approaches 0.

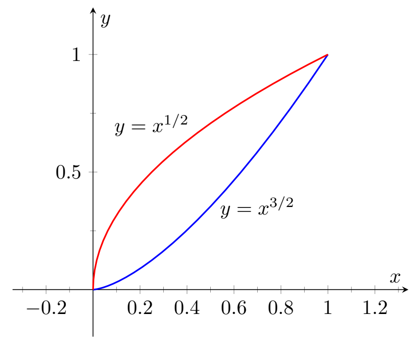


Figure 2.1.35 A graph of $y=x^{1/2}$ and $y=x^{3/2}$ in Example 2.1.34

Most calculus textbooks omit this topic and simply avoid specific cases where it could be applied. We choose in this text to not make use of the topic unless it is "needed." Many theorems in later sections require a function f to be differentiable on an *open* interval I; we could remove the word "open" and just use ". . . on an interval I," but choose to not do so in keeping with the current mathematical tradition. Our first use of differentiability on closed intervals comes in Chapter 7, where we measure the lengths of curves.

This section defined the derivative; in some sense, it answers the question of "What *is* the derivative?" The next section addresses the question "What does the derivative *mean*?"

2.1.3 Exercises

Terms and Concepts

1. (☐ True ☐ False) Let f be a position function. The average rate of change on $[a,b]$ is the slope of the line through the points $(a, f(a))$ and $(b, f(b))$.

2. (☐ True ☐ False) The definition of the derivative of a function at a point involves taking a limit.

3. In your own words, explain the difference between the average rate of change and instantaneous rate of change.

4. In your own words, explain the difference between Definitions 2.1.7 and Definition 2.1.19.

5. Let $y = f(x)$. Give three different notations equivalent to "$f'(x)$."

6. If two lines are perpendicular, what is true of their slopes?

Problems

Exercise Group. Use the definition of the derivative to compute the derivative of the given function.

7. $f(x) = 6$

8. $f(x) = 2x$

9. $f(t) = 4 - 3t$

10. $g(x) = x^2$

11. $h(x) = x^3$

12. $f(x) = 3x^2 - x + 4$

13. $r(x) = \frac{1}{x}$

14. $r(s) = \frac{1}{s-2}$

Exercise Group. A function and an x-value are given. (Note: these functions are the same as those given in Exercises 7–14.) Give the equations of the tangent line and the normal line at that x-value.

15. $f(x) = 6$ at $x = -2$

16. $f(x) = 2x$ at $x = 3$

17. $f(x) = 4 - 3x$ at $x = 7$

18. $g(x) = x^2$ at $x = 2$

19. $h(x) = x^3$ at $x = 4$

20. $f(x) = 3x^2 - x + 4$ at $x = -1$

21. $r(x) = \frac{1}{x}$ at $x = -2$

22. $r(x) = \frac{1}{x-2}$ at $x = 3$

Exercise Group. A function f and an x-value a are given. Approximate the equation of the tangent line to the graph of f at $x = a$ by numerically approximating $f'(a)$, using $h = 0.1$.

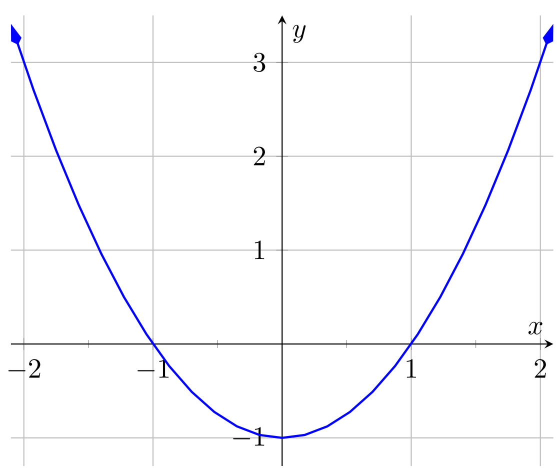
23. $f(x) = x^2 - 2x + 5$ and $a = -2$

24. $f(x) = -\frac{10}{x+8}$ and $a = -9$

25. $f(x) = e^x$ and $a = -4$

26. $f(x) = \cos(x)$ and $a = 0$

27. The graph of $f(x) = x^2 - 1$ is shown.

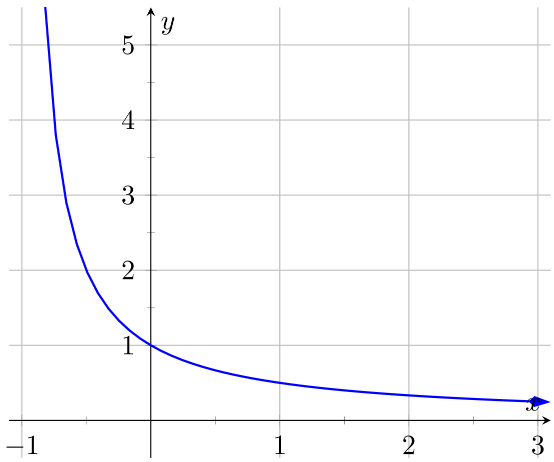


(a) Use the graph to approximate the slope of the tangent line to f at $(-1, 0)$, $(0, -1)$, and $(2, 3)$.

(b) Using the definition of the derivative, find $f'(x)$.

(c) Use the derivative to find the slope of the tangent line at the points $(-1, 0)$, $(0, -1)$ and $(2, 3)$.

28. The graph of $f(x) = \frac{1}{x+1}$ is shown.



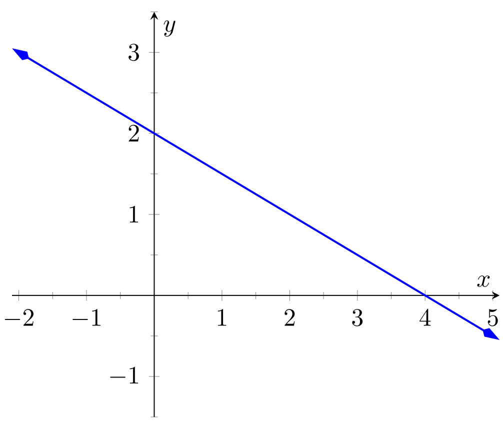
(a) Use the graph to approximate the slope of the tangent line to f at $(0, 1)$ and $(1, 0.5)$.

(b) Using the definition of the derivative, find $f'(x)$.

(c) Use the derivative to find the slope of the tangent line at the points $(0, 1)$ and $(1, 0.5)$.

Exercise Group. A graph of a function $f(x)$ is given. Using the graph, sketch $f'(x)$.

29.



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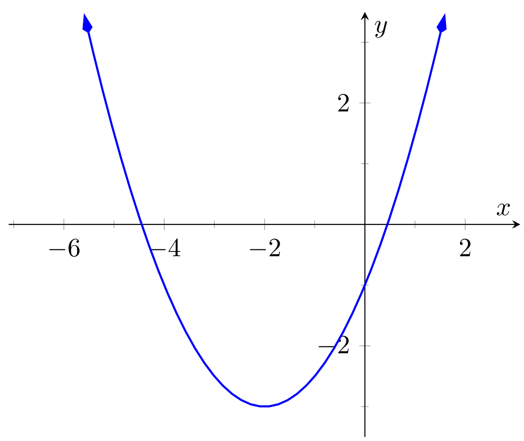

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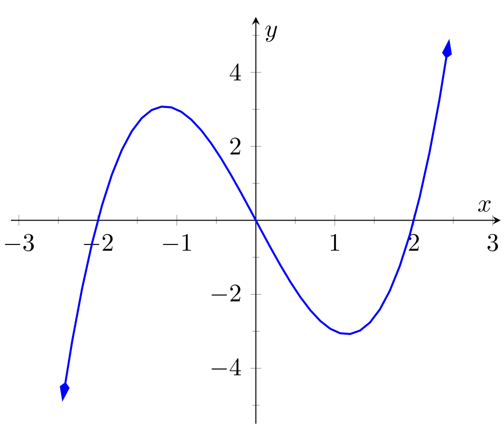

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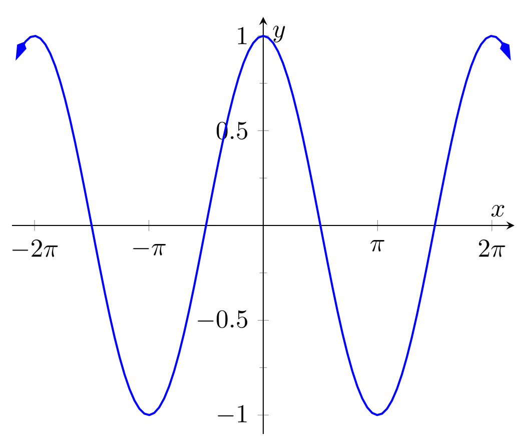


Exercise Group. Use the graph of the function to answer the following questions.

(a) Where is $g(x) > 0$?

(b) Where is $g(x) < 0$?

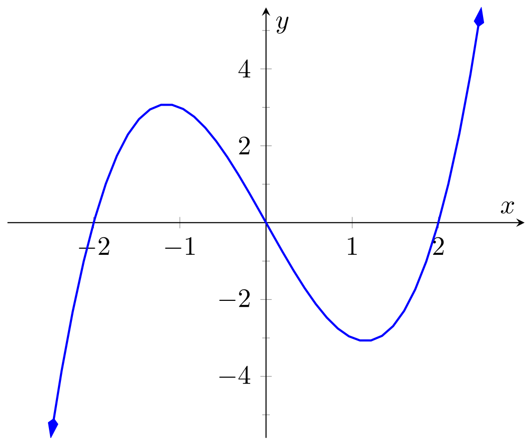
(c) Where is $g(x) = 0$?

(d) Where is $g'(x) < 0$?

(e) Where is $g'(x) > 0$?

(f) Where is $g'(x) = 0$?

33.



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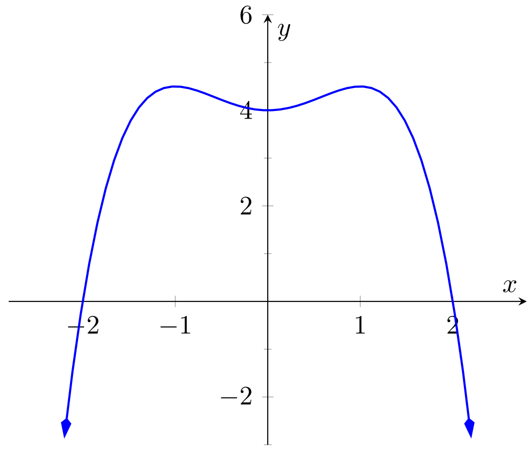


Exercise Group. A function $f(x)$ is given, along with its domain and derivative. Determine if $f(x)$ is differentiable on its domain.

35. $f(x) = \sqrt{x^5(1-x)}$, domain is $[0, 1]$, $f'(x) = \frac{(5-6x)x^{3/2}}{2\sqrt{1-x}}$
(☐ yes ☐ no)

36. $f(x) = \cos\left(\sqrt{x}\right)$, domain is $[0, \infty)$, $f'(x) = -\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$
(☐ yes ☐ no)

2.2 Interpretations of the Derivative

Section 2.1 defined the derivative of a function and gave examples of how to compute it using its definition (i.e., using limits). The section also started with a brief motivation for this definition, that is, finding the instantaneous velocity of a falling object given its position function. Section 2.3 will give us more accessible tools for computing the derivative; tools that are easier to use than repeated use of limits.

This section falls in between the "What is the definition of the derivative?" and "How do I compute the derivative?" sections. Here we are concerned with "What does the derivative mean?", or perhaps, when read with the right emphasis, "What *is* the derivative?" We offer two interconnected interpretations of the derivative, hopefully explaining why we care about it and why it is worthy of study.

2.2.1 Interpretation of the Derivative as Instantaneous Rate of Change

Section 2.1 started with an example of using the position of an object (in this case, a falling amusement park rider) to find the object's velocity. This type of example is often used when introducing the derivative because we tend to readily recognize that velocity is the *instantaneous rate of change in position*. In general, if f is a function of x, then $f'(x)$ measures the instantaneous rate of change of f with respect to x. Put another way, the derivative answers "When x changes, at what rate does f change?" Thinking back to the amusement park ride, we asked "When time changed, at what rate did the height change?" and found the answer to be "By -64 feet per second."

Now imagine driving a car and looking at the speedometer, which reads "60 mph." Five minutes later, you wonder how far you have traveled. Certainly, lots of things could have happened in those 5 minutes; you could have intentionally sped up significantly, you might have come to a complete stop, you might have slowed to 20 mph as you passed through construction. But suppose that you know, as the driver, none of these things happened. You know you maintained a fairly consistent speed over those 5 minutes. What is a good approximation of the distance traveled?

One could argue the *only* good approximation, given the information provided, would be based on "distance $=$ rate $\times$ time." In this case, we assume a constant rate of 60 mph with a time of 5 minutes or $5/60$ of an hour. Hence we would approximate the distance traveled as 5 miles.

Referring back to the falling amusement park ride, knowing that at $t = 2$ the velocity was -64 ft/s, we could reasonably approximate that 1 second later the riders' height would have dropped by about 64 feet. Knowing that the riders were *accelerating* as they fell would inform us that this is an *underapproximation*. If all we knew was that $f(2) = 86$ and $f'(2) = -64$, we'd know that we'd have to stop the riders quickly otherwise they would hit the ground.

In both of these cases, we are using the instantaneous rate of change to predict future values of the output.

2.2.2 Units of the Derivative

It is useful to recognize the *units* of the derivative function. If y is a function of x, i.e., $y = f(x)$ for some function f, and y is measured in feet and x in seconds, then the units of $y' = f'$ are "feet per second," commonly written as "ft/s." In general, if y is measured in units P and x is measured in units Q, then y' will be measured in units "P per Q", or "P/Q." Here we see the fraction-like behavior of the derivative in the notation: the units of $\frac{dy}{dx}$ are $\frac{\text{units of } y}{\text{units of } x}$.

Example 2.2.1 The meaning of the derivative: World Population.

Let $P(t)$ represent the world population t minutes after 12:00 a.m., January 1, 2012. It is fairly accurate to say that $P(0) = 7{,}028{,}734{,}178$ (www.prb.org). It is also fairly accurate to state that $P'(0) = 156$; that is, at midnight on January 1, 2012, the population of the world was growing by about 156 *people per minute* (note the units). Twenty days later (or $28{,}800$ minutes later) we could reasonably assume the population grew by about $28{,}800 \cdot 156 = 4{,}492{,}800$ people.

Example 2.2.2 The meaning of the derivative: Manufacturing.

The term *widget* is an economic term for a generic unit of manufacturing output. Suppose a company produces widgets and knows that the market supports a price of $\$10$ per widget. Let $P(n)$ give the profit, in dollars, earned by manufacturing and selling n widgets. The company likely cannot make a (positive) profit making just one widget; the start-up costs will likely exceed $\$10$. Mathematically, we would write this as $P(1) < 0$.

What do $P(1000) = 500$ and $P'(1000) = 0.25$ mean? Approximate $P(1100)$.

Solution. The equation $P(1000) = 500$ means that selling 1000 widgets returns a profit of $\$500$. We interpret $P'(1000) = 0.25$ as meaning that when we are selling 1000 widgets, the profit is increasing at rate of $\$0.25$ per widget (the units are "dollars per widget.") Since we have no other information to use, our best approximation for $P(1100)$ is:

$$\begin{aligned} P(1100) &\approx P(1000) + P'(1000) \times 100 \\ &= \$500 + (100 \text{ widgets }) \cdot \$0.25/\text{widget} \\ &= \$525. \end{aligned}$$

We approximate that selling 1100 widgets returns a profit of $\$525$.

The previous examples made use of an important approximation tool that we first used in our previous "driving a car at 60 mph" example at the beginning of this section. Five minutes after looking at the speedometer, our best approximation for distance traveled assumed the rate of change was constant. In Examples 2.2.1 and Example 2.2.2 we made similar approximations. We were given rate of change information which we used to approximate total change. Notationally, we would say that

$$f(c+h) \approx f(c) + f'(c) \cdot h.$$

This approximation is best when h is "small." "Small" is a relative term; when dealing with the world population, $h = 22$ days $= 28{,}800$ minutes is small in comparison to years. When manufacturing widgets, 100 widgets is small when one plans to manufacture thousands.

2.2.3 The Derivative and Motion

One of the most fundamental applications of the derivative is the study of motion. Let $s(t)$ be a position function, where t is time and $s(t)$ is distance. For instance, s could measure the height of a projectile or the distance an object has traveled.

Let $s(t)$ measure the distance traveled, in feet, of an object after t seconds of travel. Then $s'(t)$ has units "feet per second," and $s'(t)$ measures the *instantaneous rate of distance change with respect to time* — it measures *velocity*.

Now consider $v(t)$, a velocity function. That is, at time t, $v(t)$ gives the velocity of an object. The derivative of v, $v'(t)$, gives the *instantaneous rate of velocity change with respect to time — acceleration*. (We often think of acceleration in terms of cars: a car may "go from 0 to 60 in 4.8 seconds." This is an *average* acceleration, a measurement of how quickly the velocity changed.) If velocity is measured in feet per second, and time is measured in seconds, then the units of acceleration (i.e., the units of $v'(t)$) are "feet per second per second," or $($ft/s$)$/s. We often shorten this to "feet per second squared," or $\frac{\text{ft}}{\text{s}^2}$, but this tends to obscure the meaning of the units.

Perhaps the most well known acceleration is that of gravity. In this text, we use $g = 32\,\text{ft/s}^2$ or $g = 9.8\,\text{m/s}^2$. What do these numbers mean?

A constant acceleration of $32\,\frac{\text{ft/s}}{\text{s}}$ means that the velocity changes by $32\,\text{ft/s}$ each second. For instance, let $v(t)$ measure the velocity of a ball thrown straight up into the air, where v has units ft/s and t is measured in seconds. The ball will have a positive velocity while traveling upwards and a negative velocity while falling down. The acceleration is thus $-32\,\text{ft/s}^2$. If $v(1) = 20\,\text{ft/s}$, then 1 second later, the velocity will have decreased by $32\,\text{ft/s}$; that is, $v(2) = -12\,\text{ft/s}$. We can continue: $v(3) = -44\,\text{ft/s}$. Working backward, we can also figure that $v(0) = 52\,\text{ft/s}$.

These ideas are so important we write them out as a Key Idea.

Convention with s. Using $s(t)$ to represent position is a fairly common mathematical convention. It is also common to use s to represent arc length.

Key Idea 2.2.3 The Derivative and Motion.

1. Let $s(t)$ be the position function of an object. Then $s'(t) = v(t)$ is the velocity function of the object.

2. Let $v(t)$ be the velocity function of an object. Then $v'(t) = a(t)$ is the acceleration function of the object.

2.2.4 Interpretation of the Derivative as the Slope of the Tangent Line

We now consider the second interpretation of the derivative given in this section. This interpretation is not independent from the first by any means; many of the same concepts will be stressed, just from a slightly different perspective.

Given a function $y = f(x)$, the difference quotient $\frac{f(c+h)-f(c)}{h}$ gives a change in y values divided by a change in x values; i.e., it is a measure of the "rise over run," or "slope," of the secant line that goes through two points on the graph of f: $(c, f(c))$ and $(c+h, f(c+h))$. As h shrinks to 0, these two points come close together; in the limit we find $f'(c)$, the slope of a special line called the tangent line that intersects f only once near $x = c$.

Lines have a constant rate of change, their slope. Nonlinear functions do not have a constant rate of change, but we can measure their *instantaneous rate of change* at a given x value c by computing $f'(c)$. We can get an idea of how f is behaving by looking at the slopes of its tangent lines. We explore this idea in the following example.

Example 2.2.4 Understanding the derivative: the rate of change.

Consider $f(x) = x^2$ as shown in Figure 2.2.5. It is clear that at $x = 3$ the function is growing faster than at $x = 1$, as it is steeper at $x = 3$. How much faster is it growing at 3 compared to 1?

Solution. We can answer this exactly (and quickly) after Section 2.3, where we learn to quickly compute derivatives. For now, we will answer graphically, by considering the slopes of the respective tangent lines.
With practice, one can fairly effectively sketch tangent lines to a curve at a particular point. In Figure 2.2.6, we have sketched the tangent lines to f at $x = 1$ and $x = 3$, along with a grid to help us measure the slopes of these lines. At $x = 1$, the slope is 2; at $x = 3$, the slope is 6. Thus we can say not only is f growing faster at $x = 3$ than at $x = 1$, it is growing *three times as fast*.

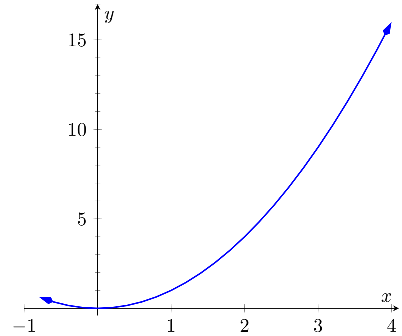


Figure 2.2.5 A graph of $f(x) = x^2$

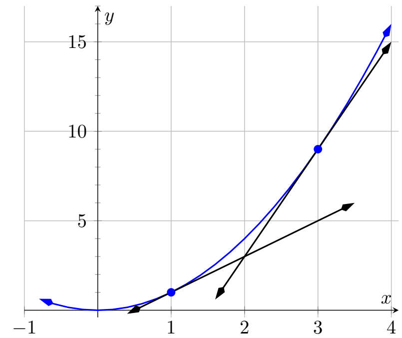


Figure 2.2.6 A graph of $f(x) = x^2$ and tangent lines at $x = 1$ and $x = 3$

Example 2.2.7 Understanding the graph of the derivative.

Consider the graph of $f(x)$ and its derivative, $f'(x)$, in Figure 2.2.8. Use these graphs to find the slopes of the tangent lines to the graph of f at $x = 1$, $x = 2$, and $x = 3$.

Solution. To find the appropriate slopes of tangent lines to the graph of f, we need to look at the corresponding values of f'.

- The slope of the tangent line to f at $x = 1$ is $f'(1)$; this looks to be about -1.
- The slope of the tangent line to f at $x = 2$ is $f'(2)$; this looks to be about 4.
- The slope of the tangent line to f at $x = 3$ is $f'(3)$; this looks to be about 3.

Using these slopes, tangent line segments to f are sketched in Figure 2.2.9. Included on the graph of f' in this figure are points where $x = 1$, $x = 2$ and $x = 3$ to help better visualize the y value of f' at those points.

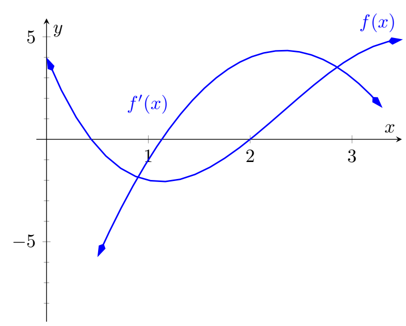


Figure 2.2.8 Graphs of f and f' in Example 2.2.7

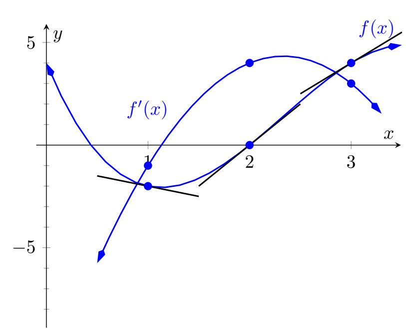


Figure 2.2.9 Graphs of f and f' in Example 2.2.7

Example 2.2.10 Approximation with the derivative.

Consider again the graph of $f(x)$ and its derivative $f'(x)$ in Example 2.2.7. Use the tangent line to f at $x = 3$ to approximate the value of $f(3.1)$.

Solution. Figure 2.2.11 shows the graph of f along with its tangent line, zoomed in at $x = 3$. Notice that near $x = 3$, the tangent line makes an excellent approximation of f. Since lines are easy to deal with, often it works well to approximate a function with its tangent line. (This is especially true when you don't actually know much about the function at hand, as we don't in this example.)
While the tangent line to f was drawn in Example 2.2.7, it was not explicitly computed. Recall that the tangent line to f at $x = c$ is $y = f'(c)(x - c) + f(c)$. While f is not explicitly given, by the graph it looks like $f(3) = 4$. Recalling that $f'(3) = 3$, we can compute the tangent line to be approximately $y = 3(x - 3) + 4$. It is often useful to

leave the tangent line in point-slope form.
To use the tangent line to approximate $f(3.1)$, we simply evaluate y at 3.1 instead of f.

$$\begin{aligned} f(3.1) &\approx y(3.1) \\ &= 3(3.1-3)+4 \\ &= 0.1\cdot 3+4 \\ &= 4.3. \end{aligned}$$

We approximate $f(3.1)\approx 4.3$.

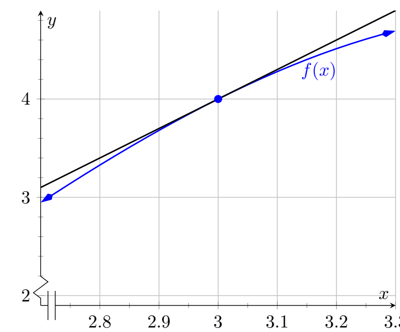


Figure 2.2.11 Zooming in on f and its tangent line at $x = 3$ for the function given in Examples 2.2.7 and Example 2.2.10

To demonstrate the accuracy of the tangent line approximation, we now state that in Example 2.2.10, $f(x) = -x^3+7x^2-12x+4$. We can evaluate $f(3.1) = 4.279$. Had we known f all along, certainly we could have just made this computation. In reality, we often only know two things:

1. what $f(c)$ is, for some value of c, and
2. what $f'(c)$ is.

For instance, we can easily observe the location of an object and its instantaneous velocity at a particular point in time. We do not have a "function f" for the location, just an observation. This is enough to create an approximating function for f.

This last example has a direct connection to our approximation method explained above after Example 2.2.2. We stated there that

$$f(c+h)\approx f(c)+f'(c)\cdot h.$$

If we know $f(c)$ and $f'(c)$ for some value $x=c$, then computing the tangent line at $(c, f(c))$ is easy: $y(x) = f'(c)(x-c)+f(c)$. In Example 2.2.10, we used the tangent line to approximate a value of f. Let's use the tangent line at $x=c$ to approximate a value of f near $x=c$; i.e., compute $y(c+h)$ to approximate $f(c+h)$, assuming again that h is "small." Note:

$$\begin{aligned} y(c+h) &= f'(c)\left((c+h)-c\right)+f(c) \\ &= f'(c)\cdot h+f(c). \end{aligned}$$

This is the exact same approximation method used above! Not only does it make intuitive sense, as explained above, it makes analytical sense, as this approximation method is simply using a tangent line to approximate a function's value.

The importance of understanding the derivative cannot be understated. When f is a function of x, $f'(x)$ measures the instantaneous rate of change of f with respect to x and gives the slope of the tangent line to f at x.

2.2.5 Exercises

Terms and Concepts

1. What is the instantaneous rate of change of position called?

2. Given a function $y = f(x)$, in your own words describe how to find the units of $f'(x)$.

3. What functions have a constant rate of change?

Problems

4. Given $f(4) = 18$ and $f'(4) = 2$, approximate $f(5)$.

5. Given $P(100) = -19$ and $P'(100) = -7$, approximate $P(110)$.

6. Given $z(60) = 106$ and $z'(60) = 3$, approximate $z(55)$.

7. Knowing $f(10) = 25$ and $f'(10) = 5$ and the methods described in this section, which approximation is likely to be most accurate? (☐ f(10.1) ☐ f(11) ☐ f(20))

8. Given $f(8) = 43$ and $f(9) = 41$, approximate $f'(8)$.

9. Given $H(5) = 12$ and $H(8) = 33$, approximate $H'(5)$.

10. Let $V(x)$ measure the volume, in decibels, measured inside a restaurant with x customers. What are the units of $V'(x)$?

11. Let $v(t)$ measure the velocity, in ft/s, of a car moving in a straight line t seconds after starting. What are the units of $v'(t)$?

12. The height H, in feet, of a river is recorded t hours after midnight, April 1. What are the units of $H'(t)$?

13. P is the profit, in thousands of dollars, of producing and selling c cars.

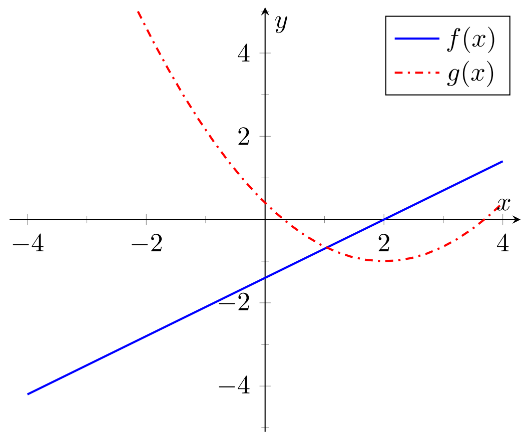
- **(a)** What are the units of $P'(c)$?
- **(b)** What is likely true of $P(0)$?

14. T is the temperature in degrees Fahrenheit, h hours after midnight on July 4 in Sidney, NE.

- **(a)** What are the units of $T'(h)$?
- **(b)** Is $T'(8)$ likely greater than or less than 0? Why?
- **(c)** Is $T(8)$ likely greater than or less than 0? Why?

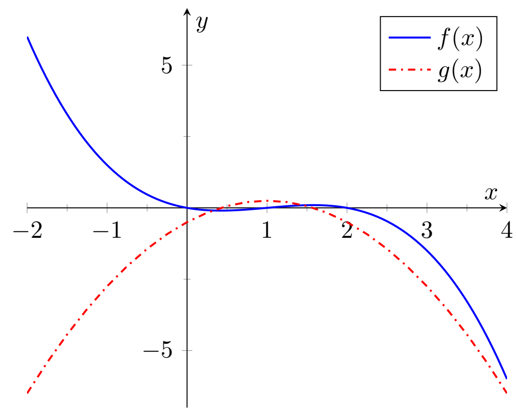
Exercise Group. Graphs of functions f and g are given. Identify which function is the derivative of the other.

15.



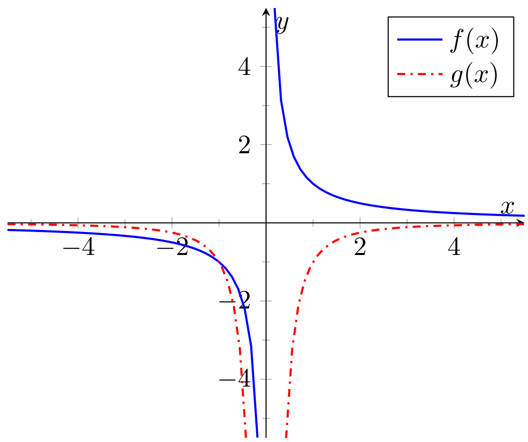
- f is the derivative of g.
- g is the derivative of f.

16.



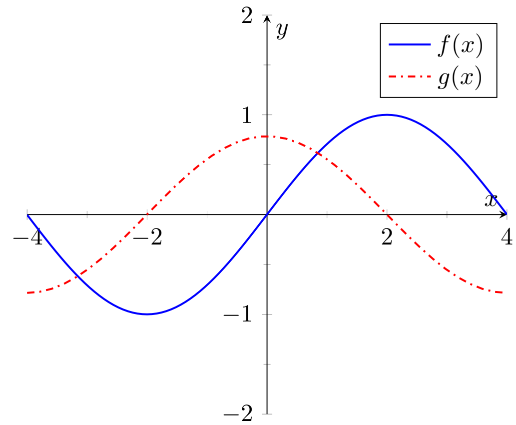
- f is the derivative of g.
- g is the derivative of f.

17.



- f is the derivative of g.
- g is the derivative of f.

18.



- f is the derivative of g.
- g is the derivative of f.

2.3 Basic Differentiation Rules

The derivative is a powerful tool but is admittedly awkward given its reliance on limits. Fortunately, one thing mathematicians are good at is *abstraction*. For instance, instead of continually finding derivatives at a point, we abstracted and found the derivative function.

Let's practice abstraction on linear functions, $y = mx + b$. What is y'? Without limits, recognize that linear functions are characterized by being functions with a constant rate of change (the slope). The derivative, y', gives the instantaneous rate of change; with a linear function, this is constant, m. Thus $y' = m$.

Let's abstract once more. Let's find the derivative of the general quadratic function, $f(x) = ax^2 + bx + c$. Using the definition of the derivative, we have:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{a(x+h)^2 + b(x+h) + c - (ax^2 + bx + c)}{h} \\ &= \lim_{h\to 0} \frac{ax^2 + 2ahx + ah^2 + bx + bh + c - ax^2 - bx - c)}{h} \\ &= \lim_{h\to 0} \frac{ah^2 + 2ahx + bh}{h} \\ &= \lim_{h\to 0} ah + 2ax + b \\ &= 2ax + b. \end{aligned}$$

So if $y = 6x^2 + 11x - 13$, we can immediately compute $y' = 12x + 11$.

In this section (and in some sections to follow) we will learn some of what mathematicians have already discovered about the derivatives of certain functions and how derivatives interact with arithmetic operations. We start with a theorem.

Theorem 2.3.1 Derivatives of Common Functions.

Constant Rule $\frac{d}{dx}(c) = 0$, *where c is a constant.*

Power Rule $\frac{d}{dx}(x^n) = nx^{n-1}$, *where n is an integer, $n > 0$.*

Other common functions
$\frac{d}{dx}(\sin(x)) = \cos(x)$
$\frac{d}{dx}(\cos(x)) = -\sin(x)$
$\frac{d}{dx}(e^x) = e^x$
$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$, *for $x > 0$.*

This theorem starts by stating an intuitive fact: constant functions have zero rate of change as they are *constant*. Therefore their derivative is 0 (they change at the rate of 0). The theorem then states some fairly amazing things. The Power Rule states that the derivatives of Power Functions (of the form $y = x^n$) are very straightforward: multiply by the power, then subtract 1 from the power. We see something incredible about the function $y = e^x$: it is its own derivative. We also see a new connection between the sine and cosine functions.

One special case of the Power Rule is when $n = 1$, i.e., when $f(x) = x$. What is $f'(x)$? According to the Power Rule,

$$f'(x) = \frac{d}{dx}(x) = \frac{d}{dx}(x^1) = 1 \cdot x^0 = 1.$$

In words, we are asking "At what rate does f change with respect to x?" Since f is x, we are asking "At what rate does x change with respect to x?"

youtu.be/watch?v=wPJ8-zKc1n0

Figure 2.3.2 Video explanation of Theorem 2.3.1

The answer is: 1. They change at the same rate. We can also interpret the derivative as the slope of the tangent line to the function at a point $(c, f(c))$. Since $f(x) = x$ is a linear function with constant slope 1, we can say that the derivative of $f(x) = x$ is $f'(x) = 1$.

Theorem 2.3.1 states that the natural exponential function has a remarkable propery: it is equal to its own derivative! The video in Figure 2.3.3 explains why this is the case.

Let's practice using this theorem.

youtu.be/watch?v=ipKTEdQFBjw

Figure 2.3.3 Determining the derivative of $f(x) = e^x$

Example 2.3.4 Using common derivative rules to find, and use, derivatives.

Let $f(x) = x^3$.

1. Find $f'(x)$.
2. Find the equation of the line tangent to the graph of f at $x = -1$.
3. Use the tangent line to approximate $(-1.1)^3$.
4. Sketch f, f' and the tangent line from Item 2 on the same axis.

Solution.

1. The Power Rule states that if $f(x) = x^3$, then $f'(x) = 3x^2$.
2. To find the equation of the line tangent to the graph of f at $x = -1$, we need a point and the slope. The point is $(-1, f(-1)) = (-1, -1)$. The slope is $f'(-1) = 3$. Thus the tangent line has equation $y = 3(x - (-1)) + (-1) = 3x + 2$.
3. We can use the tangent line to approximate $(-1.1)^3$ since -1.1 is close to -1. We have
$$(-1.1)^3 \approx 3(-1.1) + 2 = -1.3.$$
We can easily find the actual value: $(-1.1)^3 = -1.331$.
4. See Figure 2.3.5.

Video solution

youtu.be/watch?v=lyAEJSmSr-A

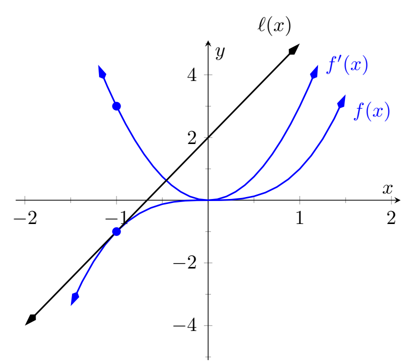


Figure 2.3.5 A graph of $f(x) = x^3$, along with its derivative $f'(x) = 3x^2$ and its tangent line at $x = -1$

Theorem 2.3.1 gives useful information, but we will need much more. For instance, using the theorem, we can easily find the derivative of $y = x^3$, but it does not tell how to compute the derivative of $y = 2x^3$, $y = x^3 + \sin(x)$ nor $y = x^3 \sin(x)$. The following theorem helps with the first two of these examples (the third is answered in the next section).

Theorem 2.3.6 Properties of the Derivative.

Let f and g be differentiable on an open interval I and let c be a real number. Then:

Sum/Difference Rule

$$\begin{aligned}\frac{d}{dx}(f(x) \pm g(x)) &= \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))\\ &= f'(x) \pm g'(x)\end{aligned}$$

Constant Multiple Rule

$$\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx}(f(x)) = c \cdot f'(x).$$

youtu.be/watch?v=Hr0sQcVhQ9A

Figure 2.3.7 Video presentation of Theorem 2.3.6

While we will be mainly focused on *using* these rules, it can also be interesting to see where they come from. Fortunately, it is not too difficult to establish these rules using the definition of the derivative. The video in Figure 2.3.8 shows why the sum rule is true.

Theorem 2.3.6 allows us to find the derivatives of a wide variety of functions. It can be used in conjunction with the Power Rule to find the derivatives of any polynomial. Recall in Example 2.1.22 that we found, using the limit definition, the derivative of $f(x) = 3x^2 + 5x - 7$. We can now find its derivative without expressly using limits:

$$\begin{aligned}\frac{d}{dx}\left(3x^2 + 5x - 7\right) &= 3\frac{d}{dx}\left(x^2\right) + 5\frac{d}{dx}(x) - \frac{d}{dx}(7)\\ &= 3 \cdot 2x + 5 \cdot 1 - 0\\ &= 6x + 5.\end{aligned}$$

youtu.be/watch?v=nVVpyilxZTw

Figure 2.3.8 Proving the sum rule

We were a bit pedantic here, showing every step. Normally we would do all the arithmetic and steps in our head and readily find $\frac{d}{dx}\left(3x^2 + 5x + 7\right) = 6x + 5$.

Example 2.3.9 Using the tangent line to approximate a function value.

Let $f(x) = \sin(x) + 2x + 1$. Approximate $f(3)$ using an appropriate tangent line.

Solution. This problem is intentionally ambiguous; we are to *approximate* using an *appropriate* tangent line. How good of an approximation are we seeking? What does "appropriate" mean?

In the "real world," people solving problems deal with these issues all time. One must make a judgment using whatever seems reasonable. In this example, the actual answer is $f(3) = \sin(3) + 7$, where the real problem spot is $\sin(3)$. What is $\sin(3)$?

Since 3 is close to π, we can assume $\sin(3) \approx \sin(\pi) = 0$. Thus one guess is $f(3) \approx 7$. Can we do better? Let's use a tangent line as instructed and examine the results; it seems best to find the tangent line at $x = \pi$.

Using Theorem 2.3.1 we find $f'(x) = \cos(x) + 2$. The slope of the tangent line is thus $f'(\pi) = \cos(\pi) + 2 = 1$. Also, $f(\pi) = 2\pi + 1 \approx 7.28$. So the tangent line to the graph of f at $x = \pi$ is $y = 1(x - \pi) + 2\pi + 1 = x + \pi + 1 \approx x + 4.14$. Evaluated at $x = 3$, our tangent line gives $y = 3 + 4.14 = 7.14$. Using the tangent line, our final approximation is that $f(3) \approx 7.14$.

Using a calculator, we get an answer accurate to four places after the decimal: $f(3) = 7.1411$. Our initial guess was 7; our tangent line approximation was more accurate, at 7.14.

The point is *not* "Here's a cool way to do some math without a calculator." Sure, that might be handy sometime, but your phone could probably give you the answer. Rather, the point is to say that tangent lines are a good way of approximating, and many scientists, engineers and mathematicians often face problems too hard to solve directly. So they approximate.

The graphs in Figure 2.3.10 shows the graph of the function $f(x)$ along with the tangent line constructed at $x = \pi$. The graph in Figure 2.3.10 shows the same tangent line and function. Once zoomed in, you can barely distinguish the tangent line from the function. This indicates that the tangent line is a good a approximation of the function so long as we are near the point of tangency.

Video solution

youtu.be/watch?v=a0hsWtT74jM

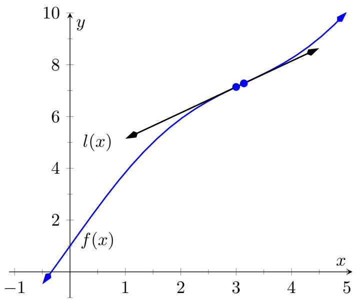


Figure 2.3.10 A graph of $f(x) = \sin(x) + 2x + 1$ along with its tangent line approximation at $x = \pi$

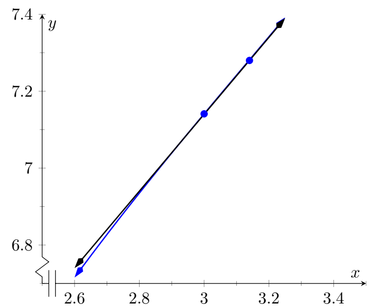


Figure 2.3.11 A graph of $f(x) = \sin(x) + 2x + 1$ along with its tangent line approximation at $x = \pi$, zoomed in

Note: The second derivative notation could be written as

$$\frac{d^2y}{dx^2} = \frac{d^2y}{(dx)^2} = \frac{d^2}{(dx)^2}(y).$$

That is, we take the derivative of y twice (hence d^2), both times with respect to x (hence $(dx)^2 = dx^2$).

2.3.1 Higher Order Derivatives

The derivative of a function f is itself a function, therefore we can take its derivative. The following definition gives a name to this concept and introduces its notation.

Definition 2.3.12 Higher Order Derivatives.

Let $y = f(x)$ be a differentiable function on I. The following are defined, provided the corresponding limits exist.

1. The *second derivative* of f is:

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y''.$$

2. The *third derivative* of f is:

$$f'''(x) = \frac{d}{dx}(f''(x)) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = y'''.$$

3. The n*th derivative* of f is:

$$f^{(n)}(x) = \frac{d}{dx}\left(f^{(n-1)}(x)\right) = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = y^{(n)}.$$

Higher Order Derivative Caveat. Definition 2.3.12 comes with the caveat "Where the corresponding limits exist." With f differentiable on I, it is possible that f' is *not* differentiable on all of I, and so on.

In general, when finding the fourth derivative and on, we resort to the $f^{(4)}(x)$ notation, not $f''''(x)$; after a while, too many ticks is confusing.

Let's practice using this new concept.

youtu.be/watch?v=usabSpUh65w

Figure 2.3.13 Video explanation of Definition 2.3.12

Example 2.3.14 Finding higher order derivatives.

Find the first four derivatives of the following functions:

1. $f(x) = 4x^2$ 2. $f(x) = \sin(x)$ 3. $f(x) = 5e^x$

Solution.

1. Using the Power and Constant Multiple Rules, we have: $f'(x) = 8x$. Continuing on, we have

$$f''(x) = \frac{d}{dx}(8x) = 8 \qquad f'''(x) = 0 \qquad f^{(4)}(x) = 0.$$

Notice how all successive derivatives will also be 0.

2. We employ Theorem 2.3.1 repeatedly.

$$f'(x) = \cos(x) \qquad f'''(x) = -\cos(x)$$
$$f''(x) = -\sin(x) \qquad f^{(4)}(x) = \sin(x)$$

Note how we have come right back to $f(x)$ again. (Can you quickly figure what $f^{(23)}(x)$ is?)

3. Employing Theorem 2.3.1 and the Constant Multiple Rule, we can see that

$$f'(x) = f''(x) = f'''(x) = f^{(4)}(x) = 5e^x.$$

Video solution

youtu.be/watch?v=nF2-IrvHmqc

2.3.2 Interpreting Higher Order Derivatives

What do higher order derivatives *mean*? What is the practical interpretation?

Our first answer is a bit wordy, but is technically correct and beneficial to understand. That is,

> The second derivative of a function f is the rate of change of the rate of change of f.

One way to grasp this concept is to let f describe a position function. Then, as stated in Key Idea 2.2.3, f' describes the rate of position change: velocity. We now consider f'', which describes the rate of velocity change. Sports car enthusiasts talk of how fast a car can go from 0 to 60 mph; they are bragging about the *acceleration* of the car.

We started this chapter with amusement park riders free-falling with position function $f(t) = -16t^2 + 150$. It is easy to compute $f'(t) = -32t$ ft/s and $f''(t) = -32$ (ft/s)/s. We may recognize this latter constant; it is the acceleration due to gravity. In keeping with the unit notation introduced in the previous section, we say the units are "feet per second per second." This is usually shortened to "feet per second squared," written as "ft/s^2."

It can be difficult to consider the meaning of the third, and higher order, derivatives. The third derivative is "the rate of change of the rate of change of the rate of change of f." That is essentially meaningless to the uninitiated. In the context of our position/velocity/acceleration example, the third derivative is the "rate of change of acceleration," commonly referred to as "jerk."

Make no mistake: higher order derivatives have great importance even if their practical interpretations are hard (or “impossible”) to understand. The mathematical topic of **series** makes extensive use of higher order derivatives.

2.3.3 Exercises

Terms and Concepts

1. What is the name of the rule which states that $\frac{d}{dx}(x^n) = nx^{n-1}$, where $n > 0$ is an integer?

2. What is $\frac{d}{dx}(\ln(x))$?

3. Give an example of a function $f(x)$ where $f'(x) = f(x)$.

4. Give an example of a function $f(x)$ where $f'(x) = 0$.

5. The derivative rules introduced in Section 2.3 explain how to compute the derivative of which of the following functions?

- $\dfrac{3}{x^2}$
- $3x^2 - x + 17$
- e^{x^2}
- $\sin(x)\cos(x)$
- $\sqrt{x}$
- $5\ln(x)$

6. Explain in your own words how to find the third derivative of a function $f(x)$.

7. Give an example of a function where $f'(x) \neq 0$ and $f''(x) = 0$.

8. Explain in your own words what the second derivative "means".

9. If $f(x)$ describes a position function, then $f'(x)$ describes what kind of function? What kind of function is $f''(x)$?

10. Let $f(x)$ be a function measured in pounds (lb), where x is measured in feet (ft). What are the units of $f''(x)$?

Problems

Exercise Group. Compute the derivative of the given function.

11. $f(x) = -\left(7x^2 + 8x + 7\right)$

12. $g(x) = 14x^2 - 16x^3 + 5x + 2$

13. $m(t) = 9t - \left(4t^5 + \frac{1}{4}t^3\right) - 6$

14. $f(\theta) = -\left(3\sin(\theta) + 19\cos(\theta)\right)$

15. $f(r) = 3e^r$

16. $g(t) = 7t^3 - 5\cos(t) - 2\sin(t)$

17. $f(x) = 6\ln(x) + 9x$

18. $p(s) = \frac{1}{4}s^4 + \frac{1}{3}s^3 + \frac{1}{2}s^2 + s + 1$

19. $h(t) = -\left(e^t + \sin(t) + \cos(t)\right)$

20. $f(x) = \ln\left(3x^8\right)$

21. $f(t) = \ln(6) + e^6 + \sin\left(\frac{\pi}{2}\right)$

22. $g(t) = (4 + 3t)^2$

23. $g(x) = (2x + 4)^3$

24. $f(x) = (3 + x)^3$

25. $f(x) = (7 + 2x)^2$

26. A property of logarithms is that $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$, for all bases $a, b > 0, \neq 1$.

(a) Rewrite this identity when $b = e$, i.e., using $\log_e(x) = \ln(x)$, with $a = 10$.

(b) Use part (a) to find the derivative of $y = \log_{10}(x)$.

(c) Find the derivative of $y = \log_a(x)$ for any $a > 0, \neq 1$.

Exercise Group. Compute the first four derivatives of the given function.

27. $f(x) = x^9$

28. $g(x) = 8\cos(x)$

29. $h(t) = -\left(4t^2 + 3t + e^t\right)$

30. $p(\theta) = \theta^2 + \theta^8$

31. $f(\theta) = -(\sin(\theta) + \cos(\theta))$

32. $f(x) = 692$

Exercise Group. Find the equations of the tangent and normal lines to the graph of the function at the given point.

33. $f(x) = x^3 + 8x$ at $x = 2$

34. $f(t) = e^t - 2$ at $t = 0$

35. $g(x) = \ln(x)$ at $x = 1$

36. $f(x) = 4\sin(x)$ at $x = \pi/6$

37. $f(x) = -2\cos(x)$ at $x = \pi/6$

38. $f(x) = 9 - 9x$ at $x = -9$

2.4 The Product and Quotient Rules

Section 2.3 showed that, in some ways, derivatives behave nicely. The Constant Multiple Rule and Sum/Difference Rule established that the derivative of $f(x) = 5x^2 + \sin(x)$ was not complicated. We neglected computing the derivative of things like $g(x) = 5x^2 \sin(x)$ and $h(x) = \frac{5x^2}{\sin(x)}$ on purpose; their derivatives are *not* as straightforward. (If you had to guess what their respective derivatives are, you would probably guess wrong.) For these, we need the Product and Quotient Rules, respectively, which are defined in this section. We begin with the Product Rule.

youtu.be/watch?v=1X3PTrkMsJ8

Figure 2.4.1 Video introduction to Section 2.4

Theorem 2.4.2 Product Rule.

Let f and g be differentiable functions on an open interval I. Then fg is a differentiable function on I, and

$$\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x).$$

Warning 2.4.3 $\frac{d}{dx}\big(f(x)g(x)\big) \neq f'(x)g'(x)$! While this would be simpler than the Product Rule, it is wrong.

We practice using this new rule in an example, followed by an example that demonstrates why this theorem is true.

Example 2.4.4 Using the Product Rule.

Use the Product Rule to compute the derivative of $y = 5x^2 \sin(x)$. Evaluate the derivative at $x = \pi/2$.

Solution. To make our use of the Product Rule explicit, let's set $f(x) = 5x^2$ and $g(x) = \sin(x)$. We easily compute/recall that $f'(x) = 10x$ and $g'(x) = \cos(x)$. Employing the rule, we have

$$\begin{aligned}\frac{d}{dx}\big(5x^2\sin(x)\big) &= \frac{d}{dx}\big(5x^2\big)\sin(x) + 5x^2\frac{d}{dx}\big(\sin(x)\big)\\ &= 10x\sin(x) + 5x^2\cos(x).\end{aligned}$$

At $x = \pi/2$, we have

$$y'(\pi/2) = 10\cdot\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) + 5\left(\frac{\pi}{2}\right)^2\cos\left(\frac{\pi}{2}\right) = 5\pi.$$

We graph y and its tangent line at $x = \pi/2$, which has a slope of 5π, in Figure 2.4.5. While this does not *prove* that the Product Rule is the correct way to handle derivatives of products, it helps validate its truth.

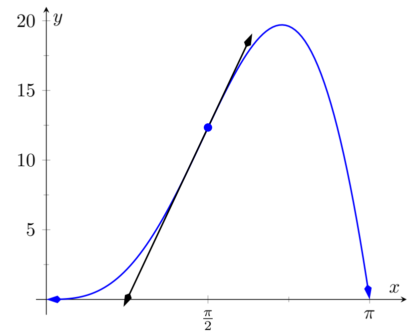


Figure 2.4.5 A graph of $y = 5x^2 \sin(x)$ and its tangent line at $x = \pi/2$

Video solution

youtu.be/watch?v=37efDywkDyE

We now investigate why the Product Rule is true.

Proof of Product Rule.
We can use the definition of the derivative to prove Theorem 2.4.2. By the limit definition, we have

$$\frac{d}{dx}\big(f(x)g(x)\big) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

We now do something a bit unexpected; add 0 to the numerator (so that nothing is changed) in the form of $-f(x)g(x+h) + f(x)g(x+h)$, then do some regrouping as shown.

youtu.be/watch?v=i791Y97O5hI

Figure 2.4.6 Video proof of product

Adding 0 in some clever form is a common mathematical proof technique.

$$\frac{d}{dx}(f(x)g(x)) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

(now add 0 to the numerator)

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

(regroup)

$$= \lim_{h\to 0} \frac{[f(x+h)g(x+h) - f(x)g(x+h)] + [f(x)g(x+h) - f(x)g(x)]}{h}$$

(split fraction)

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h}$$

(factor)

$$= \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} g(x+h)\right) + \lim_{h\to 0} \left(f(x)\frac{g(x+h) - g(x)}{h}\right)$$

(apply limit properties)

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h\to 0} g(x+h) + f(x) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

(apply limits)

$$= f'(x)g(x) + f(x)g'(x)$$

(by definition of the derivative).

We have proven the product rule as desired. (In the last step, we also relied on the fact that since g is differentiable, it is also continuous, which guarantees that $\lim_{h\to 0} g(x+h) = g(x)$.) ■

It is often true that we can recognize that a theorem is true through its proof yet somehow doubt its applicability to real problems. In the following example, we compute the derivative of a product of functions in two ways to verify that the Product Rule is indeed "right."

Example 2.4.7 Exploring alternate derivative methods.

Let $y = (x^2 + 3x + 1)(2x^2 - 3x + 1)$. Find y' two ways: first, by expanding the given product and then taking the derivative, and second, by applying the Product Rule. Verify that both methods give the same answer.

Solution. We first expand the expression for y; a little algebra shows that $y = 2x^4 + 3x^3 - 6x^2 + 1$. It is easy to compute y':

$$y' = 8x^3 + 9x^2 - 12x.$$

Instead, let's apply the Product Rule to the original factored form:

$$\begin{aligned} y' &= \frac{d}{dx}\big(x^2 + 3x + 1\big)(2x^2 - 3x + 1) + (x^2 + 3x + 1)\frac{d}{dx}\big(2x^2 - 3x + 1\big) \\ &= (2x+3)(2x^2 - 3x + 1) + (x^2 + 3x + 1)(4x - 3) \\ &= \big(4x^3 - 7x + 3\big) + \big(4x^3 + 9x^2 - 5x - 3\big) \\ &= 8x^3 + 9x^2 - 12x. \end{aligned}$$

The uninformed usually assume that "the derivative of the product is the product of the derivatives.” Thus we are tempted to say that $y' = (2x+3)(4x-3) = 8x^2+6x-9$. Obviously this is not correct.

Video solution

youtu.be/watch?v=-plvLFQ21Ig

Example 2.4.8 Using the Product Rule with a product of three functions.

Let $y = x^3 \ln(x)\cos(x)$. Find y'.

Solution. We have a product of three functions while the Product Rule only specifies how to handle a product of two functions. Our method of handling this problem is to simply group the latter two functions together, and consider $y = x^3 \cdot [\ln(x)\cos(x)]$. Following the Product Rule, we have

$$y' = \frac{d}{dx}\left(x^3\right)\ln(x)\cos(x) + (x^3)\frac{d}{dx}(\ln(x)\cos(x))$$

To evaluate $\frac{d}{dx}(\ln(x)\cos(x))$, we apply the Product Rule again:

$$\begin{aligned} y' &= 3x^2\left[\ln(x)\cos(x)\right] + (x^3)\left[\frac{1}{x}\cos(x) + \ln(x)(-\sin(x))\right] \\ &= 3x^2\ln(x)\cos(x) + x^3\frac{1}{x}\cos(x) + x^3\ln(x)(-\sin(x)). \end{aligned}$$

Video solution

youtu.be/watch?v=PYK64WB4JUg

Recognize the pattern in our answer above: when applying the Product Rule to a product of three functions, there are three terms added together in the final derivative. Each term contains only one derivative of one of the original functions, and each function’s derivative shows up in only one term. It is straightforward to extend this pattern to finding the derivative of a product of four or more functions.
Ultimately though, we would simplify our final computation to:

$$y' = 3x^2\ln(x)\cos(x) + x^2\cos(x) + -x^3\ln(x)\sin(x)$$

If you check this answer with a cas, it may factor and give the answer:

$$y' = -x^2\left[x\ln(x)\sin(x) + \cos(x) + 3\ln(x)\cos(x)\right]$$

Now that we have the hang of the product rule pattern, it’s not much more difficult to move on to products of four or more functions, as the video in Figure 2.4.9 demonstrates.

youtu.be/watch?v=QdQ14efmlMg

Figure 2.4.9 Taking the derivative of a product of four functions

We consider one more example before discussing another derivative rule.

Example 2.4.10 Using the Product Rule.

Find the derivatives of the following functions.

1. $f(x) = x\ln(x)$
2. $g(x) = x\ln(x) - x$

Solution. Recalling that the derivative of $\ln(x)$ is $1/x$, we use the Product Rule to find our answers.

1. Applying the Product Rule:

$$\frac{d}{dx}(x \ln(x)) = 1 \cdot \ln(x) + x \cdot 1/x$$
$$= \ln(x) + 1.$$

2. Using the result from above, we compute

$$\frac{d}{dx}(x \ln(x) - x) = \ln(x) + 1 - 1$$
$$= \ln(x).$$

This seems significant; if the natural log function $\ln(x)$ is an important function (it is), it seems worthwhile to know a function whose derivative is $\ln(x)$. We have found one. (We leave it to the reader to find another; a correct answer will be *very* similar to this one.)

We have learned how to compute the derivatives of sums, differences, and products of functions. We now learn how to find the derivative of a quotient of functions.

Theorem 2.4.11 Quotient Rule.

Let f and g be differentiable functions defined on an open interval I, where $g(x) \neq 0$ on I. Then f/g is differentiable on I, and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

The Quotient Rule is not hard to use, although it might be a bit tricky to remember. A useful mnemonic works as follows. Consider a fraction's numerator and denominator as "HI" and "LO", respectively. Then

$$\frac{d}{dx}\left(\frac{\text{HI}}{\text{LO}}\right) = \frac{\text{LO} \cdot \text{dHI} - \text{HI} \cdot \text{dLO}}{\text{LOLO}},$$

read "low dee high minus high dee low, over low low." Said fast, that phrase can roll off the tongue, making it easy to memorize. The "dee high" and "dee low" parts refer to the derivatives of the numerator and denominator, respectively.

Let's practice using the Quotient Rule.

youtu.be/watch?v=IlsA8342GvQ

Figure 2.4.12 Video presentation of Theorem 2.4.11

Example 2.4.13 Using the Quotient Rule.

Let $f(x) = \frac{5x^2}{\sin(x)}$. Find $f'(x)$.

Solution. Directly applying the Quotient Rule gives:

$$\frac{d}{dx}\left(\frac{5x^2}{\sin(x)}\right) = \frac{\sin(x) \cdot \frac{d}{dx}(5x^2) - 5x^2 \cdot \frac{d}{dx}(\sin(x))}{\sin^2(x)}$$
$$= \frac{10x\sin(x) - 5x^2\cos(x)}{\sin^2(x)}.$$

Video solution

youtu.be/watch?v=Hr4bt6yFwPg

The Quotient Rule allows us to fill in holes in our understanding of derivatives of the common trigonometric functions. We start with finding the derivative of the tangent function.

Example 2.4.14 Using the Quotient Rule to find $\frac{d}{dx}\big(\tan(x)\big)$.

Find the derivative of $y = \tan(x)$.

Solution. At first, one might feel unequipped to answer this question. But recall that $\tan(x) = \sin(x)/\cos(x)$, so we can apply the Quotient Rule.

$$\begin{aligned}\frac{d}{dx}\big(\tan(x)\big) &= \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)\\ &= \frac{\cos(x)\frac{d}{dx}\big(\sin(x)\big) - \sin(x)\frac{d}{dx}\big(\cos(x)\big)}{\cos^2(x)}\\ &= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}\\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}\\ &= \frac{1}{\cos^2(x)}\\ &= \sec^2(x).\end{aligned}$$

This is a beautiful result. To confirm its truth, we can find the equation of the tangent line to $y = \tan(x)$ at $x = \pi/4$. The slope is $\sec^2(\pi/4) = 2$; $y = \tan(x)$, along with its tangent line, is graphed in Figure 2.4.15.

Video solution

youtu.be/watch?v=wslbADxDg4c

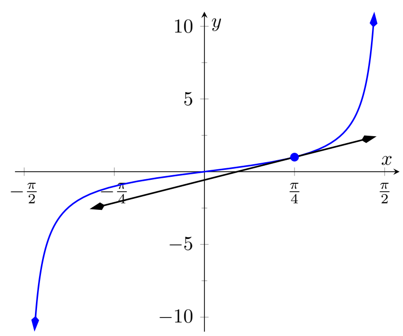


Figure 2.4.15 A graph of $y = \tan(x)$ along with its tangent line at $x = \pi/4$

We include this result in the following theorem about the derivatives of the trigonometric functions. Recall we found the derivative of $y = \sin(x)$ in Example 2.1.24 and stated the derivative of the cosine function in Theorem 2.3.1. The derivatives of the cotangent, cosecant and secant functions can all be computed directly using Theorem 2.3.1 and the Quotient Rule.

Theorem 2.4.16 Derivatives of Trigonometric Functions.

1. $\frac{d}{dx}\big(\sin(x)\big) = \cos(x)$
2. $\frac{d}{dx}\big(\cos(x)\big) = -\sin(x)$
3. $\frac{d}{dx}\big(\tan(x)\big) = \sec^2(x)$
4. $\frac{d}{dx}\big(\cot(x)\big) = -\csc^2(x)$
5. $\frac{d}{dx}\big(\sec(x)\big) = \sec(x)\tan(x)$
6. $\frac{d}{dx}\big(\csc(x)\big) = -\csc(x)\cot(x)$

To remember the above, it may be helpful to keep in mind that the derivatives of the trigonometric functions that start with "c" have a minus sign in them.

Example 2.4.17 Exploring alternate derivative methods.

In Example 2.4.13 the derivative of $f(x) = \frac{5x^2}{\sin(x)}$ was found using the Quotient Rule. Rewriting f as $f(x) = 5x^2\csc(x)$, find f' using Theorem 2.4.16 and verify the two answers are the same.

Solution. We found in Example 2.4.13 that $f'(x) = \frac{10x\sin(x) - 5x^2\cos(x)}{\sin^2(x)}$. We now find f' using the Product Rule, considering f as $f(x) =$

$5x^2\csc(x)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(5x^2\csc(x)\right) \\ &= 5x^2\frac{d}{dx}(\csc(x)) + \frac{d}{dx}\left(5x^2\right)\csc(x) \\ &= 5x^2\left(-\csc(x)\cot(x)\right) + 10x\csc(x) && \text{(now rewrite trig functions)} \\ &= 5x^2\cdot\frac{-1}{\sin(x)}\cdot\frac{\cos(x)}{\sin(x)} + \frac{10x}{\sin(x)} \\ &= \frac{-5x^2\cos(x)}{\sin^2(x)} + \frac{10x}{\sin(x)} && \text{(get common denominator)} \\ &= \frac{10x\sin(x) - 5x^2\cos(x)}{\sin^2(x)}. \end{aligned}$$

Finding f' using either method returned the same result. At first, the answers looked different, but some algebra verified they are the same. In general, there is not one final form that we seek; the immediate result from the Product Rule is fine. Work to "simplify" your results into a form that is most readable and useful to you.

The Quotient Rule gives other useful results, as shown in the next example.

Example 2.4.18 Using the Quotient Rule to expand the Power Rule.

Find the derivatives of the following functions.

1. $f(x) = \dfrac{1}{x}$
2. $f(x) = \dfrac{1}{x^n}$, where $n > 0$ is an integer.

Solution. We employ the Quotient Rule.

1.

$$\begin{aligned} f'(x) &= \frac{x\cdot 0 - 1\cdot 1}{x^2} \\ &= -\frac{1}{x^2} \end{aligned}$$

2.

$$\begin{aligned} f'(x) &= \frac{x^n\cdot 0 - 1\cdot nx^{n-1}}{(x^n)^2} \\ &= -\frac{nx^{n-1}}{x^{2n}} \\ &= -\frac{n}{x^{n+1}}. \end{aligned}$$

Video solution

youtu.be/watch?v=jPqqK-ObPm4

The derivative of $y = \frac{1}{x^n}$ turned out to be rather nice. It gets better. Consider:

$$\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}\left(x^{-n}\right) \qquad \text{(apply result from Example 2.4.18)}$$

$$= -\frac{n}{x^{n+1}} \qquad \text{(rewrite algebraically)}$$
$$= -nx^{-(n+1)}$$
$$= -nx^{-n-1} \qquad .$$

This is reminiscent of the Power Rule: multiply by the power, then subtract 1 from the power. We now add to our previous Power Rule, which had the restriction of $n > 0$.

Theorem 2.4.19 Power Rule with Integer Exponents.

Let $f(x) = x^n$, where $n \neq 0$ is an integer. Then

$$f'(x) = n \cdot x^{n-1}.$$

Taking the derivative of many functions is relatively straightforward. It is clear (with practice) what rules apply and in what order they should be applied. Other functions present multiple paths; different rules may be applied depending on how the function is treated. One of the beautiful things about calculus is that there is not "the" right way; each path, when applied correctly, leads to the same result, the derivative. We demonstrate this concept in an example.

Example 2.4.20 Exploring alternate derivative methods.

Let $f(x) = \frac{x^2-3x+1}{x}$. Find $f'(x)$ in each of the following ways:

1. By applying the Quotient Rule,
2. by viewing f as $f(x) = \left(x^2 - 3x + 1\right) \cdot x^{-1}$ and applying the Product Rule and Power Rule with Integer Exponents, and
3. by "simplifying" first through division.

Verify that all three methods give the same result.

Solution.

1. Applying the Quotient Rule gives:

$$\begin{aligned} f'(x) &= \frac{x \cdot \frac{d}{dx}\left(x^2-3x+1\right) - \left(x^2-3x+1\right)\frac{d}{dx}(x)}{x^2} \\ &= \frac{x \cdot (2x-3) - \left(x^2-3x+1\right) \cdot 1}{x^2} \\ &= \frac{x^2-1}{x^2} \\ &= 1 - \frac{1}{x^2}. \end{aligned}$$

2. By rewriting f, we can apply the Product Rule and Power Rule with Integer Exponents as follows:

$$\begin{aligned} f'(x) &= \left(x^2-3x+1\right)\frac{d}{dx}\left(x^{-1}\right) + \frac{d}{dx}\left(x^2-3x+1\right)x^{-1} \\ &= \left(x^2-3x+1\right) \cdot (-1)x^{-2} + (2x-3) \cdot x^{-1} \\ &= -\frac{x^2-3x+1}{x^2} + \frac{2x-3}{x} \end{aligned}$$

$$= -\frac{x^2 - 3x + 1}{x^2} + \frac{2x^2 - 3x}{x^2}$$
$$= \frac{x^2 - 1}{x^2} = 1 - \frac{1}{x^2},$$

the same result as above.

3. As $x \neq 0$, we can divide through by x first, giving $f(x) = x - 3 + x^{-1}$. Now apply the Power Rule with Integer Exponents.

$$f'(x) = 1 - \frac{1}{x^2},$$

the same result as before.

Video solution

youtu.be/watch?v=ESYjxNMNvh8

Example 2.4.20 demonstrates three methods of finding f'. One is hard pressed to argue for a "best method" as all three gave the same result without too much difficulty, although it is clear that using the Product Rule required more steps. Ultimately, the important principle to take away from this is: reduce the answer to a form that seems "simple" and easy to interpret. In that example, we saw different expressions for f', including:

$$1 - \frac{1}{x^2}$$
$$\frac{x \cdot (2x - 3) - \left(x^2 - 3x + 1\right) \cdot 1}{x^2}$$
$$\left(x^2 - 3x + 1\right) \cdot (-1)x^{-2} + (2x - 3) \cdot x^{-1}.$$

They are equal; they are all correct; only the first is "simple." Work to make answers simple.

In the next section we continue to learn rules that allow us to more easily compute derivatives than using the limit definition directly. We have to memorize the derivatives of a certain set of functions, such as "the derivative of sin(x) is cos(x)." The Sum/Difference Rule, Constant Multiple Rule, Power Rule with Integer Exponents, Product Rule and Quotient Rule show us how to find the derivatives of certain combinations of these functions. The next section shows how to find the derivatives when we *compose* these functions together.

2.4.1 Exercises

Terms and Concepts

1. (☐ True ☐ False) The Product Rule states that $\frac{d}{dx}\left(x^2 \sin(x)\right) = 2x\cos(x)$.

2. (☐ True ☐ False) The Quotient Rule states that $\frac{d}{dx}\left(\frac{x^2}{\sin(x)}\right) = \frac{\cos(x)}{2x}$.

3. (☐ True ☐ False) The derivatives of the trigonometric functions that start with "c" have minus signs in them.

4. What derivative rule is used to extend the Power Rule to include negative integer exponents?

5. (☐ True ☐ False) Regardless of the function, there is always exactly one right way of computing its derivative.

6. In your own words, explain what it means to make your answers "clear."

Problems

Exercise Group.

(a) Use the Product Rule to differentiate the function.

(b) Manipulate the function algebraically and differentiate without using the Product Rule.

(c) Show that the two derivatives are equivalent.

7. $f(x) = x\left(x^2 + 3x\right)$

8. $f(x) = 2x^2 \cdot 5x^3$

9. $f(s) = (2s - 1)(s + 4)$

10. $f(x) = \left(x^2 + 5\right)\left(3 - x^3\right)$

Exercise Group.

(a) Use the Quotient Rule to differentiate the function.

(b) Manipulate the function algebraically and differentiate without using the Quotient Rule.

(c) Show that the two derivatives are equivalent.

11. $f(x) = \frac{x^2+3}{x}$

12. $f(x) = \frac{x^3-2x^2}{2x^2}$

13. $f(x) = \frac{3}{4s^3}$

14. $f(x) = \frac{t^2-1}{t+1}$

Exercise Group. Compute the derivative of the given function.

15. $k(y) = y\sin(y)$

16. $k(t) = t^3\cos(t)$

17. $p(q) = e^q \ln(q)$

18. $f(y) = \frac{1}{y^6}(\csc(y) - 5)$

19. $f(t) = \frac{t+8}{t-4}$

20. $g(q) = \frac{q^3}{\sin(q)-8q^2}$

21. $h(y) = \csc(y) - e^y$

22. $h(t) = \tan(t)\ln(t)$

23. $j(q) = 7q^2 - 6q - 6$

24. $k(y) = \frac{y^6+9y^5}{y+9}$

25. $k(r) = \left(5r^2 + 7r + 3\right)e^r$

26. $p(z) = \frac{z^9+z^5}{e^z}$

27. $p(x) = \left(8x^3 - 22x^2 + 5x\right)\frac{3x-25}{8x^3-22x^2+5x}$

28. $f(r) = r^5(\tan(r) + e^r)$

29. $g(z) = \frac{\csc(z)}{\cos(z)+2}$

30. $g(\theta) = \theta^4\sec(\theta) + \frac{\sec(\theta)}{\theta^4}$

31. $h(r) = \frac{\cot(r)}{r} + \frac{r}{\tan(r)}$

32. $j(z) = e^3(\cos(\pi/6) - 1)$

33. $j(x) = 7x^5e^x - \sin(x)\cos(x)$

34. $k(r) = \frac{r^2\sin(r)-7}{r^2\cos(r)-9}$

35. $p(z) = z^4 \ln(z) \cos(z)$

36. $p(x) = 9x \cos(x) \tan(x)$

Exercise Group. Find the equations of the tangent and normal lines to the graph of g at the indicated point.

37. $g(x) = e^x \left(x^2 - 7\right)$ at $(0, -7)$

38. $g(x) = x \cos(x)$ at $\left(\frac{5\pi}{3}, \frac{5\pi}{6}\right)$

39. $g(x) = \frac{x^2}{x-(-4)}$ at $(-5, -25)$

40. $g(x) = \frac{\sin(x)-2x}{x-8}$ at $(0, 0)$

Exercise Group. Find the x-values where the graph of the function has a horizontal tangent line.

41. $f(x) = x^2 - 17x - 29$

42. $f(x) = x \sin(x)$ on $[-1, 1]$

43. $f(x) = \frac{2x}{-3x+3}$

44. $f(x) = \frac{3x^2}{x-2}$

Exercise Group. Find the requested higher order derivative.

45. $f''(x)$, where $f(x) = x \sin(x)$

46. $f^{(4)}(x)$, where $f(x) = x \sin(x)$

47. $f''(x)$, where $f(x) = \csc(x)$

48. $f^{(9)}(x)$, where
$f(x) = \left(x^3 - 4x - 3\right) \left(x^2 - 9x - 2\right)$

2.5 The Chain Rule

We have covered almost all of the derivative rules that deal with combinations of two (or more) functions. The operations of addition, subtraction, multiplication (including by a constant) and division led to the Sum/Difference Rule, the Constant Multiple Rule, the Power Rule with Integer Exponents, the Product Rule and the Quotient Rule. To complete the list of differentiation rules, we look at the last way two (or more) functions can be combined: the process of composition (i.e. one function "inside" another).

youtu.be/watch?v=k7wX-kxd7Kw

Figure 2.5.1 Video introduction to Section 2.5

One example of a composition of functions is $f(x) = \cos(x^2)$. We currently do not know how to compute this derivative. If forced to guess, one might guess $f'(x) = -\sin(2x)$, where we recognize $-\sin(x)$ as the derivative of $\cos(x)$ and $2x$ as the derivative of x^2. However, this is not the case; $f'(x) \neq -\sin(2x)$. One way to see this is to examine the graph of $y = \cos\left(x^2\right)$ in Figure 2.5.2 and its tangent line at $x = \pi/2$. Clearly the slope of the tangent line there is nonzero, but $-2\sin(2 \cdot \pi/2) = 0$. So it can't be correct to say that $y' = -\sin(2x)$.

In Example 2.5.9 we'll see the correct way to compute the derivative of $\sin\left(x^2\right)$, which employs the new rule this section introduces, the *Chain Rule*.

Before we define this new rule, recall the notation for composition of functions. We write $(f \circ g)(x)$ or $f(g(x))$, read as "f of g of x," to denote composing f with g. In shorthand, we simply write $f \circ g$ or $f(g)$ and read it as "f of g." Before giving the corresponding differentiation rule, we note that the rule extends to multiple compositions like $f(g(h(x)))$ or $f(g(h(j(x))))$, etc.

To motivate the rule, let's look at three derivatives we can already compute.

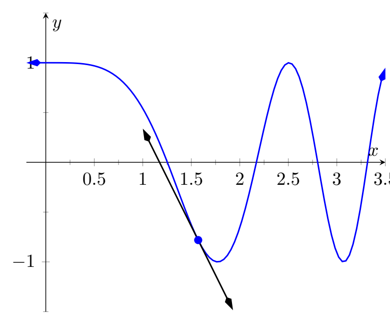


Figure 2.5.2 A graph of $y = \cos(x^2)$ and a tangent line at $\pi/2$

Example 2.5.3 Exploring similar derivatives.

Find the derivatives of $F_1(x) = (1-x)^2$, $F_2(x) = (1-x)^3$, and $F_3(x) = (1-x)^4$. (We'll see later why we are using subscripts for different functions and an uppercase F.)

Solution. In order to use the rules we already have, we must first expand each function as

$$\begin{aligned} F_1(x) &= 1 - 2x + x^2 \\ F_2(x) &= 1 - 3x + 3x^2 - x^3 \\ F_3(x) &= 1 - 4x + 6x^2 - 4x^3 + x^4 \end{aligned}$$

It is not hard to see that:

$$\begin{aligned} F_1'(x) &= -2 + 2x \\ F_2'(x) &= -3 + 6x - 3x^2 \\ F_3'(x) &= -4 + 12x - 12x^2 + 4x^3 \end{aligned}$$

An interesting fact is that these can be rewritten as:

$$\begin{aligned} F_1'(x) &= -2(1-x) \\ F_2'(x) &= -3(1-x)^2 \\ F_3'(x) &= -4(1-x)^3 \end{aligned}$$

A pattern might jump out at you; note how the we end up multiplying by the old power and the new power is reduced by 1. We also always multiply by (-1).

Recognize that each of these functions is a composition, letting $g(x) = 1 - x$:

$$F_1(x) = f_1(g(x)), \qquad \text{where } f_1(x) = x^2,$$

$$F_2(x) = f_2(g(x)), \qquad \text{where } f_2(x) = x^3,$$
$$F_3(x) = f_3(g(x)), \qquad \text{where } f_3(x) = x^4.$$

We'll come back to this example after giving the formal statements of the Chain Rule; for now, we are just illustrating a pattern.

Theorem 2.5.4 The Chain Rule.

Let g be a differentiable function on an interval I, let the range of g be a subset of the interval J, and let f be a differentiable function on J. Then $y = f(g(x))$ is a differentiable function on I, and

$$y' = f'(g(x)) \cdot g'(x).$$

Here is the Chain Rule in words:

> The derivative of the outside function, evaluated at the inside function, multiplied by the derivative of the inside function.

To help understand the Chain Rule, we return to Example 2.5.3.

Example 2.5.6 Using the Chain Rule.

Use the Chain Rule to find the derivatives of the functions $F_1(x)$, $F_2(x)$, and $F_3(x)$, as given in Example 2.5.3.

Solution. Example 2.5.3 ended with the recognition that each of the given functions was actually a composition of functions. To avoid confusion, we ignore most of the subscripts here.

$F_1(x) = (1-x)^2$ We found that

$$y = (1-x)^2 = f(g(x)),$$

where $f(x) = x^2$ and $g(x) = 1 - x$. To find y', we apply the The Chain Rule. We need to note that $f'(x) = 2x$ and $g'(x) = -1$.
Part of the The Chain Rule uses $f'(g(x))$. This means substitute $g(x)$ for x in the equation for $f'(x)$. That is, $f'(x) = 2(1-x)$. Finishing out the The Chain Rule we have

$$\begin{aligned} y' &= f'(g(x)) \cdot g'(x) \\ &= 2(1-x) \cdot (-1) \\ &= -2(1-x) \\ &= 2x - 2. \end{aligned}$$

When composing functions, we need to make sure that the new function is actually defined. For instance, consider $f(x) = \sqrt{x}$ and $g(x) = -x^2 - 1$. The domain of f excludes all negative numbers, but the range of g is only negative numbers. Therefore the composition $f(g(x)) = \sqrt{-x^2-1}$ is not defined for any x, and hence is not differentiable.

The statement of Theorem 2.5.4 takes care to ensure this problem does not arise, but our focus is more on the derivative result than on the domain/range conditions.

youtu.be/watch?v=1_Lp-ONIMuc

Figure 2.5.5 Video presentation of Theorem 2.5.4

$F_2(x) = (1-x)^3$ Let $y = (1-x)^3 = f(g(x))$, where $f(x) = x^3$ and $g(x) = (1-x)$. We have $f'(x) = 3x^2$, so $f'(g(x)) = 3(1-x)^2$. The The Chain Rule then states

$$\begin{aligned} y' &= f'(g(x)) \cdot g'(x) \\ &= 3(1-x)^2 \cdot (-1) \\ &= -3(1-x)^2. \end{aligned}$$

$F_3(x) = (1-x)^4$ Finally, when $y = (1-x)^4$, we have $f(x) = x^4$ and $g(x) = (1-x)$. Thus $f'(x) = 4x^3$ and $f'(g(x)) = 4(1-x)^3$. Thus

$$\begin{aligned} y' &= f'(g(x)) \cdot g'(x) \\ &= 4(1-x)^3 \cdot (-1) \\ &= -4(1-x)^3. \end{aligned}$$

Example 2.5.6 demonstrated a particular pattern: when $f(x) = x^n$, then $y' = n \cdot (g(x))^{n-1} \cdot g'(x)$. This is called the Generalized Power Rule.

Theorem 2.5.7 Generalized Power Rule.

Let $g(x)$ be a differentiable function and let $n \neq 0$ be an integer. Then

$$\frac{d}{dx}\left(g(x)^n\right) = n \cdot (g(x))^{n-1} \cdot g'(x).$$

This allows us to quickly find the derivative of functions like $y = (3x^2 - 5x + 7 + \sin(x))^{20}$. While it may look intimidating, the Generalized Power Rule states that

$$y' = 20(3x^2 - 5x + 7 + \sin(x))^{19} \cdot (6x - 5 + \cos(x)).$$

Treat the derivative-taking process step-by-step. In the example just given, first multiply by 20, then rewrite the inside of the parentheses, raising it all to the 19th power. Then think about the derivative of the expression inside the parentheses, and multiply by that.

We now consider more examples that employ the The Chain Rule.

Example 2.5.8 Using the Chain Rule.

Find the derivatives of the following functions:

1. $y = \sin(2x)$.
2. $y = \ln(4x^3 - 2x^2)$.
3. $y = e^{-x^2}$.

Solution.

1. Consider $y = \sin(2x)$. Recognize that this is a composition of functions, where $f(x) = \sin(x)$ and $g(x) = 2x$. Thus

$$\begin{aligned} y' &= f'(g(x)) \cdot g'(x) \\ &= \cos(2x) \cdot \frac{d}{dx}(2x) \end{aligned}$$

$$= \cos(2x) \cdot 2$$
$$= 2\cos(2x).$$

2. Recognize that $y = \ln(4x^3 - 2x^2)$ is the composition of $f(x) = \ln(x)$ and $g(x) = 4x^3 - 2x^2$. Also, recall that

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}.$$

This leads us to:

$$\begin{aligned} y' &= \frac{1}{4x^3 - 2x^2} \cdot \frac{d}{dx}(4x^3 - 2x^2) \\ &= \frac{1}{4x^3 - 2x^2} \cdot (12x^2 - 4x) \\ &= \frac{12x^2 - 4x}{4x^3 - 2x^2} \\ &= \frac{4x(3x-1)}{2x(2x^2 - x)} \\ &= \frac{2(3x-1)}{2x^2 - x}. \end{aligned}$$

Note that $\ln(4x^3 - 2x^2) = \ln(4x^2(x - 1/2))$ was only defined for $x > 1/2$, so the result of $y' = \frac{2(3x-1)}{2x^2-x}$ is only valid for $x > 1/2$ as well.

3. Recognize that $y = e^{-x^2}$ is the composition of $f(x) = e^x$ and $g(x) = -x^2$. Remembering that $f'(x) = e^x$, we have

$$\begin{aligned} y' &= e^{-x^2} \cdot \frac{d}{dx}(-x^2) \\ &= e^{-x^2} \cdot (-2x) \\ &= -2xe^{-x^2}. \end{aligned}$$

Video solution

youtu.be/watch?v=yW1BbOeDFcM

Example 2.5.9 Using the Chain Rule to find a tangent line.

Let $f(x) = \cos(x^2)$. Find the equation of the line tangent to the graph of f at $x = 1$.

Solution. The tangent line goes through the point $(1, f(1)) \approx (1, 0.54)$ with slope $f'(1)$. To find f', we need the The Chain Rule.
$f'(x) = -\sin(x^2) \cdot (2x) = -2x\sin(x^2)$. Evaluated at $x = 1$, we have $f'(1) = -2\sin(1) \approx -1.68$. Thus the equation of the tangent line is approximated by

$$y \approx -1.68(x - 1) + 0.54.$$

The tangent line is sketched along with f in Figure 2.5.10.

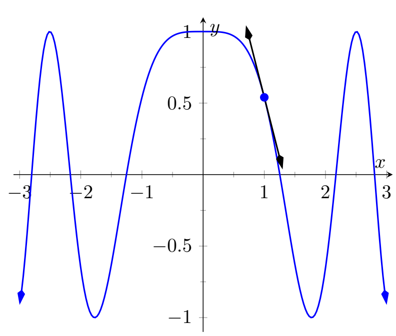


Figure 2.5.10 $f(x) = \cos(x^2)$ sketched along with its tangent line at $x = 1$

The The Chain Rule is used often in taking derivatives. Because of this, one can become familiar with the basic process and learn patterns that facilitate finding derivatives quickly. For instance,

$$\frac{d}{dx}\big(\ln(\text{anything})\big) = \frac{1}{\text{anything}} \cdot \frac{d}{dx}(\text{anything}) = \frac{\frac{d}{dx}(\text{anything})}{\text{anything}}.$$

A concrete example of this is

$$\frac{d}{dx}\Big(\ln(3x^{15} - \cos(x) + e^x)\Big) = \frac{45x^{14} + \sin(x) + e^x}{3x^{15} - \cos(x) + e^x}.$$

While the derivative may look intimidating at first, look for the pattern. The denominator is the same as what was inside the natural log function; the numerator is simply its derivative.

This pattern recognition process can be applied to lots of functions. In general, instead of writing "anything", we use u as a generic function of x. We then say

$$\frac{d}{dx}\big(\ln(u)\big) = \frac{u'}{u}.$$

The following is a short list of how the The Chain Rule can be quickly applied to familiar functions.

1. $\frac{d}{dx}(u^n) = n \cdot u^{n-1} \cdot u'$.
2. $\frac{d}{dx}(e^u) = e^u \cdot u'$.
3. $\frac{d}{dx}(\sin(u)) = \cos(u) \cdot u'$.
4. $\frac{d}{dx}(\cos(u)) = -\sin(u) \cdot u'$.
5. $\frac{d}{dx}(\tan(u)) = \sec^2(u) \cdot u'$.

Of course, the The Chain Rule can be applied in conjunction with any of the other rules we have already learned. We practice this next.

Example 2.5.11 Using the Product, Quotient and Chain Rules.

Find the derivatives of the following functions.

1. $f(x) = x^5 \sin(2x^3)$.
2. $f(x) = \dfrac{5x^3}{e^{-x^2}}$.

Solution.

1. We must use the Product Rule and The Chain Rule. Do not think that you must be able to "see" the whole answer immediately; rather, just proceed step-by-step.

$$\begin{aligned} f'(x) &= x^5 \cdot \frac{d}{dx}\big(\sin(2x^3)\big) + \sin(2x^3) \cdot \frac{d}{dx}(x^5) \\ &= x^5 \left(\cos(2x^3) \cdot \frac{d}{dx}(2x^3)\right) + 5x^4\left(\sin(2x^3)\right) \\ &= x^5\big(6x^2\cos\left(2x^3\right)\big) + 5x^4\big(\sin\left(2x^3\right)\big) \\ &= 6x^7\cos(2x^3) + 5x^4\sin(2x^3). \end{aligned}$$

2. We must employ the Quotient Rule along with the The Chain Rule. Again, proceed step-by-step.

$$f'(x) = \frac{e^{-x^2} \cdot \frac{d}{dx}\left(5x^3\right) - 5x^3 \cdot \frac{d}{dx}\left(e^{-x^2}\right)}{\left(e^{-x^2}\right)^2}$$

$$= \frac{e^{-x^2} \cdot 15x^2 - 5x^3 \cdot e^{-x^2} \cdot \frac{d}{dx}\left(-x^2\right)}{\left(e^{-x^2}\right)^2}$$
$$= \frac{e^{-x^2}\left(15x^2\right) - 5x^3\left((-2x)e^{-x^2}\right)}{\left(e^{-x^2}\right)^2}$$
$$= \frac{e^{-x^2}\left(10x^4 + 15x^2\right)}{e^{-2x^2}}$$
$$= e^{x^2}\left(10x^4 + 15x^2\right).$$

Video solution

youtu.be/watch?v=2QJLR-Y-Ht8

A key to correctly working these problems is to break the problem down into smaller, more manageable pieces. For instance, when using the Product Rule and The Chain Rule together, just consider the first part of the Product Rule at first: $f(x)g'(x)$. Just rewrite $f(x)$, then find $g'(x)$. Then move on to the $f'(x)g(x)$ part. Don't attempt to figure out both parts at once.

Likewise, using the Quotient Rule, approach the numerator in two steps and handle the denominator after completing that. Only simplify afterward.

We can also employ the The Chain Rule itself several times, as shown in the next example.

Example 2.5.12 Using the Chain Rule multiple times.

Find the derivative of $y = \tan^5(6x^3 - 7x)$.

Solution. Recognize that we have the $g(x) = \tan\left(6x^3 - 7x\right)$ function "inside" the $f(x) = x^5$ function; that is, we have $y = \left(\tan\left(6x^3 - 7x\right)\right)^5$. We begin using the Generalized Power Rule; in this first step, we do not fully compute the derivative. Rather, we are approaching this step-by-step.

$$y' = 5\left(\tan\left(6x^3 - 7x\right)\right)^4 \cdot g'(x).$$

We now find $g'(x)$. We again need the The Chain Rule;

$$g'(x) = \sec^2\left(6x^3 - 7x\right) \cdot \frac{d}{dx}\left(6x^3 - 7x\right).$$
$$= \sec^2\left(6x^3 - 7x\right) \cdot \left(18x^2 - 7\right).$$

Combine this with what we found above to give

$$y' = 5\left(\tan\left(6x^3 - 7x\right)\right)^4 \cdot \sec^2\left(6x^3 - 7x\right) \cdot \left(18x^2 - 7\right)$$
$$= \left(90x^2 - 35\right)\sec^2\left(6x^3 - 7x\right)\tan^4\left(6x^3 - 7x\right).$$

This function is frankly a ridiculous function, possessing no real practical value. It is very difficult to graph, as the tangent function has many vertical asymptotes and $6x^3 - 7x$ grows so very fast. The important thing to learn from this is that the derivative can be found. In fact, it is not "hard"; one can take several simple steps and should be careful to keep track of how to apply each of these steps.

Video solution

youtu.be/watch?v=JeLuSDqqFPA

It is a traditional mathematical exercise to find the derivatives of arbitrarily complicated functions just to demonstrate that it *can be done*. Just break everything down into smaller pieces.

Example 2.5.13 Using the Product, Quotient and Chain Rules.

Find the derivative of $f(x) = \frac{x\cos(x^{-2})-\sin^2(e^{4x})}{\ln(x^2+5x^4)}$.

Solution. This function likely has no practical use outside of demonstrating derivative skills. The answer is given below without simplification. It employs the Quotient Rule, the Product Rule, and the The Chain Rule three times.

$$\begin{aligned} &f'(x)\\ &= \Bigg(\ln\left(x^2+5x^4\right)\cdot\Bigg[\Big(x\cdot\left(-\sin\left(x^{-2}\right)\right)\cdot\left(-2x^{-3}\right)\\ &\quad +1\cdot\cos\left(x^{-2}\right)\Big)-2\sin\left(e^{4x}\right)\cdot\cos\left(e^{4x}\right)\cdot\left(4e^{4x}\right)\Bigg]\\ &\qquad -\left(x\cos\left(x^{-2}\right)-\sin^2\left(e^{4x}\right)\right)\cdot\frac{2x+20x^3}{x^2+5x^4}\Bigg)\\ &\Big/\left(\ln\left(x^2+5x^4\right)\right)^2. \end{aligned}$$

The reader is highly encouraged to look at each term and recognize why it is there. (i.e., the Quotient Rule is used; in the numerator, identify the "LOdHI" term, etc.) This example demonstrates that derivatives can be computed systematically, no matter how arbitrarily complicated the function is.

The The Chain Rule also has theoretic value. That is, it can be used to find the derivatives of functions that we have not yet learned as we do in the following example.

Example 2.5.14 The Chain Rule and exponential functions.

Use the Chain Rule to find the derivative of $y = 2^x$.

Solution. We only know how to find the derivative of one exponential function, $y = e^x$. We can accomplish our goal by rewriting 2 in terms of e. Recalling that e^x and $\ln(x)$ are inverse functions, we can write $2 = e^{\ln 2}$ and so

$$y = 2^x = \left(e^{\ln 2}\right)^x = e^{x(\ln(2))},$$

using the "power to a power" property of exponents.
The function is now the composition $y = f(g(x))$, with $f(x) = e^x$ and $g(x) = x(\ln(2))$. Since $f'(x) = e^x$ and $g'(x) = \ln(2)$, the The Chain Rule gives

$$y' = e^{x(\ln(2))}\cdot\ln 2.$$

Recall that the $e^{x(\ln(2))}$ term on the right hand side is just 2^x, our original function. Thus, the derivative contains the original function itself. We have

$$y' = y\cdot\ln(2) = 2^x\cdot\ln(2).$$

We can extend this process to use any base a, where $a > 0$ and $a \neq 1$. All we need to do is replace each "2" in our work with "a." The Chain Rule, coupled with the derivative rule of e^x, allows us to find the derivatives of all exponential functions.

The comment at the end of previous example is important and is restated formally as a theorem.

Theorem 2.5.15 Derivatives of Exponential Functions.

Let $f(x) = a^x$, for $a > 0, a \neq 1$. Then f is differentiable for all real numbers (i.e., differentiable everywhere) and

$$f'(x) = \ln(a) \cdot a^x.$$

youtu.be/watch?v=LnmwxZ5w30w

Figure 2.5.16 Derivatives of exponential and general power functions

Alternate Chain Rule Notation. It is instructive to understand what the The Chain Rule "looks like" using "$\frac{dy}{dx}$" notation instead of y' notation. Suppose that $y = f(u)$ is a function of u, where $u = g(x)$ is a function of x, as stated in Theorem 2.5.4. Then, through the composition $f \circ g$, we can think of y as a function of x, as $y = f(g(x))$. Thus the derivative of y with respect to x makes sense; we can talk about $\frac{dy}{dx}$. This leads to an interesting progression of notation:

$$y' = f'(g(x)) \cdot g'(x)$$

$$\frac{dy}{dx} = y'(u) \cdot u'(x) \qquad \text{since } y = f(u) \text{ and } u = g(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \text{(using “fractional notation” for the derivative)}$$

Here the "fractional" aspect of the derivative notation stands out. On the right hand side, it seems as though the "du" terms cancel out, leaving

$$\frac{dy}{dx} = \frac{dy}{dx}.$$

It is important to realize that we *are not* canceling these terms; the derivative notation of $\frac{dy}{du}$ is one symbol. It is equally important to realize that this notation was chosen precisely because of this behavior. It makes applying the The Chain Rule easy with multiple variables. For instance,

$$\frac{dy}{dt} = \frac{dy}{d\bigcirc} \cdot \frac{d\bigcirc}{d\triangle} \cdot \frac{d\triangle}{dt}.$$

where $\bigcirc$ and $\triangle$ are any variables you'd like to use.

One of the most common ways of "visualizing" the The Chain Rule is to consider a set of gears, as shown in Figure 2.5.17. The gears have 36, 18, and 6 teeth, respectively. That means for every revolution of the x gear, the u gear revolves twice. That is, the rate at which the u gear makes a revolution is twice as fast as the rate at which the x gear makes a revolution.

Using the terminology of calculus, the rate of u-change, with respect to x, is $\frac{du}{dx} = 2$.
Likewise, every revolution of u causes 3 revolutions of y: $\frac{dy}{du} = 3$. How does y change with respect to x? For each revolution of x, y revolves 6 times; that is,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2 \cdot 3 = 6.$$

We can then extend the The Chain Rule with more variables by adding more gears to the picture.

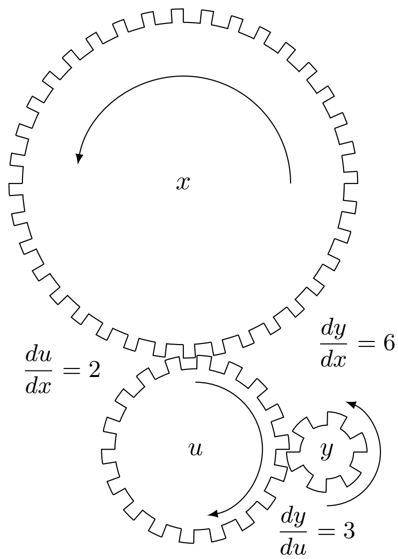


Figure 2.5.17 A series of gears to demonstrate the Chain Rule. Note how $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

It is difficult to overstate the importance of the The Chain Rule. So often the functions that we deal with are compositions of two or more functions, requiring us to use this rule to compute derivatives. It is also often used in real life when actual functions are unknown. Through measurement, we can calculate (or, approximate) $\frac{dy}{du}$ and $\frac{du}{dx}$. With our knowledge of the The Chain Rule, we can find $\frac{dy}{dx}$.

In Section 2.6, we use the The Chain Rule to justify another differentiation technique. There are many curves that we can draw in the plane that fail the "vertical line test." For instance, consider $x^2 + y^2 = 1$, which describes the unit circle. We may still be interested in finding slopes of tangent lines to the circle at various points. Section 2.6 shows how we can find $\frac{dy}{dx}$ without first "solving for y." While we can in this instance, in many other instances solving for y is impossible. In these situations, **implicit differentiation** is indispensable.

2.5.1 Exercises

Terms and Concepts

1. (☐ True ☐ False) The Chain Rule describes how to evaluate the derivative of a composition of functions.

2. (☐ True ☐ False) The Generalized Power Rule states that $\frac{d}{dx}\left(g(x)^n\right) = n\left(g(x)\right)^{n-1}$.

3. (☐ True ☐ False) $\frac{d}{dx}\left(\ln\left(x^2\right)\right) = \frac{1}{x^2}$.

4. (☐ True ☐ False) $\frac{d}{dx}\left(3^x\right) \approx 1.1 \cdot 3^x$.

5. (☐ True ☐ False) $\frac{dx}{dy} = \frac{dx}{dt} \cdot \frac{dt}{dy}$.

6. (☐ True ☐ False) Taking the derivative of $f(x) = x^2 \sin(5x)$ requires the use of both the Product and Chain Rules.

Problems

Exercise Group. Compute the derivative of the given function.

7. $f(x) = \left(4x^3 - x\right)^{10}$

8. $f(t) = (3t-2)^5$

9. $g(\theta) = (\sin(\theta) + \cos(\theta))^3$

10. $h(t) = e^{3t^2+t-1}$

11. $j(x) = \left(\ln(x) - x^4\right)^4$

12. $j(q) = 2^{q^5+4q}$

13. $k(y) = \left(y + \frac{1}{y}\right)^5$

14. $p(t) = \cos(5t)$

15. $p(q) = \tan(2q)$

16. $f(\theta) = \cot\left(\theta^2 + 3\right)$

17. $g(t) = \sin\left(t^6 + \frac{1}{t^3}\right)$

18. $g(q) = \cos^5(7q)$

19. $h(y) = \cos^3\left(y^2 + 3y - 3\right)$

20. $j(t) = \ln(\cos(t))$

21. $j(q) = \ln\left(q^8\right)$

22. $k(y) = 3\ln(y)$

23. $p(t) = 6^t$

24. $p(z) = 2^{\csc(z)}$

25. $f(x) = 8^{10}$

26. $g(t) = \frac{4^t}{9^t}$

27. $h(w) = \frac{6^w+5}{5^w+6}$

28. $h(y) = \frac{7^y+8}{5^y}$

29. $j(r) = \frac{5^{r^2}-r}{6^{r^2}}$

30. $k(w) = w^3\cot(5w)$

31. $p(x) = \left(x^2+4x\right)^6\left(7x^4+x\right)^3$

32. $m(r) = \sin(8-4r)\cos\left(6r+r^2\right)$

33. $m(w) = \cos(4w-5)\sin(9+7w)$

34. $f(x) = e^{8x^2}\sin\left(\frac{1}{x}\right)$

35. $g(r) = \frac{\cos(6r+4)}{(3r+1)^3}$

36. $h(z) = \frac{(3z+5)^2}{\sin(9z)}$

Exercise Group. Find the equations of tangent and normal lines to the graph of the function at the given point. Note: the functions here are the same as in Exercises 7–10.

37. $f(x) = \left(4x^3 - x\right)^{10}$ at $x = 0$

38. $f(x) = (3x-2)^5$ at $x = 1$

39. $g(x) = (\sin(x) + \cos(x))^3$ at $x = \pi/2$.

40. $h(x) = e^{3x^2+x-1}$ at $x = -1$

41. Compute $\frac{d}{dx}(\ln(kx))$ two ways. First by using the Chain Rule. Second, by using the logarithm rule $\ln(ab) = \ln(a) + \ln(b)$ and then taking the derivative.

42. Compute $\frac{d}{dx}\left(\ln\left(x^k\right)\right)$ two ways. First by using the Chain Rule. Second, by using the logarithm rule $\ln(a^p) = p\ln(a)$ (for positive a) and then taking the derivative.

2.6 Implicit Differentiation

In the previous sections we learned to find the derivative, $\frac{dy}{dx}$, or y', when y is given *explicitly* as a function of x. That is, if we know $y = f(x)$ for some function f, we can find y'. For example, given $y = 3x^2 - 7$, we can easily find $y' = 6x$. (Here we explicitly state how y depends on x. Knowing x, we can directly find y.)

Sometimes the relationship between y and x is not explicit; rather, it is *implicit*. For instance, we might know that $x^2 - y = 4$. This equality defines a relationship between x and y; if we know x, we could figure out y. Can we still find y'? In this case, sure; we solve for y to get $y = x^2 - 4$ (hence we now know y explicitly) and then differentiate to get $y' = 2x$.

Sometimes the *implicit* relationship between x and y is complicated. Suppose we are given $\sin(y) + y^3 = 6 - x^3$. A graph of this implicit relationship is given in Figure 2.6.2. In this case there is absolutely no way to solve for y in terms of elementary functions. The surprising thing is, however, that we can still find y' via a process known as **implicit differentiation**.

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Figure 2.6.1 Video introduction to Section 2.6

2.6.1 The method of implicit differentiation

Implicit differentiation is a technique based on the The Chain Rule that is used to find a derivative when the relationship between the variables is given implicitly rather than explicitly (solved for one variable in terms of the other).

We begin by reviewing the Chain Rule. Let f and g be functions of x. Then

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x).$$

Suppose now that $y = g(x)$. We can rewrite the above as

$$\frac{d}{dx}(f(y)) = f'(y) \cdot y', \qquad \text{or} \qquad \frac{d}{dx}(f(y)) = f'(y) \cdot \frac{dy}{dx}. \qquad (2.6.1)$$

These equations look strange; the key concept to learn here is that we can find y' even if we don't exactly know how y and x relate.

We demonstrate this process in the following example.

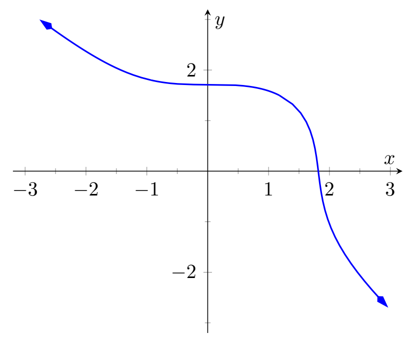

Figure 2.6.2 A graph of the implicit relationship $\sin(y) + y^3 = 6 - x^3$

Example 2.6.3 Using Implicit Differentiation.

Find y' given that $\sin(y) + y^3 = 6 - x^3$.

Solution. We start by taking the derivative of both sides (thus maintaining the equality.) We have:

$$\frac{d}{dx}\left(\sin(y) + y^3\right) = \frac{d}{dx}\left(6 - x^3\right).$$

The right hand side is easy; it returns $-3x^2$.

The left hand side requires more consideration. We take the derivative term-by-term. Using the technique derived from Equation (2.6.1) above, we can see that

$$\frac{d}{dx}(\sin(y)) = \cos(y) \cdot y'.$$

We apply the same process to the y^3 term.

$$\frac{d}{dx}\left(y^3\right) = \frac{d}{d(y)^3}(=)3(y)^2 \cdot y'.$$

Putting this together with the right hand side, we have

$$\cos(y)y' + 3y^2y' = -3x^2.$$

Now solve for y'. It's important to treat y' as an algebraically independent variable from y and x.

$$\begin{aligned}\cos(y)y' + 3y^2y' &= -3x^2\\ \left(\cos(y) + 3y^2\right)y' &= -3x^2\\ y' &= \frac{-3x^2}{\cos(y) + 3y^2}\end{aligned}$$

This equation for y' probably seems unusual for it contains both x and y terms. How is it to be used? We'll address that next.

Video solution

youtu.be/watch?v=0baXlbhup0o

Implicit functions are generally harder to deal with than explicit functions. With an explicit function, given an x value, we have an explicit formula for computing the corresponding y value. With an implicit function, one often has to find x and y values *at the same time* that satisfy the equation. It is much easier to demonstrate that a given point satisfies the equation than to actually find such a point.

For instance, we can affirm easily that the point $\left(\sqrt[3]{6}, 0\right)$ lies on the graph of the implicit function $\sin(y) + y^3 = 6 - x^3$. Plugging in 0 for y, we see the left hand side is 0. Setting $x = \sqrt[3]{6}$, we see the right hand side is also 0; the equation is satisfied. The following example finds the equation of the tangent line to this function at this point.

Example 2.6.4 Using implicit differentiation to find a tangent line.

Find the equation of the line tangent to the curve of the implicitly defined function $\sin(y) + y^3 = 6 - x^3$ at the point $\left(\sqrt[3]{6}, 0\right)$.

Solution. In Example 2.6.3 we found that

$$y' = \frac{-3x^2}{\cos(y) + 3y^2}.$$

We find the slope of the tangent line at the point $\left(\sqrt[3]{6}, 0\right)$ by substituting $\sqrt[3]{6}$ for x and 0 for y. Thus at the point $\left(\sqrt[3]{6}, 0\right)$, we have the slope as

$$y' = \frac{-3\left(\sqrt[3]{6}\right)^2}{\cos(0) + 3\cdot 0^2} = \frac{-3\sqrt[3]{36}}{1} \approx -9.91.$$

Therefore the equation of the tangent line to the implicitly defined function $\sin(y) + y^3 = 6 - x^3$ at the point $\left(\sqrt[3]{6}, 0\right)$ is

$$y = -3\sqrt[3]{36}\left(x - \sqrt[3]{6}\right) + 0 \approx -9.91x + 18.$$

The curve and this tangent line are shown in Figure 2.6.5.

Video solution

youtu.be/watch?v=qYrcm4ObwOM

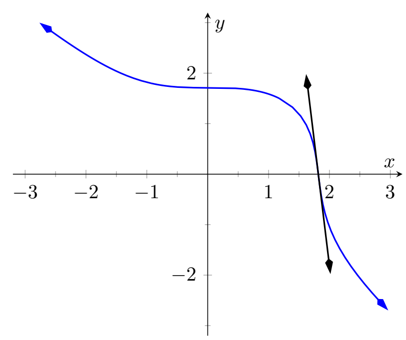

Figure 2.6.5 The function $\sin(y) + y^3 = 6 - x^3$ and its tangent line at the point $\left(\sqrt[3]{6}, 0\right)$

This suggests a general method for implicit differentiation. For the steps below assume y is a function of x.

1. Take the derivative of each term in the equation. Treat the x terms like normal. When taking the derivatives of y terms, the usual rules apply

except that, because of the Theorem 2.5.4, we need to multiply each term by y'.

2. Get all the y' terms on one side of the equal sign and put the remaining terms on the other side.

3. Factor out y'; solve for y' by dividing.

(Practical Note: when working by hand, it may be beneficial to use the symbol $\frac{dy}{dx}$ instead of y', as the latter can be easily confused for y or y^1.)

Example 2.6.6 Using Implicit Differentiation.

Given the implicitly defined function $y^3 + x^2y^4 = 1 + 2x$, find y'.

Solution. We will take the implicit derivatives term by term. The derivative of y^3 is $3y^2y'$.

The second term, x^2y^4, is a little tricky. It requires the Product Rule as it is the product of two functions of x: x^2 and y^4. Its derivative is $x^2(4y^3y') + 2xy^4$. The first part of this expression requires a y' because we are taking the derivative of a y term. The second part does not require it because we are taking the derivative of x^2.

The derivative of the right hand side is easily found to be 2. In all, we get:

$$3y^2y' + 4x^2y^3y' + 2xy^4 = 2.$$

Move terms around so that the left side consists only of the y' terms and the right side consists of all the other terms:

$$3y^2y' + 4x^2y^3y' = 2 - 2xy^4.$$

Factor out y' from the left side and solve to get

$$y' = \frac{2 - 2xy^4}{3y^2 + 4x^2y^3}.$$

To confirm the validity of our work, let's find the equation of a tangent line to this function at a point. It is easy to confirm that the point $(0, 1)$ lies on the graph of this function. At this point, $y' = 2/3$. So the equation of the tangent line is $y = 2/3(x - 0) + 1$. The function and its tangent line are graphed in Figure 2.6.7.

Notice how our curve looks much different than for functions we have seen. For one, it fails the vertical line test, and so the complete curve is not truly representing y as a function of x. But when we indicate we are interested in the derivative at $(0, 1)$, we are indicating that we want the function defined by the small portion of the curve that passes through $(0, 1)$, and that small portion does pass the vertical line test. Such functions are important in many areas of mathematics, so developing tools to deal with them is also important.

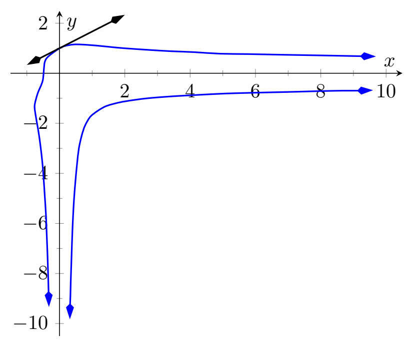

Figure 2.6.7 A graph of the implicitly defined function $y^3 + x^2y^4 = 1 + 2x$ along with its tangent line at the point $(0, 1)$

Video solution

youtu.be/watch?v=O5OqJ7a_Ovo

Example 2.6.8 Using Implicit Differentiation.

Given the implicitly defined function $\sin(x^2y^2) + y^3 = x + y$, find y'.

Solution. Differentiating term by term, we find the most difficulty in the first term. It requires both the The Chain Rule and Product Rule.

$$\begin{aligned}\frac{d}{dx}\left(\sin\left(x^2y^2\right)\right) &= \cos\left(x^2y^2\right)\cdot\frac{d}{dx}\left(x^2y^2\right)\\ &= \cos\left(x^2y^2\right)\cdot\left(x^2(2yy')+2xy^2\right)\\ &= 2\left(x^2yy'+xy^2\right)\cos\left(x^2y^2\right).\end{aligned}$$

We leave the derivatives of the other terms to the reader. After taking the derivatives of both sides, we have

$$2\left(x^2yy'+xy^2\right)\cos\left(x^2y^2\right)+3y^2y'=1+y'.$$

We now have to be careful to properly solve for y', particularly because of the product on the left. It is best to multiply out the product. Doing this, we get

$$2x^2y\cos\left(x^2y^2\right)y'+2xy^2\cos\left(x^2y^2\right)+3y^2y'=1+y'.$$

From here we can safely move around terms to get the following:

$$2x^2y\cos\left(x^2y^2\right)y'+3y^2y'-y'=1-2xy^2\cos\left(x^2y^2\right).$$

Then we can solve for y' to get

$$y'=\frac{1-2xy^2\cos\left(x^2y^2\right)}{2x^2y\cos(x^2y^2)+3y^2-1}.$$

A graph of this implicit function is given in Figure 2.6.9.
It is easy to verify that the points $(0,0)$, $(0,1)$ and $(0,-1)$ all lie on the graph. We can find the slopes of the tangent lines at each of these points using our formula for y'.

- At $(0,0)$, the slope is -1.
- At $(0,1)$, the slope is $1/2$.
- At $(0,-1)$, the slope is also $1/2$.

The tangent lines have been added to the graph of the function in Figure 2.6.10.

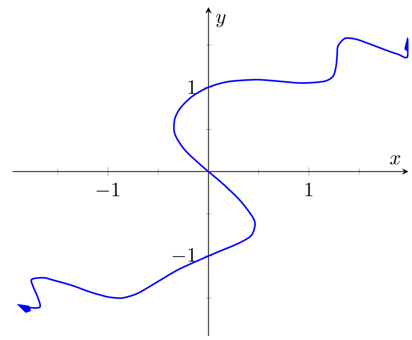

Figure 2.6.9 A graph of the implicitly defined curve $\sin\left(x^2y^2\right)+y^3=x+y$

Video solution

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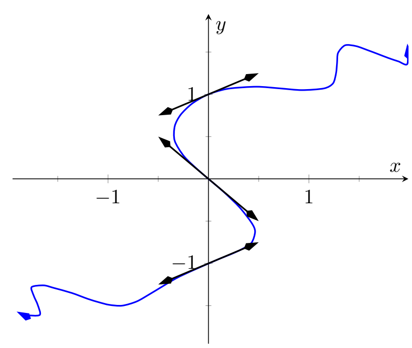


Figure 2.6.10 A graph of the implicitly defined curve $\sin\left(x^2y^2\right)+y^3=x+y$ and certain tangent lines

Quite a few "famous" curves have equations that are given implicitly. We can use implicit differentiation to find the slope at various points on those curves. We investigate two such curves in the next examples.

Example 2.6.11 Finding slopes of tangent lines to a circle.

Find the slope of the tangent line to the circle $x^2+y^2=1$ at the point $(1/2,\sqrt{3}/2)$.

Solution. Taking derivatives, we get $2x+2yy'=0$. Solving for y' gives:

$$y'=\frac{-x}{y}.$$

This is a clever formula. Recall that the slope of the line through the origin and the point (x,y) on the circle will be y/x. We have found that the slope of the tangent line to the circle at that point is the opposite

reciprocal of y/x, namely, $-x/y$. Hence these two lines are always perpendicular.
At the point $(1/2, \sqrt{3}/2)$, we have the tangent line's slope as

$$y' = \frac{-1/2}{\sqrt{3}/2} = \frac{-1}{\sqrt{3}} \approx -0.577.$$

A graph of the circle and its tangent line at $(1/2, \sqrt{3}/2)$ is given in Figure 2.6.12, along with a thin dashed line from the origin that is perpendicular to the tangent line. (It turns out that all normal lines to a circle pass through the center of the circle.)

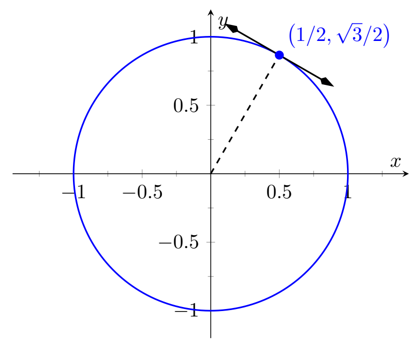


Figure 2.6.12 The unit circle with its tangent line at $(1/2, \sqrt{3}/2)$

This section has shown how to find the derivatives of implicitly defined functions, whose graphs include a wide variety of interesting and unusual shapes. Implicit differentiation can also be used to further our understanding of "regular" differentiation.

One hole in our current understanding of derivatives is this: what is the derivative of the square root function? That is,

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{1/2}\right) = ?$$

We allude to a possible solution, as we can write the square root function as a power function with a rational (or, fractional) power. We are then tempted to apply the Power Rule with Integer Exponents and obtain

$$\frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

The trouble with this is that the Power Rule with Integer Exponents was initially defined only for positive integer powers, $n > 0$. While we did not justify this at the time, generally the Power Rule with Integer Exponents is proved using something called the Binomial Theorem, which deals only with positive integers. The Quotient Rule allowed us to extend the Power Rule with Integer Exponents to negative integer powers. Implicit Differentiation allows us to extend the Power Rule with Integer Exponents to rational powers, as shown below.

Let $y = x^{m/n}$, where m and n are integers with no common factors (so $m = 2$ and $n = 5$ is fine, but $m = 2$ and $n = 4$ is not). We can rewrite this explicit function implicitly as $y^n = x^m$. Now apply implicit differentiation.

$$\begin{aligned}
y &= x^{m/n} \\
y^n &= x^m \\
\frac{d}{dx}\left(y^n\right) &= \frac{d}{dx}\left(x^m\right) \\
n \cdot y^{n-1} \cdot y' &= m \cdot x^{m-1} \\
y' &= \frac{m}{n}\frac{x^{m-1}}{y^{n-1}} && \text{(now substitute } x^{m/n} \text{ for } y\text{)} \\
&= \frac{m}{n}\frac{x^{m-1}}{(x^{m/n})^{n-1}} && \text{(apply lots of algebra)} \\
&= \frac{m}{n}x^{(m-n)/n} \\
&= \frac{m}{n}x^{m/n-1}.
\end{aligned}$$

The above derivation is the key to the proof extending the Power Rule with Integer Exponents to rational powers. Using limits, we can extend this once more to include *all* powers, including irrational (even transcendental!) powers, giving the following theorem.

Theorem 2.6.13 Power Rule for Differentiation.

Let $f(x) = x^n$, where $n \neq 0$ is a real number. Then f is differentiable on its domain, except possibly at $x = 0$, and $f'(x) = n \cdot x^{n-1}$.

This theorem allows us to say the derivative of x^π is $\pi x^{\pi-1}$.

We now apply this final version of the Power Rule for Differentiation in the next example, the second investigation of a "famous" curve.

Example 2.6.14 Using the Power Rule.

Find the slope of $x^{2/3} + y^{2/3} = 8$ at the point $(8, 8)$.

Solution. This is a particularly interesting curve called an **astroid**. It is the shape traced out by a point on the edge of a circle that is rolling around inside of a larger circle, as shown in Figure 2.6.15.

To find the slope of the astroid at the point $(8, 8)$, we take the derivative implicitly.

$$\begin{aligned} \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' &= 0 \\ \frac{2}{3}y^{-1/3}y' &= -\frac{2}{3}x^{-1/3} \\ y' &= -\frac{x^{-1/3}}{y^{-1/3}} \\ y' &= -\frac{y^{1/3}}{x^{1/3}} = -\sqrt[3]{\frac{y}{x}}. \end{aligned}$$

Plugging in $x = 8$ and $y = 8$, we get a slope of -1. The astroid, with its tangent line at $(8, 8)$, is shown in Figure 2.6.16.

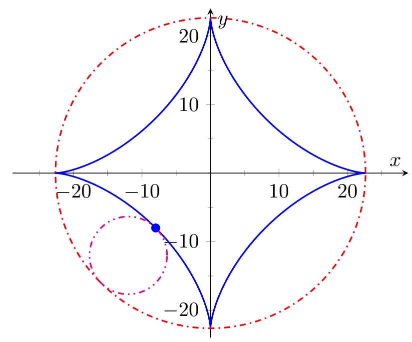


Figure 2.6.15 An astroid, traced out by a point on the smaller circle as it rolls inside the larger circle

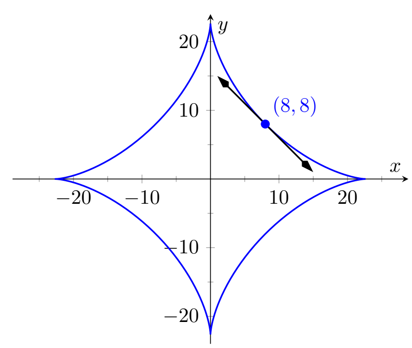


Figure 2.6.16 An astroid with a tangent line

2.6.2 Implicit Differentiation and the Second Derivative

We can use implicit differentiation to find higher order derivatives. In theory, this is simple: first find $\frac{dy}{dx}$, then take its derivative with respect to x. In practice, it is not hard, but it often requires a bit of algebra. We demonstrate this in an example.

Example 2.6.17 Finding the second derivative.

Given $x^2 + y^2 = 1$, find $\frac{d^2y}{dx^2} = y''$.

Solution. We found that $y' = \frac{dy}{dx} = -x/y$ in Example 2.6.11. To find y'', we apply implicit differentiation to y'.

$$\begin{aligned} y'' &= \frac{d}{dx}(y') \\ &= \frac{d}{dx}\left(-\frac{x}{y}\right) && \text{(Now use the Quotient Rule.)} \\ &= -\frac{y \cdot 1 - x(y')}{y^2} && \text{replace } y' \text{ with } -x/y\text{:} \end{aligned}$$

$$= -\frac{y - x(-x/y)}{y^2}$$
$$= -\frac{y + x^2/y}{y^2}.$$

While this is not a particularly simple expression, it is usable. We can see that $y'' > 0$ when $y < 0$ and $y'' < 0$ when $y > 0$. In Section 3.4, we will see how this relates to the shape of the graph.
Also, if we remember that we are only considering points on the curve $x^2 + y^2 = 1$, then we know that $x^2 = 1 - y^2$. So we can replace the x^2 in the expression for y'' to get

$$y'' = -\frac{y + \left(1 - y^2\right)/y}{y^2} = -\frac{1}{y^3}$$

which is a simpler expression. Recognizing when simplifications like this are possible is not always easy.

Video solution

youtu.be/watch?v=V6piqsjn2mk

2.6.3 Logarithmic Differentiation

Consider the function $y = x^x$; it is graphed in Figure 2.6.18. It is well-defined for $x > 0$ and we might be interested in finding equations of lines tangent and normal to its graph. How do we take its derivative?

The function is not a power function: it has a "power" of x, not a constant. It is not an exponential function either: it has a "base" of x, not a constant.

A differentiation technique known as **logarithmic differentiation** becomes useful here. The basic principle is this: take the natural log of both sides of an equation $y = f(x)$, then use implicit differentiation to find y'. We demonstrate this in the following example.

In calculus the expression 0^0 is also considered well-defined and equal to 1. This is easily confused with a limit of the *form* 0^0, which is indeterminate. We skirt the issue here.

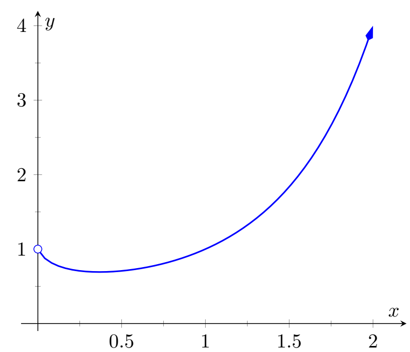


Figure 2.6.18 A plot of $y = x^x$

Example 2.6.19 Using Logarithmic Differentiation.

Given $y = x^x$, use logarithmic differentiation to find y'.

Solution. As suggested above, we start by taking the natural log of both sides then applying implicit differentiation.

$$\begin{aligned}
y &= x^x \\
\ln(y) &= \ln(x^x) && \text{(apply logarithm rule)} \\
\ln(y) &= x\ln(x) && \text{(now use implicit differentiation)} \\
\frac{d}{dx}(\ln(y)) &= \frac{d}{dx}(x\ln(x)) \\
\frac{y'}{y} &= \ln(x) + x\cdot\frac{1}{x} \\
\frac{y'}{y} &= \ln(x) + 1 \\
y' &= y\left(\ln(x) + 1\right) && \text{(substitute } y = x^x\text{)} \\
y' &= x^x\left(\ln(x) + 1\right) && .
\end{aligned}$$

To "test" our answer, let's use it to find the equation of the tangent line at $x = 1.5$. The point on the graph our tangent line must pass through is $\left(1.5, 1.5^{1.5}\right) \approx (1.5, 1.837)$. Using the equation for y', we find the slope as

$$y' = 1.5^{1.5}\left(\ln(1.5)+1\right) \approx 1.837(1.405) \approx 2.582.$$

Thus the equation of the tangent line is (approximately) $y \approx 2.582(x - 1.5) + 1.837$. Figure 2.6.20 graphs $y = x^x$ along with this tangent line.

We would not have been able to compute the derivative of the function in Example 2.6.19 without logarithmic differentiation. But the method is also useful in cases where the product and quotient rules could be used, but logarithmic differentiation is simpler. The video in Figure 2.6.21 provides such an example.

Implicit differentiation proves to be useful as it allows us to find the instantaneous rates of change of a variety of functions. In particular, it extended the Power Rule for Differentiation to rational exponents, which we then extended to all real numbers. In Section 2.7, implicit differentiation will be used to find the derivatives of **inverse** functions, such as $y = \sin^{-1}(x)$.

Video solution

youtu.be/watch?v=6eL6WBlmItk

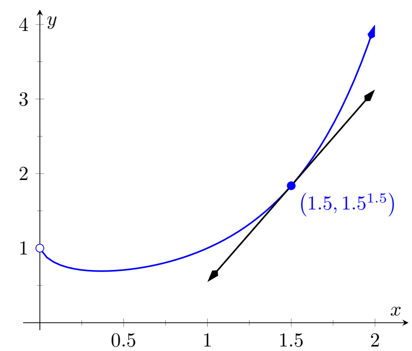


Figure 2.6.20 A graph of $y = x^x$ and its tangent line at $x = 1.5$

youtu.be/watch?v=3Cv2EgjH9ZE

Figure 2.6.21 Using logarithmic differentiation

2.6.4 Exercises

Terms and Concepts

1. In your own words, explain the difference between implicit functions and explicit functions.

2. Implicit differentiation is based on what other differentiation rule?

3. (☐ True ☐ False) Implicit differentiation can be used to find the derivative of $y = \sqrt{x}$.

4. (☐ True ☐ False) Implicit differentiation can be used to find the derivative of $y = x^{3/4}$.

Problems

Exercise Group. Compute the derivative of the given function.

5. $j(w) = \sqrt{w} - \frac{1}{\sqrt{w}}$

6. $k(y) = \sqrt[6]{y} + y^{\left(\frac{5}{6}\right)}$

7. $p(t) = \sqrt{9 + t^2}$

8. $m(w) = \sqrt{w}\tan(w)$

9. $m(y) = y^{1.2}$

10. $f(r) = r^{\pi} + r^{3.8} + \pi^{3.8}$

11. $g(w) = \frac{w+(-8)}{\sqrt{w}}$

12. $h(x) = \sqrt[6]{x}(\cos(x) + e^{x})$

Exercise Group. Find $\frac{dy}{dx}$ using implicit differentiation.

13. $x^4 + y^2 + y = 7$

14. $x^{2/5} + y^{2/5} = 1$

15. $\cos(x) + \sin(y) = 1$

16. $\dfrac{x}{y} = 10$

17. $\dfrac{y}{x} = 10$

18. $x^2e^x + 2^y = 5$

19. $x^2\tan(y) = 50$

20. $\left(3x^2 + 2y^3\right)^4 = 2$

21. $\left(y^2 + 2y - x\right)^2 = 200$

22. $\frac{x^2+y}{x+y^2} = 17$

23. $\frac{\sin(x)+y}{\cos(y)+x} = 1$

24. $\ln\left(x^2 + y^2\right) = e$

25. $\ln\left(x^2 + xy + y^2\right) = 1$

26. Show that $\frac{dy}{dx}$ is the same for each of the following implicitly defined functions.

(a) $xy = 1$

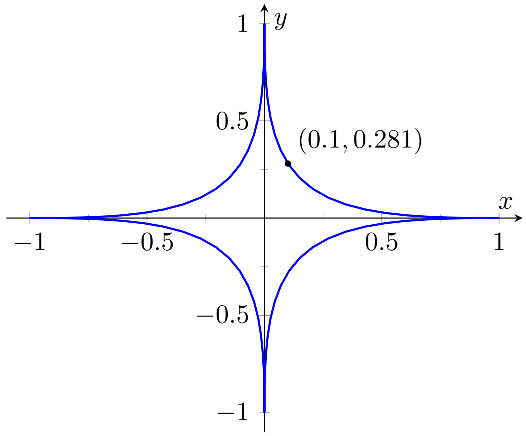
(b) $x^2y^2 = 1$

(c) $\sin(xy) = 1$

(d) $\ln(xy) = 1$

Exercise Group. Find the equation of the tangent line to the graph of the implicitly defined function at the indicated points. As a visual aid, the function is graphed.

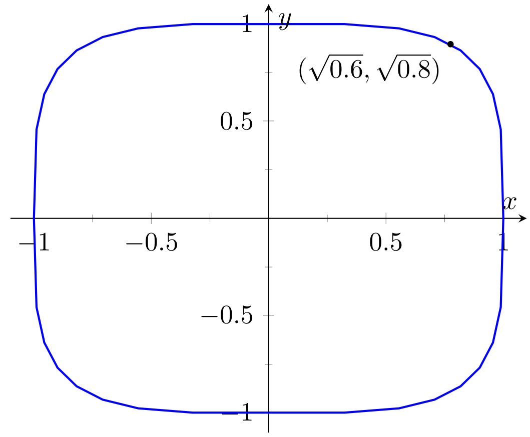
27. On the curve $x^{2/5} + y^{2/5} = 1$.



(a) At $(1, 0)$.

(b) At $(0.1, 0.2811)$ (which does not *exactly* lie on the curve, but is very close).

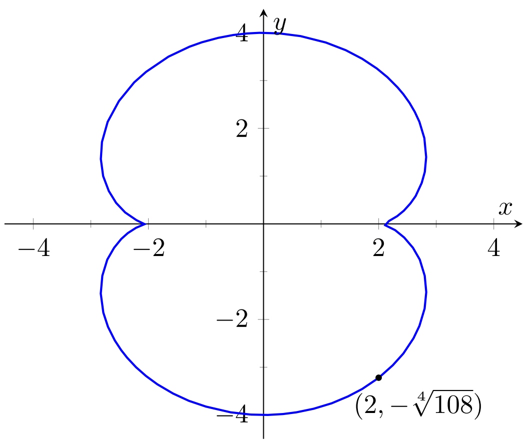
28. On the curve $x^4 + y^4 = 1$.



(a) At $(1, 0)$.

(b) At $\left(\sqrt{0.6}, \sqrt{0.8}\right)$.

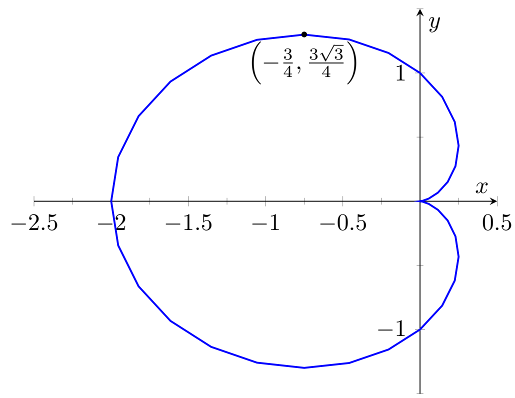
29. On the curve $(x^2 + y^2 - 4)^3 = 108y^2$.



(a) At $(0, 4)$.

(b) At $\left(2, -\sqrt[4]{108}\right)$.

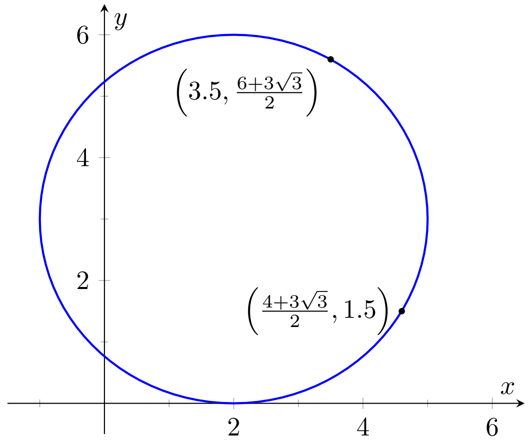
30. On the curve $(x^2 + y^2 + x)^2 = x^2 + y^2$.



(a) At $(0, 1)$.

(b) At $\left(-\frac{3}{4}, \frac{3\sqrt{3}}{4}\right)$.

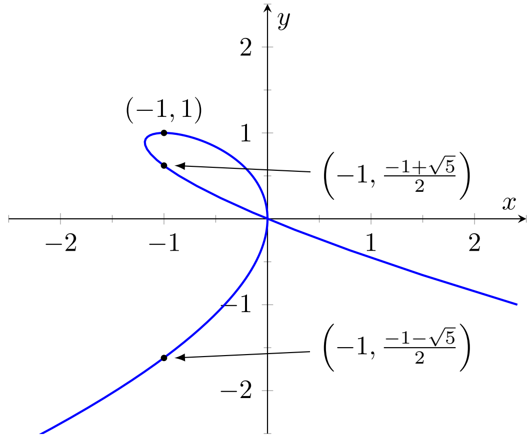
31. On the curve $(x-2)^2+(y-3)^2=9$.



(a) At $\left(\frac{7}{2}, \frac{6+3\sqrt{3}}{2}\right)$.

(b) At $\left(\frac{4+3\sqrt{3}}{2}, \frac{3}{2}\right)$.

32. On the curve $x^2+y^3+2xy=0$.



(a) At $(-1,1)$.

(b) At $\left(-1, \frac{1}{2}(-1+\sqrt{5})\right)$.

(c) At $\left(-1, \frac{1}{2}(-1-\sqrt{5})\right)$.

Exercise Group. An implicitly defined function is given. Find $\frac{d^2y}{dx^2}$. Note: these are the same functions used in Exercises 13 through 16.

33. $x^4+y^2+y=7$

34. $x^{2/5}+y^{2/5}=1$

35. $\cos(x)+\sin(y)=1$

36. $\dfrac{x}{y}=10$

Exercise Group. Use logarithmic differentiation to find $\frac{dy}{dx}$, then find the equation of the tangent line at the indicated x-value.

37. $y=(1+x)^{1/x}$ at $x=1$

38. $y=(2x)^{x^2}$ at $x=1$

39. $y=\dfrac{x^x}{x+1}$ at $x=1$

40. $y=x^{\sin(x)+2}$ at $x=\pi/2$

41. $y=\dfrac{x+1}{x+2}$ at $x=1$

42. $y=\dfrac{(x+1)(x+2)}{(x+3)(x+4)}$ at $x=0$

2.7 Derivatives of Inverse Functions

Recall that a function $y = f(x)$ is said to be **one-to-one** if it passes the horizontal line test; that is, for two different x values x_1 and x_2, we do *not* have $f(x_1) = f(x_2)$. In some cases the domain of f must be restricted so that it is one-to-one. For instance, consider $f(x) = x^2$. Clearly, $f(-1) = f(1)$, so f is not one-to-one on its regular domain, but by restricting f to $(0, \infty)$, f is one-to-one.

Now recall that one-to-one functions have **inverses**. That is, if f is one-to-one, it has an inverse function, denoted by f^{-1}, such that if $f(a) = b$, then $f^{-1}(b) = a$. The domain of f^{-1} is the range of f, and vice-versa. For ease of notation, we set $g = f^{-1}$ and treat g as a function of x.

Since $f(a) = b$ implies $g(b) = a$, when we compose f and g we get a nice result:

$$f\big(g(b)\big) = f(a) = b.$$

In general, $f\big(g(x)\big) = x$ and $g\big(f(x)\big) = x$. This gives us a convenient way to check if two functions are inverses of each other: compose them and if the result is x (on the appropriate domains), then they are inverses.

When the point (a, b) lies on the graph of f, the point (b, a) lies on the graph of g. This leads us to discover that the graph of g is the reflection of f across the line $y = x$. In Figure 2.7.3 we see a function graphed along with its inverse. See how the point $(1, 1.5)$ lies on one graph, whereas $(1.5, 1)$ lies on the other. Because of this relationship, whatever we know about f can quickly be transferred into knowledge about g.

For example, consider Figure 2.7.4 where the tangent line to f at the point $(1, 1.5)$ is drawn. That line has slope 3. Through reflection across $y = x$, we can see that the tangent line to g at the point $(1.5, 1)$ has slope $1/3$. Their slopes are reciprocals. This should make sense since reflecting a line (such as a tangent line) across the line $y = x$ switches the x and y values. Also consider the point $(0, 0.5)$ on the graph of f, where the tangent line is horizontal. At the point $(0.5, 0)$ on g, the tangent line is vertical.

More generally, consider the tangent line to f at the point (a, b). That line has slope $f'(a)$. Through reflection across $y = x$, we can extend our above observation to say that the tangent line to g at the point (b, a) should have slope $1/f'(a)$. This then tells us that $g'(b) = 1/f'(a)$.

The information from these two graphs is summarized in Table 2.7.5 below:

Table 2.7.5

Information about f	Information about $g = f^{-1}$
$(1, 1.5)$ lies on f	$(1.5, 1)$ lies on g
Slope of tangent line to f at $x = 1$ is 3	Slope of tangent line to g at $x = 1.5$ is $1/3$
$f'(1) = 3$	$g'(1.5) = 1/3$

We have discovered a relationship between f' and g' in a mostly graphical way. We can realize this relationship analytically as well. Let $y = g(x)$, where again $g = f^{-1}$. We want to find y'. Since $y = g(x)$, we know that $f(y) = x$. Using the The Chain Rule and Implicit Differentiation, take the derivative of both sides of this last equality.

$$\frac{d}{dx}\big(f(y)\big) = \frac{d}{dx}(x)$$
$$f'(y) \cdot y' = 1$$
$$y' = \frac{1}{f'(y)}$$

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Figure 2.7.1 Video introduction to Section 2.7

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Figure 2.7.2 Properties of inverse functions

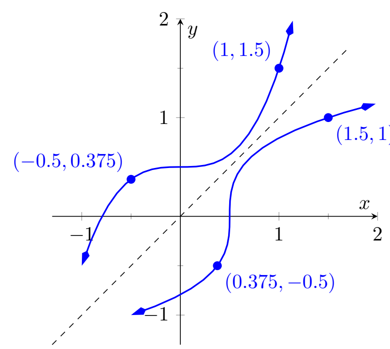


Figure 2.7.3 A function f along with its inverse f^{-1}. (Note how it does not matter which function we refer to as f; the other is f^{-1}.)

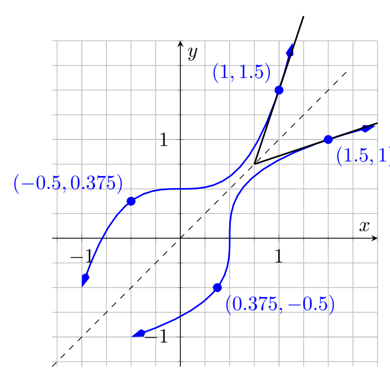


Figure 2.7.4 Corresponding tangent lines drawn to f and f^{-1}

$$y' = \frac{1}{f'(g(x))}.$$

This leads us to the following theorem.

Theorem 2.7.6 Derivatives of Inverse Functions.

Let f be differentiable and one-to-one on an open interval I, where $f'(x) \neq 0$ for all x in I, let J be the range of f on I, let g be the inverse function of f, and let $f(a) = b$ for some a in I. Then g is a differentiable function on J, and in particular,

1. $\left(f^{-1}\right)'(b) = g'(b) = \dfrac{1}{f'(a)}$
2. $\left(f^{-1}\right)'(x) = g'(x) = \dfrac{1}{f'(g(x))}$

The results of Theorem 2.7.6 are not trivial; the notation may seem confusing at first. Careful consideration, along with examples, should earn understanding.

In the next example we apply Theorem 2.7.6 to the arcsine function.

A word of caution is required here. The function $\sin(x)$ is clearly not one-to-one. How can we say that $\arcsin(x)$ is the inverse of $\sin(x)$? To make sense of this, we employ a technique known as *restriction of domain*: instead of considering the entire domain of the sine function, we consider a portion of it, on which the function is one-to-one, as explained in Figure 2.7.8.

youtu.be/watch?v=dOtVBJd75h8

Figure 2.7.7 Video presentation of Theorem 2.7.6

youtu.be/watch?v=lCNZPbfiono

Figure 2.7.8 Restricting the domain of $\sin(x)$

Example 2.7.9 Finding the derivative of an inverse trigonometric function.

Let $y = \arcsin(x) = \sin^{-1}(x)$. Find y' using Theorem 2.7.6.

Solution. Adopting our previously defined notation, let $g(x) = \arcsin(x)$ and $f(x) = \sin(x)$. Thus $f'(x) = \cos(x)$. Applying the theorem, we have

$$\begin{aligned} g'(x) &= \frac{1}{f'(g(x))} \\ &= \frac{1}{\cos(\arcsin(x))}. \end{aligned}$$

This last expression is not immediately illuminating. Drawing a figure will help, as shown in Figure 2.7.10. Recall that the sine function can be viewed as taking in an angle and returning a ratio of sides of a right triangle, specifically, the ratio "opposite over hypotenuse." This means that the arcsine function takes as input a ratio of sides and returns an angle. The equation $y = \arcsin(x)$ can be rewritten as $y = \arcsin(x/1)$; that is, consider a right triangle where the hypotenuse has length 1 and the side opposite of the angle with measure y has length x. This means the final side has length $\sqrt{1-x^2}$, using the Pythagorean Theorem. Therefore

$$\begin{aligned} \cos\left(\sin^{-1}(x)\right) &= \cos(y) \\ &= \frac{\sqrt{1-x^2}}{1} \\ &= \sqrt{1-x^2}, \end{aligned}$$

resulting in

$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}.$$

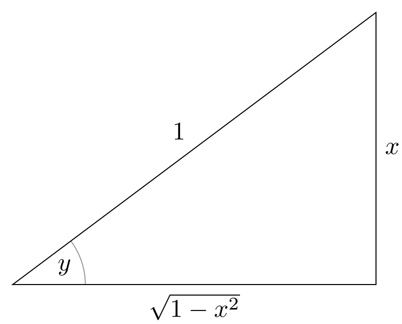


Figure 2.7.10 A right triangle defined by $y = \sin^{-1}(x/1)$ with the length of the third leg found using the Pythagorean Theorem

Video solution

youtu.be/watch?v=xBZqkvQRSG4

Remember that the input x of the arcsine function is a ratio of a side of a right triangle to its hypotenuse; the absolute value of this ratio will never be greater than 1. Therefore the inside of the square root will never be negative.

In order to make $y = \sin(x)$ one-to-one, we restrict its domain to $[-\pi/2, \pi/2]$; on this domain, the range is $[-1, 1]$. Therefore the domain of $y = \arcsin(x)$ is $[-1, 1]$ and the range is $[-\pi/2, \pi/2]$. When $x = \pm 1$, note how the derivative of the arcsine function is undefined; this corresponds to the fact that as $x \to \pm 1$, the tangent lines to arcsine approach vertical lines with undefined slopes.

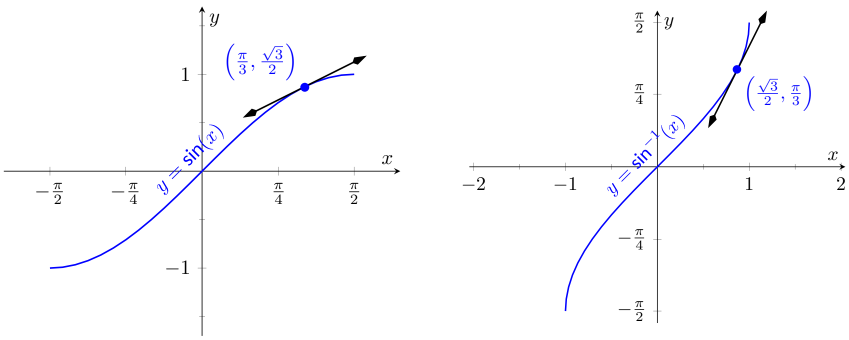


Figure 2.7.11 Graphs of $\sin(x)$ and $\sin^{-1}(x)$ along with corresponding tangent lines

In Figure 2.7.11 we see $f(x) = \sin(x)$ and $f^{-1}(x) = \sin^{-1}(x)$ graphed on their respective domains. The line tangent to $\sin(x)$ at the point $\left(\pi/3, \sqrt{3}/2\right)$ has slope $\cos(\pi)/3 = 1/2$. The slope of the corresponding point on $\sin^{-1}(x)$, the point $\left(\sqrt{3}/2, \pi/3\right)$, is

$$\begin{aligned}\frac{1}{\sqrt{1-\left(\sqrt{3}/2\right)^2}} &= \frac{1}{\sqrt{1-3/4}}\\ &= \frac{1}{\sqrt{1/4}}\\ &= \frac{1}{1/2} = 2,\end{aligned}$$

verifying yet again that at corresponding points, a function and its inverse have reciprocal slopes.

Using similar techniques, we can find the derivatives of all the inverse trigonometric functions. In Table 2.7.12 we show the restrictions of the domains of the standard trigonometric functions that allow them to be invertible.

Table 2.7.12 Domains and ranges of the trigonometric and inverse trigonometric functions

Function	Domain	Range
$\sin(x)$	$[-\pi/2, \pi/2]$	$[-1, 1]$
$\sin^{-1}(x)$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos(x)$	$[0, \pi]$	$[-1, 1]$
$\cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$
$\tan(x)$	$(-\pi/2, \pi/2)$	$(-\infty, \infty)$
$\tan^{-1}(x)$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$
$\csc(x)$	$[-\pi/2, 0) \cup (0, \pi/2]$	$(-\infty, -1] \cup [1, \infty)$
$\csc^{-1}(x)$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$
$\sec(x)$	$[0, \pi/2) \cup (\pi/2, \pi]$	$(-\infty, -1] \cup [1, \infty)$
$\sec^{-1}(x)$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
$\cot(x)$	$(0, \pi)$	$(-\infty, \infty)$
$\cot^{-1}(x)$	$(-\infty, \infty)$	$(0, \pi)$

Theorem 2.7.13 Derivatives of Inverse Trigonometric Functions.

The inverse trigonometric functions are differentiable on all open sets contained in their domains (as listed in Table 2.7.12) and their derivatives are as follows:

1. $\dfrac{d}{dx}\left(\sin^{-1}(x)\right) = \dfrac{1}{\sqrt{1-x^2}}$

2. $\dfrac{d}{dx}\left(\cos^{-1}(x)\right) = -\dfrac{1}{\sqrt{1-x^2}}$

3. $\dfrac{d}{dx}\left(\tan^{-1}(x)\right) = \dfrac{1}{1+x^2}$

4. $\dfrac{d}{dx}\left(\csc^{-1}(x)\right) = -\dfrac{1}{|x|\sqrt{x^2-1}}$

5. $\dfrac{d}{dx}\left(\sec^{-1}(x)\right) = \dfrac{1}{|x|\sqrt{x^2-1}}$

6. $\dfrac{d}{dx}\left(\cot^{-1}(x)\right) = -\dfrac{1}{1+x^2}$

Note how each derivative is the negative of the derivative of its "co" function. Because of this, derivatives of $\sin^{-1}(x)$, $\tan^{-1}(x)$, and $\sec^{-1}(x)$ are used almost exclusively throughout this text.

In Section 2.3, we stated without proof or explanation that $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$. We can justify that now using Theorem 2.7.6, as shown in the example.

youtu.be/watch?v=yO-BT5vEZ9A

Figure 2.7.14 Computing the derivative of $\arctan(x)$

Example 2.7.15 Finding the derivative of $y = \ln(x)$.

Use Theorem 2.7.6 to compute $\frac{d}{dx}(\ln(x))$.

Solution. View $y = \ln(x)$ as the inverse of $y = e^x$. Therefore, using our standard notation, let $f(x) = e^x$ and $g(x) = \ln(x)$. We wish to find

$g'(x)$. Theorem 2.7.6 gives:

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}.$$

In this chapter we have defined the derivative, given rules to facilitate its computation, and given the derivatives of a number of standard functions. We restate the most important of these in the following theorem, intended to be a reference for further work.

Theorem 2.7.16 Glossary of Derivatives of Elementary Functions.

Let f and g be differentiable functions, and let a, c and n be real numbers, $a > 0$, $n \neq 0$.

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx}(x) = 1$
3. $\frac{d}{dx}(x^n) = nx^{n-1}$
4. $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$
5. $\frac{d}{dx}(c \cdot f(x)) = c \cdot f'(x)$
6. $\frac{d}{dx}(f(x) \cdot g(x)) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
7. $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$
8. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$
9. $\frac{d}{dx}(e^x) = e^x$
10. $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$
11. $\frac{d}{dx}(a^x) = \ln(a) \cdot a^x$
12. $\frac{d}{dx}(\log_a x) = \frac{1}{\ln(a)} \cdot \frac{1}{x}$
13. $\frac{d}{dx}(\sin(x)) = \cos(x)$
14. $\frac{d}{dx}(\cos(x)) = -\sin(x)$
15. $\frac{d}{dx}(\tan(x)) = \sec^2(x)$
16. $\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$
17. $\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$
18. $\frac{d}{dx}(\cot(x)) = -\csc^2(x)$
19. $\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$
20. $\frac{d}{dx}\left(\cos^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}}$
21. $\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$
22. $\frac{d}{dx}\left(\csc^{-1}(x)\right) = -\frac{1}{|x|\sqrt{x^2-1}}$
23. $\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{|x|\sqrt{x^2-1}}$
24. $\frac{d}{dx}\left(\cot^{-1}(x)\right) = -\frac{1}{1+x^2}$

2.7.1 Exercises

Terms and Concepts

1. (☐ True ☐ False) Every function has an inverse.

2. In your own words explain what it means for a function to be "one-to-one."

3. If $(1, 10)$ lies on the graph of $y = f(x)$, what can be said about the graph of $y = f^{-1}(x)$?

4. If $(1, 10)$ lies on the graph of $y = f(x)$ and $f'(1) = 5$, what can be said about $y = f^{-1}(x)$?

Problems

Exercise Group. Verify that the given functions are inverses.

5. $f(x) = 2x + 6$ and $g(x) = \frac{1}{2}x - 3$

6. $f(x) = x^2 + 6x + 11, x \geq 3$ and $g(x) = \sqrt{x-2} - 3, x \geq 2$

7. $f(x) = \frac{3}{x-5}, x \neq 5$ and $g(x) = \frac{3+5x}{x}, x \neq 0$

8. $f(x) = \frac{x+1}{x-1}, x \neq 1$ and $g(x) = f(x)$

Exercise Group. An invertible function $f(x)$ is given along with a point that lies on its graph. Using Theorem 2.7.6, evaluate $\left(f^{-1}\right)'(x)$ at the indicated value.

9. The point $(9, 65)$ is on the graph of $f(x) = 7x + 2$. Find $\left(f^{-1}\right)'(65)$.

10. The point $(-6, 51)$ is on the graph of $f(x) = x^2 - 2x + 3, x \geq 1$. Find $\left(f^{-1}\right)'(51)$.

11. The point $\left(\frac{\pi}{24}, \frac{\sqrt{3}}{2}\right)$ is on the graph of $f(x) = \cos(4x), 0 \leq x \leq \frac{\pi}{4}$. Find $\left(f^{-1}\right)'\left(\frac{\sqrt{3}}{2}\right)$.

12. The point $(3, 576)$ is on the graph of $f(x) = x^3 - 27x^2 + 267x - 9$. Find $\left(f^{-1}\right)'(576)$.

13. The point $\left(2, \frac{1}{5}\right)$ is on the graph of $f(x) = \frac{1}{1+x^2}, x \geq 0$. Find $\left(f^{-1}\right)'\left(\frac{1}{5}\right)$.

14. The point $(0, 3)$ is on the graph of $f(x) = 3e^{4x}$. Find $\left(f^{-1}\right)'(3)$.

Exercise Group. Compute the derivative of the given function.

15. $h(w) = \cos^{-1}(4w)$

16. $h(x) = \csc^{-1}(7x)$

17. $j(r) = \tan^{-1}(2r)$

18. $k(w) = w\cos^{-1}(w)$

19. $p(x) = \tan(x)\cos^{-1}(x)$

20. $f(t) = \ln(t)e^t$

21. $m(z) = \frac{\tan^{-1}(z)}{\sin^{-1}(z)}$

22. $f(x) = \tan(\sqrt[4]{x})$

23. $g(q) = \csc\left(\frac{1}{q^3}\right)$

24. $g(z) = \sin\left(\sin^{-1}(z)\right)$

Exercise Group. Compute the derivative of the given function in two ways:

(a) By simplifying first, then taking the derivative, and

(b) by using the Chain Rule first then simplifying.

Verify that the two answers are the same.

25. $f(x) = \sin(\sin^{-1}(x))$

26. $f(x) = \tan^{-1}(\tan(x))$

27. $f(x) = \sin(\cos^{-1}(x))$

28. $f(x) = \sin(2\sin^{-1}(x))$

Exercise Group. Find the equation of the line tangent to the graph of f at the indicated x value.

29. $f(x) = \sin^{-1}(x)$ at $x = \frac{-\sqrt{3}}{2}$

30. $f(x) = \cos^{-1}(2x)$ at $x = \frac{\sqrt{3}}{4}$

Chapter 3

The Graphical Behavior of Functions

Our study of limits led to continuous functions, a certain class of functions that behave in a particularly nice way. Limits then gave us an even nicer class of functions, functions that are differentiable.

This chapter explores many of the ways we can take advantage of the information that continuous and differentiable functions provide.

3.1 Extreme Values

Given any quantity described by a function, we are often interested in the largest and/or smallest values that quantity attains. For instance, if a function describes the speed of an object, it seems reasonable to want to know the fastest/slowest the object traveled. If a function describes the value of a stock, we might want to know the highest/lowest values the stock attained over the past year. We call such values *extreme values*.

youtu.be/watch?v=srE7xUmQtCQ

Figure 3.1.2 Video presentation of Definition 3.1.1

Definition 3.1.1 Extreme Values.

Let f be defined on an interval I containing c.

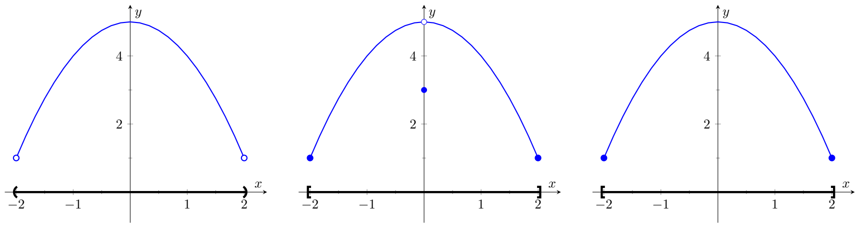
1. $f(c)$ is the *minimum* (also, *absolute minimum*) of f on I if $f(c) \leq f(x)$ for all x in I.

2. $f(c)$ is the *maximum* (also, *absolute maximum*) of f on I if $f(c) \geq f(x)$ for all x in I.

The maximum and minimum values are the **extreme values**, or **extrema**, of f on I.

Note: The extreme values of a function are "y" values, values the function attains, not the input values. However we often say there is an extreme value *at* certain input values. For example, "sin(x) has a maximum *at* $\pi/2$, and the maximum of sin(x) *is* 1."

Consider Figure 3.1.3. The function displayed in Figure 3.1.3(a) has a maximum, but no minimum, as the interval over which the function is defined is open. In Figure 3.1.3(b), the function has a minimum, but no maximum; there is a discontinuity in the "natural" place for the maximum to occur. Finally, the function shown in Figure 3.1.3(c)has both a maximum and a minimum; note that the function is continuous and the interval on which it is defined is closed.



(a) **(b)** **(c)**

Figure 3.1.3 Graphs of functions with and without extreme values

It is possible for discontinuous functions defined on an open interval to have both a maximum and minimum value, but we have just seen examples where they did not. On the other hand, continuous functions on a closed interval *always* have a maximum and minimum value.

Theorem 3.1.4 The Extreme Value Theorem.

Let f be a continuous function defined on a closed interval $I = [a, b]$. Then f has both a maximum and minimum value on I.

youtu.be/watch?v=GIcjxu8dTnY

Figure 3.1.5 Video presentation of Theorem 3.1.4

This theorem states that f has extreme values, but it does not offer any advice about how/where to find these values. The process can seem to be fairly easy, as the next example illustrates. After the example, we will draw on lessons learned to form a more general and powerful method for finding extreme values.

Example 3.1.6 Approximating extreme values.

Consider $f(x) = 2x^3 - 9x^2$ on $I = [-1, 5]$, as graphed in Figure 3.1.7. Approximate the extreme values of f.

Solution. The graph is drawn in such a way to draw attention to certain points. It certainly seems that the smallest y-value is -27, found when $x = 3$. It also seems that the largest y-value is 25, found at the endpoint of I, $x = 5$. We use the word *seems*, for by the graph alone we cannot be sure the smallest value is not less than -27. Since the problem asks for an approximation, we approximate the extreme values to be 25 and -27.

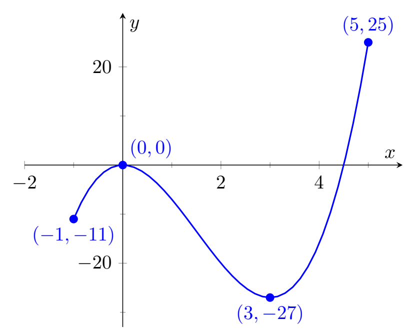


Figure 3.1.7 A graph of $f(x) = 2x^3 - 9x^2$ as in Example 3.1.6

Notice how the minimum value came at "the bottom of a hill," and the maximum value came at an endpoint. Also note that while 0 is not an extreme value, it would be if we narrowed our interval to $[-1, 4]$. The idea that the point $(0, 0)$ is the location of an extreme value for some interval is important, leading us to a definition of a *relative maximum*. In short, a "relative max" is a y-value that's the largest y-value "nearby."

Definition 3.1.9 Relative Minimum and Relative Maximum.

Let f be defined on an interval I containing c.

1. If there is a $\delta > 0$ such that $f(c) \leq f(x)$ for all x in I where $|x - c| < \delta$, then $f(c)$ is a **relative minimum** of f. We also say that f has a relative minimum at $(c, f(c))$.

2. If there is a $\delta > 0$ such that $f(c) \geq f(x)$ for all x in I where $|x - c| < \delta$, then $f(c)$ is a **relative maximum** of f. We also say that f has a relative maximum at $(c, f(c))$.

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Figure 3.1.8 Video presentation of Definition 3.1.9

The relative maximum and minimum values comprise the **relative extrema** of f.

We briefly practice using these definitions.

Alternative Vocabulary. The terms *local minimum* and *local maximum* are often used as synonyms for *relative minimum* and *relative maximum*.

As it makes intuitive sense that an absolute maximum is also a relative maximum, Definition 3.1.9 allows a relative maximum to occur at an interval's endpoint.

Example 3.1.10 Approximating relative extrema.

Consider $f(x) = (3x^4 - 4x^3 - 12x^2 + 5)/5$, as shown in Figure 3.1.11. Approximate the relative extrema of f. At each of these points, evaluate f'.

Solution. We still do not have the tools to exactly find the relative extrema, but the graph does allow us to make reasonable approximations. It seems f has relative minima at $x = -1$ and $x = 2$, with values of $f(-1) = 0$ and $f(2) = -5.4$. It also seems that f has a relative maximum at the point $(0, 1)$.
We approximate the relative minima to be 0 and -5.4; we approximate the relative maximum to be 1.
It is straightforward to evaluate $f'(x) = \frac{1}{5}\left(12x^3 - 12x^2 - 24x\right)$ at $x = 0, 1$ and 2. In each case, $f'(x) = 0$.

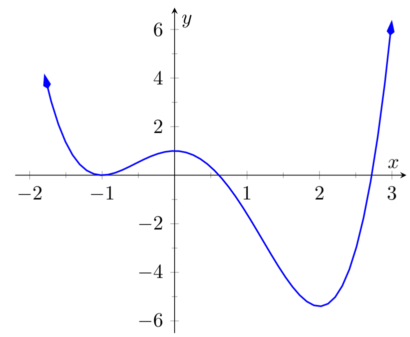

Figure 3.1.11 A graph of $f(x) = (3x^4 - 4x^3 - 12x^2 + 5)/5$ as in Example 3.1.10

Example 3.1.12 Approximating relative extrema.

Approximate the relative extrema of $f(x) = (x-1)^{2/3} + 2$, shown in Figure 3.1.13. At each of these points, evaluate f'.

Solution. The figure implies that f does not have any relative maxima, but has a relative minimum at $(1, 2)$. In fact, the graph suggests that not only is this point a relative minimum, $y = f(1) = 2$ is *the* minimum value of the function.
We compute $f'(x) = \frac{2}{3}(x-1)^{-1/3}$. When $x = 1$, f' is undefined.

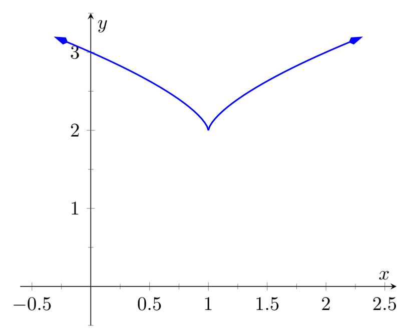

Figure 3.1.13 A graph of $f(x) = (x-1)^{2/3} + 2$ as in Example 3.1.12

What can we learn from the previous two examples? We were able to visually approximate relative extrema, and at each such point, the derivative was either 0 or it was not defined. This observation holds for all functions, leading to a definition and a theorem.

Definition 3.1.14 Critical Numbers and Critical Points.

Let f be defined at c. The value c is a **critical number** (or **critical value**) of f if $f'(c) = 0$ or $f'(c)$ is not defined.
If c is a critical number of f, then the point $(c, f(c))$ is a **critical point** of f.

In this text we use "critical number" and "critical value" interchangeably. Other textbooks reserve the term *critical value* for the function value $f(c)$, when c is a critical number.

Theorem 3.1.15 Relative Extrema and Critical Points.

Let a function f be defined on an open interval I containing c, and let f have a relative extremum at the point $(c, f(c))$. Then c is a critical number of f.

Be careful to understand that this theorem states "Relative extrema on open intervals occur at critical points." It does not say "All critical numbers produce relative extrema." For instance, consider $f(x) = x^3$. Since $f'(x) = 3x^2$, it is straightforward to determine that $x = 0$ is a critical number of f. However, f has no relative extrema, as illustrated in Figure 3.1.17.

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Figure 3.1.16 Video presentation of Definition 3.1.14 and Theorem 3.1.15

Theorem 3.1.4 states that a continuous function on a closed interval will have both an absolute maximum and an absolute minimum. Common sense tells us "extrema occur either at the endpoints or somewhere in between." It is easy to check for extrema at endpoints, but there are infinitely many points to check that are "in between." Theorem 3.1.15 tells us we need only check at the critical points that are in between the endpoints. We combine these concepts to offer a strategy for finding extrema.

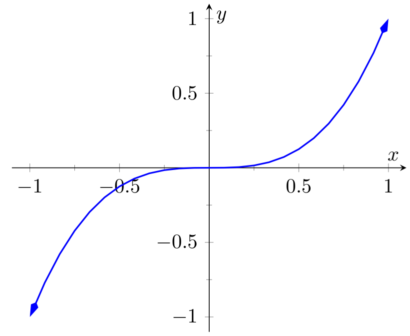


Figure 3.1.17 A graph of $f(x) = x^3$ which has a critical value of $x = 0$, but no relative extrema

Key Idea 3.1.18 Finding Extrema on a Closed Interval.

Let f be a continuous function defined on a closed interval $[a, b]$. To find the maximum and minimum values of f on $[a, b]$:

1. Evaluate f at the endpoints a and b of the interval.
2. Find the critical numbers of f in $[a, b]$.
3. Evaluate f at each critical number.
4. The absolute maximum of f is the largest of these values, and the absolute minimum of f is the least of these values.

We practice these ideas in the next examples.

Example 3.1.19 Finding extreme values.

Find the extreme values of $f(x) = 2x^3 + 3x^2 - 12x$ on $[0, 3]$, graphed in Figure 3.1.20.

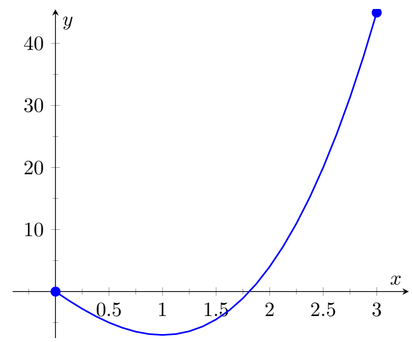


Figure 3.1.20 A graph of $f(x) = 2x^3 + 3x^2 - 12x$ on $[0, 3]$ as in Example 3.1.19

Video solution

youtu.be/watch?v=LyhHlreZvhc

Solution. We follow the steps outlined in Key Idea 3.1.18. We first evaluate f at the endpoints:

$$f(0) = 0 \qquad f(3) = 45.$$

Next, we find the critical values of f on $[0, 3]$. $f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1)$; therefore the critical values of f are $x = -2$ and $x = 1$. Since $x = -2$ does not lie in the interval $[0, 3]$, we ignore it. Evaluating f at the only critical number in our interval gives: $f(1) = -7$.
Figure 3.1.21 gives f evaluated at the "important" x values in $[0, 3]$. We can easily see the maximum and minimum values of f: the maximum value is 45 and the minimum value is -7.

x	$f(x)$
0	0
1	−7
3	45

Figure 3.1.21 Finding the extreme values of $f(x) = 2x^3 + 3x^2 - 12x$ in Example 3.1.19

Note that all this was done without the aid of a graph; this work followed an analytic algorithm and did not depend on any visualization. Figure 3.1.20 shows f and we can confirm our answer, but it is important to understand that these answers can be found without graphical assistance.
We practice again.

Example 3.1.22 Finding extreme values.

Find the maximum and minimum values of f on $[-4, 2]$, where

$$f(x) = \begin{cases} (x-1)^2 & x \le 0 \\ x + 1 & x > 0 \end{cases}.$$

Solution. Here f is piecewise-defined, but we can still apply Key Idea 3.1.18 as it is continuous on $[-4, 2]$ (one should check to verify that

$\lim_{x\to 0} f(x) = f(0)$).
Evaluating f at the endpoints gives:

$$f(-4) = 25 \qquad\qquad f(2) = 3.$$

We now find the critical numbers of f. We have to define f' in a piecewise manner; it is

$$f'(x) = \begin{cases} 2(x-1) & x < 0 \\ 1 & x > 0 \end{cases}.$$

Note that while f is defined for all of $[-4, 2]$, f' is not, as the derivative of f does not exist when $x = 0$. (From the left, the derivative approaches -2; from the right the derivative is 1.) Thus one critical number of f is $x = 0$.
We now set $f'(x) = 0$. When $x > 0$, $f'(x)$ is never 0. When $x < 0$, $f'(x)$ is also never 0, so we find no critical values from setting $f'(x) = 0$. So we have three important x-values to consider: $x = -4, 2$ and 0. Evaluating f at each gives, respectively, 25, 3 and 1, shown in Figure 3.1.23. Thus the absolute minimum of f is 1, the absolute maximum of f is 25. Our answer is confirmed by the graph of f in Figure 3.1.24.

x	$f(x)$
-4	25
0	1
2	3

Figure 3.1.23 Finding the extreme values of a piecewise-defined function in Example 3.1.22

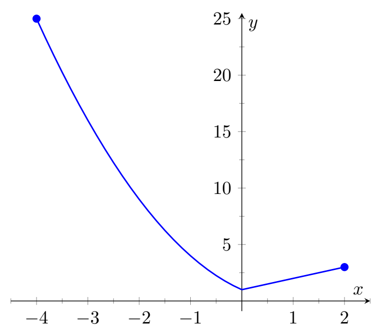


Figure 3.1.24 A graph of $f(x)$ on $[-4, 2]$ as in Example 3.1.22

Video solution

youtu.be/watch?v=zn--ShSSMqk

Example 3.1.25 Finding extreme values.

Find the extrema of $f(x) = \cos(x^2)$ on $[-2, 2]$.

Solution. We again use Key Idea 3.1.18. Evaluating f at the endpoints of the interval gives: $f(-2) = f(2) = \cos(4) \approx -0.6536$. We now find the critical values of f.
Applying the The Chain Rule, we find $f'(x) = -2x\sin(x^2)$. Set $f'(x) = 0$ and solve for x to find the critical values of f.
We have $f'(x) = 0$ when $x = 0$ and when $\sin(x^2)$. In general, $\sin(t) = 0$ when $t = \ldots -2\pi, -\pi, 0, \pi, \ldots$ Thus $\sin(x^2) = 0$ when $x^2 = 0, \pi, 2\pi, \ldots$ (x^2 is always nonnegative so we ignore $-\pi$, etc.) So $\sin(x^2) = 0$ when $x = 0, \pm\sqrt{\pi}, \pm\sqrt{2\pi}, \ldots$. The only values to fall in the given interval of $[-2, 2]$ are 0 and $\pm\sqrt{\pi}$, where $\sqrt{\pi} \approx 1.77$.
We again construct a table of important values in Figure 3.1.26. In this example we have five values to consider: $x = 0, \pm 2, \pm\sqrt{\pi}$. From the table it is clear that the maximum value of f on $[-2, 2]$ is 1; the minimum

value is -1. The graph in Figure 3.1.27 confirms our results.

x	$f(x)$
-2	-0.65
$-\sqrt{\pi}$	-1
0	1
$\sqrt{\pi}$	-1
2	-0.65

Figure 3.1.26 Finding the extrema of $f(x) = \cos(x^2)$ in Example 3.1.25

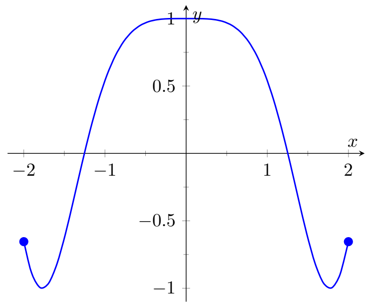


Figure 3.1.27 A graph of $f(x) = \cos(x^2)$ on $[-2, 2]$ as in Example 3.1.25

Video solution

youtu.be/watch?v=sT_3kVSsbz4

We consider one more example.

Example 3.1.28 Finding extreme values.

Find the extreme values of $f(x) = \sqrt{1-x^2}$.

Solution. A closed interval is not given, so we find the extreme values of f on its domain. f is defined whenever $1-x^2 \geq 0$; thus the domain of f is $[-1, 1]$. Evaluating f at either endpoint returns 0.

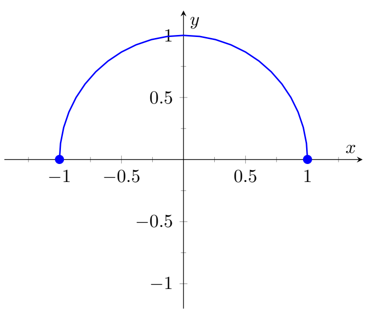


Figure 3.1.29 A graph of $f(x) = \sqrt{1-x^2}$ on $[-1, 1]$ as in Example 3.1.28

x	$f(x)$
-1	0
0	1
1	0

Figure 3.1.30 Finding the extrema of the half-circle in Example 3.1.28

Using the The Chain Rule, we find $f'(x) = -x/\sqrt{1-x^2}$. The critical points of f are found when $f'(x) = 0$ or when f' is undefined. It is straightforward to find that $f'(x) = 0$ when $x = 0$, and f' is undefined when $x = \pm 1$, the endpoints of the interval (which *are* in the domain of f.) The table of important values is given in Figure 3.1.30. The maximum value is 1, and the minimum value is 0.

Circle Revisited. We implicitly found the derivative of $x^2 + y^2 = 1$, the unit circle, in Section 2.6 Example 2.6.11 as $\frac{dy}{dx} = -x/y$. In Example 3.1.28, half of the unit circle is given as $y = f(x) = \sqrt{1-x^2}$.

We found $f'(x) = -x/\sqrt{1-x^2}$. Recognize that the denominator of this fraction is y; that is, we again found $f'(x) = \frac{dy}{dx} = -x/y$.

We have seen that continuous functions on closed intervals always have a maximum and minimum value, and we have also developed a technique to find these values. In Section 3.2, we further our study of the information we can glean from "nice" functions with the Mean Value Theorem. On a closed interval, we can find the *average rate of change* of a function (as we did at the beginning of Chapter 2). We will see that differentiable functions always have a point at which their *instantaneous* rate of change is same as the *average* rate of change. This is surprisingly useful, as we'll see.

3.1.1 Exercises

Terms and Concepts

1. Describe what an "extreme value" of a function is in your own words.

2. Sketch the graph of a function f on $(-1,1)$ that has both a maximum and minimum value.

3. Describe the difference between absolute and relative maxima in your own words.

4. Sketch the graph of a function f where f has a relative maximum at $x=1$ and $f'(1)$ is undefined.

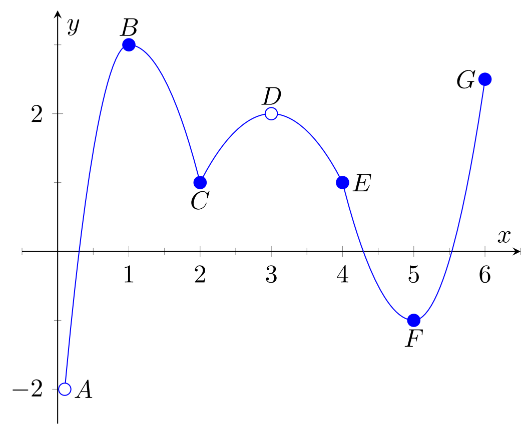
5. (☐ True ☐ False) If c is a critical value of a function f, then f has either a relative maximum or relative minimum at $x=c$.

6. Fill in the blanks: The critical points of a function f are found where $f'(x)$ is equal to ________ or where $f'(x)$ is ________.

Problems

Exercise Group. Identify each of the marked points as being an absolute maximum or minimum, a relative maximum or minimum, or none of the above.

7.



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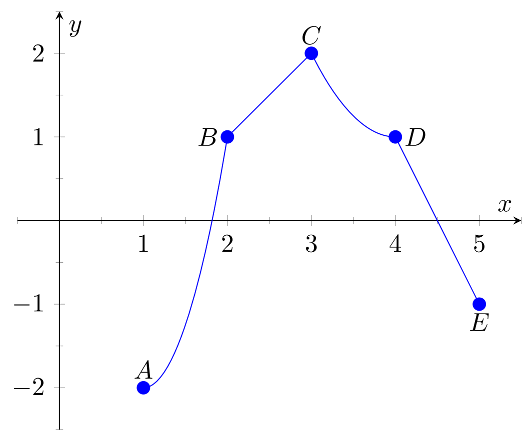
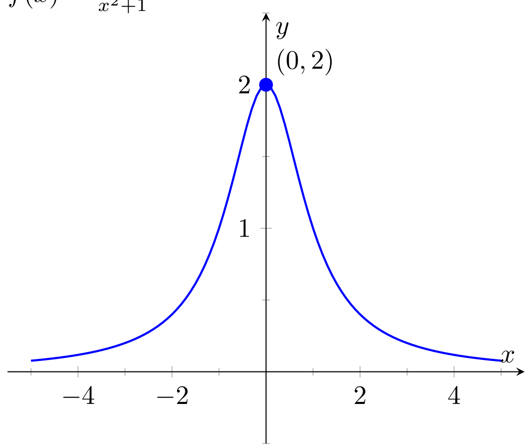
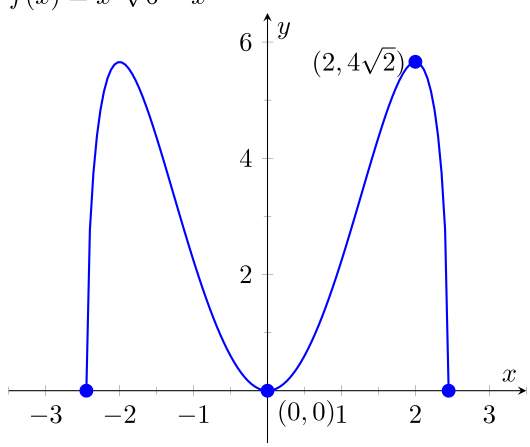


Exercise Group. Evaluate $f'(x)$ at the points indicated in the graph.

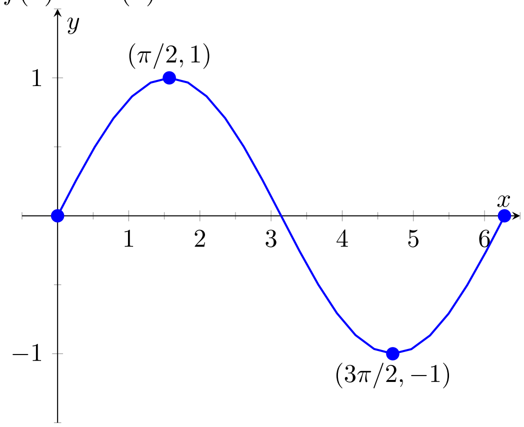
9. $f(x)=\frac{2}{x^2+1}$



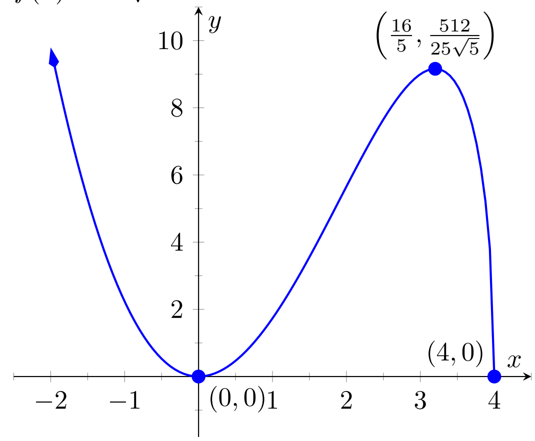
10. $f(x)=x^2\sqrt{6-x^2}$



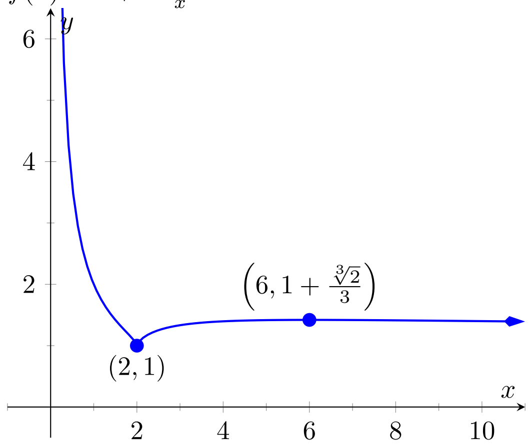
11. $f(x) = \sin(x)$



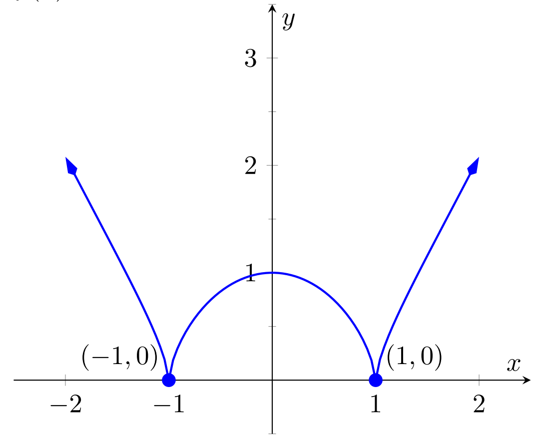
12. $f(x) = x^2\sqrt{4-x}$



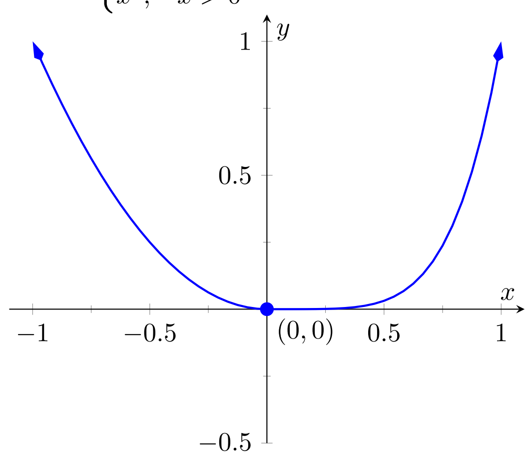
13. $f(x) = 1 + \frac{(x-2)^{\frac{2}{3}}}{x}$



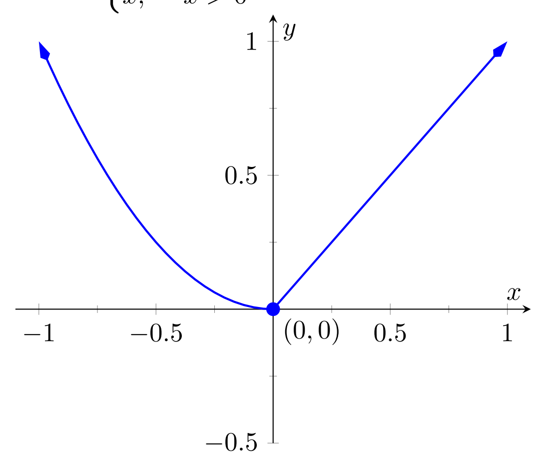
14. $f(x) = \sqrt[3]{x^4 - 2x^2 + 1}$



15. $f(x) = \begin{cases} x^2, & x \le 0 \\ x^5, & x > 0 \end{cases}$



16. $f(x) = \begin{cases} x^2, & x \le 0 \\ x, & x > 0 \end{cases}$



Exercise Group. Find the extreme values of the function on the given interval.

17. $f(x) = x^2 + 2x - 1$ on $[-5, 1]$

18. $f(x) = x^3 + \left(\frac{3}{2}\right)x^2 - 18x - 6$ on $[0, 3]$

19. $f(x) = 4\cos(x)$ on $\left[\frac{3\pi}{4}, \frac{7\pi}{6}\right]$

20. $f(x) = x^6\sqrt{4 - x^2}$ on $[-2, 2]$

21. $f(x) = x + \frac{2}{x}$ on $[1, 4]$

22. $f(x) = \frac{x^2}{x^2+7}$ on $[-2, 2]$

23. $f(x) = e^x\cos(x)$ on $[0, \pi]$

24. $f(x) = e^x\sin(x)$ on $[0, \pi]$

25. $f(x) = \frac{\ln(x)}{x^2}$ on $[1, 7]$

26. $f(x) = x^{\left(\frac{3}{4}\right)} - x^3$ on $[0, 2]$

3.2 The Mean Value Theorem

We motivate this section with the following question: Suppose you leave your house and drive to your friend's house in a city 100 miles away, completing the trip in two hours. At any point during the trip do you necessarily have to be going 50 miles per hour?

In answering this question, it is clear that the *average* speed for the entire trip is 50 mph (i.e. 100 miles in 2 hours), but the question is whether or not your *instantaneous* speed is ever exactly 50 mph. More simply, does your speedometer ever read exactly 50 mph? The answer, under some very reasonable assumptions, is "yes."

Let's now see why this situation is in a calculus text by translating it into mathematical symbols.

First assume that the function $y = f(t)$ gives the distance (in miles) traveled from your home at time t (in hours) where $0 \leq t \leq 2$. In particular, this gives $f(0) = 0$ and $f(2) = 100$. The slope of the secant line connecting the starting and ending points $(0, f(0))$ and $(2, f(2))$ is therefore

$$\begin{aligned}\frac{\Delta f}{\Delta t} &= \frac{f(2) - f(0)}{2 - 0}\\ &= \frac{100 - 0}{2}\\ &= 50 \text{ mph.}\end{aligned}$$

The slope at any point on the graph itself is given by the derivative $f'(t)$. So, since the answer to the question above is "yes," this means that at some time during the trip, the derivative takes on the value of 50 mph. Symbolically,

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = 50$$

for some time $0 \leq c \leq 2$.

How about more generally? Given any function $y = f(x)$ and a range $a \leq x \leq b$ does the value of the derivative at some point between a and b have to match the slope of the secant line connecting the points $(a, f(a))$ and $(b, f(b))$? Or equivalently, does the equation $f'(c) = \frac{f(b)-f(a)}{b-a}$ have to hold for some $a < c < b$?

Let's look at two functions in an example.

youtu.be/watch?v=GvdxKh6RpT0

Figure 3.2.1 Video introduction to Section 3.2

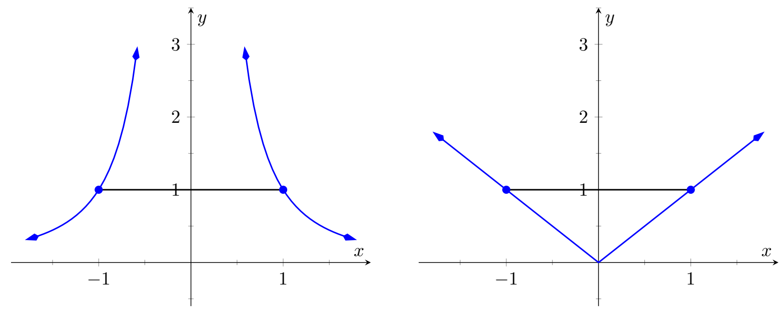
Example 3.2.2 Comparing average and instantaneous rates of change.

Consider functions

$$f_1(x) = \frac{1}{x^2} \qquad\qquad f_2(x) = |x|$$

with $a = -1$ and $b = 1$ as shown in Figure 3.2.3. Both functions have a value of 1 at a and b. Therefore the slope of the secant line connecting the end points is 0 in each case. But if you look at the plots of each, you can see that there are no points on either graph where the tangent lines have slope zero. Therefore we have found that there is no c in $[-1, 1]$ such that

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = 0.$$



(a) A graph of $f_1(x) = 1/x^2$ **(b)** A graph of $f_2(x) = |x|$

Figure 3.2.3 Graphs of two "misbehaving" functions

So what went "wrong"? It may not be surprising to find that the discontinuity of f_1 and the corner of f_2 play a role. If our functions had been continuous and differentiable, would we have been able to find that special value c? This is our motivation for the following theorem.

youtu.be/watch?v=E05H1f8TByI

Figure 3.2.6 Video presentation of Theorem 3.2.5

Theorem 3.2.4 The Mean Value Theorem of Differentiation.

Let $y = f(x)$ be a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b). There exists a value c, $a < c < b$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

That is, there is a value c in (a, b) where the instantaneous rate of change of f at c is equal to the average rate of change of f on $[a, b]$.

Note that the reasons that the functions in Example 3.2.2 fail are indeed that f_1 has a discontinuity on the interval $[-1, 1]$ and f_2 is not differentiable at the origin.

We will give a proof of the Mean Value Theorem below. To do so, we use a fact, called Rolle's Theorem, stated here.

Theorem 3.2.5 Rolle's Theorem.

Let f be continuous on $[a, b]$ and differentiable on (a, b), where $f(a) = f(b)$. There is some c in (a, b) such that $f'(c) = 0$.

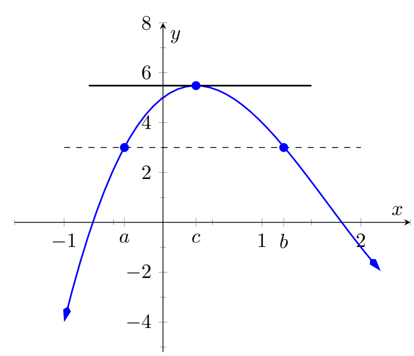


Figure 3.2.7 A graph of $f(x) = x^3 - 5x^2 + 3x + 5$, where $f(a) = f(b)$. Note the existence of c, where $a < c < b$, where $f'(c) = 0$.

Consider Figure 3.2.7 where the graph of a function f is given, where $f(a) = f(b)$. It should make intuitive sense that if f is differentiable (and hence, continuous) that there would be a value c in (a, b) where $f'(c) = 0$; that is, there would be a relative maximum or minimum of f in (a, b). Rolle's Theorem guarantees at least one; there may be more.

Rolle's Theorem is presented here as a stepping stone toward the Mean Value Theorem, but it's a useful result in its own right. It often turns up as a tool in mathematical problem solving. The video in Figure 3.2.8 illustrates one such use of Rolle's Theorem.

Rolle's Theorem is really just a special case of the Mean Value Theorem. If $f(a) = f(b)$, then the *average* rate of change on (a, b) is 0, and the theorem guarantees some c where $f'(c) = 0$. We will prove Rolle's Theorem, then use it to prove the Mean Value Theorem.

youtu.be/watch?v=le-5zsb6O7o

Figure 3.2.8 Using Rolle's Theorem to show a polynomial has at most one real root

Proof of Rolle's Theorem. Let f be differentiable on (a, b) where $f(a) = f(b)$. We consider two cases.

Case. Consider the case when f is constant on $[a, b]$; that is, $f(x) = f(a) = f(b)$ for all x in $[a, b]$. Then $f'(x) = 0$ for all x in $[a, b]$, showing there is at least one value c in (a, b) where $f'(c) = 0$.

Case. Now assume that f is not constant on $[a, b]$. The Extreme Value Theorem guarantees that f has a maximal and minimal value on $[a, b]$, found either at the endpoints or at a critical value in (a, b). Since $f(a) = f(b)$ and f is not constant, it is clear that the maximum and minimum cannot *both* be found at the endpoints. Assume, without loss of generality, that the maximum of f is not found at the endpoints. Therefore there is a c in (a, b) such that $f(c)$ is the maximum value of f. By Theorem 3.1.15, c must be a critical number of f; since f is differentiable, we have that $f'(c) = 0$, completing the proof of the theorem. ■

We can now prove the Mean Value Theorem.

Proof of the Mean Value Theorem.
Define the function

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}x.$$

We know g is differentiable on (a, b) and continuous on $[a, b]$ since f is. We can show $g(a) = g(b)$ (it is actually easier to show $g(b) - g(a) = 0$, which suffices). We can then apply Rolle's theorem to guarantee the existence of c in (a, b) such that $g'(c) = 0$. But note that

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a};$$

hence

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

which is what we sought to prove. ■

youtu.be/watch?v=1b9af8q5JMg

Figure 3.2.9 Video proof of the Mean Value Theorem

Going back to the very beginning of the section, we see that the only assumption we would need about our distance function $f(t)$ is that it be continuous and differentiable for t from 0 to 2 hours (both reasonable assumptions). By the Theorem 3.2.4, we are guaranteed a time during the trip where our instantaneous speed is 50 mph. This fact is used in practice. Some law enforcement agencies monitor traffic speeds while in aircraft. They do not measure speed with radar, but rather by timing individual cars as they pass over lines painted on the highway whose distances apart are known. The officer is able to measure the *average* speed of a car between the painted lines; if that average speed is greater than the posted speed limit, the officer is assured that the driver exceeded the speed limit at some time.

Note that the Theorem 3.2.4 is an *existence* theorem. It states that a special value c *exists*, but it does not give any indication about how to find it. It turns out that when we need the Theorem 3.2.4, existence is all we need.

Example 3.2.10 Using the Mean Value Theorem.

Consider $f(x) = x^3 + 5x + 5$ on $[-3, 3]$. Find c in $[-3, 3]$ that satisfies the Theorem 3.2.4.

Solution. The average rate of change of f on $[-3, 3]$ is:

$$\begin{aligned}\frac{f(3)-f(-3)}{3-(-3)} &= \frac{47-(-37)}{6}\\ &= \frac{84}{6}\\ &= 14.\end{aligned}$$

We want to find c such that $f'(c) = 14$. We find $f'(x) = 3x^2 + 5$. We set this equal to 14 and solve for x.

$$\begin{aligned}f'(x) &= 14\\ 3x^2 + 5 &= 14\\ x^2 &= 3\\ x &= \pm\sqrt{3} \approx \pm 1.732\end{aligned}$$

We have found two values c in $[-3, 3]$ where the instantaneous rate of change is equal to the average rate of change; the Theorem 3.2.4 guaranteed at least one. In Figure 3.2.11, f is graphed with a line representing the average rate of change; the lines tangent to f at $x = \pm\sqrt{3}$ are also given. Note how these lines are parallel (i.e., have the same slope) to the secant line.

Video solution

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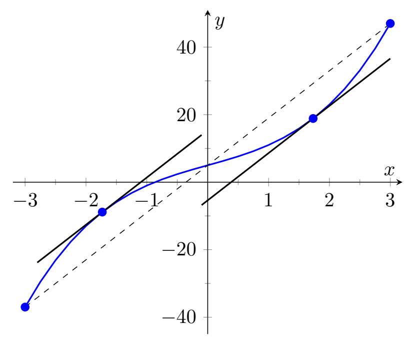

Figure 3.2.11 Demonstrating the Mean Value Theorem in Example 3.2.10

While the Theorem 3.2.4 has practical use (for instance, the speed monitoring application mentioned before), it is mostly used to advance other theory. We will use it in the next section to relate the shape of a graph to its derivative.

Before ending this section, we give two important consequences of the Mean Value Theorem. Each of these consequences has important applications to mathematical theory, and can be easily understood in the context of the position and velocity of objects in motion.

First, we recall that the derivative of any constant function is zero. Is the converse true? That is, are constant functions the only ones whose derivative is zero? The Mean Value Theorem says *yes*. This officially establishes our intuition about objects in (or, actually, *not* in) motion: if the velocity of an object is 0, then the object's position is unchanged; it is constant. Second, if two functions f and g have the same derivative, what does this tell us about f and g? The Mean Value Theorem implies that these functions must only differ by a constant; that is, $f(x) = g(x) + C$, for some constant C.

This has an application to motion that is not intuitive to some. Suppose two objects start moving while 5 ft apart, and always move with the same velocity. Then the two objects will *always* be 5 ft apart. (If two pennies are dropped from the 30th and 31st stories of a tall building at the same time, they will always be 1 story apart as they fall.)

Theorem 3.2.12 Consequences of the Mean Value Theorem.

Let f, g, and h be differentiable (and therefore continuous) functions on an in terval I.

1. *If $f'(x) = 0$ for all x in the interval I, then f is a constant function on I.*
2. *If $g'(x) = g'(x)$ for all x in I, then there is a constant C such that*

$g(x) = h(x) + C$ *for all* x *in* I.

Proof.

1. Choose any two points a and b in the interval I. By the Mean Value Theorem, we must have
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$
for some c between a and b. But $f'(c) = 0$, so $f(b) - f(a) = 0$, or $f(a) = f(b)$. Since a and b were any two points, this tells us that f must have the same value at every point; that is, f must be constant.

2. Suppose $g'(x) = h'(x)$ for each point x in I, and consider the function $f(x) = g(x) - h(x)$. By the difference rule for derivatives, we have
$$f'(x) = g'(x) - h'(x) = 0,$$
since $g'(x) = h'(x)$.

   By the previous result, this means that $f(x)$ is a constant function. That is, $f(x) = C$ for each x in I, giving us $g(x) - h(x) = C$, or $g(x) = h(x) + C$.

■

Using this result, we can establish another result which will be useful when we study antiderivatives: if two functions have the same derivative, then they differ by a constant.

We end this section with one more proof involving the Mean Value Theorem; this time, establishing a property of the sine function.

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Figure 3.2.13 Showing that a function with zero derivative is constant

youtu.be/watch?v=BzvxKeZMNtE

Figure 3.2.14 Showing that two functions with the equal derivatives differ by a constant

youtu.be/watch?v=CGBTZtM9mFY

Figure 3.2.15 Demonstrating a property of the sine function

3.2.1 Exercises

Terms and Concepts

1. Explain in your own words what the Mean Value Theorem states.

2. Explain in your own words what Rolle's Theorem states.

Problems

Exercise Group. A function $f(x)$ and interval $[a, b]$ are given. Check if Rolle's Theorem can be applied to f on $[a, b]$; if so, find c in (a, b) such that $f'(c) = 0$.

3. $f(x) = 6$ on $[-1, 1]$

4. $f(x) = 6x$ on $[-1, 1]$

5. $f(x) = x^2 + x - 6$ on $[-3, 2]$

6. $f(x) = x^2 + x - 2$ on $[-3, 2]$

7. $f(x) = x^2 + x$ on $[-2, 2]$

8. $f(x) = \sin(x)$ on $[\pi/6, 5\pi/6]$

9. $f(x) = \cos(x)$ on $[0, \pi]$

10. $f(x) = \frac{1}{x^2-2x+1}$ on $[0, 2]$

Exercise Group. A function $f(x)$ and interval $[a, b]$ are given. Check if The Mean Value Theorem of Differentiation can be applied to f on $[a, b]$; if so, find c in (a, b) guaranteed by the Mean Value Theorem.

11. $f(x) = x^2 + 3x - 1$ on $[-2, 2]$

12. $f(x) = 5x^2 - 6x + 8$ on $[0, 5]$

13. $f(x) = \sqrt{9 - x^2}$ on $[0, 3]$

14. $f(x) = \sqrt{25 - x}$ on $[0, 9]$

15. $f(x) = \frac{x^2-9}{x^2-1}$ on $[0, 2]$

16. $f(x) = \ln(x)$ on $[1, 5]$

17. $f(x) = \tan(x)$ on $[-\pi/4, \pi/4]$

18. $f(x) = x^3 - 2x^2 + x + 1$ on $[-2, 2]$

19. $f(x) = 2x^3 - 5x^2 + 6x + 1$ on $[-5, 2]$

20. $f(x) = \sin^{-1}(x)$ on $[-1, 1]$

3.3 Increasing and Decreasing Functions

Our study of "nice" functions f in this chapter has so far focused on individual points: points where f is maximal/minimal, points where $f'(x) = 0$ or f' does not exist, and points c where $f'(c)$ is the average rate of change of f on some interval.

In this section we begin to study how functions behave *between* special points; we begin studying in more detail the shape of their graphs.

We start with an intuitive concept. Given the graph in Figure 3.3.1, where would you say the function is *increasing*? *Decreasing*? Even though we have not defined these terms mathematically, one likely answered that f is increasing when $x > 1$ and decreasing when $x < 1$. We formally define these terms here.

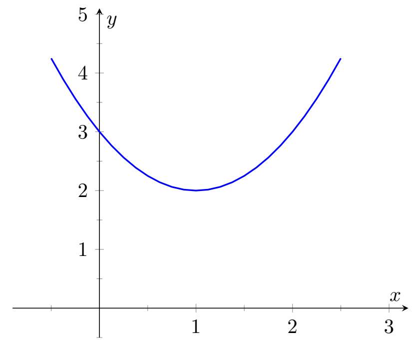

Figure 3.3.1 A graph of a function f used to illustrate the concepts of *increasing* and *decreasing*

Definition 3.3.2 Increasing and Decreasing Functions.

Let f be a function defined on an interval I.

1. f is *increasing* on I if for every $a < b$ in I, $f(a) < f(b)$.
2. f is *decreasing* on I if for every $a < b$ in I, $f(a) > f(b)$.

youtu.be/watch?v=NtV0R-JxrmM

Figure 3.3.3 Video presentation of Definition 3.3.2

Informally, a function is increasing if as x gets larger (i.e., looking left to right) $f(x)$ gets larger.

Our interest lies in finding intervals in the domain of f on which f is either increasing or decreasing. Such information should seem useful. For instance, if f describes the speed of an object, we might want to know when the speed was increasing or decreasing (i.e., when the object was accelerating vs. decelerating). If f describes the population of a city, we should be interested in when the population is growing or declining.

To find such intervals, we again consider secant lines. Let f be an increasing, differentiable function on an open interval I, such as the one shown in Figure 3.3.4, and let $a < b$ be given in I. The secant line on the graph of f from $x = a$ to $x = b$ is drawn; it has a slope of $(f(b) - f(a))/(b - a)$.

But note, since $b > a$ and f is increasing, $f(b) > f(a)$. And these facts imply $b - a > 0$ and $f(b) - f(a) > 0$. Therefore:

$$\frac{f(b) - f(a)}{b - a} > 0$$

$\Longrightarrow$ slope of the secant line > 0

$\Longrightarrow$ Average rate of change of f on $[a, b]$ is > 0.

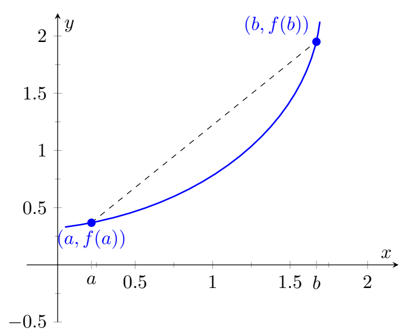

Figure 3.3.4 Examining the secant line of an increasing function

We have shown mathematically what may have already been obvious: when f is increasing, its secant lines will have a positive slope. Now recall that the Mean Value Theorem guarantees that there is a number c, where $a < c < b$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} > 0.$$

By considering all such secant lines in I, we strongly imply that $f'(x) > 0$ on I. A similar statement can be made for decreasing functions.

Caution: the definition we give in Definition 3.3.2 is not the one you will find in formal mathematics textbooks. Such texts define a function to be increasing on I if, for every $a < b$ in I, $f(a) \leq f(b)$. (Notice how equality is allowed.) The condition $f(a) < f(b)$ is then referred to as *strictly increasing*. Similar definitions are made for decreasing and strictly decreasing.

While this definition has certain technical advantages in a proof-based course, it is also conceptually counterintuitive for many students. For example, with this definition a constant function would be both increasing and decreasing!

Our above logic can be summarized as "If f is increasing, then f' is probably positive." Theorem 3.3.5 below turns this around by stating "If f' is positive, then f is increasing." This leads us to a method for finding when functions are increasing and decreasing.

Theorem 3.3.5 Test For Increasing/Decreasing Functions.

Let f be a continuous function on $[a, b]$ and differentiable on (a, b).

1. *If $f'(c) > 0$ for all c in (a, b), then f is increasing on $[a, b]$.*
2. *If $f'(c) < 0$ for all c in (a, b), then f is decreasing on $[a, b]$.*
3. *If $f'(c) = 0$ for all c in (a, b), then f is constant on $[a, b]$.*

The conclusions of Item 1 and Item 2 also hold if $f'(c) = 0$ for a finite number of nonadjacent values of c in I.

Let f be differentiable on an interval I and let a and b be in I where $f'(a) > 0$ and $f'(b) < 0$. If f' is continuous on $[a, b]$, it follows from the Intermediate Value Theorem that there must be some value c between a and b where $f'(c) = 0$. (It turns out that this is still true even if f' is not continuous on $[a, b]$.) This leads us to the following method for finding intervals on which a function is increasing or decreasing.

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Figure 3.3.6 Video presentation of Theorem 3.3.5

Key Idea 3.3.7 Finding Intervals on Which f is Increasing or Decreasing.

Let f be a continuous function on an interval I. To find intervals on which f is increasing and decreasing:

1. If not stated, find the domain of f, D. Begin a number line that only includes D.
2. Find the critical values of f. That is, find all c in the domain of f where $f'(c) = 0$ or f' is not defined. (Note: Any values of c *not* in the domain of f where $f'(c)$ is undefined should already be marked on your number line from Step 1).
3. Use the critical values to divide D into subintervals.
4. Pick any point p in each subinterval, and find the sign of $f'(p)$.
   (a) If $f'(p) > 0$, then f is increasing on that subinterval.
   (b) If $f'(p) < 0$, then f is decreasing on that subinterval.

Note that although Theorem 3.3.5 allows us to use determine that a function is increasing or decreasing on a closed interval, it is conventional to state the intervals of increase and decrease as *open* intervals. We will follow this convention in the examples that follow, but it is also acceptable to answer using closed intervals.
In particular, one should note the following:

- If $f'(x) > 0$ on (a, b) and on (b, c), with $f'(b) = 0$, then we should say that f is increasing on (a, c) (or on $[a, c]$) — the zero of the derivative should be included.
- If $f'(x) > 0$ on (a, b) and on (b, c), but $f(b)$ is undefined (or f is discontinuous at b), then we should *not* include the point b in our

interval. Instead, we say that f is increasing on (a, b) and (b, c), or on $[a, b)$ and $(b, c]$.

We demonstrate using this process in the following example.

Example 3.3.8 Finding intervals of increasing/decreasing.

Let $f(x) = x^3 + x^2 - x + 1$. Find intervals on which f is increasing or decreasing.

Solution. Since an interval was not specified for us to consider, using Key Idea 3.3.7, the domain of f is $\mathbb{R}$ or $(-\infty, \infty)$. Next, we find the critical values of f. We have $f'(x) = 3x^2 + 2x - 1 = (3x - 1)(x + 1)$, so $f'(x) = 0$ when $x = -1$ and when $x = 1/3$. f' is never undefined.

We thus break the domain (in this case the $(-\infty, \infty)$) into three subintervals based on the two critical values we just found: $(-\infty, -1)$, $(-1, 1/3)$ and $(1/3, \infty)$. This is shown in Figure 3.3.9.

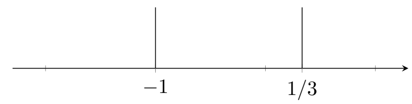


Figure 3.3.9 Number line for f in Example 3.3.8

We now pick a value p in each subinterval and find the sign of $f'(p)$. All we care about is the sign, so we do not actually have to fully compute $f'(p)$; pick "nice" values that make this simple.

Subinterval 1: $(-\infty, -1)$ — We (arbitrarily) pick $p = -2$. We can compute $f'(-2)$ directly: $f'(-2) = 3(-2)^2 + 2(-2) - 1 = 7 > 0$. We conclude that f is increasing on $(-\infty, -1)$.
Note we can arrive at the same conclusion without computation. For instance, we could choose $p = -100$. The first term in $f'(-100)$, i.e., $3(-100)^2$ is clearly positive and very large. The other terms are small in comparison, so we know $f'(-100) > 0$. All we need is the sign.

Subinterval 2: $(-1, 1/3)$ — We pick $p = 0$ since that value seems easy to deal with. $f'(0) = -1 < 0$. We conclude f is decreasing on $(-1, 1/3)$.

Subinterval 3: $(1/3, \infty)$ — Pick an arbitrarily large value for $p > 1/3$ and note that $f'(p) = 3p^2 + 2p - 1 > 0$. We conclude that f is increasing on $(1/3, \infty)$.

Video solution

youtu.be/watch?v=DW6qcgE1TZ0

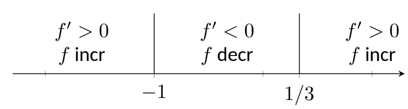


Figure 3.3.10 Completed number line for f in Example 3.3.8

Figure 3.3.10 summarizes our work.

We can verify our calculations by considering Figure 3.3.11, where f is graphed. The graph also presents f'; note how $f' > 0$ when f is increasing and $f' < 0$ when f is decreasing.

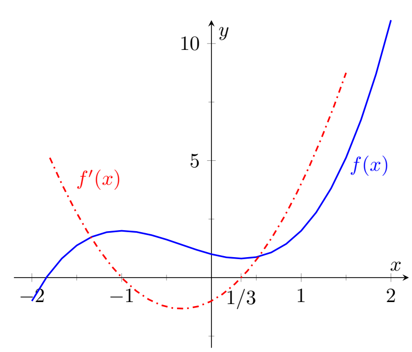


Figure 3.3.11 A graph of $f(x)$ in Example 3.3.8, showing where f is increasing and decreasing

One is justified in wondering why so much work is done when the graph seems to make the intervals very clear. We give three reasons why the above work is worthwhile.

First, the points at which f switches from increasing to decreasing are not precisely known given a graph. The graph shows us something significant happens near $x = -1$ and $x = 0.3$, but we cannot determine exactly where from the graph.

One could argue that just finding critical values is important; once we know the significant points are $x = -1$ and $x = 1/3$, the graph shows the increasing/decreasing traits just fine. That is true. However, the technique prescribed here helps reinforce the relationship between increasing/decreasing and the sign of f'. Once mastery of this concept (and several others) is obtained, one finds that either (a) just the critical points are computed and the graph shows all else that is desired, or (b) a graph is never produced, because determining increasing/ decreasing using f' is straightforward and the graph is unnecessary. So our second reason why the above work is worthwhile is this: once mastery of a subject is gained, one has *options* for finding needed information. We are working to develop mastery.

Finally, our third reason: many problems we face "in the real world" are very complex. Solutions are tractable only through the use of computers to do many calculations for us. Computers do not solve problems "on their own," however; they need to be taught (i.e., *programmed*) to do the right things. It would be beneficial to give a function to a computer and have it return maximum and minimum values, intervals on which the function is increasing and decreasing, the locations of relative maxima, etc. The work that we are doing here is easily programmable. It is hard to teach a computer to "look at the graph and see if it is going up or down." It is easy to teach a computer to "determine if a number is greater than or less than 0."

In Section 3.1 we learned the definition of relative maxima and minima and found that they occur at critical points. We are now learning that functions can switch from increasing to decreasing (and vice-versa) at critical points. This new understanding of increasing and decreasing creates a great method of determining whether a critical point corresponds to a maximum, minimum, or neither. Imagine a function increasing until a critical point at $x = c$, after which it decreases. A quick sketch helps confirm that $f(c)$ must be a relative maximum. A similar statement can be made for relative minimums. We formalize this concept in a theorem.

youtu.be/watch?v=_HjE4urOM4Y

Figure 3.3.13 Video presentation of Theorem 3.3.12

Theorem 3.3.12 First Derivative Test.

Let f be continuous on an interval I, and differentiable on I, except possibly at c, where c is a critical number in I.

1. *If the sign of f' switches from positive to negative at c, then $f(c)$ is a relative maximum of f.*
2. *If the sign of f' switches from negative to positive at c, then $f(c)$ is a relative minimum of f.*
3. *If f' is positive (or, negative) before and after c, then $f(c)$ is not a relative extrema of f.*

Remark 3.3.14 Importance of Continuity. The continuity of f when using the first derivative test is very important. Without continuity, almost anything can happen at a critical number. For example, we can construct a piecewise function where the sign of f' switches to positive to negative at c and $f(c)$ is *not* a local maximum. This is shown in Figure 3.3.15.

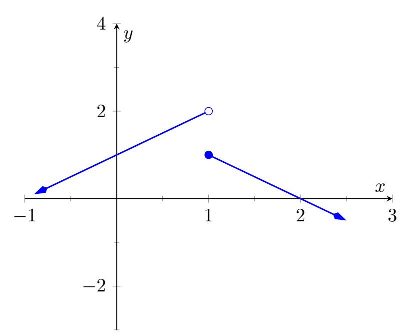


Figure 3.3.15 A discontinuous function where f' changes sign at 1, but $f(1)$ is not a local maximum

Example 3.3.16 Using the First Derivative Test.

Find the intervals on which f is increasing and decreasing, and use the First Derivative Test to determine the relative extrema of f, where

$$f(x) = \frac{x^2+3}{x-1}.$$

Solution. We start by noting the domain of f: $(-\infty, 1) \cup (1, \infty)$. Since f is not defined at $x = 1$ (it has a vertical asymptote), the increasing/decreasing nature of f could switch at this value. We know that $f'(1)$ will be undefined since f is discontinuous at 1. We do not formally consider $x = 1$ to be a critical value of f, but we will use 1 to subdivide the real number line.
Using the Quotient Rule, we find

$$f'(x) = \frac{x^2 - 2x - 3}{(x-1)^2}.$$

We need to find the critical values of f; we want to know when $f'(x) = 0$ and when f' is not defined. That latter is straightforward: when the denominator of $f'(x)$ is 0, f' is undefined. That occurs when $x = 1$, which we've already recognized as an important value, but not a critical number.
$f'(x) = 0$ when the numerator of $f'(x)$ is 0. That occurs when $x^2 - 2x - 3 = (x-3)(x+1) = 0$; i.e., when $x = -1, 3$.
We have found that f has two critical numbers, $x = -1, 3$, and at $x = 1$ something important might also happen. These three numbers divide the real number line into four subintervals:

$$(-\infty, -1), (-1, 1), (1, 3), \text{ and } (3, \infty).$$

Pick a number p from each subinterval and test the sign of f' at p to determine whether f is increasing or decreasing on that interval. Again, we do well to avoid complicated computations; notice that the denominator of f' is *always* positive so we can ignore it during our work.

Interval 1: $(-\infty, -1)$	Choosing a very small number (i.e., a negative number with a large magnitude) p returns $p^2 - 2p - 3$ in the numerator of f'; that will be positive. Hence f is increasing on $(-\infty, -1)$.
Interval 2: $(-1, 1)$	Choosing 0 seems simple: $f'(0) = -3 < 0$. We conclude f is decreasing on $(-1, 1)$.
Interval 3: $(1, 3)$	Choosing 2 seems simple: $f'(2) = -3 < 0$. Again, f is decreasing.
Interval 4: $(3, \infty)$	Choosing an very large number p from this subinterval will give a positive numerator and (of course) a positive denominator. So f is increasing on $(3, \infty)$.

In summary, f is increasing on the intervals $(-\infty, -1)$ and $(3, \infty)$ and is decreasing on the intervals $(-1, 1)$ and $(1, 3)$. Since at $x = -1$, the sign of f' switched from positive to negative, Theorem 3.3.12 states that $f(-1)$ is a relative maximum of f. At $x = 3$, the sign of f' switched from negative to positive, meaning $f(3)$ is a relative minimum. At $x = 1$, f is not defined, so there is no relative extremum at $x = 1$. As previously stated, $x = 1$ is a vertical asymptote of f.
This is summarized in the number line shown in Figure 3.3.17. Also, Figure 3.3.18 shows a graph of f, confirming our calculations. This figure also shows f', again demonstrating that f is increasing when $f' > 0$ and decreasing when $f' < 0$.

Video solution

youtu.be/watch?v=94iCiIX07R0

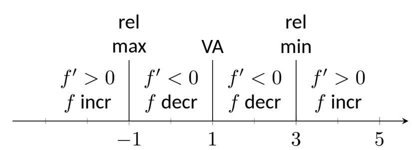


Figure 3.3.17 Number line for f in Example 3.3.16

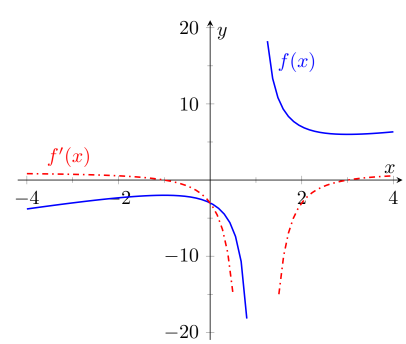


Figure 3.3.18 A graph of $f(x)$ in Example 3.3.16, showing where f is increasing and decreasing

One is often tempted to think that functions always alternate "increasing, decreasing, increasing, decreasing,..." around critical values. Our previous example demonstrated that this is not always the case. While $x = 1$ was not technically a critical value, it was an important value we needed to consider. We found that f was decreasing on "both sides of $x = 1$."

We examine one more example.

Example 3.3.19 Using the First Derivative Test.

Find the intervals on which $f(x) = x^{8/3} - 4x^{2/3}$ is increasing and decreasing and identify the relative extrema.

Solution. The domain of f is $\mathbb{R}$ (you can take the odd root of both positive and negative nubmers). Next, we take the first derivative. Since we know we want to solve $f'(x) = 0$, we will do some algebra after taking the derivative.

$$\begin{aligned} f(x) &= x^{\frac{8}{3}} - 4x^{\frac{2}{3}} \\ f'(x) &= \frac{8}{3}x^{\frac{5}{3}} - \frac{8}{3}x^{-\frac{1}{3}} \\ &= \frac{8}{3}x^{-\frac{1}{3}}\left(x^{\frac{6}{3}} - 1\right) \\ &= \frac{8}{3}x^{-\frac{1}{3}}\left(x^2 - 1\right) \\ &= \frac{8}{3}x^{-\frac{1}{3}}(x-1)(x+1). \end{aligned}$$

This derivation of f' shows that $f'(x) = 0$ when $x = \pm 1$ and $f'$is not defined when $x = 0$. Thus we have three critical values, breaking the number line into four subintervals as shown in Figure 3.3.20.

Interval 1: $(\infty, -1)$ We choose $p = -2$; we can easily verify that $f'(-2) < 0$. So f is decreasing on $(-\infty, -1)$.

Interval 2: $(-1, 0)$ Choose $p = -1/2$. Once more we practice finding the sign of $f'(p)$ without computing an actual value. We have $f'(p) = (8/3)p^{-1/3}(p-1)(p+1)$; find the sign of each of the three terms at the chosen value of p.

$$f'(p) = \frac{8}{3} \cdot \underbrace{p^{-\frac{1}{3}}}_{<0} \cdot \underbrace{(p-1)}_{<0}\underbrace{(p+1)}_{>0}.$$

We have a "negative × negative × positive" giving a positive number; f is increasing on $(-1, 0)$.

Interval 3: $(0, 1)$ We do a similar sign analysis as before, using p in $(0, 1)$.

$$f'(p) = \frac{8}{3} \cdot \underbrace{p^{-\frac{1}{3}}}_{>0} \cdot \underbrace{(p-1)}_{<0}\underbrace{(p+1)}_{>0}.$$

We have two positive factors and one negative factor; $f'(p) < 0$ and so f is decreasing on $(0, 1)$.

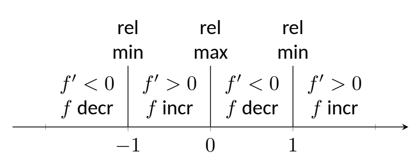


Figure 3.3.20 Number line for f in Example 3.3.19

Interval 4: $(1, \infty)$ Similar work to that done for the other three intervals shows that $f'(x) > 0$ on $(1, \infty)$, so f is increasing on this interval.

We conclude by stating that f is increasing on the intervals $(-1, 0)$ and $(1, \infty)$ and decreasing on the intervals $(-\infty, -1)$ and $(0, 1)$. The sign of f' changes from negative to positive around $x = -1$ and $x = 1$, meaning by Theorem 3.3.12 that $f(-1)$ and $f(1)$ are relative minima of f. As the sign of f' changes from positive to negative at $x = 0$, we have a relative maximum at $f(0)$. Figure 3.3.21 shows a graph of f, confirming our result. We also graph f', highlighting once more that f is increasing when $f' > 0$ and is decreasing when $f' < 0$.

Video solution

youtu.be/watch?v=T4RxcQnNotc

We have seen how the first derivative of a function helps determine when the graph of a function is going "up" or "down." In the next section, we will see how the second derivative helps determine how the graph of a function curves.

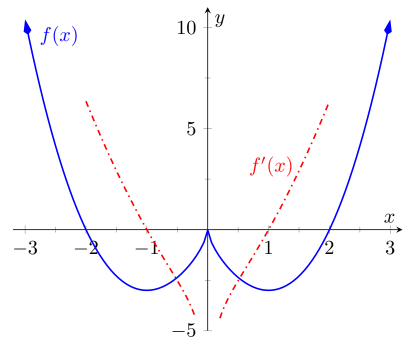


Figure 3.3.21 A graph of $f(x)$ in Example 3.3.19, showing where f is increasing and decreasing

3.3.1 Exercises

Terms and Concepts

1. In your own words describe what it means for a function to be increasing.

2. What does a decreasing function "look like"?

3. Sketch a graph of a function on $[0,2]$ that is increasing, where it is increasing "quickly" near $x=0$ and increasing "slowly" near $x=2$.

4. Give an example of a function describing a situation where it is "bad" to be increasing and "good" to be decreasing.

5. (☐ True ☐ False) Functions always switch from increasing to decreasing, or decreasing to increasing, at critical points.

6. A function f has derivative $f'(x)=(\sin x+2)e^{x^2+1}$, where $f'(x)>1$ for all x. Is f increasing, decreasing, or can we not tell from the given information? Why or why not?

Problems

Exercise Group. A function $f(x)$ is given. Graph f and f' on the same axes (using technology is permitted) and verify Theorem 3.3.5.

7. $f(x)=2x+3$

8. $f(x)=x^2-3x+5$

9. $f(x)=\cos(x)$

10. $f(x)=\tan(x)$

11. $f(x)=x^3-5x^2+7x-1$

12. $f(x)=2x^3-x^2+x-1$

13. $f(x)=x^4-5x^2+4$

14. $f(x)=\frac{1}{x^2+1}$

Exercise Group. A function $f(x)$ is given.

(a) Give the domain of f.

(b) Find the critical numbers of f.

(c) Find the intervals on which f is increasing.

(d) Find the intervals on which f is decreasing.

(e) Use the First Derivative Test to determine which critical points are a relative maximum.

(f) Use the First Derivative Test to determine which critical points are a relative minimum.

15. $f(x)=x^2+4x$

16. $f(x)=x^3+2x^2+9$

17. $f(x)=7x^3-17x^2-35x+1$

18. $f(x)=x^3-9x^2+27x-27$

19. $f(x)=\frac{1}{x^2-10x+34}$

20. $f(x)=\frac{x^2-1}{x^2-36}$

21. $f(x)=\frac{x}{x^2+12x+35}$

22. $f(x)=\frac{(x-(-5))^{\frac{2}{3}}}{x}$

23. $f(x)=\sin(x)\cos(x)$ on $(-\pi,\pi)$

24. $f(x)=x^6+192x$

3.4 Concavity and the Second Derivative

Our study of "nice" functions continues. The previous section showed how the first derivative of a function, f', can relay important information about f. We now apply the same technique to f' itself, and learn what this tells us about f.

The key to studying f' is to consider its derivative, namely f'', which is the second derivative of f. When $f'' > 0$, f' is increasing. When $f'' < 0$, $f'$is decreasing. f' has relative maxima and minima where $f'' = 0$ or is undefined.

This section explores how knowing information about f'' gives information about f.

youtu.be/watch?v=0uCjI5J4ew4

Figure 3.4.1 Video introduction to Section 3.4

3.4.1 Concavity

We begin with a definition, then explore its meaning.

Definition 3.4.2 Concave Up and Concave Down.

Let f be continuous on an interval I. The graph of f is **concave up** on I if for any $a < b$ in I,

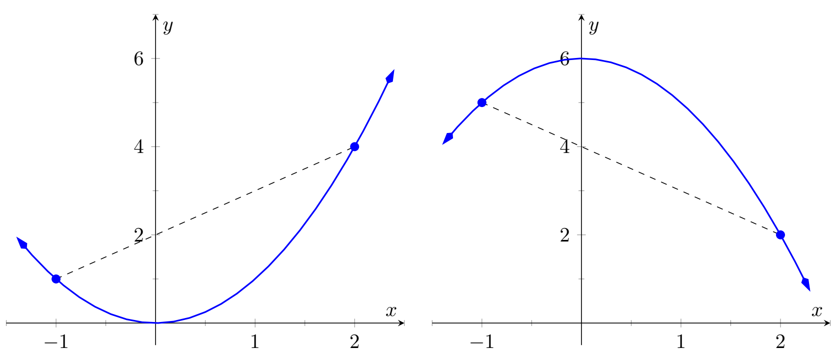
$$f\left(\frac{a+b}{2}\right) < \frac{f(a)+f(b)}{2}. \tag{3.4.1}$$

The graph of f is **concave down** on I if for any $a < b$ in I,

$$f\left(\frac{a+b}{2}\right) > \frac{f(a)+f(b)}{2}. \tag{3.4.2}$$

Geometrically, the condition in Equation (3.4.1) states that a graph is concave up if the midpoint of the secant line from $(a, f(a))$ to $(b, f(b))$ (and hence, the secant line itself) is above the graph $y = f(x)$. Similarly, Equation (3.4.2) states that the secant line lies below the graph.

In order for equality to hold instead of Equation (3.4.1) or Equation (3.4.2), the function would have to be of the form $f(x) = mx + c$, in which case the graph is a straight line. Straight lines are considered to have **no concavity**.



(a) A graph that is concave up. Notice how the secant line lies above the graph.

(b) A graph that is concave down. Notice how the secant line lies below the graph.

Figure 3.4.3 Illustrating the nature of concave up and concave down

Loose Language. We often state that "f is concave up" instead of "the graph of f is concave up" for simplicity.

Consider a function f such that f is continuous on $[a,b]$ and differentiable on (a,b). Note that $\frac{a+b}{2}$ is the midpoint of the interval $[a,b]$. By the The Mean Value Theorem of Differentiation, there must be a point c_1 in $\left[a, \frac{a+b}{2}\right]$ such that

$$f'(c_1) = \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a} = \frac{2}{b-a}\left(f\left(\frac{a+b}{2}\right) - f(a)\right).$$

Similarly, there must be a point c_2 in $\left[\frac{a+b}{2}, b\right]$ such that

$$f'(c_2) = \frac{f(b) - f\left(\frac{a+b}{2}\right)}{b - \frac{a+b}{2}} = \frac{2}{b-a}\left(f(b) - f\left(\frac{a+b}{2}\right)\right).$$

But then we have

$$\begin{aligned} f'(c_2) - f'(c_1) &= \frac{2}{b-a}\left(f(b) - f\left(\frac{a+b}{2}\right) - f\left(\frac{a+b}{2}\right) + f(a)\right) \\ &= \frac{4}{b-a}\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right). \end{aligned}$$

Now, let us suppose that $f'(x)$ is an increasing function on (a,b). In that case, $f'(c_2) - f'(c_1) > 0$, and since $b - a > 0$, this implies that

$$\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) > 0,$$

which, by Definition 3.4.2 means that the graph of f is concave up.

Similarly, if $f'(x)$ is a decreasing function on (a,b), then the graph of f will be concave down. Using Theorem 3.3.5, we arrive at the following theorem.

As with Theorem 3.3.5, Theorem 3.4.4 lets us conclude that the graph of a function is concave up (or down) on a *closed* interval, assuming that the function is continuous on that interval. Again, we follow the convention that when a problem asks us to give the intervals on which the graph is concave up or down, we give *open* intervals, even if a closed interval is technically correct.

If a function has the same concavity on adjacent intervals (a,b) and (b,c), and the function is continuous at b, we should combine the intervals, and state the result as (a,c). However, if b is a point of discontinuity, we must omit it from our intervals.

Theorem 3.4.4

Let f be a continuous function on $[a,b]$ and differentiable on (a,b).

1. *If $f''(c) > 0$ for all c in (a,b), then f is concave up on $[a,b]$.*
2. *If $f''(c) < 0$ for all c in (a,b), then f is concave down on $[a,b]$.*
3. *If $f''(c) = 0$ for all c in (a,b), then f is linear on $[a,b]$.*

The graph of a function f is *concave up* when $f'$is *increasing*. That means as one looks at a concave up graph from left to right, the slopes of the tangent lines will be increasing. Consider Figure 3.4.5, where a concave up graph is shown along with some tangent lines. Notice how the tangent line on the left is steep, downward, corresponding to a lesser (large negative) value of f'. On the right, the tangent line is steep, upward, corresponding to a greater (large positive) value of f'.

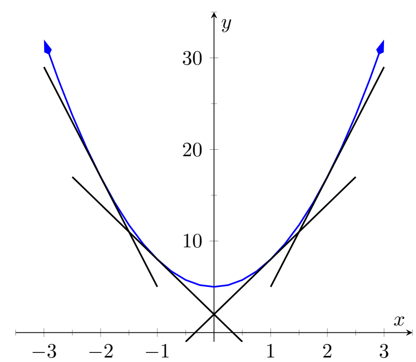


Figure 3.4.5 A function f with a concave up graph. Notice how the slopes of the tangent lines, when looking from left to right, are increasing. (The slope values pictured are $-12, -6, 6$ and 12).

If a function is decreasing and concave up, then its rate of decrease is slowing; it is "leveling off." You can see this in the left side of Figure 3.4.5. If the function is increasing and concave up, then the *rate* of increase is increasing. The function is increasing at a faster and faster rate. You can see this in the right side of Figure 3.4.5.

Now consider a function which is concave down. We essentially repeat the above paragraphs with slight variation.

The graph of a function f is *concave down* when $f'$is *decreasing*. That means as one looks at a concave down graph from left to right, the slopes of the tangent lines will be decreasing. Consider Figure 3.4.6, where a concave down graph is shown along with some tangent lines. Notice how the tangent line on the left is steep, upward, corresponding to a greater (large positive) value of f'. On the right, the tangent line is steep, downward, corresponding to a lesser (large negative) value of f'.

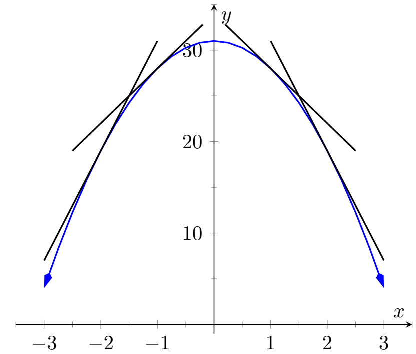


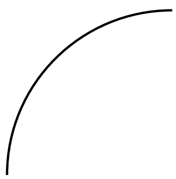
Figure 3.4.6 A function f with a concave down graph. Notice how the slopes of the tangent lines, when looking from left to right, are decreasing.

If a function is increasing and concave down, then its rate of increase is slowing; it is "leveling off." If the function is decreasing and concave down, then the *rate* of decrease is decreasing. The function is decreasing at a faster and faster rate.

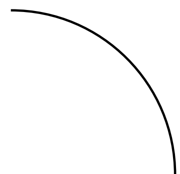
Our definition of concave up and concave down is given in terms of when the first derivative is increasing or decreasing. We can apply the results of the previous section to find intervals on which a graph is concave up or down. That is, we recognize that f' is increasing when $f'' > 0$, etc.

Theorem 3.4.7 Test for Concavity.

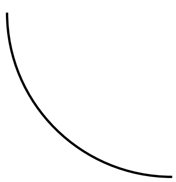
Let f be twice differentiable on an interval I. The graph of f is concave up if $f'' > 0$ on I, and is concave down if $f'' < 0$ on I.



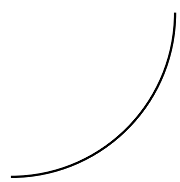
(a) $f' > 0$, f increasing; $f'' < 0$, f is concave down



(b) $f' < 0$, f decreasing; $f'' < 0$, f is concave down



(c) $f' < 0$, f decreasing; $f'' > 0$, f is concave up



(d) $f' > 0$, f increasing; $f'' > 0$, f is concave up

Figure 3.4.8 Demonstrating the four ways that concavity interacts with increasing/decreasing, along with the relationships with the first and second derivatives

Concavity Depravity. A mnemonic for remembering what concave up/down means is: "Concave up is like a cup; concave down is like a frown." It is admittedly terrible, but it works.

Geometric Concavity. Geometrically speaking, a function is concave up if its graph lies above its tangent lines and below secant line segments. A function is concave down if its graph lies below its tangent lines and above secant line segments.

If knowing where a graph is concave up/down is important, it makes sense that the places where the graph changes from one to the other is also important. This leads us to a definition.

Definition 3.4.9 Point of Inflection.

A **point of inflection** is a point on the graph of f at which the concavity of f changes.

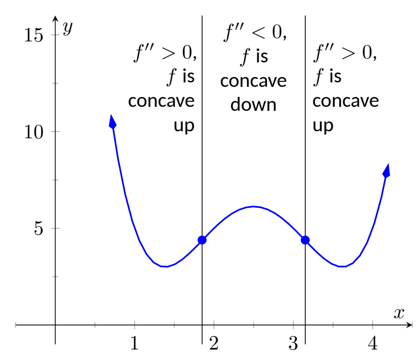


Figure 3.4.10 A graph of a function with its inflection points marked. The intervals where concave up/down are also indicated.

Figure 3.4.10 shows a graph of a function with inflection points labeled.

If the concavity of f changes at a point $(c, f(c))$, then $f'$is changing from increasing to decreasing (or, decreasing to increasing) at $x = c$. That means that the sign of $f''$is changing from positive to negative (or, negative to positive) at $x = c$. A sign change *may* occur when $f'' = 0$ or f'' is undefined. This leads to the following theorem.

Theorem 3.4.11 Points of Inflection.

If $(c, f(c))$ is a point of inflection on the graph of f, then either $f''(c) = 0$ or f'' is not defined at c.

We have identified the concepts of concavity and points of inflection. It is now time to practice using these concepts; given a function, we should be able to find its points of inflection and identify intervals on which it is concave up or down. We do so in the following examples.

Example 3.4.12 Finding intervals of concave up/down, inflection points.

Let $f(x) = x^3 - 3x + 1$. Find the inflection points of f and the intervals on which it is concave up/down.

Solution. We start by finding $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$. To find the inflection points, we use Theorem 3.4.11 and find where $f''(x) = 0$ or where $f''$is undefined. We find $f''$is always defined, and is 0 only when $x = 0$. So the point $(0, f(0)) = (0, 1)$ is the only possible point of inflection.

This possible inflection point divides the real line into two intervals, $(-\infty, 0)$ and $(0, \infty)$. We use a process similar to the one used in the previous section to determine increasing/decreasing. Pick any $c < 0$; $f''(c) < 0$ so f is concave down on $(-\infty, 0)$. Pick any $c > 0$; $f''(c) > 0$ so f is concave up on $(0, \infty)$. Since the concavity changes at $x = 0$, the point $(0, 1)$ is an inflection point.

The number line in Figure 3.4.13 illustrates the process of determining concavity; Figure 3.4.14 shows a graph of f and f'', confirming our results. Notice how f is concave down precisely when $f''(x) < 0$ and concave up when $f''(x) > 0$.

Video solution

youtu.be/watch?v=n8TRVD_8sY0

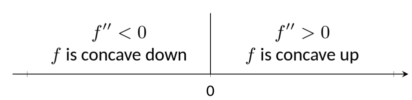


Figure 3.4.13 A number line determining the concavity of f in Example 3.4.12

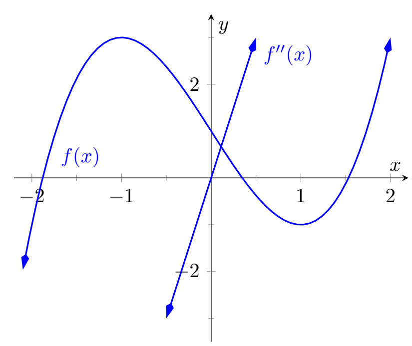


Figure 3.4.14 A graph of $f(x)$ used in Example 3.4.12

Example 3.4.15 Finding intervals of concave up/down, inflection points.

Let $f(x) = x/(x^2 - 1)$. Find the inflection points of f and the intervals on which it is concave up/down.

Solution. We need to find $f'$and f''. Using the Quotient Rule and simplifying, we find

$$f'(x) = \frac{-(1 + x^2)}{(x^2 - 1)^2} \qquad f''(x) = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}.$$

To find the possible points of inflection, we seek to find where $f''(x) = 0$ and where f'' is not defined. Solving $f''(x) = 0$ reduces to solving $2x(x^2 + 3) = 0$; we find $x = 0$. We find that $f''$is not defined when $x = \pm 1$, for then the denominator of $f''$is 0. We also note that f itself is not defined at $x = \pm 1$, having a domain of $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. Since the domain of f is the union of three intervals, it makes sense that the concavity of f could switch across intervals. We technically cannot say that f has a point of inflection at $x = \pm 1$ as they are not part of the domain, but we must still consider these x-values to be important and will include them in our number line.

The important x-values at which concavity might switch are $x = -1$, $x = 0$ and $x = 1$, which split the number line into four intervals as shown in Figure 3.4.16. We determine the concavity on each. Keep in mind that all we are concerned with is the *sign* of f''on the interval.

Interval 1: $(-\infty, -1)$ Select a number c in this interval with a large magnitude (for instance, $c = -100$). The denominator of $f''(x)$ will be positive. In the numerator, the $\left(c^2 + 3\right)$ factor will be positive and the $2c$ factor will be negative. Thus the numerator is negative and $f''(c)$ is negative. We conclude f is concave down on $(-\infty, -1)$.

Interval 2: $(-1, 0)$ For any number c in this interval, the factor $2c$ in the numerator will be negative, the factor $\left(c^2 + 3\right)$ in the numerator will be positive, and the factor $\left(c^2 - 1\right)^3$ in the denominator will be negative. Thus $f''(c) > 0$ and f is concave up on this interval.

Interval 3: $(0, 1)$ Any number c in this interval will be positive and "small." Thus the numerator is positive while the denominator is negative. Thus $f''(c) < 0$ and f is concave down on this interval.

Interval 4: $(1, \infty)$ Choose a large value for c. It is evident that $f''(c) > 0$, so we conclude that f is concave up on $(1, \infty)$.

We conclude that f is concave up on $(-1, 0)$ and $(1, \infty)$ and concave down on $(-\infty, -1)$ and $(0, 1)$. There is only one point of inflection, $(0, 0)$, as f is not defined at $x = \pm 1$. Our work is confirmed by the graph of f in Figure 3.4.17. Notice how f is concave up whenever $f''$is positive, and concave down when $f''$is negative. The inflection in f occurs where f'' changes sign.

Recall that relative maxima and minima of f are found at critical points of f; that is, they are found when $f'(x) = 0$ or when f' is undefined. Likewise, the relative maxima and minima of $f'$are found when $f''(x) = 0$ or when $f''$is undefined; note that these are the inflection points of f.

What does a "relative maximum of f'" *mean*? The derivative measures the rate of change of f; maximizing f' means finding where f is increasing the most — where f has the steepest tangent line. A similar statement can be made for minimizing f'; it corresponds to where f has the steepest negatively-sloped tangent line.

We utilize this concept in the next example.

Example 3.4.18 Understanding inflection points.

The sales of a certain product over a three-year span are modeled by $S(t) = t^4 - 8t^2 + 20$, where t is the time in years, shown in Figure 3.4.19. Over the first two years, sales are decreasing. Find the point at which sales are decreasing at their greatest rate.

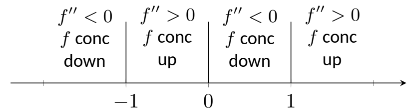


Figure 3.4.16 Number line for f in Example 3.4.15

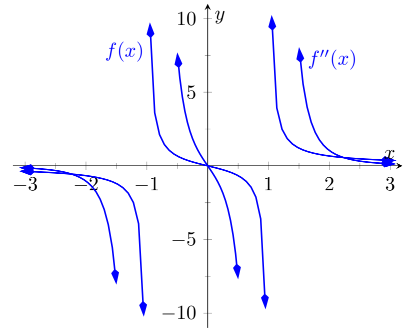


Figure 3.4.17 A graph of $f(x)$ and $f''(x)$ in Example 3.4.15

Video solution

youtu.be/watch?v=eC6QLbsuRVs

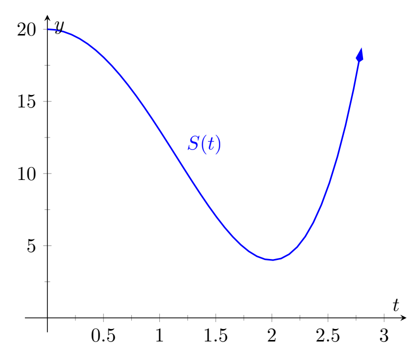


Figure 3.4.19 A graph of $S(t)$ in Example 3.4.18, modeling the sale of a product over time

Solution. We want to maximize the rate of decrease, which is to say, we want to find where S' has a minimum. To do this, we find where S'' is 0 and S'' changes from negative to positive. We find $S'(t) = 4t^3 - 16t$ and $S''(t) = 12t^2 - 16$. Setting $S''(t) = 0$ and solving, we get $t = \sqrt{4/3} \approx 1.16$ (we ignore the negative solution for t since it does not lie in the domain of our function S).
Since $S''(1) = -4 < 0$ and $S''(2) = 32 > 0$, we can say $S'(\sqrt{4/3})$ is a local minimum of S'. This is both the inflection point and the point of maximum decrease. This is the point at which things first start looking up for the company. After the inflection point, sales are still decreasing, but not decreasing quite as quickly as they had been.
A graph of $S(t)$ and $S'(t)$ is given in Figure 3.4.20. When $S'(t) < 0$, sales are decreasing; note how at $t \approx 1.16$, $S'(t)$ is minimized. That is, sales are decreasing at the fastest rate at $t \approx 1.16$. On the interval of $(1.16, 2)$, S is decreasing but concave up, so the decline in sales is "leveling off."

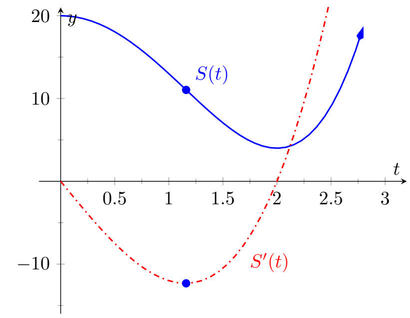


Figure 3.4.20 A graph of $S(t)$ in Example 3.4.18, along with $S'(t)$

Not every critical point corresponds to a relative extrema; $f(x) = x^3$ has a critical point at $(0, 0)$ but no relative maximum or minimum. Likewise, just because $f''(x) = 0$ we cannot conclude concavity changes at that point. We were careful before to use terminology "*possible* point of inflection" since we needed to check to see if the concavity changed. The canonical example of $f''(x) = 0$ *without* concavity changing is $f(x) = x^4$. At $x = 0$, $f''(x) = 0$ but f is always concave up, as shown in Figure 3.4.21.

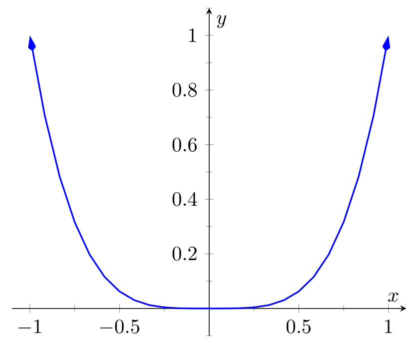


Figure 3.4.21 A graph of $f(x) = x^4$. Clearly f is always concave up, despite the fact that $f''(x) = 0$ when $x = 0$. In this example, the *possible* point of inflection $(0, 0)$ is not a point of inflection.

3.4.2 The Second Derivative Test

The first derivative of a function gave us a test to find if a critical value corresponded to a relative maximum, minimum, or neither. The second derivative gives us another way to test if a critical point is a local maximum or minimum. The following theorem officially states something that is intuitive: if a critical value occurs in a region where a function f is concave up, then that critical value must correspond to a relative minimum of f, etc. See Figure 3.4.22 for a visualization of this.

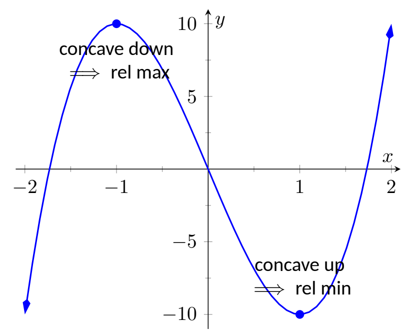


Figure 3.4.22 Demonstrating the fact that relative maxima occur when the graph is concave down and relative minima occur when the graph is concave up

Theorem 3.4.23 The Second Derivative Test.

Let c be a critical value of f where $f''(c)$ is defined.

1. *If $f''(c) > 0$, then f has a local minimum at $(c, f(c))$.*
2. *If $f''(c) < 0$, then f has a local maximum at $(c, f(c))$.*

The Second Derivative Test relates to the First Derivative Test in the following way. If $f''(c) > 0$, then the graph is concave up at a critical point c and f' itself is growing. Since $f'(c) = 0$ and f' is growing at c, then it must go from negative to positive at c. This means the function goes from decreasing to increasing, indicating a local minimum at c.

youtu.be/watch?v=4hWldEUoG2U

Figure 3.4.24 Video presentation of Theorem 3.4.23

Example 3.4.25 Using the Second Derivative Test.

Let $f(x) = 100/x + x$. Find the critical points of f and use the The Second Derivative Test to label them as relative maxima or minima.

Solution. We find $f'(x) = -100/x^2 + 1$ and $f''(x) = 200/x^3$. We set $f'(x) = 0$ and solve for x to find the critical values (note that f' is not defined at $x = 0$, but neither is f so this is not a critical value.) We find the critical values are $x = \pm 10$. We now evaluate the second derivative at these critical numbers. Evaluating $f''(10) = 0.1 > 0$, so there is a local minimum at $x = 10$. Evaluating $f''(-10) = -0.1 < 0$, determining a relative maximum at $x = -10$. These results are confirmed in Figure 3.4.26.

Video solution

youtu.be/watch?v=_DhmPXRZfi8

We have been learning how the first and second derivatives of a function relate information about the graph of that function. We have found intervals of increasing and decreasing, intervals where the graph is concave up and down, along with the locations of relative extrema and inflection points. In Chapter 1 we saw how limits explained asymptotic behavior. In the next section we combine all of this information to produce accurate sketches of functions.

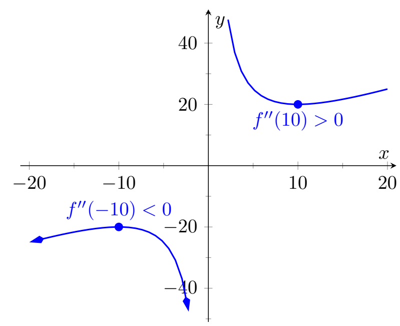


Figure 3.4.26 A graph of $f(x)$ in Example 3.4.25. The second derivative is evaluated at each critical point. When the graph is concave up, the critical point represents a local minimum; when the graph is concave down, the critical point represents a local maximum.

Use Wisely. The second derivative test can only be used on a function that is twice differentiable at c. For functions that are not twice differentiable at c, you will need to use the First Derivative Test. If you've already determined the sign diagram for f', the First Derivative Test is usually easier to apply, and it applies in cases when First Derivative Test does not.

3.4.3 Exercises

Terms and Concepts

1. Sketch a graph of a function $f(x)$ that is concave up on $(0, 1)$ and is concave down on $(1, 2)$.

2. Sketch a graph of a function $f(x)$ that is:

- increasing, concave up on $(0, 1)$,
- increasing, concave down on $(1, 2)$,
- decreasing, concave down on $(2, 3)$, and
- increasing, concave down on $(3, 4)$.

3. Is is possible for a function to be increasing and concave down on $(0, \infty)$ with a horizontal asymptote of $y = 1$? If so, give a sketch of such a function.

4. Is is possible for a function to be increasing and concave up on $(0, \infty)$ with a horizontal asymptote of $y = 1$? If so, give a sketch of such a function.

Problems

Exercise Group. A function $f(x)$ is given. Graph f and f'' on the same axes (using technology is permitted) and verify Theorem 3.4.7.

5. $f(x) = -7x + 3$

6. $f(x) = -4x^2 + 3x - 8$

7. $f(x) = 4x^2 + 3x - 8$

8. $f(x) = x^3 - 3x^2 + x - 1$

9. $f(x) = -x^3 + x^2 - 2x + 5$

10. $f(x) = \sin(x)$

11. $f(x) = \tan(x)$

12. $f(x) = \dfrac{1}{x^2 + 1}$

13. $f(x) = \frac{1}{x}$

14. $f(x) = \frac{1}{x^2}$

Exercise Group. A function $f(x)$ is given.

(a) Find the possible points of inflection of f.

(b) Find the intervals on which the graph of f is concave up.

(c) Find the intervals on which the graph of f is concave down.

15. $f(x) = x^2 - 4x + 4$

16. $f(x) = -x^2 + 4x - 1$

17. $f(x) = x^3 - 8x - 7$

18. $f(x) = 8x^3 + 6x^2 + 9x - 5$

19. $f(x) = \frac{x^4}{4} + 16\frac{x^3}{3} - 72x - 6$

20. $f(x) = 2x^4 - 40x^3 + 296x^2 - 960x + 7$

21. $f(x) = x^4 + 8x^3 + 24x^2 + 32x + 16$

22. $f(x) = \sec(x)$ on $(-3\pi/2, 3\pi/2)$

23. $f(x) = \frac{1}{x^2+1}$

24. $f(x) = \frac{1}{x^2-7x+10}$

25. $f(x) = \sin(x) + \cos(x)$ on $(-\pi, \pi)$

26. $f(x) = x^2 e^x$

27. $f(x) = x^2 \ln(x)$

28. $f(x) = e^{-x^2}$

Exercise Group. A function $f(x)$ is given. Find the critical points of f and use the Second Derivative Test, when possible, to determine the relative extrema. (Note: these are the same functions as in Exercise Group 15–28.)

29. $f(x) = x^2 + 14x + 49$

30. $f(x) = -x^2 - 5x + 3$

31. $f(x) = x^3 - 4x - 4$

32. $f(x) = -x^3 + 8x^2 - 25x - 3$

33. $f(x) = \frac{x^4}{4} + 64x - 9$

34. $f(x) = 2x^4 - 8x^3 - 16x^2 + 96x + 9$

35. $f(x) = x^4 - 12x^3 + 54x^2 - 108x + 81$

36. $f(x) = \sec(x)$ on $(-3\pi/2, 3\pi/2)$

37. $f(x) = \frac{1}{x^2+18x+83}$

38. $f(x) = \frac{1}{x^2-49}$

39. $f(x) = \sin(x) + \cos(x)$ on $(-\pi, \pi)$

40. $f(x) = x^2 e^x$

41. $f(x) = x^2 \ln(x)$

42. $f(x) = e^{-x^2}$

Exercise Group. A function $f(x)$ is given. Find the x values where $f'(x)$ has a relative maximum or minimum. (Note: these are the same functions as in Exercise Group 15–28.)

43. $f(x) = x^2 - 8x + 16$

44. $f(x) = -x^2 + 6x + 4$

45. $f(x) = x^3 - 9x - 2$

46. $f(x) = -9x^3 - 8x^2 - 7x - 1$

47. $f(x) = \frac{x^4}{4} + 14\frac{x^3}{3} + 7$

48. $f(x) = 3x^4 - 24x^3 + 66x^2 - 72x - 6$

49. $f(x) = x^4 + 4x^3 + 6x^2 + 4x + 1$

50. $f(x) = \sec(x)$ on $(-3\pi/2, 3\pi/2)$

51. $f(x) = \frac{1}{x^2-2x+4}$

52. $f(x) = \frac{1}{x^2-13x+36}$

53. $f(x) = \sin(x) + \cos(x)$ on $(-\pi, \pi)$

54. $f(x) = x^2 e^x$

55. $f(x) = x^2 \ln(x)$

56. $f(x) = e^{-x^2}$

3.5 Curve Sketching

We have been learning how we can understand the behavior of a function based on its first and second derivatives. While we have been treating the properties of a function separately (increasing and decreasing, concave up and concave down, etc.), we combine them here to produce an accurate graph of the function without plotting lots of extraneous points.

Why bother? Graphing utilities are very accessible, whether on a computer, a hand-held calculator, or a smartphone. These resources are usually very fast and accurate. We will see that our method is not particularly fast — it will require time (but it is not *hard*). So again: why bother?

We are attempting to understand the behavior of a function f based on the information given by its derivatives. While all of a function's derivatives relay information about it, it turns out that "most" of the behavior we care about is explained by $f'$and f''. Understanding the interactions between the graph of f and $f'$and $f''$is important. To gain this understanding, one might argue that all that is needed is to look at lots of graphs. This is true to a point, but is somewhat similar to stating that one understands how an engine works after looking only at pictures. It is true that the basic ideas will be conveyed, but "hands-on" access increases understanding.

Key Idea 3.5.1 summarizes what we have learned so far that is applicable to sketching graphs of functions and gives a framework for putting that information together. It is followed by several examples.

Key Idea 3.5.1 Curve Sketching.

To produce an accurate sketch a given function f, consider the following steps.

1. Find the domain of f. Generally, we assume that the domain is the entire real line then find restrictions, such as where a denominator is 0 or where negatives appear under the radical.
2. Find the critical values of f.
3. Find the possible points of inflection of f.
4. Find the location of any vertical asymptotes of f (usually done in conjunction with Item 1).
5. Consider the limits $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$ to determine the end behavior of the function.
6. Create a number line that includes all critical points, possible points of inflection, and locations of vertical asymptotes. For each interval created, determine whether f is increasing or decreasing, concave up or down.
7. Evaluate f at each critical point and possible point of inflection. Plot these points on a set of axes. Connect these points with curves exhibiting the proper concavity. Sketch asymptotes and x and y intercepts where applicable.

Example 3.5.2 Curve sketching.

Use Key Idea 3.5.1 to sketch $f(x) = 3x^3 - 10x^2 + 7x + 5$.

Solution. We follow the steps outlined in Key Idea 3.5.1.

1. The domain of f is the entire real line; there are no values x for which $f(x)$ is not defined.

2. Find the critical values of f. We compute $f'(x) = 9x^2 - 20x + 7$. Use the Quadratic Formula to find the roots of f':

$$\begin{aligned} x &= \frac{20 \pm \sqrt{(-20)^2 - 4(9)(7)}}{2(9)} \\ &= \frac{1}{9}\left(10 \pm \sqrt{37}\right) \\ x &\approx 0.435, 1.787. \end{aligned}$$

3. Find the possible points of inflection of f. Compute $f''(x) = 18x - 20$. We have

$$\begin{aligned} f''(x) &= 0 \\ 18x - 20 &= 0 \\ x &= 10/9 \\ &\approx 1.111. \end{aligned}$$

4. There are no vertical asymptotes.

5. We determine the end behavior using limits as x approaches $\pm\infty$.

$$\lim_{x\to-\infty} f(x) = -\infty \qquad \lim_{x\to\infty} f(x) = \infty.$$

   We do not have any horizontal asymptotes.

6. We place the values $x = (10 \pm \sqrt{37})/9$ and $x = 10/9$ on a number line, as shown in Figure 3.5.3. We mark each subinterval as increasing or decreasing, concave up or down, using the techniques used in Sections 3.3–3.4.

7. Evaluate f at each critical number and possible inflection point.

$$f(0.435) \approx 6.400 \qquad f(1.111) \approx 4.547 \qquad f(1.787) \approx 2.695$$

   We plot the appropriate points on axes as shown in Figure 3.5.4(a) and connect the points with straight lines (to show increasing/decreasig behavior). In Figure 3.5.4(b) we adjust these lines to demonstrate the proper concavity. In Figure 3.5.4(c) we show a graph of f drawn with a computer program, verifying the accuracy of our sketch.

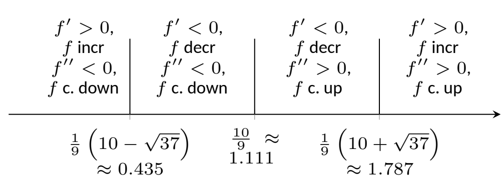
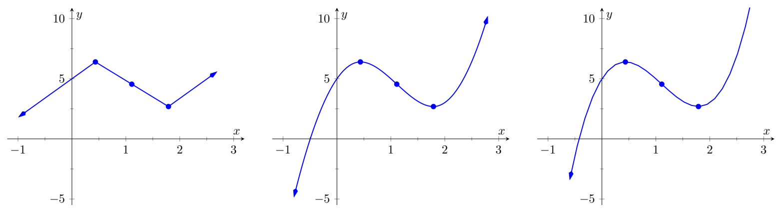


Figure 3.5.3 Number line for f in Example 3.5.2



(a) (b) (c)

Figure 3.5.4 Sketching f in Example 3.5.2

Video solution

youtu.be/watch?v=41XMvSHgl-Y

Example 3.5.5 Curve sketching.

Sketch $f(x) = \dfrac{x^2 - x - 2}{x^2 - x - 6}$.

Solution. We again follow the steps outlined in Key Idea 3.5.1.

1. In determining the domain, we assume it is all real numbers and look for restrictions. We find that at $x = -2$ and $x = 3$, $f(x)$ is not defined. So the domain of f is $D = \{x \mid x \neq -2, 3\}$.

2. To find the critical values of f, we first find $f'(x)$. Using the Quotient Rule, we find

$$f'(x) = \frac{-8x + 4}{(x^2 + x - 6)^2} = \frac{-8x + 4}{(x - 3)^2(x + 2)^2}.$$

   We get $f'(x) = 0$ when $x = 1/2$, and f' is undefined when $x = -2, 3$. Since $f'$is undefined only when f is also undefined, these are not critical values. The only critical value is $x = 1/2$.

3. To find the possible points of inflection, we find $f''(x)$, again employing the Quotient Rule:

$$f''(x) = \frac{24x^2 - 24x + 56}{(x - 3)^3(x + 2)^3}.$$

   We find that $f''(x)$ is never 0 (setting the numerator equal to 0 and solving for x, we find the only roots to this quadratic are not real numbers) and $f''$is undefined when $x = -2, 3$. Thus concavity will possibly only change at $x = -2$ and $x = 3$ (which are not in the domain of f, so these won't be inflection points).

4. The vertical asymptotes of f are at $x = -2$ and $x = 3$, the places where f is undefined.

5. There is a horizontal asymptote of $y = 1$, as $\lim\limits_{x \to -\infty} f(x) = 1$ and $\lim\limits_{x \to \infty} f(x) = 1$.

6. We place the values $x = 1/2$, $x = -2$ and $x = 3$ on a number line as shown in Figure 3.5.6. We mark in each interval whether f is increasing or decreasing, concave up or down. We see that f has a relative maximum at $x = 1/2$; concavity changes only at the vertical asymptotes.

7. Evaluate f at each critical number.

$$f(0) = 1/3 \qquad\qquad f(1/2) = 9/25$$

In Figure 3.5.7(a), we plot the points from the number line on a set of axes and connect the points with straight lines to get a general idea of what the function looks like (these lines effectively only convey increasing/decreasing information). In Figure 3.5.7(b), we adjust the graph with the appropriate concavity. We also show f crossing the x-axis at $x = -1$ and $x = 2$ and crossing the y-axis at $y = 1/3$. Finally, Figure 3.5.7(c) shows a computer generated graph of f, which verifies the accuracy of our sketch.

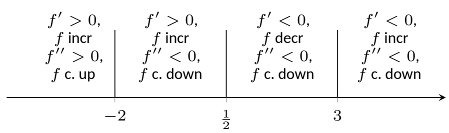
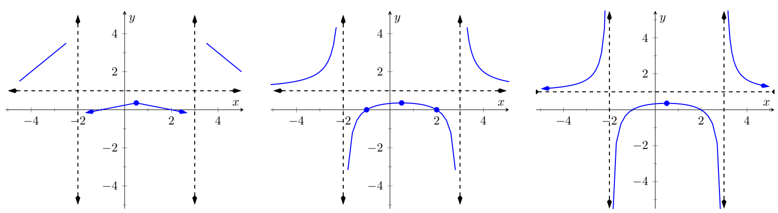


Figure 3.5.6 Number line for f in Example 3.5.5



(a) **(b)** **(c)**

Figure 3.5.7 Sketching f in Example 3.5.5

Video solution

youtu.be/watch?v=t3VI2oTmOiA

Example 3.5.8 Curve sketching.

Sketch $f(x) = \dfrac{5(x-2)(x+1)}{x^2+2x+4}$.

Solution. We again follow Key Idea 3.5.1.

1. We assume that the domain of f is all real numbers and consider restrictions. The only restrictions could come when the denominator is 0, but this never occurs because the denominator is a quadratic polynomial with no real roots. Therefore the domain of f is all real numbers, $\mathbb{R}$.

2. We find the critical values of f by setting $f'(x) = 0$ and solving for x. We find

$$\begin{aligned} f'(x) &= \frac{15x(x+4)}{(x^2+2x+4)^2} \\ 0 &= \frac{15x(x+4)}{(x^2+2x+4)^2} \\ x &= -4, 0. \end{aligned}$$

Since the denominator of f' is just the square of the denominator of f, there are no values of x for which f' is undefined.

3. We find the possible points of inflection by solving $f''(x) = 0$ for x (again, there are no values of x for which f'' is undefined.) We find
$$f''(x) = -\frac{30x^3 + 180x^2 - 240}{(x^2 + 2x + 4)^3}.$$
The cubic in the numerator does not factor very "nicely." We instead approximate the roots (using a cas) at $x = -5.759$, $x = -1.305$ and $x = 1.064$.

4. There are no vertical asymptotes as the denominator never equals zero.

5. We have a horizontal asymptote of $y = 5$, as $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 5$.

6. We place the critical points and possible points on a number line as shown in Figure 3.5.9 and mark each interval as increasing/decreasing, concave up/down appropriately.

7. Evaluate f at each critical number, possible inflection point.
$$f(-5.759) \approx 7.200 \qquad f(-4) = 7.5$$
$$f(-1.305) \approx 1.630 \qquad f(0) = 2.5$$
$$f(1.064) \approx -1.331$$

In Figure 3.5.10(a) we plot the significant points from the number line as well as the x- and y-intercepts, and connect the points with straight lines to get a general impression about the graph (this graph only includes increasing/decreasing information). In Figure 3.5.10(b), we add concavity, drawing the function so that it is smooth (since f is differentiable everywhere, there should be no kinks or corners). Figure 3.5.10(c) shows a computer generated graph of f, affirming our results.

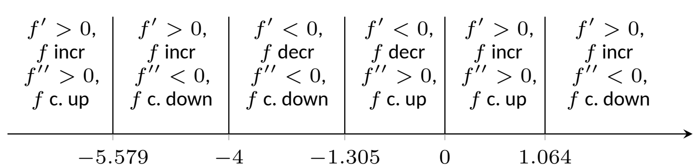
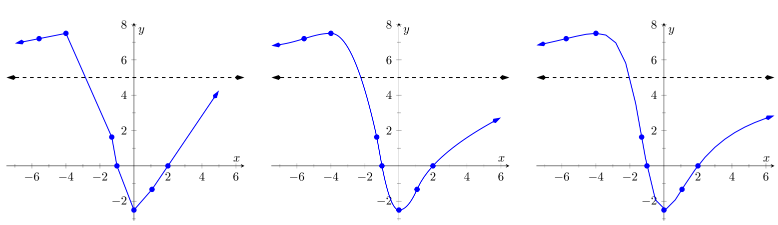


Figure 3.5.9 Number line for f in Example Example 3.5.8



(a) **(b)** **(c)**

Figure 3.5.10 Sketching f in Example 3.5.8

To get some more practice with curve sketching, we include a few more video examples to illustrate the process. (The last of these could be considered "archival footage": it was from a first run at using our new lightboard.)

In each of our examples, we found a few significant points on the graph of f that corresponded to changes in increasing/decreasing or concavity. We connected these points with straight lines, then adjusted for concavity, and finished by showing a very accurate, computer generated graph.

youtu.be/watch?v=S3j-vuUZPjE

Figure 3.5.11 Sketching the polynomial $f(x) = x^2(5-x)^3$

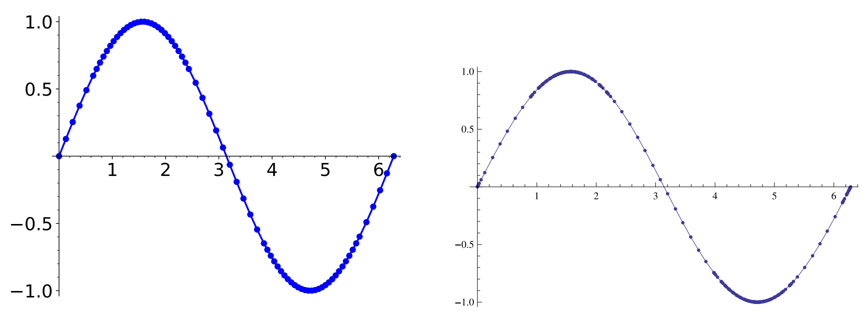
Why are computer graphics so good? It is not because computers are "smarter" than we are. Rather, it is largely because computers are much faster at computing than we are. In general, computers graph functions much like most students do when first learning to draw graphs: they plot equally spaced points, then connect the dots using lines. By using lots of points, the connecting lines are short and the graph looks smooth.

This does a fine job of graphing in most cases (in fact, this is the method used for many graphs in this text). However, in regions where the graph is very "curvy," this can generate noticeable sharp edges on the graph unless a large number of points are used. High quality computer algebra systems, such as *Mathematica* and *Sage*, use special algorithms to plot lots of points only where the graph is "curvy."

youtu.be/watch?v=fdnoec_9Yw4

Figure 3.5.12 Sketching the graph of the trigonometric function $f(x) = \sin(2x) - 2\sin(x)$

In Figure 3.5.14, two graph of $y = \sin(x)$ is given, generated by *Sage* and *Mathematica*. The small points represent each of the places where each cas sampled the function. Notice how at the "bends" of $\sin(x)$, lots of points are used; where $\sin(x)$ is relatively straight, fewer points are used. (In the *Mathematica* plot, many points are also used at the endpoints to ensure the "end behavior" is accurate.)



(a) *Sage* output **(b)** *Mathematica* output

Figure 3.5.14 CAS plots of $y = \sin(x)$ illustrating the sample points

youtu.be/watch?v=JR31YX5N3M8

Figure 3.5.13 Sketching the graph of $f(x) = x^{4/3} - 4x^{1/3}$

How does *Sage* know where the graph is "curvy"? Calculus. When we study *curvature* in a later chapter, we will see how the first and second derivatives of a function work together to provide a measurement of "curviness." *Sage* employs algorithms to determine regions of "high curvature" and plots extra points there.

Again, the goal of this section is not "How to graph a function when there is no computer to help." Rather, the goal is "Understand that the shape of the graph of a function is largely determined by understanding the behavior of the function at a few key places." In Example 3.5.8, we were able to accurately sketch a complicated graph using only five points and knowledge of asymptotes!

There are many applications of our understanding of derivatives beyond curve sketching. The next chapter explores some of these applications, demonstrating just a few kinds of problems that can be solved with a basic knowledge of differentiation.

3.5.1 Exercises

Terms and Concepts

1. Why is sketching curves by hand beneficial even though technology is ubiquitous?

2. What does "ubiquitous" mean?

3. T/F: When sketching graphs of functions, it is useful to find the critical points. (☐ True ☐ False)

4. T/F: When sketching graphs of functions, it is useful to find the possible points of inflection. (☐ True ☐ False)

5. T/F: When sketching graphs of functions, it is useful to find the horizontal and vertical asymptotes. (☐ True ☐ False)

Problems

Exercise Group. In the following exercises, practice using Key Idea 3.5.1 by applying the principles to the given functions with familiar graphs.

6. Use Key Idea 3.5.1 to sketch a graph of $f(x) = 2x + 4$

7. Use Key Idea 3.5.1 to sketch a graph of $f(x) = -x^2 + 1$

8. Use Key Idea 3.5.1 to sketch a graph of $f(x) = \sin(x)$

9. Use Key Idea 3.5.1 to sketch a graph of $f(x) = e^x$

10. Use Key Idea 3.5.1 to sketch a graph of $f(x) = \dfrac{1}{x}$

11. Use Key Idea 3.5.1 to sketch a graph of $f(x) = \dfrac{1}{x^2}$

Exercise Group. In the following exercises, sketch a graph of the given function using Key Idea 3.5.1. Show all work; check your answer with technology.

12. Use Key Idea 3.5.1 to sketch a graph of $f(x) = x^3 - 2x^2 + 4x + 1$

13. Use Key Idea 3.5.1 to sketch a graph of $f(x) = -x^3 + 5x^2 - 3x + 2$

14. Use Key Idea 3.5.1 to sketch a graph of $f(x) = x^3 + 3x^2 + 3x + 1$

15. Use Key Idea 3.5.1 to sketch a graph of $f(x) = x^3 - x^2 - x + 1$

16. Use Key Idea 3.5.1 to sketch a graph of $f(x) = (x - 2)\ln(x - 2)$

17. Use Key Idea 3.5.1 to sketch a graph of $f(x) = (x - 2)^2 \ln(x - 2)$

18. Use Key Idea 3.5.1 to sketch a graph of $f(x) = \dfrac{x^2 - 4}{x^2}$

19. Use Key Idea 3.5.1 to sketch a graph of $f(x) = \dfrac{x^2 - 4x + 3}{x^2 - 6x + 8}$

20. Use Key Idea 3.5.1 to sketch a graph of $f(x) = \dfrac{x^2 - 2x + 1}{x^2 - 6x + 8}$

21. Use Key Idea 3.5.1 to sketch a graph of $f(x) = x\sqrt{x + 1}$

22. Use Key Idea 3.5.1 to sketch a graph of $f(x) = x^2 e^x$

23. Use Key Idea 3.5.1 to sketch a graph of $f(x) = \sin(x)\cos(x)$ on $[-\pi, \pi]$

24. Use Key Idea 3.5.1 to sketch a graph of $f(x) = (x - 3)^{2/3} + 2$

25. Use Key Idea 3.5.1 to sketch a graph of $f(x) = \dfrac{(x - 1)^{2/3}}{x}$

Exercise Group. In the following exercises, a function with the parameters a and b are given. Describe the critical points and possible points of inflection of f in terms of a and b.

26. $f(x) = \dfrac{a}{x^2 + b^2}$

(a) Find the critical points of f.

(b) Find the inflection points of f.

27. $f(x) = \sin(ax + b)$

(a) Find the critical points of f.

(b) Find the inflection points of f.

28. $f(x) = (x - a)(x - b)$

(a) Find the critical points of f.

(b) Find the inflection points of f.

29. Given $x^2 + y^2 = 1$, use implicit differentiation to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Use this information to justify the sketch of the unit circle.

Chapter 4

Applications of the Derivative

In Chapter 3, we learned how the first and second derivatives of a function influence its graph. In this chapter we explore other applications of the derivative.

4.1 Newton's Method

Solving equations is one of the most important things we do in mathematics, yet we are surprisingly limited in what we can solve analytically. For instance, equations as simple as $x^5 + x + 1 = 0$ or $\cos(x) = x$ cannot be solved by algebraic methods in terms of familiar functions. Fortunately, there are methods that can give us *approximate* solutions to equations like these. These methods can usually give an approximation correct to as many decimal places as we like. In Section 1.5 we learned about the Bisection Method. This section focuses on another technique (which generally works faster), called Newton's Method.

Newton's Method is built around tangent lines. The main idea is that if x is sufficiently close to a root of $f(x)$, then the tangent line to the graph at $(x, f(x))$ will cross the x-axis at a point closer to the root than x.

We start Newton's Method with an initial guess about roughly where the root is. Call this x_0. (See Figure 4.1.1(a).) Draw the tangent line to the graph at $(x_0, f(x_0))$ and see where it meets the x-axis. Call this point x_1. Then repeat the process — draw the tangent line to the graph at $(x_1, f(x_1))$ and see where it meets the x-axis. (See Figure 4.1.1(b).) Call this point x_2. Repeat the process again to get x_3, x_4, etc. This sequence of points will often converge rather quickly to a root of f.

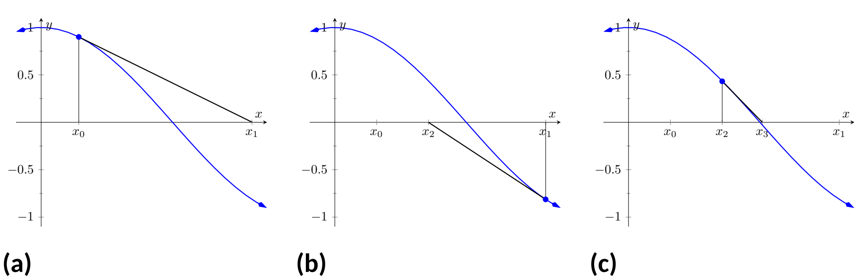


Figure 4.1.1 Demonstrating the geometric concept behind Newton's Method

We can use this *geometric* process to create an *algebraic* process. Let's look at how we found x_1. We started with the tangent line to the graph at $(x_0, f(x_0))$. The slope of this tangent line is $f'(x_0)$ and the equation of the line is

$$y = f'(x_0)(x - x_0) + f(x_0).$$

This line crosses the x-axis when $y = 0$, and the x-value where it crosses is what we called x_1. So let $y = 0$ and replace x with x_1, giving the equation:

$$0 = f'(x_0)(x_1 - x_0) + f(x_0).$$

Now solve for x_1:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Since we repeat the same geometric process to find x_2 from x_1, we have

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, given an approximation x_n, we can find the next approximation, x_{n+1} as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

We summarize this process as follows.

Key Idea 4.1.2 Newton's Method.

Let f be a differentiable function on an interval I with a root in I. To approximate the value of the root, accurate to d decimal places:

1. Choose a value x_0 as an initial approximation of the root. (This is often done by looking at a graph of f.)

2. Create successive approximations iteratively; given an approximation x_n, compute the next approximation x_{n+1} as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

3. Stop the iterations when successive approximations do not differ in the first d places after the decimal point.

Let's practice Newton's Method with a concrete example.

Newton's Method is not Infallible. The sequence of approximate values may not converge, or it may converge so slowly that one is "tricked" into thinking a certain approximation is better than it actually is. These issues will be discussed at the end of the section.

Example 4.1.3 Using Newton's Method.

Approximate the real root of $x^3 - x^2 - 1 = 0$, accurate to the first three places after the decimal, using Newton's Method and an initial approximation of $x_0 = 1$.

Solution. To begin, we compute $f'(x) = 3x^2 - 2x$. Then we apply the Newton's Method algorithm, outlined in Key Idea 4.1.2.

$$\begin{aligned} x_1 &= 1 - \frac{f(1)}{f'(1)} \\ &= 1 - \frac{1^3 - 1^2 - 1}{3\cdot 1^2 - 2\cdot 1} \\ &= 2 \end{aligned}$$

$$\begin{aligned} x_2 &= 2 - \frac{f(2)}{f'(2)} \\ &= 2 - \frac{2^3 - 2^2 - 1}{3\cdot 2^2 - 2\cdot 2} \\ &= 1.625 \end{aligned}$$

$$\begin{aligned} x_3 &= 1.625 - \frac{f(1.625)}{f'(1.625)} \\ &= 1.625 - \frac{1.625^3 - 1.625^2 - 1}{3\cdot 1.625^2 - 2\cdot 1.625} \\ &\approx 1.48579 \end{aligned}$$

$$\begin{aligned} x_4 &= 1.48579 - \frac{f(1.48579)}{f'(1.48579)} \\ &\approx 1.46596 \end{aligned}$$

$$\begin{aligned} x_5 &= 1.46596 - \frac{f(1.46596)}{f'(1.46596)} \\ &\approx 1.46557 \end{aligned}$$

We performed five iterations of Newton's Method to find a root accurate to the first three places after the decimal; our final approximation is 1.465. The exact value of the root, to six decimal places, is 1.465571; It turns out that our x_5 is accurate to more than just three decimal places. A graph of $f(x)$ is given in Figure 4.1.4. We can see from the graph that our initial approximation of $x_0 = 1$ was not particularly accurate; a closer guess would have been $x_0 = 1.5$. Our choice was based on ease of initial calculation, and shows that Newton's Method can be robust enough that we do not have to make a very accurate initial approximation.

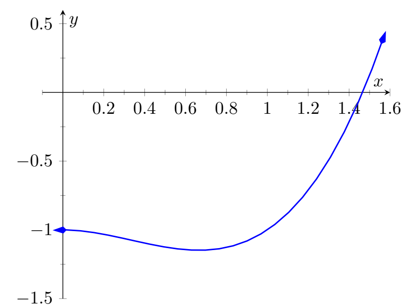


Figure 4.1.4 A graph of $f(x) = x^3 - x^2 - 1$ in Example 4.1.3

We can automate this process on a calculator that has an ANS key that returns the result of the previous calculation. Start by pressing 1 and then Enter. (We have just entered our initial guess, $x_0 = 1$.) Now compute

$$\text{ANS} - \frac{f(\text{ ANS })}{f'(\text{ ANS })}$$

by entering the following and repeatedly press the Enter key.

```
ANS-(ANS^3-ANS^2-1)/(3*ANS^2-2*ANS)
```

Each time we press the Enter key, we are finding the successive approximations, x_1, x_2, ..., and each one is getting closer to the root. In fact, once we get past around x_7 or so, the approximations don't appear to be changing. They actually are changing, but the change is far enough to the right of the decimal point that it doesn't show up on the calculator's display. When this happens, we can be pretty confident that we have found an accurate approximation.

Using a calculator in this manner makes the calculations simple; many iterations can be computed very quickly.

Example 4.1.5 Using Newton's Method to find where functions intersect.

Use Newton's Method to approximate a solution to $\cos(x) = x$, accurate to five places after the decimal.

Solution. Newton's Method provides a method of solving $f(x) = 0$; it is not (directly) a method for solving equations like $f(x) = g(x)$. However, this is not a problem; we can rewrite the latter equation as $f(x) - g(x) = 0$ and then use Newton's Method.

So we rewrite cos$(x) = x$ as cos$(x) - x = 0$. Written this way, we are finding a root of $f(x) = \cos(x) - x$. We compute $f'(x) = -\sin(x) - 1$. Next we need a starting value, x_0. Consider Figure 4.1.6, where $f(x) = \cos(x) - x$ is graphed. It seems that $x_0 = 0.75$ is pretty close to the root, so we will use that as our x_0. (The figure also shows the graphs of $y = \cos(x)$ and $y = x$. Note how they intersect at the same x value as when $f(x) = 0$.)

We now compute x_1, x_2, etc. The formula for x_1 is

$$x_1 = 0.75 - \frac{\cos(0.75) - 0.75}{-\sin(0.75) - 1} \approx 0.7391111388.$$

Apply Newton's Method again to find x_2:

$$x_2 = 0.7391111388 - \frac{\cos(0.7391111388) - 0.7391111388}{-\sin(0.7391111388) - 1} \approx 0.7390851334.$$

We can continue this way, but it is really best to automate this process. On a calculator with an `ANS` key, we would start by entering `0.75`, then `Enter`, inputting our initial approximation. We then enter:

```
ANS - (cos(ANS)-ANS)/(-sin(ANS)-1)
```

Repeatedly pressing the `Enter` key gives successive approximations. We quickly find:

$$x_3 = 0.7390851332$$
$$x_4 = 0.7390851332.$$

Our approximations x_2 and x_3 did not differ for at least the first five places after the decimal, so we could have stopped. However, using our calculator in the manner described is easy, so finding x_4 was not hard. It is interesting to see how we found an approximation, accurate to as many decimal places as our calculator displays, in just four iterations.

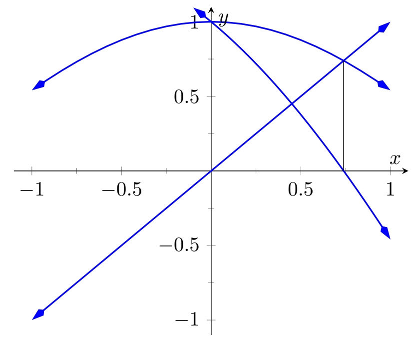


Figure 4.1.6 A graph of $f(x) = \cos(x) - x$ used to find an initial approximation of its root

If you know how to program, you can translate the following pseudocode into your favorite language to perform the computation in this problem.

```
x = 0.75
while true
    oldx = x
    x = x - (cos(x)-x)/(-sin(x)-1)
    print x
    if abs(x-oldx) < 0.0000000001
        break
```

This code calculates x_1, x_2, etc., storing each result in the variable `x`. The previous approximation is stored in the variable `oldx`. We continue looping until the difference between two successive approximations, `abs(x-oldx)`, is less than some small tolerance, in this case, `0.0000000001`.

Convergence of Newton's Method. What should one use for the initial guess, x_0? Generally, the closer to the actual root the initial guess is, the better. However, some initial guesses should be avoided. For instance, consider Example 4.1.3 where we sought the root to $f(x) = x^3 - x^2 - 1$. Choosing $x_0 = 0$ would have been a particularly poor choice. Consider Figure 4.1.7, where $f(x)$ is graphed along with its tangent line at $x = 0$. Since $f'(0) = 0$, the tangent line is horizontal and does not intersect the x-axis. Graphically, we see that Newton's Method fails.

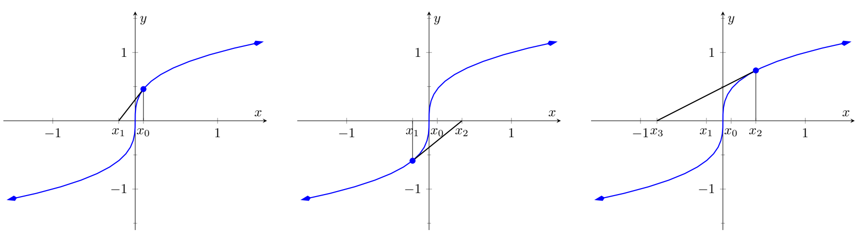
We can also see analytically that it fails. Since

$$x_1 = 0 - \frac{f(0)}{f'(0)}$$

and $f'(0) = 0$, we see that x_1 is not well defined.

This problem can also occur if, for instance, it turns out that $f'(x_5) = 0$. Adjusting the initial approximation x_0 by a very small amount will likely fix the problem.

It is also possible for Newton's Method to not converge while each successive approximation is well defined. Consider $f(x) = x^{1/3}$, as shown in Figure 4.1.8. It is clear that the root is $x = 0$, but let's approximate this with $x_0 = 0.1$. Figure 4.1.8(a) shows graphically the calculation of x_1; notice how it is farther from the root than x_0. Figure 4.1.8(b) and Figure 4.1.8(c) show the calculation of x_2 and x_3, which are even farther away; our successive approximations are getting worse. (It turns out that in this particular example, each successive approximation is twice as far from the true answer as the previous approximation.)



(a) **(b)** **(c)**

Figure 4.1.8 Newton's Method fails to find a root of $f(x) = x^{1/3}$, regardless of the choice of x_0.

There is no "fix" to this problem; Newton's Method simply will not work and another method must be used. (In this case the particular reason Newton's Method fails is that the tangent line is vertical at the root).

While Newton's Method does not always work, it does work "most of the time," and it is generally very fast. Once the approximations get close to the root, Newton's Method can as much as double the number of correct decimal places with each successive approximation. A course in Numerical Analysis will introduce the reader to more iterative root finding methods, as well as give greater detail about the strengths and weaknesses of Newton's Method.

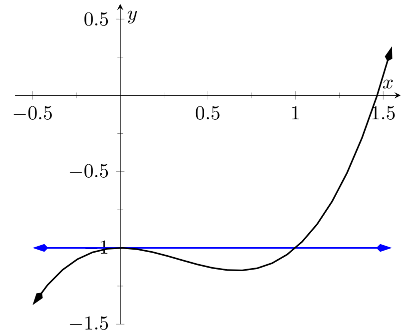


Figure 4.1.7 A graph of $f(x) = x^3 - x^2 - 1$, showing why an initial approximation of $x_0 = 0$ with Newton's Method fails

4.1.1 Exercises

Terms and Concepts

1. (☐ True ☐ False) Given a function $f(x)$, Newton's Method produces an exact solution to $f(x) = 0$.

2. (☐ True ☐ False) In order to get a solution to $f(x) = 0$ accurate to d places after the decimal, at least $d + 1$ iterations of Newton's Method must be used.

Problems

Exercise Group. The roots of the function $f(x)$ are known or are easily found. Use five iterations of Newton's Method with the given initial approximation to approximate the root. Compare it to the known value of the root.

3. $f(x) = \cos(x)$, $x_0 = 1.5$

4. $f(x) = \sin(x)$, $x_0 = 1$

5. $f(x) = x^2 + x - 2$, $x_0 = 0$

6. $f(x) = x^2 - 2$, $x_0 = 1.5$

7. $f(x) = \ln(x)$, $x_0 = 2$

8. $f(x) = x^3 - x^2 + x - 1$, $x_0 = 2$

Exercise Group. Use Newton's Method to approximate all roots of the given function accurate to three places after the decimal. If an interval is given, find only the roots that lie within that interval. Use technology to obtain good initial approximations.

9. $f(x) = x^3 + 5x^2 - x - 1$

10. $f(x) = x^4 + 2x^3 - 7x^2 - x + 5$

11. $f(x) = x^{17} - 2x^{13} - 10x^8 + 10$ on $(-2, 2)$

12. $f(x) = x^2 \cos(x) + (x - 1)\sin(x)$ on $(-3, 3)$

Exercise Group. Use Newton's Method to approximate when the given functions are equal, accurate to 3 places after the decimal. Use technology to obtain good initial approximations.

13. $f(x) = x^2$, $g(x) = \cos(x)$

14. $f(x) = x^2 - 1$, $g(x) = \sin(x)$

15. $f(x) = e^{x^2}$, $g(x) = \cos(x)$

16. $f(x) = x$, $g(x) = \tan(x)$ on $[-6, 6]$

17. Why does Newton's Method fail in finding a root of $f(x) = x^3 - 3x^2 + x + 3$ when $x_0 = 1$?

18. Why does Newton's Method fail in finding a root of $f(x) = -17x^4 + 130x^3 - 301x^2 + 156x + 156$ when $x_0 = 1$?

4.2 Related Rates

youtu.be/watch?v=TKqYEDaAidQ

Figure 4.2.1 Video introduction to Section 4.2

When two quantities are related by an equation, knowing the value of one quantity can determine the value of the other. For instance, the circumference and radius of a circle are related by $C = 2\pi r$; knowing that C is 6π in determines the radius must be 3 in.

But what if both variables are changing with time? If we know how two variables are related and we know how one of them changes with time, can we find how the other variable changes with time?

The topic of **related rates** allows us to answer this question: knowing the *rate* at which one quantity is changing can determine the *rate* at which another changes.

Remark 4.2.2 This section relies heavily on implicit differentiation, so referring back to Section 2.6 may help.

We demonstrate the concepts of related rates through examples.

Example 4.2.3 Understanding related rates.

The radius of a circle is growing at a rate of $5\,\frac{\text{in}}{\text{h}}$. At what rate is the circumference growing?

Solution (a). The circumference and radius of a circle are related by $C = 2\pi r$. We are given information about how the length of r changes with respect to time; that is, we are told $\frac{dr}{dt}$ is $5\,\frac{\text{in}}{\text{h}}$. We want to know how the length of C changes with respect to time, i.e., we want to know $\frac{dC}{dt}$.

Implicitly differentiate both sides of $C = 2\pi r$ with respect to t:

$$\begin{aligned} C &= 2\pi r \\ \frac{d}{dt}(C) &= \frac{d}{dt}(2\pi r) \\ \frac{dC}{dt} &= 2\pi\frac{dr}{dt}. \end{aligned}$$

Video solution

youtu.be/watch?v=Qg3GStrQ8pY

As we know $\frac{dr}{dt}$ is $5\,\frac{\text{in}}{\text{h}}$, we know

$$\frac{dC}{dt} = 2\pi 5 = 10\pi \approx 31.4 \text{ in/hr}.$$

This problem was relatively straightforward, owing to the linear relationship between radius and circumference. The video in Figure 4.2.4 explores what would happen if we had instead been asked for the rate at which the area is changing.

youtu.be/watch?v=RDPxSmxqUBs

Figure 4.2.4 Trying to find the rate at which area is changing for the circle in Example 4.2.3

In related rates problems, we will be presented with an application problem that involves two or more variables and one or more rate. It is the job of the reader to construct the appropriate model that can be used to answer the posed question. Key Idea 4.2.5 outlines the basic steps for solving a related rates problem.

Key Idea 4.2.5 Related Rates.

1. Read the problem carefully and identify the quantities that are changing with time. (There may be many quantities that change with time, try to identify which variables are important to your goal and only focus on these quantities.)

2. Draw a diagram (if applicable) and assign mathematical variables to each quantity that is changing with time. (If you are given a particular value of a quantity that is also changing with time, do not include these values on your diagram. We will call these "instantaneous values" of the variable.)

3. Relate the important variables using a mathematical model. (Typical models are known formulas for area, perimeter, the Pythagorean Theorem or Trigonometric Ratios.) It may be necessary to use more than one technique (such as similar triangles) to reduce your model down to one that only involves the variables of interest.

4. Implicitly differentiate both sides of the equation found in Step 3 with respect to t.

5. Substitute in the known values of rates and known instantaneous values of the variables.

6. Solve for the unknown rate.

7. Write a full sentence conclusion.

Consider another, similar example.

Example 4.2.6 Finding related rates.

Water streams out of a faucet at a rate of $2\,\frac{\text{in}^3}{\text{s}}$ onto a flat surface at a constant rate, forming a circular puddle that is $1/8$ in deep.

1. At what rate is the area of the puddle growing?

2. At what rate is the radius of the circle growing?

Solution.

1. We can answer this question two ways: using "common sense" or related rates. The common sense method states that the volume of the puddle is growing by $2\,\frac{\text{in}^3}{\text{s}}$, where

$$\text{volume of puddle} = \text{area of circle} \times \text{depth}.$$

Since the depth is constant at $1/8$ in, the area must be growing by $16\,\frac{\text{in}^2}{\text{s}}$ since $16 \cdot \frac{1}{8} = 2$. This approach reveals the underlying related rates principle.

Now let's solve the problem using Key Idea 4.2.5. Based on the problem description, the quantities that change with time are the volume of water (the volume of the puddle), the area of the circular puddle and the radius of the circle. We don't need a diagram for this problem. The important variables for this part of the problem are the volume and area.

Let V and A represent the Volume and Area of the puddle. We know $V = A \times \frac{1}{8}$. Take the derivative of both sides with respect to t, employing implicit differentiation.

$$V = \frac{1}{8}A$$

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{1}{8}A\right)$$
$$\frac{dV}{dt} = \frac{1}{8}\frac{dA}{dt}$$

We know the change in volume, $\frac{dV}{dt} = 2$, so we substitute this value into our related rates equation: $2 = \frac{1}{8}\frac{dA}{dt}$, and hence $\frac{dA}{dt} = 16$. Thus the area is growing by $16\ \frac{\text{in}^2}{\text{s}}$.

2. We already identified the quantities that are changing in Part 1. The variables of interest in this problem are the radius and the volume. We need an equation that relates the volume of the circle to the radius. Since the puddle is a right circular cylinder, we will use a known volume formula, $V = \pi r^2 h$ where V is the volume of the puddle (in in^3, r is the radius (in inches) and h is the height (i.e. depth) of the puddle in inches. (Notice that this formula is equivalent to $V =$ area$\times$depth.) We know that the height (depth) is a constant $1/8$ inch. Since this quantity does not change in the problem, we can safely substitute this value now.

   Implicitly derive both sides of $V = \pi r^2 \frac{1}{8}$ with respect to t:

$$V = \frac{1}{8}\pi r^2$$
$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{1}{8}\pi r^2\right)$$
$$\frac{dV}{dt} = \frac{1}{8}2\pi r\frac{dr}{dt}$$
$$\frac{dV}{dt} = \frac{1}{4}\pi r\frac{dr}{dt}$$

   We know that $\frac{dV}{dt}$ is $2\ \frac{\text{in}^3}{\text{s}}$. So we have:

$$2 = \frac{1}{4}\pi r\frac{dr}{dt}$$

   Solving for $\frac{dr}{dt}$, we have

$$\frac{dr}{dt} = \frac{8}{\pi r}.$$

   Note how our answer is not a number, but rather a function of r. In other words, *the rate at which the radius is growing depends on how big the circle already is.* If the circle is very large, adding $2\ \frac{\text{in}^3}{\text{s}}$ of water will not make the circle much bigger at all. If the circle is dime-sized, adding the same amount of water will make a radical change in the radius of the circle.

   In some ways, our problem was (intentionally) ill-posed. We need to specify a current (instantaneous) value of the radius in order to know a rate of change. When the puddle has a radius of 10 in, the radius is growing at a rate of

$$\frac{dr}{dt} = \frac{8}{10\pi} = \frac{4}{5\pi} \approx 0.25 \text{ in/s}.$$

Video solution

youtu.be/watch?v=8ctKxMoFWkU

Example 4.2.7 Studying related rates.

Radar guns measure the rate of distance change between the gun and the object it is measuring. For instance, a reading of "55 mph" means the object is moving away from the gun at a rate of 55 miles per hour, whereas a measurement of "-25 mph" would mean that the object is approaching the gun at a rate of 25 miles per hour.

If the radar gun is moving (say, attached to a police car) then radar readouts are only immediately understandable if the gun and the object are moving along the same line. If a police officer is traveling 60 mph and gets a readout of 15 mph, he knows that the car ahead of him is moving away at a rate of 15 miles an hour, meaning the car is traveling 75 mph. (This straight-line principle is one reason officers park on the side of the highway and try to shoot straight back down the road. It gives the most accurate reading.)

Suppose an officer is driving due north at 30 mph and sees a car moving due east, as shown in Figure 4.2.8. Using his radar gun, he measures a reading of 20 mph. By using landmarks, he believes both he and the other car are about $1/2$ mile from the intersection of their two roads.

If the speed limit on the other road is 55 mph, is the other driver speeding?

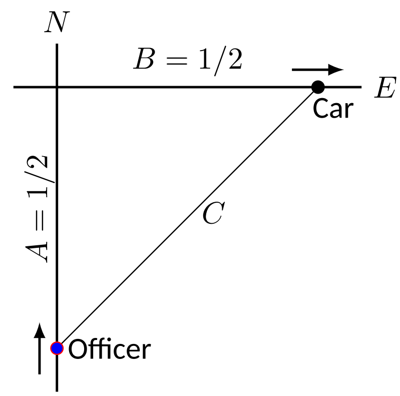


Figure 4.2.8 A sketch of a police car (at bottom) attempting to measure the speed of a car (at right) in Example 4.2.7

Solution. The important quantities that are changing are: the distance of the officer to the intersection, the distance of the car to the intersection, and the distance of the officer to the car. (There are other quantities that are changing as well such as the angles and area of the triangle, but these are not important to this problem.)

Using the diagram in Figure 4.2.8, let's label what we know about the situation. As both the police officer and other driver are $1/2$ mile from the intersection, we have $A = 1/2$, $B = 1/2$, and through the Pythagorean Theorem, $C = 1/\sqrt{2} \approx 0.707$. These values are "instantaneous" values for our variables, so we won't use them until the *end* of the problem. Instead, we will use the variables A, B, and C.

We need an equation that relates A, B, and C. The Pythagorean Theorem is a good choice: $A^2 + B^2 = C^2$. Differentiate both sides with respect to t:

$$\begin{aligned} A^2 + B^2 &= C^2 \\ \frac{d}{dt}\left(A^2 + B^2\right) &= \frac{d}{dt}\left(C^2\right) \\ 2A\frac{dA}{dt} + 2B\frac{dB}{dt} &= 2C\frac{dC}{dt} \end{aligned}$$

We know the police officer is traveling at 30 mph; that is, $\frac{dA}{dt} = -30$. The reason this rate of change is negative is that A is getting smaller; the distance between the officer and the intersection is shrinking. The radar measurement is $\frac{dC}{dt} = 20$. We want to find $\frac{dB}{dt}$.

We have values for everything except $\frac{dB}{dt}$. Solving for this we have:

$$\frac{dB}{dt} = \frac{C\frac{dC}{dt} - A\frac{dA}{dt}}{B}.$$

Now we substitue in our known rates and instantaneous values of our variables:

$$\frac{dB}{dt} \approx \frac{0.707(20) - 0.5(-30)}{(0.5)}$$

$$= 58.28 \text{ mph} .$$

The other driver appears to be speeding slightly.

Example 4.2.9 Studying related rates.

A camera is placed on a tripod 10 ft from the side of a road. The camera is to turn to track a car that is to drive by at 100 mph for a promotional video. The video's planners want to know what kind of motor the tripod should be equipped with in order to properly track the car as it passes by. Figure 4.2.10 shows the proposed setup.

How fast must the camera be able to turn to track the car?

Solution. The quantities that changing are x and θ as drawn on Figure 4.2.10. (The hypotenuse of the triangle is also changing, but this isn't important to the problem). We seek information about how fast the camera is to *turn*; therefore, we need an equation that will relate an angle θ to the position of the camera and the speed and position of the car.

Figure 4.2.10 suggests we use a trigonometric equation. Letting x represent the distance the car is from the point on the road directly in front of the camera, we have

$$\tan(\theta) = \frac{x}{10}. \tag{4.2.1}$$

Now take the derivative of both sides of Equation (4.2.1) using implicit differentiation:

$$\begin{aligned} \tan(\theta) &= \frac{x}{10} \\ \frac{d}{dt}(\tan(\theta)) &= \frac{d}{dt}\left(\frac{x}{10}\right) \\ \sec^2(\theta)\frac{d\theta}{dt} &= \frac{1}{10}\frac{dx}{dt} \end{aligned}$$

Now we solve for $\frac{d\theta}{dt}$:

$$\frac{d\theta}{dt} = \frac{\cos^2(\theta)}{10}\frac{dx}{dt} \tag{4.2.2}$$

As the car is moving at 100 mph, we have that $\frac{dx}{dt}$ is -100 mph (as in the last example, since x is getting smaller as the car travels, $\frac{dx}{dt}$ is negative). We need to convert the measurements so they use the same units (we chose ft); rewrite -100 mph in terms of $\frac{\text{ft}}{\text{s}}$:

$$\begin{aligned} \frac{dx}{dt} &= -100\frac{\text{mi}}{\text{hr}} \\ &= -100\frac{\text{mi}}{\text{hr}} \cdot 5280\frac{\text{ft}}{\text{mi}} \cdot \frac{1}{3600}\frac{\text{hr}}{\text{s}} \\ &= -146.\overline{6} \text{ ft/s} . \end{aligned}$$

We want to know the fastest the camera has to turn. Common sense tells us this is when the car is directly in front of the camera (i.e., when $\theta = 0$). Our mathematics bears this out. In Equation (4.2.2) we see this is when $\cos^2(\theta)$ is largest; this is when $\cos(\theta) = 1$, or when $\theta = 0$. We also know that we should get an answer that is in $\frac{\text{rad}}{\text{s}}$. Since $\cos(\theta)$ is a "dimensionless" measure, it won't contribute to the units. However,

Practicality. Example 4.2.7 is both interesting and impractical. It highlights the difficulty in using radar in a nonlinear fashion, and explains why "in real life" the police officer would follow the other driver to determine their speed, and not pull out pencil and paper.

The principles here are important, though. Many automated vehicles make judgments about other moving objects based on perceived distances, radar-like measurements and the concepts of related rates.

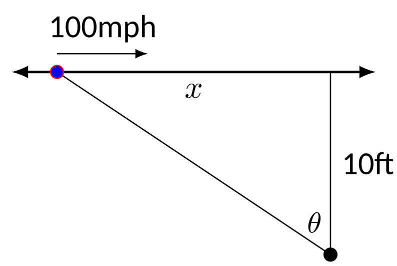


Figure 4.2.10 Tracking a speeding car (at left) with a rotating camera

radians are also dimensionless. This means we can write (or erase) the word "radian" without any unit consequences. (The same is not true of degrees — always convert degress to radians).

With $\frac{dx}{dt}$ approximately $-146.7\,\frac{\text{ft}}{\text{s}}$, we have

$$\begin{aligned}\frac{d\theta}{dt} &\approx -\frac{1}{10\text{ ft}}146.67\text{ ft/s}\\ &= -14.667\text{ radians/s}\end{aligned}$$

We find that $\frac{d\theta}{dt}$ is negative; this matches our diagram in Figure 4.2.10 for θ is getting smaller as the car approaches the camera.

What is the practical meaning of $-14.667\,\frac{\text{rad}}{\text{s}}$? Recall that 1 circular revolution goes through 2π radians, thus $14.667\,\frac{\text{rad}}{\text{s}}$ means $14.667/(2\pi) \approx 2.33$ revolutions per second. The negative sign indicates the camera is rotating in a clockwise fashion.

Video solution

youtu.be/watch?v=B85LlGHgVQo

We introduced the derivative as a function that gives the slopes of tangent lines of functions. This chapter emphasizes using the derivative in other ways. Newton's Method uses the derivative to approximate roots of functions; this section stresses the "rate of change" aspect of the derivative to find a relationship between the rates of change of two related quantities.

In the next section we use Extreme Value concepts to *optimize* quantities.

4.2.1 Exercises

Terms and Concepts

1. (☐ True ☐ False) Implicit differentiation is often used when solving "related rates" type problems.

2. (☐ True ☐ False) A study of related rates is part of the standard police officer training.

Problems

3. Water flows onto a flat surface at a rate of $4\ \frac{\text{cm}^3}{\text{s}}$ forming a circular puddle 8 mm deep. How fast is the radius growing when the radius is:

(a) 2 cm

(b) 20 cm

(c) 200 cm

4. A spherical balloon is inflated with air flowing at a rate of $5\ \frac{\text{cm}^3}{\text{s}}$. How fast is the radius of the balloon increasing when the radius is:

(a) 1 cm

(b) 10 cm

(c) 100 cm

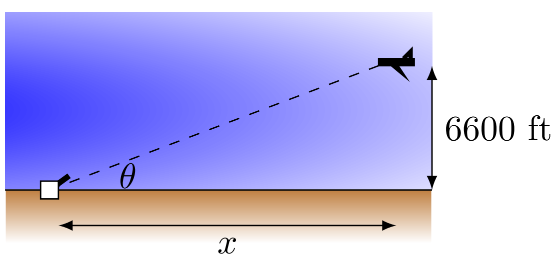
5. Consider the traffic situation introduced in Example 4.2.7. How fast is the "other car" traveling if the officer and the other car are each $\frac{3}{4}$ mile from the intersection, the other car is traveling *due west*, the officer is traveling north at 55 mph, and the radar reading is -75 mph?

6. Consider the traffic situation introduced in Example 4.2.7. Calculate how fast the "other car" is traveling in each of the following situations.

(a) The officer is traveling due north at 50 mph and is $\frac{3}{4}$ mile from the intersection, while the other car is 1 mile from the intersection traveling west and the radar reading is -85 mph?

(b) The officer is traveling due north at 50 mph and is 1 mile from the intersection, while the other car is $\frac{3}{4}$ mile from the intersection traveling west and the radar reading is -85 mph?

7. An F-22 aircraft is flying at 530 mph with an elevation of 6600 ft on a straight-line path that will take it directly over an anti-aircraft gun.



How fast (in radians per second) must the gun be able to turn to accurately track the aircraft when the plane is:

(a) 1 mile away?

(b) $1/5$ mile away?

(c) Directly overhead?

8. An F-22 aircraft is flying at 500 mi/h with an elevation of 100 ft on a straight-line path that will take it directly over an anti-aircraft gun as in Exercise 4.2.7 (note the lower elevation here).

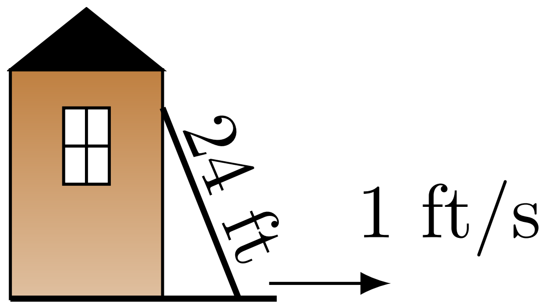
How fast must the gun be able to turn to accurately track the aircraft when the plane is:

(a) 1800 ft away?

(b) 350 ft away?

(c) Directly overhead?

9. A 24 ft ladder is leaning against a house while the base is pulled away at a constant rate of 1 ft/s.



At what rate is the top of the ladder sliding down the side of the house when the base is:

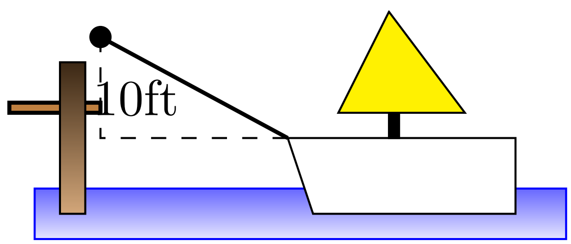
(a) 1 foot from the house?

(b) 10 feet from the house?

(c) 23 feet from the house?

(d) 24 feet from the house?

10. A boat is being pulled into a dock at a constant rate of 30 ft/min by a winch located 10 ft above the deck of the boat.



At what rate is the boat approaching the dock when the boat is:

(a) 50 feet out?

(b) 15 feet out?

(c) 1 foot from the dock?

(d) What happens when the length of rope pulling in the boat is less than 10 feet long?

11. An inverted cylindrical cone, 28 ft deep and 25 ft across at the top, is being filled with water at a rate of $12\ \frac{\text{ft}^3}{\text{s}}$. At what rate is the water rising in the tank when the depth of the water is:

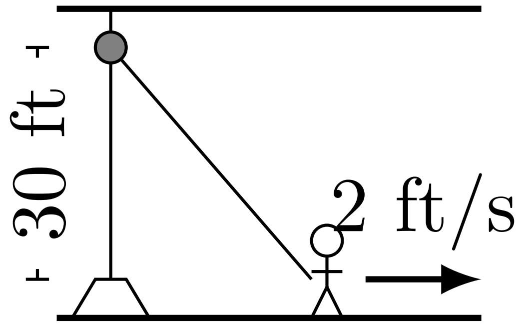
(a) 1 foot?

(b) 10 feet?

(c) 22 feet?

(d) How long will the tank take to fill when starting at empty?

12. A rope, attached to a weight, goes up through a pulley at the ceiling and back down to a worker. The man holds the rope at the same height as the connection point between rope and weight.



Suppose the man stands directly next to the weight (i.e., a total rope length of 60 feet) and begins to walk away at a rate of 2 ft/s. How fast is the weight rising when the man has walked:

(a) 10 feet?

(b) 40 feet?

(c) How far must the man walk to raise the weight all the way to the pulley?

13. Consider the situation described in Exercise 4.2.12. Suppose the man starts 40 ft from the weight and begins to walk away at a rate of $2\ \frac{\text{ft}}{\text{s}}$.

(a) How long is the rope?

(b) How fast is the weight rising after the man has walked 10 feet?

(c) How fast is the weight rising after the man has walked 30 feet?

(d) How far must the man walk to raise the weight all the way to the pulley?

14. A hot air balloon lifts off from ground rising vertically. From 90 feet away, a 6 ft tall woman tracks the path of the balloon. When her sightline with the balloon makes a $45°$ angle with the horizontal, she notes the angle is increasing at about $3°$ per minute.

(a) What is the elevation of the balloon?

(b) How fast is it rising?

15. A company that produces landscaping materials is dumping sand into a conical pile. The sand is being poured at a rate of $5\ \frac{\text{ft}^3}{\text{s}}$. The physical properties of the sand, in conjunction with gravity, ensure that the cone's height is roughly $\frac{4}{7}$ the length of the diameter of the circular base.

How fast is the cone rising when it has a height of 30 feet?

4.3 Optimization

In Section 3.1 we learned about extreme values — the largest and smallest values a function attains on an interval. We motivated our interest in such values by discussing how it made sense to want to know the highest/lowest values of a stock, or the fastest/slowest an object was moving. In this section we apply the concepts of extreme values to solve "word problems," i.e., problems stated in terms of situations that require us to create the appropriate mathematical framework in which to solve the problem.

We start with a classic example which is followed by a discussion of the topic of optimization.

youtu.be/watch?v=nlLf3CcgbII

Figure 4.3.1 A simple optimization problem

Example 4.3.2 Optimization: perimeter and area.

A man has 100 feet of fencing, a large yard, and a small dog. He wants to create a rectangular enclosure for his dog with the fencing that provides the maximal area. What dimensions provide the maximal area?

Solution. One can likely guess the correct answer — that is great. We will proceed to show how calculus can provide this answer in a context that proves this answer is correct.

It helps to make a sketch of the situation. Our enclosure is sketched twice in Figure 4.3.3, either with treetop grass and nice fence boards or as a simple rectangle. Either way, drawing a rectangle forces us to realize that we need to know the dimensions of this rectangle so we can create an area function — after all, we are trying to maximize the area.

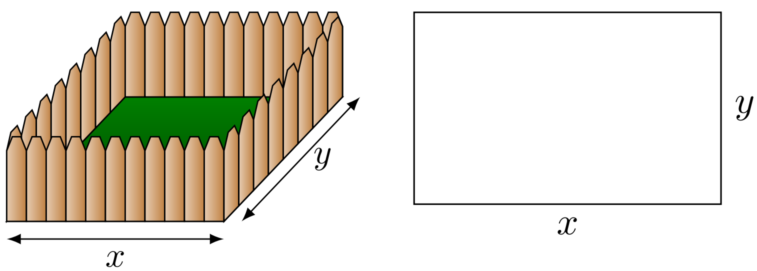


Figure 4.3.3 A sketch of the enclosure in Example 4.3.2.

We let x and y denote the lengths of the sides of the rectangle. Clearly,

$$\text{Area } = xy.$$

We do not yet know how to handle functions with two variables; we need to reduce this down to a single variable. We know more about the situation: the man has 100 feet of fencing. By knowing the perimeter of the rectangle must be 100, we can create another equation:

$$\text{Perimeter } = 100 = 2x + 2y.$$

We now have two equations and two unknowns. In the latter equation, we solve for y:

$$y = 50 - x.$$

Now substitute this expression for y in the area equation:

$$\text{Area } = A(x) = x(50 - x).$$

Note we now have an equation of one variable; we can truly call the Area a function of x.

This function only makes sense when $0 \leq x \leq 50$, otherwise we get negative values of area. So we find the extreme values of $A(x)$ on the interval $[0, 50]$ using Key Idea 3.1.18.
To find the critical points, we take the derivative of $A(x)$ and set it equal to 0, then solve for x.

$$\begin{aligned} A(x) &= x(50 - x) \\ &= 50x - x^2 \\ A'(x) &= 50 - 2x \end{aligned}$$

We solve $50 - 2x = 0$ to find $x = 25$; this is the only critical point. We evaluate $A(x)$ at the endpoints of our interval and at this critical point to find the extreme values; in this case, all we care about is the maximum. Clearly $A(0) = 0$ and $A(50) = 0$, whereas $A(25) = 625\text{ft}^2$. This is the maximum. Since we earlier found $y = 50 - x$, we find that y is also 25. Thus the dimensions of the rectangular enclosure with perimeter of 100 ft. with maximum area is a square, with sides of length 25 ft.

This example is very simplistic and a bit contrived. (After all, most people create a design then buy fencing to meet their needs, and not buy fencing and plan later.) But it models well the necessary process: create equations that describe a situation, reduce an equation to a single variable, then find the needed extreme value.

"In real life," problems are much more complex. The equations are often *not* reducible to a single variable (hence multi-variable calculus is needed) and the equations themselves may be difficult to form. Understanding the principles here will provide a good foundation for the mathematics you will likely encounter later.

We outline here the basic process of solving these optimization problems.

Key Idea 4.3.4 Solving Optimization Problems.

1. Understand the problem. Clearly identify what quantity is to be maximized or minimized. Make a sketch if helpful.

2. Create equations relevant to the context of the problem, using the information given. (One of these should describe the quantity to be optimized. We'll call this the *fundamental equation*.)

3. If the fundamental equation defines the quantity to be optimized as a function of more than one variable, reduce it to a single variable function using substitutions derived from the other equations (we'll call these **constraint** equations).

4. Identify the domain of this function, keeping in mind the context of the problem.

5. Find the extreme values of this function on the determined domain.

6. Identify the values of all relevant quantities of the problem.

We will use Key Idea 4.3.4 in a variety of examples.

Example 4.3.5 Optimization: perimeter and area.

Here is another classic calculus problem: A woman has a 100 feet of fencing, a small dog, and a large yard that contains a stream (that is mostly straight). She wants to create a rectangular enclosure with maximal area that uses the stream as one side. (Apparently her dog won't swim away.) What dimensions provide the maximal area?

Solution. We will follow the steps outlined by Key Idea 4.3.4.

1. We are maximizing *area*. A sketch of the region will help; Figure 4.3.6 gives two sketches of the proposed enclosed area. A key feature of the sketches is to acknowledge that one side is not fenced.

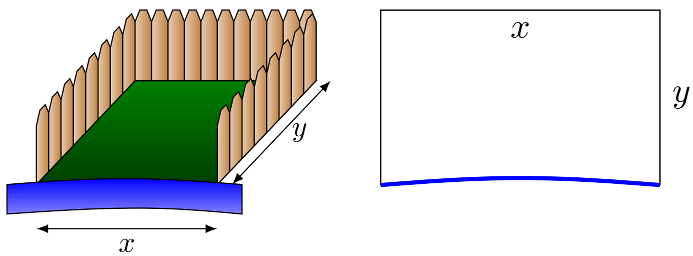


Figure 4.3.6 A sketch of the enclosure in Example 4.3.5

2. We want to maximize the area; as in the example before,

$$\text{Area } = xy.$$

This is our fundamental equation. This defines area as a function of two variables, so we need another equation to reduce it to one variable.

We again appeal to the perimeter; here the perimeter is

$$\text{Perimeter } = 100 = x + 2y.$$

The perimeter is our constraint equation. Note how this is a different equation for perimeter than in Example 4.3.2, since one of the sides does not need to be fenced.

3. We now reduce the fundamental equation to a single variable using our constraint equation. In the perimeter equation, solve for y: $y = 50 - x/2$. We can now write Area as

$$\begin{aligned}\text{Area } &= A(x) = x(50 - x/2)\\ &= 50x - \frac{1}{2}x^2.\end{aligned}$$

Area is now defined as a function of one variable.

4. We want the area to be non-negative. Since $A(x) = x(50 - x/2)$, we want $x \geq 0$ and $50 - x/2 \geq 0$. The latter inequality implies that $x \leq 100$, so $0 \leq x \leq 100$.

5. We now find the extreme values. At the endpoints, the minimum is found, giving an area of 0.

Find the critical points. We have $A'(x) = 50 - x$; setting this equal to 0 and solving for x returns $x = 50$. This gives an area of

$$A(50) = 50(25) = 1250.$$

6. We earlier set $y = 50 - x/2$; thus $y = 25$. Thus our rectangle will have two sides of length 25 and one side of length 50, with a total area of 1250 ft^2.

Video solution

youtu.be/watch?v=wIs5N5HOCrc

Keep in mind as we do these problems that we are practicing a *process*; that is, we are learning to turn a situation into a system of equations. These equations allow us to write a certain quantity as a function of one variable, which we then optimize.

Example 4.3.7 Optimization: minimizing cost.

A power line needs to be run from a power station located on the beach to an offshore facility. Figure 4.3.8 shows the distances between the power station to the facility.
It costs $\$50/$ ft to run a power line along the land, and $\$130/$ ft to run a power line under water. How much of the power line should be run along the land to minimize the overall cost? What is the minimal cost?

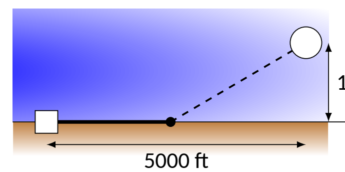


Figure 4.3.8 Running a power line from the power station to an offshore facility with minimal cost in Example 4.3.7

Solution. We will follow the strategy of Key Idea 4.3.4 implicitly, without specifically numbering steps.
There are two immediate solutions that we could consider, each of which we will reject through "common sense." First, we could minimize the distance by directly connecting the two locations with a straight line. However, this requires that all the wire be laid underwater, the most costly option. Second, we could minimize the underwater length by running a wire all 5000 ft along the beach, directly across from the offshore facility. This has the undesired effect of having the longest distance of all, probably ensuring a non-minimal cost.
The optimal solution likely has the line being run along the ground for a while, then underwater, as the figure implies. We need to label our unknown distances — the distance run along the ground and the distance run underwater. Recognizing that the underwater distance can be measured as the hypotenuse of a right triangle, we choose to label the distances as shown in Figure 4.3.9.
By choosing x as we did (instead of letting x be the distance along the land), we make the expression under the square root simple. We now create the cost function.

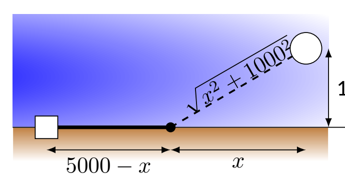


Figure 4.3.9 Labeling unknown distances in Example 4.3.7

$$\begin{aligned}\text{Cost} = \text{land cost} &\quad + \text{water cost}\\ \$50 \times \text{land distance} &+ \$130 \times \text{water distance}\\ 50(5000 - x) &\quad + 130\sqrt{x^2 + 1000^2}.\end{aligned}$$

So we have $c(x) = 50(5000 - x) + 130\sqrt{x^2 + 1000^2}$. This function only makes sense on the interval $[0, 5000]$. While we are fairly certain the endpoints will not give a minimal cost, we still evaluate $c(x)$ at each to verify.

$$c(0) = 380{,}000 \qquad\qquad c(5000) \approx 662{,}873.$$

(Notice that if $x = 0$, the line is run the full 5000 ft along land and a full 1000 ft under water. If $x = 5000$, the line is run the maximum distance underwater.)

We now find the critical values of $c(x)$. We compute $c'(x)$ as

$$c'(x) = -50 + \frac{130x}{\sqrt{x^2 + 1000^2}}.$$

Recognize that this is never undefined. Setting $c'(x) = 0$ and solving for x, we have:

$$\begin{aligned}
-50 + \frac{130x}{\sqrt{x^2 + 1000^2}} &= 0\\
\frac{130x}{\sqrt{x^2 + 1000^2}} &= 50\\
\frac{130^2x^2}{x^2 + 1000^2} &= 50^2\\
130^2x^2 &= 50^2(x^2 + 1000^2)\\
130^2x^2 - 50^2x^2 &= 50^2 \cdot 1000^2\\
(130^2 - 50^2)x^2 &= 50,000^2\\
x^2 &= \frac{50,000^2}{130^2 - 50^2}\\
x &= \frac{50,000}{\sqrt{130^2 - 50^2}}\\
x &= \frac{50,000}{120} = \frac{1250}{3} \approx 416.67.
\end{aligned}$$

Evaluating $c(x)$ at $x = 416.67$ gives a minimal cost of about $\$370{,}000$. The distance the power line is laid along land is $5000 - 416.67 = 4583.33$ ft., and the underwater distance is $\sqrt{416.67^2 + 1000^2} \approx 1083$ ft.

Video solution

youtu.be/watch?v=qDK9rqIoKRs

In the exercises you will see a variety of situations that require you to combine problem-solving skills with calculus. Focus on the *process*; learn how to form equations from situations that can be manipulated into what you need. Eschew memorizing how to do “this kind of problem” as opposed to “that kind of problem.” Learning a process will benefit one far longer than memorizing a specific technique.

Before you begin the exercises, here is one more example, presented in video form in Figure 4.3.10.

Section 4.4 introduces our final application of the derivative: *differentials*. Given $y = f(x)$, they offer a method of approximating the change in y after x changes by a small amount.

youtu.be/watch?v=XJYDMZe8JUk

Figure 4.3.10 Optimizing construction of a box with no top

4.3.1 Exercises

Terms and Concepts

1. (☐ True ☐ False) An “optimization problem” is essentially an “extreme values” problem in a “story problem” setting.

2. (☐ True ☐ False) This section teaches one to find the extreme values of a function that has more than one variable.

Problems

3. Find the maximum product of two numbers (not necessarily integers) that have a sum of 150.

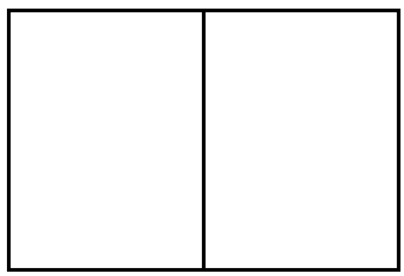
4. Find the minimum sum of two positive numbers whose product is 560.

5. Find the maximum sum of two positive numbers whose product is 580.

6. Find the maximum sum of two numbers, each of which is less than or equal to 290, whose product is 400.

7. Find the maximal area of a right triangle with hypotenuse of length 2.

8. A rancher has 900 feet of fencing in which to construct adjacent, equally sized rectangular pens. What dimensions should these pens have to maximize the enclosed area?



9. A standard soda can is roughly cylindrical and holds 355 cm^3 of liquid. What dimensions should the cylinder have to minimize the material needed to produce the can? Based on your dimensions, determine whether or not the standard can is produced to minimize the material costs.

10. Find the dimensions of a cylindrical can with a volume of 206 in^3 that minimizes the surface area.

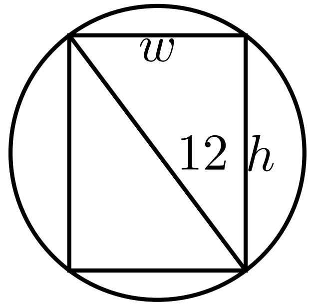
The “#10 can”is a standard sized can used by the restaurant industry that holds about 206 in^3 with a diameter of $6\,\frac{3}{16}$ in and height of 7 in. Does it seem these dimensions where chosen with minimization in mind?

11. A standard soda can is roughly cylindrical and holds 355 cm^3 of liquid. A real-world soda can has material on the top and bottom that is thicker than the material around the side. Assume that the top/bottom material is twice as thick as the material around the side. What dimensions should the cylinder have to minimize the material needed to produce the can? Based on your dimensions and the assumption about material thickness, determine whether or not the standard can is produced to minimize the material costs.

12. The United States Postal Service charges more for boxes whose combined length and girth exceeds 108 inches. (The “length” of a package is the length of its longest side; the girth is the perimeter of the cross section, i.e., $2w + 2h$).

What is the maximum volume of a package with a square cross section ($w = h$) that does not exceed the 108 inch standard?

13. The strength S of a wooden beam is directly proportional to its cross sectional width w and the square of its height h. that is, $S = kwh^2$ for some constant k.



Given a circular log with diameter of 18 inches, what sized beam can be cut from the log with maximum strength?

14. A power line is to be run to an offshore facility in the manner described in Example 4.3.7. The offshore facility is 6 miles at sea and 4 miles along the shoreline from the power plant. It costs $\$35{,}000$ per mile to lay a power line underground and $\$70{,}000$ to run the line underwater.

How much of the power line should be run underground? What is the minimum overall cost?

15. A power line is to be run to an offshore facility in the manner described in Example 4.3.7. The offshore facility is 6 miles at sea and 2 miles along the shoreline from the power plant. It costs $\$45{,}000$ per mile to lay a power line underground and $\$75{,}000$ to run the line underwater.

How much of the power line should be run underground? What is the minimum overall cost?

16. A woman throws a stick into a lake for her dog to fetch; the stick is 35 feet down the shore line and 13 feet into the water from there. The dog may jump directly into the water and swim, or run along the shore line to get closer to the stick before swimming. The dog runs about $19\ \frac{\text{ft}}{\text{s}}$ and swims about $2\ \frac{\text{ft}}{\text{s}}$.

How far along the shore should the dog run to minimize the time it takes to get to the stick? (Hint: the figure from Example 4.3.7 can be useful.)

17. A woman throws a stick into a lake for her dog to fetch; the stick is 25 feet down the shore line and 16 feet into the water from there. The dog may jump directly into the water and swim, or run along the shore line to get closer to the stick before swimming. The dog runs about $22\ \frac{\text{ft}}{\text{s}}$ and swims about $1.7\ \frac{\text{ft}}{\text{s}}$.

How far along the shore should the dog run to minimize the time it takes to get to the stick? (Google “calculus dog” to learn more about a dog’s ability to minimize times.)

18. What are the dimensions of the rectangle with largest area that can be drawn inside the unit circle?

4.4 Differentials

In Section 2.2 we explored the meaning and use of the derivative. This section starts by revisiting some of those ideas.

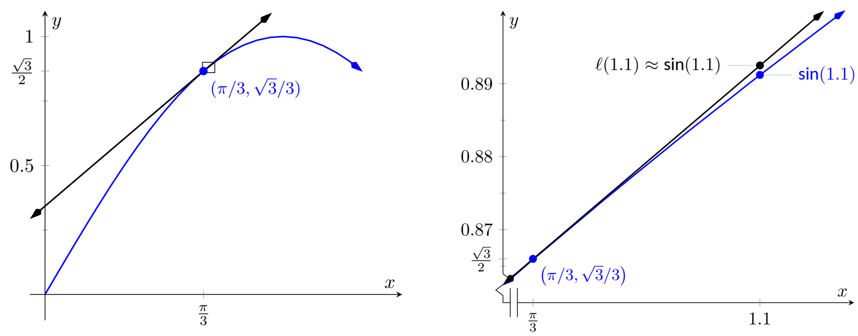
Recall that the derivative of a function f can be used to find the slopes of lines tangent to the graph of f. At $x = c$, the tangent line to the graph of f has equation

$$y = f'(c)(x - c) + f(c).$$

The tangent line can be used to find good approximations of $f(x)$ for values of x near c.

For instance, we can approximate $\sin(1.1)$ using the tangent line to the graph of $f(x) = \sin(x)$ at $x = \pi/3 \approx 1.05$. Recall that $\sin(\pi/3) = \sqrt{3}/2 \approx 0.866$, and $f'(\pi/3) = \cos(\pi/3) = 1/2$. Thus the tangent line to $f(x) = \sin(x)$ at $x = \pi/3$ is:

$$\ell(x) = \frac{1}{2}(x - \pi/3) + 0.866.$$



(a) **(b)**

Figure 4.4.2 Graphing $f(x) = \sin(x)$ and its tangent line at $x = \pi/3$ in order to estimate $\sin(1.1)$

In Figure 4.4.2(a), we see a graph of $f(x) = \sin(x)$ graphed along with its tangent line at $x = \pi/3$. The small rectangle shows the region that is displayed in Figure 4.4.2(b). In this figure, we see how we are approximating $\sin(1.1)$ with the tangent line, evaluated at 1.1. Together, the two figures show how close these values are.

Using this line to approximate $\sin(1.1)$, we have:

$$\begin{aligned}\ell(1.1) &= \frac{1}{2}(1.1 - \pi/3) + 0.866\\ &= \frac{1}{2}(0.053) + 0.866 = 0.8925.\end{aligned}$$

(We leave it to the reader to see how good of an approximation this is.)

We now generalize this concept. Given $f(x)$ and an x-value c, the tangent line is $y = \ell(x)$, where $\ell(x) = f'(c)(x - c) + f(c)$. Clearly, $f(c) = \ell(c)$. Let Δx be a small number, representing a small change in the x-value. We assert that:

$$f(c + \Delta x) \approx \ell(c + \Delta x),$$

since the tangent line to a function approximates well the values of that function near $x = c$. This tangent line approximation is used frequently enough in applications that we give it a name.

youtu.be/watch?v=YmODT2PolKY

Figure 4.4.1 Video introduction to Section 4.4

youtu.be/watch?v=mQRelmurD-w

Figure 4.4.3 Approximating the value of $\sin(1.1)$

Definition 4.4.4

The function $\ell(x)$ is often referred to as the **linearization**, or **linear approximation** of f at c. It is the linear function that best approximates the value of $f(x)$ when x is close to c.

As the x-value changes from c to $c+\Delta x$, the y-value of f changes from $f(c)$ to $f(c+\Delta x)$. We call this change of y-value Δy. That is:

$$\Delta y = f(c+\Delta x) - f(c).$$

Replacing $f(c+\Delta x)$ with its tangent line approximation, we have

$$\begin{aligned}\Delta y &\approx \ell(c+\Delta x) - f(c)\\ &= f'(c)\big((c+\Delta x)-c\big) + f(c) - f(c)\\ &= f'(c)\Delta x.\end{aligned} \tag{4.4.1}$$

This final equation is important; it becomes the basis of Definition 4.4.5 and Key Idea 4.4.7. In short, it says that when the x-value changes from c to $c+\Delta x$, the y value of a function f changes by about $f'(c)\Delta x$.

We introduce two new variables, dx and dy in the context of a formal definition.

youtu.be/watch?v=y9lTgdHD8wI

Figure 4.4.6 Video presentation of Definition 4.4.5

Definition 4.4.5 Differentials of x and y.

Let $y = f(x)$ be differentiable. The **differential** of x, denoted dx, is any nonzero real number (usually taken to be a small number). The **differential** of y, denoted dy, is

$$dy = f'(x)dx.$$

We can solve for $f'(x)$ in the above equation: $f'(x) = dy/dx$. This states that the derivative of f with respect to x is the differential of y divided by the differential of x; this is *not* the alternate notation for the derivative, $\frac{dy}{dx}$. This latter notation was chosen because of the fraction-like qualities of the derivative, but again, it is one symbol and not a fraction.

It is helpful to organize our new concepts and notations in one place.

Differentials and linearization. The relationship between the differential and the linearization given in Definition 4.4.4 is as follows:

$$\ell(x) = f(c) + dy,$$

if we take dy to be evaluated at $x = c$.

It is often useful to think of dy is the *linear change* in f, while Δy represents the true change in f.

Key Idea 4.4.7 Differential Notation.

Let $y = f(x)$ be a differentiable function.

1. Let Δx represent a small, nonzero change in x value.
2. Let dx represent a small, nonzero change in x value (i.e., $\Delta x = dx$).
3. Let Δy be the change in y value as x changes by Δx; hence

   $$\Delta y = f(x+\Delta x) - f(x).$$

4. Let $dy = f'(x)dx$ which, by Equation (4.4.1), is an *approximation* of the change in y-value as x changes by Δx; $dy \approx \Delta y$.

When students first encounter differentials, they are often left wondering why dy and Δy are different, while dx and Δx are the same. The video in Figure 4.4.8 attempts to offer an explanation.

youtu.be/watch?v=XxpcZw702nA

Figure 4.4.8 Why is it that $dx = \Delta x$?

What is the value of differentials? Like many mathematical concepts, differentials provide both practical and theoretical benefits. We explore both here.

Example 4.4.9 Finding and using differentials.

Consider $f(x) = x^2$. Knowing $f(3) = 9$, approximate $f(3.1)$.

Solution. The x-value is changing from $x = 3$ to $x = 3.1$; therefore, we see that $dx = 0.1$. If we know how much the y-value changes from $f(3)$ to $f(3.1)$ (i.e., if we know Δy), we will know exactly what $f(3.1)$ is (since we already know $f(3)$). We can approximate Δy with dy.

$$\begin{aligned}\Delta y &\approx dy\\ &= f'(3)dx\\ &= 2\cdot 3\cdot 0.1 = 0.6.\end{aligned}$$

We expect the y-value to change by about 0.6, so we approximate $f(3.1) \approx 9.6$.

We leave it to the reader to verify this, but the preceding discussion links the differential to the tangent line of $f(x)$ at $x = 3$. One can verify that the tangent line, evaluated at $x = 3.1$, also gives $y = 9.6$.

Video solution

youtu.be/watch?v=KCDezzvfDKA

Of course, it is easy to compute the actual answer (by hand or with a calculator): $3.1^2 = 9.61$. (Before we get too cynical and say "Then why bother?", note our approximation is *really* good!)

So why bother?

In "most" real life situations, we do not know the function that describes a particular behavior. Instead, we can only take measurements of how things change — measurements of the derivative.

Imagine water flowing down a winding channel. It is easy to measure the speed and direction (i.e., the *velocity*) of water at any location. It is very hard to create a function that describes the overall flow, hence it is hard to predict where a floating object placed at the beginning of the channel will end up. However, we can *approximate* the path of an object using differentials. Over small intervals, the path taken by a floating object is essentially linear. Differentials allow us to approximate the true path by piecing together lots of short, linear paths. This technique is called Euler's Method, studied in introductory Differential Equations courses.

We use differentials once more to approximate the value of a function. Even though calculators are very accessible, it is neat to see how these techniques can sometimes be used to easily compute something that looks rather hard.

PID controllers. Another place differentials are used is in a PID controller, which stands for "Proportional Integral Derivative". A PID controller uses concepts of both derivative and integral calculus to very accurately control a process (such as maintaining a stable temperature on an espresso machine).

Example 4.4.10 Using differentials to approximate a function value.

Approximate $\sqrt{4.5}$.

Solution. We expect $\sqrt{4.5} \approx 2$, yet we can do better. Let $f(x) = \sqrt{x}$, and let $c = 4$. Thus $f(4) = 2$. We can compute $f'(x) = 1/(2\sqrt{x})$, so $f'(4) = 1/4$.

We approximate the difference between $f(4.5)$ and $f(4)$ using differentials, with $dx = 0.5$:

$$\begin{aligned}f(4.5) - f(4) = \Delta y &\approx dy\\ &= f'(4)\cdot dx\\ &= 1/4\cdot 1/2\end{aligned}$$

$$= 1/8$$
$$= 0.125.$$

The approximate change in f from $x = 4$ to $x = 4.5$ is 0.125, so we approximate $\sqrt{4.5} \approx 2.125$.

Video solution

youtu.be/watch?v=nFaq1O_wWso

Differentials are important when we discuss *integration*. When we study that topic, we will use notation such as

$$\int f(x)\,dx$$

quite often. While we don't discuss here what all of that notation means, note the existence of the differential dx. Proper handling of *integrals* comes with proper handling of differentials.

In light of that, we practice finding differentials in general.

Example 4.4.11 Finding differentials.

In each of the following, find the differential dy.

1. $y = \sin(x)$
2. $y = e^x\left(x^2+2\right)$
3. $y = \sqrt{x^2+3x-1}$

Solution.

1. $y = \sin(x)$: As $f(x) = \sin(x)$, $f'(x) = \cos(x)$. Thus
$$dy = \cos(x)dx.$$
2. $y = e^x\left(x^2+2\right)$: Let $f(x) = e^x\left(x^2+2\right)$. We need $f'(x)$, requiring the Theorem 2.4.2.
We have $f'(x) = e^x\left(x^2+2\right) + 2xe^x$, so
$$dy = \left(e^x\left(x^2+2\right) + 2xe^x\right)dx.$$
3. $y = \sqrt{x^2+3x-1}$: Let $f(x) = \sqrt{x^2+3x-1}$; we need $f'(x)$, requiring the Theorem 2.5.4.
We have $f'(x) = \frac{1}{2}\left(x^2+3x-1\right)^{-\frac{1}{2}}(2x+3) = \frac{2x+3}{2\sqrt{x^2+3x-1}}$. Thus
$$dy = \frac{(2x+3)dx}{2\sqrt{x^2+3x-1}}.$$

Finding the differential dy of $y = f(x)$ is really no harder than finding the derivative of f; we just *multiply* $f'(x)$ by dx. It is important to remember that we are not simply adding the symbol "dx" at the end.

We have seen a practical use of differentials as they offer a good method of making certain approximations. Another use is *error propagation*. Suppose a length is measured to be x, although the actual value is $x+\Delta x$ (where Δx is the error, which we hope is small). This measurement of x may be used to compute some other value; we can think of this latter value as $f(x)$ for some function f. As the true length is $x+\Delta x$, one really should have computed $f(x+\Delta x)$. The difference between $f(x)$ and $f(x+\Delta x)$ is the propagated error.

How close are $f(x)$ and $f(x+\Delta x)$? This is a difference in "y" values:

$$f(x+\Delta x) - f(x) = \Delta y \approx dy.$$

We can approximate the propagated error using differentials.

Example 4.4.12 Using differentials to approximate propagated error.

A steel ball bearing is to be manufactured with a diameter of $2\,\text{cm}$. The manufacturing process has a tolerance of ± 0.1mm in the diameter. Given that the density of steel is about $7.85\,\frac{\text{g}}{\text{cm}^3}$, estimate the propagated error in the mass of the ball bearing.

Solution. The mass of a ball bearing is found using the equation "mass = volume × density." In this situation the mass function is a product of the radius of the ball bearing, hence it is $m = 7.85\frac{4}{3}\pi r^3$. The differential of the mass is

$$dm = 31.4\pi r^2 dr.$$

The radius is to be $1\,\text{cm}$; the manufacturing tolerance in the radius is ± 0.05mm, or ± 0.005cm. The propagated error is approximately:

$$\begin{aligned}\Delta m &\approx dm\\ &= 31.4\pi(1)^2(\pm 0.005)\\ &= \pm 0.493\text{g}\end{aligned}$$

Is this error significant? It certainly depends on the application, but we can get an idea by computing the *relative error*. The ratio between amount of error to the total mass is

$$\begin{aligned}\frac{dm}{m} &= \pm\frac{0.493}{7.85\frac{4}{3}\pi}\\ &= \pm\frac{0.493}{32.88}\\ &= \pm 0.015,\end{aligned}$$

or ± 1.5%.

We leave it to the reader to confirm this, but if the diameter of the ball was supposed to be $10\,\text{cm}$, the same manufacturing tolerance would give a propagated error in mass of ± 12.33g, which corresponds to a *percent error* of ± 0.188%. While the amount of error is much greater $(12.33 > 0.493)$, the percent error is much lower.

Video solution

youtu.be/watch?v=0_tSaBZZR1s

4.4.1 Exercises

Terms and Concepts

1. (☐ True ☐ False) Given a differentiable function $y = f(x)$, we are generally free to choose a value for dx, which then determines the value of dy.

2. (☐ True ☐ False) The symbols "dx" and "Δx" represent the same concept.

3. (☐ True ☐ False) The symbols "dy" and "Δy" represent the same concept.

4. (☐ True ☐ False) Differentials are important in the study of integration.

5. How are differentials and tangent lines related?

6. (☐ True ☐ False) In real life, differentials are used to approximate function values when the function itself is not known.

Problems

Exercise Group. Use differentials to approximate the given value by hand.

7. 2.07^2

8. 2.95^2

9. 4.4^3

10. 4.7^3

11. $\sqrt{25.5}$

12. $\sqrt{34.6}$

13. $\sqrt[3]{124}$

14. $\sqrt[3]{216.6}$

15. $\sin(3)$

16. $e^{0.1}$

Exercise Group. Compute the differential dy.

17. $y = x^2 - 5x - 6$

18. $y = x^5 + x^9$

19. $y = \dfrac{1}{4x^6}$

20. $y = (6x + \sin(x))^2$

21. $y = x^7 + e^{8x}$

22. $y = \dfrac{8}{x^5}$

23. $y = \dfrac{9x}{\tan(x) + 2}$

24. $y = \ln(9x)$

25. $y = e^x \sin(x)$

26. $y = \cos(\sin(x))$

27. $y = \dfrac{x-4}{x+5}$

28. $y = 5^x \ln(x)$

29. $y = x \tan^{-1}(x) - 0.5 \ln\left(1 + x^2\right)$

30. $y = \ln(\sin(x))$

31. A set of plastic spheres are to be made with a diameter of 4 cm. If the manufacturing process is accurate to 2 mm, what is the propagated error in volume of the spheres?

32. The distance, in feet, a stone drops in t seconds is given by $d(t) = 16t^2$. The depth of a hole is to be approximated by dropping a rock and listening for it to hit the bottom. What is the propagated error if the time measurement is accurate to $4/10$ of a second and the measured time is:

(a) 4 seconds?

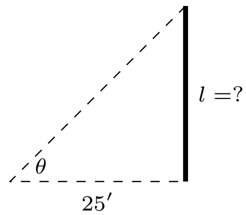
(b) 6 seconds?

33. What is the propagated error in the measurement of the cross sectional area of a circular log if the diameter is measured at $20''$, accurate to $1/8''$?

34. A wall is to be painted that is $8'$ high and is measured to be $13'$, $2''$ long. Find the propagated error in the measurement of the wall's surface area if the measurement is accurate to $1/-2''$.

Exercise Group. The following exercises explore some issues related to surveying in which distances are approximated using other measured distances and measured angles. *(Hint: Convert all angles to radians before computing.)*

35. The length L of a long wall is to be approximated. The angle θ, as shown in the diagram (not to scale), is measured at a distance of 25 feet from the wall, and found to be $85.2°$, accurate to $1°$. Assume that the triangle formed is a right triangle.

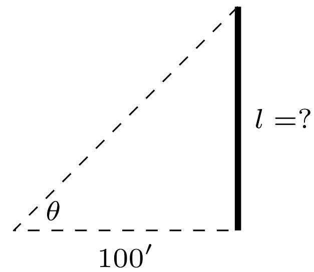


(a) What is the measured length L of the wall?

(b) What is the propagated error?

(c) What is the percent error?

36. The length L of a long wall is to be approximated. The angle θ, as shown in the diagram (not to scale), is measured at a distance of 100 feet from the wall, and found to be $71.5°$, accurate to $1°$. Assume that the triangle formed is a right triangle.

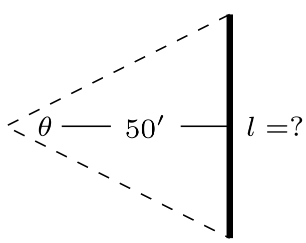


(a) What is the measured length L of the wall?

(b) What is the propagated error?

(c) What is the percent error?

37. The length L of a long wall is to be calculated by measuring the angle θ shown in the diagram (not to scale) at a distance of 50 feet from the wall. Assume the formed triangle is an isosceles triangle. The measured angle is $143°$, accurate to $1°$.



(a) What is the measured length L of the wall?

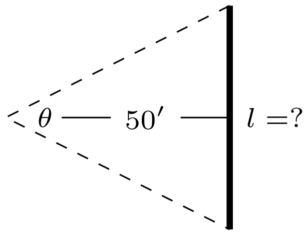
(b) What is the propagated error?

(c) What is the percent error?

38. The length of the walls in Exercise 4.4.35–4.4.37 are essentially the same. Which setup gives the most accurate result?

- Right triangle at 25 feet
- Right triangle at 100 feet
- Isosceles triangle at 50 feet

39. Consider the setup in Exercise 4.4.37. This time, assume the angle measurement of $143°$ is exact but the measured $50'$ from the wall is accurate to $6''$.



What is the approximate percent error?

4.5 Taylor Polynomials

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Figure 4.5.1 Video introduction to Section 4.5

Consider a function $y = f(x)$ and a point $\big(c, f(c)\big)$. The derivative, $f'(c)$, gives the instantaneous rate of change of f at $x = c$. Of all lines that pass through the point $\big(c, f(c)\big)$, the line that best approximates f at this point is the tangent line; that is, the line whose slope (rate of change) is $f'(c)$.

In Figure 4.5.2, we see a function $y = f(x)$ graphed. The table in Figure 4.5.3 shows that $f(0) = 2$ and $f'(0) = 1$; therefore, the tangent line to f at $x = 0$ is $p_1(x) = 1(x - 0) + 2 = x + 2$. The tangent line is also given in the figure. Note that "near" $x = 0$, $p_1(x) \approx f(x)$; that is, the tangent line approximates f well.

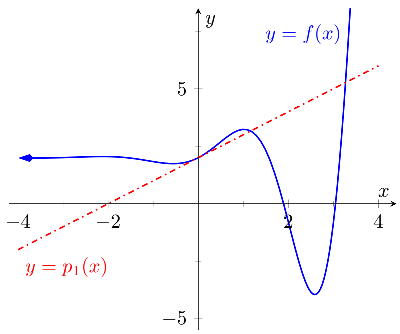

Figure 4.5.2 A graph of $f(x)$ and its tangent line at 0

$f(0) = 2$	$f'''(0) = -1$
$f'(0) = 1$	$f^{(4)}(0) = -12$
$f''(0) = 2$	$f^{(5)}(0) = -19$

Figure 4.5.3 Derivatives of f evaluated at 0

One shortcoming of this approximation is that the tangent line only matches the slope of f; it does not, for instance, match the concavity of f. We can find a polynomial, $p_2(x)$, that does match the concavity near 0 without much difficulty, though. The table in Figure 4.5.3 gives the following information:

$$f(0) = 2 \qquad f'(0) = 1 \qquad f''(0) = 2.$$

Therefore, we want our polynomial $p_2(x)$ to have these same properties. That is, we need

$$p_2(0) = 2 \qquad p_2'(0) = 1 \qquad p_2''(0) = 2.$$

Let's start with a general quadratic function

$$p(x) = a_0 + a_1x + a_2x^2.$$

We find the following:

$$\begin{aligned} p_2(x) &= a_0 + a_1x + a_2x^2 & p_2(0) &= a_0 \\ p_2{}'(x) &= a_1 + 2a_2x & p_2{}'(0) &= a_1 \\ p_2{}''(x) &= 2a_2 & p_2{}''(0) &= 2a_2. \end{aligned}$$

To get the desired properties above, we must have

$$a_0 = f(0) = 2,\ a_1 = f'(0) = 1,\ 2a_2 = f''(0) = 2,$$

so $a_0 = 2$, $a_1 = 1$, and $a_2 = 2/2 = 1$, giving us the polynomial

$$p_2(x) = 2 + x + x^2.$$

We can repeat this approximation process by creating polynomials of higher degree that match more of the derivatives of f at $x = 0$. In general, a polynomial of degree n can be created to match the first n derivatives of f. Figure 4.5.4

shows $p_4(x) = -x^4/2 - x^3/6 + x^2 + x + 2$, whose first four derivatives at 0 match those of f.

How do we ensure that the derivatives of our polynomial match those of f? We simply begin with a polynomial of the desired degree, compute its derivatives, and compare them to those of f! Recall that each term in a polynomial consists of a power of x, and a coefficient, like so: $a_n x^n$. Our goal is to determine the value for each coefficient a_n so that the derivatives of our polynomial match those of our function f. If we take k derivatives of the term $a_n x^n$, with $k \le n$, we obtain

$$\frac{d^k}{dx^k}(a_n x^n) = n(n-1)\cdots(n-k+1)a_n x^{n-k}.$$

For $k < n$, the expression above vanishes when we set $x = 0$. However, for $n = k$, we obtain the constant value

$$\frac{d^k}{dx^k}(a_k x^k) = k \cdot (k-1) \cdots 2 \cdot 1 a_k. \qquad (4.5.1)$$

Consider a polynomial

$$p_n(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots + a_n x^n$$

of degree n. If we take k derivatives, all of the terms involving powers of x less than k disappear, and when we set $x = 0$, all of the terms involving powers of x larger than k disappear, leaving us with the single constant given in Equation (4.5.1).

Recalling the notation $k! = 1 \cdot 2 \cdot 3 \cdots k$ for the product of the first k integers, we have shown that

$$p_n^{(k)}(0) = k! a_k.$$

If we want the derivatives of p_n to agree with some unknown function f when $x = 0$, then we must have

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

As we use more and more derivatives, our polynomial approximation to f gets better and better. In this example, the interval on which the approximation is "good" gets bigger and bigger. Figure 4.5.6 shows $p_{13}(x)$; we can visually affirm that this polynomial approximates f very well on $[-2, 3]$. (The polynomial $p_{13}(x)$ is not particularly "nice". It is

$$p_{13}(x) = \frac{16901x^{13}}{6227020800} + \frac{13x^{12}}{1209600} - \frac{1321x^{11}}{39916800} - \frac{779x^{10}}{1814400} - \frac{359x^9}{362880} + \frac{x^8}{240} + \frac{139x^7}{5040} + \frac{11x^6}{360} - \frac{19x^5}{120} - \frac{x^4}{2} - \frac{x^3}{6} + x^2 + x + 2.$$

The polynomials we have created are examples of *Taylor polynomials*, named after the British mathematician Brook Taylor who made important discoveries about such functions. In the discussion above, we concentrated on evaluating the derivatives of f at 0; however, there is nothing special about this point. Just as we can consider the linear approximation of a function near any point, so too can we determine a polynomial approximation about any value c in the domain of f. The only catch is that our polynomial will then be given in terms of powers of $x - c$, rather than powers of x, as we see in the following definition.

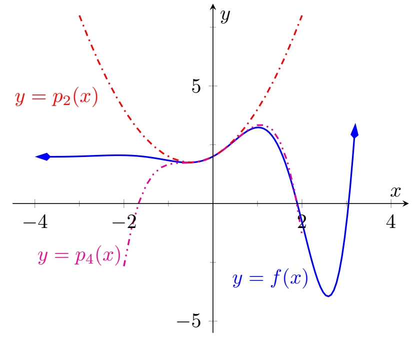

Figure 4.5.4 Plotting f, p_2 and p_4

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Figure 4.5.5 Determining the coefficients of a Taylor polynomial

The notation $k!$ is read as "k factorial". By convention, we also define $0! = 1$, mostly because it makes our formulas look a lot nicer.

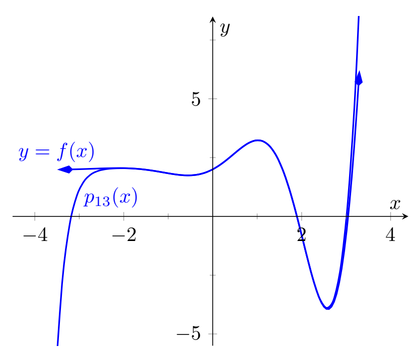

Figure 4.5.6 Plotting f and p_{13}

Definition 4.5.7 Taylor Polynomial, Maclaurin Polynomial.

Let f be a function whose first n derivatives exist at $x = c$.

1. The *Taylor polynomial of degree* n *of* f *at* $x = c$ is

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

2. A special case of the Taylor polynomial is the Maclaurin polynomial, where $c = 0$. That is, the *Maclaurin polynomial of degree* n *of* f is

$$p_n(x) = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots+\frac{f^{(n)}(0)}{n!}x^n.$$

We will practice creating Taylor and Maclaurin polynomials in the following examples.

Historical note: Colin Maclaurin was a Scottish mathematician, born in 1698. He lived until 1746, and made a number of contributions to the development of mathematics and physics. His election as professor of mathematics at the University of Aberdeen at the age of 19 made him the world's youngest professor, a record he held until 2008! He was also a staunch foe of the Jacobite Rebellion, and was instrumental in the defence of Edinburgh against the army of Bonnie Prince Charlie. (For more details, see Wikipedia[1].)

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Figure 4.5.8 Video presentation of Definition 4.5.7

Example 4.5.9 Finding and using Maclaurin polynomials.

1. Find the nth Maclaurin polynomial for $f(x) = e^x$.
2. Use $p_5(x)$ to approximate the value of e.

Solution.

1. We start with creating a table of the derivatives of e^x evaluated at $x = 0$. In this particular case, this is relatively simple, as shown in Figure 4.5.10.

   By the definition of the Maclaurin polynomial, we have

$$\begin{aligned} p_n(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\ &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots + \frac{1}{n!}x^n. \end{aligned}$$

2. Using our answer from part 1, we have

$$e^x \approx p_5(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5.$$

   To approximate the value of e, note that $e = e^1 = f(1) \approx p_5(1)$. It is very straightforward to evaluate $p_5(1)$:

$$p_5(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{163}{60} \approx 2.71667.$$

   A plot of $f(x) = e^x$ and $p_5(x)$ is given in Figure 4.5.11. To 5 decimal places, the actual value of e is 2.71828. So this approximation agrees to two decimal places.

$$\begin{array}{ll} f(x) = e^x & f(0) = 1 \\ f'(x) = e^x & f'(0) = 1 \\ f''(x) = e^x & f''(0) = 1 \\ \vdots & \vdots \\ f^{(n)}(x) = e^x & f^{(n)}(0) = 1 \end{array}$$

Figure 4.5.10 The derivatives of $f(x) = e^x$ evaluated at $x = 0$

Video solution

youtu.be/watch?v=ENf-Z2pLrJg

Example 4.5.12 Finding and using Taylor polynomials.

1. Find the nth Taylor polynomial of $y = \ln(x)$ at $x = 1$.
2. Use $p_6(x)$ to approximate the value of $\ln(1.5)$.
3. Use $p_6(x)$ to approximate the value of $\ln(2)$.

Solution.

1. We begin by creating a table of derivatives of $\ln(x)$ evaluated at $x = 1$. While this is not as straightforward as it was in the previous example, a pattern does emerge (for $n \geq 1$), as shown in Figure 4.5.13. Notice in the table below that each time we take a derivative (starting at the second derivative), we apply the power rule and "bring down" the exponent to multiply by the previous coefficent. So the 6 in the 4^{th} derivative is actually $1 \cdot 2 \cdot 3 = 3!$.

   Notice that the coefficients alternate in sign starting at $n = 1$. Using Definition 4.5.7, we have

$$\begin{aligned} p_n(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots \\ &\dots \frac{f'''(c)}{3!}(x-c)^3 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &= 0 + \frac{0!}{1!}(x-1) - \frac{1!}{2!}(x-1)^2 + \dots \\ &\dots \frac{2!}{3!}(x-1)^3 + \dots + \frac{(-1)^{n+1}\cdot(n-1)!}{n!}(x-1)^n \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots \\ &\dots \frac{1}{4}(x-1)^4 + \dots + \frac{(-1)^{n+1}}{n}(x-1)^n. \end{aligned}$$

   Note how the coefficients of the $(x-1)$ terms turn out to be "nice."

2. We can compute $p_6(x)$ using our work above:

$$\begin{aligned} p_6(x) = (x-1) &- \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \\ &- \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6. \end{aligned}$$

   Since $p_6(x)$ approximates $\ln(x)$ well near $x = 1$, we approximate $\ln(1.5) \approx p_6(1.5)$:

$$\begin{aligned} p_6(1.5) &= (1.5-1) - \frac{1}{2}(1.5-1)^2 + \frac{1}{3}(1.5-1)^3 + \dots \\ &\dots - \frac{1}{4}(1.5-1)^4 + \frac{1}{5}(1.5-1)^5 - \frac{1}{6}(1.5-1)^6 \\ &= \frac{259}{640} \\ &\approx 0.404688. \end{aligned}$$

   This is a good approximation as a calculator shows that $\ln(1.5) \approx 0.4055$. Figure 4.5.14 below plots $y = \ln(x)$ with $y = p_6(x)$. We can see that $\ln(1.5) \approx p_6(1.5)$.

$f(x) = \ln(x)$	$f(1) = 0$
$f'(x) = \frac{1}{x}$	$f'(1) = 1$
$f''(x) = -\frac{1}{x^2}$	$f''(1) = -1$
$f'''(x) = \frac{2}{x^3}$	$f'''(1) = 2$
$f^{(4)}(x) = -\frac{6}{x^4}$	$f^{(4)}(1) = -6$
$\vdots$	$\vdots$
$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$	$f^{(n)}(1) = (-1)^{n+1}(n-1)!$

Figure 4.5.13 Derivatives of $\ln(x)$ evaluated at $x = 1$

3. We approximate $\ln 2$ with $p_6(2)$:

$$\begin{aligned} p_6(2) &= (2-1) - \frac{1}{2}(2-1)^2 + \frac{1}{3}(2-1)^3 - \frac{1}{4}(2-1)^4 + \cdots \\ &\quad \cdots + \frac{1}{5}(2-1)^5 - \frac{1}{6}(2-1)^6 \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \\ &= \frac{37}{60} \\ &\approx 0.616667. \end{aligned}$$

This approximation is not terribly impressive: a hand held calculator shows that $\ln(2) \approx 0.693147$. The graph in Figure 4.5.14 shows that $p_6(x)$ provides less accurate approximations of $\ln(x)$ as x gets close to 0 or 2. Surprisingly enough, even the 20th degree Taylor polynomial fails to approximate $\ln(x)$ for $x > 2$ very well, as shown in Figure 4.5.15. We'll soon discuss why this is.

Video solution

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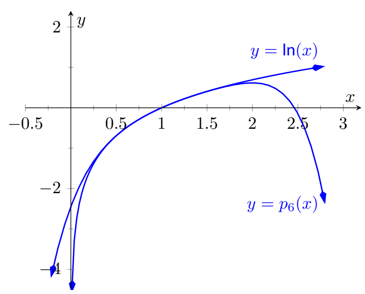

Figure 4.5.14 A plot of $y = \ln(x)$ and its 6th degree Taylor polynomial at $x = 1$

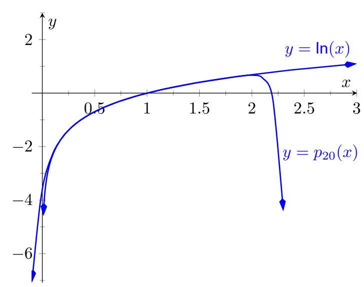

Figure 4.5.15 A plot of $y = \ln(x)$ and its 20^{th} degree Taylor polynomial at $x = 1$

As always in calculus, angles are measured in radians, so the 2 in $\cos(2)$ is an angle of 2 radians.

Taylor polynomials are used to approximate functions $f(x)$ in mainly two situations:

1. When $f(x)$ is known, but perhaps "hard" to compute directly. For instance, we can define the cosine of an angle as either the ratio of sides of a right triangle ("adjacent over hypotenuse") or using the definition in terms of the unit circle. However, neither of these provides a convenient way of computing $\cos(2)$. A Taylor polynomial of sufficiently high degree can provide a reasonable method of computing such values using only operations usually hard-wired into a computer ($+$, $-$, × and $\div$).

2. When $f(x)$ is not known, but information about its derivatives is known. This occurs more often than one might think, especially in the study of differential equations.

In both situations, a critical piece of information to have is "How good is my approximation?" If we use a Taylor polynomial to compute $\cos(2)$, how do we know how accurate the approximation is?

Although much of the content presented in Calculus concerns the search for exact answers to problems such as integration and differentiation, many practical applications of calculus involve attempts to find *approximations*; for example, using Newton's Method to approximate the zeros of a function, or numerical integration to approximate the value of an integral that cannot be solved exactly. Whenever an approximation is used, one naturally wishes to know how good

Even though Taylor polynomials *could* be used in calculators and computers to calculate values of trigonometric functions, in practice they generally aren't. Other more efficient and accurate methods have been developed, such as the CORDIC algorithm. However, understanding how Taylor polynomials could be used is important to developing an understanding of various approximating techniques.

the approximation is. In other words, we look for a bound on the error introduced by working with an approximation. The following theorem gives bounds on the error introduced in using a Taylor (and hence Maclaurin) polynomial to approximate a function.

Theorem 4.5.16 Taylor's Theorem.

1. *Let f be a function whose $(n+1)$th derivative exists on an interval I and let c be in I. Then, for each x in I, there exists z_x between x and c such that*

$$f(x) = f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+R_n(x),$$

where $R_n(x) = \dfrac{f^{(n+1)}(z_x)}{(n+1)!}(x-c)^{(n+1)}$.

2. $|R_n(x)| \leq \dfrac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|(x-c)^{(n+1)}\right|$*, where z is in I.*

The first part of Taylor's Theorem states that $f(x) = p_n(x) + R_n(x)$, where $p_n(x)$ is the nth order Taylor polynomial and $R_n(x)$ is the remainder, or error, in the Taylor approximation. The second part gives bounds on how big that error can be. If the $(n+1)$th derivative is large on I, the error may be large; if x is far from c, the error may also be large. However, the $(n+1)!$ term in the denominator tends to ensure that the error gets smaller as n increases.

The following example computes error estimates for the approximations of $\ln(1.5)$ and $\ln(2)$ made in Example 4.5.12.

Example 4.5.18 Finding error bounds of a Taylor polynomial.

Use Theorem 4.5.16 to find error bounds when approximating $\ln(1.5)$ and $\ln(2)$ with $p_6(x)$, the Taylor polynomial of degree 6 of $f(x) = \ln(x)$ at $x = 1$, as calculated in Example 4.5.12.

Solution.

1. We start with the approximation of $\ln(1.5)$ with $p_6(1.5)$. The theorem references an open interval I that contains both x and c. The smaller the interval we use the better; it will give us a more accurate (and smaller!) approximation of the error. We let $I = (0.9, 1.6)$, as this interval contains both $c = 1$ and $x = 1.5$. The theorem references $\max\left|f^{(n+1)}(z)\right|$. In our situation, this is asking "How big can the 7th derivative of $y = \ln(x)$ be on the interval $(0.9, 1.6)$?" The seventh derivative is $y = -6!/x^7$. The largest absolute value it attains on I is about 1506. (There are no critical numbers of $f^{(7)}$ in the interval so we evaluate the endpoints: $f^{(7)}(0.9) \approx 1506$ and $f^{(7)}(1.6) \approx 27$.) In particular, we are evaluating at $x = 1.5$, so we let $x = 1.5$. Thus we can bound the error as:

$$\begin{aligned}|R_6(1.5)| &\leq \frac{\max\left|f^{(7)}(z)\right|}{7!}\left|(1.5-1)^7\right| \\ &\leq \frac{1506}{5040}\cdot\frac{1}{2^7} \\ &\approx 0.0023.\end{aligned}$$

One way of quantifying the extent to which one function approximates another is using the *order* to which they agree. We say that two functions f and g *agree to order n at c* if n is the *largest* integer for which

$$\lim_{x\to c}\frac{f(x)-g(x)}{(x-c)^n} = 0.$$

Taylor's Theorem tells us that a function and its degree n Taylor polynomial agree to order n. Roughly speaking, this means that their difference goes to zero faster than the nth power of $x-c$ as x approaches c.

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Figure 4.5.17 Video presentation of Theorem 4.5.16

We computed $p_6(1.5) = 0.404688$; using a calculator, we find $\ln(1.5) \approx 0.405465$, so the actual error is about 0.000778, which is less than our bound of 0.0023. This affirms Taylor's Theorem; the theorem states that our approximation would be within about 2 thousandths of the actual value, whereas the approximation was actually closer. Taylor's Theorem only gives an upper bound on the error.

2. We again find an interval I that contains both $c = 1$ and $x = 2$; we choose $I = (0.9, 2.1)$. The maximum value of the seventh derivative of f on this interval is again about 1506 (as the largest values come near $x = 0.9$). Thus

$$\begin{aligned} |R_6(2)| &\leq \frac{\max\left|f^{(7)}(z)\right|}{7!}\left|(2-1)^7\right| \\ &\leq \frac{1506}{5040}\cdot 1^7 \\ &\approx 0.30. \end{aligned}$$

This bound is not as nearly as good as before. Using the degree 6 Taylor polynomial at $x = 1$ will bring us within 0.3 of the correct answer. As $p_6(2) \approx 0.61667$, our error estimate guarantees that the actual value of $\ln(2)$ is somewhere between 0.31667 and 0.91667. These bounds are not particularly useful. In reality, our approximation was only off by about 0.07. However, we are approximating ostensibly because we do not know the real answer. In order to be assured that we have a good approximation, we would have to resort to using a polynomial of higher degree.

Video solution

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We practice again. This time, we use Taylor's theorem to find n that guarantees our approximation is within a certain amount.

Example 4.5.19 Finding sufficiently accurate Taylor polynomials.

Find n such that the nth Taylor polynomial of $f(x) = \cos(x)$ at $x = 0$ approximates $\cos(2)$ to within 0.001 of the actual answer. What is $p_n(2)$?

Solution. Following Taylor's theorem, we need bounds on the size of the derivatives of $f(x) = \cos(x)$. In the case of this trigonometric function, this is easy. All derivatives of cosine are $\pm\sin(x)$ or $\pm\cos(x)$. In all cases, these functions are never greater than 1 in absolute value. We want the error to be less than 0.001. To find the appropriate n, consider the following inequalities:

$$\begin{aligned} \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|(2-0)^{(n+1)}\right| &\leq 0.001 \\ \frac{1}{(n+1)!}\cdot 2^{(n+1)} &\leq 0.001. \end{aligned}$$

We find an n that satisfies this last inequality with trial-and-error. When $n = 8$, we have $\dfrac{2^{8+1}}{(8+1)!} \approx 0.0014$; when $n = 9$, we have $\dfrac{2^{9+1}}{(9+1)!} \approx 0.000282 < 0.001$. Thus we want to approximate $\cos(2)$ with $p_9(2)$.

We now set out to compute $p_9(x)$. We again need a table of the derivatives of $f(x) = \cos(x)$ evaluated at $x = 0$. A table of these values is given in Figure 4.5.20.
Notice how the derivatives, evaluated at $x = 0$, follow a certain pattern. All the odd powers of x in the Taylor polynomial will disappear as their coefficient is 0. While our error bounds state that we need $p_9(x)$, our work shows that this will be the same as $p_8(x)$.
Since we are forming our polynomial at $x = 0$, we are creating a Maclaurin polynomial, and:

$$\begin{aligned} p_8(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(8)}(0)}{8!}x^8 \\ &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8. \end{aligned}$$

We finally approximate $\cos(2)$:

$$\cos(2) \approx p_8(2) = -\frac{131}{315} \approx -0.41587.$$

Our error bound guarantee that this approximation is within 0.001 of the correct answer. Technology shows us that our approximation is actually within about 0.0003 of the correct answer.
Figure 4.5.21 shows a graph of $y = p_8(x)$ and $y = \cos(x)$. Note how well the two functions agree on about $(-\pi, \pi)$.

$f(x) = \cos(x)$	$f(0) = 1$
$f'(x) = -\sin(x)$	$f'(0) = 0$
$f''(x) = -\cos(x)$	$f''(0) = -1$
$f'''(x) = \sin(x)$	$f'''(0) = 0$
$f^{(4)}(x) = \cos(x)$	$f^{(4)}(0) = 1$
$f^{(5)}(x) = -\sin(x)$	$f^{(5)}(0) = 0$
$f^{(6)}(x) = -\cos(x)$	$f^{(6)}(0) = -1$
$f^{(7)}(x) = \sin(x)$	$f^{(7)}(0) = 0$
$f^{(8)}(x) = \cos(x)$	$f^{(8)}(0) = 1$
$f^{(9)}(x) = -\sin(x)$	$f^{(9)}(0) = 0$

Figure 4.5.20 A table of the derivatives of $f(x) = \cos(x)$ evaluated at $x = 0$

Video solution

youtu.be/watch?v=zg1W9miUCB4

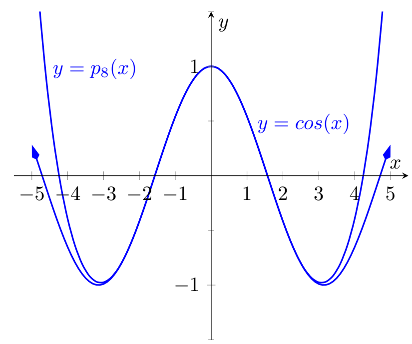

Figure 4.5.21 A graph of $f(x) = \cos(x)$ and its degree 8 Maclaurin polynomial

Example 4.5.22 Finding and using Taylor polynomials.

1. Find the degree 4 Taylor polynomial, $p_4(x)$, for $f(x) = \sqrt{x}$ at $x = 4$.
2. Use $p_4(x)$ to approximate $\sqrt{3}$.
3. Find bounds on the error when approximating $\sqrt{3}$ with $p_4(3)$.

Solution.

1. We begin by evaluating the derivatives of f at $x = 4$. This is done in Figure 4.5.23.

   These values allow us to form the Taylor polynomial $p_4(x)$:

$$\begin{aligned} p_4(x) = 2 + \frac{1}{4}(x-4) + \frac{-1/32}{2!}(x-4)^2 + \ldots \\ \ldots \frac{3/256}{3!}(x-4)^3 + \frac{-15/2048}{4!}(x-4)^4. \end{aligned}$$

2. As $p_4(x) \approx \sqrt{x}$ near $x = 4$, we approximate $\sqrt{3}$ with $p_4(3) = 1.73212$.
3. To find a bound on the error, we need an open interval that contains $x = 3$ and $x = 4$. We set $I = (2.9, 4.1)$. The largest value the fifth derivative of $f(x) = \sqrt{x}$ takes on this interval is near $x = 2.9$, at about 0.0273. (We often graph the $(n+1)^{th}$ derivative to find its extrema. In this case is $f^{(5)}(x) = 105/(32x^{9/2})$ is

$f(x) = \sqrt{x}$	$f(4) = 2$
$f'(x) = \dfrac{1}{2\sqrt{x}}$	$f'(4) = \dfrac{1}{4}$
$f''(x) = \dfrac{-1}{4x^{3/2}}$	$f''(4) = \dfrac{-1}{32}$
$f'''(x) = \dfrac{3}{8x^{5/2}}$	$f'''(4) = \dfrac{3}{256}$
$f^{(4)}(x) = \dfrac{-15}{16x^{7/2}}$	$f^{(4)}(4) = \dfrac{-15}{2048}$

Figure 4.5.23 A table of the derivatives of $f(x) = \sqrt{x}$ evaluated at $x = 4$

always decreasing, so the maximum occurs at 2.9.) Thus

$$|R_4(3)| \leq \frac{0.0273}{5!}\left|(3-4)^5\right| \approx 0.00023.$$

This shows our approximation is accurate to at least the first 2 places after the decimal. (It turns out that our approximation is actually accurate to 4 places after the decimal.) A graph of $f(x) = \sqrt{x}$ and $p_4(x)$ is given in Figure 4.5.24. Note how the two functions are nearly indistinguishable on $(2, 7)$.

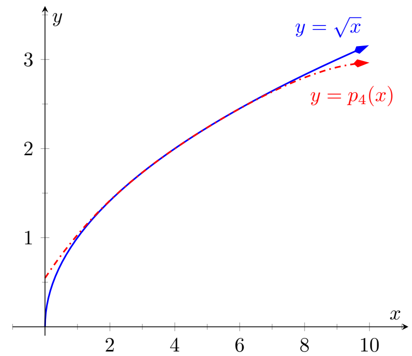

Figure 4.5.24 A graph of $f(x) = \sqrt{x}$ and its degree 4 Taylor polynomial at $x = 4$

Our final example gives a brief introduction to using Taylor polynomials to solve differential equations.

Example 4.5.25 Approximating an unknown function.

A function $y = f(x)$ is unknown save for the following two facts.

1. $y(0) = f(0) = 1$, and
2. $y' = y^2$

(This second fact says that amazingly, the derivative of the function is actually the function squared!)
Find the degree 3 Maclaurin polynomial $p_3(x)$ of $y = f(x)$.

Solution. One might initially think that not enough information is given to find $p_3(x)$. However, note how the second fact above actually lets us know what $y'(0)$ is:

$$y' = y^2 \Rightarrow y'(0) = y^2(0).$$

Since $y(0) = 1$, we conclude that $y'(0) = 1$.
Now we find information about y''. Starting with $y' = y^2$, take derivatives of both sides, *with respect to* x. That means we must use implicit differentiation.

$$\begin{aligned} y' &= y^2 \\ \frac{d}{dx}(y') &= \frac{d}{dx}(y^2) \\ y'' &= 2y \cdot y'. \end{aligned}$$

Now evaluate both sides at $x = 0$:

$$\begin{aligned} y''(0) &= 2y(0) \cdot y'(0) \\ y''(0) &= 2. \end{aligned}$$

We repeat this once more to find $y'''(0)$. We again use implicit differentiation; this time the Product Rule is also required.

$$\begin{aligned} \frac{d}{dx}(y'') &= \frac{d}{dx}(2yy') \\ y''' &= 2y' \cdot y' + 2y \cdot y''. \end{aligned}$$

Now evaluate both sides at $x = 0$:

$$y'''(0) = 2y'(0)^2 + 2y(0)y''(0)$$

$$y'''(0) = 2 + 4 = 6.$$

In summary, we have:

$$y(0) = 1 \qquad y'(0) = 1 \qquad y''(0) = 2 \qquad y'''(0) = 6.$$

We can now form $p_3(x)$:

$$\begin{aligned} p_3(x) &= 1 + x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 \\ &= 1 + x + x^2 + x^3. \end{aligned}$$

It turns out that the differential equation we started with, $y' = y^2$, where $y(0) = 1$, can be solved without too much difficulty:

$$y = \frac{1}{1-x}.$$

Figure 4.5.26 shows this function plotted with $p_3(x)$. Note how similar they are near $x = 0$.

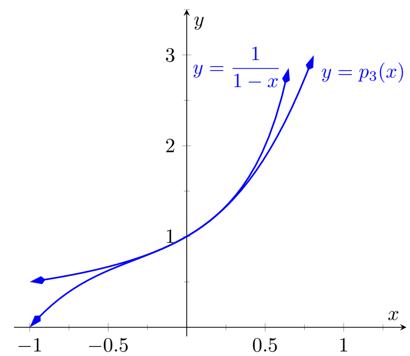

Figure 4.5.26 A graph of $y = -1/(x-1)$ and $y = p_3(x)$ from Example 4.5.25

It is beyond the scope of this text to pursue error analysis when using Taylor polynomials to approximate solutions to differential equations. This topic is often broached in introductory Differential Equations courses and usually covered in depth in Numerical Analysis courses. Such an analysis is very important; one needs to know how good their approximation is. We explored this example simply to demonstrate the usefulness of Taylor polynomials.

We first learned of the derivative in the context of instantaneous rates of change and slopes of tangent lines. We furthered our understanding of the power of the derivative by studying how it relates to the graph of a function (leading to ideas of increasing/decreasing and concavity).

4.5.1 Exercises

Terms and Concepts

1. What is the difference between a Taylor polynomial and a Maclaurin polynomial?

2. True or False? In general, $p_n(x)$ approximates $f(x)$ better and better as n gets larger. (☐ True ☐ False)

3. For some function $f(x)$, the Maclaurin polynomial of degree 4 is $p_4(x) = 6 + 3x - 4x^2 + 5x^3 - 7x^4$. What is $p_2(x)$?

4. For some function $f(x)$, the Maclaurin polynomial of degree 4 is $p_4(x) = 6 + 3x - 4x^2 + 5x^3 - 7x^4$. What is $f'''(0)$?

Problems

Exercise Group. In the following exercises, find the Maclaurin polynomial of degree n for the given function.

5. Find the Maclaurin polynomial of degree $n = 3$ for $f(x) = e^{-x}$.

6. Find the Maclaurin polynomial of degree $n = 8$ for $f(x) = \sin(x)$.

7. Find the Maclaurin polynomial of degree $n = 5$ for $f(x) = x \cdot e^x$.

8. Find the Maclaurin polynomial of degree $n = 6$ for $f(x) = \tan(x)$.

9. Find the Maclaurin polynomial of degree $n = 4$ for $f(x) = e^{2x}$.

10. Find the Maclaurin polynomial of degree $n = 4$ for $f(x) = \dfrac{1}{1-x}$.

11. Find the Maclaurin polynomial of degree $n = 4$ for $f(x) = \dfrac{1}{1+x}$.

12. Find the Maclaurin polynomial of degree $n = 7$ for $f(x) = \dfrac{1}{1+x}$.

Exercise Group. In the following exercises, find the Taylor polynomial of degree n, at $x = c$, for the given function.

13. Find the Taylor polynomial for $f(x) = \sqrt{x}$ of degree $n = 4$, at $c = 1$.

14. Find the degree $n = 4$ Taylor polynomial for $f(x) = \ln(x + 1)$, at $c = 1$.

15. Find the degree $n = 6$ Taylor polynomial for $f(x) = \cos(x)$, at $c = \pi/4$.

16. Find the degree $n = 5$ Taylor poplynomial for $f(x) = \sin(x)$, at $c = \pi/6$.

17. Find the degree $n = 5$ Taylor poplynomial for $f(x) = \frac{1}{x}$, at $c = 2$.

18. Find the degree $n = 8$ Taylor poplynomial for $f(x) = \dfrac{1}{x^2}$, at $c = 1$.

19. Find the degree $n = 3$ Taylor poplynomial for $f(x) = \dfrac{1}{x^2+1}$, at $c = -1$.

20. Find the degree $n = 2$ Taylor polynomial for $f(x) = x^2 \cos(x)$, at $c = \pi$.

Exercise Group. In the following exercises, approximate the function value with the indicated Taylor polynomial and give approximate bounds on the error.

21. Approximate $\sin(0.1)$ with the Maclaurin polynomial of degree 3.

22. Approximate $\cos(1)$ with the Maclaurin polynomial of degree 4.

23. Approximate $\sqrt{10}$ with the Taylor polynomial of degree 2 centered at $x = 9$.

24. Approximate $\ln(1.5)$ with the Taylor polynomial of degree 3 centered at $x = 1$.

Exercise Group. The following exercises ask for an n to be found such that $p_n(x)$ approximates $f(x)$ within a certain bound of accuracy.

25. Find n such that the Maclaurin polynomial of degree n of $f(x) = e^x$ approximates e within 0.0001 of the actual value.

26. Find n such that the Taylor polynomial of degree n of $f(x) = \sqrt{x}$, centered at $x = 4$, approximates $\sqrt{3}$ within 0.0001 of the actual value.

27. Find n such that the Maclaurin polynomial of degree n of $f(x) = \cos(x)$ approximates $\cos(\pi/3)$ within 0.0001 of the actual value.

28. Find n such that the Maclaurin polynomial of degree n of $f(x) = \sin(x)$ approximates $\cos(\pi)$ within 0.0001 of the actual value.

Exercise Group. In the following exercises, find the nth term of the indicated Taylor polynomial.

29. Find a formula for the nth term of the Maclaurin polynomial for $f(x) = e^x$.

30. Find a formula for the nth term of the Maclaurin polynomial for $f(x) = \cos(x)$.

31. Find a formula for the nth term of the Maclaurin polynomial for $f(x) = \sin x$.

32. Find a formula for the nth term of the Maclaurin polynomial for $f(x) = \dfrac{1}{1-x}$.

33. Find a formula for the nth term of the Maclaurin polynomial for $f(x) = \dfrac{1}{1+x}$.

34. Find a formula for the nth term of the Taylor polynomial for $f(x) = \ln(x)$ centered at $x = 1$.

Exercise Group. In the following exercises, approximate the solution to the given differential equation with a degree 4 Maclaurin polynomial.

35. $y' = y, y(0) = 1$

36. $y' = 5y, y(0) = 3$

37. $y' = \dfrac{2}{y}, y(0) = 1$

We first learned of the derivative in the context of instantaneous rates of change and slopes of tangent lines. We furthered our understanding of the power of the derivative by studying how it relates to the graph of a function (leading to ideas of increasing/decreasing and concavity). This chapter has put the derivative to yet more uses:

- Equation solving (Newton's Method),
- Related Rates (furthering our use of the derivative to find instantaneous rates of change),
- Optimization (applied extreme values), and
- Differentials (useful for various approximations and for something called integration).

In the next chapters, we will consider the "reverse" problem to computing the derivative: given a function f, can we find a function whose derivative is f? Being able to do so opens up an incredible world of mathematics and applications.

Chapter 5

Integration

We have spent considerable time considering the derivatives of a function and their applications. In the following chapters, we are going to starting thinking in "the other direction." That is, given a function $f(x)$, we are going to consider functions $F(x)$ such that $F'(x) = f(x)$. There are numerous reasons this will prove to be useful: these functions will help us compute area, volume, mass, force, pressure, work, and much more.

5.1 Antiderivatives and Indefinite Integration

Given a function $y = f(x)$, a **differential equation** is an equation that incorporates y, x, and the derivatives of y. For instance, a simple differential equation is:

$$y' = 2x.$$

Solving a differential equation amounts to finding a function y that satisfies the given equation. Take a moment and consider that equation; can you find a function y such that $y' = 2x$?

Can you find another?

And yet another?

Hopefully you were able to come up with at least one solution: $y = x^2$. "Finding another" may have seemed impossible until one realizes that a function like $y = x^2 + 1$ also has a derivative of $2x$. Once that discovery is made, finding "yet another" is not difficult; the function $y = x^2 + 123{,}456{,}789$ also has a derivative of $2x$. The differential equation $y' = 2x$ has many solutions. This leads us to some definitions.

youtu.be/watch?v=z1XH1JTUKTU

Figure 5.1.1 Video introduction to Section 5.1

Definition 5.1.2 Antiderivatives and Indefinite Integrals.

Let a function $f(x)$ be given. An **antiderivative** of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

The set of all antiderivatives of $f(x)$ is the **indefinite integral of** f, denoted by

$$\int f(x)\, dx.$$

Make a note about our definition: we refer to *an* antiderivative of f, as opposed to *the* antiderivative of f, since there is *always* an infinite number of them. We often use upper-case letters to denote antiderivatives.

youtu.be/watch?v=BRAjDVVn4H4

Figure 5.1.3 Video presentation of Definition 5.1.2

When f is continuous, knowing one antiderivative of f allows us to find infinitely more, simply by adding a constant. Not only does this give us *more* antiderivatives, it gives us *all* of them.

Theorem 5.1.4 Antiderivative Forms.

Let $F(x)$ and $G(x)$ be antiderivatives of a continuous function $f(x)$ on an interval I. Then there exists a constant C such that, on I,

$$G(x) = F(x) + C.$$

Given a continuous function f defined on an interval I and one of its antiderivatives F, we know *all* antiderivatives of f on I have the form $F(x) + C$ for some constant C. Using Definition 5.1.2, we can say that

$$\int f(x)\,dx = F(x) + C.$$

Note that we are abusing notation somewhat: when we write $F(x) + C$ on the right-hand side, we really mean the set of all such functions, for each real number value of C. Let's analyze this indefinite integral notation.

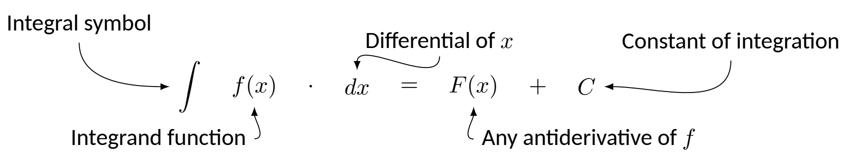

Figure 5.1.5 Antiderivative notation

Figure 5.1.5 shows the typical notation of the indefinite integral. The integration symbol, $\int$, is in reality an "elongated S," representing "take the sum." We will later see how *sums* and *antiderivatives* are related.

The function we want to find an antiderivative of is called the *integrand*. It contains the differential of the variable we are integrating with respect to. The $\int$ symbol and the differential dx are not "bookends" with a function sandwiched in between; rather, the symbol $\int$ means "find all antiderivatives of what follows," and the function $f(x)$ and dx are multiplied together; the dx does not "just sit there."

Another way of looking at the notation is that it tells us that $f(x)\,dx$ is the differential of $F(x)$: $dF(x) = f(x)\,dx$, confirming that $F'(x) = f(x)$, as required of an antiderivative. The integral symbol can then be viewed as an instruction to "undo" the differential and recover the antiderivative $F(x)$.

Another important aspect of the dx is that it tells us which variable we're taking the antiderivative with respect to, much like how $\frac{d}{dx}$ would mean to take the derivative with respect to x, while $\frac{d}{dt}$ would be the derivative with respect to t.

Let's practice using this notation.

Example 5.1.6 Evaluating indefinite integrals.

Evaluate $\int \sin(x)\,dx$.

Solution. We are asked to find all functions $F(x)$ such that $F'(x) = \sin(x)$. Some thought will lead us to one solution: $F(x) = -\cos(x)$, because $\frac{d}{dx}(-\cos(x)) = \sin(x)$.
The indefinite integral of $\sin(x)$ is thus $-\cos(x)$, plus a constant of inte-

Video solution

youtu.be/watch?v=W-FUL0ApGL8

gration. So:

$$\int \sin(x)\,dx = -\cos(x) + C.$$

A commonly asked question is "What happened to the dx?" The unenlightened response is "Don't worry about it. It just goes away." A full understanding includes the following.

This process of *antidifferentiation* is really solving a *differential* question. The integral

$$\int \sin(x)\,dx$$

presents us with a differential, $dy = \sin(x)\,dx$. It is asking: "What is y?" We found lots of solutions, all of the form $y = -\cos(x) + C$.

Letting $dy = \sin(x)\,dx$, rewrite

$$\int \sin(x)\,dx \text{ as } \int dy.$$

This is asking: "What functions have a differential of the form dy?" The answer is "Functions of the form $y+C$, where C is a constant." What is y? We have lots of choices, all differing by a constant; the simplest choice is $y = -\cos(x)$.

Understanding all of this is more important later as we try to find antiderivatives of more complicated functions. In this section, we will simply explore the rules of indefinite integration, and one can succeed for now with answering "What happened to the dx?" with "It went away."

Let's practice once more before stating integration rules.

Example 5.1.7 Evaluating indefinite integrals.

Evaluate $\int \left(3x^2 + 4x + 5\right)\,dx$.

Solution. We seek a function $F(x)$ whose derivative is $3x^2 + 4x + 5$. When taking derivatives, we can consider functions term-by-term, so we can likely do that here.

What functions have a derivative of $3x^2$? Some thought will lead us to a cubic, specifically $x^3 + C_1$, where C_1 is a constant.

What functions have a derivative of $4x$? Here the x term is raised to the first power, so we likely seek a quadratic. Some thought should lead us to $2x^2 + C_2$, where C_2 is a constant.

Finally, what functions have a derivative of 5? Functions of the form $5x + C_3$, where C_3 is a constant.

Our answer appears to be

$$\int \left(3x^2 + 4x + 5\right)\,dx = x^3 + C_1 + 2x^2 + C_2 + 5x + C_3.$$

Video solution

youtu.be/watch?v=3DFPqGHX7Yw

We do not need three separate constants of integration; combine them as one constant, giving the final answer of

$$\int \left(3x^2 + 4x + 5\right)\,dx = x^3 + 2x^2 + 5x + C.$$

It is easy to verify our answer; take the derivative of $x^3 + 2x^2 + 5x + C$ and see we indeed get $3x^2 + 4x + 5$.

This final step of "verifying our answer" is important both practically and theoretically. In general, taking derivatives is easier than finding antiderivatives so checking our work is easy and vital as we learn.

We also see that taking the derivative of our answer returns the function in the integrand. Thus we can say that:

$$\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x).$$

Differentiation "undoes" the work done by antidifferentiation.

Theorem 2.7.16 gave a list of the derivatives of common functions we had learned at that point. We restate part of that list here to stress the relationship between derivatives and antiderivatives. This list will also be useful as a glossary of common antiderivatives as we learn.

Theorem 5.1.8 Derivatives and Antiderivatives.

Here are the Common Differentiation Rules and their Common Indefinite Integral Rule counterparts.

$\frac{d}{dx}(cf(x)) = c\cdot f'(x)$	$\int c\cdot f(x)\,dx = c\cdot\int f(x)\,dx$
$\frac{d}{dx}(f(x)\pm g(x)) = f'(x)\pm g'(x)$	$\int \big(f(x)\pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
$\frac{d}{dx}(C) = 0$	$\int 0\,dx = C$
$\frac{d}{dx}(x) = 1$	$\int 1\,dx = \int dx = x + C$
$\frac{d}{dx}(x^n) = n\cdot x^{n-1}$	$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad (n\neq -1)$
$\frac{d}{dx}(\sin(x)) = \cos(x)$	$\int \cos(x)\,dx = \sin(x) + C$
$\frac{d}{dx}(\cos(x)) = -\sin(x)$	$\int \sin(x)\,dx = -\cos(x) + C$
$\frac{d}{dx}(\tan(x)) = \sec^2(x)$	$\int \sec^2(x)\,dx = \tan(x) + C$
$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$	$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$
$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$	$\int \sec(x)\tan(x)\,dx = \sec(x) + C$
$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$	$\int \csc^2(x)\,dx = -\cot(x) + C$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}(a^x) = \ln(a)\cdot a^x$	$\int a^x\,dx = \frac{1}{\ln(a)}\cdot a^x + C$
$\frac{d}{dx}(\ln(x)) = \frac{1}{x},\ x>0$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$

We highlight a few important points from Theorem 5.1.8.

- $$\int c\cdot f(x)\,dx = c\cdot\int f(x)\,dx$$

This is the Constant Multiple Rule: we can temporarily ignore constants when finding antiderivatives, just as we did when computing derivatives (i.e., $\frac{d}{dx}\big(3x^2\big)$ is just as easy to compute as $\frac{d}{dx}\big(x^2\big)$). An example:

$$\int 5\cos(x)\,dx = 5\cdot\int\cos(x)\,dx = 5\cdot(\sin(x)+C) = 5\sin(x)+C.$$

In the last step we can consider the constant as also being multiplied by 5, but "5 times a constant" is still a constant, so we just write "C".

- $$\int \big(f(x)\pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

  This is the Sum/Difference Rule: we can split integrals apart when the integrand contains terms that are added/subtracted, as we did in Example 5.1.7. So:

$$\begin{aligned}\int(3x^2+4x+5)\,dx &= \int 3x^2\,dx + \int 4x\,dx + \int 5\,dx\\ &= 3\int x^2\,dx + 4\int x\,dx + \int 5\,dx\\ &= 3\cdot\frac{1}{3}x^3 + 4\cdot\frac{1}{2}x^2 + 5x + C\\ &= x^3+2x^2+5x+C\end{aligned}$$

  In practice we generally do not write out all these steps, but we demonstrate them here for completeness.

- $$\int x^n\,dx = \frac{1}{n+1}x^{n+1}+C \quad (n\neq -1)$$

  This is the Power Rule of indefinite integration. There are two important things to keep in mind:

  1. Notice the restriction that $n\neq -1$. This is important: $\int\frac{1}{x}\,dx \neq$ "$\frac{1}{0}x^0+C$"; rather, see the last rule from the list.
  2. We are presenting antidifferentiation as the "inverse operation" of differentiation. Here is a useful quote to remember:

     > "Inverse operations do the opposite things in the opposite order."

     When taking a derivative using the Power Rule, we *first multiply* by the power, then *second subtract* 1 from the power. To find the antiderivative, do the opposite things in the opposite order: *first add* 1 to the power, then *second divide* by the power.

- $$\int\frac{1}{x}\,dx = \ln|x|+C$$

  Note that this rule uses the absolute value of x. The exercises will work the reader through why this is the case; for now, know the absolute value is important and cannot be ignored.

Initial Value Problems. In Section 2.3 we saw that the derivative of a position function gave a velocity function, and the derivative of a velocity function describes acceleration. We can now go "the other way:" the antiderivative of an acceleration function gives a velocity function, etc.. While there is just one derivative of a given function, there are infinitely many antiderivatives. Therefore we cannot ask "What is *the* velocity of an object whose acceleration is $-32\,\frac{\text{ft}}{\text{s}^2}$?", since there is more than one answer.

We can find *the* answer if we provide more information with the question, as done in the following example. Often the additional information comes in the form of an *initial value*, a value of the function that one knows beforehand.

youtu.be/watch?v=Oo6OHiiGbOc

Figure 5.1.9 Introducing initial value problems

Example 5.1.10 Solving initial value problems.

The acceleration due to gravity of a falling object is $-32\,\frac{\text{ft}}{\text{s}^2}$. At time $t = 3$, a falling object had a velocity of $-10\,\frac{\text{ft}}{\text{s}}$. Find the equation of the object's velocity.

Solution. We want to know a velocity function, $v(t)$. We know two things:

- The acceleration, i.e., $v'(t) = -32$, and
- the velocity at a specific time, i.e., $v(3) = -10$.

Using the first piece of information, we know that $v(t)$ is an antiderivative of $v'(t) = -32$. So we begin by finding the indefinite integral of -32:

$$\int (-32)\,dt = -32t + C = v(t).$$

Now we use the fact that $v(3) = -10$ to find C:

$$\begin{aligned} v(t) &= -32t + C \\ v(3) &= -10 \\ -32(3) + C &= -10 \\ C &= 86 \end{aligned}$$

Thus $v(t) = -32t + 86$. We can use this equation to understand the motion of the object: when $t = 0$, the object had a velocity of $v(0) = 86\frac{\text{ft}}{\text{s}}$. Since the velocity is positive, the object was moving upward.
When did the object begin moving down? Immediately after $v(t) = 0$:

$$-32t + 86 = 0 \implies t = \frac{43}{16} \approx 2.69\text{s}.$$

Recognize that we are able to determine quite a bit about the path of the object knowing just its acceleration and its velocity at a single point in time.

Video solution

youtu.be/watch?v=3K_gWY4lmOs

Example 5.1.11 Solving initial value problems.

Find $f(t)$, given that $f''(t) = \cos(t)$, $f'(0) = 3$ and $f(0) = 5$.

Solution. We start by finding $f'(t)$, which is an antiderivative of $f''(t)$:

$$\int f''(t)\,dt = \int \cos(t)\,dt$$

$$= \sin(t) + C$$
$$= f'(t).$$

So $f'(t) = \sin(t) + C$ for the correct value of C. We are given that $f'(0) = 3$, so:

$$\sin(0) + C = 3$$
$$C = 3.$$

Using the initial value, we have found $f'(t) = \sin(t) + 3$. We now find $f(t)$ by integrating again. We will use a different integration constant since we have already defined C to equal 3 above.

$$f(t) = \int f'(t)\,dt = \int (\sin(t) + 3)\,dt = -\cos(t) + 3t + D.$$

We are given that $f(0) = 5$, so

$$-\cos(0) + 3(0) + D = 5$$
$$-1 + C = 5$$
$$C = 6$$

Thus $f(t) = -\cos(t) + 3t + 6$.

Video solution

youtu.be/watch?v=MB1dLY4lOew

This section introduced antiderivatives and the indefinite integral. We found they are needed when finding a function given information about its derivative(s). For instance, we found a velocity function given an acceleration function.

In the next section, we will see how position and velocity are unexpectedly related by the areas of certain regions on a graph of the velocity function. Then, in Section 5.4, we will see how areas and antiderivatives are closely tied together. This connection is incredibly important, as indicated by the name of the theorem that describes it: The Fundamental Theorem of Calculus.

5.1.1 Exercises

Terms and Concepts

1. Define the term "antiderivative" in your own words.

2. Is it more accurate to refer to "the" antiderivative of $f(x)$ or "an" antiderivative of $f(x)$?

3. Use your own words to define the indefinite integral of $f(x)$.

4. Fill in the blanks: "Inverse operations do the ___________ things in the ___________ order."

5. What is an "initial value problem"?

6. The derivative of a position function is a/an ___________ function.

7. An antiderivative of an acceleration function is a/an ___________ function.

8. If $F(x)$ is an antiderivative of $f(x)$, and $G(x)$ is an antiderivative of $g(x)$, give an antiderivative of $f(x) + g(x)$.

Problems

Exercise Group. Evaluate the indefinite integral. Don't forget your constant of integration!

9. $\int 8x^5\,dx$

10. $\int x^9\,dx$

11. $\int \left(5x^8 - 6\right)\,dx$

12. $\int dt$

13. $\int 1\,ds$

14. $\int \frac{1}{5t^8}\,dt$

15. $\int \frac{6}{t^4}\,dt$

16. $\int \frac{1}{\sqrt{x}}\,dx$

17. $\int \sec(\theta)\tan(\theta)\,d\theta$

18. $\int \sin(\theta)\,d\theta$

19. $\int (\sec(x)\tan(x) - \csc(x)\cot(x))\,dx$

20. $\int 2e^\theta\,d\theta$

21. $\int 3^t\,dt$

22. $\int \frac{4^t}{9}\,dt$

23. $\int (5t+2)^2\,dt$

24. $\int \left(t^4 - 5\right)\left(t^5 + 2t\right)dt$

25. $\int x^7 x^9\,dx$

26. $\int 1.41421^e\,dx$

27. $\int r\,dx$

28. Consider the two integrals, $\displaystyle\int s^n\,ds$ and $\displaystyle\int s^n\,dn$.

(a) What is the difference between these two indefinite integrals?

(b) Evaluate $\displaystyle\int s^n\,ds$.

(c) Evaluate $\displaystyle\int s^n\,dn$.

29. This problem investigates why Theorem 5.1.8 states that $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$.

(a) What is the domain of $y = \ln(x)$?

(b) Find $\frac{d}{dx}(\ln(x))$.

(c) What is the domain of $y = \ln(-x)$?

(d) Find $\frac{d}{dx}(\ln(-x))$.

(e) You should find that $1/x$ has two types of antiderivatives, depending on whether $x > 0$ or $x < 0$. In one expression, give a formula for $\displaystyle\int \frac{1}{x}\,dx$ that takes these different domains into account, and explain your answer.

Exercise Group. Find the function determined by the given initial value problem.

30. $f'(x) = \sin(x)$ and $f(0) = 7$

31. $f'(x) = 2e^x$ and $f(0) = 8$

32. $f'(x) = 3x^3 - 6x$ and $f(-2) = 9$

33. $f'(x) = \sec(x)\tan(x)$ and $f\left(\frac{\pi}{3}\right) = 6$

34. $f'(x) = 5^x$ and $f(2) = 5$

35. $f''(x) = 6$ and $f'(0) = 2$, $f(0) = 5$

36. $f''(x) = 4x$ and $f'(1) = 7$, $f(1) = 6$

37. $f''(x) = 7e^x$ and $f'(0) = -3$, $f(0) = -8$

38. $f''(\theta) = \cos(\theta)$ and $f'(0) = 6$, $f(0) = 9$

39. $f''(x) = 30x^4 + 2^x + \cos(x)$ and $f'(0) = 0$, $f(0) = 2$

40. $f''(x) = 0$ and $f'(-5) = -2$, $f(-5) = -1$

5.2 The Definite Integral

youtu.be/watch?v=__Xh37Qw4UE

Figure 5.2.1 Video introduction to Section 5.2

We start with an easy problem. An object travels in a straight line at a constant velocity of $5\,\frac{\text{ft}}{\text{s}}$ for 10 seconds. How far away from its starting point is the object?

We approach this problem with the familiar "Distance $=$ Rate $\times$ Time" equation. In this case, the distance traveled is $5\,\frac{\text{ft}}{\text{s}} \times 10\,\text{s} = 50$ feet.

It is interesting to note that this solution of 50 feet can be represented graphically. Consider Figure 5.2.2, where the constant velocity of $5\,\frac{\text{ft}}{\text{s}}$ is graphed on the axes. Shading the area under the line from $t=0$ to $t=10$ gives a rectangle with an area of 50 square units; when one considers the units of the axes, we can say this area represents $50\,$ft.

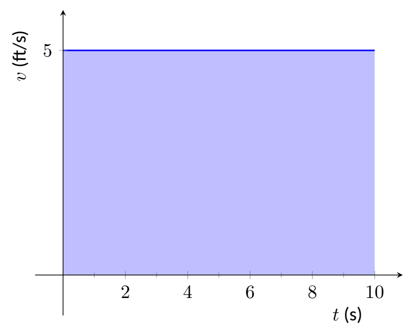

Figure 5.2.2 The area under a constant velocity function corresponds to distance traveled

Now consider a slightly harder situation (and not particularly realistic): an object travels in a straight line with a constant velocity of $5\,\frac{\text{ft}}{\text{s}}$ for 10 seconds, then instantly reverses course at a rate of $2\,\frac{\text{ft}}{\text{s}}$ for 4 seconds. (Since the object is traveling in the opposite direction when reversing course, we say the velocity is a constant $-2\,\frac{\text{ft}}{\text{s}}$.) How far away from the starting point is the object — what is its *displacement*?

Here we use "Distance $=$ Rate$_1$ $\times$ Time$_1$ $+$ Rate$_2$ $\times$ Time$_2$," which is

$$\text{Distance } = 5\cdot 10 + (-2)\cdot 4 = 42 \text{ ft}.$$

Hence the object is 42 feet from its starting location.

We can again depict this situation graphically. In Figure 5.2.3 we have the velocities graphed as straight lines on $[0,10]$ and $[10,14]$, respectively. The displacement of the object is

"Area above the t-axis $-$Area below the t-axis,"

which is easy to calculate as $50-8=42$ feet.

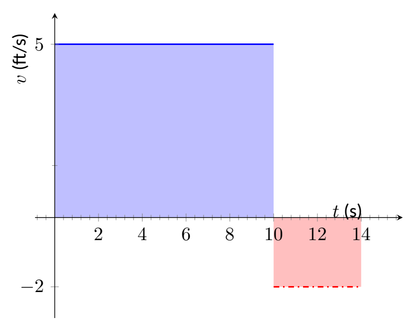

Figure 5.2.3 The total displacement is the area above the t-axis minus the area below the t-axis

Now consider a more difficult problem.

Example 5.2.4 Finding position using velocity.

The velocity of an object moving straight up/down under the acceleration of gravity is given as $v(t) = -32t+48$, where time t is given in seconds and velocity is in $\frac{\text{ft}}{\text{s}}$. When $t=0$, the object had a height of $0\,$ft.

1. What was the initial velocity of the object?
2. What was the maximum height of the object?
3. What was the height of the object at time $t=2$?

Solution. It is straightforward to find the initial velocity; at time $t=0$,

$$\begin{aligned} v(0) &= -32\cdot 0 + 48 \\ &= 48 \end{aligned}$$

The initial velocity was $48\,\frac{\text{ft}}{\text{s}}$.
To answer questions about the height of the object, we need to find the object's position function $s(t)$. This is an initial value problem, which we studied in the previous section. We are told the initial height is 0, i.e., $s(0)=0$. We know $s'(t)=v(t)=-32t+48$. To find s, we find the indefinite integral of $v(t)$:

$$s(t) = \int v(t)\,dt$$

$$= \int (-32t + 48)\, dt$$
$$= -16t^2 + 48t + C.$$

Since $s(0) = 0$, we conclude that $C = 0$ and $s(t) = -16t^2 + 48t$.
To find the maximum height of the object, we need to find the maximum of s. Recalling our work finding extreme values, we find the critical points of s by setting its derivative (the velocity function) equal to 0 and solving for t:

$$0 = -32t + 48$$
$$t = 48/32$$
$$= 1.5 \text{ s}.$$

(Notice how we ended up just finding when the velocity was 0ft/s!) The first derivative test shows this is a maximum, so the maximum height of the object is found at

$$s(1.5) = -16(1.5)^2 + 48(1.5) = 36 \text{ ft}.$$

The height at time $t = 2$ is now straightforward to compute:

$$s(2) = -16(2)^2 + 48(2)$$
$$= 32.$$

The height is 32 ft after 2 seconds.
While we have answered all three questions (using derivatives and antiderivatives), let's look at them again graphically, using the concepts of area that we explored earlier.
Figure 5.2.5 shows a graph of $v(t)$ on axes from $t = 0$ to $t = 3$. It is again straightforward to find $v(0)$. How can we use the graph to find the maximum height of the object?
Recall how in our previous work that the displacement of the object (in this case, its height) was found as the area under the velocity curve, as shaded in the figure. Moreover, the area between the curve and the t-axis that is below the t-axis counted as "negative" area. That is, it represents the object coming back toward its starting position. So to find the maximum distance from the starting point — the maximum height — we find the area under the velocity line that is above the t-axis, i.e., from $t = 0$ to $t = 1.5$. This region is a triangle; its area is

$$\text{Area} = \frac{1}{2}\text{Base} \times \text{Height}$$
$$= \frac{1}{2} \times 1.5 \text{ s} \times 48 \text{ ft/s}$$
$$= 36 \text{ ft}$$

which matches our previous calculation of the maximum height.
Finally, to find the height of the object at time $t = 2$ we calculate the total "signed area" (where some area is negative) under the velocity function from $t = 0$ to $t = 2$. This signed area is equal to $s(2)$, the displacement (i.e., signed distance) from the starting position at $t = 0$ to the position at time $t = 2$. That is,
Displacement = Area above the t-axis $-$ Area below t-axis.

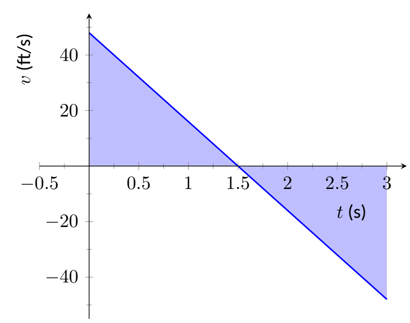

Figure 5.2.5 A graph of $v(t) = -32t + 48$; the shaded areas help determine displacement

The regions are triangles, and we find

$$\begin{aligned}\text{Displacement} &= \frac{1}{2}(1.5\text{s})(48 \text{ ft/s}) - \frac{1}{2}(0.5\text{s})(16 \text{ ft/s}) \\ &= 32 \text{ ft}.\end{aligned}$$

This also matches our previous calculation of the height at $t = 2$.

Notice how we answered each question in this example in two ways. Our first method was to manipulate equations using our understanding of antiderivatives and derivatives. Our second method was geometric: we answered questions looking at a graph and finding the areas of certain regions of this graph.

Video solution

youtu.be/watch?v=MUx3n9511e8

The above example does not *prove* a relationship between area under a velocity function and displacement, but it does imply a relationship exists. Section 5.4 will fully establish fact that the area under a velocity function is displacement.

Given a graph of a function $y = f(x)$, we will find that there is great use in computing the area between the curve $y = f(x)$ and the x-axis. Because of this, we need to define some terms.

Definition 5.2.6 The Definite Integral, Total Signed Area.

Let $y = f(x)$ be defined on a closed interval $[a, b]$. The **total signed area** from $x = a$ to $x = b$ under f is:
(area under $y = f(x)$ and above the x-axis on $[a, b]$) $-$ (area above $y = f(x)$ and under the x-axis on $[a, b]$).
The **definite integral** of f on $[a, b]$ is the total signed area of f on $[a, b]$, denoted

$$\int_a^b f(x)\, dx,$$

where a and b are the **bounds of integration**.

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Figure 5.2.7 Video presentation of Definition 5.2.6

By our definition, the definite integral gives the "signed area under f." We usually drop the word "signed" when talking about the definite integral, and simply say the definite integral gives "the area under f" or, more commonly, "the area under the curve."

The previous section introduced the indefinite integral, which related to antiderivatives. We have now defined the definite integral, which relates to areas under a function. The two are very much related, as we'll see when we learn the Fundamental Theorem of Calculus in Section 5.4. Recall that earlier we said that the "$\int$" symbol was an "elongated S" that represented finding a "sum." In the context of the definite integral, this notation makes a bit more sense, as we are adding up areas under the function f.

We practice using this notation.

Example 5.2.8 Evaluating definite integrals.

Consider the function f given in Figure 5.2.9.
Find:

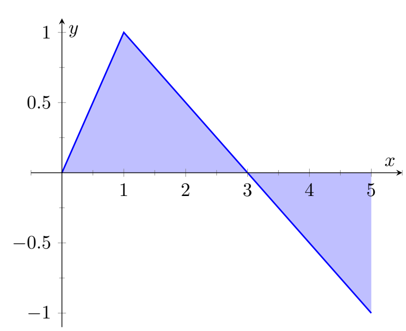

Figure 5.2.9 A graph of $f(x)$ in Example 5.2.8

1. $\int_0^3 f(x)\,dx$
2. $\int_3^5 f(x)\,dx$
3. $\int_0^5 f(x)\,dx$
4. $\int_0^3 5f(x)\,dx$
5. $\int_1^1 f(x)\,dx$

Solution.

1. $\int_0^3 f(x)\,dx$ is the area under f on the interval $[0,3]$. This region is a triangle, so the area is $\int_0^3 f(x)\,dx = \frac{1}{2}(3)(1) = 1.5$.

2. $\int_3^5 f(x)\,dx$ represents the area of the triangle found under the x-axis on $[3,5]$. The area is $\frac{1}{2}(2)(1) = 1$; since it is found *under* the x-axis, this is "negative area." Therefore $\int_3^5 f(x)\,dx = -1$.

3. $\int_0^5 f(x)\,dx$ is the total signed area under f on $[0,5]$. This is $1.5 + (-1) = 0.5$.

4. $\int_0^3 5f(x)\,dx$ is the area under $5f$ on $[0,3]$. This is sketched in Figure 5.2.10. Again, the region is a triangle, with height 5 times that of the height of the original triangle. Thus the area is $\int_0^3 5f(x)\,dx = \frac{1}{2}(15)(1) = 7.5$.

5. $\int_1^1 f(x)\,dx$ is the area under f on the "interval" $[1,1]$. This describes a line segment, not a region; it has no width. Therefore the area is 0.

Video solution

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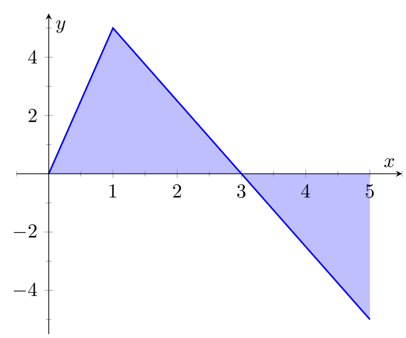

Figure 5.2.10 A graph of $5f$ in Example 5.2.8. (Yes, it looks just like the graph of f in Figure 5.2.9, just with a different y-scale.)

This example illustrates some of the properties of the definite integral, given here.

Theorem 5.2.11 Properties of the Definite Integral.

Let f and g be defined on a closed interval I that contains the values a, b and c, and let k be a constant. The following hold:

1. $\int_a^a f(x)\,dx = 0$
2. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
3. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
4. $\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. $\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$

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Figure 5.2.12 Video presentation of Theorem 5.2.11

We give a brief justification of Theorem 5.2.11 here.

1. As demonstrated in Example 5.2.8, there is no "area under the curve" when the region has no width; hence this definite integral is 0.

2. This states that total area is the sum of the areas of subregions. It is easily considered when we let $a < b < c$. We can break the interval $[a, c]$ into two subintervals, $[a, b]$ and $[b, c]$. The total area over $[a, c]$ is the area over $[a, b]$ plus the area over $[b, c]$. It is important to note that this still holds true even if $a < b < c$ is not true. We discuss this in the next point.

3. This property can be viewed a merely a convention to make other properties work well. (Later we will see how this property has a justification all its own, not necessarily in support of other properties.) Suppose $b < a < c$. The discussion from the previous point clearly justifies

$$\int_b^a f(x)\,dx + \int_a^c f(x)\,dx = \int_b^c f(x)\,dx. \quad (5.2.1)$$

However, we still claim that, as originally stated,

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx. \quad (5.2.2)$$

How do Equations (5.2.1) and (5.2.2) relate? Start with Equation (5.2.1):

$$\begin{aligned}\int_b^a f(x)\,dx + \int_a^c f(x)\,dx &= \int_b^c f(x)\,dx\\ \int_a^c f(x)\,dx &= -\int_b^a f(x)\,dx + \int_b^c f(x)\,dx\end{aligned}$$

Property (3) justifies changing the sign and switching the bounds of integration on the $-\displaystyle\int_b^a f(x)\,dx$ term; when this is done, Equations (5.2.1) and (5.2.2) are equivalent. The conclusion is this: by adopting the convention of Property (3), Property (2) holds no matter the order of a, b and c. Again, in the next section we will see another justification for this property.

4,5. Each of these may be non-intuitive. Property (5) states that when one scales a function by, for instance, 7, the area of the enclosed region also is scaled by a factor of 7. Both Properties (4) and (5) can be proved using geometry. The details are not complicated but are not discussed here.

Example 5.2.13 Evaluating definite integrals using Theorem 5.2.11.

Consider the graph of a function $f(x)$ shown in Figure 5.2.14.
Answer the following:

1. Which value is greater: $\int_a^b f(x)\,dx$ or $\int_b^c f(x)\,dx$?
2. Is $\int_a^c f(x)\,dx$ greater or less than 0?
3. Which value is greater: $\int_a^b f(x)\,dx$ or $\int_c^b f(x)\,dx$?

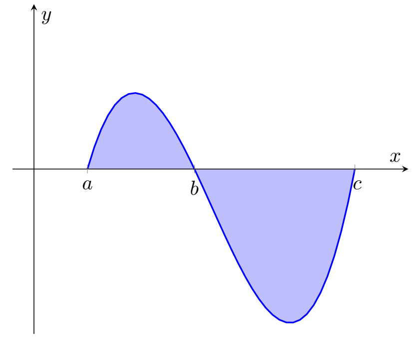

Figure 5.2.14 A graph of a function in Example 5.2.13

Solution.

1. $\int_a^b f(x)\,dx$ has a positive value (since the area is above the x-axis) whereas $\int_b^c f(x)\,dx$ has a negative value. Hence $\int_a^b f(x)\,dx$ is bigger.
2. $\int_a^c f(x)\,dx$ is the total signed area under f between $x = a$ and $x = c$. Since the region below the x-axis looks to be larger than the region above, we conclude that the definite integral has a value less than 0.
3. Note how the second integral has the bounds "reversed." Therefore $\int_c^b f(x)\,dx = -\int_b^c f(x)\,dx$ represents a positive number, greater than the area described by the first definite integral. Hence $\int_c^b f(x)\,dx$ is greater.

The area definition of the definite integral allows us to use geometry to compute the definite integral of some simple functions.

Example 5.2.15 Evaluating definite integrals using geometry.

Evaluate the following definite integrals:

$$1.\ \int_{-2}^{5} (2x-4)\,dx \qquad 2.\ \int_{-3}^{3} \sqrt{9-x^2}\,dx.$$

Solution.

1. It is useful to sketch the function in the integrand, as shown in Figure 5.2.16. We see we need to compute the areas of two regions, which we have labeled R_1 and R_2. Both are triangles, so the area computation is straightforward:

$$R_1 : \frac{1}{2}(4)(8) = 16 \qquad R_2 : \frac{1}{2}(3)6 = 9.$$

Region R_1 lies under the x-axis, hence it is counted as negative area (we can think of the triangle's height as being "-8"), so

$$\int_{-2}^{5} (2x-4)\,dx = -16 + 9 = -7.$$

2. Recognize that the integrand of this definite integral describes a half circle, as sketched in Figure 5.2.17, with radius 3. Thus the

area is:

$$\int_{-3}^{3}\sqrt{9-x^2}\,dx = \frac{1}{2}\pi r^2 = \frac{9}{2}\pi.$$

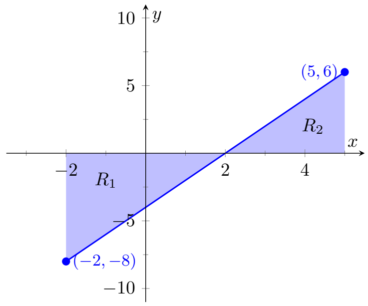

Figure 5.2.16 $f(x) = 2x - 4$

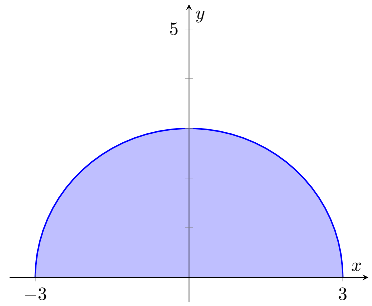

Figure 5.2.17 $f(x) = \sqrt{9-x^2}$

Video solution

youtu.be/watch?v=AuGASVXd3qA

Example 5.2.18 Understanding motion given velocity.

Consider the graph of a velocity function of an object moving in a straight line, given in Figure 5.2.19, where the numbers in the given regions gives the area of that region. Assume that the definite integral of a velocity function gives displacement. Find the maximum speed of the object and its maximum displacement from its starting position.

Solution. Since the graph gives velocity, finding the maximum speed is simple: it looks to be 15ft/s.
At time $t = 0$, the displacement is 0; the object is at its starting position. At time $t = a$, the object has moved backward 11 feet. Between times $t = a$ and $t = b$, the object moves forward 38 feet, bringing it into a position 27 feet forward of its starting position. From $t = b$ to $t = c$ the object is moving backwards again, hence its maximum displacement is 27 feet from its starting position.

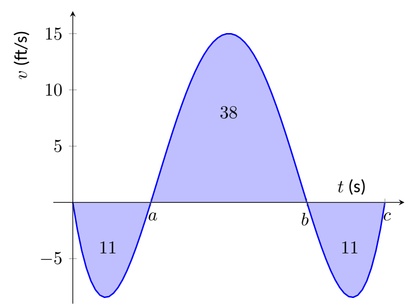

Figure 5.2.19 A graph of a velocity in Example 5.2.18

Video solution

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In our examples, we have either found the areas of regions that have nice geometric shapes (such as rectangles, triangles and circles) or the areas were given to us. Consider Figure 5.2.20, where a region below $y = x^2$ is shaded. What is its area? The function $y = x^2$ is relatively simple, yet the shape it defines has an area that is not simple to find geometrically.

In Section 5.3 we will explore how to find the areas of such regions.

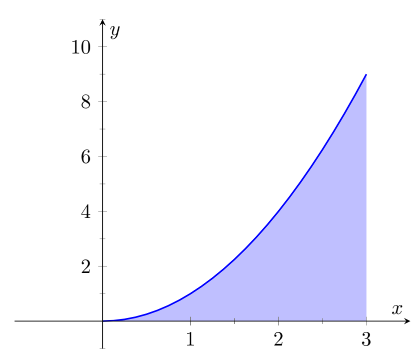


Figure 5.2.20 What is the area below $y = x^2$ on $[0, 3]$? The region is not a usual geometric shape.

5.2.1 Exercises

Terms and Concepts

1. What is "total signed area"?

2. What is "displacement"?

3. What is $\int_3^3 \sin(x)\,dx$?

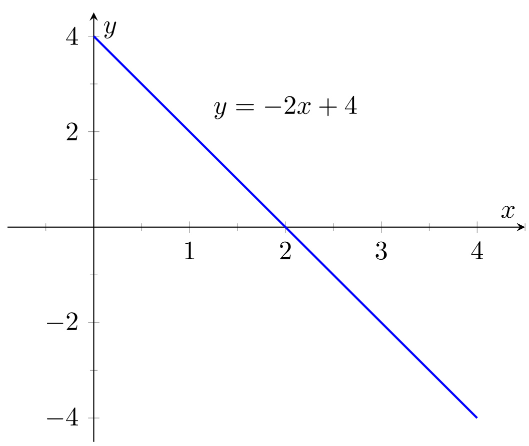
4. Give a single definite integral that has the same value as

$$I = \int_0^1 (2x+3)\,dx + \int_1^2 (2x+3)\,dx.$$

Problems

Exercise Group. A graph of a function $f(x)$ is given. Using the geometry of the graph, evaluate the definite integrals.

5.



(a) $\int_0^1 (-2x+4)\,dx$

(b) $\int_0^2 (-2x+4)\,dx$

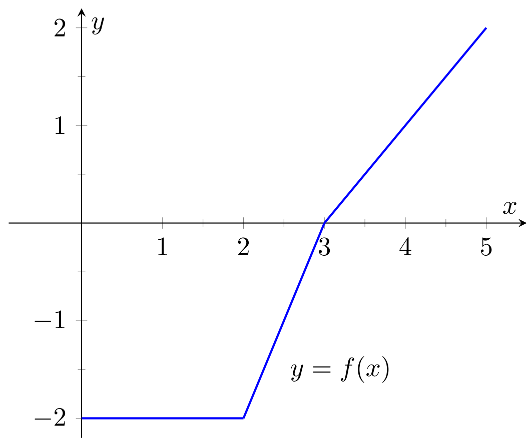
(c) $\int_0^3 (-2x+4)\,dx$

(d) $\int_1^3 (-2x+4)\,dx$

(e) $\int_2^4 (-2x+4)\,dx$

(f) $\int_0^1 (-6x+12)\,d$

6.



(a) $\int_0^2 f(x)\,dx$

(b) $\int_0^3 f(x)\,dx$

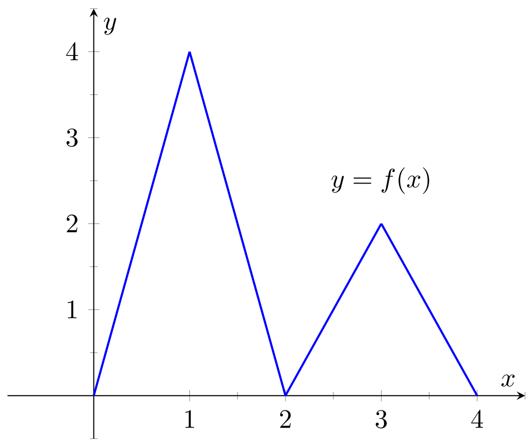
(c) $\int_0^5 f(x)\,dx$

(d) $\int_2^5 f(x)\,dx$

(e) $\int_5^3 f(x)\,dx$

(f) $\int_0^3 -2f(x)\,dx$

7.



(a) $\int_0^2 f(x)\,dx$

(b) $\int_2^4 f(x)\,dx$

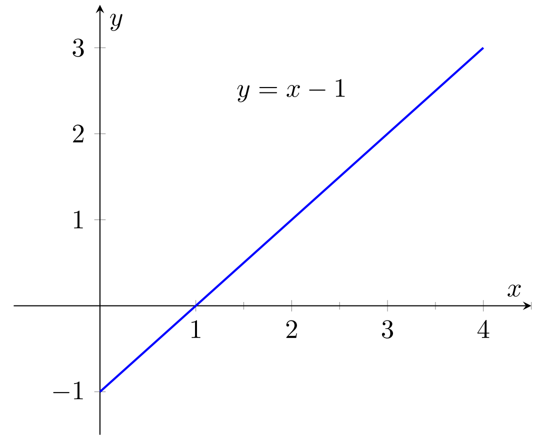
(c) $\int_2^4 2f(x)\,dx$

(d) $\int_0^1 4x\,dx$

(e) $\int_2^3 (2x-4)\,dx$

(f) $\int_2^3 (4x-8)\,dx$

8.



(a) $\int_0^1 (x-1)\,dx$

(b) $\int_0^2 (x-1)\,dx$

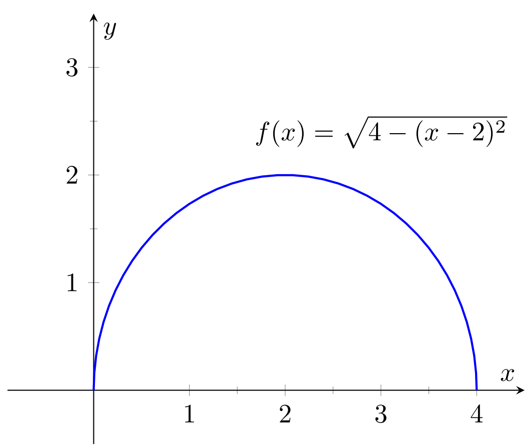
(c) $\int_0^3 (x-1)\,dx$

(d) $\int_2^3 (x-1)\,dx$

(e) $\int_1^4 (x-1)\,dx$

(f) $\int_1^4 \big((x-1)+1\big)\,dx$

9.



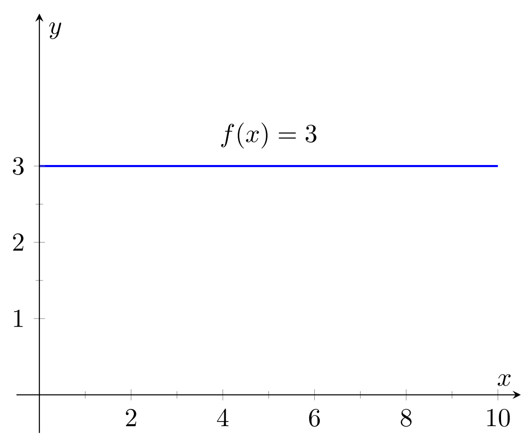
(a) $\int_0^2 f(x)\,dx$

(b) $\int_2^4 f(x)\,dx$

(c) $\int_0^4 f(x)\,dx$

(d) $\int_0^4 5f(x)\,dx$

10.



(a) $\int_0^5 f(x)\,dx$

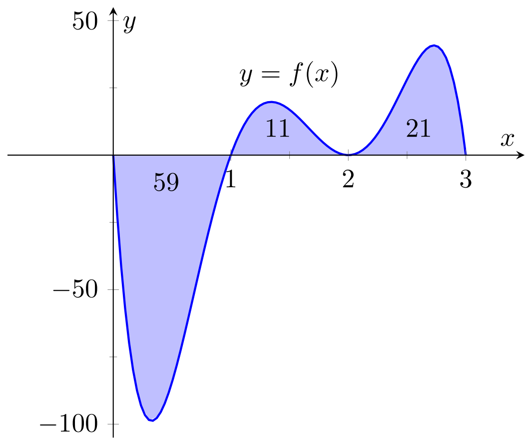
(b) $\int_3^7 f(x)\,dx$

(c) $\int_0^0 f(x)\,dx$

(d) $\displaystyle\int_a^b f(x)\,dx$, where $0 \le a \le b \le 10$

Exercise Group. A graph of a function $f(x)$ is given; the numbers inside the shaded regions give the area of that region. Evaluate the definite integrals using this area information.

11.



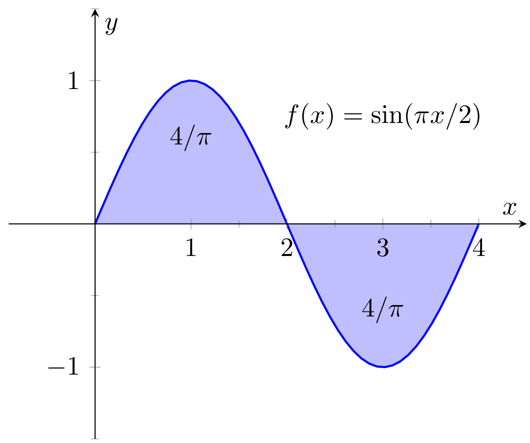
(a) $\int_0^1 f(x)\,dx$

(b) $\int_0^2 f(x)\,dx$

(c) $\int_0^3 f(x)\,dx$

(d) $\int_1^2 -3f(x)\,dx$

12.



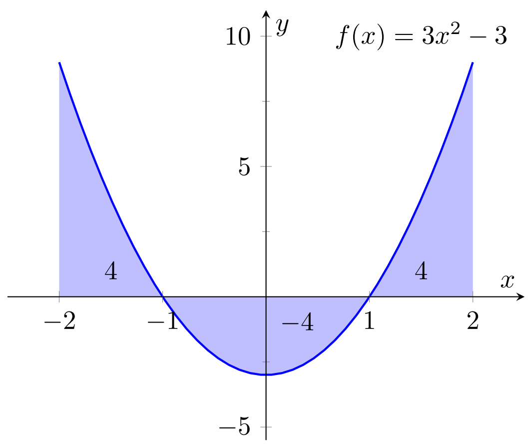
(a) $\int_0^2 f(x)\,dx$

(b) $\int_2^4 f(x)\,dx$

(c) $\int_0^4 f(x)\,dx$

(d) $\int_0^1 f(x)\,dx$

13.



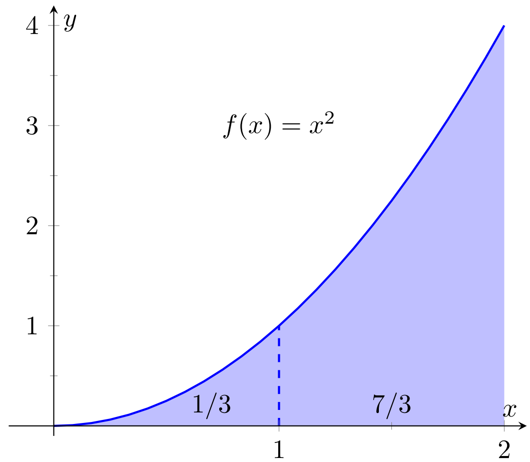
(a) $\int_{-2}^{-1} f(x)\,dx$

(b) $\int_1^2 f(x)\,dx$

(c) $\int_{-1}^1 f(x)\,dx$

(d) $\int_0^1 f(x)\,dx$

14.



(a) $\int_0^2 5x^2\,dx$

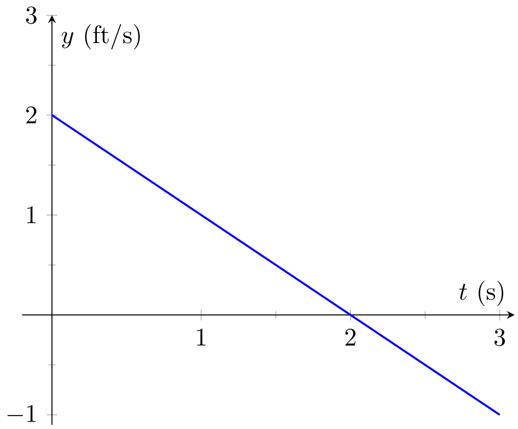
(b) $\int_0^2 (x^2+3)\,dx$

(c) $\int_1^3 (x-1)^2\,dx$

(d) $\int_2^4 \big((x-2)^2+5\big)\,dx$

Exercise Group. A graph is given of the velocity function of an object moving in a straight line. Answer the questions based on the graph.

15.

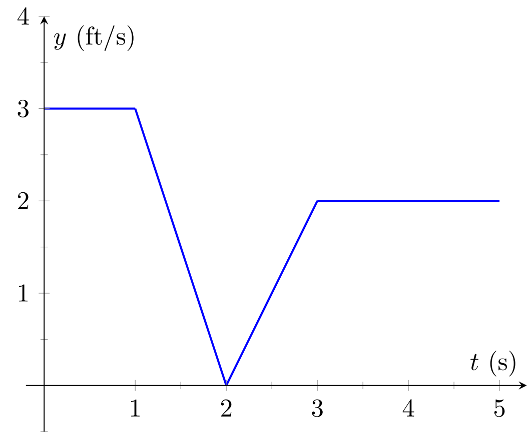


(a) What is the object's maximum velocity?

(b) What is the object's maximum displacement?

(c) What is the object's total displacement on $[0, 3]$?

16.



(a) What is the object's maximum velocity?

(b) What is the object's maximum displacement?

(c) What is the object's total displacement on $[0, 5]$?

17. An object is thrown straight up with a velocity, in ft/s, given by $v(t) = -32t + 64$, where t is in seconds, from a height of 48 feet.

(a) What is the object's maximum velocity?

(b) What is the object's maximum displacement?

(c) When does the maximum displacement occur?

(d) When will the object reach a height of 0? (Hint: find when the displacement is -48ft.)

18. An object is thrown straight up with a velocity, in ft/s, given by $v(t) = -32t + 96$, where t is in seconds, from a height of 64 feet.

(a) What is the object's initial velocity?

(b) When is the object's displacement 0?

(c) How long does it take for the object to return to its initial height?

(d) What is the maximum height the object reaches?

Exercise Group. The values of several definite integrals are given as follows:

$$\int_0^2 f(x)\,dx = 5 \qquad \int_0^3 f(x)\,dx = 7 \qquad \int_0^2 g(x)\,dx = -3 \qquad \int_2^3 g(x)\,dx = 5$$

Use these values and properties of definite integrals to evaluate the indicated definite integral.

19. $\int_0^2 \big(f(x) + g(x)\big)\,dx$

20. $\int_0^3 \big(f(x) - g(x)\big)\,dx$

21. $\int_2^3 \big(3f(x) + 2g(x)\big)\,dx$

22. Find a formula for a in terms of b such that

$$\int_0^3 \big(af(x) + bg(x)\big)\,dx = 0.$$

Exercise Group. The values of several definite integrals are given as follows:

$$\int_0^3 s(t)\,dt = 10 \qquad \int_3^5 s(t)\,dt = 8 \qquad \int_3^5 r(t)\,dt = -1 \qquad \int_0^5 r(t)\,dt = 11$$

Use these values and properties of definite integrals to evaluate the indicated definite integral.

23. $\int_0^3 \big(s(t) + r(t)\big)\,dt$

24. $\int_5^0 \big(s(t) - r(t)\big)\,dt$

25. $\int_3^3 \big(\pi s(t) - 7r(t)\big)\,dt$

26. Find a formula for a in terms of b such that
$$\int_0^5 \big(ar(t) + bs(t)\big)\,dt = 0.$$

5.3 Riemann Sums

In the previous section we defined the definite integral of a function on $[a,b]$ to be the signed area between the curve and the x-axis. Some areas were simple to compute; we ended the section with a region whose area was not simple to compute. In this section we develop a technique to find such areas.

A fundamental calculus technique is to first answer a given problem with an approximation, then refine that approximation to make it better, then use limits in the refining process to find the exact answer. That is what we will do here.

Consider the region given in Figure 5.3.2, which is the area under $y = 4x - x^2$ on $[0,4]$. What is the signed area of this region — i.e., what is $\int_0^4 (4x - x^2)\,dx$?

We start by approximating. We can surround the region with a rectangle with height and width of 4 and find the area is approximately 16 square units. This is obviously an *over-approximation*; we are including area in the rectangle that is not under the parabola.

We have an approximation of the area, using one rectangle. How can we refine our approximation to make it better? The key to this section is this answer: *use more rectangles.*

Let's use four rectangles with an equal width of 1. This *partitions* the interval $[0,4]$ into 4 *subintervals*, $[0,1]$, $[1,2]$, $[2,3]$ and $[3,4]$. On each subinterval we will draw a rectangle.

There are three common ways to determine the height of these rectangles: the *Left Hand Rule*, the *Right Hand Rule*, and the *Midpoint Rule*. The *Left Hand Rule* says to evaluate the function at the left-hand endpoint of the subinterval and make the rectangle that height. In Figure 5.3.4, the rectangle drawn on the interval $[2,3]$ has height determined by the Left Hand Rule; it has a height of $f(2)$. (The rectangle is labeled “LHR.”)

The *Right Hand Rule* says the opposite: on each subinterval, evaluate the function at the right endpoint and make the rectangle that height. In the figure, the rectangle drawn on $[0,1]$ is drawn using $f(1)$ as its height; this rectangle is labeled “RHR.”.

The *Midpoint Rule* says that on each subinterval, evaluate the function at the midpoint and make the rectangle that height. The rectangle drawn on $[1,2]$ was made using the Midpoint Rule, with a height of $f(1.5)$. That rectangle is labeled “MPR.”

These are the three most common rules for determining the heights of approximating rectangles, but one is not forced to use one of these three methods. The rectangle on $[3,4]$ has a height of approximately $f(3.53)$, very close to the Midpoint Rule. It was chosen so that the area of the rectangle is *exactly* the area of the region under f on $[3,4]$. (Later you'll be able to figure how to do this, too.)

The following example will approximate the value of $\int_0^4 (4x - x^2)\,dx$ using these rules.

Example 5.3.5 Using the Left Hand, Right Hand and Midpoint Rules.

Approximate the value of $\int_0^4 (4x - x^2)\,dx$ using the Left Hand Rule, the Right Hand Rule, and the Midpoint Rule, using 4 equally spaced subintervals.

Solution. We break the interval $[0,4]$ into four subintervals as before. In Figure 5.3.6(a) we see 4 rectangles drawn on $f(x) = 4x - x^2$ using the Left Hand Rule. (The areas of the rectangles are given in each figure.) Note how in the first subinterval, $[0,1]$, the rectangle has height $f(0) = 0$. We add up the areas of each rectangle (height× width) for our Left

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Figure 5.3.1 Video introduction to Section 5.3

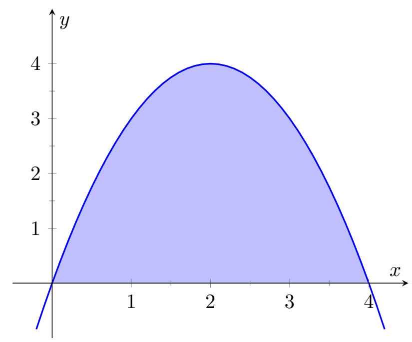


Figure 5.3.2 A graph of $f(x) = 4x - x^2$. What is the area of the shaded region?

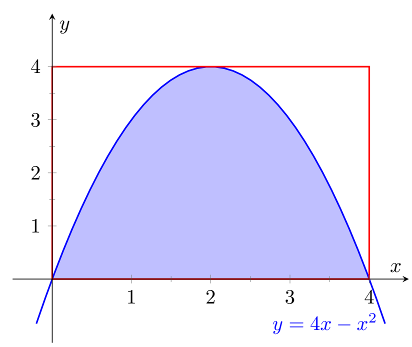


Figure 5.3.3 Approximating area under a curve with one rectangle

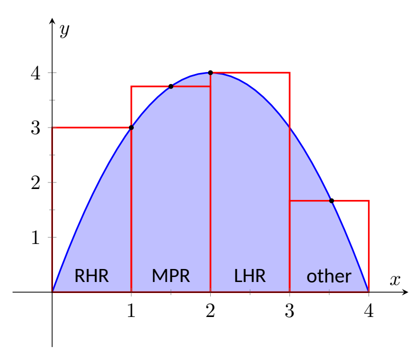


Figure 5.3.4 Approximating $\int_0^4 (4x - x^2)\,dx$ using rectangles. The heights of the rectangles are determined using different rules.

Hand Rule approximation:

$$\begin{aligned} &f(0)\cdot 1+f(1)\cdot 1+f(2)\cdot 1+f(3)\cdot 1\\ =&0+3+4+3=10. \end{aligned}$$

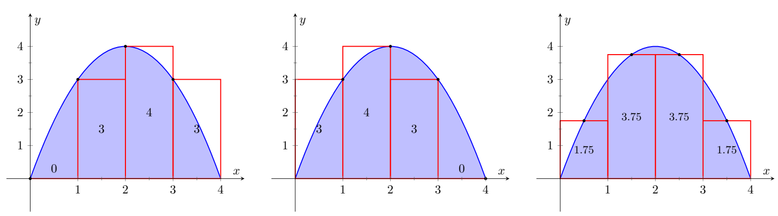
Figure 5.3.6(b) shows 4 rectangles drawn under f using the Right Hand Rule; note how the $[3,4]$ subinterval has a rectangle of height 0.
In this example, these rectangles seem to be the mirror image of those found in Figure 5.3.6(a). This is because of the symmetry of our shaded region. Our approximation gives the same answer as before, though calculated a different way:

$$\begin{aligned} &f(1)\cdot 1+f(2)\cdot 1+f(3)\cdot 1+f(4)\cdot 1\\ &=3+4+3+0=10. \end{aligned}$$

Figure 5.3.6(c) shows 4 rectangles drawn under f using the Midpoint Rule.
This gives an approximation of $\int_0^4(4x-x^2)\,dx$ as:

$$\begin{aligned} &f(0.5)\cdot 1+f(1.5)\cdot 1+f(2.5)\cdot 1+f(3.5)\cdot 1\\ &\qquad =1.75+3.75+3.75+1.75=11. \end{aligned}$$

Our three methods provide two approximations of $\int_0^4(4x-x^2)\,dx$: 10 and 11.



(a) using the Left Hand Rule **(b)** using the Right Hand Rule **(c)** using the Midpoint Rule

Figure 5.3.6 Approximating $\int_0^4(4x-x^2)\,dx$ in Example 5.3.5

Video solution

youtu.be/watch?v=qn8Q1i8s5Ng

5.3.1 Summation Notation

It is hard to tell at this moment which is a better approximation: 10 or 11? We can continue to refine our approximation by using more rectangles. The notation can become unwieldy, though, as we add up longer and longer lists of numbers. We introduce *summation notation* to ameliorate this problem.

Suppose we wish to add up a list of numbers a_1, a_2, a_3, ..., a_9. Instead of writing

$$a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8+a_9,$$

we use summation notation and write $\sum_{i=1}^{9} a_i$. The upper case sigma, $Sigma$ represents the term "sum". The index (counter) of summation in this example is i; any symbol can be used. By convention, the index takes on only the integer values between (and including) the lower and upper bounds. To the right of Σ, the expression a_i is called the summand. It tells us what we are summing. This

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Figure 5.3.7 Explaining summation notation is summarized in Equation (5.3.1).

$$\sum_{\underbrace{i=1}_{i\text{-index of summation}}}^{\overbrace{9}^{\text{upper bound}}} \underbrace{a_i}_{\text{summand}} \tag{5.3.1}$$

Let's practice using this notation.

Example 5.3.8 Using summation notation.

Let the numbers $\{a_i\}$ be defined as $a_i = 2i - 1$ for integers i, where $i \geq 1$. So $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, etc. (The output is the positive odd integers). Evaluate the following summations:

1. $\displaystyle\sum_{i=1}^{6} a_i$
2. $\displaystyle\sum_{i=3}^{7} (3a_i - 4)$
3. $\displaystyle\sum_{i=1}^{4} (a_i)^2$

Solution.

1. $$\begin{aligned}\sum_{i=1}^{6} a_i &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6\\ &= 1 + 3 + 5 + 7 + 9 + 11\\ &= 36.\end{aligned}$$

2. Note the starting value is different than 1:

$$\begin{aligned}\sum_{i=3}^{7} (3a_i - 4) &= (3a_3 - 4) + (3a_4 - 4) + (3a_5 - 4) + (3a_6 - 4) + (3a_7 - 4)\\ &= 11 + 17 + 23 + 29 + 35\\ &= 115.\end{aligned}$$

3. $$\begin{aligned}\sum_{i=1}^{4} (a_i)^2 &= (a_1)^2 + (a_2)^2 + (a_3)^2 + (a_4)^2\\ &= 1^2 + 3^2 + 5^2 + 7^2\\ &= 84.\end{aligned}$$

Video solution

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It might seem odd to stress a new, concise way of writing summations only to write each term out as we add them up. It is. The following theorem gives some of the properties of summations that allow us to work with them without writing individual terms. Examples will follow.

Theorem 5.3.9 Properties of Summations.

1. $\sum_{i=1}^{n} c = c \cdot n$, where c is a constant.
2. $\sum_{i=m}^{n} (a_i \pm b_i) = \sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i$
3. $\sum_{i=m}^{n} c \cdot a_i = c \cdot \sum_{i=m}^{n} a_i$
4. $\sum_{i=m}^{j} a_i + \sum_{i=j+1}^{n} a_i = \sum_{i=m}^{n} a_i$
5. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
6. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
7. $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Example 5.3.11 Evaluating summations using Theorem 5.3.9.

Revisit Example 5.3.8 and, using Theorem 5.3.9, evaluate

$$\sum_{i=1}^{6} a_i = \sum_{i=1}^{6} (2i-1).$$

Solution.

$$\begin{aligned}\sum_{i=1}^{6}(2i-1) &= \sum_{i=1}^{6} 2i - \sum_{i=1}^{6}(1)\\ &= \left(2\sum_{i=1}^{6} i\right) - 6\\ &= 2\frac{6(6+1)}{2} - 6\\ &= 42 - 6 = 36\end{aligned}$$

We obtained the same answer without writing out all six terms. When dealing with small sizes of n, it may be faster to write the terms out by hand. However, Theorem 5.3.9 is incredibly important when dealing with large sums as we'll soon see.

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Figure 5.3.10 Video presentation of Theorem 5.3.9

5.3.2 Riemann Sums

Consider again $\int_0^4 (4x - x^2)\, dx$. We will approximate this definite integral using 16 equally spaced subintervals and the Right Hand Rule in Example 5.3.13. Before doing so, it will pay to do some careful preparation.

Figure 5.3.12 shows a number line of $[0, 4]$ divided, or *partitioned*, into 16 equally spaced subintervals. We denote 0 as x_0; we have marked the values of x_4, x_8, x_{12} and x_{16}. We could mark them all, but the figure would get crowded. While it is easy to figure that $x_9 = 2.25$, in general, we want a method of determining the value of x_i without consulting the figure. Consider:

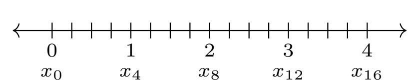

Figure 5.3.12 Dividing $[0, 4]$ into 16 equally spaced subintervals

number of subintervals between x_0 and x_i

$$x_i = x_0 + i\Delta x$$

starting value — subinterval size

So $x_9 = x_0 + 9(4/16) = 2.25$.

If we had partitioned $[0, 4]$ into 100 equally spaced subintervals, each subinterval would have length $\Delta x = 4/100 = 0.04$. We could compute x_{31} as

$$x_{31} = x_0 + 31(4/100) = 1.24.$$

(That was far faster than creating a sketch first.)

Given any subdivision of $[0, 4]$, the first subinterval is $[x_0, x_1]$; the second is $[x_1, x_2]$; the ith subinterval is $[x_{i-1}, x_i]$.

When using the Left Hand Rule, the height of the ith rectangle will be $f(x_{i-1})$.

When using the Right Hand Rule, the height of the ith rectangle will be $f(x_i)$.

When using the Midpoint Rule, the height of the ith rectangle will be $f\left(\frac{x_{i-1}+x_i}{2}\right)$.

Thus approximating $\int_0^4 (4x - x^2)\, dx$ with 16 equally spaced subintervals can be expressed as follows, where $\Delta x = 4/16 = 1/4$:

Left Hand Rule $$\sum_{i=1}^{16} f(x_{i-1})\Delta x$$

Right Hand Rule $$\sum_{i=1}^{16} f(x_i)\Delta x$$

Midpoint Rule $$\sum_{i=1}^{16} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$$

We use these formulas in the next two examples. The following example lets us practice using the Right Hand Rule and the summation formulas introduced in Theorem 5.3.9.

Example 5.3.13 Approximating definite integrals using sums.

Approximate $\int_0^4 (4x - x^2)\, dx$ using the Right Hand Rule and summation formulas with 16 and 1000 equally spaced intervals.

Solution. Using the formula derived before, using 16 equally spaced intervals and the Right Hand Rule, we can approximate the definite integral as

$$\sum_{i=1}^{16} f(x_i)\Delta x.$$

We have $\Delta x = 4/16 = 0.25$. Since $x_i = 0 + i\Delta x$, we have

$$x_i = 0 + i\Delta x = i\Delta x.$$

Using the summation formulas, consider:

$$\int_0^4 (4x - x^2)\, dx \approx \sum_{i=1}^{16} f(x_i)\Delta x$$

$$\begin{aligned}
&= \sum_{i=1}^{16} f(i\Delta x)\Delta x \\
&= \sum_{i=1}^{16} \big(4i\Delta x - (i\Delta x)^2\big)\Delta x \\
&= \sum_{i=1}^{16} (4i\Delta x^2 - i^2\Delta x^3) \\
&= (4\Delta x^2)\sum_{i=1}^{16} i - \Delta x^3 \sum_{i=1}^{16} i^2 \qquad (5.3.2) \\
&= (4\Delta x^2)\frac{16\cdot 17}{2} - \Delta x^3 \frac{16(17)(33)}{6} \\
&= 4\cdot 0.25^2\cdot 136 - 0.25^3\cdot 1496 \\
&= 10.625
\end{aligned}$$

We were able to sum up the areas of 16 rectangles with very little computation. In Figure 5.3.14 the function and the 16 rectangles are graphed. While some rectangles over-approximate the area, other under-approximate the area (by about the same amount). Thus our approximate area of 10.625 is likely a fairly good approximation.

Notice Equation (5.3.2); by replacing 16 by 1,000 (and appropriately changing the value of Δx), we can use that equation to sum up 1000 rectangles!

We do so here, skipping from the original summand to the equivalent of Equation (5.3.2) to save space. Note that $\Delta x = 4/1000 = 0.004$.

$$\begin{aligned}
\int_0^4 (4x - x^2)\,dx &\approx \sum_{i=1}^{1000} f(x_i)\Delta x \\
&= (4\Delta x^2)\sum_{i=1}^{1000} i - \Delta x^3\sum_{i=1}^{1000} i^2 \\
&= (4\Delta x^2)\frac{1000\cdot 1001}{2} - \Delta x^3\frac{1000(1001)(2001)}{6} \\
&= 10.666656
\end{aligned}$$

Using many, many rectangles, we have a likely good approximation of $\int_0^4 (4x - x^2)\Delta x$. That is,

$$\int_0^4 (4x - x^2)\,dx \approx 10.666656.$$

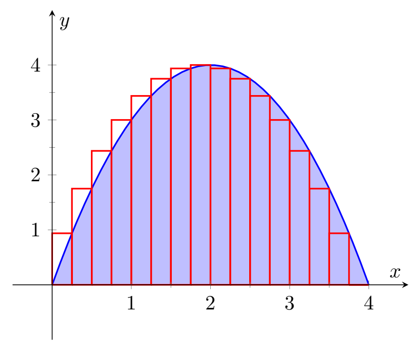

Figure 5.3.14 Approximating $\int_0^4 (4x - x^2)\,dx$ with the Right Hand Rule and 16 evenly spaced subintervals

Video solution

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Before the above example, we stated what the summations for the Left Hand, Right Hand and Midpoint Rules looked like. Each had the same basic structure, which was:

1. each rectangle has the same width, which we referred to as Δx, and

2. each rectangle's height is determined by evaluating f at a particular point in each subinterval. For instance, the Left Hand Rule states that each rectangle's height is determined by evaluating f at the left hand endpoint of the subinterval the rectangle lives on.

One could partition an interval $[a, b]$ with subintervals that do not have the same size. We refer to the length of the ith subinterval as Δx_i. Also, one could determine each rectangle's height by evaluating f at *any* point c_i in the ith subinterval. Thus the height of the ith subinterval would be $f(c_i)$, and the area of the ith rectangle would be $f(c_i)\Delta x_i$. These ideas are formally defined below.

Definition 5.3.15 Partition.

A **partition** Δx of a closed interval $[a, b]$ is a set of numbers $x_0, x_1, \ldots x_n$ where

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b.$$

The length of the ith subinterval, $[x_{i-1}, x_i]$, is $\Delta x_i = x_i - x_{i-1}$. If $[a, b]$ is partitioned into subintervals of equal length, we let Δx represent the length of each subinterval.
The **size of the partition**, denoted $\|\Delta x\|$, is the length of the largest subinterval of the partition.

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Figure 5.3.16 Video presentation of Definition 5.3.15

Summations of rectangles with area $f(c_i)\Delta x_i$ are named after mathematician Georg Friedrich Bernhard Riemann, as given in the following definition.

Definition 5.3.17 Riemann Sum.

Let f be defined on a closed interval $[a, b]$, let Δx be a partition of $[a, b]$ as given in Definition 5.3.15, and let c_i denote any value in the ith subinterval.
The sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

is a **Riemann sum** of f on $[a, b]$.

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Figure 5.3.18 Video presentation of Definition 5.3.17

Figure 5.3.19 shows the approximating rectangles of a Riemann sum of $\int_0^4 (4x - x^2)\,dx$. While the rectangles in this example do not approximate well the shaded area, they demonstrate that the subinterval widths may vary and the heights of the rectangles can be determined without following a particular rule.

"Usually" Riemann sums are calculated using one of the three methods we have introduced. The uniformity of construction makes computations easier. Before working another example, let's summarize some of what we have learned in a convenient way.

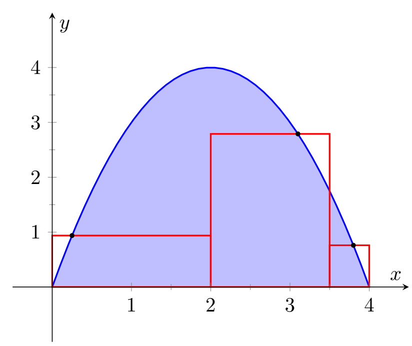


Figure 5.3.19 An example of a general Riemann sum to approximate $\int_0^4 (4x - x^2)\,dx$

Key Idea 5.3.20 Riemann Sum Concepts.

Consider $\displaystyle\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(c_i)\Delta x_i$.

1. When the n subintervals have equal length, $\Delta x_i = \Delta x = \dfrac{b-a}{n}$.
2. The ith term of an equally spaced partition is $x_i = a + i\Delta x$. (Thus $x_0 = a$ and $x_n = b$.)
3. The Left Hand Rule summation is: $\displaystyle\sum_{i=1}^{n} f(x_{i-1})\Delta x$.

4. The Right Hand Rule summation is: $\sum_{i=1}^{n} f(x_i)\Delta x$.

5. The Midpoint Rule summation is: $\sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$.

Let's do another example.

Example 5.3.21 Approximating definite integrals with sums.

Approximate $\int_{-2}^{3}(5x+2)\,dx$ using the Midpoint Rule and 10 equally spaced intervals.

Solution. Following Key Idea 5.3.20, we have

$$\Delta x = \frac{3-(-2)}{10} = 1/2 \text{ and } x_i = (-2) + (1/2)(i) = i/2 - 2.$$

As we are using the Midpoint Rule, we will also need x_{i-1} and $\frac{x_{i-1}+x_i}{2}$. Since $x_i = i/2 - 2$,$x_{i-1} = (i-1)/2 - 2 = i/2 - 5/2$. This gives

$$\frac{x_{i-1}+x_i}{2} = \frac{(i/2-5/2)+(i/2-2)}{2} = \frac{i-9/2}{2} = i/2 - 9/4.$$

We now construct the Riemann sum and compute its value using summation formulas.

$$\begin{aligned}
\int_{-2}^{3}(5x+2)\,dx &\approx \sum_{i=1}^{10} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x \\
&= \sum_{i=1}^{10} f(i/2-9/4)\Delta x \\
&= \sum_{i=1}^{10} (5(i/2-9/4)+2)\Delta x \\
&= \Delta x \sum_{i=1}^{10}\left[\left(\frac{5}{2}\right)i - \frac{37}{4}\right] \\
&= \Delta x\left(\frac{5}{2}\sum_{i=1}^{10}(i) - \sum_{i=1}^{10}\left(\frac{37}{4}\right)\right) \\
&= \frac{1}{2}\left(\frac{5}{2}\cdot\frac{10(11)}{2} - 10\cdot\frac{37}{4}\right) \\
&= \frac{45}{2} = 22.5
\end{aligned}$$

Note the graph of $f(x) = 5x+2$ in Figure 5.3.22. The regions whose area is computed by the definite integral are triangles, meaning we can find the exact answer without summation techniques. We find that the exact answer is indeed 22.5. One of the strengths of the Midpoint Rule is that often each rectangle includes area that should not be counted, but misses other area that should. When the partition size is small, these two amounts are about equal and these errors almost "cancel each other out." In this example, since our function is a line, these errors are exactly equal and they do cancel each other out, giving us the exact answer.

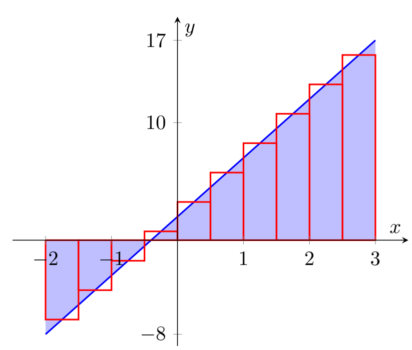


Figure 5.3.22 Approximating $\int_{-2}^{3}(5x+2)\,dx$ using the Midpoint Rule and 10 evenly spaced subintervals in Example 5.3.21

Note too that when the function is negative, the rectangles have a "negative" height. When we compute the area of the rectangle, we use $f(c_i)\Delta x$; when f is negative, the area is counted as negative.

Notice in the previous example that while we used 10 equally spaced intervals, the number "10" didn't play a big role in the calculations until the very end. Mathematicians love to abstract ideas; let's approximate the area of another region using n subintervals, where we do not specify a value of n until the very end.

Video solution

youtu.be/watch?v=O6S1f6-D8Ls

Example 5.3.23 Approximating definite integrals with a formula, using sums.

Revisit $\int_0^4 (4x-x^2)\,dx$ yet again. Approximate this definite integral using the Right Hand Rule with n equally spaced subintervals.

Solution. Using Key Idea 5.3.20, we know $\Delta x = \frac{4-0}{n} = 4/n$. We also find $x_i = 0 + i\Delta x = 4i/n$.
We construct the Right Hand Rule Riemann sum as follows. Be sure to follow each step carefully. If you get stuck, and do not understand how one line proceeds to the next, you may skip to the result and consider how this result is used. You should come back, though, and work through each step for full understanding.

$$\begin{aligned}\int_0^4 (4x-x^2)\,dx &\approx \sum_{i=1}^{n} f(x_i)\Delta x\\ &= \sum_{i=1}^{n} f\left(\frac{4i}{n}\right)\Delta x\\ &= \sum_{i=1}^{n}\left[4\frac{4i}{n} - \left(\frac{4i}{n}\right)^2\right]\Delta x\\ &= \sum_{i=1}^{n}\left(\frac{16\Delta x}{n}\right) i - \sum_{i=1}^{n}\left(\frac{16\Delta x}{n^2}\right) i^2\\ &= \left(\frac{16\Delta x}{n}\right)\sum_{i=1}^{n} i - \left(\frac{16\Delta x}{n^2}\right)\sum_{i=1}^{n} i^2\\ &= \left(\frac{16\Delta x}{n}\right)\cdot\frac{n(n+1)}{2} - \left(\frac{16\Delta x}{n^2}\right)\frac{n(n+1)(2n+1)}{6}\\ &= \frac{32(n+1)}{n} - \frac{32(n+1)(2n+1)}{3n^2} (\text{ recall } \Delta x = 4/n)\\ &= \frac{32}{3}\left(1-\frac{1}{n^2}\right) \quad \text{(after simplifying)}\end{aligned}$$

The result is an amazing, easy to use formula. To approximate the definite integral with 10 equally spaced subintervals and the Right Hand Rule, set $n = 10$ and compute

$$\int_0^4 (4x-x^2)\,dx \approx \frac{32}{3}\left(1-\frac{1}{10^2}\right) = 10.56.$$

Recall how earlier we approximated the definite integral with 4 subintervals; with $n = 4$, the formula gives 10, our answer as before.

It is now easy to approximate the integral with 1,000,000 subintervals! Hand-held calculators will round off the answer a bit prematurely giving an answer of 10.66666667. (The actual answer is 10.666666666656.)
We now take an important leap. Up to this point, our mathematics has been limited to geometry and algebra (finding areas and manipulating expressions). Now we apply *calculus*. For any *finite* n, we know that

$$\int_0^4 (4x - x^2)\,dx \approx \frac{32}{3}\left(1 - \frac{1}{n^2}\right).$$

Both common sense and high-level mathematics tell us that as n gets large, the approximation gets better. In fact, if we take the *limit* as $n \to \infty$, we get the *exact area* described by $\int_0^4 (4x - x^2)\,dx$. That is,

$$\begin{aligned}\int_0^4 (4x - x^2)\,dx &= \lim_{n\to\infty} \frac{32}{3}\left(1 - \frac{1}{n^2}\right)\\ &= \frac{32}{3}(1 - 0)\\ &= \frac{32}{3} = 10.\overline{6}\end{aligned}$$

This is a fantastic result. By considering n equally-spaced subintervals, we obtained a formula for an approximation of the definite integral that involved our variable n. As n grows large — without bound — the error shrinks to zero and we obtain the exact area.

Video solution

youtu.be/watch?v=LjMZOVNdRxQ

This section started with a fundamental calculus technique: make an approximation, refine the approximation to make it better, then use limits in the refining process to get an exact answer. That is precisely what we just did.

Let's practice this again.

Example 5.3.24 Approximating definite integrals with a formula, using sums.

Find a formula that approximates $\int_{-1}^5 x^3\,dx$ using the Right Hand Rule and n equally spaced subintervals, then take the limit as $n \to \infty$ to find the exact area.

Solution. Following Key Idea 5.3.20, we have $\Delta x = \frac{5-(-1)}{n} = 6/n$. We have $x_i = (-1) + i\Delta x$, which is the right endpoint of the ith subinterval. The Riemann sum corresponding to the Right Hand Rule is (followed by simplifications):

$$\begin{aligned}\int_{-1}^5 x^3\,dx &\approx \sum_{i=1}^n f(x_i)\Delta x\\ &= \sum_{i=1}^n f(-1 + i\Delta x)\Delta x\\ &= \sum_{i=1}^n (-1 + i\Delta x)^3 \Delta x\\ &= \sum_{i=1}^n \big((i\Delta x)^3 - 3(i\Delta x)^2 + 3i\Delta x - 1\big)\Delta x \text{ (now distribute } \Delta x)\end{aligned}$$

$$= \sum_{i=1}^{n} \left(i^3\Delta x^4 - 3i^2\Delta x^3 + 3i\Delta x^2 - \Delta x\right) \text{ (now split up summation)}$$

$$= \Delta x^4 \sum_{i=1}^{n} i^3 - 3\Delta x^3 \sum_{i=1}^{n} i^2 + 3\Delta x^2 \sum_{i=1}^{n} i - \sum_{i=1}^{n} \Delta x$$

$$= \Delta x^4 \left(\frac{n(n+1)}{2}\right)^2 - 3\Delta x^3 \frac{n(n+1)(2n+1)}{6} + 3\Delta x^2 \frac{n(n+1)}{2} - n\Delta x$$

(use $\Delta x = 6/n$)

$$= \frac{1296}{n^4} \cdot \frac{n^2(n+1)^2}{4} - 3\frac{216}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + 3\frac{36}{n^2}\frac{n(n+1)}{2} - 6$$

(now do a sizable amount of algebra to simplify)

$$= 156 + \frac{378}{n} + \frac{216}{n^2}$$

Once again, we have found a compact formula for approximating the definite integral with n equally spaced subintervals and the Right Hand Rule. Using 10 subintervals, we have an approximation of 195.96 (these rectangles are shown in Figure 5.3.25). Using $n = 100$ gives an approximation of 159.802.

Now find the exact answer using a limit:

$$\int_{-1}^{5} x^3\,dx = \lim_{n\to\infty} \left(156 + \frac{378}{n} + \frac{216}{n^2}\right) = 156.$$

Video solution

youtu.be/watch?v=yvWSszI0Xvc

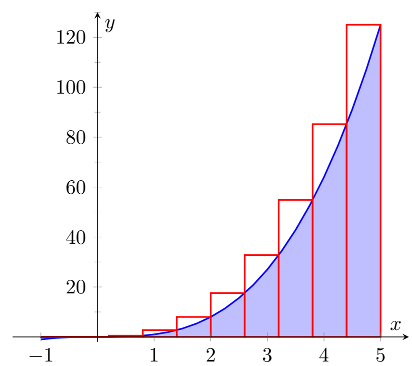


Figure 5.3.25 Approximating $\int_{-1}^{5} x^3\,dx$ using the Right Hand Rule and 10 evenly spaced subintervals

5.3.3 Limits of Riemann Sums

We have used limits to evaluate given definite integrals. Will this always work? It can be shown, given not-very-restrictive conditions, that yes, it will always work — this is the content of Theorem 5.3.26 below.

The previous two examples demonstrated how an expression such as

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

can be rewritten as an expression explicitly involving n, such as $32/3(1 - 1/n^2)$.

Viewed in this manner, we can think of the summation as a function of n. An n value is given (where n is a positive integer), and the sum of areas of n equally spaced rectangles is returned, using the Left Hand, Right Hand, or Midpoint Rules.

Given a definite integral $\int_a^b f(x)\,dx$, let:

- $S_L(n) = \sum_{i=1}^{n} f(x_{i-1})\Delta x$, the sum of equally spaced rectangles formed using the Left Hand Rule,
- $S_R(n) = \sum_{i=1}^{n} f(x_i)\Delta x$, the sum of equally spaced rectangles formed using the Right Hand Rule, and

- $S_M(n) = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$, the sum of equally spaced rectangles formed using the Midpoint Rule.

Recall the definition of a limit as $n \to \infty$: $\lim_{n\to\infty} S_L(n) = K$ if, given any $\varepsilon > 0$, there exists $N > 0$ such that

$$|S_L(n) - K| < \varepsilon \text{ when } n \geq N.$$

The following theorem states that we can use any of our three rules to find the exact value of a definite integral $\int_a^b f(x)\,dx$. It also goes two steps further. The theorem states that the height of each rectangle doesn't have to be determined following a specific rule, but could be $f(c_i)$, where c_i is any point in the ith subinterval, as discussed before Riemann Sums were defined in Definition 5.3.17.

The theorem goes on to state that the rectangles do not need to be of the same width. Using the notation of Definition 5.3.15, let Δx_i denote the length of the ith subinterval in a partition of $[a, b]$ and let $\|\Delta x\|$ represent the length of the largest subinterval in the partition: that is, $\|\Delta x\|$ is the largest of all the Δx_i. If $\|\Delta x\|$ is small, then $[a, b]$ must be partitioned into many subintervals, since all subintervals must have small lengths. "Taking the limit as $\|\Delta x\|$ goes to zero" implies that the number n of subintervals in the partition is growing to infinity, as the largest subinterval length is becoming arbitrarily small. We then interpret the expression

$$\lim_{\|\Delta x\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

as "the limit of the sum of the areas of rectangles, where the width of each rectangle can be different but getting small, and the height of each rectangle is not necessarily determined by a particular rule." The theorem states that this Riemann Sum also gives the value of the definite integral of f over $[a, b]$.

Theorem 5.3.26 Definite Integrals and the Limit of Riemann Sums.

Let f be continuous on the closed interval $[a, b]$ and let $S_L(n)$, $S_R(n)$, $S_M(n)$, Δx, Δx_i and c_i be defined as before. Then:

1. $$\begin{aligned}\lim_{n\to\infty} S_L(n) &= \lim_{n\to\infty} S_R(n)\\ &= \lim_{n\to\infty} S_M(n)\\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x\end{aligned}$$

2. $$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\,dx$$

3. $$\lim_{\|\Delta x\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,dx$$

youtu.be/watch?v=A-WLvclVMC0

Figure 5.3.27 Video presentation of Theorem 5.3.26

We summarize what we have learned over the past few sections here.

- Knowing the "area under the curve" can be useful. One common example: the area under a velocity curve is displacement.

One of the things Theorem 5.3.26 tells us is that if f is continuous on $[a, b]$, then the definite integral $\int_a^b f(x)\,dx$ is guaranteed to exist.

Knowing that every continuous function can be integrated is useful, since most of the functions we work with are continuous. However, it turns out that a function can be integrated even if it has a finite number of discontinuities, as long as these are removable or jump discontinuities.

- We have defined the definite integral, $\int_a^b f(x)\,dx$, to be the signed area under f on the interval $[a,b]$.
- While we can approximate a definite integral many ways, we have focused on using rectangles whose heights can be determined using the Left Hand Rule, the Right Hand Rule and the Midpoint Rule.
- Sums of rectangles of this type are called Riemann sums.
- The exact value of the definite integral can be computed using the limit of a Riemann sum. We generally use one of the above methods as it makes the algebra simpler.

We first learned of derivatives through limits then learned rules that made the process simpler. We know of a way to evaluate a definite integral using limits; in the next section we will see how the Fundamental Theorem of Calculus makes the process simpler. The key feature of this theorem is its connection between the indefinite integral and the definite integral.

5.3.4 Exercises

Terms and Concepts

1. A fundamental calculus technique is to use ___________ to refine approximations to get an exact answer.

2. What is the upper bound in the summation $\sum_{i=2}^{12}(50i+195)$?

3. This section approximates definite integrals using what geometric shape?

4. (☐ True ☐ False) A sum using the Right Hand Rule is an example of a Riemann Sum.

Problems

Exercise Group. Write out each term of the summation and compute the sum.

5. $\sum_{i=3}^{6} i^2$

6. $\sum_{i=-2}^{2} (3i+2)$

7. $\sum_{i=-2}^{1} \sin\left(\frac{\pi i}{2}\right)$

8. $\sum_{i=1}^{8} 9$

9. $\sum_{i=1}^{6} \frac{1}{i}$

10. $\sum_{i=1}^{8} (-1)^i\, i$

11. $\sum_{i=1}^{3} \left(\frac{1}{i}-\frac{1}{i+1}\right)$

12. $\sum_{i=0}^{5} (-1)^i \cos(\pi i)$

Exercise Group. Write the sum in summation notation.

13. $3+6+9+12$

14. $2+3+6+11+18+27+38$

15. $\frac{1}{4}+\frac{2}{5}+\frac{3}{6}+\frac{4}{7}+\frac{5}{8}$

16. $e-e^2+e^3-e^4+e^5$

Exercise Group. Evaluate the summation using Theorem 5.3.9.

17. $\sum_{i=1}^{8} 9$

18. $\sum_{i=1}^{29} i$

19. $\sum_{i=1}^{12} \left(2i^2+2i\right)$

20. $\sum_{i=1}^{12} \left(5i^3-7\right)$

21. $\sum_{i=1}^{8} \left(-3i^3+3i^2+2i-2\right)$

22. $\sum_{i=1}^{9} \left(i^3-8i^2-9i-3\right)$

23. $1+2+3+\cdots+94+95$

24. $1+4+9+\cdots+484+529$

Exercise Group. Theorem 5.3.9 states $\sum_{i=1}^{n} a_i = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{n} a_i$, so $\sum_{i=k+1}^{n} a_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{k} a_i$. Use this fact, along with other parts of Theorem 5.3.9, to evaluate the summation.

25. $\sum_{i=11}^{19} i$

26. $\sum_{i=17}^{28} i^3$

27. $\sum_{i=8}^{14} 3$

28. $\sum_{i=8}^{16} 6i^3$

Exercise Group. In the following exercises, a definite integral $\int_a^b f(x)\,dx$ is given.

(a) Graph $f(x)$ on $[a,b]$.

(b) Add to the sketch rectangles using the provided rule.

(c) Approximate $\int_a^b f(x)\,dx$ by summing the areas of the rectangles.

29. $\int_{-3}^{3} x^2\,dx$, with 6 rectangles using the Left Hand Rule.

30. $\int_0^2 (5-x^2)\,dx$, with 4 rectangles using the Midpoint Rule.

31. $\int_0^{\pi} \sin(x)\,dx$, with 6 rectangles using the Right Hand Rule.

32. $\int_0^3 2^x\,dx$, with 5 rectangles using the Left Hand Rule.

33. $\int_1^2 \ln(x)\,dx$, with 3 rectangles using the Midpoint Rule.

34. $\int_1^9 \frac{1}{x}\,dx$, with 4 rectangles using the Right Hand Rule.

Exercise Group. A definite integral is given below. As demonstrated in Examples 5.3.23 and 5.3.24, do the following:

(a) Find a formula to approximate the definite integral using n subintervals and the provided rule.

(b) Evaluate the formula using $n = 10, 100$, and 1000.

(c) Find the limit of the formula, as $n \to \infty$, to find the exact value of the definite integral.

35. $\int_0^1 x^3\,dx$, using the Left Hand Rule.

36. $\int_{-2}^{1} 2x^2\,dx$, using the Left Hand Rule.

37. $\int_{-2}^{4} (5x+1)\,dx$, using the Midpoint Rule.

38. $\int_2^4 \left(4x^2-2\right)\,dx$, using the Left Hand Rule.

39. $\int_{-11}^{11} (6-x)\,dx$, using the Left Hand Rule.

40. $\int_0^1 \left(x^3-x^2\right)\,dx$, using the Right Hand Rule.

5.4 The Fundamental Theorem of Calculus

youtu.be/watch?v=8d3R9MSwKuk

Figure 5.4.1 Video introduction to Section 5.4

Let $f(t)$ be a continuous function defined on $[a, b]$. The definite integral $\int_a^b f(x)\,dx$ is the "area under f" on $[a, b]$. We can turn this concept into a function by letting the upper (or lower) bound vary.

Let $F(x) = \int_a^x f(t)\,dt$. It computes the area under f on $[a, x]$ as illustrated in Figure 5.4.2. We can study this function using our knowledge of the definite integral. For instance, $F(a) = 0$ since $\int_a^a f(t)\,dt = 0$.

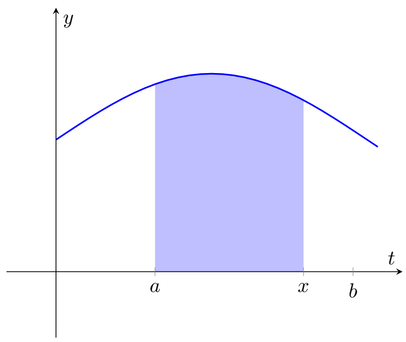


Figure 5.4.2 The area of the shaded region is $F(x) = \int_a^x f(t)\,dt$

Example 5.4.3 Exploring the "Area so far" function.

Consider $f(t) = 2t$ pictured in Figure Figure 5.4.4 and its associated "area so far" function, $F(x) = \int_1^x 2t\,dt$. Using the graph of f and geometry, find an explicit formula for F.

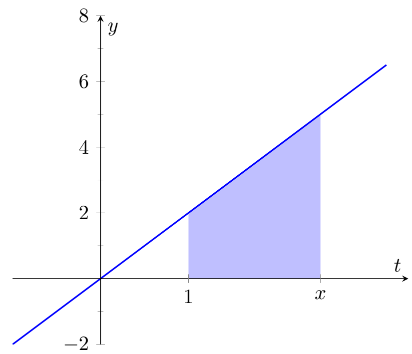


Figure 5.4.4 The area of the shaded region is $F(x) = \int_1^x 2t\,dt$

Solution. We can see from Figure 5.4.5 that for $x \geq 1$, the area under the curve can be found by subtracting the area of two triangles. The larger triangle will have a base of x and a height of $f(x) = 2x$, while the smaller triangle will have a base of 1 and a height of 2. Therefore, the area under the curve for $x \geq 1$ is given by $A(x) = \frac{1}{2}(x)(2x) - \frac{1}{2}(1)(2) = x^2 - 1$.

Note that this same formula holds for $x < 1$. If $x < 1$, then $F(x) = \int_1^x 2t\,dt = -\int_x^1 2t\,dt$. The areas to the left of $x = 1$ will have opposite signs (since they areas are accumulated *before* $x = 1$). For example, when $x = 0$, $F(0) = -\int_0^1 2t\,dt = -\frac{1}{2}(1)(2) = -1$. This is the same value we get from evaluating $x^2 - 1$ for $x = 0$. Also notice that $F(-1) = \int_1^{-1} 2t\,dt = -\int_{-1}^1 2t\,dt$. This integral is clearly 0 since the areas over $[-1, 0]$ and $[0, 1]$ will sum to zero. Again, this is the same answer obtained by evaluating $x^2 - 1$ for $x = -1$.

Therefore, we can reasonably say that $F(x) = x^2 - 1$. A plot of both $f(x) = 2x$ and $F(x) = x^2 - 1$ are given in Figure Figure 5.4.6. You should notice a familiar relationship between these two functions. This relationship is formally stated in Theorem 5.4.7.

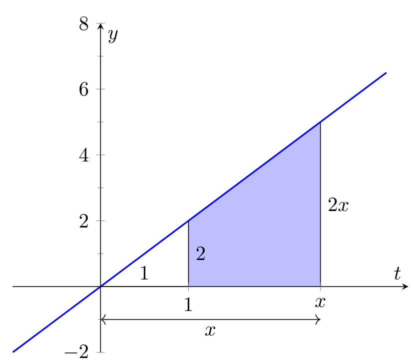


Figure 5.4.5 The area of the shaded region is $F(x) = \int_1^x 2t\,dt$

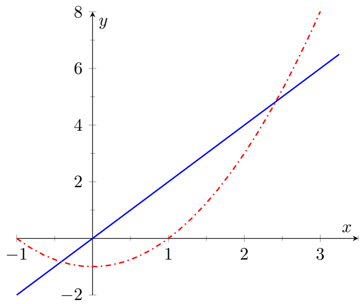


Figure 5.4.6 Graphs of $f(x) = 2x$ and $F(x) = x^2 - 1$

Video solution

youtu.be/watch?v=zVyMghQRLcI

5.4.1 Fundamental Theorem of Calculus, Parts 1 and 2

As Example 5.4.3 hinted, we can apply calculus ideas to $F(x)$; in particular, we can compute its derivative. In Example 5.4.3, $F(x) = x^2 - 1$, so $F'(x) = 2x = f(x)$. While this may seem like an innocuous thing to do, it has far-reaching implications, as demonstrated by the fact that the result is given as an important theorem.

Theorem 5.4.7 The Fundamental Theorem of Calculus, Part 1.

Let f be continuous on $[a, b]$ and let $F(x) = \int_a^x f(t)\,dt$. Then F is continuous on $[a, b]$, differentiable on (a, b), and

$$F'(x) = f(x).$$

In other words:

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

youtu.be/watch?v=TE3kZRIso-Q

Figure 5.4.8 Video presentation of Theorem 5.4.7

Initially this seems simple, as demonstrated in the following example.

Example 5.4.9 Using the Fundamental Theorem of Calculus, Part 1.

Let $F(x) = \displaystyle\int_{-5}^{x} (t^2 + \sin(t))\,dt$. What is $F'(x)$?

Solution. Using the Fundamental Theorem of Calculus, we have $F'(x) = x^2 + \sin(x)$. That is, the derivative of the “area so far” function, is simply the integrand replacing x with t.

This simple example reveals something incredible: $F(x)$ is an antiderivative of $x^2 + \sin(x)$! Therefore, $F(x) = \frac{1}{3}x^3 - \cos(x) + C$ for some value of C. (We can find C, but generally we do not care. We know that $F(-5) = 0$, which allows us to compute C. In this case, $C = \cos(-5) + \frac{125}{3}$.)

Video solution

youtu.be/watch?v=7tHmgPcUZG4

What we have done in Example 5.4.9 was more than finding a complicated way of computing an antiderivative. Consider a function f defined on an open interval containing a, b and c. Suppose we want to compute $\int_a^b f(t)\,dt$. First, let

$$F(x) = \int_c^x f(t)\,dt. \qquad (5.4.1)$$

Using the properties of the definite integral found in Theorem 5.2.11, we know

$$\begin{aligned}\int_a^b f(t)\,dt &= \int_a^c f(t)\,dt + \int_c^b f(t)\,dt\\ &= -\int_c^a f(t)\,dt + \int_c^b f(t)\,dt\end{aligned}$$

Using Equation (5.4.1), let $x = a$ in the first integral and $x = b$ in the second integral so that $\int_c^a f(t)\,dt = F(a)$ and $\int_c^b f(t)\,dt = F(b)$. Therefore:

$$\begin{aligned}\int_a^b f(t)\,dt &= -F(a) + F(b)\\ &= F(b) - F(a).\end{aligned}$$

We now see how indefinite integrals and definite integrals are related: we can evaluate a definite integral using antiderivatives! In fact, this is exactly what we noticed in Example 5.4.3. The "area so far" function was indeed an antiderivative of the integrand. This is the second part of the Fundamental Theorem of Calculus.

Theorem 5.4.10 Fundamental Theorem of Calculus, Part 2.

Let f be continuous on $[a, b]$ and let F be any *antiderivative of f. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

youtu.be/watch?v=jU_WUPjamFQ

Figure 5.4.11 Video presentation of Theorem 5.4.10

As its name suggests, the Fundamental Theorem of Calculus is an important result. In fact, it's sufficiently important that it's worth taking a moment to understand why it's true. A proof is given in Figure 5.4.12.

Example 5.4.13 Using the Fundamental Theorem of Calculus, Part 2.

We spent a great deal of time in the previous section studying $\int_0^4 (4x - x^2)\,dx$. Using the Fundamental Theorem of Calculus, evaluate this definite integral.

Solution. We need an antiderivative of $f(x) = 4x - x^2$. All antiderivatives of f have the form $F(x) = 2x^2 - \frac{1}{3}x^3 + C$; for simplicity, choose $C = 0$.

The Fundamental Theorem of Calculus states

$$\begin{aligned}\int_0^4 (4x - x^2)\,dx &= F(4) - F(0)\\ &= \left(2(4)^2 - \frac{1}{3}4^3\right) - (0 - 0)\\ &= 32 - \frac{64}{3} = 32/3.\end{aligned}$$

This is the same answer we obtained using limits in the previous section, just with much less work.

youtu.be/watch?v=8doi_Al_Img

Figure 5.4.12 Proving the Fundamental Theorem of Calculus

Notation: A special notation is often used in the process of evaluating definite integrals using the Fundamental Theorem of Calculus. Instead of explicitly writing $F(b) - F(a)$, the notation $F(x)\Big|_a^b$ is used. Thus the solution to Example 5.4.13 would be written as:

$$\int_0^4 (4x - x^2)\,dx = \left(2x^2 - \frac{1}{3}x^3\right)\Bigg|_0^4$$
$$= \big(2(4)^2 - \frac{1}{3}4^3\big) - \big(0 - 0\big) = 32/3.$$

The Constant C: Any antiderivative $F(x)$ can be chosen when using the Fundamental Theorem of Calculus to evaluate a definite integral, meaning any value of C can be picked. The constant *always* cancels out of the expression when evaluating $F(b) - F(a)$, so it does not matter what value is picked. This being the case, we might as well let $C = 0$.

Example 5.4.14 Using the Fundamental Theorem of Calculus, Part 2.

Evaluate the following definite integrals.

1. $\displaystyle\int_{-2}^{2} x^3\,dx$
2. $\displaystyle\int_0^{\pi} \sin(x)\,dx$
3. $\displaystyle\int_0^5 e^t\,dt$
4. $\displaystyle\int_4^9 \sqrt{u}\,du$
5. $\displaystyle\int_1^5 2\,dx$

Solution.

1.

$$\int_{-2}^{2} x^3\,dx = \frac{1}{4}x^4\Bigg|_{-2}^{2}$$
$$= \left(\frac{1}{4}2^4\right) - \left(\frac{1}{4}(-2)^4\right)$$
$$= 0.$$

2.

$$\int_0^{\pi} \sin(x)\,dx = -\cos(x)\Big|_0^{\pi}$$
$$= -\cos(\pi) - \big(-\cos(0)\big)$$
$$= 1 + 1 = 2.$$

(This is interesting; it says that the area under one "hump" of a sine curve is 2.)

3.

$$\int_0^5 e^t\,dt = e^t\Big|_0^5$$
$$= e^5 - e^0$$
$$= e^5 - 1 \approx 147.41.$$

4.

$$\int_4^9 \sqrt{u}\,du = \int_4^9 u^{\frac{1}{2}}\,du$$

$$= \frac{2}{3}u^{\frac{3}{2}}\Big|_4^9$$
$$= \frac{2}{3}\left(9^{\frac{3}{2}} - 4^{\frac{3}{2}}\right)$$
$$= \frac{2}{3}(27 - 8) = \frac{38}{3}.$$

5.

$$\int_1^5 2\,dx = 2x\Big|_1^5$$
$$= 2(5) - 2$$
$$= 2(5 - 1) = 8.$$

This integral is interesting; the integrand is a constant function, hence we are finding the area of a rectangle with width $(5-1) = 4$ and height 2. Notice how the evaluation of the definite integral led to $2(4) = 8$. In general, if c is a constant, then $\int_a^b c\,dx = c(b - a)$.

Video solution

youtu.be/watch?v=YxQyFln5UIQ

5.4.2 Understanding Motion with the Fundamental Theorem of Calculus

We established, starting with Key Idea 2.2.3, that the derivative of a position function is a velocity function, and the derivative of a velocity function is an acceleration function. Now consider definite integrals of velocity and acceleration functions. Specifically, if $v(t)$ is a velocity function, what does $\displaystyle\int_a^b v(t)\,dt$ mean?

The Fundamental Theorem of Calculus states that

$$\int_a^b v(t)\,dt = V(b) - V(a),$$

where $V(t)$ is any antiderivative of $v(t)$. Since $v(t)$ is a velocity function, $V(t)$ must be a position function, and $V(b) - V(a)$ measures a change in position, or *displacement*.

Example 5.4.15 Finding displacement and distance.

A ball is thrown straight up with velocity given by $v(t) = -32t + 20$ft/s, where t is measured in seconds. Find, and interpret, $\int_0^1 v(t)\,dt$ and $\int_0^1 |v(t)|\,dt$.

Solution. Using the Fundamental Theorem of Calculus, we have

$$\int_0^1 v(t)\,dt = \int_0^1 (-32t + 20)\,dt$$
$$= \left(-16t^2 + 20t\right)\Big|_0^1$$
$$= 4.$$

Thus if a ball is thrown straight up into the air with velocity $v(t) = -32t + 20$, the height of the ball, 1 second later, will be 4 feet above the initial height.

Note that the ball has *traveled* much farther. It has gone up to its peak and is falling down, but the difference between its height at $t = 0$ and $t = 1$ is 4ft.

If we wish to find the total distance traveled, we must evaluate $\int_0^1 |v(t)|\ dt$ (noting that negative velocities will reduce the diplacement, but we want *distance*, not displacement). In this case, we know that the velocity changes sign once when $v(t) = 0$,so $t = 20/32 = 5/8$ seconds. The velocity is positive over $[0, 5/8]$ and negative over $[5/8, 1]$. Therefore

$$\begin{aligned}\int_0^1 |v(t)|\ dt &= \int_0^{5/8} v(t)\,dt + \int_{5/8}^1 -v(t)\,dt \\ &= \int_0^{5/8} (-32t+20)\,dt - \int_{5/8}^1 (-32t+20)\,dt \\ &= \left(-16t^2+20t\right)\Big|_0^{5/8} - \left(-16t^2+20t\right)\Big|_0^1 \\ &= \frac{25}{4} - \left(-\frac{9}{4}\right) = 9.\end{aligned}$$

So the total distance traveled over $[0, 1]$ is $\int_0^1 |-32t+20|\ dt = 9$ feet . As we can see in Figure 5.4.16, the positive area between $v(t)$ and the t-axis, $A_1 = 25/4$, while the negative area, $A_2 = -9/4$. When we add these two areas, we get the displacement of 4 ft. But when we add the absolute value of both of these areas (as in Figure 5.4.17), we get the total distance of 9 ft.

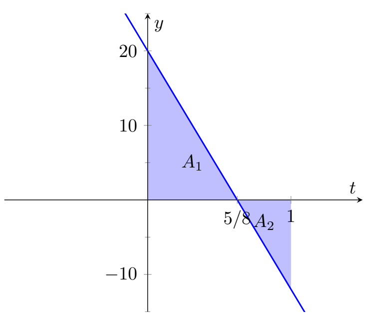


Figure 5.4.16 The area between $v(t)$ and the t-axis can be used to represent displacement

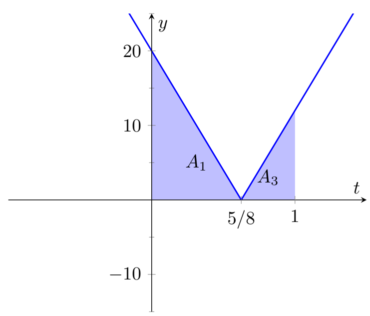


Figure 5.4.17 The area between $|v(t)|$ and the t-axis can be used to represent distance

Integrating a rate of change function gives total change. Velocity is the rate of position change; integrating velocity gives the total change of position, i.e., displacement.

Integrating a speed function gives a similar, though different, result. Speed is also the rate of position change, but does not account for direction. That is, the speed an object is the absolute value of its velocity. This is what we saw in Example 5.4.15 when we evaluated $\int_0^1 |v(t)|\ dt$. So integrating a speed function gives total change of position, without the possibility of "negative position change." Hence the integral of a speed function gives *distance traveled*.

As acceleration is the rate of velocity change, integrating an acceleration function gives total change in velocity. We do not have a simple term for this analogous to displacement. If $a(t) = 5$miles/h^2 and t is measured in hours, then

$$\int_0^3 a(t)\,dt = 15$$

means the velocity has increased by 15m/h from $t = 0$ to $t = 3$.

5.4.3 The Fundamental Theorem of Calculus and the Chain Rule

Part 1 of the Fundamental Theorem of Calculus (FTC) states that given

$$F(x) = \int_a^x f(t)\,dt,$$

we have $F'(x) = f(x)$. Using other notation,

$$\frac{d}{dx}(F(x)) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

While we have just practiced evaluating definite integrals, sometimes finding antiderivatives is impossible and we need to rely on other techniques to approximate the value of a definite integral. Functions written as $F(x) = \int_a^x f(t)\,dt$ are useful in such situations.

It may be of further use to compose such a function with another. As an example, we may compose $F(x)$ with $g(x)$ to get

$$F\big(g(x)\big) = \int_a^{g(x)} f(t)\,dt.$$

What is the derivative of such a function? The Chain Rule can be employed to state

$$\frac{d}{dx}\Big(F\big(g(x)\big)\Big) = F'\big(g(x)\big)g'(x) = f\big(g(x)\big)g'(x).$$

An example will help us understand this.

youtu.be/watch?v=fywjn8-evpE

Figure 5.4.18 Video presentation of Subsection 5.4.3 and Example 5.4.19

Example 5.4.19 The FTC, Part 1, and the Chain Rule.

Find the derivative of $F(x) = \displaystyle\int_2^{x^2} \ln(t)\,dt$.

Solution. We can view $F(x)$ as being the function $G(x) = \int_2^x \ln(t)\,dt$ composed with $g(x) = x^2$; that is, $F(x) = G\big(g(x)\big)$. The Fundamental Theorem of Calculus states that $G'(x) = \ln(x)$. The Chain Rule gives us

$$\begin{aligned} F'(x) &= G'\big(g(x)\big)g'(x)\\ &= \ln(g(x))g'(x)\\ &= \ln(x^2)2x\\ &= 2x\ln(x^2) \end{aligned}$$

Normally, the steps defining $G(x)$ and $g(x)$ are skipped.

Let's practice this once more.

Example 5.4.20 The FTC, Part 1, and the Chain Rule.

Find the derivative of $F(x) = \displaystyle\int_{\cos(x)}^{5} t^3\,dt$.

Solution. Note that $F(x) = -\int_5^{\cos(x)} t^3\,dt$. Viewed this way, the derivative of F is straightforward:

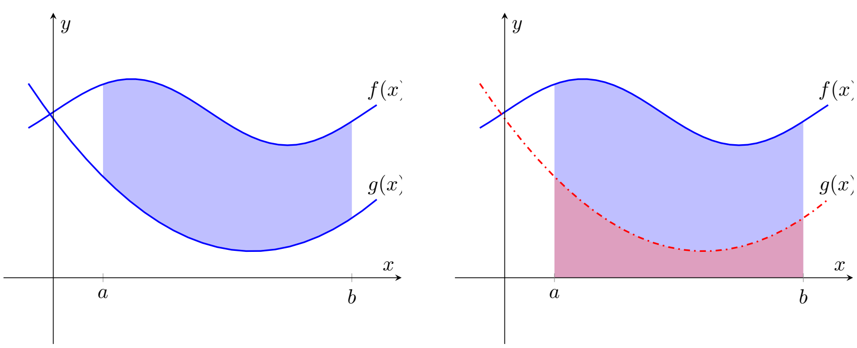
$$\begin{aligned} F'(x) &= -\cos^3(x)\,(-\sin(x)) \\ &= \cos^3(x)\sin(x). \end{aligned}$$

Video solution

youtu.be/watch?v=nGS4ENM8arI

5.4.4 Area Between Curves

Consider continuous functions $f(x)$ and $g(x)$ defined on $[a, b]$, where $f(x) \geq g(x)$ for all x in $[a, b]$, as demonstrated in Figure 5.4.22. What is the area of the shaded region bounded by the two curves over $[a, b]$?



(a) **(b)**

Figure 5.4.22 Finding the area bounded by two functions on an interval by subtracting the area under g from the area under f

youtu.be/watch?v=UufnFHBnv88

Figure 5.4.21 Video introduction to Subsection 5.4.4

The area can be found by recognizing that this area is "the area under f − the area under g." Using mathematical notation, the area is

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

Properties of the definite integral allow us to simplify this expression to

$$\int_a^b \big(f(x) - g(x)\big)\,dx.$$

Theorem 5.4.23 Area Between Curves.

Let $f(x)$ and $g(x)$ be continuous functions defined on $[a, b]$ where $f(x) \geq g(x)$ for all x in $[a, b]$. The area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is

$$\int_a^b \big(f(x) - g(x)\big)\,dx.$$

Example 5.4.24 Finding area between curves.

Find the area of the region enclosed by $y = x^2 + x - 5$ and $y = 3x - 2$.

Solution. It will help to sketch these two functions, as done in Figure 5.4.25.

The region whose area we seek is completely bounded by these two functions; they seem to intersect at $x = -1$ and $x = 3$. To check, set $x^2 + x - 5 = 3x - 2$ and solve for x:

$$\begin{aligned} x^2 + x - 5 &= 3x - 2 \\ (x^2 + x - 5) - (3x - 2) &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \\ x &= -1,\, 3. \end{aligned}$$

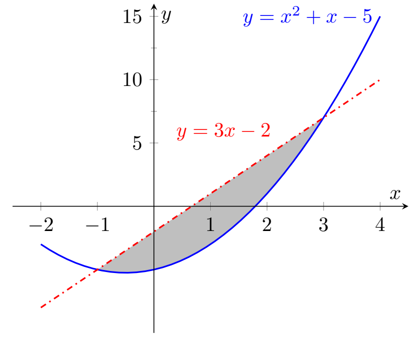


Figure 5.4.25 Sketching the region enclosed by $y = x^2 + x - 5$ and $y = 3x - 2$ in Example 5.4.24

Following Theorem 5.4.23, the area is

$$\begin{aligned} \int_{-1}^{3} \left(3x - 2 - (x^2 + x - 5)\right) dx &= \int_{-1}^{3} (-x^2 + 2x + 3)\, dx \\ &= \left(-\frac{1}{3}x^3 + x^2 + 3x\right)\Bigg|_{-1}^{3} \\ &= -\frac{1}{3}(27) + 9 + 9 - \left(\frac{1}{3} + 1 - 3\right) \\ &= 10\frac{2}{3} = 10.\overline{6} \end{aligned}$$

Video solution

youtu.be/watch?v=su2CXdpYPdo

One of the things we have to be careful about when finding the area between curves is that the curves might cross, so that the distinction between "upper curve" and "lower curve" can change. The video example in Figure 5.4.26 illustrates this phenomenon.

youtu.be/watch?v=Bgji1b7Wdr4

Figure 5.4.26 Finding the area between curves that intersect multiple times

5.4.5 The Mean Value Theorem and Average Value

Consider the graph of a function f in Figure 5.4.27 and the area defined by $\int_1^4 f(x)\, dx$. Three rectangles are drawn in Figure 5.4.28; in Figure 5.4.28(a), the height of the rectangle is greater than f on $[1, 4]$, hence the area of this rectangle is is greater than $\int_1^4 f(x)\, dx$.

In Figure 5.4.28(b), the height of the rectangle is smaller than f on $[1, 4]$, hence the area of this rectangle is less than $\int_1^4 f(x)\, dx$.

Finally, in Figure 5.4.28(c) the height of the rectangle is such that the area of the rectangle is *exactly* that of $\int_1^4 f(x)\, dx$. Since rectangles that are "too big", as in (a), and rectangles that are "too little," as in (b), give areas greater/lesser than $\int_1^4 f(x)\, dx$, it makes sense that there is a rectangle, whose top intersects $f(x)$ somewhere on $[1, 4]$, whose area is *exactly* that of the definite integral.

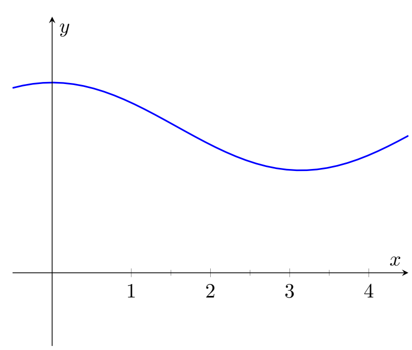
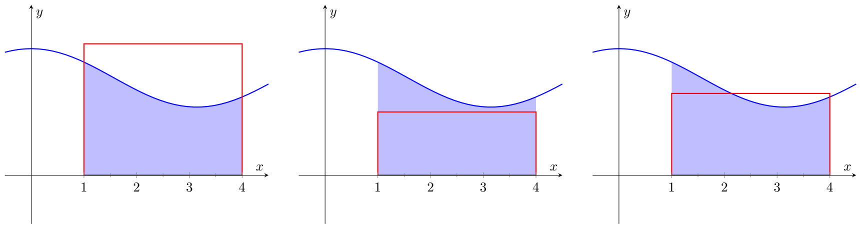


Figure 5.4.27 A graph of a function f to introduce the Mean Value Theorem



(a) (b) (c)

Figure 5.4.28 Differently sized rectangles give upper and lower bounds on $\int_1^4 f(x)\,dx$; the last rectangle matches the area exactly

We state this idea formally in a theorem.

Theorem 5.4.29 The Mean Value Theorem of Integration.

Let f be continuous on $[a,b]$. There exists a value c in $[a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

This is an *existential* statement; c exists, but we do not provide a method of finding it. Theorem 5.4.29 is directly connected to the Mean Value Theorem of Differentiation, given as Theorem 3.2.4; we leave it to the reader to see how.

We demonstrate the principles involved in this version of the Mean Value Theorem in the following example.

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Figure 5.4.30 Video presentation of Theorem 5.4.29

Example 5.4.31 Using the Mean Value Theorem.

Consider $\int_0^\pi \sin(x)\,dx$. Find a value c guaranteed by the Mean Value Theorem.

Solution. We first need to evaluate $\int_0^\pi \sin(x)\,dx$. (This was previously done in Example 5.4.14.)

$$\int_0^\pi \sin(x)\,dx = -\cos(x)\Big|_0^\pi = 2.$$

Thus we seek a value c in $[0,\pi]$ such that $\pi\sin(c) = 2$.

$$\pi\sin(c) = 2 \;\Rightarrow\; \sin(c) = 2/\pi \;\Rightarrow\; c = \arcsin(2/\pi) \approx 0.69.$$

In Figure 5.4.32 $\sin(x)$ is sketched along with a rectangle with height $\sin(0.69)$. The area of the rectangle is the same as the area under $\sin(x)$ on $[0,\pi]$.

The Theorem 5.4.29 simply says that there is a rectangle with height $f(c)$ and width $b-a$, the area of which is the same as the area between f and the x-axis over $[a,b]$. Furthermore, we know that c will be in the interval $[a,b]$.

We now turn our attention to a related topic —average value. Let f be a function on $[a,b]$ with c such that $f(c)(b-a) = \int_a^b f(x)\,dx$. Consider $\int_a^b \big(f(x) - f(c)\big)\,dx$:

$$\begin{aligned}\int_a^b \big(f(x)-f(c)\big)\,dx &= \int_a^b f(x) - \int_a^b f(c)\,dx\\ &= f(c)(b-a) - f(c)(b-a)\\ &= 0.\end{aligned}$$

When $f(x)$ is shifted by $-f(c)$, the amount of area under f above the x-axis on $[a,b]$ is the same as the amount of area below the x-axis above f; see

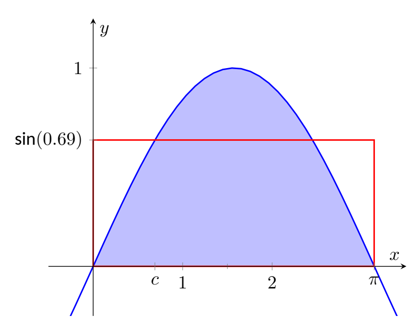


Figure 5.4.32 A graph of $y = \sin(x)$ on $[0,\pi]$ and the rectangle guaranteed by the Mean Value Theorem

Figure 5.4.33 for an illustration of this. In this sense, we can say that $f(c)$ is the *average value* of f on $[a, b]$.

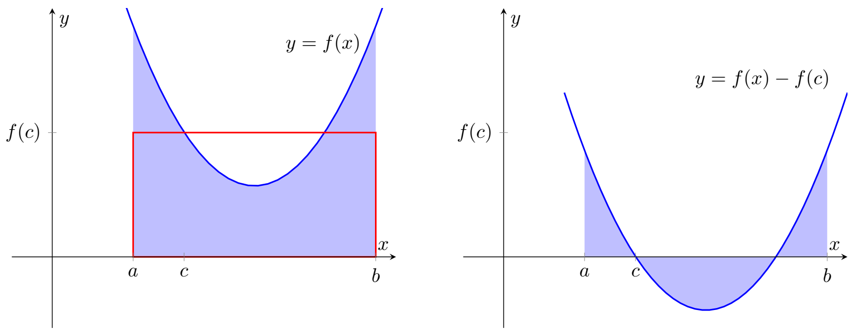


Figure 5.4.33 On the left, a graph of $y = f(x)$ and the rectangle guaranteed by the Mean Value Theorem. On the right, $y = f(x)$ is shifted down by $f(c)$; the resulting "area under the curve" is 0

The value $f(c)$ is the average value in another sense. First, recognize that the Mean Value Theorem can be rewritten as

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

for some value of c in $[a, b]$. Replacing the integral with the limit of a Riemann sum (as in Theorem 5.3.26):

$$\begin{aligned} f(c) &= \frac{1}{b-a}\int_a^b f(x)\,dx \\ &= \frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^n f(c_i)\,\Delta x && \text{Using Theorem 5.3.26} \\ &= \frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^n f(c_i)\,\frac{b-a}{n} && \Delta x = \frac{b-a}{n} \\ &= \lim_{n\to\infty}\sum_{i=1}^n f(c_i)\frac{1}{n} && \text{Cancelling the common factor of } b-a. \end{aligned}$$

Examining this last line closely, the expression $\sum_{i=1}^n f(c_i)\frac{1}{n}$ represents adding up n sample values of $f(x)$and then dividing by n. This is *exactly* what we do when we calculate the average of a set of n numbers. Now when we consider taking the limit as n goes to ∞, $\lim_{n\to\infty}\sum_{i=1}^n f(c_i)\frac{1}{n}$, we are adding up *all* of the function's output values over $[a, b]$ and dividing by the "number of numbers". In a sense, we are adding up an infinite number of output values and then dividing by the number of terms we summed (which is again infinite).

This leads us to a definition.

Definition 5.4.34 The Average Value of f on $[a, b]$.

Let f be continuous on $[a, b]$. The **average value** of f on $[a, b]$ is $f(c)$, where c is a value in $[a, b]$ guaranteed by the Mean Value Theorem. i.e.,

$$\text{Average Value of } f \text{ on } [a, b] = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

An application of this definition is given in the following example.

youtu.be/watch?v=Gz9r9zF5asU

Figure 5.4.35 Video presentation of Definition 5.4.34

Example 5.4.36 Finding the average value of a function.

An object moves back and forth along a straight line with a velocity given by $v(t) = (t-1)^2$ on $[0,3]$, where t is measured in seconds and $v(t)$ is measured in ft/s.
What is the average velocity of the object?

Solution. By our definition, the average velocity is:

$$\begin{aligned}\frac{1}{3-0}\int_0^3 (t-1)^2\,dt &= \frac{1}{3}\int_0^3 \left(t^2-2t+1\right)dt\\ &= \frac{1}{3}\left(\frac{1}{3}t^3 - t^2 + t\right)\Big|_0^3\\ &= \frac{1}{3}\left[\left(\frac{1}{3}(3)^3 - (3)^2 + (3)\right) - \left(\frac{1}{3}(0)^3 - (0)^2 + (0)\right)\right]\\ &= 1 \text{ ft/s}.\end{aligned}$$

We can understand the above example through a simpler situation. Suppose you drove 100 miles in 2 hours. What was your average speed? The answer is simple: displacement/time = 100 miles/2 hours = 50 mph.

What was the displacement of the object in Example 5.4.36? We calculate this by integrating its velocity function: $\int_0^3 (t-1)^2\,dt = 3$ ft. Its final position was 3 feet from its initial position after 3 seconds: its average velocity was 1 ft/s.

This section has laid the groundwork for a lot of great mathematics to follow. The most important lesson is this: definite integrals can be evaluated using antiderivatives. Since Section 5.3 established that definite integrals are the limit of Riemann sums, we can later create Riemann sums to approximate values other than “area under the curve,” convert the sums to definite integrals, then evaluate these using the Theorem 5.4.10. This will allow us to compute the work done by a variable force, the volume of certain solids, the arc length of curves, and more.

The downside is this: generally speaking, computing antiderivatives is much more difficult than computing derivatives. Chapter 6 is devoted to techniques of finding antiderivatives so that a wide variety of definite integrals can be evaluated. Before that, Section 5.5 explores techniques of approximating the value of definite integrals beyond using the Left Hand, Right Hand and Midpoint Rules. These techniques are invaluable when antiderivatives cannot be computed, or when the actual function f is unknown and all we know is the value of f at certain x-values.

5.4.6 Exercises

Terms and Concepts

1. How are definite and indefinite integrals related?

2. What constant of integration is most commonly used when evaluating definite integrals?

3. (☐ True ☐ False) If f is a continuous function, then $F(x) = \int_a^x f(t)\,dt$ is also a continuous function.

4. The definite integral can be used to find "the area under a curve." Give two other uses for definite integrals.

Problems

Exercise Group. Evaluate the definite integral.

5. $\int_1^3 \left(3x^2 - 2x - 7\right)\,dx$

6. $\int_0^5 (x-4)^2\,dx$

7. $\int_{-3}^3 \left(x^7 - x^3\right)\,dx$

8. $\int_0^{\frac{\pi}{2}} \sin(x)\,dx$

9. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec(x)\tan(x)\,dx$

10. $\int_1^{e^7} \frac{1}{x}\,dx$

11. $\int_{-3}^2 8^x\,dx$

12. $\int_{-2}^{-1} \left(3x^2 - 9\right)\,dx$

13. $\int_0^\pi \left(-\left(9\cos(x) + 2\sin(x)\right)\right)\,dx$

14. $\int_1^2 e^x\,dx$

15. $\int_1^{16} \sqrt{t}\,dt$

16. $\int_4^9 \frac{1}{\sqrt{t}}\,dt$

17. $\int_0^{256} \sqrt[4]{x}\,dx$

18. $\int_1^6 \frac{1}{x}\,dx$

19. $\int_1^7 \frac{1}{x^2}\,dx$

20. $\int_1^9 \frac{1}{x^6}\,dx$

21. $\int_0^1 x\,dx$

22. $\int_0^1 x^2\,dx$

23. $\int_0^1 x^3\,dx$

24. $\int_0^1 x^{90}\,dx$

25. $\int_{-7}^7 dx$

26. $\int_{-5}^{-2} 8\,dx$

27. $\int_{-5}^5 0\,dx$

28. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \csc(x)\cot(x)\,dx$

29.

(a) Explain why $\int_{-1}^1 x^n\,dx = 0$ when n is a positive, odd integer.

(b) Explain why $\int_{-1}^1 x^n\,dx = 2\int_0^1 x^n\,dx$ when n is a positive, even integer.

30. Explain why $\int_a^{a+2\pi} \sin t\,dt = 0$ for all values of a.

Exercise Group. Find all values c such that $\int_a^b f(x)\,dx = f(c)(b-a)$, as guaranteed by the Theorem 5.4.29.

31. $\int x^2\,dx$

32. $\int x^2\,dx$

33. $\int e^x\,dx$

34. $\int \sqrt{x}\,dx$

Exercise Group. Find the average value of the function on the given interval.

35. $f(x) = \sin(x)$ on $\left[\frac{\pi}{2}, \pi\right]$

36. $y = \cos(x)$ on $[0, \pi]$

37. $y = x$ on $[0, 7]$

38. $y = x^2$ on $[0, 8]$

39. $y = x^3$ on $[0, 9]$

40. $y = \dfrac{1}{t}$ on $[1, e]$

Exercise Group. A velocity function is given for an object moving along a straight line. Find the displacement of the object over the given time interval.

41. $v(t) = -32t + 22\ \frac{\text{ft}}{\text{s}}$ on $[0, 4]$

42. $v(t) = -32t + 160\ \frac{\text{ft}}{\text{s}}$ on $[0, 9]$

43. $v(t) = 19\ \frac{\text{ft}}{\text{s}}$ on $[0, 4]$

44. $v(t) = 4^t$ mph on $[-2, 2]$

45. $v(t) = \cos(t)\ \frac{\text{ft}}{\text{s}}$ on $[0, \pi]$

46. $v(t) = \sqrt[5]{t}\ \frac{\text{ft}}{\text{s}}$ on $[0, 1024]$

Exercise Group. An acceleration function of an object moving along a straight line is given. Find the change of the object's velocity over the given time interval.

47. $a(t) = -32\ \frac{\text{ft}}{\text{s}^2}$ on $[0, 8]$

48. $a(t) = 8\ \frac{\text{ft}}{\text{s}^2}$ on $[0, 9]$

49. $a(t) = t\ \frac{\text{ft}}{\text{s}^2}$ on $[0, 1]$

50. $a(t) = \sin(t)\ \frac{\text{ft}}{\text{s}^2}$ on $\left[\frac{\pi}{2}, \pi\right]$

Exercise Group. Sketch the given relations and find the area of the enclosed region.

51. $y = 2x$, $y = 5x$, and $x = 3$

52. $y = -x + 1$, $y = 3x + 6$, $x = 2$ and $x = -1$

53. $y = x^2 - 2x + 5$, $y = 5x - 5$

54. $y = 2x^2 + 2x - 5$, $y = x^2 + 3x + 7$,

Exercise Group. Find $F'(x)$.

55. $F(x) = \displaystyle\int_3^{x^3-7x} \frac{1}{t}\,dt$

56. $F(x) = \displaystyle\int_{x^3}^{6} t^3\,dt$

57. $F(x) = \displaystyle\int_x^{x^3} (t-1)\,dt$

58. $F(x) = \displaystyle\int_{\sin(x)}^{e^x} \cos(t)\,dt$

59. $F(x) = \displaystyle\int_9^{x^4} (\sin(4t^2))\,dt$

60. $F(x) = \displaystyle\int_{\sin(x)}^{\ln(x)} (\sqrt{t^4 + 6t^2})\,dt$

5.5 Numerical Integration

The Fundamental Theorem of Calculus gives a concrete technique for finding the exact value of a definite integral. That technique is based on computing antiderivatives. Despite the power of this theorem, there are still situations where we must *approximate* the value of the definite integral instead of finding its exact value. The first situation we explore is where we *cannot* compute the antiderivative of the integrand. The second case is when we actually do not know the function in the integrand, but only its value when evaluated at certain points.

An *elementary function* is any function that is a combination of polynomial, nth root, rational, exponential, logarithmic and trigonometric functions. We can compute the derivative of any elementary function, but there are many elementary functions of which we cannot compute an antiderivative. For example, the following functions do not have antiderivatives that we can express with elementary functions:

$$e^{x^2}, \qquad \sin(x^3), \qquad \frac{\sin(x)}{x}.$$

The simplest way to refer to the antiderivatives of e^{-x^2} is to simply write $\int e^{-x^2}\,dx$.

This section outlines three common methods of approximating the value of definite integrals. We describe each as a systematic method of approximating area under a curve. By approximating this area accurately, we find an accurate approximation of the corresponding definite integral.

We will apply the methods we learn in this section to the following definite integrals:

$$\int_0^1 e^{-x^2}\,dx, \qquad \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx, \qquad \int_{0.5}^{4\pi} \frac{\sin(x)}{x}\,dx,$$

as pictured in Figure 5.5.1.

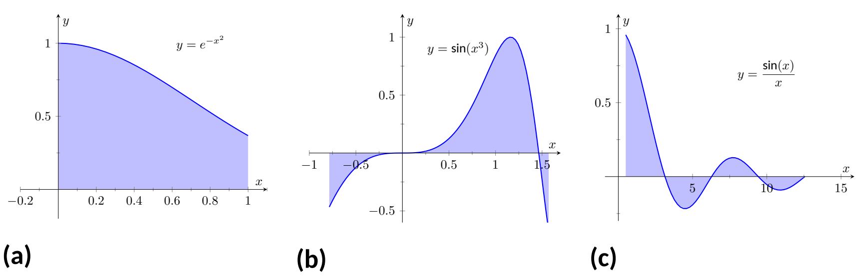


Figure 5.5.1 Graphically representing three definite integrals that cannot be evaluated using antiderivatives

5.5.1 The Left and Right Hand Rule Methods

In Section 5.3 we addressed the problem of evaluating definite integrals by approximating the area under the curve using rectangles. We revisit those ideas here before introducing other methods of approximating definite integrals.

We start with a review of notation. Let f be a continuous function on the interval $[a, b]$. We wish to approximate $\displaystyle\int_a^b f(x)\,dx$. We partition $[a, b]$ into n equally spaced subintervals, each of length $\Delta x = \dfrac{b-a}{n}$. The endpoints of these subintervals are labeled as

$$x_0 = a,\ x_1 = a + \Delta x,\ x_2 = a + 2\Delta x,\ \ldots,\ x_i = a + i\Delta x,\ \ldots,\ x_n = b.$$

Key Idea 5.3.20 states that to use the Left Hand Rule we use the summation $\sum_{i=1}^{n} f(x_{i-1})\Delta x$ and to use the Right Hand Rule we use $\sum_{i=1}^{n} f(x_i)\Delta x$. We review the use of these rules in the context of examples.

Example 5.5.2 Approximating definite integrals with rectangles.

Approximate $\int_0^1 e^{-x^2}\,dx$ using the Left and Right Hand Rules with 5 equally spaced subintervals.

Solution. We begin by partitioning the interval $[0,1]$ into 5 equally spaced intervals. We have $\Delta x = \frac{1-0}{5} = 1/5 = 0.2$, so

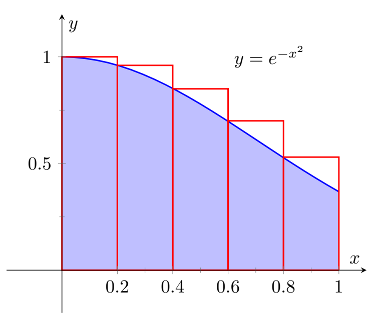
$$x_0 = 0,\ x_1 = 0.2,\ x_2 = 0.4,\ x_3 = 0.6,\ x_4 = 0.8,\ \text{and}\ x_5 = 1.$$

Using the Left Hand Rule, we have:

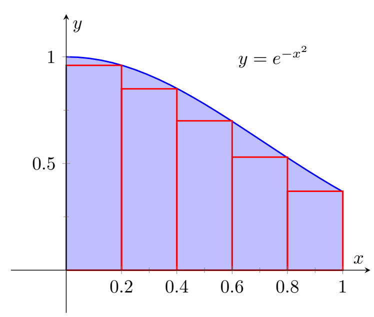
$$\begin{aligned}\sum_{i=1}^{n} f(x_{i-1})\Delta x &= \big(f(x_0)+f(x_1)+f(x_2)+f(x_3)+f(x_4)\big)\Delta x\\ &= \big(f(0)+f(0.2)+f(0.4)+f(0.6)+f(0.8)\big)\Delta x\\ &\approx (1+0.9608+0.8521+0.6977+0.5273)(0.2)\\ &\approx 0.8076.\end{aligned}$$

Using the Right Hand Rule, we have:

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)\Delta x &= \big(f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)\big)\Delta x\\ &= \big(f(0.2)+f(0.4)+f(0.6)+f(0.8)+f(1)\big)\Delta x\\ &\approx (0.9608+0.8521+0.6977+0.5273+0.3678)(0.2)\\ &\approx 0.6812.\end{aligned}$$



(a) Using the Left Hand Rule



(b) Using the Right Hand Rule

Figure 5.5.3 Approximating $\int_0^1 e^{-x^2}\,dx$ in Example 5.5.2

Figure 5.5.3 shows the rectangles used in each method to approximate the definite integral. These graphs show that in this particular case, the Left Hand Rule is an over approximation and the Right Hand Rule is an under approximation. To get a better approximation, we could use more rectangles, as we did in Section 5.3. We could also average the Left and Right Hand Rule results together, giving

$$\frac{0.8076+0.6812}{2} = 0.7444.$$

The actual answer, accurate to 4 places after the decimal, is 0.7468, showing our average is a good approximation.

Example 5.5.4 Approximating definite integrals with rectangles.

Approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ using the Left and Right Hand Rules with 10 equally spaced subintervals.

Solution. We begin by finding Δx:

$$\frac{b-a}{n} = \frac{\pi/2-(-\pi/4)}{10} = \frac{3\pi}{40} \approx 0.2356.$$

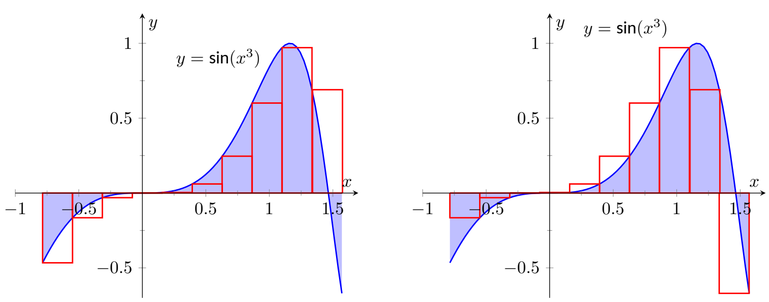
It is useful to write out the endpoints of the subintervals in a table; in Figure 5.5.5, we give the exact values of the endpoints, their decimal approximations, and decimal approximations of $\sin(x^3)$ evaluated at these points.

Once this table is created, it is straightforward to approximate the definite integral using the Left and Right Hand Rules. (Note: the table itself is easy to create, especially with a standard spreadsheet program on a computer. The last two columns are all that are needed.) The Left Hand Rule sums the first 10 values of $\sin(x_i^3)$ and multiplies the sum by Δx; the Right Hand Rule sums the last 10 values of $\sin(x_i^3)$ and multiplies by Δx. Therefore we have:

Left Hand Rule: $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx \approx (1.9093)(0.2356) \approx 0.4498.$

Right Hand Rule: $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx \approx (1.705)(0.2356) \approx 0.4017.$

Average of the Left and Right Hand Rules: 0.4258.



(a) **(b)**

Figure 5.5.6 Approximating $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ in Example 5.5.4

The actual answer, accurate to 4 places after the decimal, is 0.4609. Our approximations were once again fairly good. The rectangles used in each approximation are shown in Figure 5.5.6(a). It is clear from the graphs that using more rectangles (and hence, narrower rectangles) should result in a more accurate approximation.

x_i	Exact	Approx.	$\sin(x_i^3)$
x_0	$-\pi/4$	-0.7854	-0.4657
x_1	$-7\pi/40$	-0.5498	-0.1654
x_2	$-\pi/10$	-0.3142	-0.0310
x_3	$-\pi/40$	-0.0785	-0.0005
x_4	$\pi/20$	0.1571	0.0039
x_5	$\pi/8$	0.3927	0.0605
x_6	$\pi/5$	0.6283	0.2455
x_7	$11\pi/40$	0.8639	0.6011
x_8	$7\pi/20$	1.0996	0.9710
x_9	$17\pi/40$	1.3352	0.6899
x_{10}	$\pi/2$	1.5708	-0.6700

Figure 5.5.5 Values used to approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ in Example 5.5.4

5.5.2 The Trapezoidal Rule

In Example 5.5.2 we approximated the value of $\displaystyle\int_0^1 e^{-x^2}\,dx$ with 5 rectangles of equal width. Figure 5.5.3 shows the rectangles used in the Left and Right Hand Rules. These graphs clearly show that rectangles do not match the shape of the graph all that well, and that accurate approximations will only come by using lots of rectangles.

Instead of using rectangles to approximate the area, we can instead use *trapezoids*. In Figure 5.5.7, we show the region under $f(x) = e^{-x^2}$ on $[0,1]$ approximated with 5 trapezoids of equal width; the top "corners" of each trapezoid lies on the graph of $f(x)$. It is clear from this figure that these trapezoids more accurately approximate the area under f and hence should give a better approximation of $\int_0^1 e^{-x^2}\,dx$. (In fact, these trapezoids seem to give a *great* approximation of the area!)

The formula for the area of a trapezoid is given in Figure 5.5.8. We approximate $\int_0^1 e^{-x^2}\,dx$ with these trapezoids in the following example.

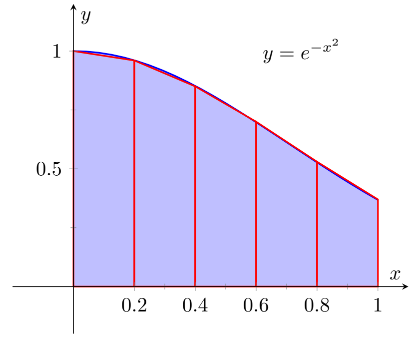


Figure 5.5.7 Approximating $\int_0^1 e^{-x^2}\,dx$ using 5 trapezoids of equal widths

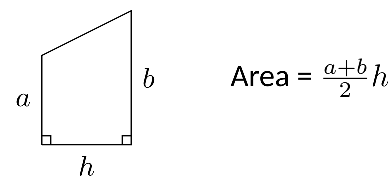


Figure 5.5.8 The area of a trapezoid

Example 5.5.9 Approximating definite integrals using trapezoids.

Use 5 trapezoids of equal width to approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$.

Solution. To compute the areas of the 5 trapezoids in Figure 5.5.7, it will again be useful to create a table of values as shown in Figure 5.5.10. The leftmost trapezoid has legs of length 1 and 0.9607 and a height of 0.2. Thus, by our formula, the area of the leftmost trapezoid is:

$$\frac{1+0.9608}{2}(0.2) = 0.1961.$$

Moving right, the next trapezoid has legs of length 0.9607 and 0.8521 and a height of 0.2. Thus its area is:

$$\frac{0.9608+0.8521}{2}(0.2) = 0.1813.$$

The sum of the areas of all 5 trapezoids is:

$$\frac{1+0.9608}{2}(0.2) + \frac{0.9608+0.8521}{2}(0.2) + \frac{0.8521+0.6977}{2}(0.2) + \frac{0.6977+0.5273}{2}(0.2) + \frac{0.5273+0.3679}{2}(0.2) = 0.7444.$$

We approximate $\int_0^1 e^{-x^2}\,dx \approx 0.7444$.

x_i	$e^{-x_i^2}$
0	1
0.2	0.9608
0.4	0.8521
0.6	0.6977
0.8	0.5273
1	0.3679

Figure 5.5.10 A table of values of e^{-x^2}

There are many things to observe in this example. Note how each term in the final summation was multiplied by both 1/2 and by $\Delta x = 0.2$. We can factor these coefficients out, leaving a more concise summation as:

$$\frac{1}{2}(0.2)\Big[(1+0.9608) + (0.9608+0.8521) + (0.8521+0.6977) + (0.6977+0.5273) + (0.5273+0.3679)\Big].$$

Now notice that all numbers except for the first and the last are added twice. Therefore we can write the summation even more concisely as

$$\frac{0.2}{2}\Big[1 + 2(0.9608+0.8521+0.6977+0.5273) + 0.3679\Big].$$

This is the heart of the *Trapezoidal Rule*, wherein a definite integral $\int_a^b f(x)\,dx$ is approximated by using trapezoids of equal widths to approximate the corresponding area under f. Using n equally spaced subintervals with endpoints x_0, $x_1, \ldots, x_n$, we again have $\Delta x = \dfrac{b-a}{n}$. Thus:

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \sum_{i=1}^{n} \frac{f(x_{i-1})+f(x_i)}{2}\Delta x\\ &= \frac{\Delta x}{2}\sum_{i=1}^{n}\big(f(x_{i-1})+f(x_i)\big)\\ &= \frac{\Delta x}{2}\Big[f(x_0)+\left(2\sum_{i=1}^{n-1} f(x_i)\right)+f(x_n)\Big].\end{aligned}$$

Example 5.5.11 Using the Trapezoidal Rule.

Revisit Example 5.5.4 and approximate $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ using the Trapezoidal Rule and 10 equally spaced subintervals.

Solution. We refer back to Figure 5.5.5 for the table of values of $\sin(x^3)$. Recall that $\Delta x = 3\pi/40 \approx 0.236$. Thus we have:

$$\begin{aligned}&\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx\\ &\approx \frac{0.236}{2}\Big[-0.4657+2\Big(-0.1654+(-0.031)+\ldots+0.68999\Big)+(-0.67)\Big]\\ &= 0.4258.\end{aligned}$$

The actual answer, accurate to 4 decimal places is 0.4609. So the Trapezoidal Rule with 10 subintervals is an under-approximation by about 0.0351.

Notice how "quickly" the Trapezoidal Rule can be implemented once the table of values is created. This is true for all the methods explored in this section; the real work is creating a table of x_i and $f(x_i)$ values. Once this is completed, approximating the definite integral is not difficult. Again, using technology is wise. Spreadsheets can make quick work of these computations and make using lots of subintervals easy.

Also notice the approximations the Trapezoidal Rule gives. It is the average of the approximations given by the Left and Right Hand Rules! This effectively renders the Left and Right Hand Rules obsolete. They are useful when first learning about definite integrals, but if a real approximation is needed, one is generally better off using the Trapezoidal Rule instead of either the Left or Right Hand Rule. However, there are two other methods that are also generally more accurate than the Left or Right Hand Rule.

5.5.3 The Midpoint Rule

Another method that can be more accurate than the Trapezoidal Rule is the Midpoint Rule:

$$S_M(n) = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$$

$$= \sum_{i=1}^{n} f\left(\overline{x_i}\right) \Delta x$$

where $\overline{x_i}$ is the midpoint of each subinterval,

$$\overline{x_i} = a + \Delta x \left(i - \frac{1}{2}\right)$$

Example 5.5.12 Using the Midpoint Rule.

Use the Midpoint Rule with $n = 5$ to approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$.

Solution. We cannot use the table in Figure 5.5.10 that we used for the Trapezoidal, Right and Left Hand Rules when using the Midpoint Rule. The Trapezoidal rule averages the *outputs* of the function to obtain a more accurate estimate of the definite integral. The Midpoint Rule averages the *inputs* of each subinterval to create a rectangle with height $f\left(\frac{x_{i-1}+x_i}{2}\right)$. Generally $f\left(\frac{x_{i-1}+x_i}{2}\right) \neq \frac{f(x_{i-1})+f(x_i)}{2}$.
So we will create a new table of values as shown in Figure 5.5.13. We have $\Delta x = (1-0)/5 = 0.2$. The midpoint of the first subinteval is at $0 + 0.2(1/2) = 0.1$ and each successive midpoint is 0.2 from the last. So we have

$$\int_0^1 e^{-x^2}\,dx \approx 0.2(0.99 + 0.9139 + 0.7788 + 0.6126 + 0.4449)$$
$$\approx 0.7480$$

We approximate $\displaystyle\int_0^1 e^{-x^2}\,dx \approx 0.7480.$

x_i	$e^{-x_i^2}$
0.1	0.9900
0.3	0.9139
0.5	0.7788
0.7	0.6126
0.9	0.4449

Figure 5.5.13 A table of values of e^{-x^2}

Example 5.5.14 Using the Midpoint Rule.

Revisit Example 5.5.11 and approximate $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ using the Midpoint Rule and 10 equally spaced subintervals.

Solution. Again, a table will be useful. Recall that $\Delta x = 3\pi/40 \approx 0.2356$. The midpoint of the first subinterval is $\overline{x_1} = a + \Delta x/2 = -\pi/4 + 3\pi/40(1/2) = -17\pi/80$ (notice that $\overline{x_1}$ is half of a subinterval width to the right of a). Each successive midpoint is $\Delta x = 3\pi/40 = 6\pi/80$ to the right of the last. So we have:
Thus we have:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$$
$$\approx 0.2356\Big[-0.2932 + (-0.0805) + (-0.0076) + \cdots + 0.9729 + 0.0740\Big]$$
$$= 0.2356 \cdot 2.0339$$
$$\approx 0.4792.$$

The actual answer, accurate to 4 decimal places is 0.4609. So the Midpoint Rule with 10 subintervals is an overapproximation by about 0.0183. Notice that this error is about half of the error in using the Trapezoidal Rule.

$\overline{x_i}$	Exact	Approx.	$\sin(x_i^3)$
$\overline{x_1}$	$-17\pi/80$	-0.6676	-0.2932
$\overline{x_2}$	$-11\pi/80$	-0.4320	-0.0805
$\overline{x_3}$	$-5\pi/80$	-0.1963	-0.0076
$\overline{x_4}$	$1\pi/80$	-0.0393	0.0001
$\overline{x_5}$	$7\pi/80$	0.2749	0.0208
$\overline{x_6}$	$13\pi/80$	0.5105	0.1327
$\overline{x_7}$	$19\pi/80$	0.7461	0.4035
$\overline{x_8}$	$25\pi/80$	0.9817	0.8112
$\overline{x_9}$	$31\pi/80$	1.2174	0.9729
$\overline{x_{10}}$	$37\pi/80$	1.4530	0.0740

Figure 5.5.15 Values used to approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ in Example 5.5.14

In many cases, the Midpoint Rule will more accurate than the Trapezoidal Rule. You may wonder though, how can we improve on the Trapezoidal and Midpoint Rules, apart from using more and more subintervals? The answer is clear once we look back and consider what we have *really* done so far. The Left Hand Rule, Right Hand Rule and Midpoint Rules are not *really* about using rectangles to approximate area. Instead, they approximate a function f with constant functions on small subintervals and then compute the definite integral of these constant functions. The Trapezoidal Rule is really approximating a function f with a linear function on a small subinterval, then computing the definite integral of this linear function. In all of these cases the definite integrals are easy to compute in geometric terms.

So we have a progression: we start by approximating f with a constant function and then with a linear function. What is next? A quadratic function. By approximating the curve of a function with lots of parabolas, we generally get an even better approximation of the definite integral. We call this process *Simpson's Rule*, named after Thomas Simpson (1710-1761), even though others had used this rule as much as 100 years prior.

5.5.4 Simpson's Rule

Given one point, we can create a constant function that goes through that point. Given two points, we can create a linear function that goes through those points. Given three points, we can create a quadratic function that goes through those three points (given that no two have the same x-value).

Consider three points (x_0, y_0), (x_1, y_1) and (x_2, y_2) whose x-values are equally spaced and $x_0 < x_1 < x_2$. Let f be the quadratic function that goes through these three points. It is not hard to show that

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{x_2 - x_0}{6}\big(y_0 + 4y_1 + y_2\big). \qquad (5.5.1)$$

While it's not *hard* to show the results of Equation (5.5.1), it's also not exactly easy. This video might help: `youtu.be/uc4xJsi99bk`

Consider Figure 5.5.16. A function f goes through the 3 points shown and the parabola g that also goes through those points is graphed with a dashed line. Using our equation from above, we know exactly that

$$\int_1^3 g(x)\,dx = \frac{3-1}{6}\big(3 + 4(1) + 2\big) = 3.$$

Since g is a good approximation for f on $[1, 3]$, we can state that

$$\int_1^3 f(x)\,dx \approx 3.$$

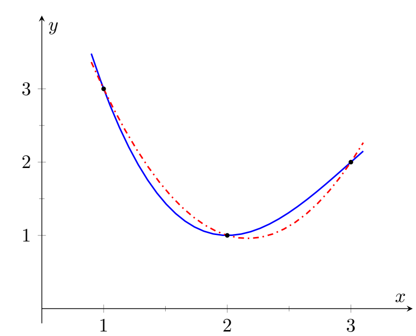


Figure 5.5.16 A graph of a function f and a parabola that approximates it well on $[1, 3]$

Notice how the interval $[1, 3]$ was split into two subintervals as we needed 3 points. Because of this, whenever we use Simpson's Rule, we need to break the interval into an even number of subintervals.

In general, to approximate $\int_a^b f(x)\,dx$ using Simpson's Rule, subdivide $[a, b]$ into n subintervals, where n is even and each subinterval has width $\Delta x = (b-a)/n$. We approximate f with $n/2$ parabolic curves, using Equation (5.5.1) to compute the area under these parabolas. Adding up these areas gives the formula:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\ldots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\Big].$$

Note how the coefficients of the terms in the summation have the pattern 1, 4, 2, 4, 2, 4, . . ., 2, 4, 1.

Figure 5.5.17 illustrates how the area calculated by Simpson's Rule approximates $\int_0^5 f(x)\,dx$ for the function $f(x) = \sin(\pi x)$. In this case, 8 subintervals were used, resulting in 4 quadratic curves (dashed lines) being fitted to each pair of subintervals. The actual answer (accurate to 4 decimal places) is about 10.6366, while Simpson's rule gives 10.7294. Of course more subintervals would result in better accuracy. However 8 intervals were chosen specifically so that you could see how the parabolas compare to the original function. With larger values of n, it becomes difficult to distinguish the function and its quadratic approximations on each subinterval.

Let's demonstrate Simpson's Rule with a concrete example.

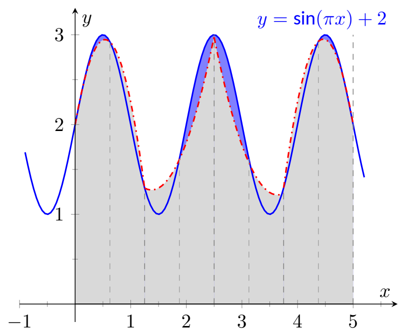


Figure 5.5.17 An illustration of Simpson's rule on $f(x) = \sin(\pi x) + 2$ over $[0, 5]$ using 8 subintervals, resulting in 4 quadratic approximations

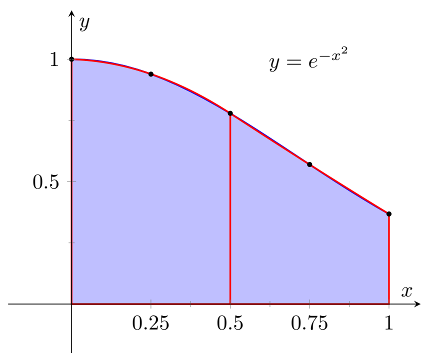
Example 5.5.18 Using Simpson's Rule.

Approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$ using Simpson's Rule and 4 equally spaced subintervals.

Solution. We begin by making a table of values as we have in the past, as shown in Figure 5.5.19(a).

x_i	$e^{-x_i^2}$
0	1
0.25	0.939
0.5	0.779
0.75	0.570
1	0.368

(a)



(b)

Figure 5.5.19 A table of values to approximate $\int_0^1 e^{-x^2}\,dx$, along with a graph of the function

Simpson's Rule states that

$$\int_0^1 e^{-x^2}\,dx \approx \frac{0.25}{3}\Big[1{+}4(0.939){+}2(0.779){+}4(0.570){+}0.368\Big] = 0.7468\overline{3}.$$

Recall in Example 5.5.2 we stated that the correct answer, accurate to 4 places after the decimal, was 0.7468. Our approximation with Simpson's Rule, with 4 subintervals, is better than our approximation with the Trapezoidal Rule using 5!

Figure 5.5.19(b) shows $f(x) = e^{-x^2}$ along with its approximating parabolas, demonstrating how good our approximation is. The approximating curves are nearly indistinguishable from the actual function.

Example 5.5.20 Using Simpson's Rule.

Approximate $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ using Simpson's Rule and 10 equally spaced intervals.

Solution. Figure 5.5.21 shows the table of values that we used in the past for this problem, shown here again for convenience. Again, $\Delta x = (\pi/2 + \pi/4)/10 \approx 0.236$.
Simpson's Rule states that

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx \approx \frac{0.2356}{3}\Big[(-0.4657) + 4(-0.1654) + 2(-0.0310) + \ldots$$
$$\ldots + 2(0.9710) + 4(0.6899) + (-0.6700)\Big]$$
$$\approx 0.4701$$

Recall that the actual value, accurate to 3 decimal places, is 0.4609. Our approximation is within one $1/100$th of the correct value. The graph in Figure 5.5.22 shows how closely the parabolas match the shape of the graph.

x_i	$\sin(x_i^3)$
-0.7854	-0.4657
-0.5498	-0.1654
-0.3142	-0.0310
-0.0785	-0.0005
0.1571	0.0039
0.3927	0.0605
0.6283	0.2455
0.8639	0.6011
1.0996	0.9710
1.3352	0.6899
1.5708	-0.6700

Figure 5.5.21 Values used to approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ in Example 5.5.20

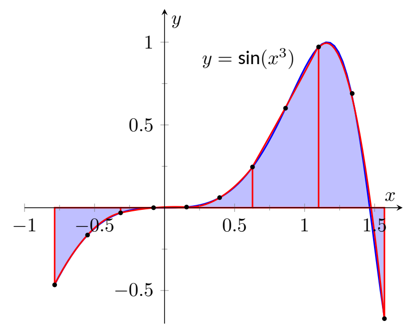


Figure 5.5.22 Approximating $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ in Example 5.5.20 with Simpson's Rule and 10 equally spaced intervals

5.5.5 Summary and Error Analysis

We summarize the key concepts of this section thus far in the following Key Idea.

Key Idea 5.5.23 Numerical Integration.

Let f be a continuous function on $[a, b]$, let n be a positive integer, and let $\Delta x = \dfrac{b-a}{n}$.
Set $x_0 = a$, $x_1 = a + \Delta x$, …, $x_i = a + i\Delta x$, $x_n = b$.
Consider $\displaystyle\int_a^b f(x)\,dx$.

Left Hand Rule: $\displaystyle\int_a^b f(x)\,dx \approx \Delta x\big[f(x_0) + f(x_1) + \ldots + f(x_{n-1})\big]$.

Right Hand Rule: $\displaystyle\int_a^b f(x)\,dx \approx \Delta x\big[f(x_1) + f(x_2) + \ldots + f(x_n)\big]$.

Trapezoidal Rule: $\displaystyle\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\big[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\big]$.

Midpoint Rule: $\displaystyle\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$.

Simpson's Rule: $\displaystyle\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\big[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 4f(x_{n-1}) + f(x_n)\big]$ for n even.

In our examples, we approximated the value of a definite integral using a given method then compared it to the "right" answer. This should have raised several questions in the reader's mind, such as:

1. How was the "right" answer computed?
2. If the right answer can be found, what is the point of approximating?
3. If there is value to approximating, how are we supposed to know if the approximation is any good?

These are good questions, and their answers are educational. In the examples, *the* right answer was never computed. Rather, an approximation accurate to a certain number of places after the decimal was given. In Example 5.5.2, we do not know the *exact* answer, but we know it starts with 0.7468. These more accurate approximations were computed using numerical integration but with more precision (i.e., more subintervals and the help of a computer).

Since the exact answer cannot be found, approximation still has its place. How are we to tell if the approximation is any good?

"Trial and error" provides one way. Using technology, make an approximation with, say, 10, 100, and 200 subintervals. This likely will not take much time at all, and a trend should emerge. If a trend does not emerge, try using yet more subintervals. Keep in mind that trial and error is never foolproof; you might stumble upon a problem in which a trend will not emerge.

A second method is to use Error Analysis. While the details are beyond the scope of this text, there are some formulas that give *bounds* for how good your approximation will be. For instance, the formula might state that the approximation is within 0.1 of the correct answer. If the approximation is 1.58, then one knows that the correct answer is between 1.48 and 1.68. By using lots of subintervals, one can get an approximation as accurate as one likes. Theorem 5.5.24 states what these bounds are.

Theorem 5.5.24 Error Bounds in the Trapezoidal Rule and Simpson's Rule.

1. *Let E_T and E_M be the error in approximating $\int_a^b f(x)\,dx$ using the Trapezoidal and Midpoint Rules respectively, with n subintervals. If f has a continuous second derivative on $[a,b]$ and K is any upper bound of $|f''(x)|$ on $[a,b]$, then*

$$E_T \le \frac{(b-a)^3}{12n^2}K.$$

and

$$E_M \le \frac{(b-a)^3}{24n^2}K.$$

2. *Let E_S be the error in approximating $\int_a^b f(x)\,dx$ using Simpson's Rule with n subintervals.. If f has a continuous 4th derivative on $[a,b]$ and K is any upper bound of $\left|f^{(4)}(x)\right|$ on $[a,b]$, then*

$$E_S \le \frac{(b-a)^5}{180n^4}K.$$

There are some key things to note about this theorem.

1. The larger the interval, the larger the error. This should make sense intuitively.

2. The error shrinks as more subintervals are used (i.e., as n gets larger).

3. The maximum error in the Midpoint Rule is half of the maximum error in the Trapezoidal Rule. (*Usually* the errors in these two rules have opposite signs as well, that is one will be an under approximation and the other will be an over approximation).

4. The error in Simpson's Rule has a term relating to the 4th derivative of f. Consider a cubic polynomial: its 4th derivative is 0. Therefore, the error in

approximating the definite integral of a cubic polynomial with Simpson's Rule is 0 — Simpson's Rule computes the exact answer!

We revisit Examples 5.5.9 and 5.5.18 and compute the error bounds using Theorem 5.5.24 in the following example.

Example 5.5.25 Computing error bounds.

Find the error bounds when approximating $\int_0^1 e^{-x^2}\,dx$ using the Trapezoidal and Midpoint Rules and 5 subintervals, and using Simpson's Rule with 4 subintervals.

Solution. *Trapezoidal and Midpoints Rules with $n = 5$:*
We start by computing the 2nd derivative of $f(x) = e^{-x^2}$:

$$f''(x) = e^{-x^2}(4x^2 - 2).$$

Figure 5.5.26 shows a graph of $f''(x)$ on $[0, 1]$. It is clear that the largest value of f'', in absolute value, is 2.
Thus we let $K = 2$ and apply the error formula from Theorem 5.5.24.

$$E_T \leq \frac{(1-0)^3}{12 \cdot 5^2} \cdot 2 = 0.00\overline{6}.$$

Since the maximum error in the Midpoint rule is half the error in the Trapezoidal Rule, we can say: $E_M \leq 0.00\overline{3}$
Our error estimation formula states that our approximation of 0.7444 found in Example 5.5.9 is within 0.0067 of the correct answer. Hence we know that the actual value is within $[0.7444-0.0067, 0.7444+0.0067] = [0.7377, 0.7511]$. So:

$$0.7377 \leq \int_0^1 e^{-x^2}\,dx \leq 0.7511$$

But we can do better than this with the Midpoint Rule since its error is at most half of the error of the Trapezoidal Rule. Our error estimate formula state that our approximate of 0.7480 found in Example 5.5.12 is within 0.0034 of the correct answer. Hence Hence we know that the actual value is within $[0.7480 - 0.0034, 0.7480 + 0.0033] = [0.7447, 0.7513]$.
We had earlier stated the actual answer, correct to 4 decimal places, to be 0.7468, affirming the validity of Theorem 5.5.24.
Simpson's Rule with $n = 4$:
We start by computing the 4th derivative of $f(x) = e^{-x^2}$:

$$f^{(4)}(x) = e^{-x^2}(16x^4 - 48x^2 + 12).$$

Figure 5.5.27 shows a graph of $f^{(4)}(x)$ on $[0, 1]$. It is clear that the largest value of $f^{(4)}$, in absolute value, is 12. Thus we let $K = 12$ and apply the error formula from Theorem 5.5.24.

$$E_s = \leq \frac{(1-0)^5}{180 \cdot 4^4} \cdot 12 = 0.00026.$$

Our error estimation formula states that our approximation of $0.7468\overline{3}$ found in Example 5.5.18 is within 0.00026 of the correct answer,

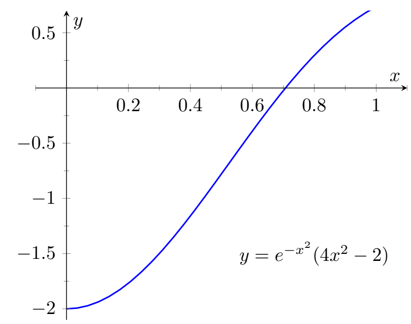


Figure 5.5.26 Graphing $f''(x)$ in Example 5.5.25 to help establish error bounds

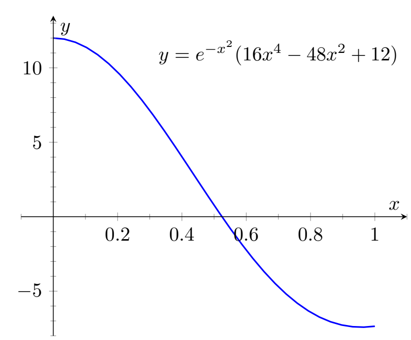


Figure 5.5.27 Graphing $f^{(4)}(x)$ in Example 5.5.25 to help establish error bounds

hence we know that the correct answer is in the interval $[0.74683 - 0.00026, 0.74683 + 0.00026] = [0.74657, 0.74709]$. So:

$$0.74657 \leq \int_0^1 e^{-x^2}\, dx \leq 0.74709.$$

Once again we affirm the validity of Theorem 5.5.24 since the answer to 4 decimal places is actually 0.7468.

At the beginning of this section we mentioned two main situations where numerical integration was desirable. We have considered the case where an antiderivative of the integrand cannot be computed. We now investigate the situation where the integrand is not known. This is, in fact, the most widely used application of Numerical Integration methods. "Most of the time" we observe behavior but do not know "the" function that describes it. We instead collect data about the behavior and make approximations based on this data. We demonstrate this in an example.

Example 5.5.28 Approximating distance traveled.

One of the authors drove his daughter home from school while she recorded their speed every 30 seconds. The data is given in Figure 5.5.29. Approximate the distance they traveled.

Solution. Recall that by integrating a speed function we get distance traveled. We have information about $v(t)$; we will use Simpson's Rule to approximate $\int_a^b v(t)\, dt$.

The most difficult aspect of this problem is converting the given data into the form we need it to be in. The speed is measured in miles per hour, whereas the time is measured in minutes.

We need to compute $\Delta x = (b-a)/n$. With 25 data points collected, there are $n = 24$ subintervals. What are a and b? Since we start at time $t = 0$, we have $a = 0$. The final recorded time was $t = 12$ minutes, which is 1/5 of an hour. Thus we have

$$\Delta x = \frac{b-a}{n} = \frac{1/5 - 0}{24} = \frac{1}{120}; \frac{\Delta x}{3} = \frac{1}{360}.$$

Thus the distance traveled is approximately:

$$\begin{aligned}\int_0^{0.2} v(t)\, dt &\approx \frac{1}{360}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)\Big]\\ &= \frac{1}{360}\Big[0 + 4\cdot 25 + 2\cdot 22 + \cdots + 2\cdot 40 + 4\cdot 23 + 0\Big]\\ &\approx 6.2167 \text{ miles.}\end{aligned}$$

We approximate the author drove 6.2 miles. (Because we are sure the reader wants to know, the author's odometer recorded the distance as about 6.05 miles.)

Time (min)	Speed (mph)
0	0
1	25
2	22
3	19
4	39
5	0
6	43
7	59
8	54
9	51
10	43
11	35
12	40
13	43
14	30
15	0
16	0
17	28
18	40
19	42
20	40
21	39
22	40
23	23
24	0

Figure 5.5.29 Speed data collected at 30 second intervals for Example 5.5.28

5.5.6 Exercises

Terms and Concepts

1. T/F: Simpson's Rule is a method of approximating antiderivatives. (☐ True ☐ False)

2. What are the two basic situations where approximating the value of a definite integral is necessary?

3. Why are the Left and Right Hand Rules rarely used?

4. Simpson's Rule is based on approximating portions of a function with what type of function?

Problems

Exercise Group. In the following exercises, approximate the definite integral with the Trapezoidal Rule and Simpson's Rule, with $n = 4$. Then find the exact value.

5. For the integral $\int_{-1}^{1} x^2\, dx$:

- **(a)** Approximate using the trapezoidal rule.
- **(b)** Approximate using Simpson's rule.
- **(c)** Find the exact value.

6. For the integral $\int_0^{10} 5x\, dx$:

- **(a)** Approximate using the trapezoidal rule.
- **(b)** Approximate using Simpson's rule.
- **(c)** Find the exact value.

7. For the integral $\int_0^{\pi} \sin(x)\, dx$:

- **(a)** Approximate using the trapezoidal rule.
- **(b)** Approximate using Simpson's rule.
- **(c)** Find the exact value.

8. For the integral $\int_0^{4} \sqrt{x}\, dx$:

- **(a)** Approximate using the trapezoidal rule.
- **(b)** Approximate using Simpson's rule.
- **(c)** Find the exact value.

9. For the integral $\int_0^{3} (x^3 + 2x^2 - 5x + 7)\, dx$:

- **(a)** Approximate using the trapezoidal rule.
- **(b)** Approximate using Simpson's rule.
- **(c)** Find the exact value.

10. For the integral $\int_0^{1} x^4\, dx$:

- **(a)** Approximate using the trapezoidal rule.
- **(b)** Approximate using Simpson's rule.
- **(c)** Find the exact value.

11. For the integral $\int_0^{2\pi} \cos(x)\, dx$:

- **(a)** Approximate using the trapezoidal rule.
- **(b)** Approximate using Simpson's rule.
- **(c)** Find the exact value.

12. For the integral $\int_{-3}^{3} \sqrt{9 - x^2}\, dx$:

- **(a)** Approximate using the trapezoidal rule.
- **(b)** Approximate using Simpson's rule.
- **(c)** Find the exact value.

Exercise Group. In the following exercises, approximate the definite integral with the Trapezoidal Rule and Simpson's Rule, with $n = 6$.

13. For the integral $\int_0^{1} \cos\left(x^2\right)\, dx$:

- **(a)** Approximate using the trapezoidal rule.
- **(b)** Approximate using Simpson's rule.

14. For the integral $\int_{-1}^{1} e^{x^2}\, dx$:

- **(a)** Approximate using the trapezoidal rule.
- **(b)** Approximate using Simpson's rule.

15. For the integral $\int_0^{5} \sqrt{x^2 + 1}\, dx$:

- **(a)** Approximate using the trapezoidal rule.
- **(b)** Approximate using Simpson's rule.

16. For the integral $\int_0^{\pi} x \sin(x)\, dx$:

- **(a)** Approximate using the trapezoidal rule.
- **(b)** Approximate using Simpson's rule.

17. For the integral $\int_0^{\pi/2} \sqrt{\cos(x)}\,dx$:

(a) Approximate using the trapezoidal rule.

(b) Approximate using Simpson's rule.

18. For the integral $\int_1^4 \ln(x)\,dx$:

(a) Approximate using the trapezoidal rule.

(b) Approximate using Simpson's rule.

19. For the integral $\int_{-1}^{1} \frac{1}{\sin(x)+2}\,dx$:

(a) Approximate using the trapezoidal rule.

(b) Approximate using Simpson's rule.

20. For the integral $\int_0^6 \frac{1}{\sin(x)+2}\,dx$:

(a) Approximate using the trapezoidal rule.

(b) Approximate using Simpson's rule.

Exercise Group. In the following exercises, find n such that the error in approximating the given definite integral is less than 0.0001 when using the Trapezoidal Rule and Simpson's Rule.

21. For the integral $\int_0^{\pi} \sin(x)\,dx$:

(a) Using the trapezoid rule.

(b) Using Simpson's rule.

22. For the integral $\int_1^4 \frac{1}{\sqrt{x}}\,dx$:

(a) Using the trapezoid rule.

(b) Using Simpson's rule.

23. For the integral $\int_0^{\pi} \cos\left(x^2\right)\,dx$:

(a) Using the trapezoid rule.

(b) Using Simpson's rule.

24. For the integral $\int_0^5 x^4\,dx$:

(a) Using the trapezoid rule.

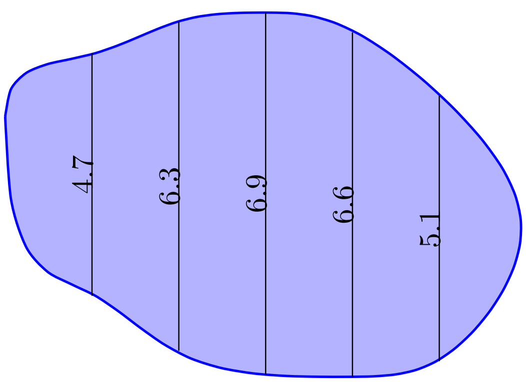
(b) Using Simpson's rule.

Exercise Group. In the following exercises, a region is given. Find the area of the region using Simpson's Rule:

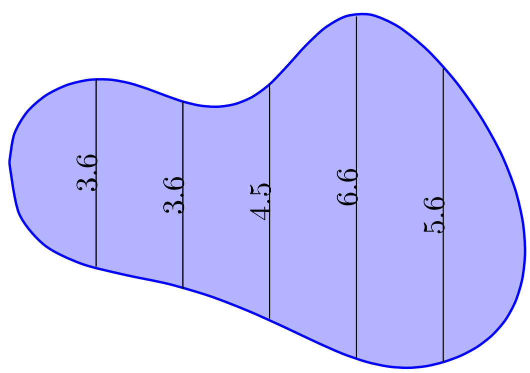
(a) where the measurements are in centimeters, taken in 1 cm increments, and

(b) where the measurements are in hundreds of feet, taken in 100 ft increments.

25.



26.



We started this chapter learning about antiderivatives and indefinite integrals. We then seemed to change focus by looking at areas between the graph of a function and the x-axis. We defined these areas as the definite integral of the function, using a notation very similar to the notation of the indefinite integral. The Fundamental Theorem of Calculus tied these two seemingly separate concepts together: we can find areas under a curve, i.e., we can evaluate a definite integral, using antiderivatives.

We ended the chapter by noting that antiderivatives are sometimes more than difficult to find: they are impossible. Therefore we developed numerical techniques that gave us good approximations of definite integrals.

We used the definite integral to compute areas, and also to compute displacements and distances traveled. There is far more we can do than that. In Chapter 7 we'll see more applications of the definite integral. Before that, in Chapter 6 we'll learn advanced techniques of integration, analogous to learning rules like the Product, Quotient and Chain Rules of differentiation.

Appendices

Appendix A

Answers to Selected Exercises

I • Math 1560: Calculus I
1 • Limits
1.1 • An Introduction To Limits
1.1 • Exercises

Terms and Concepts

1.1.2. an indeterminate form
1.1.3. False
1.1.6. 1

Problems

1.1.7. 5
1.1.8. 3
1.1.9. DNE
1.1.10. $\frac{2}{3}$
1.1.11. -4
1.1.12. DNE or ∞
1.1.13. DNE
1.1.14. 6
1.1.15. 1
1.1.16. DNE
1.1.17. 1
1.1.18. DNE
1.1.19. DNE
1.1.20. 1
1.1.21. -7
1.1.22. 9
1.1.23. 5
1.1.24. -0.111111
1.1.25. 29
1.1.26. 0.2
1.1.27. -1
1.1.28. 0

1.2 • Epsilon-Delta Definition of a Limit
1.2 • Exercises

Terms and Concepts

1.2.2. y-tolerance
1.2.3. True
1.2.4. True

1.3 • Finding Limits Analytically
1.3 • Exercises

Terms and Concepts

1.3.6. True

Problems

1.3.7. 9
1.3.8. 6
1.3.9. 0
1.3.10. DNE
1.3.11. 3
1.3.12. not possible to know
1.3.13. 3
1.3.14. -45
1.3.15. 0
1.3.16. $\cos(3.14159)$
1.3.17. π
1.3.18. 1
1.3.19. 23
1.3.20. $\left(\frac{\pi-5}{\pi-8}\right)^4$
1.3.21. $\frac{\sqrt{3}}{4}$
1.3.22. $-\frac{16}{5}$
1.3.23. DNE
1.3.24. 256
1.3.25. $\frac{2\sqrt{3}}{3}$
1.3.26. $\ln(4)$
1.3.27. $\frac{\pi^2-4\pi-2}{2\pi^2-2\pi+1}$
1.3.28. $\frac{2\pi-4}{5\pi-5}$
1.3.29. $\frac{1}{4}$
1.3.30. $-\frac{7}{2}$
1.3.31. $\frac{17}{4}$
1.3.32. $\frac{13}{3}$
1.3.33. $\frac{4}{9}$
1.3.34. $\frac{5}{4}$
1.3.35. 0
1.3.36. 0
1.3.37. 1
1.3.38. 9
1.3.39. 8
1.3.40. $\frac{9}{8}$
1.3.41. 1
1.3.42. $\frac{\pi}{180}$

1.4 · One-Sided Limits

1.4 · Exercises

Terms and Concepts

1.4.2. False
1.4.3. False
1.4.4. True

Problems

1.4.5.

(a) 2

(b) 2

(c) 2

(d) 1

(e) DNE

(f) 4

1.4.6.

(a) 0

(b) 4

(c) DNE

(d) 4

(e) DNE

(f) 1

1.4.7.

- **(a)** DNE or ∞
- **(b)** DNE or ∞
- **(c)** DNE or ∞
- **(d)** DNE
- **(e)** 5
- **(f)** 4

1.4.8.

- **(a)** 2
- **(b)** 3
- **(c)** DNE
- **(d)** 4

1.4.9.

- **(a)** 1
- **(b)** 1
- **(c)** 1
- **(d)** 1

1.4.10.

- **(a)** -5
- **(b)** 1
- **(c)** DNE
- **(d)** 3

1.4.11.

- **(a)** 2
- **(b)** 2
- **(c)** 2
- **(d)** 0
- **(e)** 2
- **(f)** 2
- **(g)** 2
- **(h)** DNE

1.4.12.

- **(a)** $a-1$
- **(b)** a
- **(c)** DNE
- **(d)** a

1.4.13.

- **(a)** 2
- **(b)** 6
- **(c)** DNE
- **(d)** 2

1.4.14.

- **(a)** -17
- **(b)** 0
- **(c)** DNE
- **(d)** 0

1.4.15.

- **(a)** 9
- **(b)** 9
- **(c)** 9
- **(d)** 9
- **(e)** 126
- **(f)** 126
- **(g)** 126
- **(h)** 126

1.4.16.

- **(a)** -1
- **(b)** 0
- **(c)** DNE
- **(d)** 0

1.4.17.

(a) $1 - \cos^2(a)$

(b) $\sin^2(a)$

(c) $1 - \cos^2(a)$ or $\sin^2(a)$

(d) $\sin^2(a)$

1.4.18.

(a) 0

(b) 1

(c) DNE

(d) -2

1.4.19.

(a) -4

(b) -4

(c) -4

(d) -2

1.4.20.

(a) c

(b) c

(c) c

(d) c

1.4.21.

(a) -1

(b) 1

(c) DNE

(d) 0

1.5 • Continuity

1.5 • Exercises

Terms and Concepts

1.5.5. False
1.5.6. True
1.5.7. True
1.5.8. False
1.5.9. False
1.5.10. True

Problems

1.5.11. No.
1.5.12. No.
1.5.13. No.
1.5.14. Yes.
1.5.15. Yes.
1.5.16. Yes.
1.5.17. **(a).** No.
(b). Yes.
(c). No.
1.5.18. Yes.

1.5.19.

(a) Yes.

(b) Yes.

1.5.20.

(a) Yes.

(b) No.

1.5.21.

- **(a)** Yes.
- **(b)** Yes.

1.5.22.

- **(a)** Yes.
- **(b)** No.

1.5.23. $(-\infty, \infty)$

1.5.24. $(-\infty, -2], [2, \infty)$

1.5.25. $[-2, 2]$

1.5.26. $[-3, 3]$

1.5.27. $(-\infty, -1.73205], [1.73205, \infty)$

1.5.28. $(-7, 7)$

1.5.29. $(-\infty, \infty)$

1.5.30. $(-\infty, \infty)$

1.5.31. $(0, \infty)$

1.5.32. $(-\infty, \infty)$

1.5.33. $(-\infty, 1.09861]$

1.5.34. $(-\infty, \infty)$

1.5.39. 1.23633

1.5.40. 0.523633

1.5.41. 0.693164

1.5.42. 0.785547

1.6 · Limits Involving Infinity

1.6 · Exercises

Terms and Concepts

1.6.1. False

1.6.2. True

1.6.3. False

1.6.4. True

1.6.5. True

Problems

1.6.9.

- **(a)** $-\infty$
- **(b)** ∞

1.6.10.

- **(a)** $-\infty$
- **(b)** ∞
- **(c)** DNE
- **(d)** ∞
- **(e)** ∞
- **(f)** ∞

1.6.11.

- **(a)** 0
- **(b)** 3
- **(c)** 1.5
- **(d)** 1.5

1.6.12.

- **(a)** DNE
- **(b)** DNE
- **(c)** 0
- **(d)** 0

1.6.13.

(a) DNE

(b) DNE

1.6.14.

(a) -9

(b) ∞

1.6.15.

(a) $-\infty$

(b) ∞

(c) DNE

1.6.16.

(a) $-\infty$

(b) $-\infty$

(c) $-\infty$

1.6.17.

(a) ∞

(b) ∞

(c) ∞

1.6.18.

(a) 1.8

(b) 1.8

(c) 1.8

1.6.19. $y = 2, x = -2, x = 9$

1.6.20. $y = \frac{5}{-2}, x = -9$

1.6.21. $y = 0, x = 0, x = 4$

1.6.22. $x = -3$

1.6.23. NONE

1.6.24. $y = \frac{4}{-1}$

1.6.25. ∞

1.6.26. ∞

1.6.27. ∞

1.6.28. ∞

2 · Derivatives

2.1 · Instantaneous Rates of Change: The Derivative

2.1 · Exercises

Terms and Concepts

2.1.1. True

2.1.2. True

Problems

2.1.7. 0

2.1.8. 2

2.1.9. -3

2.1.10. $2x$

2.1.11. $3x^2$

2.1.12. $6x - 1$

2.1.13. $\frac{-1}{x^2}$

2.1.14. $\frac{-1}{(s-2)^2}$

2.1.15. **(a).** $y = 6$

(b). $x = -2$

2.1.16. **(a).** $y - 2x = 0$

(b). $0.5x + y = 7.5$

2.1.17. **(a).** $3x + y = 4$

(b). $y - 0.333333x = -19.3333$

2.1.18. **(a).** $y - 4x = -4$

(b). $0.25x + y = 4.5$

2.1.19. **(a).** $y - 48x = -128$

(b). $0.0208333x + y = 64.0833$

2.1.20. **(a).** $7x + y = 1$

(b). $y - 0.142857x = 8.14286$

2.1.21. **(a).** $0.25x + y = -1$

(b). $y - 4x = 7.5$

2.1.22. **(a).** $x + y = 4$

(b). $y - x = -2$

2.1.23. $5.9x + y = 1.2$

2.1.24. $y - 11.1111x = 110$

2.1.25. $y - 0.0192627x = 0.0953664$

2.1.26. $0.04996x + y = 1$

2.1.27.

(a) $-2, 0, 4$

(b) $2x$

(c) $-2, 0, 4$

2.1.28.

(a) $-1, -0.25$

(b) $\frac{-1}{(x+1)^2}$

(c) $-1, -0.25$

2.1.33. **(a).** $(-2, 0) \cup (2, \infty)$
(b). $(-\infty, -2) \cup (0, 2)$
(c). $\{-2, 0, 2\}$
(d). $(-1, 1)$
(e). $(-\infty, -1) \cup (1, \infty)$
(f). $\{-1, 1\}$

2.1.34. **(a).** $(-2, 2)$
(b). $(-\infty, -2) \cup (2, \infty)$
(c). $\{-2, 2\}$
(d). $(-1, 0) \cup (1, \infty)$
(e). $(-\infty, -1) \cup (0, 1)$
(f). $\{-1, 0, 1\}$

2.1.35. no

2.1.36. yes

2.2 · Interpretations of the Derivative

2.2 · Exercises

Terms and Concepts

2.2.1. velocity

2.2.3. linear functions

Problems

2.2.4. 20

2.2.5. -89

2.2.6. 91

2.2.7. f(10.1)

2.2.8. -2

2.2.9. 7

2.2.10. decibels per customer

2.2.11. foot per second squared

2.2.12. foot per hour

2.2.15. Choice 1

2.2.16. Choice 2

2.2.17. Choice 2

2.2.18. Choice 2

2.3 · Basic Differentiation Rules

2.3 • Exercises

Terms and Concepts

2.3.1. the power rule
2.3.2. $\frac{1}{x}$
2.3.3. e^x
2.3.4. 10
2.3.5. Choice 1, Choice 2, Choice 5, Choice 6
2.3.7. $17x - 205$
2.3.9. **(a).** a velocity function
(b). an acceleration function
2.3.10. pound per foot squared

Problems

2.3.11. $-(14x+8)$
2.3.12. $28x - 48x^2 + 5$
2.3.13. $9 - \left(20t^4 + \frac{3}{4}t^2\right)$
2.3.14. $19\sin(\theta) - 3\cos(\theta)$
2.3.15. $3e^r$
2.3.16. $21t^2 + 5\sin(t) - 2\cos(t)$
2.3.17. $\frac{6}{x} + 9$
2.3.18. $s^3 + s^2 + s + 1$
2.3.19. $\sin(t) - (e^t + \cos(t))$
2.3.20. $\frac{8}{x}$
2.3.21. 0
2.3.22. $18t + 24$
2.3.23. $24x^2 + 96x + 96$
2.3.24. $3x^2 + 18x + 27$
2.3.25. $8x + 28$

2.3.27. **(a).** $9x^8$
(b). $9 \cdot 8x^7$
(c). $9 \cdot 8 \cdot 7x^6$
(d). $9 \cdot 8 \cdot 7 \cdot 6x^5$

2.3.28. **(a).** $-8\sin(x)$
(b). $-(8\cos(x))$
(c). $8\sin(x)$
(d). $8\cos(x)$

2.3.29. **(a).** $-(4 \cdot 2t + 3 + e^t)$
(b). $-(8 + e^t)$
(c). $-e^t$
(d). $-e^t$

2.3.30. **(a).** $2\theta + 8\theta^7$
(b). $2 + 8 \cdot 7\theta^6$
(c). $8 \cdot 7 \cdot 6\theta^5$
(d). $8 \cdot 7 \cdot 6 \cdot 5\theta^4$

2.3.31. **(a).** $-(\cos(\theta) - \sin(\theta))$
(b). $\sin(\theta) + \cos(\theta)$
(c). $\cos(\theta) - \sin(\theta)$
(d). $-(\sin(\theta) + \cos(\theta))$

2.3.32. **(a).** 0
(b). 0
(c). 0
(d). 0

2.3.33. **(a).** $y = 20(x-2) + 24$
(b). $y = -\frac{1}{20}(x-2) + 24$

2.3.34. **(a).** $y = e^0 \ln(e)\,(t-0) + e^0 - 2$
(b). $y = \frac{-1}{e^0 \ln(e)}(t-0) + e^0 - 2$

2.3.35. **(a).** $y = x - 1$
(b). $y = -(x-1)$

2.3.36. **(a).** $y = \frac{4\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) + \frac{4 \cdot 1}{2}$
(b). $y = -\left(\frac{1}{4}\frac{2\sqrt{3}}{3}\right)\left(x - \frac{\pi}{6}\right) + \frac{4 \cdot 1}{2}$

2.3.37. **(a).** $y = \frac{2 \cdot 1}{2}\left(x - \frac{\pi}{6}\right) + \frac{-2\sqrt{3}}{2}$
(b). $y = -\left(\frac{1}{2} \cdot 2\right)\left(x - \frac{\pi}{6}\right) + \frac{-2\sqrt{3}}{2}$

2.3.38. **(a).** $9 - 9x$
(b). $y = \frac{-1}{-9}(x - (-9)) + 90$

2.4 · The Product and Quotient Rules

2.4 · Exercises

Terms and Concepts

2.4.1. False
2.4.2. False
2.4.3. True
2.4.4. the quotient rule
2.4.5. False

Problems

2.4.15. $\sin(y) + y\cos(y)$

2.4.16. $3t^2\cos(t) - t^3\sin(t)$

2.4.17. $e^q\ln(q) + e^q\frac{1}{q}$

2.4.18. $-\left(\frac{6y^5}{(y^6)^2}(\csc(y)-5) + \frac{1}{y^6}\csc(y)\cot(y)\right)$

2.4.19. $\frac{t-4-(t+8)}{(t-4)^2}$

2.4.20. $\frac{3q^2\left(\sin(q)-8q^2\right)-q^3(\cos(q)-8\cdot 2q)}{(\sin(q)-8q^2)^2}$

2.4.21. $-(\csc(y)\cot(y)+e^y)$

2.4.22. $\sec^2(t)\ln(t) + \frac{1}{t}\tan(t)$

2.4.23. $7\cdot 2q - 6$

2.4.24. $5y^4$

2.4.25. $\left(5r^2+17r+10\right)e^r$

2.4.26. $\frac{9z^8-z^9-z^5+5z^4}{e^z}$

2.4.27. 3

2.4.28. $5r^4(\tan(r)+e^r) + r^5\left(\sec^2(r)+e^r\right)$

2.4.29. $\frac{\csc(z)\sin(z)-\csc(z)\cot(z)(\cos(z)+2)}{(\cos(z)+2)^2}$

2.4.30. $4\theta^3\sec(\theta) + \theta^4\sec(\theta)\tan(\theta) + \frac{\sec(\theta)\tan(\theta)\theta^4-4\theta^3\sec(\theta)}{(\theta^4)^2}$

2.4.31. $\frac{\tan(r)-r\sec^2(r)}{\tan^2(r)} - \frac{\csc^2(r)r+\cot(r)}{r^2}$

2.4.32. 0

2.4.33. $7\cdot 5x^4e^x + 7x^5e^x - (\cos(x)\cos(x) - \sin(x)\sin(x))$

2.4.34. $\frac{\left(2r\sin(r)+r^2\cos(r)\right)\left(r^2\cos(r)-9\right)-\left(r^2\sin(r)-7\right)\left(2r\cos(r)-r^2\sin(r)\right)}{(r^2\cos(r)-9)^2}$

2.4.35. $\left(4z^3\ln(z)+z^4\frac{1}{z}\right)\cos(z) - z^4\ln(z)\sin(z)$

2.4.36. $(9\cos(x)-9x\sin(x))\tan(x) + 9x\cos(x)\sec^2(x)$

2.4.37. **(a).** $y = -(7x+7)$
(b). $y = \left(\frac{1}{7}\right)x - 7$

2.4.38. **(a).** $y = 5.0345\left(x-\frac{5\pi}{3}\right) + \frac{5\pi}{6}$
(b). $y = \frac{5\pi}{6} - \left(\frac{12837432}{64630031}\right)\left(x-\frac{5\pi}{3}\right)$

2.4.39. **(a).** $y = -(15(x+5)+25)$
(b). $y = \left(\frac{1}{15}\right)(x+5) - 25$

2.4.40. **(a).** $y = \left(\frac{1}{8}\right)x$
(b). $y = -8x$

2.4.41. $\frac{17}{2}$

2.4.42. 0

2.4.43. NONE

2.4.44. $0, 4$

2.4.45. $2\cos(x) - x\sin(x)$

2.4.46. $-4\cos(x) + x\sin(x)$

2.4.47. $\csc(x)\cot(x)\cot(x) + \csc^2(x)\csc(x)$

2.4.48. 0

2.5 · The Chain Rule

2.5 · Exercises

Terms and Concepts

2.5.1. True
2.5.2. False
2.5.3. False
2.5.4. True
2.5.5. True
2.5.6. True

Problems

2.5.7. $10\left(4x^3-x\right)^9\left(12x^2-1\right)$

2.5.8. $15(3t-2)^4$

2.5.9. $3(\sin(\theta)+\cos(\theta))^2(\cos(\theta)-\sin(\theta))$

2.5.10. $(6t+1)\,e^{3t^2+t-1}$

2.5.11. $4\left(\ln(x)-x^4\right)^3\left(\frac{1}{x}-4x^3\right)$

2.5.12. $0.693147\cdot 2^{q^5+4q}\left(5q^4+4\right)$

2.5.13. $5\left(y+\frac{1}{y}\right)^4\left(1-\frac{1}{y^2}\right)$

2.5.14. $-5\sin(5t)$

2.5.15. $2\sec^2(2q)$

2.5.16. $-\csc^2\left(\theta^2+3\right)\cdot 2\theta$

2.5.17. $\left(6t^5-\frac{3t^2}{(t^3)^2}\right)\cos\left(t^6+\frac{1}{t^3}\right)$

2.5.18. $-5\cos^4(7q)\cdot 7\sin(7q)$

2.5.19. $-3\cos^2\left(y^2+3y-3\right)(2y+3)\sin\left(y^2+3y-3\right)$

2.5.20. $-\frac{1}{\cos(t)}\sin(t)$

2.5.21. $\frac{1}{q^8}\cdot 8q^7$

2.5.22. $3\frac{1}{y}$

2.5.23. $1.79176\cdot 6^t$

2.5.24. $-0.693147\cdot 2^{\csc(z)}\csc(z)\cot(z)$

2.5.25. 0

2.5.26. $\frac{1.38629\cdot 4^t\cdot 9^t-4^t\cdot 2.19722\cdot 9^t}{(9^t)^2}$

2.5.27. $\frac{1.79176\cdot 6^w(5^w+6)-(6^w+5)\cdot 1.60944\cdot 5^w}{(5^w+6)^2}$

2.5.28. $\frac{1.94591\cdot 7^y\cdot 5^y-(7^y+8)\cdot 1.60944\cdot 5^y}{(5^y)^2}$

2.5.29. $\frac{\left(1.60944\cdot 5^{r^2}\cdot 2r-1\right)\cdot 6^{r^2}-\left(5^{r^2}-r\right)\cdot 1.79176\cdot 6^{r^2}\cdot 2r}{\left(6^{r^2}\right)^2}$

2.5.30. $3w^2\cot(5w)-w^3\cdot 5\csc^2(5w)$

2.5.31. $6\left(x^2+4x\right)^5(2x+4)\left(7x^4+x\right)^3+\left(x^2+4x\right)^6\cdot 3\left(7x^4+x\right)^2\left(7\cdot 4x^3+1\right)$

2.5.32. $-\left(4\cos(8-4r)\cos\left(6r+r^2\right)+(6+2r)\sin\left(6r+r^2\right)\sin(8-4r)\right)$

2.5.33. $7\cos(9+7w)\cos(4w-5)-4\sin(4w-5)\sin(9+7w)$

2.5.34. $e^{8x^2}\cdot 8\cdot 2x\sin\left(\frac{1}{x}\right)-e^{8x^2}\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$

2.5.35. $-\frac{6\sin(6r+4)(3r+1)^3+3\cdot 3(3r+1)^2\cos(6r+4)}{\left((3r+1)^3\right)^2}$

2.5.36. $\frac{3\cdot 2(3z+5)\sin(9z)-(3z+5)^2\cdot 9\cos(9z)}{\sin^2(9z)}$

2.5.37. **(a).** $y=0$
(b). $x=0$

2.5.38. **(a).** $y=15(x-1)+1$
(b). $y=\frac{-1}{15}(x-1)+1$

2.5.39. **(a).** $y=-3\left(x-\frac{\pi}{2}\right)+1$
(b). $y=\frac{1}{3}\left(x-\frac{\pi}{2}\right)+1$

2.5.40. **(a).** $y=-5e(x+1)+e$
(b). $y=\frac{1}{5e}(x+1)+e$

2.5.41. $\frac{1}{x}$

2.5.42. $\frac{k}{x}$

2.6 · Implicit Differentiation

2.6 · Exercises

Terms and Concepts

2.6.2. the chain rule
2.6.3. True
2.6.4. True

Problems

2.6.5. $\frac{1}{2\sqrt{w}} + \frac{\frac{1}{2\sqrt{w}}}{\left(\sqrt{w}\right)^2}$

2.6.6. $\frac{1}{6}\frac{1}{\left(\sqrt[6]{y}\right)^5} + \left(\frac{5}{6}\right)\frac{1}{y^{0.166667}}$

2.6.7. $\frac{1}{2\sqrt{9+t^2}} \cdot 2t$

2.6.8. $\frac{1}{2\sqrt{w}}\tan(w) + \sec^2(w)\sqrt{w}$

2.6.9. $1.2y^{0.2}$

2.6.10. $\pi r^{\pi-1} + 3.8r^{2.8}$

2.6.11. $\frac{\sqrt{w}-(w-8)\frac{1}{2\sqrt{w}}}{\left(\sqrt{w}\right)^2}$

2.6.12. $\frac{1}{6}\frac{1}{\left(\sqrt[6]{x}\right)^5}(\cos(x)+e^x) + (e^x - \sin(x))\sqrt[6]{x}$

2.6.13. $\frac{-4x^3}{2y+1}$

2.6.14. $\frac{-y^{0.6}}{x^{0.6}}$

2.6.15. $\sin(x)\sec(y)$

2.6.16. $\frac{y}{x}$

2.6.17. $\frac{y}{x}$

2.6.18. $\frac{-\left(e^x x(x+2)\cdot 2^{-y}\right)}{\ln(2)}$

2.6.19. $\frac{-2\sin(y)\cos(y)}{x}$

2.6.20. $-\frac{x}{y^2}$

2.6.21. $\frac{1}{2y+2}$

2.6.22. $\frac{y-x^2-2xy^2}{x-y^2-2x^2y}$

2.6.23. $\frac{1-\cos(x)}{\sin(y)+1}$

2.6.24. $\frac{-x}{y}$

2.6.25. $\frac{-(2x+y)}{2y+x}$

2.6.27.

(a) $y = 0$

(b) $y = -1.859(x-0.1) + 0.2811$

2.6.28.

(a) $x = 1$

(b) $y = \frac{-3\sqrt{3}}{8}\left(x - \sqrt{0.6}\right) + \sqrt{0.8}$

2.6.29.

(a) $y = 4$

(b) $y = \frac{3}{108^{\frac{1}{4}}}(x-2) - 108^{\frac{1}{4}}$

2.6.30.

(a) $y = -x + 1$

(b) $y = \frac{3\sqrt{3}}{4}$

2.6.31.

(a) $y = \frac{-1}{\sqrt{3}}\left(x - \frac{7}{2}\right) + \frac{6+3\sqrt{3}}{2}$

(b) $y = \frac{\sqrt{3}\left(x-\left(4+3\sqrt{3}\right)\right)}{2} + \frac{3}{2}$

2.6.32.

(a) $y = 1$

(b) $y = \frac{-2}{\sqrt{5}}(x+1) + \frac{1}{2}\left(-1+\sqrt{5}\right)$

(c) $y = \frac{2}{\sqrt{5}}(x+1) + \frac{1}{2}\left(-1-\sqrt{5}\right)$

2.6.33. $\frac{-\left((2y+1)\cdot 12x^2 - 4x^3\frac{2\left(-\left(4x^3\right)\right)}{2y+1}\right)}{(2y+1)^2}$

2.6.34. $\frac{-\left(\frac{x^{0.6}\cdot 3}{5}y^{-0.4}\frac{-y^{0.6}}{x^{0.6}} - \frac{y^{0.6}\cdot 3}{5}x^{-0.4}\right)}{x^{1.2}}$

2.6.35. $\sin^2(x)\sec^2(y)\tan(y) + \cos(x)\sec(y)$

2.6.36. 0

2.6.37. **(a).** $(1+x)^{\frac{1}{x}}\left(\frac{1}{x(x+1)} - \frac{\ln(1+x)}{x^2}\right)$
(b). $y = (1 - 2\ln(2))(x-1) + 2$

2.6.38. **(a).** $(2x)^{x^2}(2x\ln(2x) + x)$
(b). $y = (2 + 4\ln(2))(x-1) + 2$

2.6.39. **(a).** $\frac{x^x}{x+1}\left(\ln(x)+1-\frac{1}{x+1}\right)$

(b). $y=\frac{1}{4}(x-1)+\frac{1}{2}$

2.6.40. **(a).** $x^{\sin(x)+2}\left(\cos(x)\ln(x)+\frac{\sin(x)+2}{x}\right)$

(b). $y=\frac{3\pi^2}{4}\left(x-\frac{\pi}{2}\right)+\left(\frac{\pi}{2}\right)^3$

2.6.41. **(a).** $\frac{x+1}{x+2}\left(\frac{1}{x+1}-\frac{1}{x+2}\right)$

(b). $y=\frac{1}{9}(x-1)+\frac{2}{3}$

2.6.42. **(a).** $\frac{(x+1)(x+2)}{(x+3)(x+4)}\left(\frac{1}{x+1}+\frac{1}{x+2}-\frac{1}{x+3}-\frac{1}{x+4}\right)$

(b). $y=\frac{11}{72}x+\frac{1}{6}$

2.7 · Derivatives of Inverse Functions

2.7 · Exercises

Terms and Concepts

2.7.1. False

Problems

2.7.9. $\frac{1}{7}$

2.7.10. $-\frac{1}{14}$

2.7.11. -0.5

2.7.12. $\frac{1}{132}$

2.7.13. $-\frac{25}{4}$

2.7.14. $\frac{1}{12}$

2.7.15. $-\frac{1}{\sqrt{1-(4w)^2}}\cdot 4$

2.7.16. $-\frac{1}{|7x|\sqrt{(7x)^2-1}}\cdot 7$

2.7.17. $\frac{1}{1+(2r)^2}\cdot 2$

2.7.18. $\cos^{-1}(w)-w\frac{1}{\sqrt{1-w^2}}$

2.7.19. $(\sec(x))^2\cos^{-1}(x)-\frac{1}{\sqrt{1-x^2}}\tan(x)$

2.7.20. $\frac{e^t}{t}+\ln(t)\,e^t$

2.7.21. $\frac{\frac{1}{1+z^2}\sin^{-1}(z)-\frac{1}{\sqrt{1-z^2}}\tan^{-1}(z)}{\left(\sin^{-1}(z)\right)^2}$

2.7.22. $(\sec(\sqrt[4]{x}))^2\,\frac{1}{4}\frac{1}{\left(\sqrt[4]{x}\right)^3}$

2.7.23. $\csc\left(\frac{1}{q^3}\right)\cot\left(\frac{1}{q^3}\right)\frac{3q^2}{(q^3)^2}$

2.7.24. 1

2.7.29. $y=2\left(x-\frac{-\sqrt{3}}{2}\right)+\left(-\frac{\pi}{3}\right)$

2.7.30. $y=-4\left(x-\frac{\sqrt{3}}{4}\right)+\frac{\pi}{6}$

3 · The Graphical Behavior of Functions

3.1 · Extreme Values

3.1 · Exercises

Terms and Concepts

3.1.2. Answers will vary.

3.1.4. Answers will vary.

3.1.5. False

3.1.6. **(a).** 0

(b). undefined

Problems

3.1.7. (a). B
(b). NONE
(c). B, G
(d). C, F

3.1.8. (a). C
(b). A
(c). C
(d). A, E

3.1.9. 0

3.1.10. (a). 0
(b). 0

3.1.11. (a). 0
(b). 0

3.1.12. (a). 0
(b). 0
(c). DNE

3.1.13. (a). DNE
(b). 0

3.1.14. (a). DNE
(b). DNE

3.1.15. 0

3.1.16. DNE

3.1.17. (a). 14
(b). -2

3.1.18. (a). -6
(b). -28

3.1.19. (a). -2.82843
(b). -4

3.1.20. (a). 30.4664
(b). 0

3.1.21. (a). $\frac{9}{2}$
(b). 2.82843

3.1.22. (a). $\frac{4}{11}$
(b). 0

3.1.23. (a). $\frac{e^{\frac{\pi}{4}}}{\sqrt{2}}$
(b). $-e^{\pi}$

3.1.24. (a). $\frac{e^{\frac{3\pi}{4}}}{\sqrt{2}}$
(b). 0

3.1.25. (a). $\frac{1}{2e}$
(b). 0

3.1.26. (a). 0.47247
(b). -6.31821

3.2 • The Mean Value Theorem

3.2 • Exercises

Problems

3.2.3. $(-1, 1)$

3.2.4. does not apply

3.2.5. $-\frac{1}{2}$

3.2.6. $-\frac{1}{2}$

3.2.7. does not apply

3.2.8. $\frac{\pi}{2}$

3.2.9. does not apply

3.2.10. does not apply

3.2.11. 0

3.2.12. $\frac{5}{2}$

3.2.13. $3\frac{\sqrt{2}}{2}$

3.2.14. $\frac{19}{4}$

3.2.15. does not apply

3.2.16. $\frac{4}{\ln(5)}$

3.2.17. $-\sec^{-1}\left(\frac{2}{\sqrt{\pi}}\right), \sec^{-1}\left(\frac{2}{\sqrt{\pi}}\right)$

3.2.18. $-\frac{2}{3}$

3.2.19. $5+7\frac{\sqrt{7}}{6}, 5-7\frac{\sqrt{7}}{6}$

3.2.20. $\frac{\sqrt{\pi^2-4}}{\pi}, \frac{-\sqrt{\pi^2-4}}{\pi}$

3.3 • Increasing and Decreasing Functions

3.3 • Exercises

Terms and Concepts

3.3.3. Answers will vary; graphs should be steeper near $x = 0$ than near $x = 2$.

3.3.5. False

Problems

3.3.15. **(a).** $(-\infty, \infty)$
(b). -2
(c). $[-2, \infty)$
(d). $(-\infty, -2]$
(e). NONE
(f). -2

3.3.16. **(a).** $(-\infty, \infty)$
(b). $-\frac{4}{3}, 0$
(c). $(-\infty, -1.33333], [0, \infty)$
(d). $[-1.33333, 0]$
(e). -1.33333
(f). 0

3.3.17. **(a).** $(-\infty, \infty)$
(b). $-\frac{5}{7}, \frac{7}{3}$
(c). $(-\infty, -0.714286], [2.33333, \infty)$
(d). $[-0.714286, 2.33333]$
(e). $-\frac{5}{7}$
(f). $\frac{7}{3}$

3.3.18. **(a).** $(-\infty, \infty)$
(b). 3
(c). $(-\infty, \infty)$
(d). NONE
(e). NONE
(f). NONE

3.3.19. **(a).** $(-\infty, \infty)$
(b). 5
(c). $(-\infty, 5]$
(d). $[5, \infty)$
(e). 5
(f). NONE

3.3.20. **(a).** $(-\infty, -6) \cup (-6, 6) \cup (6, \infty)$
(b). 0
(c). $(-\infty, -6), (-6, 0]$
(d). $[0, 6), (6, \infty)$
(e). 0
(f). NONE

3.3.21. **(a).** $(-\infty, -7) \cup (-7, -5) \cup (-5, \infty)$
(b). $-5.91608, 5.91608$
(c). $[-5.91608, -5), (-5, 5.91608]$
(d). $(-\infty, -7), (-7, -5.91608], [5.91608, \infty)$
(e). 5.91608
(f). -5.91608

3.3.22. **(a).** $(-\infty, 0) \cup (0, \infty)$
(b). $-5, -15$
(c). $[-15, -5]$
(d). $(-\infty, -15], [-5, 0), (0, \infty)$
(e). -5
(f). -15

3.3.23. **(a).** $(-\pi, \pi)$
(b). $-2.35619, -0.785398, 0.785398, 2.35619$
(c). $(-3.14159, -2.35619), (-0.785398, 0.785398), (2.35619, 3.14159)$
(d). $(-2.35619, -0.785398), (0.785398, 2.35619)$
(e). $-2.35619, 0.785398$
(f). $-0.785398, 2.35619$

3.3.24. **(a).** $(-\infty, \infty)$
(b). -2
(c). $[-2, \infty)$
(d). $(-\infty, -2]$
(e). NONE
(f). -2

3.4 • Concavity and the Second Derivative

3.4 • Exercises

Terms and Concepts

3.4.1. Answers will vary.

3.4.2. Answers will vary.

3.4.3. Yes; Answers will vary.

3.4.4. No.

Problems

3.4.15. **(a).** NONE
(b). $(-\infty, \infty)$
(c). NONE

3.4.16. **(a).** NONE
(b). NONE
(c). $(-\infty, \infty)$

3.4.17. **(a).** 0
(b). $[0, \infty)$
(c). $(-\infty, 0]$

3.4.18. **(a).** $-\frac{1}{4}$
(b). $[-0.25, \infty)$
(c). $(-\infty, -0.25]$

3.4.19. **(a).** $-\frac{32}{3}, 0$
(b). $(-\infty, -10.6667], [0, \infty)$
(c). $[-10.6667, 0]$

3.4.20. **(a).** $4.42265, 5.57735$
(b). $(-\infty, 4.42265], [5.57735, \infty)$
(c). $[4.42265, 5.57735]$

3.4.21. **(a).** -2
(b). $(-\infty, \infty)$
(c). NONE

3.4.22. **(a).** NONE
(b). $(-1.5708, 1.5708)$
(c). $(-4.71239, -1.5708), (1.5708, 4.71239)$

3.4.23. **(a).** $-0.57735, 0.57735$
(b). $(-\infty, -0.57735], [0.57735, \infty)$
(c). $[-0.57735, 0.57735]$

3.4.24. **(a).** NONE
(b). $(-\infty, 2), (5, \infty)$
(c). $(2, 5)$

3.4.25. **(a).** $-0.785398, 2.35619$
(b). $(-3.14159, -0.785398], [2.35619, 3.14159)$
(c). $[-0.785398, 2.35619]$

3.4.26. **(a).** $-0.585786, -3.41421$
(b). $(-\infty, -3.41421], [-0.585786, \infty)$
(c). $[-3.41421, -0.585786]$

3.4.27. **(a).** 0.22313
(b). $[0.22313, \infty)$
(c). $(0, 0.22313]$

3.4.28. **(a).** $0.707107, -0.707107$
(b). $(-\infty, -0.707107], [0.707107, \infty)$
(c). $[-0.707107, 0.707107]$

3.4.29. **(a).** -7
(b). NONE
(c). -7

3.4.30. **(a).** $-\frac{5}{2}$
(b). $-\frac{5}{2}$
(c). NONE

3.4.31. **(a).** $-1.1547, 1.1547$
(b). -1.1547
(c). 1.1547

3.4.32. **(a).** NONE
(b). NONE
(c). NONE

3.4.33. **(a).** -4
(b). NONE
(c). -4

3.4.34. **(a).** $-3, -2, 2$
(b). -2
(c). $-3, 2$

3.4.35. **(a).** 3
(b). NONE
(c). NONE

3.4.36. **(a).** $-3.14159, 0, 3.14159$
(b). $-3.14159, 3.14159$
(c). 0

3.4.37. (a). -9
(b). -9
(c). NONE

3.4.38. (a). 0
(b). 0
(c). NONE

3.4.39. (a). $-2.35619, 0.785398$
(b). 0.785398
(c). -2.35619

3.4.40. (a). $-2, 0$
(b). -2
(c). 0

3.4.41. (a). 0.606531
(b). NONE
(c). 0.606531

3.4.42. (a). 0
(b). 0
(c). NONE

3.4.43. (a). NONE
(b). NONE

3.4.44. (a). NONE
(b). NONE

3.4.45. (a). NONE
(b). 0

3.4.46. (a). $-\frac{8}{27}$
(b). NONE

3.4.47. (a). $-\frac{28}{3}$
(b). 0

3.4.48. (a). 1.42265
(b). 2.57735

3.4.49. (a). NONE
(b). NONE

3.4.50. (a). NONE
(b). NONE

3.4.51. (a). 0
(b). 2

3.4.52. (a). NONE
(b). NONE

3.4.53. (a). -0.785398
(b). 2.35619

3.4.54. (a). -3.41421
(b). -0.585786

3.4.55. (a). NONE
(b). 0.22313

3.4.56. (a). -0.707107
(b). 0.707107

3.5 · Curve Sketching

3.5 · Exercises

Terms and Concepts

3.5.3. True
3.5.4. True
3.5.5. True

4 · Applications of the Derivative

4.1 · Newton's Method

4.1 · Exercises

Terms and Concepts

4.1.1. False
4.1.2. False

Problems

4.1.3. (a). 1.57091
(b). 1.5708
(c). 1.5708
(d). 1.5708
(e). 1.5708

4.1.4. (a). -0.557408
(b). 0.0659365
(c). -9.57219×10^{-5}
(d). 0
(e). 0

4.1.5. (a). 2
(b). 1.2
(c). 1.01176
(d). 1.00005
(e). 1

4.1.6. (a). 1.41667
(b). 1.41422
(c). 1.41421
(d). 1.41421
(e). 1.41421

4.1.7. (a). 0.613706
(b). 0.913341
(c). 0.996132
(d). 0.999993
(e). 1

4.1.8. (a). 1.44444
(b). 1.13057
(c). 1.01498
(d). 1.00022
(e). 1

4.1.9. $\{-5.15633, -0.369102, 0.525428\}$
4.1.10. $\{-3.71448, -0.856723, 1, 1.5712\}$
4.1.11. $\{-1.0134, 0.988312, 1.39341\}$
4.1.12. $\{-2.16477, 0, 0.524501, 1.81328\}$

4.1.13. $\{-0.824132, 0.824132\}$
4.1.14. $\{-0.636733, 1.40962\}$
4.1.15. $\{0\}$
4.1.16. $\{-4.49341, 0, 4.49341\}$

4.2 · Related Rates
4.2 · Exercises

Terms and Concepts

4.2.1. True
4.2.2. False

Problems

4.2.3.

(a) $0.198944\ \frac{\text{cm}}{\text{s}}$

(b) $0.0198944\ \frac{\text{cm}}{\text{s}}$

(c) $0.00198944\ \frac{\text{cm}}{\text{s}}$

4.2.4.

(a) $0.397887\ \frac{\text{cm}}{\text{s}}$

(b) $0.00397887\ \frac{\text{cm}}{\text{s}}$

(c) $3.97887 \times 10^{-5}\ \frac{\text{cm}}{\text{s}}$

4.2.5. $51.066\ \frac{\text{mi}}{\text{h}}$

4.2.6.

(a) $68.75\ \frac{\text{mi}}{\text{h}}$

(b) $75\ \frac{\text{mi}}{\text{h}}$

4.2.7.

(a) $258.537\ \frac{\text{rad}}{\text{hr}}$

(b) $413.417\ \frac{\text{rad}}{\text{hr}}$

(c) $424\ \frac{\text{rad}}{\text{hr}}$

4.2.8.

(a) $0.0225641\ \frac{\text{rad}}{\text{s}}$

(b) $0.553459\ \frac{\text{rad}}{\text{s}}$

(c) $7.33333\ \frac{\text{rad}}{\text{s}}$

4.2.9.

(a) $0.0417029\ \frac{\text{ft}}{\text{s}}$

(b) $0.458349\ \frac{\text{ft}}{\text{s}}$

(c) $3.35489\ \frac{\text{ft}}{\text{s}}$

(d) ∞

4.2.10.

(a) $30.5941\ \frac{\text{ft}}{\text{min}}$

(b) $36.0555\ \frac{\text{ft}}{\text{min}}$

(c) $301.496\ \frac{\text{ft}}{\text{min}}$

4.2.11.

(a) $19.1658\ \frac{\text{ft}}{\text{s}}$

(b) $0.191658\ \frac{\text{ft}}{\text{s}}$

(c) $0.0395988\ \frac{\text{ft}}{\text{s}}$

(d) 381.791 s

4.2.12.

(a) $0.632456\ \frac{\text{ft}}{\text{s}}$

(b) $1.6\ \frac{\text{ft}}{\text{s}}$

(c) 51.9615 ft

4.2.13.

(a) 80 ft

(b) $1.71499\ \frac{\text{ft}}{\text{s}}$

(c) $1.83829\ \frac{\text{ft}}{\text{s}}$

(d) 74.162 ft

4.2.14.

(a) 96 ft

(b) $9.42478\ \frac{\text{ft}}{\text{s}}$

4.2.15. $0.00230973\ \frac{\text{ft}}{\text{s}}$

4.3 · Optimization

4.3 · Exercises

Terms and Concepts

4.3.1. True
4.3.2. False

Problems

4.3.3. 5625
4.3.4. $2\sqrt{560}$
4.3.5. DNE
4.3.6. $\frac{8450}{29}$
4.3.7. 1
4.3.8. 150 ft; $\left(\frac{225}{2}\right)$ ft
4.3.9. **(a).** 3.83722 cm
(b). 7.67443 cm
4.3.10. **(a).** 3.20058 in
(b). 6.40117 in
4.3.11. **(a).** 3.0456 cm
(b). 12.1824 cm
4.3.12. 11664 in^3
4.3.13. 10.3923 in; 14.6969 in
4.3.14. **(a).** 0.535898 mi
(b). $\$503{,}730.67$

4.3.15. **(a).** 0 mi

(b). $\$474{,}341.65$

4.3.16. 33.6239 ft

4.3.17. 23.7599 ft

4.3.18. $\sqrt{2}$; $\sqrt{2}$

4.4 · Differentials

4.4 · Exercises

Terms and Concepts

4.4.1. True
4.4.2. True
4.4.3. False
4.4.4. True
4.4.6. True

Problems

4.4.7. 4.28
4.4.8. 8.7
4.4.9. 83.2
4.4.10. 102.5
4.4.11. 5.05
4.4.12. 5.88333
4.4.13. 4.98667
4.4.14. 6.00556
4.4.15. 0.141593
4.4.16. 1.1

4.4.17. $(2x-5)\,dx$

4.4.18. $\left(5x^4+9x^8\right)dx$

4.4.19. $-\frac{24x^5}{(4x^6)^2}dx$

4.4.20. $2(6x+\sin(x))\,(6+\cos(x))\,dx$

4.4.21. $\left(7x^6+8e^{8x}\right)dx$

4.4.22. $-\frac{40x^4}{(x^5)^2}dx$

4.4.23. $\frac{9(\tan(x)+2)-9x\sec^2(x)}{(\tan(x)+2)^2}dx$

4.4.24. $\frac{9}{9x}dx$

4.4.25. $(e^x\sin(x)+e^x\cos(x))\,dx$

4.4.26. $-\sin(\sin(x))\cos(x)\,dx$

4.4.27. $\frac{x+5-(x-4)}{(x+5)^2}dx$

4.4.28. $\left(1.60944\cdot 5^x\ln(x)+\frac{5^x\cdot 1}{x}\right)dx$

4.4.29. $\tan^{-1}(x)\,dx$

4.4.30. $\cot(x)\,dx$

4.4.31. 5.02655 cm^3

4.4.32.

(a) 51.2

(b) 76.8

4.4.33. 3.92699

4.4.34. -4 ft^2

4.4.35.

(a) 297.717 ft

(b) 62.3155 ft

(c) 20.9%

4.4.36.

(a) 298.868 ft

(b) 17.335 ft

(c) 5.8%

4.4.37.

(a) 298.868 ft

(b) 8.66751 ft

(c) 2.9%

4.4.39. 1%

4.4.38. Isosceles ... feet

4.5 · Taylor Polynomials

4.5 · Exercises

Terms and Concepts

4.5.2. True
4.5.3. $6 + 3x - 4x^2$
4.5.4. 30

Problems

4.5.5. $1 - x + 0.5x^2 - 0.166667x^3$

4.5.6. $x - 0.166667x^3 + 0.00833333x^5 - 0.000198413x^7$

4.5.7. $x + x^2 + 0.5x^3 + 0.166667x^4 + 0.0416667x^5$

4.5.8. $x + 0.333333x^3 + 0.133333x^5$

4.5.9. $1 + 2x + 2x^2 + 1.33333x^3 + 0.666667x^4$

4.5.10. $1 + x + x^2 + x^3 + x^4$

4.5.11. $1 - x + x^2 - x^3 + x^4$

4.5.12. $1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7$

4.5.13. $1 + 0.5(x-1) - 0.125(x-1)^2 + 0.0625(x-1)^3 - 0.0390625(x-1)^4$

4.5.14. $0.693147 + 0.5(x-1) - 0.125(x-1)^2 + 0.0416667(x-1)^3 - 0.015625(x-1)^4$

4.5.15. $0.707107 - 0.707107\left(x - \frac{\pi}{4}\right) - 0.353553\left(x - \frac{\pi}{4}\right)^2 + 0.117851\left(x - \frac{\pi}{4}\right)^3 + 0.0294628\left(x - \frac{\pi}{4}\right)^4 - 0.00589256\left(x - \frac{\pi}{4}\right)^5 - 0.000982093\left(x - \frac{\pi}{4}\right)^6$

4.5.16. $0.5 + 0.866025\left(x - \frac{\pi}{6}\right) - 0.25\left(x - \frac{\pi}{6}\right)^2 - 0.144338\left(x - \frac{\pi}{6}\right)^3 + 0.0208333\left(x - \frac{\pi}{6}\right)^4 + 0.00721688\left(x - \frac{\pi}{6}\right)^5$

4.5.17. $0.5 - 0.25(x-2) + 0.125(x-2)^2 - 0.0625(x-2)^3 + 0.03125(x-2)^4 + 0.015625(x-2)^5$

4.5.18. $1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4 - 6(x-1)^5 + 7(x-1)^6 - 8(x-1)^7 + 9(x-1)^8$

4.5.19. $0.5 + 0.5(x+1) + 0.25(x+1)^2$

4.5.20. $-\pi^2 - 2\pi(x-\pi) + \frac{\pi^2-2}{2}(x-\pi)^2$

4.5.31. The nth term is: when n even, 0; when n is odd, $\frac{(-1)^{(n-1)/2}}{n!}x^n$.

5 · Integration

5.1 · Antiderivatives and Indefinite Integration

5.1 · Exercises

Terms and Concepts

5.1.2. an antiderivative

5.1.4. **(a).** opposite

(b). opposite

5.1.6. velocity

5.1.7. velocity

5.1.8. $F(x) + G(x)$

Problems

5.1.9. $\left(\frac{4}{3}\right) x^6 + C$

5.1.10. $\frac{1}{10} x^{10} + C$

5.1.11. $\left(\frac{5}{9}\right) x^9 - 6x + C$

5.1.12. $t + C$

5.1.13. $s + C$

5.1.14. $C - \frac{1}{35t^7}$

5.1.15. $C - \frac{2}{t^3}$

5.1.16. $2\sqrt{x} + C$

5.1.17. $\sec(\theta) + C$

5.1.18. $-\cos(\theta) + C$

5.1.19. $\sec(x) + \csc(x) + C$

5.1.20. $2e^\theta + C$

5.1.21. $\frac{3^t}{\ln(3)} + C$

5.1.22. $\frac{4^t}{9\ln(4)} + C$

5.1.23. $\left(\frac{25}{3}\right) t^3 + 10t^2 + 4t + \left(\frac{8}{15}\right) + C$

5.1.24. $\frac{t^{10}}{10} - \frac{t^6}{2} - 5t^2 + C$

5.1.25. $\frac{x^{17}}{17} + C$

5.1.26. $1.41421^e x + C$

5.1.27. $rx + C$

5.1.30. $8 - \cos(x)$

5.1.31. $2e^x + 6$

5.1.32. $3\frac{x^4}{4} - 3x^2 + 9$

5.1.33. $\sec(x) + 4$

5.1.34. $\frac{5^x}{\ln(5)} - \frac{25}{\ln(5)} + 5$

5.1.35. $3x^2 + 2x + 5$

5.1.36. $\left(\frac{2}{3}\right) x^3 + 7x + \left(-\frac{5}{3}\right)$

5.1.37. $7e^x - 10x - 15$

5.1.38. $6\theta - \cos(\theta) + 10$

5.1.39. $x^6 + \frac{2^x}{0.480453} - \cos(x) - 1.4427x + 0.918631$

5.1.40. $-(2x + 11)$

5.2 · The Definite Integral

5.2 · Exercises

Terms and Concepts

5.2.3. 0

5.2.4. $\int 0^2 (2x + 3)\, dx$

Problems

5.2.5.

- **(a)** 3
- **(b)** 4
- **(c)** 3
- **(d)** 0
- **(e)** -4
- **(f)** 9

5.2.6.

- **(a)** -4
- **(b)** -5
- **(c)** -3
- **(d)** 1
- **(e)** -2
- **(f)** 10

5.2.7.

- **(a)** 4
- **(b)** 2
- **(c)** 4
- **(d)** 2
- **(e)** 1
- **(f)** 2

5.2.8.

- **(a)** $-\frac{1}{2}$
- **(b)** 0
- **(c)** $\frac{3}{2}$
- **(d)** $\frac{3}{2}$
- **(e)** $\frac{9}{2}$
- **(f)** $\frac{15}{2}$

5.2.9.

- **(a)** π
- **(b)** π
- **(c)** 2π
- **(d)** 10π

5.2.10.

- **(a)** 15
- **(b)** 12
- **(c)** 0
- **(d)** $3(b-a)$

5.2.11.

- **(a)** -59
- **(b)** -48
- **(c)** -27
- **(d)** -33

5.2.12.

- **(a)** $\frac{4}{\pi}$
- **(b)** $\frac{-4}{\pi}$
- **(c)** 0
- **(d)** $\frac{2}{\pi}$

5.2.13.

- **(a)** 4
- **(b)** 4
- **(c)** -4
- **(d)** -2

5.2.14.

- **(a)** $\frac{40}{3}$
- **(b)** $\frac{26}{3}$
- **(c)** $\frac{8}{3}$
- **(d)** $\frac{38}{3}$

5.2.15.

- **(a)** 2 $\frac{\text{ft}}{\text{s}}$
- **(b)** 2 ft
- **(c)** 1.5 ft

5.2.16.

- **(a)** 3 $\frac{\text{ft}}{\text{s}}$
- **(b)** 9.5 ft
- **(c)** 9.5 ft

5.2.17.

(a) $64\ \frac{\text{ft}}{\text{s}}$

(b) 64 ft

(c) 2 s

(d) 4.64575 s

5.2.18.

(a) $96\ \frac{\text{ft}}{\text{s}}$

(b) 6 s

(c) 6 s

(d) 208 ft

5.2.19. 2

5.2.20. 5

5.2.21. 16

5.2.22. $a = -\frac{2}{7}b$

5.2.23. 22

5.2.24. -7

5.2.25. 0

5.2.26. $a = -\frac{18}{11}b$

5.3 · Riemann Sums

5.3 · Exercises

Terms and Concepts

5.3.1. limits
5.3.2. 12
5.3.3. rectangles
5.3.4. True

Problems

5.3.5. **(a).** $9 + 16 + 25 + 36$
(b). 86

5.3.6. **(a).** $-4 + (-1) + 2 + 5 + 8$
(b). 10

5.3.7. **(a).** $0 + (-1) + 0 + 1$
(b). 0

5.3.8. **(a).** $9 + 9 + 9 + 9 + 9 + 9 + 9 + 9$
(b). 72

5.3.9. **(a).** $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$
(b). $\frac{49}{20}$

5.3.10.
(a). $-1 + 2 + (-3) + 4 + (-5) + 6 + (-7) + 8$
(b). 4

5.3.11. **(a).** $\frac{1}{2} + \frac{1}{6} + \frac{1}{12}$
(b). $\frac{3}{4}$

5.3.12. **(a).** $1 + 1 + 1 + 1 + 1 + 1$
(b). 6

5.3.13. $1;\ 4;\ 3i$

5.3.14. $0;\ 6;\ i^2 + 2$

5.3.15. $1;\ 5;\ \frac{i}{i+3}$

5.3.16. $1;\ 5;\ -(-e)^i$

5.3.17. 72

5.3.18. 435

5.3.19. 1456

5.3.20. 30336

5.3.21. -3220
5.3.22. -687
5.3.23. 4560
5.3.24. 4324

5.3.25. 135
5.3.26. 146340
5.3.27. 21
5.3.28. 106272

5.3.35. **(a).** $\frac{(n-1)^2}{4n^2}$
(b). 0.2025
(c). 0.245025
(d). 0.2495
(e). $\frac{1}{4}$

5.3.36. **(a).** $6+\frac{9}{1n}+\frac{9}{1n^2}$
(b). 6.99
(c). 6.0909
(d). 6.00901
(e). 6

5.3.37. **(a).** 36
(b). 36
(c). 36
(d). 36
(e). 36

5.3.38. **(a).** $\left(\frac{212}{3}\right)+\frac{-48}{1n}+\frac{16}{3n^2}$
(b). 65.92
(c). 70.1872
(d). 70.6187
(e). $\frac{212}{3}$

5.3.39. **(a).** $132-\frac{242}{n}$
(b). 107.8
(c). 129.58
(d). 131.758
(e). 132

5.3.40. **(a).** $-\frac{1}{12}+\frac{1}{12n^2}$
(b). -0.0825
(c). -0.083325
(d). -0.0833332
(e). $-\frac{1}{12}$

5.4 · The Fundamental Theorem of Calculus

5.4 · Exercises

Terms and Concepts

5.4.2. 0
5.4.3. True

Problems

5.4.5. 4
5.4.6. $\frac{65}{3}$
5.4.7. 0
5.4.8. 1
5.4.9. $2-\sqrt{2}$
5.4.10. 7
5.4.11. $\frac{\left(\frac{32767}{512}\right)}{\ln(8)}$
5.4.12. -2
5.4.13. -4
5.4.14. e^2-e^1
5.4.15. 42
5.4.16. 2
5.4.17. $\frac{4096}{5}$
5.4.18. $\ln(6)$
5.4.19. $\frac{6}{7}$
5.4.20. $\frac{59048}{295245}$
5.4.21. $\frac{1}{2}$
5.4.22. $\frac{1}{3}$
5.4.23. $\frac{1}{4}$
5.4.24. $\frac{1}{91}$
5.4.25. 14
5.4.26. 24
5.4.27. 0
5.4.28. $2-\sqrt{2}$

5.4.31. 1.1547

5.4.32. $-4.6188, 4.6188$

5.4.33. 0.541325

5.4.34. 4

5.4.35. $\frac{\frac{1}{\pi-\frac{\pi}{2}}\cdot 3.14159}{\pi}$

5.4.36. $\frac{\frac{0}{\pi-0}\cdot 3.14159}{\pi}$

5.4.37. $\frac{7}{2}$

5.4.38. $\frac{64}{3}$

5.4.39. $\frac{729}{4}$

5.4.40. $\frac{1}{e^1-1}$

5.4.41. -168 ft

5.4.42. 144 ft

5.4.43. 76 ft

5.4.44. 11.4965 mi

5.4.45. 0 ft

5.4.46. $\frac{10240}{3}$ ft

5.4.47. $-256\ \frac{\text{ft}}{\text{s}}$

5.4.48. $72\ \frac{\text{ft}}{\text{s}}$

5.4.49. $\frac{1}{2}\ \frac{\text{ft}}{\text{s}}$

5.4.50. $1\ \frac{\text{ft}}{\text{s}}$

5.4.55. $\frac{3x^2-7}{x^3-7x}$

5.4.56. $-3x^{11}$

5.4.57. $3x^2\left(x^3-1\right)-(x-1)$

5.4.58. $e^x\cos(e^x)-\cos(x)\cos(\sin(x))$

5.4.59. $4x^3\sin\left(4x^8\right)$

5.4.60. $\frac{1}{x}\sqrt{\ln^4(x)+6\ln^2(x)}-\cos(x)\sqrt{\sin^4(x)+6\sin^2(x)}$

5.5 · Numerical Integration

5.5 · Exercises

Terms and Concepts

5.5.1. False

5.5.4. A quadratic function (i.e., parabola)

Problems

5.5.5.

- **(a)** 0.75
- **(b)** 0.666667
- **(c)** 0.666667

5.5.6.

- **(a)** 250
- **(b)** 250
- **(c)** 250

5.5.7.

- **(a)** 1.89612
- **(b)** 2.00456
- **(c)** 2

5.5.8.

- **(a)** 5.14626
- **(b)** 5.25221
- **(c)** 5.33333

5.5.9.

- **(a)** 38.5781
- **(b)** 36.75
- **(c)** 36.75

5.5.10.

- **(a)** 0.220703
- **(b)** 0.200521
- **(c)** 0.2

5.5.11.

(a) 0

(b) 0

(c) 0

5.5.12.

(a) 12.2942

(b) 13.3923

(c) 14.1372

5.5.13.

(a) 0.900628

(b) 0.904523

5.5.14.

(a) 3.02419

(b) 2.93151

5.5.15.

(a) 13.9604

(b) 13.9066

5.5.16.

(a) 3.06949

(b) 3.14295

5.5.17.

(a) 1.17029

(b) 1.18728

5.5.18.

(a) 2.52971

(b) 2.54465

5.5.19.

(a) 1.08025

(b) 1.07699

5.5.20.

(a) 3.46822

(b) 3.4985

5.5.21.

(a) 161

(b) 12

5.5.22.

(a) 130

(b) 18

5.5.23.

(a) 994

(b) 62

5.5.24.

(a) 5591

(b) 46

5.5.25. **(a).** 30.8667 cm^2
(b). 308667 ft^2

5.5.26. **(a).** 25.0667 cm^2
(b). 250667 ft^2

Appendix B

Quick Reference

B.1 Differentiation Formulas

List B.1.1 Derivative Rules

1. $\frac{d}{dx}(cx) = c$
2. $\frac{d}{dx}(u \pm v) = u' \pm v'$
3. $\frac{d}{dx}(u \cdot v) = uv' + u'v$
4. $\frac{d}{dx}(\frac{u}{v}) = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}(u(v)) = u'(v)v'$
6. $\frac{d}{dx}(c) = 0$
7. $\frac{d}{dx}(x) = 1$

List B.1.2 Derivatives of Elementary Functions

1. $\frac{d}{dx}(x^n) = nx^{n-1}$
2. $\frac{d}{dx}(e^x) = e^x$
3. $\frac{d}{dx}(a^x) = \ln a \cdot a^x$
4. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
5. $\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$
6. $\frac{d}{dx}(\sin x) = \cos x$
7. $\frac{d}{dx}(\cos x) = -\sin x$
8. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
9. $\frac{d}{dx}(\sec x) = \sec x \tan x$
10. $\frac{d}{dx}(\tan x) = \sec^2 x$
11. $\frac{d}{dx}(\cot x) = -\csc^2 x$
12. $\frac{d}{dx}(\cosh x) = \sinh x$
13. $\frac{d}{dx}(\sinh x) = \cosh x$
14. $\frac{d}{dx}(\text{sech}\, x) = -\text{sech}\, x \tanh x$
15. $\frac{d}{dx}(\tanh x) = \text{sech}^2 x$
16. $\frac{d}{dx}(\text{csch}\, x) = -\text{csch}\, x \coth x$
17. $\frac{d}{dx}(\coth x) = -\text{csch}^2 x$

List B.1.3 Derivatives of Inverse Functions

1. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
2. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
3. $\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$
4. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
5. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
6. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
7. $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$
8. $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$
9. $\frac{d}{dx}(\text{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$
10. $\frac{d}{dx}(\text{csch}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}$
11. $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$
12. $\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$

B.2 Integration Formulas

List B.2.1 Basic Rules

1. $\int c \cdot f(x)\,dx = c\int f(x)\,dx$
2. $\int \big(f(x) \pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
3. $\int 0\,dx = C$
4. $\int 1\,dx = x + C$

List B.2.2 Integrals of Elementary (non-Trig) Functions

1. $\int e^x\,dx = e^x + C$
2. $\int \ln x\,dx = x\ln x - x + C$
3. $\int a^x\,dx = \frac{1}{\ln a} \cdot a^x + C$
4. $\int \frac{1}{x}\,dx = \ln|x| + C$
5. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, n \neq -1$

List B.2.3 Integrals Involving Trigonometric Functions

1. $\int \cos x\,dx = \sin x + C$
2. $\int \sin x\,dx = -\cos x + C$
3. $\int \tan x\,dx = -\ln|\cos x| + C$
4. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
5. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

6. $\int \cot x\,dx = \ln|\sin x| + C$

7. $\int \sec^2 x\,dx = \tan x + C$

8. $\int \csc^2 x\,dx = -\cot x + C$

9. $\int \sec x \tan x\,dx = \sec x + C$

10. $\int \csc x \cot x\,dx = -\csc x + C$

11. $\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$

12. $\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$

13. $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

14. $\int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

15. $\int \frac{1}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$

List B.2.4 Integrals Involving Hyperbolic Functions

1. $\int \cosh x\,dx = \sinh x + C$

2. $\int \sinh x\,dx = \cosh x + C$

3. $\int \tanh x\,dx = \ln(\cosh x) + C$

4. $\int \coth x\,dx = \ln|\sinh x| + C$

5. $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|x + \sqrt{x^2-a^2}\right| + C$

6. $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left|x + \sqrt{x^2+a^2}\right| + C$

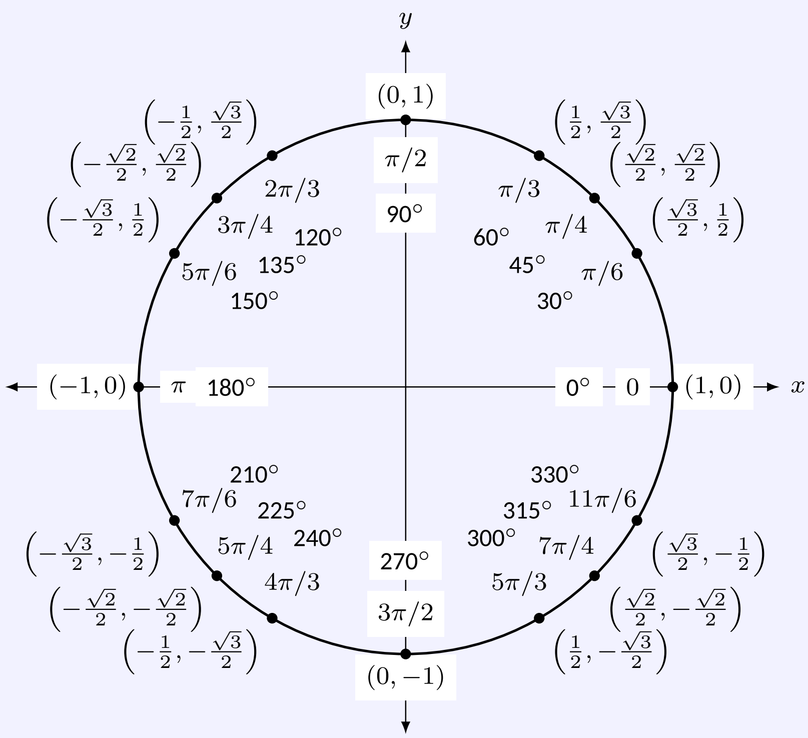
7. $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

8. $\int \frac{1}{x\sqrt{a^2-x^2}}\,dx = \frac{1}{a}\ln\left(\frac{x}{a+\sqrt{a^2-x^2}}\right) + C$

9. $\int \frac{1}{x\sqrt{x^2+a^2}} = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{x^2+a^2}}\right| + C$

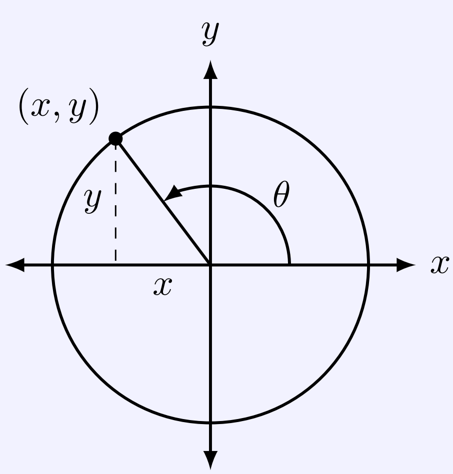
B.3 Trigonometry Reference

The Unit Circle.



B.3.1 Definitions of the Trigonometric Functions

Unit Circle Definition.

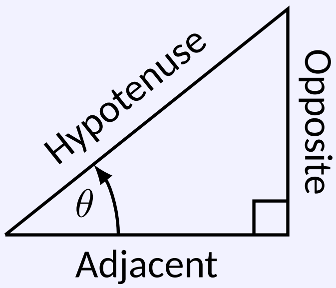


$\sin\theta = y \qquad \cos\theta = x$

$\csc\theta = \dfrac{1}{y} \qquad \sec\theta = \dfrac{1}{x}$

$\tan\theta = \dfrac{y}{x} \qquad \cot\theta = \dfrac{x}{y}$

Right Triangle Definition.



$\sin\theta = \frac{\text{O}}{\text{H}}$ $\quad \csc\theta = \frac{\text{H}}{\text{O}}$

$\cos\theta = \frac{\text{A}}{\text{H}}$ $\quad \sec\theta = \frac{\text{H}}{\text{A}}$

$\tan\theta = \frac{\text{O}}{\text{A}}$ $\quad \cot\theta = \frac{\text{A}}{\text{O}}$

B.3.2 Common Trigonometric Identities

1. $\sin^2 x + \cos^2 x = 1$
2. $\tan^2 x + 1 = \sec^2 x$
3. $1 + \cot^2 x = \csc^2 x$

List B.3.1 Pythagorean Identities

1. $\sin 2x = 2\sin x\cos x$
2. $$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x\\ &= 2\cos^2 x - 1\\ &= 1 - 2\sin^2 x\end{aligned}$$
3. $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

List B.3.2 Double Angle Formulas

1. $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
2. $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
3. $\tan\left(\frac{\pi}{2} - x\right) = \cot x$
4. $\csc\left(\frac{\pi}{2} - x\right) = \sec x$
5. $\sec\left(\frac{\pi}{2} - x\right) = \csc x$
6. $\cot\left(\frac{\pi}{2} - x\right) = \tan x$

List B.3.3 Cofunction Identities

1. $\sin(-x) = -\sin x$
2. $\cos(-x) = \cos x$
3. $\tan(-x) = -\tan x$
4. $\csc(-x) = -\csc x$
5. $\sec(-x) = \sec x$
6. $\cot(-x) = -\cot x$

List B.3.4 Even/Odd Identities

1. $\sin^2 x = \dfrac{1-\cos 2x}{2}$
2. $\cos^2 x = \dfrac{1+\cos 2x}{2}$
3. $\tan^2 x = \dfrac{1-\cos 2x}{1+\cos 2x}$

List B.3.5 Power-Reducing Formulas

1. $\sin x + \sin y = 2\sin\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$
2. $\sin x - \sin y = 2\sin\left(\dfrac{x-y}{2}\right)\cos\left(\dfrac{x+y}{2}\right)$
3. $\cos x + \cos y = 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$
4. $\cos x - \cos y = -2\sin\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$

List B.3.6 Sum to Product Formulas

List B.3.7 Product to Sum Formulas

1. $\sin x \sin y = \dfrac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$
2. $\cos x \cos y = \dfrac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$
3. $\sin x \cos y = \dfrac{1}{2}\big(\sin(x+y) + \sin(x-y)\big)$

List B.3.8 Angle Sum/Difference Formulas

1. $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
2. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
3. $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

B.4 Areas and Volumes

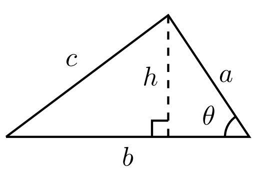
Triangles

$h = a\sin\theta$

Area = $\frac{1}{2}bh$

Law of Cosines:

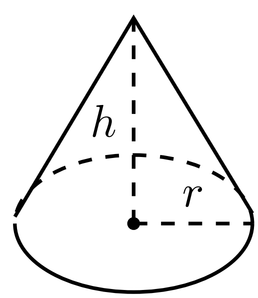
$c^2 = a^2 + b^2 - 2ab\cos\theta$



Right Circular Cone

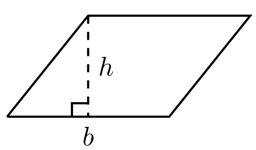
Volume = $\frac{1}{3}\pi r^2 h$

Surface Area = $\pi r\sqrt{r^2+h^2} + \pi r^2$



Parallelograms

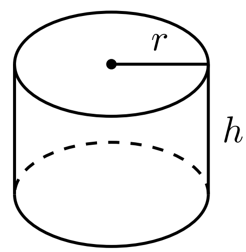
Area = bh



Right Circular Cylinder

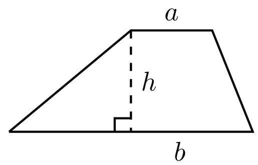
Volume = $\pi r^2 h$

Surface Area = $2\pi rh + 2\pi r^2$



Trapezoids

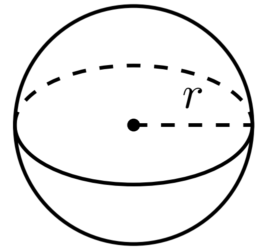
Area = $\frac{1}{2}(a+b)h$



Sphere

Volume = $\frac{4}{3}\pi r^3$

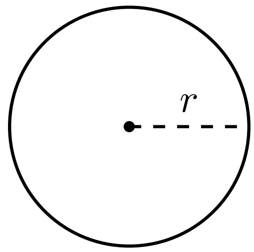
Surface Area =$4\pi r^2$



Circles

Area = πr^2

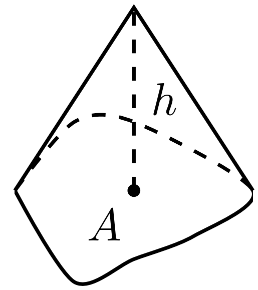
Circumference = $2\pi r$



General Cone

Area of Base = A

Volume = $\frac{1}{3}Ah$

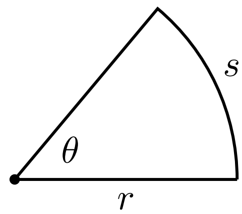


Sectors of Circles

θ in radians

Area = $\frac{1}{2}\theta r^2$

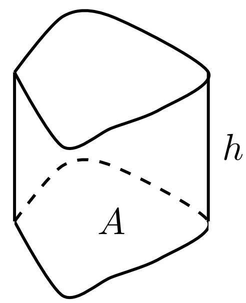
$s = r\theta$



General Right Cylinder

Area of Base = A

Volume = Ah



B.5 Algebra

Factors and Zeros of Polynomials.

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. If $p(a) = 0$, then a is a $zero$ of the polynomial and a solution of the equation $p(x) = 0$. Furthermore, $(x - a)$ is a $factor$ of the polynomial.

Fundamental Theorem of Algebra.

An nth degree polynomial has n (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

Quadratic Formula.

If $p(x) = ax^2 + bx + c$, and $0 \le b^2 - 4ac$, then the real zeros of p are $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

Special Factors.

$$
\begin{aligned}
x^2 - a^2 &= (x-a)(x+a) \\
x^3 - a^3 &= (x-a)(x^2+ax+a^2) \\
x^3 + a^3 &= (x+a)(x^2-ax+a^2) \\
x^4 - a^4 &= (x^2-a^2)(x^2+a^2)
\end{aligned}
$$

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n$$
$$(x-y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \cdots \pm nxy^{n-1} \mp y^n$$

Binomial Theorem.

$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x-y)^2 = x^2 - 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$

Rational Zero Theorem.

If $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ has integer coefficients, then every $rational\ zero$ of p is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Factoring by Grouping.

$$acx^3 + adx^2 + bcx + bd = ax^2(cx+d) + b(cx+d) = (ax^2+b)(cx+d)$$

Arithmetic Operations.

$$ab + ac = a(b+c) \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \qquad \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc} \qquad \frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc} \qquad \frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c} \qquad \frac{a-b}{c-d} = \frac{b-a}{d-c} \qquad \frac{ab+ac}{a} = b+c$$

Exponents and Radicals.

$$a^0 = 1,\ a \neq 0 \qquad (ab)^x = a^xb^x \qquad a^xa^y = a^{x+y} \qquad \sqrt{a} = a^{1/2}$$

$$\frac{a^x}{a^y} = a^{x-y} \qquad \sqrt[n]{a} = a^{1/n} \qquad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \sqrt[n]{a^m} = a^{m/n}$$

$$a^{-x} = \frac{1}{a^x} \qquad \sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b} \qquad (a^x)^y = a^{xy} \qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

B.6 Additional Formulas

Summation Formulas:.

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Trapezoidal Rule:.

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\big[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\big]$$

with Error $\leq \dfrac{(b-a)^3}{12n^2}\big[\,\text{max}\,|f\,''(x)|\,\big]$

Simpson's Rule:.

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\big]$$

with Error $\leq \dfrac{(b-a)^5}{180n^4}\left[\,\text{max}\left|f^{\,(4)}(x)\right|\,\right]$

Arc Length:.

$$L = \int_a^b \sqrt{1+f\,'(x)^2}\,dx$$

Surface of Revolution:.

$$2\pi\int_a^b f(x)\sqrt{1+f\,'(x)^2}dx$$

(where $f(x) \geq 0$)

$$S = 2\pi\int_a^b x\sqrt{1+f\,'(x)^2}dx$$

(where $a, b \geq 0$)

Work Done by a Variable Force:.

$$W = \int_a^b F(x)dx$$

Force Exerted by a Fluid:.

$$F = \int_a^b w\, d(y)\, \ell(y) dy$$

Taylor Series Expansion for $f(x)$:.

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

Maclaurin Series Expansion for $f(x)$, where $c = 0$:.

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

B.7 Summary of Tests for Series

Table B.7.1

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	Cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} r^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum $= \frac{1}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+a})$	$\lim_{n\to\infty} b_n = L$		Sum $= \left(\sum_{n=1}^{a} b_n\right) - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$	$p > 1$	$p \leq 1$	
Integral Test	$\sum_{n=0}^{\infty} a_n$	$\int_1^{\infty} a(n)\, dn$ converges	$\int_1^{\infty} a(n)\, dn$ diverges	$a_n = a(n)$ must be continuous
Direct Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$	$\sum_{n=0}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$	
Limit Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n\to\infty} \frac{a_n}{b_n} \geq 0$	$\sum_{n=0}^{\infty} b_n$ diverges and $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$	Also diverges if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$
Ratio Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$ Also diverges if	$\{a_n\}$ must be positive $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$
Root Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} (a_n)^{1/n} < 1$	$\lim_{n\to\infty} (a_n)^{1/n} > 1$ Also diverges if	$\{a_n\}$ must be positive $\lim_{n\to\infty} (a_n)^{1/n} = \infty$

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