APEX Calculus

for University of Lethbridge

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Website: apexcalculus.com[1]

[1] www.apexcalculus.com

Thanks

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Preface

A Note on Using this Text. Thank you for reading this short preface. Allow us to share a few key points about the text so that you may better understand what you will find beyond this page.

This text comprises a three—volume series on Calculus. The first part covers material taught in many "Calc 1" courses: limits, derivatives, and the basics of integration, found in Chapters 1 through 6.1. The second text covers material often taught in "Calc 2:" integration and its applications, including an introduction to differential equations, along with an introduction to sequences, series and Taylor Polynomials, found in Chapters 5 through 8. The third text covers topics common in "Calc 3" or "multivariable calc:" parametric equations, polar coordinates, vector-valued functions, and functions of more than one variable, found in Chapters 10 through 15. All three are available separately for free at apexcalculus.com[2], and HTML versions of the book can be found at opentext.uleth.ca[3].

These three texts are intended to work together and make one cohesive text, *APEX Calculus*, which can also be downloaded from the website.

Printing the entire text as one volume makes for a large, heavy, cumbersome book. One can certainly only print the pages they currently need, but some prefer to have a nice, bound copy of the text. Therefore this text has been split into these three manageable parts, each of which can be purchased for about $15 at Amazon.com[4].

For Students: How to Read this Text. Mathematics textbooks have a reputation for being hard to read. High—level mathematical writing often seeks to say much with few words, and this style often seeps into texts of lower—level topics. This book was written with the goal of being easier to read than many other calculus textbooks, without becoming too verbose.

Each chapter and section starts with an introduction of the coming material, hopefully setting the stage for "why you should care," and ends with a look ahead to see how the just—learned material helps address future problems.

- *Please read the text.*

  It is written to explain the concepts of Calculus. There are numerous examples to demonstrate the meaning of definitions, the truth of theorems, and the application of mathematical techniques. When you encounter a sentence you don't understand, read it again. If it still doesn't make sense, read on anyway, as sometimes confusing sentences are explained by later sentences.

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[2] apexcalculus.com
[3] opentext.uleth.ca/calculus.html
[4] amazon.com

- *You don't have to read every equation.*

  The examples generally show "all" the steps needed to solve a problem. Sometimes reading through each step is helpful; sometimes it is confusing. When the steps are illustrating a new technique, one probably should follow each step closely to learn the new technique. When the steps are showing the mathematics needed to find a number to be used later, one can usually skip ahead and see how that number is being used, instead of getting bogged down in reading how the number was found.

- *Most proofs have been omitted.*

  In mathematics, *proving* something is always true is extremely important, and entails much more than testing to see if it works twice. However, students often are confused by the details of a proof, or become concerned that they should have been able to construct this proof on their own. To alleviate this potential problem, we do not include the proofs to most theorems in the text. The interested reader is highly encouraged to find proofs online or from their instructor. In most cases, one is very capable of understanding what a theorem *means* and *how to apply it* without knowing fully *why* it is true.

Interactive, 3D Graphics. Versions 3.0 and 4.0 of the textbook include interactive, 3D graphics in the pdf version. Nearly all graphs of objects in space can be rotated, shifted, and zoomed in/out so the reader can better understand the object illustrated. However, the only pdf viewers that support these 3D graphics are Adobe Reader Acrobat (and only the versions for PC/Mac/Unix/Linux computers, not tablets or smartphones).

The latest version of the book, which is authored in PreTeXt, is available in html. In html, the 3D graphics are rendered using WebGL, and should work in any modern web browser.

Interactive graphics are no longer supported within the pdf, but clicking on any 3D graphic within the pdf will take you directly to the interactive version on the web.

APEX – Affordable Print and Electronic teXts. *APEX* is a consortium of authors who collaborate to produce high quality, low cost textbooks. The current textbook—writing paradigm is facing a potential revolution as desktop publishing and electronic formats increase in popularity. However, writing a good textbook is no easy task, as the time requirements alone are substantial. It takes countless hours of work to produce text, write examples and exercises, edit and publish. Through collaboration, however, the cost to any individual can be lessened, allowing us to create texts that we freely distribute electronically and sell in printed form for an incredibly low cost. Having said that, nothing is entirely free; someone always bears some cost. This text "cost" the authors of this book their time, and that was not enough. **APEX Calculus** would not exist had not the Virginia Military Institute, through a generous Jackson—Hope grant, given the lead author significant time away from teaching so he could focus on this text.

Each text is available as a free .pdf, protected by a Creative Commons Attribution - Noncommercial 4.0 copyright. That means you can give the .pdf to anyone you like, print it in any form you like, and even edit the original content and redistribute it. If you do the latter, you must clearly reference this work and you cannot sell your edited work for money.

We encourage others to adapt this work to fit their own needs. One might add sections that are "missing" or remove sections that your students won't need. The source files can be found at github.com/APEXCalculus[5].

You can learn more at www.vmi.edu/APEX[6].

First PreTeXt Edition (Version 5.0). Key changes from Version 4.0 to 5.0:

- The underlying source code has been completely rewritten, to use the PreTeXt[7] language, instead of the original LaTeX.
- Using PreTeXt allows us to produce the books in multiple formats, including html, which is both more accessible and more interactive than the original pdf. html versions of the book can be found at opentext.uleth.ca[8].
- The appendix on differential equations from the "Calculus for Quarters" version of the book has been included as Chapter 8, just after applications of integration. Chapters 8 — 14 are now numbered 9 — 15 as a result.
- In the html version of the book, many of the exercises are now interactive, and powered by WeBWorK.

Key changes from Version 3.0 to 4.0:

- Numerous typographical and "small" mathematical corrections (again, thanks to all my close readers!).
- "Large" mathematical corrections and adjustments. There were a number of places in Version 3.0 where a definition/theorem was not correct as stated. See www.apexcalculus.com[9] for more information.
- More useful numbering of Examples, Theorems, etc. . "Definition 11.4.2" refers to the second definition of Chapter 11, Section 4.
- The addition of Section 13.7: Triple Integration with Cylindrical and Spherical Coordinates
- The addition of Chapter 14: Vector Analysis.

[5] github.com/APEXCalculus
[6] www.vmi.edu/APEX
[7] pretextbook.org
[8] opentext.uleth.ca/calculus.html
[9] apexcalculus.com

A Brief History of Calculus

Calculus means "a method of calculation or reasoning." When one computes the sales tax on a purchase, one employs a simple calculus. When one finds the area of a polygonal shape by breaking it up into a set of triangles, one is using another calculus. Proving a theorem in geometry employs yet another calculus.

Despite the wonderful advances in mathematics that had taken place into the first half of the 17th century, mathematicians and scientists were keenly aware of what they *could not do*. (This is true even today.) In particular, two important concepts eluded mastery by the great thinkers of that time: area and rates of change.

Area seems innocuous enough; areas of circles, rectangles, parallelograms, etc., are standard topics of study for students today just as they were then. However, the areas of *arbitrary* shapes could not be computed, even if the boundary of the shape could be described exactly.

Rates of change were also important. When an object moves at a constant rate of change, then "distance $=$ rate $\times$ time." But what if the rate is not constant—can distance still be computed? Or, if distance is known, can we discover the rate of change?

It turns out that these two concepts were related. Two mathematicians, Sir Isaac Newton and Gottfried Leibniz, are credited with independently formulating a system of computing that solved the above problems and showed how they were connected. Their system of reasoning was "a" calculus. However, as the power and importance of their discovery took hold, it became known to many as "the" calculus. Today, we generally shorten this to discuss "calculus."

Contents

Part II

Math 2565: Accelerated Calculus II

Chapter 6

Techniques of Antidifferentiation

The previous chapter introduced the antiderivative and connected it to signed areas under a curve through the Fundamental Theorem of Calculus. The next chapter explores more applications of definite integrals than just area. As evaluating definite integrals will become important, we will want to find antiderivatives of a variety of functions.

This chapter is devoted to exploring techniques of antidifferentiation. While not every function has an antiderivative in terms of elementary functions (a concept introduced in the section on Numerical Integration), we can still find antiderivatives of a wide variety of functions.

6.1 Integration by Parts

Here's a simple integral that we can't yet evaluate:

$$\int x \cos(x)\, dx.$$

It's a simple matter to take the derivative of the integrand using the Product Rule, but there is no Product Rule for integrals. However, this section introduces *Integration by Parts*, a method of integration that is based on the Product Rule for derivatives. It will enable us to evaluate this integral.

The Product Rule says that if u and v are functions of x, then $(uv)' = u'v + uv'$. For simplicity, we've written u for $u(x)$ and v for $v(x)$. Suppose we integrate both sides with respect to x. This gives

$$\int (uv)'\, dx = \int (u'v + uv')\, dx.$$

By the Fundamental Theorem of Calculus, the left side integrates to uv. The right side can be broken up into two integrals, and we have

$$uv = \int u'v\, dx + \int uv'\, dx.$$

Solving for the second integral we have

$$\int uv'\, dx = uv - \int u'v\, dx.$$

youtu.be/watch?v=v7KGuoM-cgU

Figure 6.1.1 Video introduction to Section 6.1

Using differential notation, we can write $du = u'(x)dx$ and $dv = v'(x)dx$ and the expression above can be written as follows:

$$\int u\,dv = uv - \int v\,du.$$

This is the Integration by Parts formula. For reference purposes, we state this in a theorem.

Theorem 6.1.2 Integration by Parts.

Let u and v be differentiable functions of x on an interval I containing a and b. Then

$$\int u\,dv = uv - \int v\,du,$$

and

$$\int_{x=a}^{x=b} u\,dv = uv\Big|_a^b - \int_{x=a}^{x=b} v\,du.$$

The integration by parts formula can also be written as

$$\int f(x)\,g'(x)\,dx = f(x)g(x) - \int f'(x)\,g(x)\,dx$$

for differentiable functions f and g.

Let's try an example to understand our new technique.

Example 6.1.3 Integrating using Integration by Parts.

Evaluate $\displaystyle\int x\cos(x)\,dx$.

Solution. The key to Integration by Parts is to identify part of the integrand as "u" and part as "dv." Regular practice will help one make good identifications, and later we will introduce some principles that help. For now, let $u = x$ and $dv = \cos(x)\,dx$.
It is generally useful to make a small table of these values as done below. Right now we only know u and dv as shown on the left of Figure 6.1.4; on the right we fill in the rest of what we need. If $u = x$, then $du = dx$. Since $dv = \cos(x)\,dx$, v is an antiderivative of $\cos(x)$. We choose $v = \sin(x)$.

$$\begin{array}{cc} u = x & v = ? \\ du = ? & dv = \cos(x)\,dx \end{array} \quad \Longrightarrow \quad \begin{array}{cc} u = x & v = \sin(x) \\ du = dx & dv = \cos(x)\,dx \end{array}$$

Figure 6.1.4 Setting up Integration by Parts

Now substitute all of this into the Integration by Parts formula, giving

$$\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx.$$

We can then integrate $\sin(x)$ to get $-\cos(x) + C$ and overall our answer is

$$\int x\cos(x)\,dx = x\sin(x) + \cos(x) + C.$$

Note how the antiderivative contains a product, $x\sin(x)$. This product is what makes Integration by Parts necessary.
We can check our work by taking the derivative:

$$\begin{aligned}\frac{d}{dx}(x\sin(x) + \cos(x) + C) &= x\cos(x) + \sin(x) - \sin(x) + 0\\ &= x\cos(x).\end{aligned}$$

Video solution

youtu.be/watch?v=gKtzlaH2EPo

You may wonder what would have happened in Example 6.1.3 if we had chosen our u and dv differently. If we had chosen $u = \cos(x)$ and $dv = x\,dx$ then $du = -\sin(x)\,dx$ and $v = x^2/2$. Our second integral is not simpler than the first; we would have

$$\int x\cos(x)\,dx = \cos(x)\frac{x^2}{2} - \int \frac{x^2}{2}\left(-\sin(x)\right)\,dx.$$

The only way to approach this second integral would be yet another integration by parts.

Example 6.1.3 demonstrates how Integration by Parts works in general. We try to identify u and dv in the integral we are given, and the key is that we usually want to choose u and dv so that du is simpler than u and v is hopefully not too much more complicated than dv. This will mean that the integral on the right side of the Integration by Parts formula, $\int v\,du$ will be simpler to integrate than the original integral $\int u\,dv$.

In the example above, we chose $u = x$ and $dv = \cos(x)\,dx$. Then $du = dx$ was simpler than u and $v = \sin(x)$ is no more complicated than dv. Therefore, instead of integrating $x\cos(x)\,dx$, we could integrate $\sin(x)\,dx$, which we knew how to do.

A useful mnemonic for helping to determine u is "liate," where

l = *L*ogarithmic, i = *I*nverse Trig., a = *A*lgebraic (polynomials, roots, power functions), t = *T*rigonometric, and e = *E*xponential.

If the integrand contains both a logarithmic and an algebraic term, in general letting u be the logarithmic term works best, as indicated by l coming before a in liate.

We now consider another example.

Example 6.1.5 Integrating using Integration by Parts.

Evaluate $\displaystyle\int xe^x\,dx$.

Solution. The integrand contains an *A*lgebraic term (x) and an *E*xponential term (e^x). Our mnemonic suggests letting u be the algebraic term, so we choose $u = x$ and $dv = e^x\,dx$. Then $du = dx$ and $v = e^x$ as indicated by the tables below.

$$\begin{array}{cc} u = x & v = ? \\ du = ? & dv = e^x\,dx \end{array} \quad \Longrightarrow \quad \begin{array}{cc} u = x & v = e^x \\ du = dx & dv = e^x\,dx \end{array}$$

Figure 6.1.6 Setting up Integration by Parts

We see du is simpler than u, while there is no change in going from dv to v. This is good. The Integration by Parts formula gives

$$\int xe^x\,dx = xe^x - \int e^x\,dx.$$

The integral on the right is simple; our final answer is

$$\int xe^x\,dx = xe^x - e^x + C.$$

Note again how the antiderivatives contain a product term.

Example 6.1.7 Integrating using Integration by Parts.

Evaluate $\int x^2 \cos(x)\, dx$.

Solution. The mnemonic suggests letting $u = x^2$ instead of the trigonometric function, hence $dv = \cos(x)\, dx$. Then $du = 2x\, dx$ and $v = \sin(x)$ as shown below.

$$\begin{array}{cc} u = x^2 & v = ? \\ du = ? & dv = \cos(x)\, dx \end{array} \quad \Longrightarrow \quad \begin{array}{cc} u = x^2 & v = \sin(x) \\ du = 2x\, dx & dv = \cos(x)\, dx \end{array}$$

Figure 6.1.8 Setting up Integration by Parts

The Integration by Parts formula gives

$$\int x^2 \cos(x)\, dx = x^2 \sin(x) - \int 2x \sin(x)\, dx.$$

At this point, the integral on the right is indeed simpler than the one we started with, but to evaluate it, we need to do Integration by Parts again. Here we choose $r = 2x$ and $ds = \sin(x)$ and fill in the rest below. (We are choosing new names since we have already used u and v. Our integration by parts formula is now $\int r\, ds = rs - \int s\, dr$.)

$$\begin{array}{cc} u = 2x & v = ? \\ du = ? & dv = \sin(x)\, dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = 2x & v = -\cos(x) \\ du = 2\, dx & dv = \sin(x)\, dx \end{array}$$

Figure 6.1.9 Setting up Integration by Parts (again)

Video solution

youtu.be/watch?v=j9pCcQMSjbg

$$\int x^2 \cos(x)\, dx = x^2 \sin(x) - \left(-2x \cos(x) - \int -2 \cos(x)\, dx \right).$$

The integral all the way on the right is now something we can evaluate. It evaluates to $-2\sin(x)$. Then going through and simplifying, being careful to keep all the signs straight, our answer is

$$\int x^2 \cos(x)\, dx = x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + C.$$

Example 6.1.10 Integrating using Integration by Parts.

Evaluate $\int e^x \cos(x)\, dx$.

Solution. This is a classic problem. Our mnemonic suggests letting u be the trigonometric function instead of the exponential. In this particular example, one can let u be either $\cos(x)$ or e^x; to demonstrate that we do not have to follow liate, we choose $u = e^x$ and hence $dv = \cos(x)\, dx$. Then $du = e^x\, dx$ and $v = \sin(x)$ as shown below.

$$\begin{array}{ll} u = e^x & v = ? \\ du = ? & dv = \cos(x)\,dx \end{array} \quad \Longrightarrow \quad \begin{array}{ll} u = e^x & v = \sin(x) \\ du = e^x & dv = \cos(x)\,dx \end{array}$$

Figure 6.1.11 Setting up Integration by Parts

Notice that du is no simpler than u, going against our general rule (but bear with us). The Integration by Parts formula yields

$$\int e^x \cos(x)\,dx = e^x \sin(x) - \int e^x \sin(x)\,dx.$$

The integral on the right is not much different than the one we started with, so it seems like we have gotten nowhere. Let's keep working and apply Integration by Parts to the new integral, using $u = e^x$ and $dv = \sin(x)\,dx$. This leads us to the following:

$$\begin{array}{ll} r = e^x & s = ? \\ dr = ? & ds = \sin(x)\,dx \end{array} \quad \Longrightarrow \quad \begin{array}{ll} r = e^x & s = -\cos(x) \\ dr = e^x\,dx & ds = \sin(x)\,dx \end{array}$$

Figure 6.1.12 Setting up Integration by Parts (again)

The Integration by Parts formula then gives:

$$\begin{aligned} \int e^x \cos(x)\,dx &= e^x \sin(x) - \left(-e^x \cos(x) - \int -e^x \cos(x)\,dx\right) \\ &= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx. \end{aligned}$$

It seems we are back right where we started, as the right hand side contains $\int e^x \cos(x)\,dx$. But this is actually a good thing.

Add $\displaystyle\int e^x \cos(x)\,dx$ to both sides. This gives

$$2\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x)$$

Now divide both sides by 2 and then add the integration constant:

$$\int e^x \cos(x)\,dx = \frac{1}{2}\big(e^x \sin(x) + e^x \cos(x)\big) + C.$$

Simplifying a little, our answer is thus

$$\int e^x \cos(x)\,dx = \frac{1}{2}e^x\left(\sin(x) + \cos(x)\right) + C.$$

Video solution

youtu.be/watch?v=z0A1v2Zkfns

Example 6.1.13 Integrating using Integration by Parts: antiderivative of $\ln(x)$**.**

Evaluate $\int \ln(x)\,dx$.

Solution. One may have noticed that we have rules for integrating the familiar trigonometric functions and e^x, but we have not yet given a rule for integrating $\ln(x)$. That is because $\ln(x)$ can't easily be integrated with any of the rules we have learned up to this point. But we can find its antiderivative by a clever application of Integration by Parts. Set $u = \ln(x)$ and $dv = dx$. This is a good, sneaky trick to learn as it can help in other situations. This determines $du = (1/x)\,dx$ and $v = x$ as shown below.

$$\begin{array}{ll} u = \ln(x) & v = ? \\ du = ? & dv = 1\,dx \end{array} \quad \Longrightarrow \quad \begin{array}{ll} u = \ln(x) & v = x \\ du = 1/x\,dx & dv = 1\,dx \end{array}$$

Figure 6.1.14 Setting up Integration by Parts

Putting this all together in the Integration by Parts formula, things work out very nicely:

$$\int \ln(x)\,dx = x\ln(x) - \int x\,\frac{1}{x}\,dx.$$

The new integral simplifies to $\int 1\,dx$, which is about as simple as things get. Its integral is $x + C$ and our answer is

$$\int \ln(x)\,dx = x\ln(x) - x + C.$$

Video solution

youtu.be/watch?v=NGkLj7djFSw

Example 6.1.15 Integrating using Int. by Parts: antiderivative of $\arctan x$**.**

Evaluate $\displaystyle\int \arctan x\,dx.$

Solution. The same sneaky trick we used above works here. Let $u = \arctan x$ and $dv = dx$. Then $du = 1/(1+x^2)\,dx$ and $v = x$. The Integration by Parts formula gives

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx.$$

The integral on the right can be solved by substitution. Taking $w = 1+x^2$, we get $dw = 2x\,dx$. The integral then becomes

$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\int \frac{1}{w}\,dw.$$

The integral on the right evaluates to $\ln|w| + C$, which becomes $\ln(1+x^2) + C$ (we can drop the absolute values as $1+x^2$ is always positive). Therefore, the answer is

$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C.$$

Video solution

youtu.be/watch?v=md3-8bv5E5M

Substitution Before Integration. When taking derivatives, it was common to employ multiple rules (such as using both the Quotient and the Chain Rules). It should then come as no surprise that some integrals are best evaluated by combining integration techniques. In particular, here we illustrate making an “unusual” substitution first before using Integration by Parts.

Example 6.1.16 Integration by Parts after substitution.

Evaluate $\int \cos(\ln(x))\, dx$.

Solution. The integrand contains a composition of functions, leading us to think Substitution would be beneficial. Letting $u = \ln(x)$, we have $du = 1/x\, dx$. This seems problematic, as we do not have a $1/x$ in the integrand. But consider:

$$du = \frac{1}{x}\, dx \quad \Rightarrow \quad x \cdot du = dx.$$

Since $u = \ln(x)$, we can use inverse functions and conclude that $x = e^u$. Therefore we have that

$$\begin{aligned} dx &= x \cdot du \\ &= e^u\, du. \end{aligned}$$

We can thus replace $\ln(x)$ with u and dx with $e^u\, du$. Thus we rewrite our integral as

$$\int \cos(\ln(x))\, dx = \int e^u \cos u\, du.$$

We evaluated this integral on the right in Example 6.1.10. (This integral can also be found in a table of integrals). Using the result there, we have:

$$\begin{aligned} \int \cos(\ln(x))\, dx &= \int e^u \cos(u)\, du \\ &= \frac{1}{2} e^u \big(\sin(u) + \cos(u)\big) + C \\ &= \frac{1}{2} e^{\ln(x)} \big(\sin(\ln(x)) + \cos(\ln(x))\big) + C \\ &= \frac{1}{2} x \big(\sin(\ln(x)) + \cos(\ln(x))\big) + C. \end{aligned}$$

Video solution

youtu.be/watch?v=0j0vM0nosYs

Definite Integrals and Integration By Parts. So far we have focused only on evaluating indefinite integrals. Of course, we can use Integration by Parts to evaluate definite integrals as well, as Theorem 6.1.2 states. We do so in the next example.

Example 6.1.17 Definite integration using Integration by Parts.

Evaluate $\int_1^2 x^2 \ln(x)\, dx$.

Solution. Our mnemonic suggests letting $u = \ln(x)$, hence $dv = x^2\, dx$. We then get $du = (1/x)\, dx$ and $v = x^3/3$ as shown below.

$$\begin{array}{llcll} u = \ln(x) & v = ? & \Rightarrow & u = \ln(x) & v = x^3/3 \\ du = ? & dv = x^2\, dx & & du = 1/x\, dx & dv = x^2\, dx \end{array}$$

Figure 6.1.18 Setting up Integration by Parts

The Integration by Parts formula then gives

$$\begin{aligned}\int_1^2 x^2 \ln(x)\,dx &= \frac{x^3}{3}\ln(x)\Big|_1^2 - \int_1^2 \frac{x^3}{3}\,\frac{1}{x}\,dx\\ &= \frac{x^3}{3}\ln(x)\Big|_1^2 - \int_1^2 \frac{x^2}{3}\,dx\\ &= \frac{x^3}{3}\ln(x)\Big|_1^2 - \frac{x^3}{9}\Big|_1^2\\ &= \left(\frac{x^3}{3}\ln(x) - \frac{x^3}{9}\right)\Big|_1^2\\ &= \left(\frac{8}{3}\ln(2) - \frac{8}{9}\right) - \left(\frac{1}{3}\ln(1) - \frac{1}{9}\right)\\ &= \frac{8}{3}\ln(2) - \frac{7}{9}\\ &\approx 1.07.\end{aligned}$$

Video solution

youtu.be/watch?v=O9_0B2gatMo

In general, Integration by Parts is useful for integrating certain products of functions, like $\int xe^x\,dx$ or $\int x^3\sin(x)\,dx$. It is also useful for integrals involving logarithms and inverse trigonometric functions.

As stated before, integration is generally more difficult than derivation. We are developing tools for handling a large array of integrals, and experience will tell us when one tool is preferable/necessary over another. For instance, consider the three similar-looking integrals

$$\int xe^x\,dx, \qquad \int xe^{x^2}\,dx \qquad \text{and} \qquad \int xe^{x^3}\,dx.$$

While the first is calculated easily with Integration by Parts, the second is best approached with Substitution. Taking things one step further, the third integral has no answer in terms of elementary functions, so none of the methods we learn in calculus will get us the exact answer.

Integration by Parts is a very useful method, second only to Substitution. In the following sections of this chapter, we continue to learn other integration techniques. Section 6.2 focuses on handling integrals containing trigonometric functions.

6.1.1 Exercises

Terms and Concepts

1. (☐ True ☐ False) Integration by Parts is useful in evaluating integrands that contain products of functions.

2. (☐ True ☐ False) Integration by Parts can be thought of as the "opposite of the Chain Rule."

3. For what is "LIATE" useful?

4. (☐ True ☐ False) If the integral that results from Integration by Parts appears to also need Integration by Parts, then a mistake was made in the original choice of "u".

Problems

Exercise Group. Evaluate the given indefinite integral.

5. $\int x \sin(x)\, dx$

6. $\int xe^{-x}\, dx$

7. $\int x^2 \sin(x)\, dx$

8. $\int x^3 \sin(x)\, dx$

9. $\int xe^{x^2}\, dx$

10. $\int x^3 e^x\, dx$

11. $\int xe^{-2x}\, dx$

12. $\int e^x \sin(x)\, dx$

13. $\int e^{2x} \cos(x)\, dx$

14. $\int e^{7x} \sin(3x)\, dx$

15. $\int e^{8x} \cos(8x)\, dx$

16. $\int \sin(x)\cos(x)\, dx$

17. $\int \sin^{-1}(x)\, dx$

18. $\int \tan^{-1}(3x)\, dx$

19. $\int x \tan^{-1}(x)\, dx$

20. $\int \cos^{-1}(x)\, dx$

21. $\int x \ln(x)\, dx$

22. $\int (x+2)\ln(x)\, dx$

23. $\int x \ln(x-4)\, dx$

24. $\int x \ln(x^2)\, dx$

25. $\int x^2 \ln(x)\, dx$

26. $\int (\ln(x))^2\, dx$

27. $\int \ln^2(x-7)\, dx$

28. $\int x \sec^2(x)\, dx$

29. $\int x \csc^2(x)\, dx$

30. $\int x\sqrt{x+2}\, dx$

31. $\int x\sqrt{x^2-6}\, dx$

32. $\int \sec(x)\tan(x)\, dx$

33. $\int x \sec(x)\tan(x)\, dx$

34. $\int x \csc(x)\cot(x)\, dx$

Exercise Group. Evaluate the indefinite integral after first making a substitution.

35. $\int \sin(\ln(x))\, dx$

36. $\int e^{2x} \sin(e^x)\, dx$

37. $\int \sin(\sqrt{x})\, dx$

38. $\int \ln(\sqrt{x})\, dx$

39. $\int e^{\sqrt{x}}\,dx$

40. $\int e^{\ln(x)}\,dx$

Exercise Group. Evaluate the definite integral. Note: the corresponding indefinite integral appears in Exercises 5–13.

41. $\int_0^{3\pi/2} x\sin(x)\,dx$

42. $\int_{-1}^{2} xe^{-x}\,dx$

43. $\int_{-\pi/2}^{\pi/2} x^2\sin(x)\,dx$

44. $\int_{-\pi/6}^{\pi/6} x^3\sin(x)\,dx$

45. $\int_0^{\sqrt{\ln(2)}} xe^{x^2}\,dx$

46. $\int_0^{1} x^3e^{x}\,dx$

47. $\int_2^{4} xe^{-2x}\,dx$

48. $\int_0^{\pi} e^{x}\sin(x)\,dx$

49. $\int_{-3\pi/2}^{3\pi/2} e^{2x}\cos(x)\,dx$

6.2 Trigonometric Integrals

Functions involving trigonometric functions are useful as they are good at describing periodic behavior. This section describes several techniques for finding antiderivatives of certain combinations of trigonometric functions.

6.2.1 Integrals of the form $\int \sin^m(x)\cos^n(x)\,dx$

In learning the technique of Substitution, we saw the integral $\int \sin(x)\cos(x)\,dx$ in Example 5.6.8. The integration was not difficult, and one could easily evaluate the indefinite integral by letting $u = \sin(x)$ or by letting $u = \cos(x)$. This integral is easy since the power of both sine and cosine is 1.

We generalize this integral and consider integrals of the form $\int \sin^m(x)\cos^n(x)\,dx$, where m, n are nonnegative integers. Our strategy for evaluating these integrals is to use the identity $\cos^2(x) + \sin^2(x) = 1$ to convert high powers of one trigonometric function into the other, leaving a single sine or cosine term in the integrand. Let's see an example of how this technique works.

Example 6.2.1 Integrating powers of sine and cosine.

Evaluate $\displaystyle\int \sin^3(x)\cos(x)\,dx$.

Solution. We have used substitution on problems similar to this problem in Section 5.6 . If we let $u = \sin(x)$, then $du = \cos(x)\,dx$, and

$$\int \sin^3(x)\cos(x)\,dx = \int u^3\,du = \frac{u^4}{4} + C = \frac{1}{4}\sin^4(x) + C.$$

But what if, for some reason, we wanted to let $u = \cos(x)$ instead? Unfortunately, we have $\sin^3(x)$ as part of our integrand, not just $\sin(x)$. The solution to this problem is to replace some of our powers of sine (two of them to be exact) with expressions that involve cosine. We will use the Pythagorean Identity $\sin^2(x) = 1 - \cos^2(x)$.

$$\begin{aligned}\int \sin^3(x)\cos(x)\,dx &= \int \sin(x)\cdot\sin^2(x)\cos(x)\,dx\\ &= \int \sin(x)\left(1-\cos^2(x)\right)\cos(x)\,dx.\end{aligned}$$

Now we let $u = \cos(x)$ so that $-du = \sin(x)\,dx$.

$$\begin{aligned}\int \sin^3(x)\cos(x)\,dx &= \int \sin(x)\left(1-\cos^2(x)\right)\cos(x)\,dx\\ &= \int -\left(1-u^2\right)u\,du\\ &= \int -\left(u-u^3\right)\,du\\ &= -\frac{u^2}{2} + \frac{u^4}{4} + C\\ &= -\frac{\cos^2(x)}{2} + \frac{\cos^4(x)}{4} + C.\end{aligned}$$

This looks like a very different answer, so you might wonder if we went wrong somewhere. But in fact, the two answers are equivalent, in the sense that they differ by a constant! (So the "$+C$" is different in each case, if you like.) Notice that

$$\begin{aligned}\frac{1}{4}\sin^4(x) &= \frac{1}{4}(1-\cos^2(x))^2\\ &= \frac{1}{4}-\frac{1}{2}\cos^2(x)+\frac{1}{4}\cos^4(x),\end{aligned}$$

so the difference between the two answers is the constant $\frac{1}{4}$.

We summarize the general technique in the following Key Idea.

Key Idea 6.2.2 Integrals Involving Powers of Sine and Cosine.

Consider $\displaystyle\int \sin^m(x)\cos^n(x)\,dx$, where m, n are nonnegative integers.

1. If m is odd, then $m = 2k+1$ for some integer k. Rewrite

$$\begin{aligned}\sin^m(x) &= \sin^{2k+1}(x)\\ &= \sin^{2k}(x)\sin(x)\\ &= (\sin^2(x))^k\sin(x)\\ &= (1-\cos^2(x))^k\sin(x).\end{aligned}$$

   Then

$$\begin{aligned}\int \sin^m(x)\cos^n(x)\,dx &= \int (1-\cos^2(x))^k\sin(x)\cos^n(x)\,dx\\ &= -\int (1-u^2)^k u^n\,du,\end{aligned}$$

   where $u = \cos(x)$ and $du = -\sin(x)\,dx$.

2. If n is odd, then using substitutions similar to that outlined above (replacing all of the even powers of $cosine$ using a Pythagorean identity) we have:

$$\int \sin^m(x)\cos^n(x)\,dx = \int u^m(1-u^2)^k\,du,$$

   where $u = \sin(x)$ and $du = \cos(x)\,dx$.

3. If both m and n are even, use the power-reducing identities:

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \text{ and } \sin^2(x) = \frac{1-\cos(2x)}{2}$$

   to reduce the degree of the integrand. Expand the result and apply the principles of this Key Idea again.

We practice applying Key Idea 6.2.2 in the next examples.

Example 6.2.3 Integrating powers of sine and cosine.

Evaluate $\displaystyle\int \sin^5(x)\cos^8(x)\,dx$.

Video solution

youtu.be/watch?v=soXjOeFRrsk

Solution. The power of the sine term is odd, so we rewrite $\sin^5(x)$ as

$$\begin{aligned}\sin^5(x) &= \sin^4(x)\sin(x)\\ &= (\sin^2(x))^2\sin(x)\\ &= (1-\cos^2(x))^2\sin(x).\end{aligned}$$

Our integral is now $\int (1-\cos^2(x))^2\cos^8(x)\sin(x)\,dx$. Let $u=\cos(x)$, hence $du = -\sin(x)\,dx$. Making the substitution and expanding the integrand gives

$$\begin{aligned}\int (1-\cos^2)^2\cos^8(x)\sin(x)\,dx &= -\int (1-u^2)^2u^8\,du\\ &= -\int \left(1-2u^2+u^4\right)u^8\,du\\ &= -\int \left(u^8-2u^{10}+u^{12}\right)du.\end{aligned}$$

This final integral is not difficult to evaluate, giving

$$\begin{aligned}-\int \left(u^8-2u^{10}+u^{12}\right)du &= -\frac{1}{9}u^9+\frac{2}{11}u^{11}-\frac{1}{13}u^{13}+C\\ &= -\frac{1}{9}\cos^9(x)+\frac{2}{11}\cos^{11}(x)-\frac{1}{13}\cos^{13}(x)+C.\end{aligned}$$

Video solution

youtu.be/watch?v=CAV4gSbw1GU

Example 6.2.4 Integrating powers of sine and cosine.

Evaluate $\int \sin^5(x)\cos^9(x)\,dx$.

Solution. The powers of both the sine and cosine terms are odd, therefore we can apply the techniques of Key Idea 6.2.2 to either power. We choose to work with the power of the cosine term since the previous example used the sine term's power.
We rewrite $\cos^9(x)$ as

$$\begin{aligned}\cos^9(x) &= \cos^8(x)\cos(x)\\ &= \left(\cos^2(x)\right)^4\cos(x)\\ &= \left(1-\sin^2(x)\right)^4\cos(x).\end{aligned}$$

We rewrite the integral as

$$\int \sin^5(x)\cos^9(x)\,dx = \int \sin^5(x)\left(1-\sin^2(x)\right)^4\cos(x)\,dx.$$

Now substitute and integrate, using $u=\sin(x)$ and $du=\cos(x)\,dx$. Expand the binomial using algebra.

$$\begin{aligned}&\int u^5(1-u^2)^4\,du\\ &= \int u^5(1-4u^2+6u^4-4u^6+u^8)\,du\end{aligned}$$

$$= \int \left(u^5 - 4u^7 + 6u^9 - 4u^{11} + u^{13}\right) du$$
$$= \frac{1}{6}u^6 - \frac{1}{2}u^8 + \frac{3}{5}u^{10} - \frac{1}{3}u^{12} + \frac{1}{14}u^{14} + C$$
$$= \frac{1}{6}\sin^6(x) - \frac{1}{2}\sin^8(x) + \frac{3}{5}\sin^{10}(x) - \frac{1}{3}\sin^{12}(x) + \frac{1}{14}\sin^{14}(x) + C.$$

Technology Note: The work we are doing here can be a bit tedious, but the skills developed (problem solving, algebraic manipulation, etc.) are important. Nowadays problems of this sort are often solved using a computer algebra system. The powerful program *Mathematica*™ integrates $\int \sin^5(x)\cos^9(x)\,dx$ as

$$f(x) = -\frac{45\cos(2x)}{16384} - \frac{5\cos(4x)}{8192} + \frac{19\cos(6x)}{49152} + \frac{\cos(8x)}{4096} - \frac{\cos(10x)}{81920} - \frac{\cos(12x)}{24576} - \frac{\cos(14x)}{114688},$$

which clearly has a different form than our answer in Example 6.2.4, which is

$$g(x) = \frac{1}{6}\sin^6(x) - \frac{1}{2}\sin^8(x) + \frac{3}{5}\sin^{10}(x) - \frac{1}{3}\sin^{12}(x) + \frac{1}{14}\sin^{14}(x).$$

Figure 6.2.5 shows a graph of f and g; they are clearly not equal, but they differ *only by a constant*. That is $g(x) = f(x) + C$ for some constant C. So we have two different antiderivatives of the same function, meaning both answers are correct.

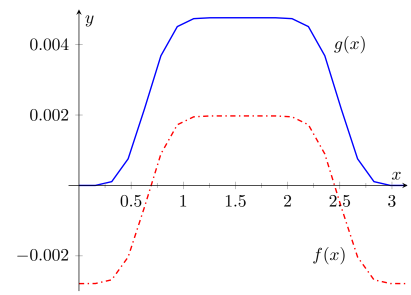

Figure 6.2.5 A plot of $f(x)$ and $g(x)$ from Example 6.2.4 and the Technology Note

Example 6.2.6 Integrating powers of sine and cosine.

Evaluate $\displaystyle\int \cos^4(x)\sin^2(x)\,dx$.

Solution. The powers of sine and cosine are both even, so we employ the power-reducing formulas and algebra as follows.

$$\int \cos^4(x)\sin^2(x)\,dx = \int \left(\frac{1+\cos(2x)}{2}\right)^2 \left(\frac{1-\cos(2x)}{2}\right) dx$$
$$= \int \frac{1+2\cos(2x)+\cos^2(2x)}{4} \cdot \frac{1-\cos(2x)}{2}\,dx$$
$$= \int \frac{1}{8}\left(1+\cos(2x)+\cos^2(2x)-\cos^3(2x)\right) dx$$
$$= \frac{1}{8}\left(\underbrace{\int 1\,dx}_{a} + \underbrace{\int \cos(2x)\,dx}_{b} - \underbrace{\int \cos^2(2x)\,dx}_{c} - \underbrace{\int \cos^3(2x)\,dx}_{d}\right)$$

The first integral labeled a is easy to integrate. The $\cos(2x)$ term is also easy to integrate, especially with Key Idea 5.6.5. The $\cos^2(2x)$ term is another trigonometric integral with an even power, requiring the power-reducing formula again. The $\cos^3(2x)$ term is a cosine function with an odd power, requiring a substitution as done before. We integrate each in turn below.

$$\underbrace{\int \cos(2x)\,dx}_{b} = \frac{1}{2}\sin(2x) + C$$

$$\underbrace{\int \cos^2(2x)\,dx}_{c} = \int \frac{1+\cos(4x)}{2}\,dx$$
$$= \frac{1}{2}\big(x + \frac{1}{4}\sin(4x)\big) + C.$$

Finally, we rewrite $\cos^3(2x)$ as

$$\begin{aligned}\cos^3(2x) &= \cos^2(2x)\cos(2x)\\ &= \big(1-\sin^2(2x)\big)\cos(2x).\end{aligned}$$

Letting $u = \sin(2x)$, we have $du = 2\cos(2x)\,dx$, hence

$$\begin{aligned}\underbrace{\int \cos^3(2x)\,dx}_{d} &= \int \big(1-\sin^2(2x)\big)\cos(2x)\,dx\\ &= \int \frac{1}{2}(1-u^2)\,du\\ &= \frac{1}{2}\Big(u - \frac{1}{3}u^3\Big) + C\\ &= \frac{1}{2}\Big(\sin(2x) - \frac{1}{3}\sin^3(2x)\Big) + C.\end{aligned}$$

Putting all the pieces together, we have

$$\begin{aligned}&\int \cos^4(x)\sin^2(x)\,dx\\ &= \int \frac{1}{8}\big(1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\big)\,dx\\ &= \frac{1}{8}\Big[x + \frac{1}{2}\sin(2x) - \frac{1}{2}\big(x+\frac{1}{4}\sin(4x)\big) - \frac{1}{2}\Big(\sin(2x) - \frac{1}{3}\sin^3(2x)\Big)\Big] + C\\ &= \frac{1}{8}\Big[\frac{1}{2}x - \frac{1}{8}\sin(4x) + \frac{1}{6}\sin^3(2x)\Big] + C.\end{aligned}$$

Video solution

youtu.be/watch?v=EXODR17otIw

The process above was a bit long and tedious, but being able to work a problem such as this from start to finish is important.

6.2.2 Integrals of the form $\int \sin(mx)\sin(nx)\,dx$, $\int \cos(mx)\cos(nx)\,dx$, and $\int \sin(mx)\cos(nx)\,dx$

Functions that contain products of sines and cosines of differing periods are important in many applications including the analysis of sound waves. Integrals of the form

$$\int \sin(mx)\sin(nx)\,dx,\ \int \cos(mx)\cos(nx)\,dx \text{ and } \int \sin(mx)\cos(nx)\,dx$$

are best approached by first applying the Product to Sum Formulas found in the back cover of this text, namely

$$\begin{aligned}\sin(mx)\sin(nx) &= \frac{1}{2}\Big[\cos\big((m-n)x\big) - \cos\big((m+n)x\big)\Big]\\ \cos(mx)\cos(nx) &= \frac{1}{2}\Big[\cos\big((m-n)x\big) + \cos\big((m+n)x\big)\Big]\end{aligned}$$

$$\sin(mx)\cos(nx) = \frac{1}{2}\Big[\sin\big((m-n)x\big) + \sin\big((m+n)x\big)\Big].$$

Example 6.2.7 Integrating products of $\sin(mx)$ and $\cos(nx)$.

Evaluate $\displaystyle\int \sin(5x)\cos(2x)\,dx$.

Solution. The application of the formula and subsequent integration are straightforward:

$$\begin{aligned}\int \sin(5x)\cos(2x)\,dx &= \int \frac{1}{2}\Big[\sin((5-2)x) + \sin((5+2)x)\Big]\,dx\\ &= \int \frac{1}{2}\Big[\sin(3x) + \sin(7x)\Big]\,dx\\ &= -\frac{1}{6}\cos(3x) - \frac{1}{14}\cos(7x) + C\end{aligned}$$

Video solution

youtu.be/watch?v=KbW-xwlTuyI

6.2.3 Integrals of the form $\int \tan^m(x)\sec^n(x)\,dx$

When evaluating integrals of the form $\int \sin^m(x)\cos^n(x)\,dx$, the Pythagorean Theorem allowed us to convert even powers of sine into even powers of cosine, and vise-versa. If, for instance, the power of sine was odd, we pulled out one $\sin(x)$ and converted the remaining even power of $\sin(x)$ into a function using powers of $\cos(x)$, leading to an easy substitution.

The same basic strategy applies to integrals of the form $\int \tan^m(x)\sec^n(x)\,dx$, albeit a bit more nuanced. The following three facts will prove useful:

- $\frac{d}{dx}(\tan(x)) = \sec^2(x)$,
- $\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$,
- $1 + \tan^2(x) = \sec^2(x)$ (the Pythagorean Theorem).

If the integrand can be manipulated to separate a $\sec^2(x)$ term with the remaining secant power even, or if a $\sec(x)\tan(x)$ term can be separated with the remaining $\tan(x)$ power even, the Pythagorean Theorem can be employed, leading to a simple substitution. This strategy is outlined in the following Key Idea.

Key Idea 6.2.8 Integrals Involving Powers of Tangent and Secant.

Consider $\displaystyle\int \tan^m(x)\sec^n(x)\,dx$, where m, n are nonnegative integers.

1. If n is even, then $n = 2k$ for some integer k. Rewrite $\sec^n(x)$ as

$$\begin{aligned}\sec^n(x) &= \sec^{2k}(x)\\ &= \sec^{2k-2}(x)\sec^2(x)\\ &= (1+\tan^2(x))^{k-1}\sec^2(x).\end{aligned}$$

   Then

$$\begin{aligned}\int \tan^m(x)\sec^n(x)\,dx &= \int \tan^m(x)(1+\tan^2(x))^{k-1}\sec^2(x)\,dx\\ &= \int u^m(1+u^2)^{k-1}\,du,\end{aligned}$$

where $u = \tan(x)$ and $du = \sec^2(x)\,dx$.

2. If m is odd, then $m = 2k+1$ for some integer k. Rewrite $\tan^m(x)\sec^n(x)$ as

$$\begin{aligned}\tan^m(x)\sec^n(x) &= \tan^{2k+1}(x)\sec^n(x)\\ &= \tan^{2k}(x)\sec^{n-1}(x)\sec(x)\tan(x)\\ &= (\sec^2(x)-1)^k\sec^{n-1}(x)\sec(x)\tan(x).\end{aligned}$$

Then

$$\begin{aligned}\int \tan^m(x)\sec^n(x)\,dx &= \int (\sec^2(x)-1)^k\sec^{n-1}(x)\sec(x)\tan(x)\,dx\\ &= \int (u^2-1)^k u^{n-1}\,du,\end{aligned}$$

where $u = \sec(x)$ and $du = \sec(x)\tan(x)\,dx$.

3. If n is odd and m is even, then $m = 2k$ for some integer k. Convert $\tan^m(x)$ to $(\sec^2(x)-1)^k$. Expand the new integrand and use Integration By Parts, with $dv = \sec^2(x)\,dx$.

4. If m is even and $n = 0$, rewrite $\tan^m(x)$ as

$$\begin{aligned}\tan^m(x) &= \tan^{m-2}(x)\tan^2(x)\\ &= \tan^{m-2}(x)(\sec^2(x)-1)\\ &= \tan^{m-2}\sec^2(x) - \tan^{m-2}(x).\end{aligned}$$

So

$$\int \tan^m(x)\,dx = \underbrace{\int \tan^{m-2}\sec^2(x)\,dx}_{\text{apply rule 1}} - \underbrace{\int \tan^{m-2}(x)\,dx}_{\text{apply rule 4 again}}.$$

The techniques described in Item 1 and Item 2 of Key Idea 6.2.8 are relatively straightforward, but the techniques in Item 3 and Item 4 can be rather tedious. A few examples will help with these methods.

Example 6.2.9 Integrating powers of tangent and secant.

Evaluate $\displaystyle\int \tan^2(x)\sec^6(x)\,dx$.

Solution. Since the power of secant is even, we use Rule 1from Key Idea 6.2.8 and pull out a $\sec^2(x)$ in the integrand. We convert the remaining powers of secant into powers of tangent.

$$\begin{aligned}\int \tan^2(x)\sec^6(x)\,dx &= \int \tan^2(x)\sec^4(x)\sec^2(x)\,dx\\ &= \int \tan^2(x)\big(1+\tan^2(x)\big)^2\sec^2(x)\,dx\end{aligned}$$

Now substitute, with $u = \tan(x)$, with $du = \sec^2(x)\,dx$.

$$= \int u^2\big(1+u^2\big)^2\,du$$

We leave the integration and subsequent substitution to the reader. The final answer is

$$= \frac{1}{3}\tan^3(x) + \frac{2}{5}\tan^5(x) + \frac{1}{7}\tan^7(x) + C.$$

Video solution

youtu.be/watch?v=yYbn6R20qTk

When we have an odd power of $\tan(x)$ (and $\sec(x)$ to any power of at least one), we can split off a factor of $\tan(x)\sec(x)$ and use the substitution $u = \sec(x)$, as the video in Figure 6.2.10 illustrates.

youtu.be/watch?v=QsdKxEr3jG8

Figure 6.2.10 An integral with odd powers of $\tan(x)$ and $\sec(x)$

Example 6.2.11 Integrating powers of tangent and secant.

Evaluate $\displaystyle\int \sec^3(x)\,dx$.

Solution. We apply Rule 3 from Key Idea 6.2.8 as the power of secant is odd and the power of tangent is even (0 is an even number). We use Integration by Parts; the rule suggests letting $dv = \sec^2(x)\,dx$, meaning that $u = \sec(x)$.

$$\begin{array}{llcll} u = \sec(x) & v = ? & \Longrightarrow & u = \sec(x) & v = \tan(x) \\ du = ? & dv = \sec^2(x)\,dx & & du = \sec(x)\tan(x)\,dx & dv = \sec^2(x)\,dx \end{array}$$

Figure 6.2.12 Setting up Integration by Parts

Employing Integration by Parts, we have

$$\begin{aligned} \int \sec^3(x)\,dx &= \int \underbrace{\sec(x)}_{u} \cdot \underbrace{\sec^2(x)\,dx}_{dv} \\ &= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx. \end{aligned}$$

This new integral also requires applying Rule 3 of Key Idea 6.2.8:

$$\begin{aligned} \int \sec^3(x)\,dx &= \sec(x)\tan(x) - \int \sec(x)\big(\sec^2(x) - 1\big)\,dx \\ &= \sec(x)\tan(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx \\ &= \sec(x)\tan(x) - \int \sec^3(x)\,dx + \ln|\sec(x) + \tan(x)| \end{aligned}$$

In previous applications of Integration by Parts, we have seen where the original integral has reappeared in our work. We resolve this by adding $\int \sec^3(x)\,dx$ to both sides, giving:

$$\begin{aligned} 2\int \sec^3(x)\,dx &= \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| \\ \int \sec^3(x)\,dx &= \frac{1}{2}\Big(\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|\Big) + C \end{aligned}$$

Video solution

youtu.be/watch?v=mPuR46ztxZQ

Integrals involving odd powers of $\sec(x)$ (and nothing else) are often among the more intimidating tasks for beginning calculus students. However, larger odd powers are best handled not by doing the integral directly, but by employing a *reduction forumula*. The video in Figure 6.2.13 shows how to obtain a reduction formula for the integral of $\sec^{2k+1}(x)$; this formula allows us to express the integral in terms of an integral where the power of sec(x) is reduced by two.

We give one more example.

Example 6.2.14 Integrating powers of tangent and secant.

Evaluate $\int \tan^6(x)\,dx$.

Solution. We employ Rule 3 of Key Idea 6.2.8.

$$\begin{aligned}
\int \tan^6(x)\,dx &= \int \tan^4(x)\tan^2(x)\,dx\\
&= \int \tan^4(x)\big(\sec^2(x)-1\big)\,dx\\
&= \int \tan^4(x)\sec^2(x)\,dx - \int \tan^4(x)\,dx
\end{aligned}$$

Integrate the first integral with substitution, $u = \tan(x)$; integrate the second by employing rule Rule 4 again.

$$\begin{aligned}
&= \frac{1}{5}\tan^5(x) - \int \tan^2(x)\tan^2(x)\,dx\\
&= \frac{1}{5}\tan^5(x) - \int \tan^2(x)\big(\sec^2(x)-1\big)\,dx\\
&= \frac{1}{5}\tan^5(x) - \underbrace{\int \tan^2(x)\sec^2(x)\,dx}_{a} + \underbrace{\int \tan^2(x)\,dx}_{b}
\end{aligned}$$

Again, use substitution ($u = \tan(x)$) for the first integral (a) and Rule 4 for the second (b).

$$\begin{aligned}
&= \frac{1}{5}\tan^5(x) - \frac{1}{3}\tan^3(x) + \int \big(\sec^2(x)-1\big)\,dx\\
\int \tan^6(x)\,dx &= \frac{1}{5}\tan^5(x) - \frac{1}{3}\tan^3(x) + \tan(x) - x + C.
\end{aligned}$$

These latter examples were admittedly long, with repeated applications of the same rule. Try to not be overwhelmed by the length of the problem, but rather admire how robust this solution method is. A trigonometric function of a high power can be systematically reduced to trigonometric functions of lower powers until all antiderivatives can be computed.

Section 6.3 introduces an integration technique known as Trigonometric Substitution, a clever combination of Substitution and the Pythagorean Theorem.

youtu.be/watch?v=Om0iOgV9IwA

Figure 6.2.13 Deriving a power reduction formula for secant integrals

Video solution

youtu.be/watch?v=MUDKKDz3_C8

6.2.4 Exercises

Terms and Concepts

1. (☐ True ☐ False) $\int \sin^2(x)\cos^2(x)\,dx$ cannot be evaluated using the techniques described in this section since both powers of $\sin(x)$ and $\cos(x)$ are even.

2. (☐ True ☐ False) $\int \sin^3(x)\cos^3(x)\,dx$ cannot be evaluated using the techniques described in this section since both powers of $\sin(x)$ and $\cos(x)$ are odd.

3. (☐ True ☐ False) This section addresses how to evaluate indefinite integrals such as $\int \sin^5(x)\tan^3(x)\,dx$.

4. (☐ True ☐ False) Sometimes computer programs evaluate integrals involving trigonometric functions differently than one would using the techniques of this section. When this is the case, the techniques of this section have failed and one should only trust the answer given by the computer.

Problems

Exercise Group. Evaluate the indefinite integral.

5. $\int \sin(x)\cos^4(x)\,dx$

6. $\int \sin^3(x)\cos(x)\,dx$

7. $\int \sin^3(x)\cos^4(x)\,dx$

8. $\int \sin^3(x)\cos^5(x)\,dx$

9. $\int \sin^6(x)\cos^5(x)\,dx$

10. $\int \sin^2(x)\cos^7(x)\,dx$

11. $\int \sin^2(x)\cos^2(x)\,dx$

12. $\int \sin(5x)\cos(3x)\,dx$

13. $\int \sin(x)\cos(6x)\,dx$

14. $\int \sin(3x)\sin(4x)\,dx$

15. $\int \sin(\pi x)\sin(8\pi x)\,dx$

16. $\int \cos(x)\cos(2x)\,dx$

17. $\int \cos\left(\frac{\pi}{3}x\right)\cos\left(\frac{\pi}{6}x\right)\,dx$

18. $\int \tan^4(x)\sec^2(x)\,dx$

19. $\int \tan^2(x)\sec^4(x)\,dx$

20. $\int \tan^8(x)\sec^4(x)\,dx$

21. $\int \tan^9(x)\sec^2(x)\,dx$

22. $\int \tan^3(x)\sec^2(x)\,dx$

23. $\int \tan^5(x)\sec^3(x)\,dx$

24. $\int \tan^4(x)\,dx$

25. $\int \sec^5(x)\,dx$

26. $\int \tan^2(x)\sec(x)\,dx$

27. $\int \tan^2(x)\sec^3(x)\,dx$

Exercise Group. Evaluate the definite integral. Note: the corresponding indefinite integrals appear in Exercises 5–27.

28. $\int_0^{\frac{3\pi}{2}} \sin(x)\cos^4(x)\,dx$

29. $\int_{-2\pi}^{2\pi} \sin^3(x)\cos(x)\,dx$

30. $\int_{-2\pi}^{2\pi} \sin^2(x)\cos^7(x)\,dx$

31. $\int_0^{\frac{\pi}{4}} \sin(5x)\cos(3x)\,dx$

32. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\cos(2x)\,dx$

33. $\displaystyle\int_{0}^{\frac{\pi}{4}} \tan^4(x)\sec^2(x)\,dx$

34. $\displaystyle\int_{-\pi/4}^{\pi/4} \tan^2(x)\sec^4(x)\,dx$

6.3 Trigonometric Substitution

In Section 5.2 we defined the definite integral as the "signed area under the curve." In that section we had not yet learned the Fundamental Theorem of Calculus, so we only evaluated special definite integrals which described nice, geometric shapes. For instance, we were able to evaluate

$$\int_{-3}^{3} \sqrt{9-x^2}\,dx = \frac{9\pi}{2} \tag{6.3.1}$$

as we recognized that $f(x) = \sqrt{9-x^2}$ described the upper half of a circle with radius 3.

We have since learned a number of integration techniques, including Substitution and Integration by Parts, yet we are still unable to evaluate the above integral without resorting to a geometric interpretation. This section introduces Trigonometric Substitution, a method of integration that fills this gap in our integration skill. This technique works on the same principle as Substitution as found in Section 5.6, though it can feel "backward." In Section 5.6, we set $u = f(x)$, for some function f, and replaced $f(x)$ with u. In this section, we will set $x = f(\theta)$, where f is a trigonometric function, then replace x with $f(\theta)$.

We start by demonstrating this method in evaluating the integral in Equation (6.3.1). After the example, we will generalize the method and give more examples.

youtu.be/watch?v=l3gtQyPLr-E

Figure 6.3.1 Video introduction to Section 6.3

Example 6.3.2 Using Trigonometric Substitution.

Evaluate $\displaystyle\int_{-3}^{3} \sqrt{9-x^2}\,dx$.

Solution. We begin by noting that $9\left(\sin^2(\theta)+\cos^2(\theta)\right) = 9$, and hence $9\cos^2(\theta) = 9 - 9\sin^2(\theta)$. If we let $x = 3\sin(\theta)$, then $9 - x^2 = 9 - 9\sin^2(\theta) = 9\cos^2(\theta)$.

Setting $x = 3\sin(\theta)$ gives $dx = 3\cos(\theta)\,d\theta$. We are almost ready to substitute. We also wish to change our bounds of integration. The bound $x = -3$ corresponds to $\theta = -\pi/2$ (for when $\theta = -\pi/2$, $x = 3\sin(\theta) = -3$). Likewise, the bound of $x = 3$ is replaced by the bound $\theta = \pi/2$. Thus

$$\begin{aligned}\int_{-3}^{3} \sqrt{9-x^2}\,dx &= \int_{-\pi/2}^{\pi/2} \sqrt{9-9\sin^2(\theta)}\,(3\cos(\theta))\,d\theta\\ &= \int_{-\pi/2}^{\pi/2} 3\sqrt{9\cos^2(\theta)}\cos(\theta)\,d\theta\\ &= \int_{-\pi/2}^{\pi/2} 3\,|3\cos(\theta)|\cos(\theta)\,d\theta.\end{aligned}$$

On $[-\pi/2, \pi/2]$, $\cos(\theta)$ is always positive, so we can drop the absolute value bars, then employ a power-reducing formula:

$$\begin{aligned}\int_{-3}^{3} \sqrt{9-x^2}\,dx &= \int_{-\pi/2}^{\pi/2} 9\cos^2(\theta)\,d\theta\\ &= \int_{-\pi/2}^{\pi/2} \frac{9}{2}\big(1+\cos(2\theta)\big)\,d\theta\\ &= \frac{9}{2}\Big(\theta + \frac{1}{2}\sin(2\theta)\Big)\Big|_{-\pi/2}^{\pi/2}\end{aligned}$$

$$= \frac{9}{2}\pi.$$

This matches our answer from before.

We now describe in detail Trigonometric Substitution. This method excels when dealing with integrands that contain $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ and $\sqrt{x^2+a^2}$. The following Key Idea outlines the procedure for each case, followed by more examples. Each right triangle acts as a reference to help us understand the relationships between x and θ.

Video solution

youtu.be/watch?v=5CKWeQvnGAU

Key Idea 6.3.3 Trigonometric Substitution.

1. *Integrands containing* $\sqrt{a^2-x^2}$.

   Let $x = a\sin(\theta)$, $dx = a\cos(\theta)\,d\theta$. Thus $\theta = \sin^{-1}(x/a)$, for $-\pi/2 \leq \theta \leq \pi/2$. On this interval, $\cos(\theta) \geq 0$, so $\sqrt{a^2-x^2} = a\cos(\theta)$.

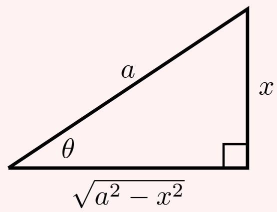

Figure 6.3.4

2. *Integrands containing* $\sqrt{x^2+a^2}$.

   Let $x = a\tan(\theta)$, $dx = a\sec^2(\theta)\,d\theta$. Thus $\theta = \tan^{-1}(x/a)$, for $-\pi/2 < \theta < \pi/2$. On this interval, $\sec(\theta) > 0$, so $\sqrt{x^2+a^2} = a\sec(\theta)$.

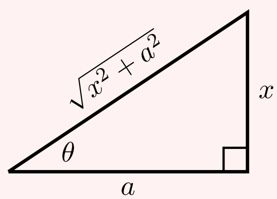

Figure 6.3.5

3. *Integrands containing* $\sqrt{x^2-a^2}$.

   Let $x = a\sec(\theta)$, $dx = a\sec(\theta)\tan(\theta)\,d\theta$. Thus $\theta = \sec^{-1}(x/a)$. If $x/a \geq 1$, then $0 \leq \theta < \pi/2$; if $x/a \leq -1$, then $\pi/2 < \theta \leq \pi$. We restrict our work to where $x \geq a$, so $x/a \geq 1$, and $0 \leq \theta < \pi/2$. On this interval, $\tan(\theta) \geq 0$, so $\sqrt{x^2-a^2} = a\tan(\theta)$.

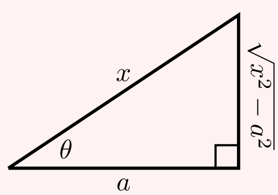

Figure 6.3.6

Example 6.3.7 Using Trigonometric Substitution.

Evaluate $\displaystyle\int \frac{1}{\sqrt{5+x^2}}\,dx$.

Solution. Using Item 2 in Key Idea 6.3.3, we recognize $a = \sqrt{5}$ and set $x = \sqrt{5}\tan(\theta)$. This makes $dx = \sqrt{5}\sec^2(\theta)\,d\theta$. We will use the fact that $\sqrt{5+x^2} = \sqrt{5+5\tan^2(\theta)} = \sqrt{5\sec^2(\theta)} = \sqrt{5}\sec(\theta)$. Substituting, we have:

$$\begin{aligned}\int \frac{1}{\sqrt{5+x^2}}\,dx &= \int \frac{1}{\sqrt{5+5\tan^2(\theta)}}\sqrt{5}\sec^2(\theta)\,d\theta\\ &= \int \frac{\sqrt{5}\sec^2(\theta)}{\sqrt{5}\sec(\theta)}\,d\theta\\ &= \int \sec(\theta)\,d\theta\\ &= \ln|\sec(\theta)+\tan(\theta)|+C.\end{aligned}$$

While the integration steps are over, we are not yet done. The original problem was stated in terms of x, whereas our answer is given in terms of θ. We must convert back to x.

The reference triangle given in Figure 6.3.5 helps. With $x=\sqrt{5}\tan(\theta)$, we have

$$\tan(\theta)=\frac{x}{\sqrt{5}} \text{ and } \sec(\theta)=\frac{\sqrt{x^2+5}}{\sqrt{5}}.$$

This gives

$$\begin{aligned}\int \frac{1}{\sqrt{5+x^2}}\,dx &= \ln|\sec(\theta)+\tan(\theta)|+C\\ &= \ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}}+\frac{x}{\sqrt{5}}\right|+C.\end{aligned}$$

We can leave this answer as is, or we can use a logarithmic identity to simplify it. Note:

$$\begin{aligned}\ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}}+\frac{x}{\sqrt{5}}\right|+C &= \ln\left|\frac{1}{\sqrt{5}}\left(\sqrt{x^2+5}+x\right)\right|+C\\ &= \ln\left|\frac{1}{\sqrt{5}}\right|+\ln\left|\sqrt{x^2+5}+x\right|+C\\ &= \ln\left|\sqrt{x^2+5}+x\right|+C,\end{aligned}$$

where the $\ln\left(1/\sqrt{5}\right)$ term is absorbed into the constant C. (In Section 5.7 we will learn another way of approaching this problem.)

Video solution

youtu.be/watch?v=2a9Oks-FCg0

Example 6.3.8 Using Trigonometric Substitution.

Evaluate $\int \sqrt{4x^2-1}\,dx$.

Solution. We start by rewriting the integrand so that it looks like $\sqrt{x^2-a^2}$ for some value of a:

$$\begin{aligned}\sqrt{4x^2-1} &= \sqrt{4\left(x^2-\frac{1}{4}\right)}\\ &= 2\sqrt{x^2-\left(\frac{1}{2}\right)^2}.\end{aligned}$$

So we have $a=1/2$, and following Part 3 of Key Idea 6.3.3, we set

$x = \frac{1}{2}\sec(\theta)$, and hence $dx = \frac{1}{2}\sec(\theta)\tan(\theta)\,d\theta$. We now rewrite the integral with these substitutions:

$$\begin{aligned}\int \sqrt{4x^2-1}\,dx &= \int 2\sqrt{x^2-\left(\frac{1}{2}\right)^2}\,dx\\ &= \int 2\sqrt{\frac{1}{4}\sec^2(\theta)-\frac{1}{4}}\left(\frac{1}{2}\sec(\theta)\tan(\theta)\right)d\theta\\ &= \int \sqrt{\frac{1}{4}(\sec^2(\theta)-1)}\Big(\sec(\theta)\tan(\theta)\Big)\,d\theta\\ &= \int \sqrt{\frac{1}{4}\tan^2(\theta)}\Big(\sec(\theta)\tan(\theta)\Big)\,d\theta\\ &= \int \frac{1}{2}\tan^2(\theta)\sec(\theta)\,d\theta\\ &= \frac{1}{2}\int \Big(\sec^2(\theta)-1\Big)\sec(\theta)\,d\theta\\ &= \frac{1}{2}\int \big(\sec^3(\theta)-\sec(\theta)\big)\,d\theta.\end{aligned}$$

We integrated $\sec^3(\theta)$ in Example 6.2.11, finding its antiderivatives to be

$$\int \sec^3(\theta)\,d\theta = \frac{1}{2}\Big(\sec(\theta)\tan(\theta)+\ln|\sec(\theta)+\tan(\theta)|\Big)+C.$$

Thus

$$\begin{aligned}&\int \sqrt{4x^2-1}\,dx = \frac{1}{2}\int \big(\sec^3(\theta)-\sec(\theta)\big)\,d\theta\\ &= \frac{1}{2}\left(\frac{1}{2}\Big(\sec(\theta)\tan(\theta)+\ln|\sec(\theta)+\tan(\theta)|\Big)-\ln|\sec(\theta)+\tan(\theta)|\right)+C\\ &= \frac{1}{4}\big(\sec(\theta)\tan(\theta)-\ln|\sec(\theta)+\tan(\theta)|\big)+C.\end{aligned}$$

We are not yet done. Our original integral is given in terms of x, whereas our final answer, as given, is in terms of θ. We need to rewrite our answer in terms of x. With $a = 1/2$, and $x = \frac{1}{2}\sec(\theta)$, the reference triangle in Figure 6.3.6 shows that

$$\tan(\theta) = \sqrt{x^2-1/4}\Big/(1/2) = 2\sqrt{x^2-1/4} \text{ and } \sec(\theta) = 2x.$$

Thus

$$\begin{aligned}&\frac{1}{4}\Big(\sec(\theta)\tan(\theta)-\ln|\sec(\theta)+\tan(\theta)|\Big)+C\\ &\quad= \frac{1}{4}\left(2x\cdot 2\sqrt{x^2-1/4}-\ln\left|2x+2\sqrt{x^2-1/4}\right|\right)+C\\ &\quad= \frac{1}{4}\left(4x\sqrt{x^2-1/4}-\ln\left|2x+2\sqrt{x^2-1/4}\right|\right)+C.\end{aligned}$$

The final answer is given in the last line above, repeated here:

$$\int \sqrt{4x^2-1}\,dx = \frac{1}{4}\left(4x\sqrt{x^2-1/4}-\ln\left|2x+2\sqrt{x^2-1/4}\right|\right)+C.$$

Video solution

youtu.be/watch?v=0oCjVzIa_t8

Example 6.3.9 Using Trigonometric Substitution.

Evaluate $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$.

Solution. We use Part 1 of Key Idea 6.3.3 with $a = 2$, $x = 2\sin(\theta)$, $dx = 2\cos(\theta)$ and hence $\sqrt{4-x^2} = 2\cos(\theta)$. This gives

$$\begin{aligned}\int \frac{\sqrt{4-x^2}}{x^2}\,dx &= \int \frac{2\cos(\theta)}{4\sin^2(\theta)}(2\cos(\theta))\,d\theta\\ &= \int \cot^2(\theta)\,d\theta\\ &= \int (\csc^2(\theta)-1)\,d\theta\\ &= -\cot(\theta)-\theta+C.\end{aligned}$$

We need to rewrite our answer in terms of x. Using the reference triangle found in Figure 6.3.4, we have $\cot(\theta) = \sqrt{4-x^2}/x$ and $\theta = \sin^{-1}(x/2)$. Thus

$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx = -\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\left(\frac{x}{2}\right)+C.$$

Video solution

youtu.be/watch?v=E37-2LvYSsg

Trigonometric Substitution can be applied in many situations, even those not of the form $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ or $\sqrt{x^2+a^2}$. In the following example, we apply it to an integral we already know how to handle.

Example 6.3.10 Using Trigonometric Substitution.

Evaluate $\displaystyle\int \frac{1}{x^2+1}\,dx$.

Solution. We know the answer already as $\tan^{-1}(x)+C$. We apply Trigonometric Substitution here to show that we get the same answer without inherently relying on knowledge of the derivative of the arctangent function.

Using Part 2 of Key Idea 6.3.3, let $x = \tan(\theta)$, $dx = \sec^2(\theta)\,d\theta$ and note that $x^2+1 = \tan^2(\theta)+1 = \sec^2(\theta)$. Thus

$$\begin{aligned}\int \frac{1}{x^2+1}\,dx &= \int \frac{1}{\sec^2(\theta)}\sec^2(\theta)\,d\theta\\ &= \int 1\,d\theta\\ &= \theta+C.\end{aligned}$$

Since $x = \tan(\theta)$, $\theta = \tan^{-1}(x)$, and we conclude that $\displaystyle\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x)+C$.

The next example is similar to the previous one in that it does not involve a square-root. It shows how several techniques and identities can be combined to obtain a solution.

Example 6.3.11 Using Trigonometric Substitution.

Evaluate $\displaystyle\int \frac{1}{(x^2+6x+10)^2}\,dx$.

Solution. We start by completing the square, then make the substitution $u = x+3$, followed by the trigonometric substitution of $u = \tan(\theta)$:

$$\int \frac{1}{(x^2+6x+10)^2}\,dx = \int \frac{1}{\left((x+3)^2+1\right)^2}\,dx = \int \frac{1}{(u^2+1)^2}\,du.$$

Now make the substitution $u = \tan(\theta)$, $du = \sec^2(\theta)\,d\theta$:

$$\begin{aligned} &= \int \frac{1}{(\tan^2(\theta)+1)^2}\sec^2(\theta)\,d\theta \\ &= \int \frac{1}{(\sec^2(\theta))^2}\sec^2(\theta)\,d\theta \\ &= \int \cos^2(\theta)\,d\theta. \end{aligned}$$

Applying a power reducing formula, we have

$$\begin{aligned} &= \int \left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)d\theta \\ &= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C. \end{aligned} \tag{6.3.2}$$

We need to return to the variable x. As $u = \tan(\theta)$, $\theta = \tan^{-1}(u)$. Using the identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and using the reference triangle found in Figure 6.3.5, we have

$$\frac{1}{4}\sin(2\theta) = \frac{1}{2}\frac{u}{\sqrt{u^2+1}}\cdot\frac{1}{\sqrt{u^2+1}} = \frac{1}{2}\frac{u}{u^2+1}.$$

Finally, we return to x with the substitution $u = x+3$. We start with the expression in Equation (6.3.2):

$$\begin{aligned} \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C &= \frac{1}{2}\tan^{-1}(u) + \frac{1}{2}\frac{u}{u^2+1} + C \\ &= \frac{1}{2}\tan^{-1}(x+3) + \frac{x+3}{2(x^2+6x+10)} + C. \end{aligned}$$

Stating our final result in one line,

$$\int \frac{1}{(x^2+6x+10)^2}\,dx = \frac{1}{2}\tan^{-1}(x+3) + \frac{x+3}{2(x^2+6x+10)} + C.$$

Video solution

youtu.be/watch?v=5JAXeV1-vCo

Our last example returns us to definite integrals, as seen in our first example. Given a definite integral that can be evaluated using Trigonometric Substitution, we could first evaluate the corresponding indefinite integral (by changing from an integral in terms of x to one in terms of θ, then converting back to x) and then evaluate using the original bounds. It is much more straightforward, though, to change the bounds as we substitute.

Example 6.3.12 Definite integration and Trigonometric Substitution.

Evaluate $\displaystyle\int_0^5 \frac{x^2}{\sqrt{x^2+25}}\,dx$.

Solution. Using Part 2 of Key Idea 6.3.3, we set $x = 5\tan(\theta)$, $dx = 5\sec^2(\theta)\,d\theta$, and note that $\sqrt{x^2+25} = 5\sec(\theta)$. As we substitute, we can also change the bounds of integration.

The lower bound of the original integral is $x = 0$. As $x = 5\tan(\theta)$, we solve for θ and find $\theta = \tan^{-1}(x/5)$. Thus the new lower bound is $\theta = \tan^{-1}(0) = 0$. The original upper bound is $x = 5$, thus the new upper bound is $\theta = \tan^{-1}(5/5) = \pi/4$.

Thus we have

$$\begin{aligned}\int_0^5 \frac{x^2}{\sqrt{x^2+25}}\,dx &= \int_0^{\pi/4} \frac{25\tan^2(\theta)}{5\sec(\theta)}5\sec^2(\theta)\,d\theta\\ &= 25\int_0^{\pi/4} \tan^2(\theta)\sec(\theta)\,d\theta.\end{aligned}$$

We encountered this indefinite integral in Example 6.3.8 where we found

$$\int \tan^2(\theta)\sec(\theta)\,d\theta = \frac{1}{2}\Big(\sec(\theta)\tan(\theta) - \ln|\sec(\theta)+\tan(\theta)|\Big).$$

So

$$\begin{aligned}25\int_0^{\pi/4} \tan^2(\theta)\sec(\theta)\,d\theta &= \frac{25}{2}\Big(\sec(\theta)\tan(\theta) - \ln|\sec(\theta)+\tan(\theta)|\Big)\Big|_0^{\pi/4}\\ &= \frac{25}{2}\big(\sqrt{2} - \ln(\sqrt{2}+1)\big)\\ &\approx 6.661.\end{aligned}$$

Video solution

youtu.be/watch?v=Pz56QfleHX4

The following equalities are very useful when evaluating integrals using Trigonometric Substitution.

Key Idea 6.3.13 Useful Equalities with Trigonometric Substitution.

1. $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
2. $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$
3. $\displaystyle\int \sec^3(\theta)\,d\theta = \frac{1}{2}\Big(\sec(\theta)\tan(\theta) + \ln|\sec(\theta)+\tan(\theta)|\Big) + C$
4. $\displaystyle\int \cos^2(\theta)\,d\theta = \int \frac{1}{2}\big(1+\cos(2\theta)\big)\,d\theta = \frac{1}{2}\big(\theta + \sin(\theta)\cos(\theta)\big) + C$.

The next section introduces Partial Fraction Decomposition, which is an algebraic technique that turns "complicated" fractions into sums of "simpler" fractions, making integration easier.

6.3.1 Exercises

Terms and Concepts

1. Trigonometric Substitution works on the same principles as Integration by Substitution, though it can feel "____________________".

2. If one uses Trigonometric Substitution on an integrand containing $\sqrt{36-x^2}$, then one should set $x =$ ____________________.

3. Consider the Pythagorean Identity $\sin^2(\theta) + \cos^2(\theta) = 1$.

a. What identity is obtained when both sides are divided by $\cos^2(\theta)$?

b. Use the new identity to simplify $9\tan^2(\theta) + 9$.

4. Why does Part 1 of Key Idea 6.3.3 state that $\sqrt{a^2-x^2} = a\cos(\theta)$, and not $|a\cos(\theta)|$?

Problems

Exercise Group. Apply Trigonometric Substitution to evaluate the indefinite integral.

5. $\displaystyle\int \sqrt{x^2+1}\,dx$

6. $\displaystyle\int \sqrt{x^2+4}\,dx$

7. $\displaystyle\int \sqrt{1-x^2}\,dx$

8. $\displaystyle\int \sqrt{9-x^2}\,dx$

9. $\displaystyle\int \sqrt{x^2-1}\,dx$

10. $\displaystyle\int \sqrt{x^2-16}\,dx$

11. $\displaystyle\int \sqrt{36x^2+1}\,dx$

12. $\displaystyle\int \sqrt{1-49x^2}\,dx$

13. $\displaystyle\int \sqrt{64x^2-1}\,dx$

14. $\displaystyle\int \frac{9}{\sqrt{x^2+6}}\,dx$

15. $\displaystyle\int \frac{2}{\sqrt{15-x^2}}\,dx$

16. $\displaystyle\int \frac{3}{\sqrt{x^2-11}}\,dx$

Exercise Group. Evaluate the indefinite integral. Trigonometric Substitution may not be required.

17. $\displaystyle\int \frac{\sqrt{x^2-5}}{x}\,dx$

18. $\displaystyle\int \frac{1}{(x^2+1)^2}\,dx$

19. $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\,dx$

20. $\displaystyle\int x^2\sqrt{1-x^2}\,dx$

21. $\displaystyle\int \frac{x}{(x^2+49)^{\left(\frac{3}{2}\right)}}\,dx$

22. $\displaystyle\int \frac{8x^2}{\sqrt{x^2-10}}\,dx$

23. $\displaystyle\int \frac{1}{(x^2-16x+68)^2}\,dx$

24. $\displaystyle\int x^2(1-x^2)^{-3/2}\,dx$

25. $\displaystyle\int \frac{\sqrt{10-x^2}}{3x^2}\,dx$

26. $\displaystyle\int \frac{x^2}{\sqrt{x^2+5}}\,dx$

Exercise Group. Evaluate the definite integral by making the proper trigonometric substitution *and* changing the bounds of integration. (Note: the corresponding indefinite integrals appeared previously in the Section 6.3 exercises.)

27. $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$

28. $\displaystyle\int_{4}^{7} \sqrt{x^2-16}\,dx$

29. $\displaystyle\int_0^6 \sqrt{x^2+4}\,dx$

30. $\displaystyle\int_{-7}^7 \frac{1}{(x^2+1)^2}\,dx$

31. $\displaystyle\int_{-2}^2 \sqrt{9-x^2}\,dx$

32. $\displaystyle\int_{-1}^1 x^2\sqrt{1-x^2}\,dx$

6.4 Partial Fraction Decomposition

In this section we investigate the antiderivatives of rational functions. Recall that rational functions are functions of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. Such functions arise in many contexts, one of which is the solving of certain fundamental differential equations.

We begin with an example that demonstrates the motivation behind this section. Consider the integral $\displaystyle\int \frac{1}{x^2-1}\,dx$. We do not have a simple formula for this (if the denominator were x^2+1, we would recognize the antiderivative as being the arctangent function). It can be solved using Trigonometric Substitution, but note how the integral is easy to evaluate once we realize:

$$\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

Thus

$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= \int \frac{1/2}{x-1}\,dx - \int \frac{1/2}{x+1}\,dx\\ &= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.\end{aligned}$$

This section teaches how to *decompose*

$$\frac{1}{x^2-1} \text{ into } \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

We start with a rational function $f(x) = \frac{p(x)}{q(x)}$, where p and q do not have any common factors and the degree of p is less than the degree of q. It can be shown that any polynomial, and hence q, can be factored into a product of linear and irreducible quadratic terms. The following Key Idea states how to decompose a rational function into a sum of rational functions whose denominators are all of lower degree than q.

youtu.be/watch?v=BoUWS_SVr8A

Figure 6.4.1 Video introduction to Section 6.4

Key Idea 6.4.2 Partial Fraction Decomposition.

Let $\dfrac{p(x)}{q(x)}$ be a rational function, where the degree of p is less than the degree of q.

1. *Linear Terms:* Let $(x-a)$ divide $q(x)$, where $(x-a)^n$ is the highest power of $(x-a)$ that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum
$$\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}.$$

2. *Quadratic Terms:* Let x^2+bx+c be an irreducible quadratic that divides $q(x)$, where $(x^2+bx+c)^n$ is the highest power of x^2+bx+c that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum
$$\frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(x^2+bx+c)^n}.$$

To find the coefficients A_i, B_i and C_i:

1. Multiply all fractions by $q(x)$, clearing the denominators. Collect like terms.

2. Equate the resulting coefficients of the powers of x and solve the resulting system of linear equations.

An irreducible quadratic is a quadratic that has no real solutions. Solving $ax^2 + bx + c = 0$ using the quadratic equation will determine if a quadratic is irreducible. Completing the square (which is a common integration technique) will also tell you if a quadratic is irreducible.

The following examples will demonstrate how to put this Key Idea into practice. Example 6.4.3 stresses the decomposition aspect of the Key Idea.

Example 6.4.3 Decomposing into partial fractions.

Decompose $f(x) = \dfrac{1}{(x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2}$ without solving for the resulting coefficients.

Solution. The denominator is already factored, as both x^2+x+2 and x^2+x+7 cannot be factored further. We need to decompose $f(x)$ properly. Since $(x+5)$ is a linear term that divides the denominator, there will be a

$$\frac{A}{x+5}$$

term in the decomposition.
As $(x-2)^3$ divides the denominator, we will have the following terms in the decomposition:

$$\frac{B}{x-2}, \frac{C}{(x-2)^2} \text{ and } \frac{D}{(x-2)^3}.$$

The x^2+x+2 term in the denominator results in a $\dfrac{Ex+F}{x^2+x+2}$ term.
Finally, the $(x^2+x+7)^2$ term results in the terms

$$\frac{Gx+H}{x^2+x+7} \text{ and } \frac{Ix+J}{(x^2+x+7)^2}.$$

All together, we have

$$\begin{aligned}&\frac{1}{(x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2}\\ &\quad= \frac{A}{x+5}+\frac{B}{x-2}+\frac{C}{(x-2)^2}+\frac{D}{(x-2)^3}\\ &\qquad+\frac{Ex+F}{x^2+x+2}+\frac{Gx+H}{x^2+x+7}+\frac{Ix+J}{(x^2+x+7)^2}\end{aligned}$$

Solving for the coefficients A, $B \ldots J$ would be a bit tedious but not "hard."

Video solution

youtu.be/watch?v=9gRlWESr8lM

Example 6.4.4 Decomposing into partial fractions.

Perform the partial fraction decomposition of $\dfrac{1}{x^2-1}$.

Solution. The denominator factors into two linear terms: $x^2 - 1 = (x-1)(x+1)$. Thus

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}.$$

To solve for A and B, first multiply through by $x^2 - 1 = (x-1)(x+1)$:

$$\begin{aligned} 1 &= \frac{A(x-1)(x+1)}{x-1} + \frac{B(x-1)(x+1)}{x+1} \\ &= A(x+1) + B(x-1) \\ &= Ax + A + Bx - B \\ &= (A+B)x + (A-B), \end{aligned}$$

by collecting like terms.
The next step is key. Note the equality we have:

$$1 = (A+B)x + (A-B).$$

For clarity's sake, rewrite the left hand side as

$$0x + 1 = (A+B)x + (A-B).$$

On the left, the coefficient of the x term is 0; on the right, it is $(A+B)$. Since both sides are equal, we must have that $0 = A + B$.
Likewise, on the left, we have a constant term of 1; on the right, the constant term is $(A - B)$. Therefore we have $1 = A - B$.
We have two linear equations with two unknowns. This one is easy to solve by hand, leading to

$$\begin{aligned} A + B &= 0 \\ A - B &= 1 \end{aligned}$$

If we add these two equations, we get $2A = 1 \Rightarrow A = 1/2$. Substitution into the first equation gives $B = -1/2$.
Thus

$$\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

Video solution

youtu.be/watch?v=u-avVoj3qR0

There is another method for finding the partial fraction decomposition called the "Heaviside" method, named after Oliver Heaviside. We show a variation of this process using the same example as in Example 6.4.3.

Example 6.4.5 Decomposing into partial fractions using the Heaviside method.

Perform the partial fraction decomposition of $\dfrac{1}{x^2-1}$.

Solution. As we saw in Example 6.4.4,

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}.$$

To solve for A and B using the Heaviside method, we will build to a common denominator:

$$\begin{aligned} \frac{1}{x^2-1} &= \frac{A(x+1)}{(x-1)(x+1)} + \frac{B(x-1)}{(x+1)(x-1)} \\ &= \frac{A(x+1) + B(x-1)}{(x-1)(x+1)} \end{aligned}$$

Now since the denomiators match, we will only consider the numerator equation (essentially if we multiply both sides of the equation by $(x-1)(x+1)$, we will clear the denominators):

$$1 = A(x+1) + B(x-1)$$

Now we substitute in "convenient" values of x. When $x=1$, we get $1=2A \Rightarrow A=1/2$. When $x=-1$, we get $1=-2B \Rightarrow B=-1/2$. You may note that $x=1$ and $x=-1$ were not in the domain of the original fraction. However,

$$\frac{1}{x^2-1} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$$

is an identity, meaning it is true for all values of x, even those for which the equation is undefined. We could have chosen any values of x to substitute. Whenever possible, we choose values of x that will make one of the factors zero. In this way, we can avoid solving a system of equations.
Thus as in Example 6.4.3, we get

$$\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

Video solution

youtu.be/watch?v=QfEgKEhkL6o

For the remaining examples, we will use a combination of systems of equations and the Heaviside method to get partial fraction decompositions.

Example 6.4.6 Integrating using partial fractions.

Use partial fraction decomposition to integrate $\displaystyle\int \frac{1}{(x-1)(x+2)^2}\,dx$.

Solution. We decompose the integrand as follows, as described by Key Idea 6.4.2:

$$\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}.$$

To solve for A, B and C, we multiply both sides by $(x-1)(x+2)^2$:

$$1 = A(x+2)^2 + B(x-1)(x+2) + C(x-1) \qquad (6.4.1)$$

Equation (6.4.1) offers a direct route to finding the values of A, B and C. Since the equation holds for all values of x, it holds in particular when $x=1$. However, when $x=1$, the right hand side simplifies to $A(1+2)^2 = 9A$. Since the left hand side is still 1, we have $1=9A$. Hence $A=1/9$.
Likewise, the equality holds when $x=-2$; this leads to the equation $1=-3C$. Thus $C=-1/3$.
Knowing A and C, we can find the value of B by choosing yet another value of x, such as $x=0$, and solving for B.

Now we collect like terms:

$$\begin{aligned} 1 &= A(x+2)^2 + B(x-1)(x+2) + C(x-1) \\ &= Ax^2 + 4Ax + 4A + Bx^2 + Bx - 2B + Cx - C \\ &= (A+B)x^2 + (4A+B+C)x + (4A-2B-C) \end{aligned}$$

We have

$$0x^2 + 0x + 1 = (A+B)x^2 + (4A+B+C)x + (4A-2B-C)$$

leading to the equations

$$A+B=0,\ 4A+B+C=0 \text{ and } 4A-2B-C=1.$$

These three equations of three unknowns lead to a unique solution:

$$A=1/9,\ B=-1/9 \text{ and } C=-1/3.$$

Thus

$$\int \frac{1}{(x-1)(x+2)^2}\,dx = \int \frac{1/9}{x-1}\,dx + \int \frac{-1/9}{x+2}\,dx + \int \frac{-1/3}{(x+2)^2}\,dx.$$

Each can be integrated with a simple substitution with $u = x - 1$ or $u = x + 2$ (or by directly applying Key Idea 5.6.5 as the denominators are linear functions). The end result is

$$\int \frac{1}{(x-1)(x+2)^2}\,dx = \frac{1}{9}\ln|x-1| - \frac{1}{9}\ln|x+2| + \frac{1}{3(x+2)} + C.$$

Video solution

youtu.be/watch?v=LqE8pvIvJco

In examples like Example 6.4.6 where there are repeated roots, there is an extension of the Heaviside method using derivatives. This method is explained in Figure 6.4.7 below.

youtu.be/watch?v=0yrzd4JhR3I

Figure 6.4.7 Alternate method for finding coefficients in Example 6.4.6

Example 6.4.8 Integrating using partial fractions.

Use partial fraction decomposition to integrate $\displaystyle\int \frac{x^3}{(x-5)(x+3)}\,dx$.

Solution. Key Idea 6.4.2 presumes that the degree of the numerator is less than the degree of the denominator. Since this is not the case here, we begin by using polynomial division to reduce the degree of the numerator. We omit the steps, but encourage the reader to verify that

$$\frac{x^3}{(x-5)(x+3)} = x + 2 + \frac{19x+30}{(x-5)(x+3)}.$$

Using Key Idea 6.4.2, we can rewrite the new rational function as:

$$\frac{19x+30}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3}$$

for appropriate values of A and B. Clearing denominators, we have

$$\begin{aligned} 19x + 30 &= A(x+3) + B(x-5) \\ &= (A+B)x + (3A - 5B). \end{aligned}$$

The values of A and B can be quickly found using the technique described in Example 6.4.6, or they can be found by equating coefficients, as we do in Example 6.4.8.

This implies that:

$$\begin{aligned} 19 &= A + B \\ 30 &= 3A - 5B. \end{aligned}$$

Solving this system of linear equations gives

$$\begin{aligned} 125/8 &= A \\ 27/8 &= B. \end{aligned}$$

We can now integrate.

$$\begin{aligned} \int \frac{x^3}{(x-5)(x+3)}\,dx &= \int \left(x + 2 + \frac{125/8}{x-5} + \frac{27/8}{x+3}\right)dx \\ &= \frac{x^2}{2} + 2x + \frac{125}{8}\ln|x-5| + \frac{27}{8}\ln|x+3| + C. \end{aligned}$$

Video solution

youtu.be/watch?v=1cszweGfYR0

Example 6.4.9 Integrating using partial fractions.

Use partial fraction decomposition to evaluate $\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx$.

Solution. The degree of the numerator is less than the degree of the denominator so we begin by applying Key Idea 6.4.2. We have:

$$\frac{7x^2+31x+54}{(x+1)(x^2+6x+11)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+6x+11}.$$

Now clear the denominators.

$$7x^2+31x+54 = A(x^2+6x+11) + (Bx+C)(x+1)$$

Now, letting $x=-1$ we have $30 = 6A \Rightarrow A = 5$. When $x=0$, $54 = 11A + C$. But we know that $A=5$, so $54 = 55 + C \Rightarrow C = -1$ Finally, we choose $x = 1$ (with $A = 5, C = -1$) we have $92 = 90 + (B-1)(2) \Rightarrow B = 2$.
Thus

$$\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx = \int \left(\frac{5}{x+1} + \frac{2x-1}{x^2+6x+11}\right)dx.$$

The first term of this new integrand is easy to evaluate; it leads to a $5\ln|x+1|$ term. The second term is not hard, but takes several steps and uses substitution techniques.
The integrand $\frac{2x-1}{x^2+6x+11}$ has a quadratic in the denominator and a linear term in the numerator. This leads us to try substitution. Let $u = x^2+6x+11$, so $du = (2x+6)\,dx$. The numerator is $2x-1$, not $2x+6$, but we can get a $2x+6$ term in the numerator by adding 0 in the form of "$7-7$."

$$\begin{aligned}\frac{2x-1}{x^2+6x+11} &= \frac{2x-1+7-7}{x^2+6x+11}\\ &= \frac{2x+6}{x^2+6x+11} - \frac{7}{x^2+6x+11}.\end{aligned}$$

We can now integrate the first term with substitution, leading to a $\ln\left|x^2+6x+11\right|$ term. The final term can be integrated using arctangent. (We can tell there is no further factoring for this quadratic since the denominator has no real solutions). First, complete the square in the denominator:

$$\frac{7}{x^2+6x+11} = \frac{7}{(x+3)^2+2}.$$

An antiderivative of the latter term can be found using Theorem 5.6.20 and substitution:

$$\int \frac{7}{x^2+6x+11}\,dx = \frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right) + C.$$

Let's start at the beginning and put all of the steps together.

$$\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx$$

$$= \int \left(\frac{5}{x+1} + \frac{2x-1}{x^2+6x+11} \right) dx$$
$$= \int \frac{5}{x+1}\, dx + \int \frac{2x+6}{x^2+6x+11}\, dx - \int \frac{7}{(x+3)^2+2}\, dx$$
$$= 5\ln|x+1| + \ln\left|x^2+6x+11\right| - \frac{7}{\sqrt{2}} \tan^{-1}\left(\frac{x+3}{\sqrt{2}} \right) + C.$$

As with many other problems in calculus, it is important to remember that one is not expected to "see" the final answer immediately after seeing the problem. Rather, given the initial problem, we break it down into smaller problems that are easier to solve. The final answer is a combination of the answers of the smaller problems.

Video solution

youtu.be/watch?v=KNN0krvf1UE

Partial Fraction Decomposition is an important tool when dealing with rational functions. Note that at its heart, it is a technique of algebra, not calculus, as we are rewriting a fraction in a new form. Regardless, it is very useful in the realm of calculus as it lets us evaluate a certain set of "complicated" integrals.

Section 5.7 introduces new functions, called the Hyperbolic Functions. They will allow us to make substitutions similar to those found when studying Trigonometric Substitution, allowing us to approach even more integration problems.

6.4.1 Exercises

Terms and Concepts

1. Partial Fraction Decomposition is a method of rewriting ___________ functions.

2. (☐ True ☐ False) It is sometimes necessary to use polynomial division before using Partial Fraction Decomposition.

Exercise Group. Decompose without solving for the coefficients, as done in Example 6.4.3.

3. $\dfrac{1}{x^2-5x}$

4. $\dfrac{4x+8}{x^2-16}$

5. $\dfrac{x-3}{x^2-7}$

6. $\dfrac{9x+7}{x^3+6x}$

Problems

Exercise Group. Evaluate the indefinite integral.

7. $\displaystyle\int \frac{5x-14}{x^2-6x+8}\,dx$

8. $\displaystyle\int -\frac{x+15}{x^2-5x}\,dx$

9. $\displaystyle\int \frac{18}{5x^2-5}\,dx$

10. $\displaystyle\int \frac{6x+28}{3x^2+28x+9}\,dx$

11. $\displaystyle\int \frac{3x+29}{(x+7)^2}\,dx$

12. $\displaystyle\int \frac{9x+48}{(x+5)^2}\,dx$

13. $\displaystyle\int \frac{13x^2+53x+54}{x(x+3)^2}\,dx$

14. $\displaystyle\int \frac{81x^2+735x+516}{(x+1)(x+6)(-5-8x)}\,dx$

15. $\displaystyle\int \frac{113x-273x^2}{(5x-4)(6x+2)(9x-1)}\,dx$

16. $\displaystyle\int \frac{x^2+x-5}{x^2-3x-4}\,dx$

17. $\displaystyle\int \frac{x^3}{x^2+x-42}\,dx$

18. $\displaystyle\int \frac{4x^2-32x+84}{x^2-8x+21}\,dx$

19. $\displaystyle\int \frac{1}{x^3+8x^2+19x}\,dx$

20. $\displaystyle\int \frac{x^2+8x+12}{x^2+6x+11}\,dx$

21. $\displaystyle\int \frac{4x-9x^2+65}{(x+4)(3x^2+5x-9)}\,dx$

22. $\displaystyle\int \frac{7x^2+9x+5}{(x+1)(x^2+2x+2)}\,dx$

23. $\displaystyle\int \frac{(8)\,x^2+(5)\,x-(9)}{(x+8)(x^2+9)}\,dx$

24. $\displaystyle\int \frac{x^2+18x-194}{(x+4)(x^2-2x+26)}\,dx$

25. $\displaystyle\int \frac{6x^2-42x+32}{(x-4)(x^2-4x+10)}\,dx$

26. $\displaystyle\int \frac{(9)\,x^2-(41)\,x-(144)}{(x+7)(x^2-10x+27)}\,dx$

Exercise Group. Evaluate the definite integral.

27. $\displaystyle\int_1^2 \frac{11x-64}{(x-4)(x-8)}\,dx$

28. $\displaystyle\int_0^2 \frac{38x-15}{(5x+3)(x+6)}\,dx$

29. $\displaystyle\int_{-1}^1 \frac{x^2+7x}{(x-10)(x^2+6x+10)}\,dx$

30. $\displaystyle\int_0^1 \frac{x}{(x+1)(x^2+2x+1)}\,dx$

6.5 Improper Integration

We begin this section by considering the following definite integrals:

- $\displaystyle\int_0^{100} \frac{1}{1+x^2}\,dx \approx 1.5608$
- $\displaystyle\int_0^{1000} \frac{1}{1+x^2}\,dx \approx 1.5698$
- $\displaystyle\int_0^{10,000} \frac{1}{1+x^2}\,dx \approx 1.5707$

youtu.be/watch?v=HhBRqV7rt4I

Figure 6.5.2 Video introduction to Section 6.5

Notice how the integrand is $1/(1+x^2)$ in each integral (which is sketched in Figure 6.5.1). As the upper bound gets larger, one would expect the "area under the curve" would also grow. While the definite integrals do increase in value as the upper bound grows, they are not increasing by much. In fact, consider:

$$\int_0^b \frac{1}{1+x^2}\,dx = \tan^{-1}(x)\Big|_0^b = \tan^{-1}(b) - \tan^{-1}(0) = \tan^{-1}(b).$$

As $b \to \infty$, $\tan^{-1}(b) \to \pi/2$. Therefore it seems that as the upper bound b grows, the value of the definite integral $\displaystyle\int_0^b \frac{1}{1+x^2}\,dx$ approaches $\pi/2 \approx 1.5708$. This should strike the reader as being a bit amazing: even though the curve extends "to infinity," it has a finite amount of area underneath it.

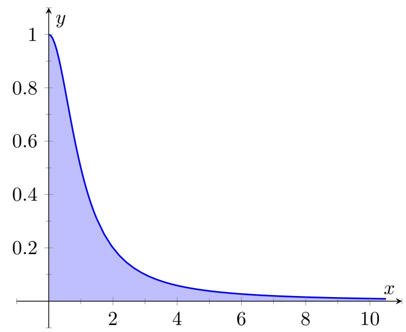

Figure 6.5.1 Graphing $f(x) = \dfrac{1}{1+x^2}$

When we defined the definite integral $\displaystyle\int_a^b f(x)\,dx$, we made two stipulations:

1. The interval over which we integrated, $[a,b]$, was a finite interval, and
2. The function $f(x)$ was continuous on $[a,b]$ (ensuring that the range of f was finite).

In this section we consider integrals where one or both of the above conditions do not hold. Such integrals are called *improper integrals*.

6.5.1 Improper Integrals with Infinite Bounds

Definition 6.5.3 Improper Integrals with Infinite Bounds; Converge, Diverge.

1. Let f be a continuous function on $[a,\infty)$. Define

$$\int_a^\infty f(x)\,dx \text{ to be } \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. Let f be a continuous function on $(-\infty,b]$. Define

$$\int_{-\infty}^b f(x)\,dx \text{ to be } \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. Let f be a continuous function on $(-\infty,\infty)$. Let c be any real number; define

$$\int_{-\infty}^\infty f(x)\,dx \text{ to be } \lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx.$$

An improper integral is said to **converge** if its corresponding limit exists; otherwise, it **diverges**. The improper integral in part 3 converges if and only if both of its limits exist.

Example 6.5.4 Evaluating improper integrals.

Evaluate the following improper integrals.

1. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$
2. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$
3. $\displaystyle\int_{-\infty}^0 e^x\,dx$
4. $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

Solution.

1.

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx &= \lim_{b\to\infty} \frac{-1}{x}\bigg|_1^b\\ &= \lim_{b\to\infty}\frac{-1}{b}+1\\ &= 1.\end{aligned}$$

A graph of the area defined by this integral is given in Figure 6.5.5.

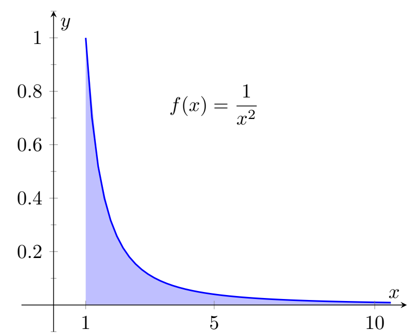

Figure 6.5.5 A graph of $f(x) = \frac{1}{x^2}$ in Example 6.5.4

2.

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx\\ &= \lim_{b\to\infty} \ln|x|\,\bigg|_1^b\\ &= \lim_{b\to\infty} \ln(b)\\ &= \infty.\end{aligned}$$

The limit does not exist, hence the improper integral $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges. Compare the graphs in Figures 6.5.5 and 6.5.6; notice how the graph of $f(x) = 1/x$ is noticeably larger. This difference is enough to cause the improper integral to diverge.

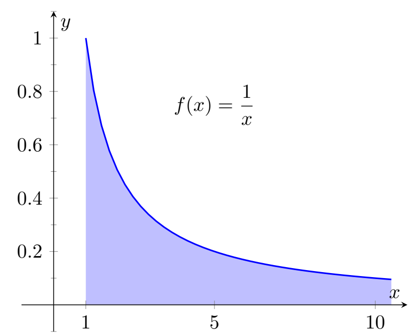

Figure 6.5.6 A graph of $f(x) = \frac{1}{x}$ in Example 6.5.4

3.

$$\begin{aligned}\int_{-\infty}^0 e^x\,dx &= \lim_{a\to-\infty}\int_a^0 e^x\,dx\\ &= \lim_{a\to-\infty} e^x\,\bigg|_a^0\\ &= \lim_{a\to-\infty} e^0 - e^a\\ &= 1.\end{aligned}$$

A graph of the area defined by this integral is given in Figure 6.5.7.

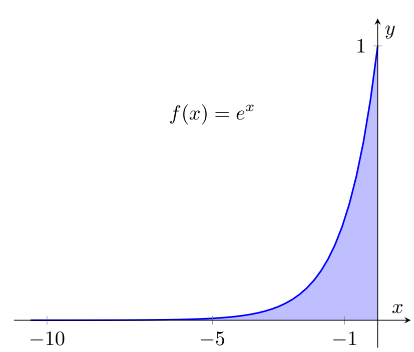

Figure 6.5.7 A graph of $f(x) = e^x$ in Example 6.5.4

4. We will need to break this into two improper integrals and choose a value of c as in part 3 of Definition 6.5.3. Any value of c is fine; we choose $c = 0$.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{a\to-\infty}\int_a^0 \frac{1}{1+x^2}\,dx + \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx\\ &= \lim_{a\to-\infty} \tan^{-1}(x)\Big|_a^0 + \lim_{b\to\infty} \tan^{-1}(x)\Big|_0^b\\ &= \lim_{a\to-\infty}\left(\tan^{-1}(0) - \tan^{-1}(a)\right)\\ &\quad + \lim_{b\to\infty}\left(\tan^{-1}(b) - \tan^{-1}(0)\right)\\ &= \left(0 - \frac{-\pi}{2}\right) + \left(\frac{\pi}{2} - 0\right).\end{aligned}$$

Each limit exists, hence the original integral converges and has value:

$$= \pi.$$

A graph of the area defined by this integral is given in Figure 6.5.8.

Video solution

youtu.be/watch?v=hXnmu7fZj-E

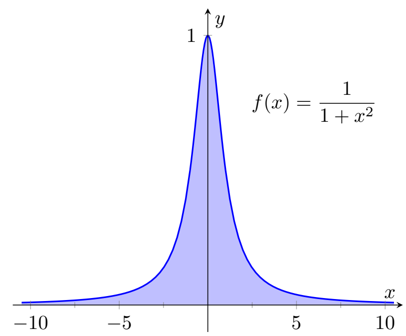

Figure 6.5.8 A graph of $f(x) = \frac{1}{1+x^2}$ in Example 6.5.4

The previous section introduced L'Hospital's Rule, a method of evaluating limits that return indeterminate forms. It is not uncommon for the limits resulting from improper integrals to need this rule as demonstrated next.

Example 6.5.9 Improper integration and L'Hospital's Rule.

Evaluate the improper integral $\displaystyle\int_1^{\infty} \frac{\ln(x)}{x^2}\,dx$.

Solution. This integral will require the use of Integration by Parts. Let $u = \ln(x)$ and $dv = 1/x^2\,dx$. Then

$$\begin{aligned}\int_1^{\infty} \frac{\ln(x)}{x^2}\,dx &= \lim_{b\to\infty}\int_1^b \frac{\ln(x)}{x^2}\,dx\\ &= \lim_{b\to\infty}\left(-\frac{\ln(x)}{x}\Big|_1^b + \int_1^b \frac{1}{x^2}\,dx\right)\\ &= \lim_{b\to\infty}\left(-\frac{\ln(x)}{x} - \frac{1}{x}\right)\Big|_1^b\\ &= \lim_{b\to\infty}\left(-\frac{\ln(b)}{b} - \frac{1}{b} - (-\ln(1) - 1)\right).\end{aligned}$$

The $1/b$ and $\ln(1)$ terms go to 0, leaving $\lim_{b\to\infty} -\frac{\ln(b)}{b} + 1$. We need to evaluate $\lim_{b\to\infty} \frac{\ln(b)}{b}$ with l'Hospital's Rule. We have:

$$\begin{aligned}\lim_{b\to\infty}\frac{\ln(b)}{b} &\overset{\text{by LHR}}{=} \lim_{b\to\infty}\frac{1/b}{1}\\ &= 0.\end{aligned}$$

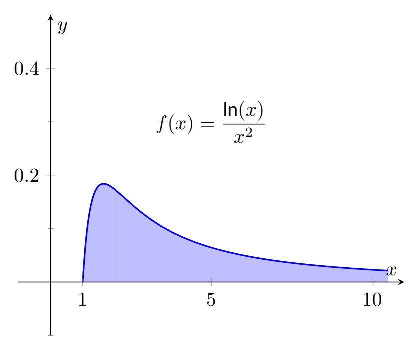

Figure 6.5.10 A graph of $f(x) = \frac{\ln(x)}{x^2}$ in Example 6.5.9

Thus the improper integral evaluates as:

$$\int_1^\infty \frac{\ln(x)}{x^2}\, dx = 1.$$

Video solution

youtu.be/watch?v=hKpXU3fFS6g

6.5.2 Improper Integrals with Infinite Range

We have just considered definite integrals where the interval of integration was infinite. We now consider another type of improper integration, where the range of the integrand is infinite.

Definition 6.5.11 Improper Integration with Infinite Range.

Let $f(x)$ be a continuous function on $[a, b]$ except at c, $a \leq c \leq b$, where $x = c$ is a vertical asymptote of f. Define

$$\int_a^b f(x)\, dx = \lim_{t\to c^-} \int_a^t f(x)\, dx + \lim_{t\to c^+} \int_t^b f(x)\, dx.$$

In Definition 6.5.11, c can be one of the endpoints (a or b). In that case, there is only one limit to consider as part of the definition.

Example 6.5.12 Improper integration of functions with infinite range.

Evaluate the following improper integrals:

1. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\, dx$

2. $\displaystyle\int_{-1}^1 \frac{1}{x^2}\, dx$

Solution.

1. A graph of $f(x) = 1/\sqrt{x}$ is given in Figure 6.5.13. Notice that f has a vertical asymptote at $x = 0$; in some sense, we are trying to compute the area of a region that has no "top." Could this have a finite value?

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\, dx &= \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\, dx\\ &= \lim_{a\to 0^+} 2\sqrt{x}\Big|_a^1\\ &= \lim_{a\to 0^+} 2\left(\sqrt{1} - \sqrt{a}\right)\\ &= 2.\end{aligned}$$

It turns out that the region does have a finite area even though it has no upper bound (strange things can occur in mathematics when considering the infinite).

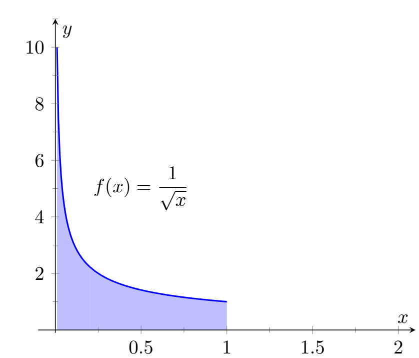

Figure 6.5.13 A graph of $f(x) = \frac{1}{\sqrt{x}}$ in Example 6.5.12

2. The function $f(x) = 1/x^2$ has a vertical asymptote at $x = 0$, as shown in Figure 6.5.14, so this integral is an improper integral. Let's eschew using limits for a moment and proceed without recognizing the improper nature of the integral. This leads to:

$$\int_{-1}^1 \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_{-1}^1$$

$$= -1 - (1)$$
$$= -2. \,(!)$$

Clearly the area in question is above the x-axis, yet the area is supposedly negative! Why does our answer not match our intuition? To answer this, evaluate the integral using Definition 6.5.11.

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{x^2}\,dx &= \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{x^2}\,dx + \lim_{t\to 0^+} \int_{t}^{1} \frac{1}{x^2}\,dx \\
&= \lim_{t\to 0^-} -\frac{1}{x}\bigg|_{-1}^{t} + \lim_{t\to 0^+} -\frac{1}{x}\bigg|_{t}^{1} \\
&= \lim_{t\to 0^-} -\frac{1}{t} - 1 + \lim_{t\to 0^+} -1 + \frac{1}{t} \\
&\Rightarrow \Big(\infty - 1\Big) + \Big(-1 + \infty\Big).
\end{aligned}$$

Neither limit converges hence the original improper integral diverges. The nonsensical answer we obtained by ignoring the improper nature of the integral is just that: nonsensical.

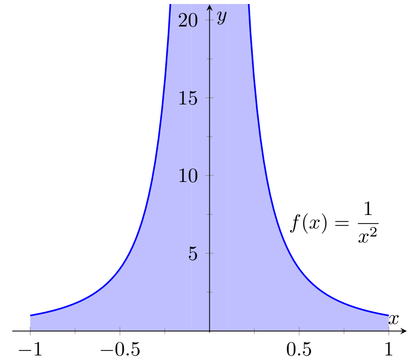

Figure 6.5.14 A graph of $f(x) = \frac{1}{x^2}$ in Example 6.5.12

Video solution

youtu.be/watch?v=F46oIXOBjAw

6.5.3 Understanding Convergence and Divergence

Oftentimes we are interested in knowing simply whether or not an improper integral converges, and not necessarily the value of a convergent integral. We provide here several tools that help determine the convergence or divergence of improper integrals without integrating.

Our first tool is to understand the behavior of functions of the form $\dfrac{1}{x^p}$.

Example 6.5.15 Improper integration of $1/x^p$.

Determine the values of p for which $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges.

Solution. We begin by integrating and then evaluating the limit.

$$\begin{aligned}
\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx \\
&= \lim_{b\to\infty} \int_1^b x^{-p}\,dx \qquad \text{(assume } p \neq 1\text{)} \\
&= \lim_{b\to\infty} \frac{1}{-p+1} x^{-p+1}\bigg|_1^b \\
&= \lim_{b\to\infty} \frac{1}{1-p}\big(b^{1-p} - 1^{1-p}\big).
\end{aligned}$$

When does this limit converge — i.e., when is this limit *not* ∞? This limit converges precisely when the power of b is less than 0: when $1 - p < 0 \Rightarrow 1 < p$.

Our analysis shows that if $p > 1$, then $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges. When $p < 1$ the improper integral diverges; we showed in Example 6.5.4 that when $p = 1$ the integral also diverges.

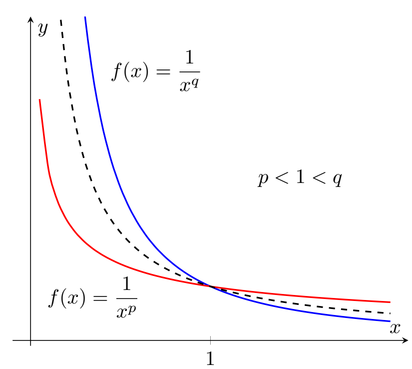

Figure 6.5.16 Plotting functions of the form $1/x^p$ in Example 6.5.15

Figure 6.5.16 graphs $y = 1/x$ with a dashed line, along with graphs of $y = 1/x^p, p < 1$, and $y = 1/x^q, q > 1$. Somehow the dashed line forms a dividing line between convergence and divergence.

Video solution

youtu.be/watch?v=-W8yESqiexA

The result of Example 6.5.15 provides an important tool in determining the convergence of other integrals. A similar result is proved in the exercises about improper integrals of the form $\int_0^1 \frac{1}{x^p}\,dx$. These results are summarized in the following Key Idea.

Key Idea 6.5.17 Convergence of Improper Integrals involving $1/x^p$.

1. The improper integral $\int_1^\infty \frac{1}{x^p}\,dx$ converges when $p > 1$ and diverges when $p \leq 1$.
2. The improper integral $\int_0^1 \frac{1}{x^p}\,dx$ converges when $p < 1$ and diverges when $p \geq 1$.

We used the upper and lower bound of "1" in Key Idea 6.5.17 for convenience. It can be replaced by any a where $a > 0$.

A basic technique in determining convergence of improper integrals is to compare an integrand whose convergence is unknown to an integrand whose convergence is known. We often use integrands of the form $1/x^p$ to compare to as their convergence on certain intervals is known. This is described in the following theorem.

Theorem 6.5.18 Direct Comparison Test for Improper Integrals.

Let f and g be continuous on $[a, \infty)$ where $0 \leq f(x) \leq g(x)$ for all x in $[a, \infty)$.

1. *If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.*
2. *If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.*

Example 6.5.19 Determining convergence of improper integrals.

Determine the convergence of the following improper integrals.

1. $\int_1^\infty e^{-x^2}\,dx$
2. $\int_3^\infty \frac{1}{\sqrt{x^2 - x}}\,dx$

Solution.

1. The function $f(x) = e^{-x^2}$ does not have an antiderivative expressible in terms of elementary functions, so we cannot integrate directly. It is comparable to $g(x) = 1/x^2$, and as demonstrated in Figure 6.5.20, $e^{-x^2} < 1/x^2$ on $[1, \infty)$. We know from Key Idea 6.5.17 that $\int_1^\infty \frac{1}{x^2}\,dx$ converges, hence $\int_1^\infty e^{-x^2}\,dx$ also converges.

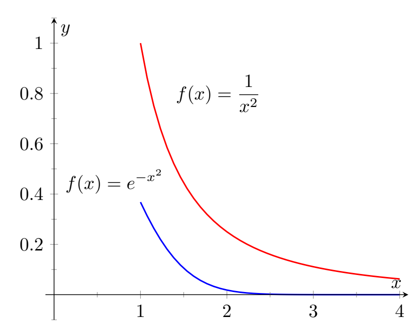

Figure 6.5.20 Graphs of $f(x) = e^{-x^2}$ and $f(x) = 1/x^2$ in Example 6.5.19

2. Note that for large values of x, $\dfrac{1}{\sqrt{x^2-x}} \approx \dfrac{1}{\sqrt{x^2}} = \dfrac{1}{x}$. We know from Key Idea 6.5.17 and the subsequent note that $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges, so we seek to compare the original integrand to $1/x$. It is easy to see that when $x > 0$, we have $x = \sqrt{x^2} > \sqrt{x^2-x}$. Taking reciprocals reverses the inequality, giving

$$\frac{1}{x} < \frac{1}{\sqrt{x^2-x}}.$$

Using Theorem 6.5.18, we conclude that since $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges, $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2-x}}\,dx$ diverges as well. Figure 6.5.21 illustrates this.

Video solution

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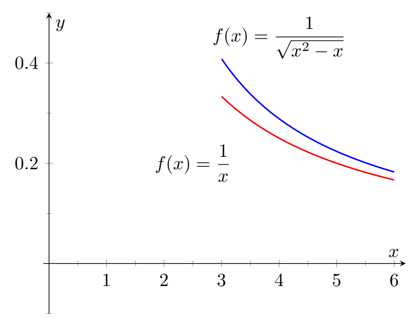


Figure 6.5.21 Graphs of $f(x) = 1/\sqrt{x^2-x}$ and $f(x) = 1/x$ in Example 6.5.19

Being able to compare "unknown" integrals to "known" integrals is very useful in determining convergence. However, some of our examples were a little "too nice." For instance, it was convenient that $\dfrac{1}{x} < \dfrac{1}{\sqrt{x^2-x}}$, but what if the "$-x$" were replaced with a "$+2x+5$"? That is, what can we say about the convergence of $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$? We have $\dfrac{1}{x} > \dfrac{1}{\sqrt{x^2+2x+5}}$, so we cannot use Theorem 6.5.18.

In cases like this (and many more) it is useful to employ the following theorem.

Theorem 6.5.22 Limit Comparison Test for Improper Integrals.

Let f and g be continuous functions on $[a,\infty)$ where $f(x) > 0$ and $g(x) > 0$ for all x. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \text{ and } \int_a^\infty g(x)\,dx$$

either both converge or both diverge.

Example 6.5.23 Determining convergence of improper integrals.

Determine the convergence of $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$.

Solution. As x gets large, the denominator of the integrand will begin to behave much like $y = x$. So we compare $\dfrac{1}{\sqrt{x^2+2x+5}}$ to $\dfrac{1}{x}$ with the Limit Comparison Test:

$$\lim_{x\to\infty} \frac{1/\sqrt{x^2+2x+5}}{1/x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+2x+5}}.$$

The immediate evaluation of this limit returns ∞/∞, an indeterminate

form. Using L'Hospital's Rule seems appropriate, but in this situation, it does not lead to useful results. (We encourage the reader to employ L'Hospital's Rule at least once to verify this.)
The trouble is the square root function. To get rid of it, we employ the following fact: If $\lim_{x\to c} f(x) = L$, then $\lim_{x\to c} f(x)^2 = L^2$. (This is true when either c or L is ∞.) So we consider now the limit

$$\lim_{x\to\infty} \frac{x^2}{x^2+2x+5}.$$

This converges to 1, meaning the original limit also converged to 1. As x gets very large, the function $\dfrac{1}{\sqrt{x^2+2x+5}}$ looks very much like $\dfrac{1}{x}$. Since we know that $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges, by the Limit Comparison Test we know that $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$ also diverges. Figure 6.5.24 graphs $f(x) = 1/\sqrt{x^2+2x+5}$ and $f(x) = 1/x$, illustrating that as x gets large, the functions become indistinguishable.

Video solution

youtu.be/watch?v=nIr1A1Tmako

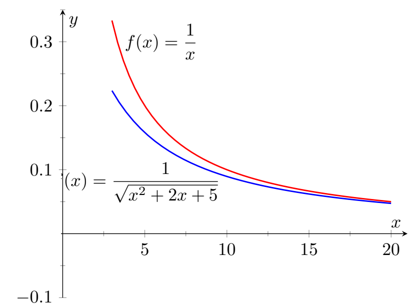


Figure 6.5.24 Graphing $f(x) = \frac{1}{\sqrt{x^2+2x+5}}$ and $f(x) = \frac{1}{x}$ in Example 6.5.23

Both the Direct and Limit Comparison Tests were given in terms of integrals over an infinite interval. There are versions that apply to improper integrals with an infinite range, but as they are a bit wordy and a little more difficult to employ, they are omitted from this text.

This chapter has explored many integration techniques. We learned Substitution, which "undoes" the Chain Rule of differentiation, as well as Integration by Parts, which "undoes" the Product Rule. We learned specialized techniques for handling trigonometric functions and introduced the hyperbolic functions, which are closely related to the trigonometric functions. All techniques effectively have this goal in common: rewrite the integrand in a new way so that the integration step is easier to see and implement.

As stated before, integration is, in general, hard. It is easy to write a function whose antiderivative is impossible to write in terms of elementary functions, and even when a function does have an antiderivative expressible by elementary functions, it may be really hard to discover what it is. The powerful computer algebra system *Mathematica*™ has approximately 1,000 pages of code dedicated to integration.

If you do need to use comparison for an improper integral with infinite range, it is generally wise to stick with direct comparison. Direct comparison will continue to work in more or less the way you expect; however, limit comparison is much more subtle, and prone to incorrect conclusions.

Do not let this difficulty discourage you. There is great value in learning integration techniques, as they allow one to manipulate an integral in ways that can illuminate a concept for greater understanding. There is also great value in understanding the need for good numerical techniques: the Trapezoidal and Simpson's Rules are just the beginning of powerful techniques for approximating the value of integration.

The next chapter stresses the uses of integration. We generally do not find antiderivatives for antiderivative's sake, but rather because they provide the solution to some type of problem. The following chapter introduces us to a number of different problems whose solution is provided by integration.

6.5.4 Exercises

Terms and Concepts

1. The definite integral was defined with what two stipulations?

2. If $\lim\limits_{b\to\infty} \int_0^b f(x)\,dx$ exists, then the integral $\displaystyle\int_0^\infty f(x)\,dx$ is said to .

3. If $\displaystyle\int_1^\infty f(x)\,dx = 10$, and $0 \leq g(x) \leq f(x)$ for all x, then we know that $\displaystyle\int_1^\infty g(x)\,dx$.

4. For what values of p will $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converge?

(a) $p < 1$

(b) $p \leq 1$

(c) $p > 1$

(d) $p \geq 1$

5. For what values of p will $\displaystyle\int_{10}^\infty \frac{1}{x^p}\,dx$ converge?

(a) $p < 1$

(b) $p \leq 1$

(c) $p > 1$

(d) $p \geq 1$

6. For what values of p will $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converge?

(a) $p < 1$

(b) $p \leq 1$

(c) $p > 1$

(d) $p \geq 1$

Problems

Exercise Group. In the following exercises, evaluate the given improper integral.

7. $\displaystyle\int_0^\infty e^{5-2x}\,dx$

8. $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$

9. $\displaystyle\int_1^\infty x^{-4}\,dx$

10. $\displaystyle\int_{-\infty}^\infty \frac{1}{x^2+9}\,dx$

11. $\displaystyle\int_{-\infty}^0 2^x\,dx$

12. $\displaystyle\int_{-\infty}^0 0.5^x\,dx$

13. $\displaystyle\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$

14. $\displaystyle\int_3^\infty \frac{x}{x^2-4}\,dx$

15. $\displaystyle\int_2^\infty \frac{1}{(x-1)^2}\,dx$

16. $\displaystyle\int_1^2 \frac{1}{(x-1)^2}\,dx$

17. $\displaystyle\int_2^\infty \frac{1}{x-1}\,dx$

18. $\displaystyle\int_1^2 \frac{1}{x-1}\,dx$

19. $\displaystyle\int_{-1}^1 \frac{1}{x}\,dx$

20. $\displaystyle\int_1^3 \frac{1}{x-2}\,dx$

21. $\displaystyle\int_0^\pi \sec^2(x)\,dx$

22. $\displaystyle\int_{-2}^1 \frac{1}{\sqrt{|x|}}\,dx$

23. $\displaystyle\int_0^\infty xe^{-x}\,dx$

24. $\displaystyle\int_0^\infty xe^{-x^2}\,dx$

25. $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\,dx$

26. $\displaystyle\int_{-\infty}^\infty \frac{1}{e^x+e^{-x}}\,dx$

27. $\displaystyle\int_0^1 x\ln(x)\,dx$

28. $\displaystyle\int_0^1 x^2\ln(x)\,dx$

29. $\displaystyle\int_1^\infty \frac{\ln(x)}{x}\,dx$

30. $\displaystyle\int_0^1 \ln(x)\,dx$

31. $\displaystyle\int_1^\infty \frac{\ln(x)}{x^2}\,dx$

32. $\displaystyle\int_1^\infty \frac{\ln(x)}{\sqrt{x}}\,dx$

33. $\displaystyle\int_0^\infty e^{-x}\sin(x)\,dx$

34. $\displaystyle\int_0^\infty e^{-x}\cos(x)\,dx$

Exercise Group. In the following exercises, use the Direct Comparison Test or the Limit Comparison Test to determine whether the given definite integral converges or diverges. Clearly state what test is being used and what function the integrand is being compared to.

35. $\displaystyle\int_{10}^\infty \frac{3}{\sqrt{3x^2+2x-5}}\,dx$

36. $\displaystyle\int_2^\infty \frac{4}{\sqrt{7x^3-x}}\,dx$

37. $\displaystyle\int_0^\infty \frac{\sqrt{x+3}}{\sqrt{x^3-x^2+x+1}}\,dx$

38. $\displaystyle\int_1^\infty e^{-x}\ln(x)\,dx$

39. $\displaystyle\int_5^\infty e^{-x^2+3x+1}\,dx$

40. $\displaystyle\int_0^\infty \frac{\sqrt{x}}{e^x}\,dx$

41. $\displaystyle\int_2^\infty \frac{1}{x^2+\sin(x)}\,dx$

42. $\displaystyle\int_0^\infty \frac{x}{x^2+\cos(x)}\,dx$

43. $\displaystyle\int_0^\infty \frac{1}{x+e^x}\,dx$

44. $\displaystyle\int_0^\infty \frac{1}{e^x-x}\,dx$

Chapter 7

Applications of Integration

We begin this chapter with a reminder of a few key concepts from Chapter 5. Let f be a continuous function on $[a, b]$ which is partitioned into n equally spaced subintervals as

$$a = x_0 < x_1 < \cdots < x_n < x_n = b.$$

Let $\Delta x = (b-a)/n$ denote the length of the subintervals, and let c_i be any x-value in the ith subinterval. Definition 5.3.17 states that the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x$$

is a *Riemann Sum*. Riemann Sums are often used to approximate some quantity (area, volume, work, pressure, etc.). The *approximation* becomes *exact* by taking the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$

Theorem 5.3.26 connects limits of Riemann Sums to definite integrals:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\,dx.$$

Finally, the Fundamental Theorem of Calculus states how definite integrals can be evaluated using antiderivatives.

This chapter employs the following technique to a variety of applications. Suppose the value Q of a quantity is to be calculated. We first approximate the value of Q using a Riemann Sum, then find the exact value via a definite integral. We spell out this technique in the following Key Idea.

Key Idea 7.0.1 Application of Definite Integrals Strategy.

Let a quantity be given whose value Q is to be computed.

1. Divide the quantity into n smaller "subquantities" of value Q_i.
2. Identify a variable x and function $f(x)$ such that each subquantity can be approximated with the product $f(c_i)\Delta x$, where Δx represents a small change in x. Thus $Q_i \approx f(c_i)\Delta x$. A sample approximation $f(c_i)\Delta x$ of Q_i is called a *differential element*.
3. Recognize that $Q = \sum_{i=1}^{n} Q_i \approx \sum_{i=1}^{n} f(c_i)\Delta x$, which is a Riemann Sum.

4. Taking the appropriate limit gives $Q = \displaystyle\int_a^b f(x)\,dx$

This Key Idea will make more sense after we have had a chance to use it several times. We begin with Area Between Curves, which we addressed briefly in Section 5.4.

7.1 Area Between Curves

youtu.be/watch?v=HAWBGypgKoQ

Figure 7.1.1 Video introduction to Section 7.1

We are often interested in knowing the area of a region. Forget momentarily that we addressed this already in Section 5.4 and approach it instead using the technique described in Key Idea 7.0.1.

Let Q be the area of a region bounded by continuous functions f and g. If we break the region into many subregions, we have an obvious equation:

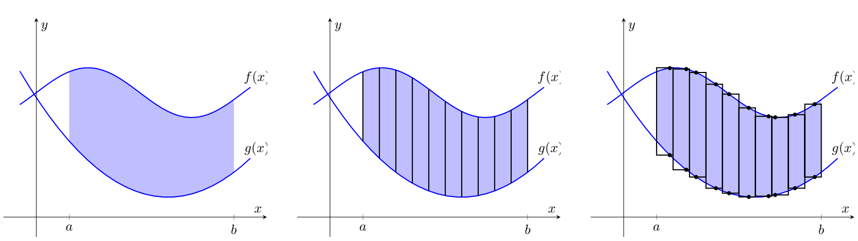
Total Area = sum of the areas of the subregions.

The issue to address next is how to systematically break a region into subregions. A graph will help. Consider Figure 7.1.2(a) where a region between two curves is shaded. While there are many ways to break this into subregions, one particularly efficient way is to “slice” it vertically, as shown in Figure 7.1.2(b), into n equally spaced slices.

We now approximate the area of a slice. Again, we have many options, but using a rectangle seems simplest. Picking any x-value c_i in the ith slice, we set the height of the rectangle to be $f(c_i) - g(c_i)$, the difference of the corresponding y-values. The width of the rectangle is a small difference in x-values, which we represent with Δx. Figure 7.1.2(c) shows sample points c_i chosen in each subinterval and appropriate rectangles drawn. (Each of these rectangles represents a differential element.) Each slice has an area approximately equal to $\big(f(c_i) - g(c_i)\big)\Delta x$; hence, the total area is approximately the Riemann Sum

$$Q = \sum_{i=1}^{n} \big(f(c_i) - g(c_i)\big)\Delta x.$$

Taking the limit as $n \to \infty$ gives the exact area as $\int_a^b \big(f(x) - g(x)\big)\,dx$.



(a) **(b)** **(c)**

Figure 7.1.2 Subdividing a region into vertical slices and approximating the areas with rectangles

Theorem 7.1.3 Area Between Curves (restatement of Theorem 5.4.23).

Let $f(x)$ and $g(x)$ be continuous functions defined on $[a, b]$ where $f(x) \geq g(x)$ for all x in $[a, b]$. The area of the region bounded by the *curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is*

$$\int_a^b \big(f(x) - g(x)\big)\,dx.$$

Example 7.1.4 Finding area enclosed by curves.

Find the area of the region bounded by $f(x) = \sin(x) + 2$, $g(x) = \frac{1}{2}\cos(2x) - 1$, $x = 0$ and $x = 4\pi$, as shown in Figure 7.1.5.

Solution. The graph verifies that the upper boundary of the region is given by f and the lower bound is given by g. Therefore the area of the region is the value of the integral

$$\begin{aligned}\int_0^{4\pi} \big(f(x) - g(x)\big)\,dx &= \int_0^{4\pi} \Big(\sin(x) + 2 - \big(\tfrac{1}{2}\cos(2x) - 1\big)\Big)\,dx \\ &= -\cos(x) - \frac{1}{4}\sin(2x) + 3x\Big|_0^{4\pi} \\ &= 12\pi \approx 37.7\,\text{units}^2.\end{aligned}$$

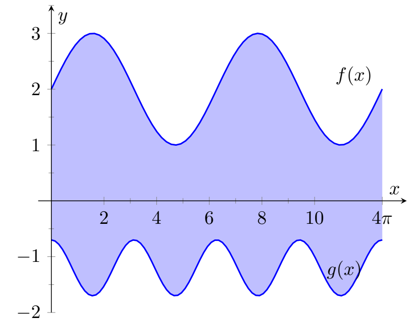


Figure 7.1.5 Graphing an enclosed region in Example 7.1.4

Video solution

youtu.be/watch?v=qbhOWW70UyM

Example 7.1.6 Finding total area enclosed by curves.

Find the total area of the region enclosed by the functions $f(x) = -2x + 5$ and $g(x) = x^3 - 7x^2 + 12x - 3$ as shown in Figure 7.1.7.

Solution. A quick calculation shows that $f = g$ at $x = 1, 2$ and 4. One can proceed thoughtlessly by computing $\int_1^4 \big(f(x) - g(x)\big)\,dx$, but this ignores the fact that on $[1, 2]$, $g(x) > f(x)$. (In fact, the thoughtless integration returns $-9/4$, hardly the expected value of an *area*.) Thus we compute the total area by breaking the interval $[1, 4]$ into two subintervals, $[1, 2]$ and $[2, 4]$ and using the proper integrand in each.

$$\begin{aligned}\text{Total Area} &= \int_1^2 \big(g(x) - f(x)\big)\,dx + \int_2^4 \big(f(x) - g(x)\big)\,dx \\ &= \int_1^2 \big(x^3 - 7x^2 + 14x - 8\big)\,dx + \int_2^4 \big(-x^3 + 7x^2 - 14x + 8\big)\,dx \\ &= 5/12 + 8/3 \\ &= 37/12 = 3.083\,\text{units}^2.\end{aligned}$$

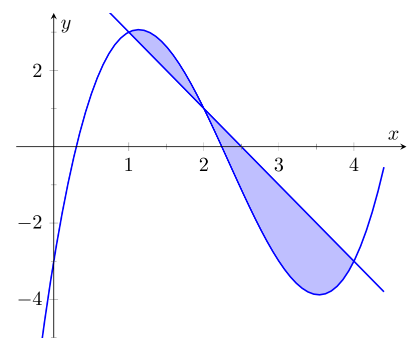


Figure 7.1.7 Graphing a region enclosed by two functions in Example 7.1.6

The previous example makes note that we are expecting area to be *positive*. When first learning about the definite integral, we interpreted it as "signed area under the curve," allowing for "negative area." That doesn't apply here; area is to be positive.

The previous example also demonstrates that we often have to break a given region into subregions before applying Theorem 7.1.3. The following example shows another situation where this is applicable, along with an alternate view of applying the Theorem.

Video solution

youtu.be/watch?v=4wap7fFasZk

Example 7.1.8 Finding area: integrating with respect to y.

Find the area of the region enclosed by the functions $y = \sqrt{x}+2$, $y = -(x-1)^2+3$ and $y=2$, as shown in Figure 7.1.9.

Solution. We give two approaches to this problem. In the first approach, we notice that the region's "top" is defined by two different curves. On $[0,1]$, the top function is $y = \sqrt{x}+2$; on $[1,2]$, the top function is $y = -(x-1)^2+3$.

Thus we compute the area as the sum of two integrals:

$$\begin{aligned}\text{Total Area} &= \int_0^1 \Big(\big(\sqrt{x}+2\big)-2\Big)\,dx + \int_1^2 \Big(\big(-(x-1)^2+3\big)-2\Big)\,dx\\ &= 2/3+2/3\\ &= 4/3.\end{aligned}$$

The second approach is clever and very useful in certain situations. We are used to viewing curves as functions of x; we input an x-value and a y-value is returned. Some curves can also be described as functions of y: input a y-value and an x-value is returned. We can rewrite the equations describing the boundary by solving for x:

$$y = \sqrt{x}+2 \Rightarrow x = (y-2)^2$$

$$y = -(x-1)^2+3 \Rightarrow x = \sqrt{3-y}+1.$$

Figure 7.1.10 shows the region with the boundaries relabeled. A differential element, a horizontal rectangle, is also pictured. The width of the rectangle is a small change in y: Δy. The height of the rectangle is a difference in x-values. The "top" x-value is the largest value, i.e., the rightmost. The "bottom" x-value is the smaller, i.e., the leftmost. Therefore the height of the rectangle is

$$\big(\sqrt{3-y}+1\big)-(y-2)^2.$$

The area is found by integrating the above function with respect to y with the appropriate bounds. We determine these by considering the y-values the region occupies. It is bounded below by $y=2$, and bounded above by $y=3$. That is, both the "top" and "bottom" functions exist on the y interval $[2,3]$. Thus

$$\begin{aligned}\text{Total Area} &= \int_2^3 \big(\sqrt{3-y}+1-(y-2)^2\big)\,dy\\ &= \Big(-\frac{2}{3}(3-y)^{3/2}+y-\frac{1}{3}(y-2)^3\Big)\Big|_2^3\\ &= 4/3.\end{aligned}$$

This calculus-based technique of finding area can be useful even with shapes that we normally think of as "easy." Example 7.1.11 computes the area of a triangle. While the formula "$\frac{1}{2}$ × base × height" is well known, in arbitrary triangles it can be nontrivial to compute the height. Calculus makes the problem simple.

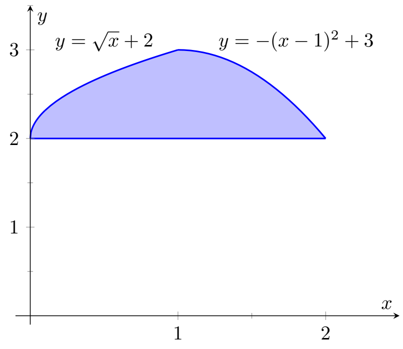


Figure 7.1.9 Graphing a region for Example 7.1.8

Video solution

youtu.be/watch?v=GhdqEHPbPm0

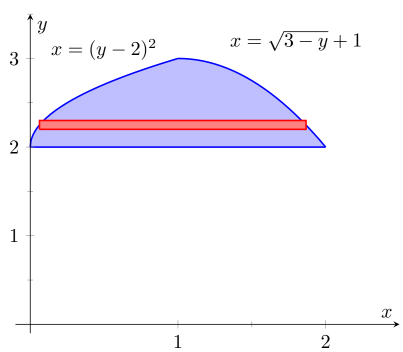


Figure 7.1.10 The region used in Example 7.1.8 with boundaries relabeled as functions of y

Example 7.1.11 Finding the area of a triangle.

Compute the area of the regions bounded by the lines $y = x + 1$, $y = -2x + 7$ and $y = -\frac{1}{2}x + \frac{5}{2}$, as shown in Figure 7.1.12.

Solution. Recognize that there are two "top" functions to this region, causing us to use two definite integrals.

$$\begin{aligned}\text{Total Area} &= \int_1^2 \left((x+1) - (-\frac{1}{2}x + \frac{5}{2})\right) dx \\ &\quad + \int_2^3 \left((-2x+7) - (-\frac{1}{2}x + \frac{5}{2})\right) dx \\ &= 3/4 + 3/4 \\ &= 3/2.\end{aligned}$$

We can also approach this by converting each function into a function of y. This also requires 2 integrals, so there isn't really any advantage to doing so. We do it here for demonstration purposes.
The "top" function is always $x = \frac{7-y}{2}$ while there are two "bottom" functions. Being mindful of the proper integration bounds, we have

$$\begin{aligned}\text{Total Area} &= \int_1^2 \left(\frac{7-y}{2} - (5-2y)\right) dy + \int_2^3 \left(\frac{7-y}{2} - (y-1)\right) dy \\ &= 3/4 + 3/4 \\ &= 3/2.\end{aligned}$$

Of course, the final answer is the same. (It is interesting to note that the area of all 4 subregions used is 3/4. This is coincidental.)

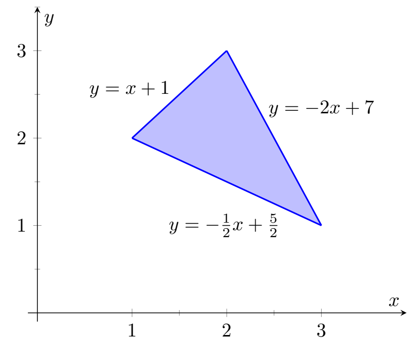


Figure 7.1.12 Graphing a triangular region in Example 7.1.11

While we have focused on producing exact answers, we are also able to make approximations using the principle of Theorem 7.1.3. The integrand in the theorem is a distance ("top minus bottom"); integrating this distance function gives an area. By taking discrete measurements of distance, we can approximate an area using numerical integration techniques developed in Section 5.5. The following example demonstrates this.

Example 7.1.13 Numerically approximating area.

To approximate the area of a lake, shown in Figure 7.1.14(a), the "length" of the lake is measured at 200-foot increments, as shown in Figure 7.1.14(b). The lengths are given in hundreds of feet. Approximate the area of the lake.

Video solution

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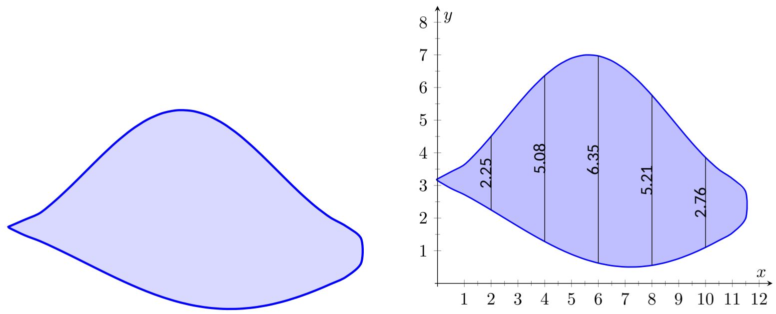


(a) **(b)**

Figure 7.1.14 (a) A sketch of a lake, and (b) the lake with length measurements

Solution. The measurements of length can be viewed as measuring "top minus bottom" of two functions. The exact answer is found by integrating $\int_0^{12} \big(f(x) - g(x)\big)\,dx$, but of course we don't know the functions f and g. Our discrete measurements instead allow us to approximate. We have the following data points:

$$(0, 0),\ (2, 2.25),\ (4, 5.08),\ (6, 6.35),\ (8, 5.21),\ (10, 2.76),\ (12, 0).$$

We also have that $\Delta x = \frac{b-a}{n} = 2$, so Simpson's Rule gives

$$\begin{aligned}\text{Area} &\approx \frac{2}{3}\Big(1\cdot 0 + 4\cdot 2.25 + 2\cdot 5.08 + 4\cdot 6.35 + 2\cdot 5.21 + 4\cdot 2.76 + 1\cdot 0\Big)\\ &= 44.01\overline{3}\ \text{units}^2.\end{aligned}$$

Since the measurements are in hundreds of feet, square units are given by $(100\text{ ft})^2 = 10,000\text{ ft}^2$, giving a total area of $440,133\text{ ft}^2$. (Since we are approximating, we'd likely say the area was about $440,000\text{ ft}^2$, which is a little more than 10 acres.)

In the next section we apply our applications of integration techniques to finding the volumes of certain solids.

7.1.1 Exercises

Terms and Concepts

1. The area between curves is always positive. (☐ True ☐ False)

2. Calculus can be used to find the area of basic geometric shapes. (☐ True ☐ False)

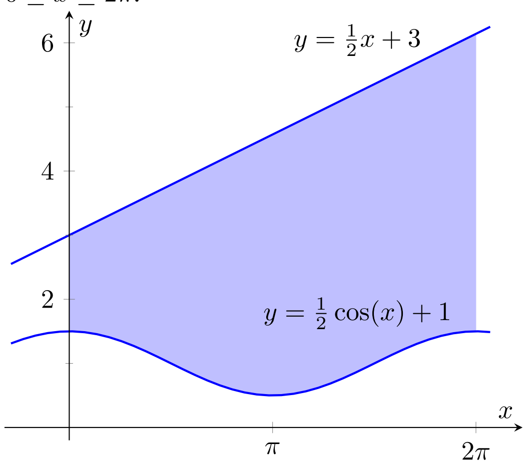
3. In your own words, describe how to find the total area enclosed by $y = f(x)$ and $y = g(x)$.

4. Describe a situation where it is advantageous to find an area enclosed by curves through integration with respect to y instead of x.

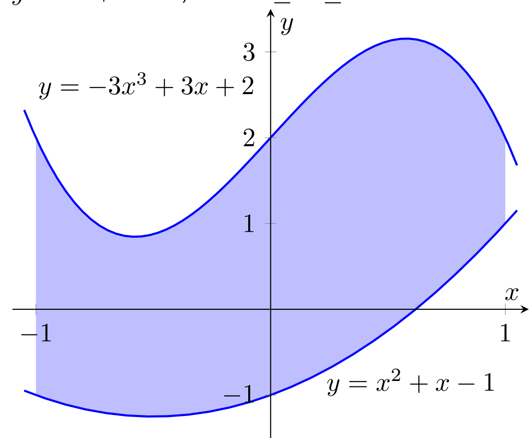
Problems

Exercise Group. In the following exercises, find the area of the shaded region in the given graph.

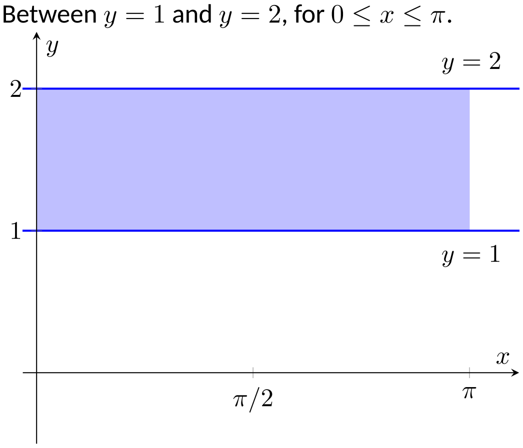
5. Between $y = \frac{1}{2}x + 3$ and $y = \frac{1}{2}\cos(x) + 1$, for $0 \leq x \leq 2\pi$.



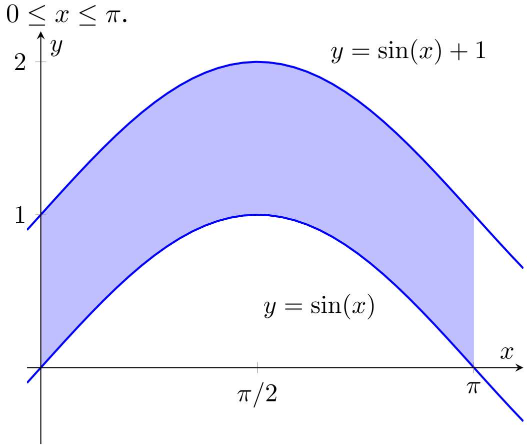
6. Between $y = -3x^3 + 3x + 2$ and $y = x^2 + x - 1$, for $-1 \leq x \leq 1$.



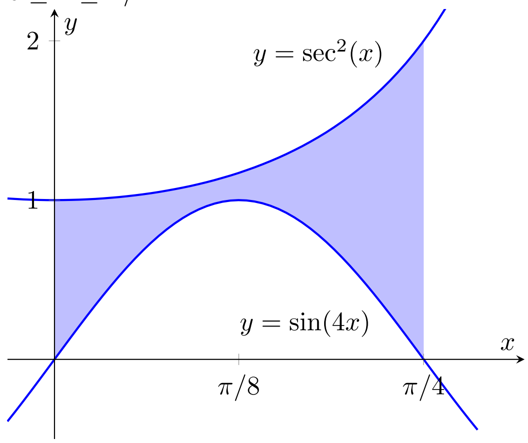
7. Between $y = 1$ and $y = 2$, for $0 \leq x \leq \pi$.



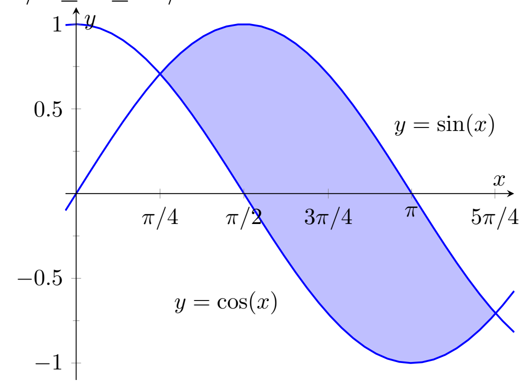
8. Between $y = \sin(x) + 1$ and $y = \sin(x)$, for $0 \leq x \leq \pi$.



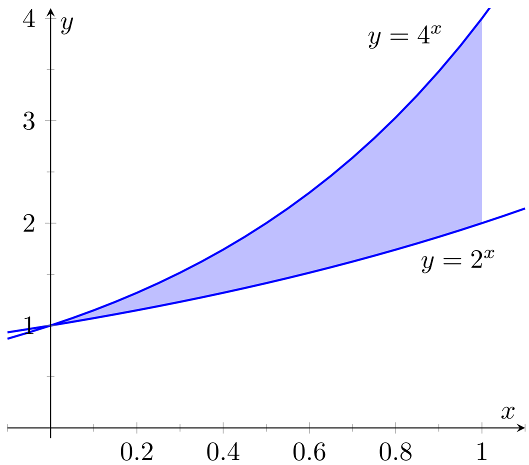
9. Between $y = \sin(4x)$ and $y = \sec^2(x)$, for $0 \leq x \leq \pi/4$.



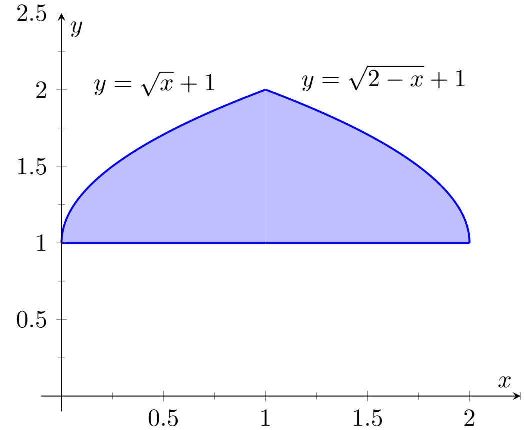
10. Between $y = \sin(x)$ and $y = \cos(x)$, for $\pi/4 \leq x \leq 5\pi/4$.



11. Between $y = 2^x$ and $y = 4^x$, for $0 \leq x \leq 1$.



12. Bounded by the curves $y = \sqrt{x} + 1$, $y = \sqrt{2 - x} + 1$, and $y = 1$.



Exercise Group. In the following exercises, find the total area enclosed by the functions f and g.

13. $f(x) = 2x^2 + 5x - 3$, $g(x) = x^2 + 4x - 1$

14. $f(x) = x^2 - 3x + 2$, $g(x) = -3x + 3$

15. $f(x) = \sin(x)$, $g(x) = 2x/\pi$

16. $f(x) = x^3 - 4x^2 + x - 1$, $g(x) = -x^2 + 2x - 4$

17. $f(x) = x$, $g(x) = \sqrt{x}$

18. $f(x) = -x^3 + 5x^2 + 2x + 1$, $g(x) = 3x^2 + x + 3$

19. The functions $f(x) = \cos(x)$ and $g(x) = \sin x$ intersect infinitely many times, forming an infinite number of repeated, enclosed regions. Find the areas of these regions.

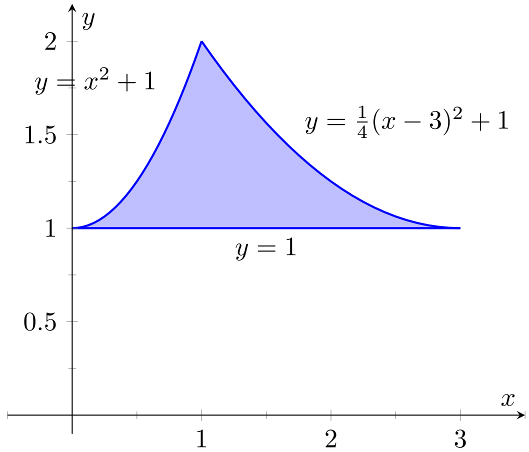
20. The functions $f(x) = \cos(2x)$ and $g(x) = \sin(x)$ intersect infinitely many times, forming an infinite number of repeated, enclosed regions. Find the areas of these regions.

Exercise Group. In the following exercises, find the area of the enclosed region in two ways:

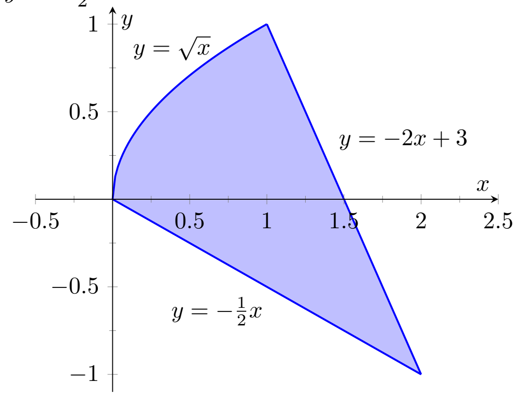
(a) by treating the boundaries as functions of x, and

(b) by treating the boundaries as functions of y.

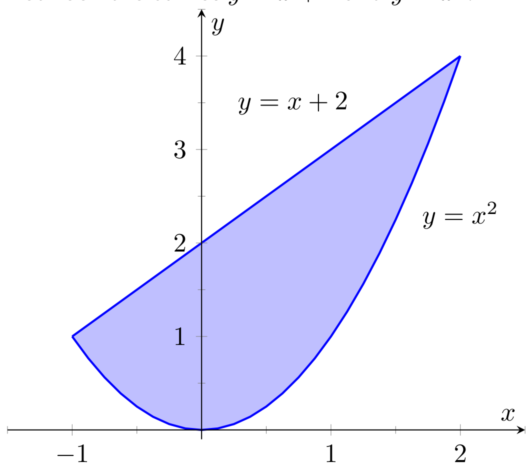
21. Bounded by $y = x^2 + 1$, $y = \frac{1}{4}(x-3)^2 + 1$, and $y = 1$.



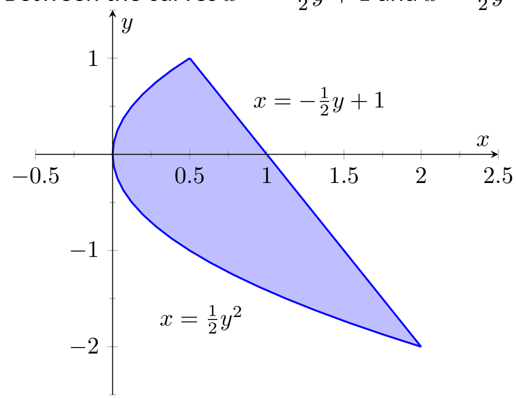
22. Bounded by $y = \sqrt{x}$, $y = -2x + 3$, and $y = -\frac{1}{2}x$.



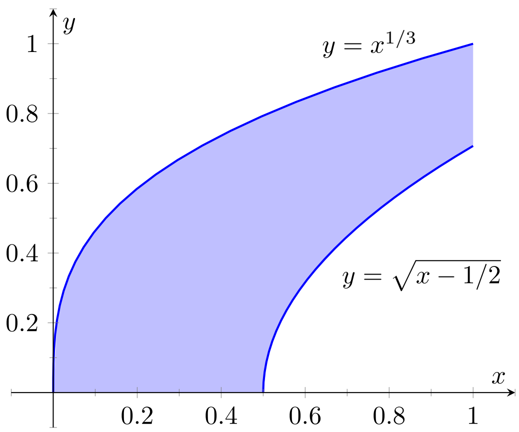
23. Between the curves $y = x + 2$ and $y = x^2$.



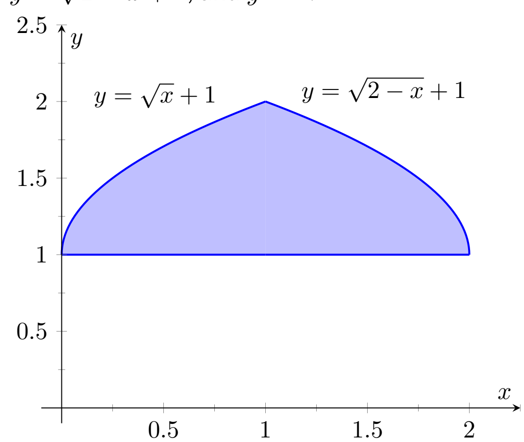
24. Between the curves $x = -\frac{1}{2}y + 1$ and $x = \frac{1}{2}y^2$.



25. Bounded by $y = x^{1/3}$, $y = \sqrt{x - 1/2}$, $y = 0$, and $x = 1$.



26. Bounded by the curves $y = \sqrt{x} + 1$, $y = \sqrt{2 - x} + 1$, and $y = 1$.



Exercise Group. In the following exercises, find the area of the triangle formed by the given three points.

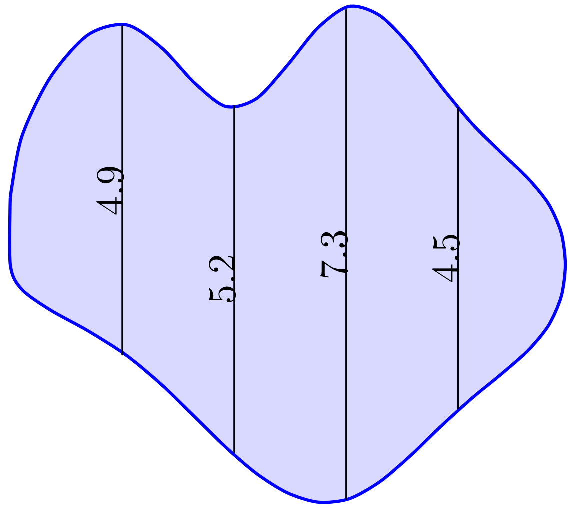
27. $(1, 1)$,$(2, 3)$, and $(3, 3)$

28. $(-1, 1)$,$(1, 3)$, and $(2, -1)$

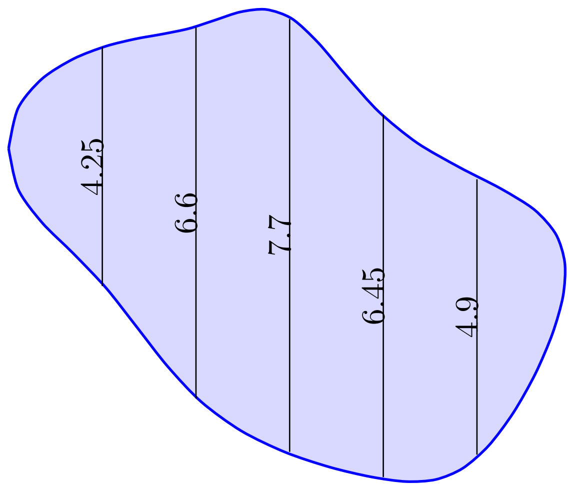
29. $(1, 1)$,$(3, 3)$, and $(0, 4)$

30. $(0, 0)$,$(2, 5)$, and $(5, 2)$

31. Use the Trapezoidal Rule to approximate the area of the pictured lake whose lengths, in hundreds of feet, are measured in 100-foot increments.



32. Use Simpson's Rule to approximate the area of the pictured lake whose lengths, in hundreds of feet, are measured in 200-foot increments.



7.2 Volume by Cross-Sectional Area; Disk and Washer Methods

The volume of a general right cylinder, as shown in Figure 7.2.1, is

Area of the base × height.

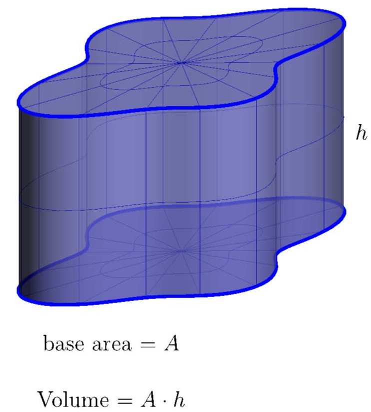


Figure 7.2.1 The volume of a general right cylinder

We can use this fact as the building block in finding volumes of a variety of shapes.

Given an arbitrary solid, we can *approximate* its volume by cutting it into n thin slices. When the slices are thin, each slice can be approximated well by a general right cylinder. Thus the volume of each slice is approximately its cross-sectional area × thickness. (These slices are the differential elements.)

By orienting a solid along the x-axis, we can let $A(x_i)$ represent the cross-sectional area of the ith slice, and let Δx_i represent the thickness of this slice (the thickness is a small change in x). The total volume of the solid is approximately:

$$\text{Volume} \approx \sum_{i=1}^{n} \Big[\text{Area} \times \text{thickness}\Big] = \sum_{i=1}^{n} A(x_i)\Delta x_i.$$

Recognize that this is a Riemann Sum. By taking a limit (as the thickness of the slices goes to 0) we can find the volume exactly.

youtu.be/watch?v=jx9XyQKDaP4

Figure 7.2.2 Video introduction to Section 7.2

Theorem 7.2.3 Volume By Cross-Sectional Area.

The volume V of a solid, oriented along the x-axis with cross-sectional area $A(x)$ from $x = a$ to $x = b$, is

$$V = \int_a^b A(x)\, dx.$$

Example 7.2.4 Finding the volume of a solid.

Find the volume of a pyramid with a square base of side length 10 in and a height of 5 in.

Solution. There are many ways to “orient” the pyramid along the x-axis; Figure 7.2.5 gives one such way, with the pointed top of the pyramid at the origin and the x-axis going through the center of the base.

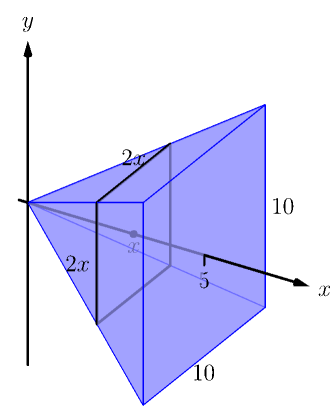


Figure 7.2.5 Orienting a pyramid along the x-axis in Example 7.2.4

Each cross section of the pyramid is a square; this is a sample differential element. To determine its area $A(x)$, we need to determine the side lengths of the square.

When $x = 5$, the square has side length 10; when $x = 0$, the square has side length 0. Since the edges of the pyramid are lines, it is easy to figure that each cross-sectional square has side length $2x$, giving $A(x) = (2x)^2 = 4x^2$.

If one were to cut a slice out of the pyramid at $x = 3$, as shown in Figure 7.2.6, one would have a shape with square bottom and top with sloped sides. If the slice were thin, both the bottom and top squares would have sides lengths of about 6, and thus the cross-sectional area of the bottom and top would be about $36\,\text{in}^2$. Letting Δx_i represent the thickness of the slice, the volume of this slice would then be about

$36\Delta x_i$ in^3.
Cutting the pyramid into n slices divides the total volume into n equally-spaced smaller pieces, each with volume $(2x_i)^2\Delta x$, where x_i is the approximate location of the slice along the x-axis and Δx represents the thickness of each slice. One can approximate total volume of the pyramid by summing up the volumes of these slices:

$$\text{Approximate volume} = \sum_{i=1}^{n}(2x_i)^2\Delta x.$$

Taking the limit as $n \to \infty$ gives the actual volume of the pyramid; recoginizing this sum as a Riemann Sum allows us to find the exact answer using a definite integral, matching the definite integral given by Theorem 7.2.3.
We have

$$\begin{aligned} V &= \lim_{n\to\infty}\sum_{i=1}^{n}(2x_i)^2\Delta x \\ &= \int_0^5 4x^2\,dx \\ &= \frac{4}{3}x^3\Big|_0^5 \\ &= \frac{500}{3}\text{ in}^3 \approx 166.67\text{ in}^3. \end{aligned}$$

We can check our work by consulting the general equation for the volume of a pyramid (see the back cover under "Volume of A General Cone"):
$\frac{1}{3}\times$ area of base $\times$ height.
Certainly, using this formula from geometry is faster than our new method, but the calculus-based method can be applied to much more than just cones.

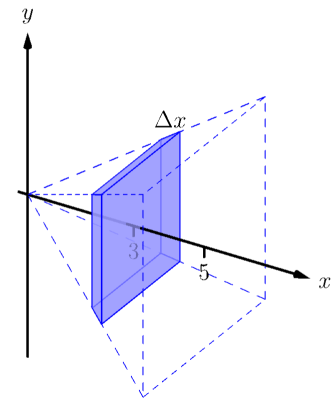


Figure 7.2.6 Cutting a slice in the pyramid in Example 7.2.4 at $x = 3$

Video solution

youtu.be/watch?v=JeQve79KVDE

An important special case of Theorem 7.2.3 is when the solid is a *solid of revolution*, that is, when the solid is formed by rotating a shape around an axis.

Start with a function $y = f(x)$ from $x = a$ to $x = b$. Revolving this curve about a horizontal axis creates a three-dimensional solid whose cross sections are disks (thin circles). Let $R(x)$ represent the radius of the cross-sectional disk at x; the area of this disk is $\pi R(x)^2$. Applying Theorem 7.2.3 gives the Disk Method.

Key Idea 7.2.7 The Disk Method.

Let a solid be formed by revolving the curve $y = f(x)$ from $x = a$ to $x = b$ around a horizontal axis, and let $R(x)$ be the radius of the cross-sectional disk at x. The volume of the solid is

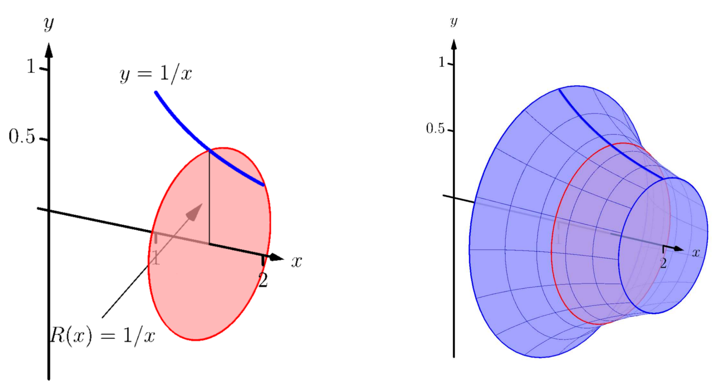
$$V = \pi\int_a^b R(x)^2\,dx.$$

Example 7.2.8 Finding volume using the Disk Method.

Find the volume of the solid formed by revolving the curve $y = 1/x$, from $x = 1$ to $x = 2$, around the x-axis.

Solution. A sketch can help us understand this problem. In Figure 7.2.9(a), the curve $y = 1/x$ is sketched along with the differential element — a disk — at x with radius $R(x) = 1/x$. In Figure 7.2.9(b) the whole solid is pictured, along with the differential element.

The volume of the differential element shown in Figure 7.2.9(a) is approximately $\pi R(x_i)^2 \Delta x$, where $R(x_i)$ is the radius of the disk shown and Δx is the thickness of that slice. The radius $R(x_i)$ is the distance from the x-axis to the curve, hence $R(x_i) = 1/x_i$.



(a) **(b)**

Figure 7.2.9 Sketching a solid in Example 7.2.8

Slicing the solid into n equally-spaced slices, we can approximate the total volume by adding up the approximate volume of each slice:

$$\text{Approximate volume} = \sum_{i=1}^{n} \pi \left(\frac{1}{x_i}\right)^2 \Delta x.$$

Taking the limit of the above sum as $n \to \infty$ gives the actual volume; recognizing this sum as a Riemann sum allows us to evaluate the limit with a definite integral, which matches the formula given in Key Idea 7.2.7:

$$\begin{aligned}
V &= \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left(\frac{1}{x_i}\right)^2 \Delta x\\
&= \pi \int_1^2 \left(\frac{1}{x}\right)^2 dx\\
&= \pi \int_1^2 \frac{1}{x^2}\, dx\\
&= \pi \left[-\frac{1}{x}\right]\Big|_1^2\\
&= \pi \left[-\frac{1}{2} - (-1)\right]\\
&= \frac{\pi}{2}\ \text{units}^3.
\end{aligned}$$

Video solution

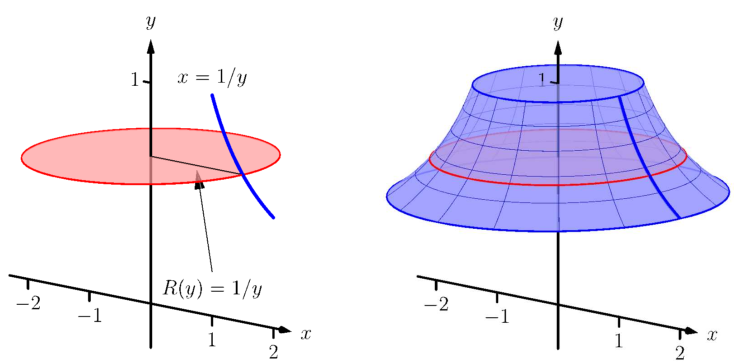
youtu.be/watch?v=_a79nyOUVTg

While Key Idea 7.2.7 is given in terms of functions of x, the principle involved can be applied to functions of y when the axis of rotation is vertical, not horizontal. We demonstrate this in the next example.

Example 7.2.10 Finding volume using the Disk Method.

Find the volume of the solid formed by revolving the curve $y = 1/x$, from $x = 1$ to $x = 2$, about the y-axis.

Solution. Since the axis of rotation is vertical, we need to convert the function into a function of y and convert the x-bounds to y-bounds. Since $y = 1/x$ defines the curve, we rewrite it as $x = 1/y$. The bound $x = 1$ corresponds to the y-bound $y = 1$, and the bound $x = 2$ corresponds to the y-bound $y = 1/2$.
Thus we are rotating the curve $x = 1/y$, from $y = 1/2$ to $y = 1$ about the y-axis to form a solid. The curve and sample differential element are sketched in Figure 7.2.11(a), with a full sketch of the solid in Figure 7.2.11(b).



(a) **(b)**

Figure 7.2.11 Sketching a solid in Example 7.2.10

We integrate to find the volume:

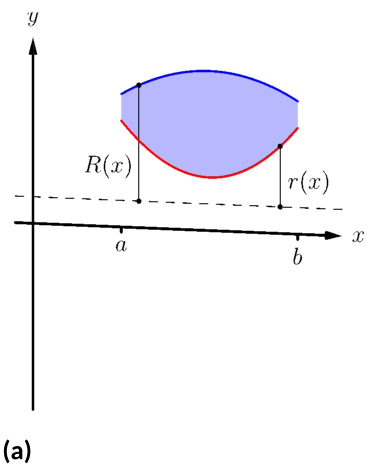
$$\begin{aligned} V &= \pi \int_{1/2}^{1} \frac{1}{y^2}\, dy \\ &= -\frac{\pi}{y}\Big|_{1/2}^{1} \\ &= \pi \text{ units}^3. \end{aligned}$$

Video solution

youtu.be/watch?v=k9vdYxWD8xc

We can also compute the volume of solids of revolution that have a hole in the center. The general principle is simple: compute the volume of the solid irrespective of the hole, then subtract the volume of the hole. If the outside radius of the solid is $R(x)$ and the inside radius (defining the hole) is $r(x)$, then the volume is

$$V = \pi \int_a^b R(x)^2\, dx - \pi \int_a^b r(x)^2\, dx = \pi \int_a^b \left(R(x)^2 - r(x)^2\right)\, dx.$$



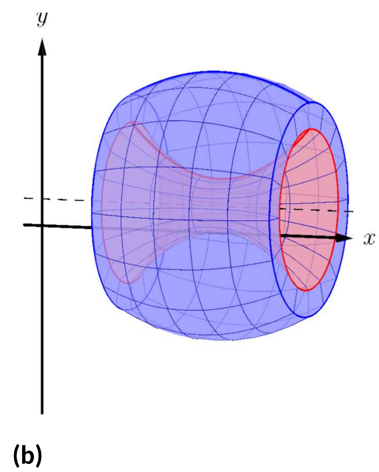


Figure 7.2.12 Establishing the Washer Method; see also Figure 7.2.13

One can generate a solid of revolution with a hole in the middle by revolving a region about an axis. Consider Figure 7.2.12(a), where a region is sketched along with a dashed, horizontal axis of rotation. By rotating the region about the axis, a solid is formed as sketched in Figure 7.2.12(b). The outside of the solid has radius $R(x)$, whereas the inside has radius $r(x)$. Each cross section of this solid will be a washer (a disk with a hole in the center) as sketched in Figure 7.2.13. This leads us to the Washer Method.

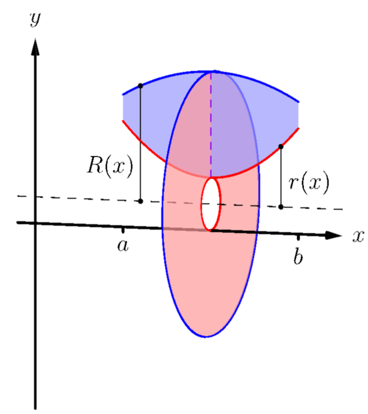


Figure 7.2.13 Establishing the Washer Method; see also Figure 7.2.12

Key Idea 7.2.14 The Washer Method.

Let a region bounded by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ be rotated about a horizontal axis that does not intersect the region, forming a solid. Each cross section at x will be a washer with outside radius $R(x)$ and inside radius $r(x)$. The volume of the solid is

$$V = \pi \int_a^b \Big(R(x)^2 - r(x)^2\Big)\, dx.$$

Even though we introduced it first, the Disk Method is just a special case of the Washer Method with an inside radius of $r(x) = 0$.

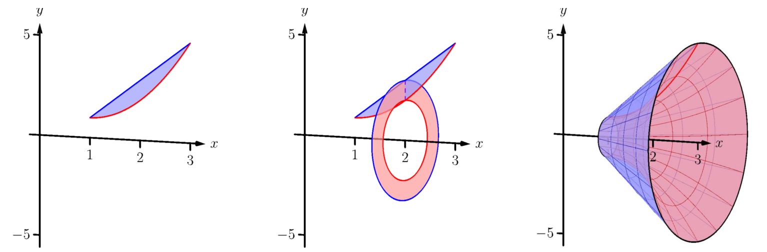
Example 7.2.15 Finding volume with the Washer Method.

Find the volume of the solid formed by rotating the region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$ about the x-axis.

Solution. A sketch of the region will help, as given in Figure 7.2.16(a). Rotating about the x-axis will produce cross sections in the shape of washers, as shown in Figure 7.2.16(b); the complete solid is shown in Figure 7.2.16(c). The outside radius of this washer is $R(x) = 2x - 1$; the inside radius is $r(x) = x^2 - 2x + 2$. As the region is bounded from $x = 1$ to $x = 3$, we integrate as follows to compute the volume.

$$\begin{aligned} V &= \pi \int_1^3 \Big((2x-1)^2 - (x^2-2x+2)^2\Big)\, dx \\ &= \pi \int_1^3 \big(-x^4 + 4x^3 - 4x^2 + 4x - 3\big)\, dx \\ &= \pi \Big[-\frac{1}{5}x^5 + x^4 - \frac{4}{3}x^3 + 2x^2 - 3x\Big]\Big|_1^3 \end{aligned}$$

$$= \frac{104}{15}\pi \approx 21.78\,\text{units}^3.$$



(a) **(b)** **(c)**

Figure 7.2.16 Sketching the differential element and solid in Example 7.2.15

Video solution

youtu.be/watch?v=EcQPsBZwpZ8

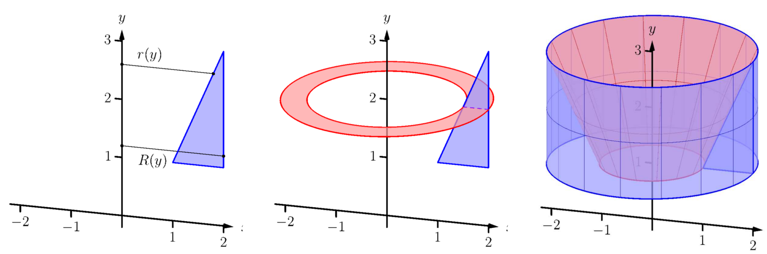
When rotating about a vertical axis, the outside and inside radius functions must be functions of y.

Example 7.2.17 Finding volume with the Washer Method.

Find the volume of the solid formed by rotating the triangular region with vertices at $(1,1)$, $(2,1)$ and $(2,3)$ about the y-axis.

Solution. The triangular region is sketched in Figure 7.2.18(a); the differential element is sketched in Figure 7.2.18(b) and the full solid is drawn in Figure 7.2.18(c). They help us establish the outside and inside radii. Since the axis of rotation is vertical, each radius is a function of y.

The outside radius $R(y)$ is formed by the line connecting $(2,1)$ and $(2,3)$; it is a constant function, as regardless of the y-value the distance from the line to the axis of rotation is 2. Thus $R(y) = 2$.



(a) **(b)** **(c)**

Figure 7.2.18 Sketching the solid in Example 7.2.17

The inside radius is formed by the line connecting $(1,1)$ and $(2,3)$. The equation of this line is $y = 2x-1$, but we need to refer to it as a function of y. Solving for x gives $r(y) = \frac{1}{2}(y+1)$.

We integrate over the y-bounds of $y=1$ to $y=3$. Thus the volume is

$$\begin{aligned} V &= \pi\int_1^3 \left(2^2 - \left(\frac{1}{2}(y+1)\right)^2\right)dy \\ &= \pi\int_1^3 \left(-\frac{1}{4}y^2 - \frac{1}{2}y + \frac{15}{4}\right)dy \\ &= \pi\left[-\frac{1}{12}y^3 - \frac{1}{4}y^2 + \frac{15}{4}y\right]\Big|_1^3 \end{aligned}$$

$$= \frac{10}{3}\pi \approx 10.47\,\text{units}^3.$$

Video solution

youtu.be/watch?v=VO7B1TRcvhM

This section introduced a new application of the definite integral. Our default view of the definite integral is that it gives "the area under the curve." However, we can establish definite integrals that represent other quantities; in this section, we computed volume.

The ultimate goal of this section is not to compute volumes of solids. That can be useful, but what is more useful is the understanding of this basic principle of integral calculus, outlined in Key Idea 7.0.1: to find the exact value of some quantity,

- we start with an approximation (in this section, slice the solid and approximate the volume of each slice),
- then make the approximation better by refining our original approximation (i.e., use more slices),
- then use limits to establish a definite integral which gives the exact value.

We practice this principle in the next section where we find volumes by slicing solids in a different way.

7.2.1 Exercises

Terms and Concepts

1. T/F: A solid of revolution is formed by revolving a shape around an axis.

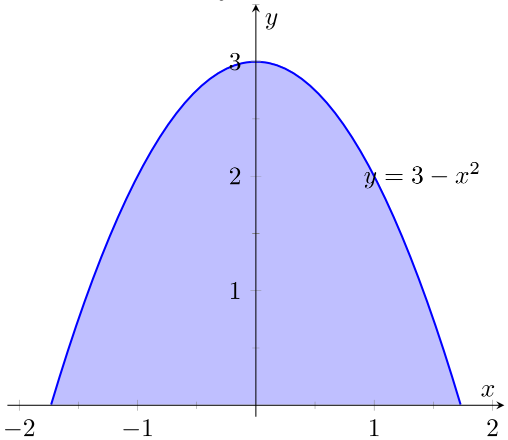
2. In your own words, explain how the Disk and Washer Methods are related.

3. Explain the how the units of volume are found in the integral of Theorem 7.2.3: if $A(x)$ has units of in^2, how does $\int A(x)\,dx$ have units of in^3?

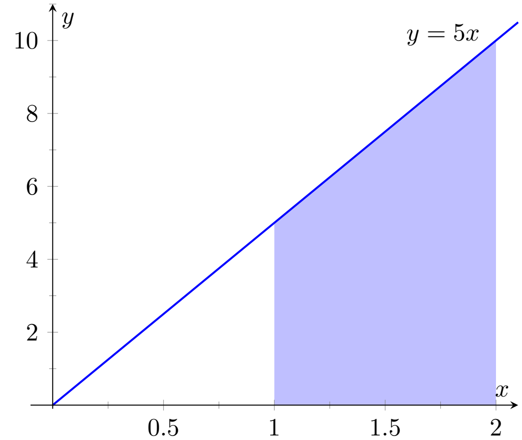
Problems

Exercise Group. Use the Disk/Washer Method to find the volume of the solid of revolution formed by revolving the given region about the x-axis.

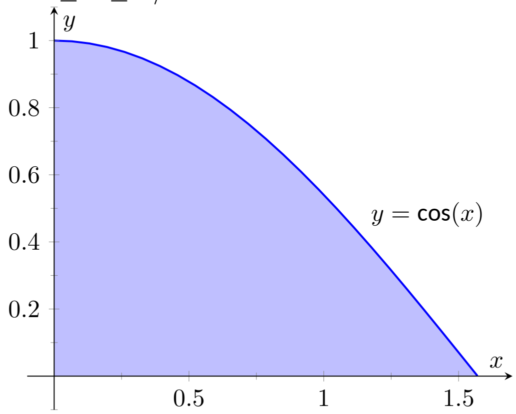
4. The region between $y = 3 - x^2$ and the x axis:



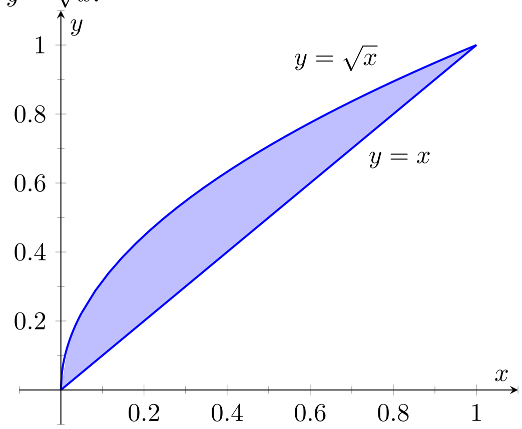
5. The region between $y = 5x$ and the x axis, for $1 \le x \le 2$:



6. The region between $y = \cos(x)$ and the x axis, for $0 \le x \le \pi/2$:

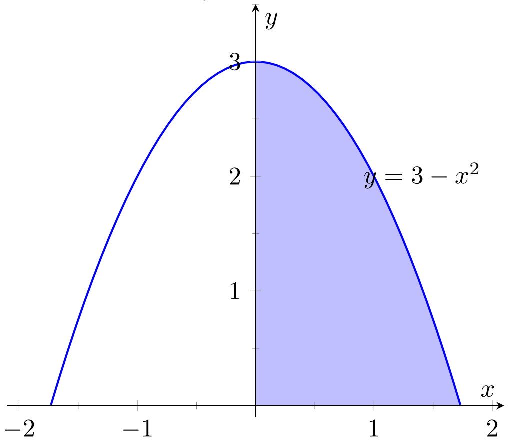


7. The region between the curves $y = x$ and $y = \sqrt{x}$:

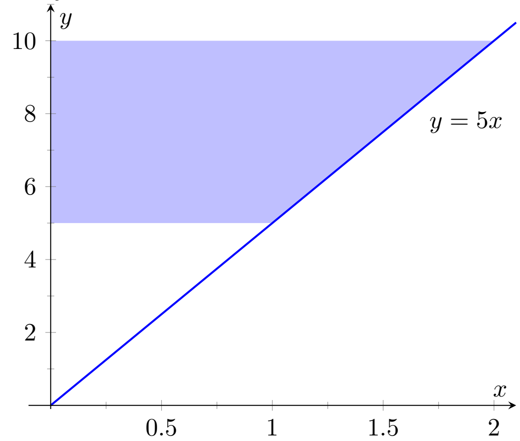


Exercise Group. Use the Disk/Washer Method to find the volume of the solid of revolution formed by revolving the given region about the y-axis.

8. The region bounded by the curve $y = 3 - x^2$, the x axis, and the y axis:

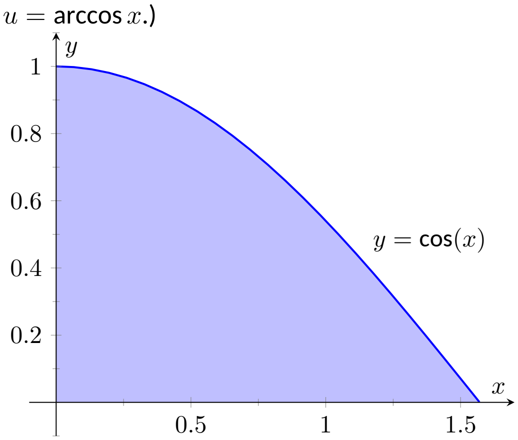


9. The region between $y = 5x$ and the y axis, for $5 \leq y \leq 10$:

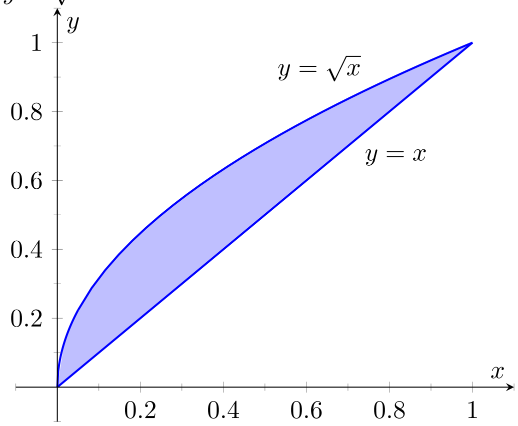


10. The region between $y = \cos(x)$ and the x axis, for $0 \leq x \leq \pi/2$:

(Hint: Integration By Parts will be necessary, twice. First let $u = \arccos^2 x$, then let $u = \arccos x$.)



11. The region between the curves $y = x$ and $y = \sqrt{x}$:



Exercise Group. Use the Disk/Washer Method to find the volume of the solid of revolution formed by rotating the given region about each of the given axes.

12. Region bounded by: $y = \sqrt{x}$, $y = 0$ and $x = 1$.

- **(a)** Rotate about the x axis.
- **(b)** Rotate about $y = 1$.
- **(c)** Rotate about the y axis.
- **(d)** Rotate about $x = 1$.

13. Region bounded by: $y = 4 - x^2$ and $y = 0$.

- **(a)** Rotate about the x axis.
- **(b)** Rotate about $y = 4$.
- **(c)** Rotate about $y = -1$.
- **(d)** Rotate about $x = 2$.

14. The triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 1)$.

- **(a)** Roate about the x axis.
- **(b)** Roate about $y = 2$.
- **(c)** Rotate about the y axis.
- **(d)** Rotate about $x = 1$.

15. Region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$.

- **(a)** Rotate about the x axis.
- **(b)** Rotate about $y = 1$.
- **(c)** Rotate about $y = 5$.

16. Region bounded by $y = 1/\sqrt{x^2+1}$, $x = -1$, $x = 1$ and the x-axis.

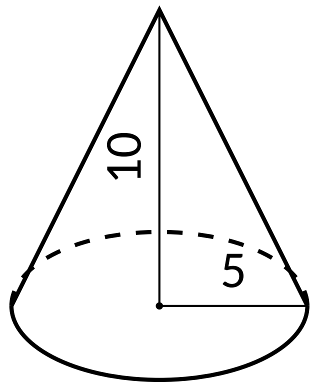
- **(a)** Rotate about the x axis.
- **(b)** Rotate about $y = 1$.
- **(c)** Rotate about $y = -1$.

17. Region bounded by $y = 2x$, $y = x$ and $x = 2$.

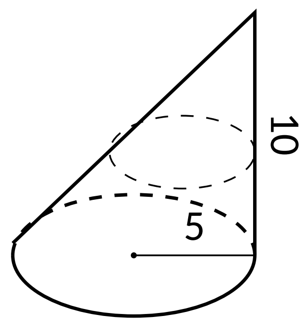
- **(a)** Rotate about the x axis.
- **(b)** Rotate about $y = 4$.
- **(c)** Rotate about the y axis.
- **(d)** Rotate about $x = 2$.

Exercise Group. Orient the given solid along the x-axis such that a cross-sectional area function $A(x)$ can be obtained, then apply Theorem 7.2.3 to find the volume of the solid.

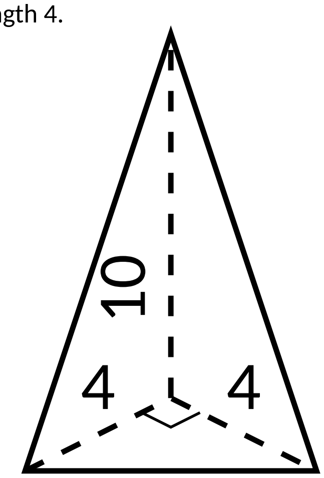
18. A right circular cone with height of 10 and base radius of 5.



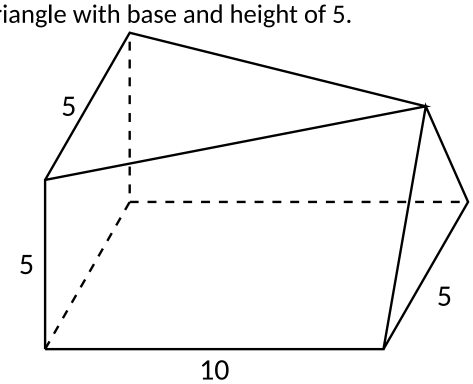
19. A skew right circular cone with height of 10 and base radius of 5. (Hint: all cross-sections are circles.)



20. A right triangular cone with height of 10 and whose base is a right, isosceles triangle with side length 4.



21. A solid with length 10 with a rectangular base and triangular top, wherein one end is a square with side length 5 and the other end is a triangle with base and height of 5.



7.3 The Shell Method

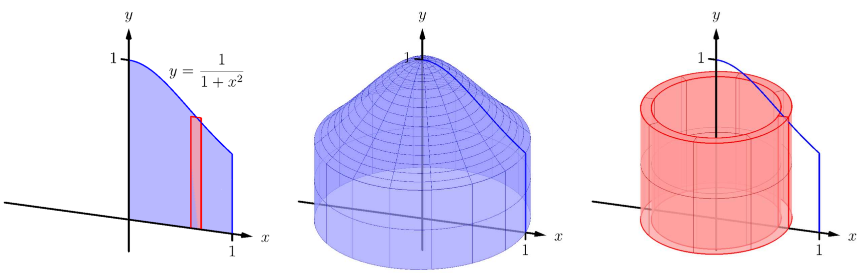
Often a given problem can be solved in more than one way. A particular method may be chosen out of convenience, personal preference, or perhaps necessity. Ultimately, it is good to have options.

The previous section introduced the Disk and Washer Methods, which computed the volume of solids of revolution by integrating the cross-sectional area of the solid. This section develops another method of computing volume, the *Shell Method*. Instead of slicing the solid perpendicular to the axis of rotation creating cross-sections, we now slice it parallel to the axis of rotation, creating "shells."

youtu.be/watch?v=YPZjBrm770g

Figure 7.3.1 Video introduction to Section 7.3

Consider Figure 7.3.2, where the region shown in Figure 7.3.2(a) is rotated around the y-axis forming the solid shown in Figure 7.3.2(b). A small slice of the region is drawn in Figure 7.3.2(a), parallel to the axis of rotation. When the region is rotated, this thin slice forms a *cylindrical shell*, as pictured in Figure 7.3.2(c). The previous section approximated a solid with lots of thin disks (or washers); we now approximate a solid with many thin cylindrical shells.



(a) **(b)** **(c)**

Figure 7.3.2 Introducing the Shell Method

To compute the volume of one shell, first consider the paper label on a soup can with radius r and height h. What is the area of this label? A simple way of determining this is to cut the label and lay it out flat, forming a rectangle with height h and length $2\pi r$. Thus the area is $A = 2\pi rh$; see Figure 7.3.3(a).

Do a similar process with a cylindrical shell, with height h, thickness Δx, and approximate radius r. Cutting the shell and laying it flat forms a rectangular solid with length $2\pi r$, height h and depth Δx. Thus the volume is $V \approx 2\pi rh\Delta x$; see Figure 7.3.3(b). (We say "approximately" since our radius was an approximation.)

By breaking the solid into n cylindrical shells, we can approximate the volume of the solid as

$$V \approx \sum_{i=1}^{n} 2\pi r_i h_i \Delta x_i,$$

where r_i, h_i and Δx_i are the radius, height and thickness of the ith shell, respectively.

This is a Riemann Sum. Taking a limit as the thickness of the shells approaches 0 leads to a definite integral.

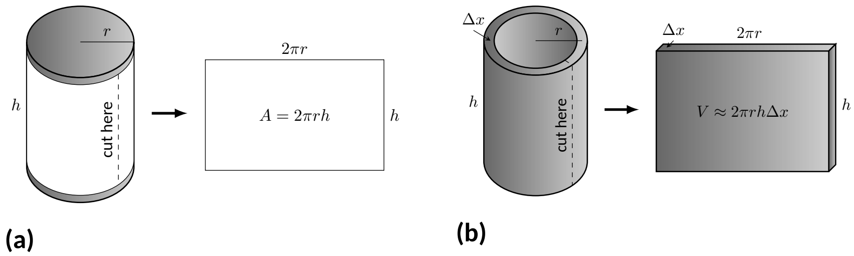


Figure 7.3.3 Determining the volume of a thin cylindrical shell

Key Idea 7.3.4 The Shell Method.

Let a solid be formed by revolving a region R, bounded by $x = a$ and $x = b$, around a vertical axis. Let $r(x)$ represent the distance from the axis of rotation to x (i.e., the radius of a sample shell) and let $h(x)$ represent the height of the solid at x (i.e., the height of the shell). The volume of the solid is

$$V = 2\pi \int_a^b r(x)h(x)\,dx.$$

Special Cases:

1. When the region R is bounded above by $y = f(x)$ and below by $y = g(x)$, then $h(x) = f(x) - g(x)$.
2. When the axis of rotation is the y-axis (i.e., $x = 0$) then $r(x) = x$.

Let's practice using the Shell Method.

Example 7.3.5 Finding volume using the Shell Method.

Find the volume of the solid formed by rotating the region bounded by $y = 0$, $y = 1/(1+x^2)$, $x = 0$ and $x = 1$ about the y-axis.

Solution. This is the region used to introduce the Shell Method in Figure 7.3.2, but is sketched again in Figure 7.3.6 for closer reference. A line is drawn in the region parallel to the axis of rotation representing a shell that will be carved out as the region is rotated about the y-axis. (This is the differential element.)

The distance this line is from the axis of rotation determines $r(x)$; as the distance from x to the y-axis is x, we have $r(x) = x$. The height of this line determines $h(x)$; the top of the line is at $y = 1/(1+x^2)$, whereas the bottom of the line is at $y = 0$. Thus $h(x) = 1/(1+x^2) - 0 = 1/(1+x^2)$. The region is bounded from $x = 0$ to $x = 1$, so the volume is

$$V = 2\pi \int_0^1 \frac{x}{1+x^2}\,dx.$$

This requires substitution. Let $u = 1 + x^2$, so $du = 2x\,dx$. We also change the bounds: $u(0) = 1$ and $u(1) = 2$. Thus we have:

$$\begin{aligned} &= \pi \int_1^2 \frac{1}{u}\,du \\ &= \pi \ln(u)\Big|_1^2 \\ &= \pi \ln(2) \approx 2.178\,\text{units}^3. \end{aligned}$$

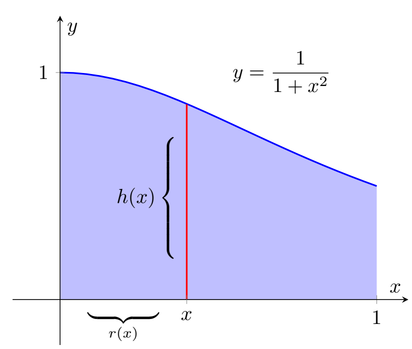


Figure 7.3.6 Graphing a region in Example 7.3.5

Note: in order to find this volume using the Disk Method, two integrals would be needed to account for the regions above and below $y = 1/2$.

With the Shell Method, nothing special needs to be accounted for to compute the volume of a solid that has a hole in the middle, as demonstrated next.

Example 7.3.7 Finding volume using the Shell Method.

Find the volume of the solid formed by rotating the triangular region determined by the points $(0,1)$, $(1,1)$ and $(1,3)$ about the line $x = 3$.

Solution. The region is sketched in Figure 7.3.8(a) along with the differential element, a line within the region parallel to the axis of rotation. In Figure 7.3.8(b), we see the shell traced out by the differential element, and in Figure 7.3.8(c) the whole solid is shown.

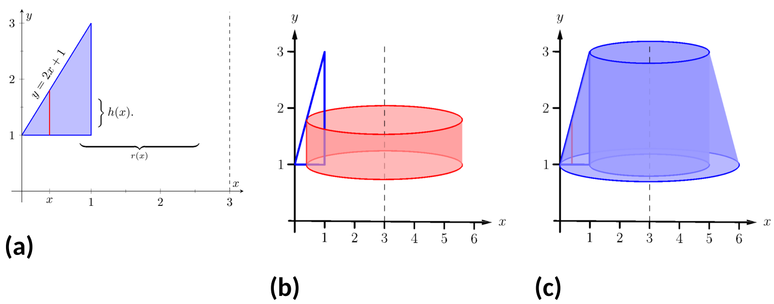


Figure 7.3.8 Graphing a region in Example 7.3.7

The height of the differential element is the distance from $y = 1$ to $y = 2x + 1$, the line that connects the points $(0,1)$ and $(1,3)$. Thus $h(x) = 2x + 1 - 1 = 2x$. The radius of the shell formed by the differential element is the distance from x to $x = 3$; that is, it is $r(x) = 3 - x$. The x-bounds of the region are $x = 0$ to $x = 1$, giving

$$\begin{aligned} V &= 2\pi \int_0^1 (3-x)(2x)\,dx \\ &= 2\pi \int_0^1 \left(6x - 2x^2\right) dx \\ &= 2\pi \left(3x^2 - \frac{2}{3}x^3\right)\Big|_0^1 \\ &= \frac{14}{3}\pi \approx 14.66\,\text{units}^3. \end{aligned}$$

When revolving a region around a horizontal axis, we must consider the radius and height functions in terms of y, not x.

Example 7.3.9 Finding volume using the Shell Method.

Find the volume of the solid formed by rotating the region given in Example 7.3.7 about the x-axis.

Solution. The region is sketched in Figure 7.3.10(a) with a sample differential element. In Figure 7.3.10(b) the shell formed by the differential element is drawn, and the solid is sketched in Figure 7.3.10(c). (Note that the triangular region looks "short and wide" here, whereas in the

Video solution

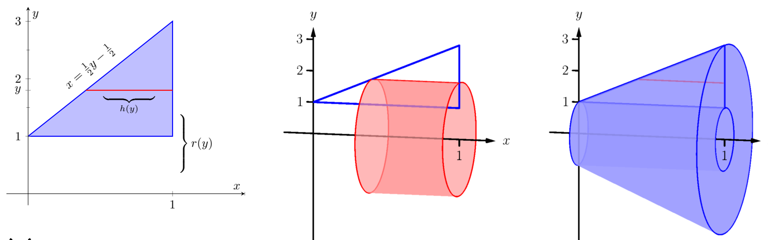
youtu.be/watch?v=WQ3rUzAhgPw

Video solution

youtu.be/watch?v=wGVmSx1TqQI

previous example the same region looked "tall and narrow." This is because the bounds on the graphs are different.)

The height of the differential element is an x-distance, between $x = \frac{1}{2}y - \frac{1}{2}$ and $x = 1$. Thus $h(y) = 1 - (\frac{1}{2}y - \frac{1}{2}) = -\frac{1}{2}y + \frac{3}{2}$. The radius is the distance from y to the x-axis, so $r(y) = y$. The y bounds of the region are $y = 1$ and $y = 3$, leading to the integral



(a) (b) (c)

Figure 7.3.10 Graphing a region in Example 7.3.9

$$\begin{aligned} V &= 2\pi \int_1^3 \left[y\left(-\frac{1}{2}y + \frac{3}{2}\right)\right] dy \\ &= 2\pi \int_1^3 \left[-\frac{1}{2}y^2 + \frac{3}{2}y\right] dy \\ &= 2\pi \left[-\frac{1}{6}y^3 + \frac{3}{4}y^2\right]\Big|_1^3 \\ &= 2\pi \left[\frac{9}{4} - \frac{7}{12}\right] \\ &= \frac{10}{3}\pi \approx 10.472\,\text{units}^3. \end{aligned}$$

Video solution

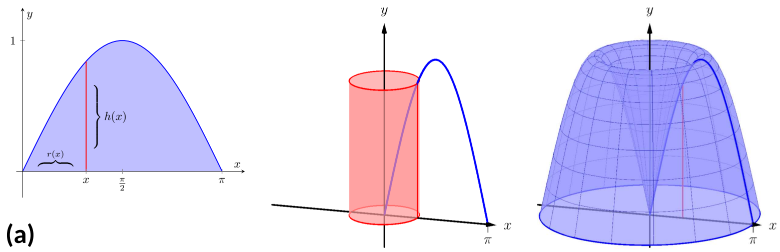
youtu.be/watch?v=pu80zsXPw5E

At the beginning of this section it was stated that "it is good to have options." The next example finds the volume of a solid rather easily with the Shell Method, but using the Washer Method would be quite a chore.

Example 7.3.11 Finding volume using the Shell Method.

Find the volume of the solid formed by revolving the region bounded by $y = \sin(x)$ and the x-axis from $x = 0$ to $x = \pi$ about the y-axis.

Solution. The region and a differential element, the shell formed by this differential element, and the resulting solid are given in Figure 7.3.12.



(a) (b) (c)

Figure 7.3.12 Graphing a region in Example 7.3.11

The radius of a sample shell is $r(x) = x$; the height of a sample shell is $h(x) = \sin(x)$, each from $x = 0$ to $x = \pi$. Thus the volume of the solid is

$$V = 2\pi \int_0^{\pi} x \sin(x)\, dx.$$

This requires Integration By Parts. Set $u = x$ and $dv = \sin(x)\, dx$; we leave it to the reader to fill in the rest. We have:

$$\begin{aligned}
&= 2\pi\Big[-x\cos(x)\Big|_0^{\pi} + \int_0^{\pi} \cos(x)\, dx\Big] \\
&= 2\pi\Big[\pi + \sin(x)\Big|_0^{\pi}\Big] \\
&= 2\pi\Big[\pi + 0\Big] \\
&= 2\pi^2 \approx 19.74\,\text{units}^3.
\end{aligned}$$

Note that in order to use the Washer Method, we would need to solve $y = \sin x$ for x, requiring the use of the arcsine function. We leave it to the reader to verify that the outside radius function is $R(y) = \pi - \arcsin y$ and the inside radius function is $r(y) = \arcsin y$. Thus the volume can be computed as

$$\pi \int_0^1 \Big[(\pi - \arcsin y)^2 - (\arcsin y)^2\Big]\, dy.$$

This integral isn't terrible given that the $\arcsin^2 y$ terms cancel, but it is more onerous than the integral created by the Shell Method.

Video solution

youtu.be/watch?v=nd16wB-0qIQ

We end this section with a table summarizing the usage of the Washer and Shell Methods.

Key Idea 7.3.13 Summary of the Washer and Shell Methods.

Let a region R be given with x-bounds $x = a$ and $x = b$ and y-bounds $y = c$ and $y = d$.

	Washer Method	Shell Method
Horizontal Axis	$\pi\int_a^b \big(R(x)^2 - r(x)^2\big)\,dx$	$2\pi\int_c^d r(y)h(y)\,dy$
Vertical Axis	$\pi\int_c^d \big(R(y)^2 - r(y)^2\big)\,dy$	$2\pi\int_a^b r(x)h(x)\,dx$

As in the previous section, the real goal of this section is not to be able to compute volumes of certain solids. Rather, it is to be able to solve a problem by first approximating, then using limits to refine the approximation to give the exact value. In this section, we approximate the volume of a solid by cutting it into thin cylindrical shells. By summing up the volumes of each shell, we get an approximation of the volume. By taking a limit as the number of equally spaced shells goes to infinity, our summation can be evaluated as a definite integral, giving the exact value.

We use this same principle again in the next section, where we find the length of curves in the plane.

7.3.1 Exercises

Terms and Concepts

1. T/F: A solid of revolution is formed by revolving a shape around an axis.

2. T/F: The Shell Method can only be used when the Washer Method fails.

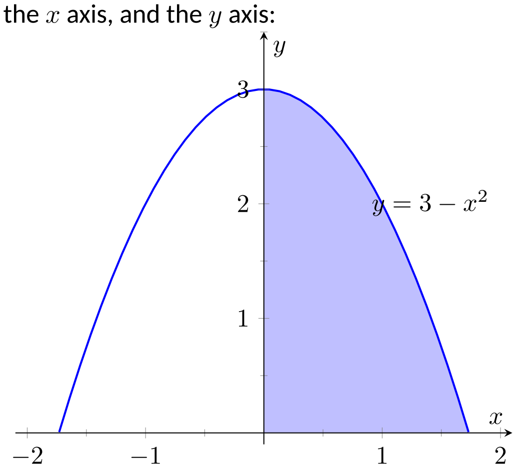
3. T/F: The Shell Method works by integrating cross-sectional areas of a solid.

4. T/F: When finding the volume of a solid of revolution that was revolved around a vertical axis, the Shell Method integrates with respect to x.

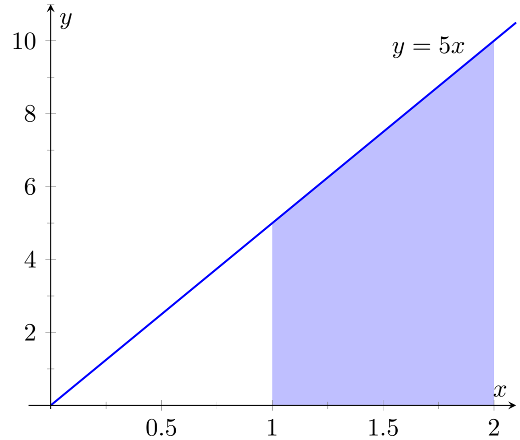
Problems

Exercise Group. Use the Shell Method to find the volume of the solid of revolution formed by revolving the given region about the y-axis.

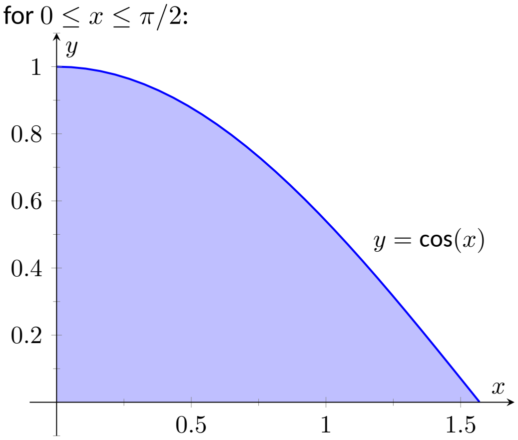
5. The region bounded by the curve $y = 3 - x^2$, the x axis, and the y axis:



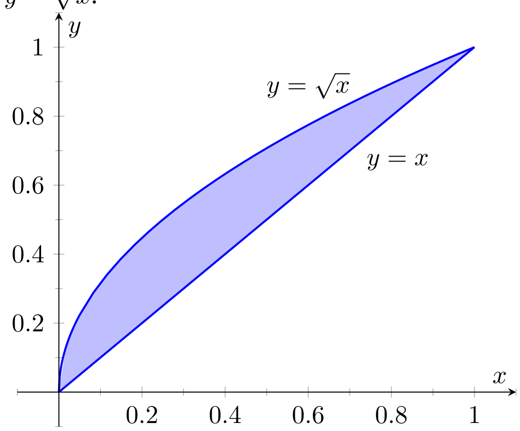
6. The region between $y = 5x$ and the x axis, for $1 \le x \le 2$:



7. The region between $y = \cos(x)$ and the x axis, for $0 \le x \le \pi/2$:

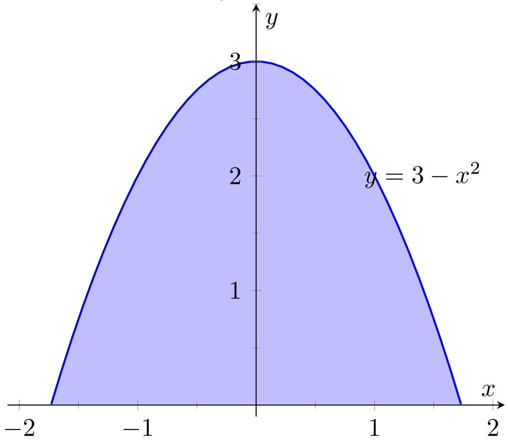


8. The region between the curves $y = x$ and $y = \sqrt{x}$:

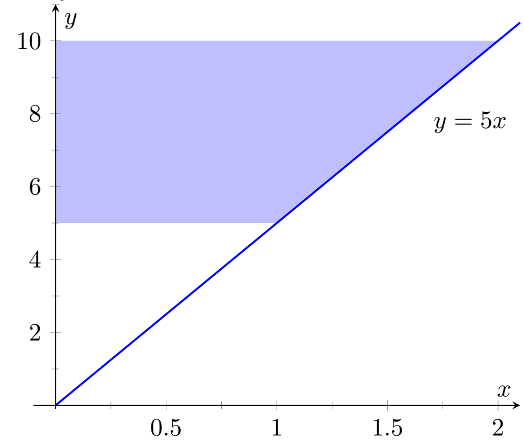


Exercise Group. Use the Shell Method to find the volume of the solid of revolution formed by revolving the given region about the x-axis.

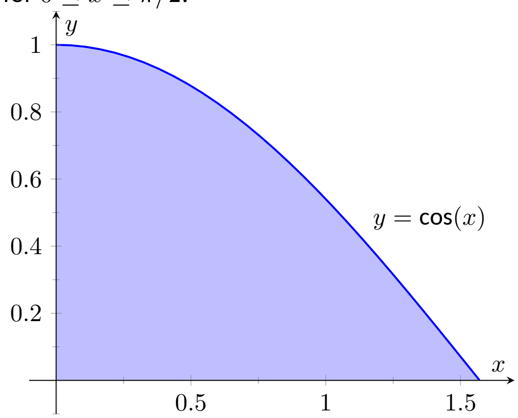
9. The region between $y = 3 - x^2$ and the x axis:



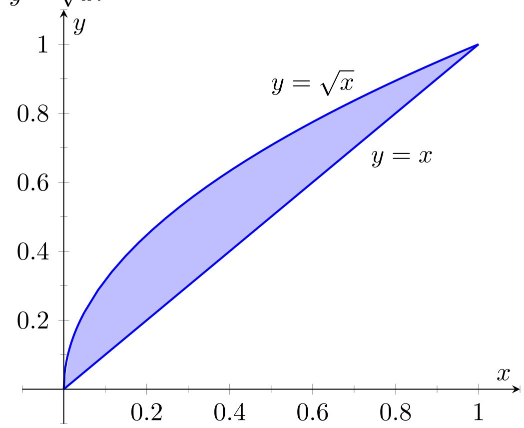
10. The region between $y = 5x$ and the y axis, for $5 \leq y \leq 10$:



11. The region between $y = \cos(x)$ and the x axis, for $0 \leq x \leq \pi/2$:



12. The region between the curves $y = x$ and $y = \sqrt{x}$:



Exercise Group. Use the Shell Method to find the volume of the solid of revolution formed by revloving the given region about each of the given axes.

13. Region bounded by: $y = \sqrt{x}$, $y = 0$ and $x = 1$.

- **(a)** Rotate about the y axis.
- **(b)** Rotate about $x = 1$.
- **(c)** Rotate about the x axis.
- **(d)** Rotate about $y = 1$.

14. Region bounded by: $y = 4 - x^2$ and $y = 0$.

- **(a)** Rotate about $x = 2$.
- **(b)** Rotate about $x = -2$.
- **(c)** Rotate about the x axis.
- **(d)** Rotate about $y = 4$.

15. The triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 1)$.

- **(a)** Rotate about the y axis.
- **(b)** Rotate about $x = 1$.
- **(c)** Rotate about the x axis.
- **(d)** Rotate about $y = 2$.

16. Region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$.

- **(a)** Rotate about the y axis.
- **(b)** Rotate about $x = 1$.
- **(c)** Rotate about $x = -1$.

17. Region bounded by $y = 1/\sqrt{x^2+1}$, $x = 1$ and the x and y axes.

(a) Rotate about the y axis.

(b) Rotate about $x = 1$.

18. Region bounded by $y = 2x$, $y = x$ and $x = 2$.

(a) Rotate about the y axis.

(b) Rotate about $x = 2$.

(c) Rotate about the x axis.

(d) Rotate about $y = 4$.

7.4 Arc Length and Surface Area

In previous sections we have used integration to answer the following questions:

1. Given a region, what is its area?
2. Given a solid, what is its volume?

In this section, we address two related questions:

1. Given a curve, what is its length? This is often referred to as **arc length**.
2. Given a solid, what is its *surface area*?

youtu.be/watch?v=r8JJru-DcAw

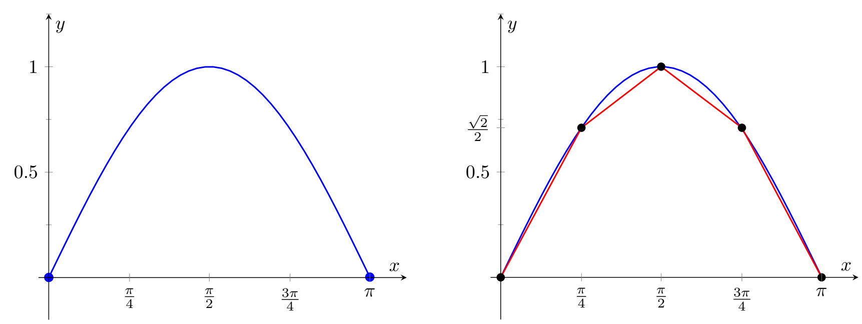
Figure 7.4.1 Video introduction to Section 7.4

7.4.1 Arc Length

Consider the graph of $y = \sin(x)$ on $[0, \pi]$ given in Figure 7.4.2(a). How long is this curve? That is, if we were to use a piece of string to exactly match the shape of this curve, how long would the string be?

As we have done in the past, we start by approximating; later, we will refine our answer using limits to get an exact solution.

The length of straight-line segments is easy to compute using the Distance Formula. We can approximate the length of the given curve by approximating the curve with straight lines and measuring their lengths.



(a) **(b)**

Figure 7.4.2 Graphing $y = \sin(x)$ on $[0, \pi]$ and approximating the curve with line segments

In Figure 7.4.2(b), the curve $y = \sin(x)$ has been approximated with 4 line segments (the interval $[0, \pi]$ has been divided into 4 subintervals of equal length). It is clear that these four line segments approximate $y = \sin(x)$ very well on the first and last subinterval, though not so well in the middle. Regardless, the sum of the lengths of the line segments is 3.79, so we approximate the arc length of $y = \sin(x)$ on $[0, \pi]$ to be 3.79.

In general, we can approximate the arc length of $y = f(x)$ on $[a, b]$ in the following manner. Let $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ be a partition of $[a, b]$ into n subintervals. Let Δx_i represent the length of the ith subinterval $[x_{i-1}, x_i]$.

Figure 7.4.3 zooms in on the ith subinterval where $y = f(x)$ is approximated by a straight line segment. The dashed lines show that we can view this line segment as the hypotenuse of a right triangle whose sides have length Δx_i and Δy_i. Using the Pythagorean Theorem, the length of this line segment is

$$\sqrt{\Delta x_i^2 + \Delta y_i^2}.$$

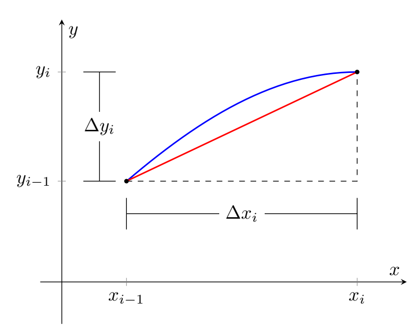


Figure 7.4.3 Zooming in on the ith subinterval $[x_{i-1}, x_i]$ of a partition of $[a, b]$

Summing over all subintervals gives an arc length approximation

$$L \approx \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}.$$

As shown here, this is *not* a Riemann Sum. While we could conclude that taking a limit as the subinterval length goes to zero gives the exact arc length, we would not be able to compute the answer with a definite integral. We need first to do a little algebra.

In the above expression factor out a Δx_i^2 term:

$$\sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sum_{i=1}^{n} \sqrt{\Delta x_i^2 \left(1 + \frac{\Delta y_i^2}{\Delta x_i^2}\right)}.$$

Now pull the Δx_i^2 term out of the square root:

$$= \sum_{i=1}^{n} \sqrt{1 + \frac{\Delta y_i^2}{\Delta x_i^2}}\, \Delta x_i.$$

This is nearly a Riemann Sum. Consider the $\Delta y_i^2/\Delta x_i^2$ term. The expression $\Delta y_i/\Delta x_i$ measures the "change in y/change in x," that is, the "rise over run" of f on the ith subinterval. The Mean Value Theorem of Differentiation (Theorem 3.2.4) states that there is a c_i in the ith subinterval where $f'(c_i) = \Delta y_i/\Delta x_i$. Thus we can rewrite our above expression as:

$$= \sum_{i=1}^{n} \sqrt{1 + f'(c_i)^2}\, \Delta x_i.$$

This *is* a Riemann Sum. As long as f' is continuous, we can invoke Theorem 5.3.26 and conclude

$$= \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Theorem 7.4.4 Arc Length.

Let f be differentiable on $[a, b]$, where f' is also continuous on $[a, b]$. Then the arc length of f from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Note: This is our first use of differentiability on a closed interval since Section 2.1.

The theorem also requires that f' be continuous on $[a, b]$; while examples are arcane, it is possible for f to be differentiable yet f' is not continuous.

As the integrand contains a square root, it is often difficult to use the formula in Theorem 7.4.4 to find the length exactly. When exact answers are difficult to come by, we resort to using numerical methods of approximating definite integrals. The following examples will demonstrate this.

Example 7.4.5 Finding arc length.

Find the arc length of $f(x) = x^{3/2}$ from $x = 0$ to $x = 4$.

Solution. We find $f'(x) = \frac{3}{2}x^{1/2}$; note that on $[0, 4]$, f is differentiable and f' is also continuous. Using the formula, we find the arc length L as

$$\begin{aligned} L &= \int_0^4 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx \\ &= \int_0^4 \sqrt{1+\frac{9}{4}x}\,dx \\ &= \int_0^4 \left(1+\frac{9}{4}x\right)^{1/2}\,dx \\ &= \frac{2}{3}\cdot\frac{4}{9}\cdot\left(1+\frac{9}{4}x\right)^{3/2}\Big|_0^4 \\ &= \frac{8}{27}\left(10^{3/2}-1\right) \approx 9.07 \text{ units.} \end{aligned}$$

A graph of f is given in Figure 7.4.6.

Video solution

youtu.be/watch?v=0NPr4wZlTi8

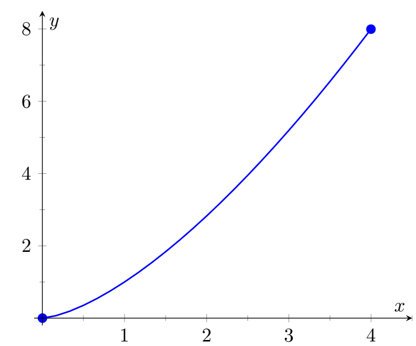


Figure 7.4.6 A graph of $f(x) = x^{3/2}$ from Example 7.4.5

Example 7.4.7 Finding arc length.

Find the arc length of $f(x) = \frac{1}{8}x^2 - \ln(x)$ from $x = 1$ to $x = 2$.

Solution. This function was chosen specifically because the resulting integral can be evaluated exactly. We begin by finding $f'(x) = x/4 - 1/x$. The arc length is

$$\begin{aligned} L &= \int_1^2 \sqrt{1+\left(\frac{x}{4}-\frac{1}{x}\right)^2}\,dx \\ &= \int_1^2 \sqrt{1+\frac{x^2}{16}-\frac{1}{2}+\frac{1}{x^2}}\,dx \\ &= \int_1^2 \sqrt{\frac{x^2}{16}+\frac{1}{2}+\frac{1}{x^2}}\,dx \\ &= \int_1^2 \sqrt{\left(\frac{x}{4}+\frac{1}{x}\right)^2}\,dx \end{aligned}$$

$$\begin{aligned} &= \int_1^2 \left(\frac{x}{4}+\frac{1}{x}\right)\,dx \\ &= \left(\frac{x^2}{8}+\ln(x)\right)\Big|_1^2 \\ &= \frac{3}{8}+\ln(2) \approx 1.07 \text{ units.} \end{aligned}$$

A graph of f is given in Figure 7.4.8; the portion of the curve measured in this problem is in bold.

Video solution

youtu.be/watch?v=mJlyz_9yiao

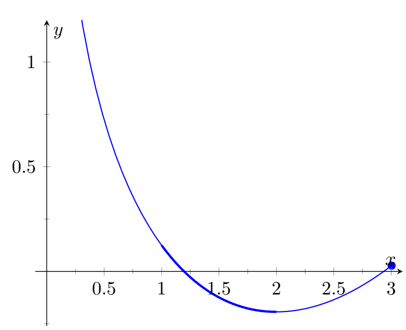


Figure 7.4.8 A graph of $f(x) = \frac{1}{8}x^2 - \ln(x)$ from Example 7.4.7

The previous examples found the arc length exactly through careful choice of the functions. In general, exact answers are much more difficult to come by and numerical approximations are necessary.

Example 7.4.9 Approximating arc length numerically.

Find the length of the sine curve from $x = 0$ to $x = \pi$.

Solution. This is somewhat of a mathematical curiosity; in Example 5.4.14 we found the area under one "hump" of the sine curve is 2 square units; now we are measuring its arc length.
The setup is straightforward: $f(x) = \sin(x)$ and $f'(x) = \cos(x)$. Thus

$$L = \int_0^{\pi} \sqrt{1 + \cos^2(x)}\, dx.$$

This integral *cannot* be evaluated in terms of elementary functions so we will approximate it with Simpson's Method with $n = 4$.
Figure 7.4.10 gives $\sqrt{1+\cos^2(x)}$ evaluated at 5 evenly spaced points in $[0, \pi]$. Simpson's Rule then states that

$$\int_0^{\pi} \sqrt{1+\cos^2(x)}\, dx \approx \frac{\pi - 0}{4 \cdot 3}\left(\sqrt{2} + 4\sqrt{3/2} + 2(1) + 4\sqrt{3/2} + \sqrt{2}\right)$$
$$= 3.82918.$$

Using a computer with $n = 100$ the approximation is $L \approx 3.8202$; our approximation with $n = 4$ is quite good.

x	$\sqrt{1+\cos^2(x)}$
0	$\sqrt{2}$
$\pi/4$	$\sqrt{3/2}$
$\pi/2$	1
$3\pi/4$	$\sqrt{3/2}$
π	$\sqrt{2}$

Figure 7.4.10 A table of values of $y = \sqrt{1+\cos^2(x)}$ to evaluate a definite integral in Example 7.4.9

7.4.2 Surface Area of Solids of Revolution

We have already seen how a curve $y = f(x)$ on $[a, b]$ can be revolved around an axis to form a solid. Instead of computing its volume, we now consider its surface area.

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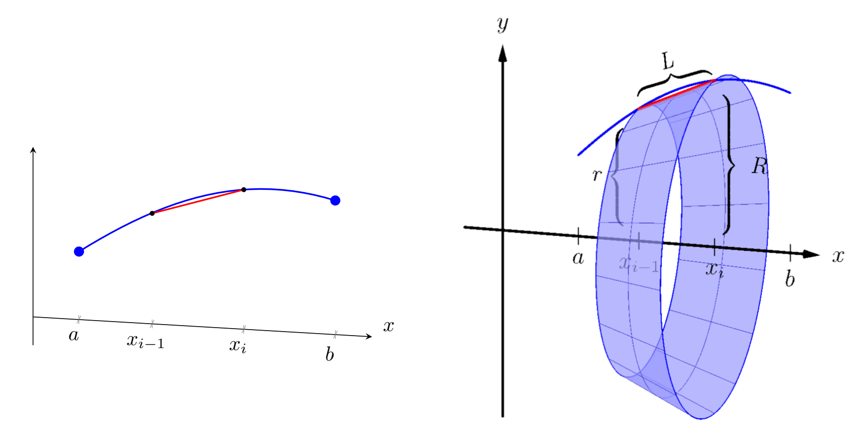
Figure 7.4.11 Video introduction to Subsection 7.4.2



(a) **(b)**

Figure 7.4.12 Establishing the formula for surface area

We begin as we have in the previous sections: we partition the interval $[a, b]$ with n subintervals, where the ith subinterval is $[x_{i-1}, x_i]$. On each subinterval, we can approximate the curve $y = f(x)$ with a straight line that connects $f(x_{i-1})$ and $f(x_i)$ as shown in Figure 7.4.12(a). Revolving this line segment about the x-axis creates part of a cone (called a *frustum* of a cone) as shown in Figure 7.4.12(b). The surface area of a frustum of a cone is

$$2\pi \cdot \text{length} \cdot \text{average of the two radii } R \text{ and } r.$$

The length is given by L; we use the material just covered by arc length to state that

$$L \approx \sqrt{1+f'(c_i)^2}\Delta x_i$$

for some c_i in the ith subinterval. The radii are just the function evaluated at the endpoints of the interval. That is,

$$R = f(x_i) \text{ and } r = f(x_{i-1}).$$

Thus the surface area of this sample frustum of the cone is approximately

$$2\pi\frac{f(x_{i-1})+f(x_i)}{2}\sqrt{1+f'(c_i)^2}\Delta x_i.$$

Since f is a continuous function, the Intermediate Value Theorem states there is some d_i in $[x_{i-1}, x_i]$ such that $f(d_i) = \dfrac{f(x_{i-1})+f(x_i)}{2}$; we can use this to rewrite the above equation as

$$2\pi f(d_i)\sqrt{1+f'(c_i)^2}\Delta x_i.$$

Summing over all the subintervals we get the total surface area to be approximately

$$\text{Surface Area } \approx \sum_{i=1}^{n} 2\pi f(d_i)\sqrt{1+f'(c_i)^2}\Delta x_i,$$

which is a Riemann Sum. Taking the limit as the subinterval lengths go to zero gives us the exact surface area, given in the following theorem.

Theorem 7.4.13 Surface Area of a Solid of Revolution.

Let f be differentiable on $[a,b]$, where f' is also continuous on $[a,b]$.

1. *The surface area of the solid formed by revolving the graph of $y = f(x)$, where $f(x) \geq 0$, about the x-axis is*

$$\text{Surface Area } = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx.$$

2. *The surface area of the solid formed by revolving the graph of $y = f(x)$ about the y-axis, where $a, b \geq 0$, is*

$$\text{Surface Area } = 2\pi\int_a^b x\sqrt{1+f'(x)^2}\,dx.$$

(When revolving $y = f(x)$ about the y-axis, the radii of the resulting frustum are x_{i-1} and x_i; their average value is simply the midpoint of the interval. In the limit, this midpoint is just x. This gives the second part of Theorem 7.4.13.)

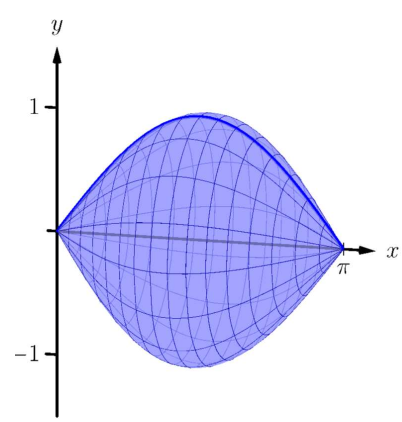


Figure 7.4.15 Revolving $y = \sin(x)$ on $[0,\pi]$ about the x-axis

Example 7.4.14 Finding surface area of a solid of revolution.

Find the surface area of the solid formed by revolving $y = \sin(x)$ on $[0,\pi]$ around the x-axis, as shown in Figure 7.4.15.

Solution. The setup is relatively straightforward. Using Theorem 7.4.13, we have the surface area SA is:

$$SA = 2\pi\int_0^\pi \sin(x)\sqrt{1+\cos^2(x)}\,dx$$

$$= -2\pi\frac{1}{2}\left(\sinh^{-1}(\cos(x)) + \cos(x)\sqrt{1+\cos^2(x)}\right)\Big|_0^{\pi}$$
$$= 2\pi\left(\sqrt{2} + \sinh^{-1}(1)\right)$$
$$\approx 14.42\,\text{units}^2.$$

The integration step above is nontrivial, utilizing the integration method of Trigonometric Substitution from Section 6.3.
It is interesting to see that the surface area of a solid, whose shape is defined by a trigonometric function, involves both a square root and an inverse hyperbolic trigonometric function.

Video solution

youtu.be/watch?v=ehC1adQ-pTs

Example 7.4.16 Finding surface area of a solid of revolution.

Find the surface area of the solid formed by revolving the curve $y = x^2$ on $[0, 1]$ about:

1. the x-axis
2. the y-axis.

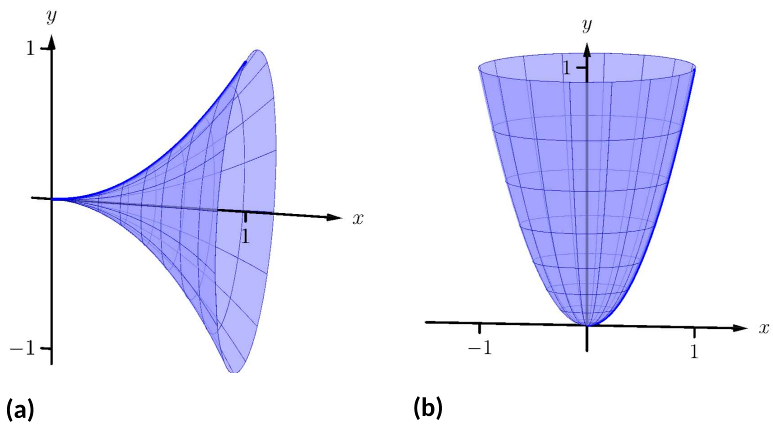


Figure 7.4.17 The solids used in Example 7.4.16

Solution.

1. The integral is straightforward to setup:

$$SA = 2\pi\int_0^1 x^2\sqrt{1+(2x)^2}\,dx.$$

Like the integral in Example 7.4.14, this requires Trigonometric Substitution.

$$= \frac{\pi}{32}\left(2(8x^3+x)\sqrt{1+4x^2} - \sinh^{-1}(2x)\right)\Big|_0^1$$
$$= \frac{\pi}{32}\left(18\sqrt{5} - \sinh^{-1}(2)\right)$$
$$\approx 3.81\,\text{units}^2.$$

The solid formed by revolving $y = x^2$ around the x-axis is graphed in Figure 7.4.17(a).

2. Since we are revolving around the y-axis, the "radius" of the solid is not $f(x)$ but rather x. Thus the integral to compute the surface area is:

$$SA = 2\pi \int_0^1 x\sqrt{1+(2x)^2}\,dx.$$

This integral can be solved using substitution. Set $u = 1+4x^2$; the new bounds are $u = 1$ to $u = 5$. We then have

$$\begin{aligned} &= \frac{\pi}{4}\int_1^5 \sqrt{u}\,du \\ &= \frac{\pi}{4}\frac{2}{3}u^{3/2}\Big|_1^5 \\ &= \frac{\pi}{6}\left(5\sqrt{5}-1\right) \\ &\approx 5.33\ \text{units}^2. \end{aligned}$$

The solid formed by revolving $y = x^2$ about the y-axis is graphed in Figure 7.4.17(b).

Video solution

youtu.be/watch?v=jK04gmbaTtE

Our final example is a famous mathematical "paradox."

Example 7.4.18 The surface area and volume of Gabriel's Horn.

Consider the solid formed by revolving $y = 1/x$ about the x-axis on $[1,\infty)$. Find the volume and surface area of this solid. (This shape, as graphed in Figure 7.4.19, is known as "Gabriel's Horn" since it looks like a very long horn that only a supernatural person, such as an angel, could play.)

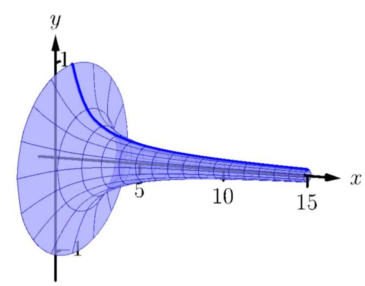


Figure 7.4.19 A graph of Gabriel's Horn

Solution. To compute the volume it is natural to use the Disk Method. We have:

$$\begin{aligned} V &= \pi\int_1^\infty \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \pi\int_1^b \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \pi\left(\frac{-1}{x}\right)\Big|_1^b \\ &= \lim_{b\to\infty} \pi\left(1-\frac{1}{b}\right) \\ &= \pi\ \text{units}^3. \end{aligned}$$

Gabriel's Horn has a finite volume of π cubic units. Since we have already seen that regions with infinite length can have a finite area, this is not too difficult to accept.

We now consider its surface area. The integral is straightforward to setup:

$$SA = 2\pi\int_1^\infty \frac{1}{x}\sqrt{1+1/x^4}\,dx.$$

Integrating this expression is not trivial. We can, however, compare it to other improper integrals. Since $1 < \sqrt{1+1/x^4}$ on $[1,\infty)$, we can state that

$$2\pi\int_1^\infty \frac{1}{x}\,dx < 2\pi\int_1^\infty \frac{1}{x}\sqrt{1+1/x^4}\,dx.$$

By Key Idea 6.5.17, the improper integral on the left diverges. Since the integral on the right is larger, we conclude it also diverges, meaning Gabriel's Horn has infinite surface area.

Hence the "paradox": we can fill Gabriel's Horn with a finite amount of paint, but since it has infinite surface area, we can never paint it.

Somehow this paradox is striking when we think about it in terms of volume and area. However, we have seen a similar paradox before, as referenced above. We know that the area under the curve $y = 1/x^2$ on $[1,\infty)$ is finite, yet the shape has an infinite perimeter. Strange things can occur when we deal with the infinite.

A standard equation from physics is "Work = force × distance", when the force applied is constant. In Section 7.5 we learn how to compute work when the force applied is variable.

Video solution

youtu.be/watch?v=L4ogGgyzmvs

7.4.3 Exercises

Terms and Concepts

1. T/F: The integral formula for computing Arc Length was found by first approximating arc length with straight line segments.

2. T/F: The integral formula for computing Arc Length includes a square-root, meaning the integration is probably easy.

Problems

Exercise Group. In the following exercises, find the arc length of the function on the given interval.

3. $f(x) = x$ on $[0,1]$.

4. $f(x) = \sqrt{8}x$ on $[-1,1]$.

5. $f(x) = \frac{1}{3}x^{3/2} - x^{1/2}$ on $[0,1]$.

6. $f(x) = \frac{1}{12}x^3 + \frac{1}{x}$ on $[1,4]$.

7. $f(x) = 2x^{3/2} - \frac{1}{6}\sqrt{x}$ on $[1,4]$.

8. $f(x) = \cosh(x)$ on $[-\ln(2), \ln(2)]$.

9. $f(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$ on $[0, \ln(5)]$.

10. $f(x) = \frac{1}{12}x^5 + \frac{1}{5x^3}$ on $[0.1, 1]$.

11. $f(x) = \ln\left(\sin(x)\right)$ on $[\pi/6, \pi/2]$.

12. $f(x) = \ln\left(\cos(x)\right)$ on $[0, \pi/4]$.

Exercise Group. In the following exercises, set up the integral to compute the arc length of the function on the given interval. Do not evaluate the integral.

13. $f(x) = x^2$ on $[0,1]$.

14. $f(x) = x^{10}$ on $[0,1]$.

15. $f(x) = \ln(x)$ on $[1,e]$.

16. $f(x) = \frac{1}{x}$ on $[1,2]$.

17. $f(x) = cos(x)$ on $[0, \pi/2]$.

18. $f(x) = \sec(x)$ on $[-\pi/4, \pi/4]$.

Exercise Group. In the following exercises, use Simpson's Rule, with $n = 4$, to approximate the arc length of the function on the given interval. Note: these are the same problems as in Exercises 13–18.

19. $f(x) = x^2$ on $[0,1]$.

20. $f(x) = x^{10}$ on $[0,1]$.

21. $f(x) = \ln(x)$ on $[1,e]$.

22. $f(x) = \frac{1}{x}$ on $[1,2]$.

23. $f(x) = \cos(x)$ on $[0, \pi/2]$.

24. $f(x) = \sec(x)$ on $[-\pi/4, \pi/4]$.

Exercise Group. In the following exercises, find the surface area of the described solid of revolution.

25. The solid formed by revolving $y = 2x$ on $[0,1]$ about the x-axis.

26. The solid formed by revolving $y = 2x$ on $[0,1]$ about the y-axis.

27. The solid formed by revolving $y = x^2$ on $[0,1]$ about the y-axis.

28. The solid formed by revolving $y = x^3$ on $[0,1]$ about the x-axis.

Exercise Group. The following arc length and surface area problems lead to improper integrals. Although the hypotheses of Theorem 7.4.4 and Theorem 7.4.13 are not satisfied, the improper integrals converge, and formulas for arc length and surface area still give the correct result.

29. Find the length of the curve $f(x) = \sqrt{x}$ on $[0,1]$. (Note: this is the same as the length of $f(x) = x^2$ on $[0,1]$. Why?)

30. Find the length of the curve $f(x) = \sqrt{1-x^2}$ on $[-1,1]$. (Note: this describes the top half of a circle with radius 1.)

31. Find the length of the curve $f(x) = \sqrt{1-x^2/9}$ on $[-3,3]$. (Note: this describes the top half of an ellipse with a major axis of length 6 and a minor axis of length 2.)

32. Find the surface area of the solid formed by revolving $y = \sqrt{x}$ on $[0,1]$ about the x-axis.

33. Find the surface area of the sphere formed by revolving $y = \sqrt{1 - x^2}$ on $[-1, 1]$ about the x-axis.

7.5 Work

Work is the scientific term used to describe the action of a force which moves an object. When a constant force F is applied to move an object a distance d, the amount of work performed is $W = F \cdot d$.

The SI unit of force is the **newton**; one newton is equal to one $\frac{\text{kg}\cdot\text{m}}{\text{s}^2}$, and the SI unit of distance is a meter (m). The fundamental unit of work is one newton–meter, or a **joule** (J). That is, applying a force of one newton for one meter performs one joule of work. In Imperial units (as used in the United States), force is measured in pounds (lb) and distance is measured in feet (ft), hence work is measured in ft–lb.

When force is constant, the measurement of work is straightforward. For instance, lifting a 200 lb object 5 ft performs $200 \cdot 5 = 1000$ ft–lb of work.

What if the force applied is variable? For instance, imagine a climber pulling a 200 ft rope up a vertical face. The rope becomes lighter as more is pulled in, requiring less force and hence the climber performs less work.

Mass and *weight* are closely related, yet different, concepts. The mass m of an object is a quantitative measure of that object's resistance to acceleration. The weight w of an object is a measurement of the force applied to the object by the acceleration of gravity g.

Since the two measurements are proportional, $w = m\cdot g$, they are often used interchangeably in everyday conversation. When computing work, one must be careful to note which is being referred to. When mass is given, it must be multiplied by the acceleration of gravity to reference the related force.

7.5.1 Work Done by a Variable Force

In general, let $F(x)$ be a force function on an interval $[a, b]$. We want to measure the amount of work done applying the force F from $x = a$ to $x = b$. We can approximate the amount of work being done by partitioning $[a, b]$ into subintervals $a = x_0 < x_1 < \cdots < x_n = b$ and assuming that F is constant on each subinterval. Let c_i be a value in the ith subinterval $[x_{i-1}, x_i]$. Then the work done on this interval is approximately $W_i \approx F(c_i) \cdot (x_i - x_{i-1}) = F(c_i)\Delta x_i$, a constant force × the distance over which it is applied. The total work is

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} F(c_i)\Delta x_i.$$

This, of course, is a Riemann sum. Taking a limit as the subinterval lengths go to zero gives an exact value of work which can be evaluated through a definite integral.

Key Idea 7.5.1 Work.

Let $F(x)$ be a continuous function on $[a, b]$ describing the amount of force being applied to an object in the direction of travel from distance $x = a$ to distance $x = b$. The total work W done on $[a, b]$ is

$$W = \int_a^b F(x)\,dx.$$

Example 7.5.2 Computing work performed: applying variable force.

A 60 m climbing rope is hanging over the side of a tall cliff. How much work is performed in pulling the rope up to the top, where the rope has a linear mass density of $66\ \frac{\text{g}}{\text{m}}$?

Solution. We need to create a force function $F(x)$ on the interval $[0, 60]$. To do so, we must first decide what x is measuring: is it the length of the rope still hanging or is it the amount of rope pulled in? As long as we are consistent, either approach is fine. We adopt for this example the convention that x is the amount of rope pulled in. This seems to match intuition better; pulling up the first 10 meters of rope involves $x = 0$ to $x = 10$ instead of $x = 60$ to $x = 50$.

As x is the amount of rope pulled in, the amount of rope still hanging is $60 - x$. This length of rope has a mass of $66\,\frac{\text{g}}{\text{m}}$ or $0.066\,\frac{\text{kg}}{\text{m}}$. The mass of the rope still hanging is $0.066(60 - x)$ kg; multiplying this mass by the acceleration of gravity, $9.8\,\frac{\text{m}}{\text{s}^2}$, gives our variable force function

$$F(x) = (9.8)(0.066)(60 - x) = 0.6468(60 - x).$$

Thus the total work performed in pulling up the rope is

$$W = \int_0^{60} 0.6468(60 - x)\,dx = 1,164.24\,\text{J}.$$

By comparison, consider the work done in lifting the entire rope 60 meters. The rope weighs $60 \times 0.066 \times 9.8 = 38.808$ N, so the work applying this force for 60 meters is $60 \times 38.808 = 2,328.48$ J. This is exactly twice the work calculated before (and we leave it to the reader to understand why.)

Example 7.5.3 Computing work performed: applying variable force.

Consider again pulling a 60 m rope up a cliff face, where the rope has a mass of $66\,\frac{\text{g}}{\text{m}}$. At what point is exactly half the work performed?

Solution. From Example 7.5.2 we know the total work performed is $1,164.24$ J. We want to find a height h such that the work in pulling the rope from a height of $x = 0$ to a height of $x = h$ is 582.12, or half the total work. Thus we want to solve the equation

$$\int_0^{h} 0.6468(60 - x)\,dx = 582.12$$

for h.

$$\begin{aligned} \int_0^{h} 0.6468(60 - x)\,dx &= 582.12 \\ \left(38.808x - 0.3234x^2\right)\Big|_0^h &= 582.12 \\ 38.808h - 0.3234h^2 &= 582.12 \\ -0.3234h^2 + 38.808h - 582.12 &= 0. \end{aligned}$$

Apply the Quadratic Formula:

$$h = 17.57 \text{ and } 102.43$$

As the rope is only 60 m long, the only sensible answer is $h = 17.57$. Thus about half the work is done pulling up the first 17.57 m; the other half of the work is done pulling up the remaining 42.43 m.

In Example 7.5.3, we find that half of the work performed in pulling up a 60 m rope is done in the last 42.43 m. Why is it not coincidental that $60/\sqrt{2} = 42.43$?

Example 7.5.4 Computing work performed: applying variable force.

A box of 100 lb of sand is being pulled up at a uniform rate a distance of 50 ft over 1 minute. The sand is leaking from the box at a rate of $1\,\frac{\text{lb}}{\text{s}}$. The box itself weighs 5 lb and is pulled by a rope weighing $0.2\,\frac{\text{lb}}{\text{ft}}$.

1. How much work is done lifting just the rope?
2. How much work is done lifting just the box and sand?
3. What is the total amount of work performed?

Solution.

1. We start by forming the force function $F_r(x)$ for the rope (where the subscript denotes we are considering the rope). As in the previous example, let x denote the amount of rope, in feet, pulled in. (This is the same as saying x denotes the height of the box.) The weight of the rope with x feet pulled in is $F_r(x) = 0.2(50 - x) = 10 - 0.2x$. (Note that we do not have to include the acceleration of gravity here, for the *weight* of the rope per foot is given, not its *mass* per meter as before.) The work performed lifting the rope is

$$W_r = \int_0^{50} (10 - 0.2x)\,dx = 250\,\text{ft–lb}.$$

2. The sand is leaving the box at a rate of $1\,\frac{\text{lb}}{\text{s}}$. As the vertical trip is to take one minute, we know that 60 lb will have left when the box reaches its final height of 50 ft. Again letting x represent the height of the box, we have two points on the line that describes the weight of the sand: when $x = 0$, the sand weight is 100 lb, producing the point $(0, 100)$; when $x = 50$, the sand in the box weighs 40 lb, producing the point $(50, 40)$. The slope of this line is $\frac{100-40}{0-50} = -1.2$, giving the equation of the weight of the sand at height x as $w(x) = -1.2x + 100$. The box itself weighs a constant 5 lb, so the total force function is $F_b(x) = -1.2x + 105$. Integrating from $x = 0$ to $x = 50$ gives the work performed in lifting box and sand:

$$W_b = \int_0^{50} (-1.2x + 105)\,dx = 3750\,\text{ft–lb}.$$

3. The total work is the sum of W_r and W_b: $250 + 3750 = 4000$ ft–lb. We can also arrive at this via integration:

$$\begin{aligned} W &= \int_0^{50} (F_r(x) + F_b(x))\,dx \\ &= \int_0^{50} (10 - 0.2x - 1.2x + 105)\,dx \\ &= \int_0^{50} (-1.4x + 115)\,dx \\ &= 4000\,\text{ft–lb}. \end{aligned}$$

7.5.2 Hooke's Law and Springs

Hooke's Law states that the force required to compress or stretch a spring x units from its natural length is proportional to x; that is, this force is $F(x) = kx$ for some constant k. For example, if a force of 1 N stretches a given spring 2 cm, then a force of 5 N will stretch the spring 10 cm. Converting the distances to meters, we have that stretching this spring 0.02 cm requires a force of $F(0.02) = k(0.02) = 1$ N, hence $k = 1/0.02 = 50\ \frac{\text{N}}{\text{m}}$.

Example 7.5.5 Computing work performed: stretching a spring.

A force of 20 lb stretches a spring from a natural length of 7 inches to a length of 12 inches. How much work was performed in stretching the spring to this length?

Solution. In many ways, we are not at all concerned with the actual length of the spring, only with the amount of its change. Hence, we do not care that 20 lb of force stretches the spring to a length of 12 inches, but rather that a force of 20 lb stretches the spring by 5 inches. This is illustrated in Figure 7.5.6; we only measure the change in the spring's length, not the overall length of the spring.
Converting the units of length to feet, we have

$$F(5/12) = 5/12k = 20\,\text{lb}.$$

Thus $k = 48\ \frac{\text{lb}}{\text{ft}}$ and $F(x) = 48x$.
We compute the total work performed by integrating $F(x)$ from $x = 0$ to $x = 5/12$:

$$\begin{aligned} W &= \int_0^{5/12} 48x\,dx \\ &= 24x^2\Big|_0^{5/12} \\ &= 25/6 \approx 4.1667\,\text{ft-lb}. \end{aligned}$$

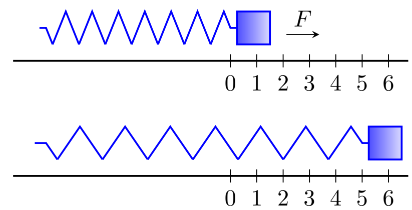


Figure 7.5.6 Illustrating the important aspects of stretching a spring in computing work in Example 7.5.5

7.5.3 Pumping Fluids

Another useful example of the application of integration to compute work comes in the pumping of fluids, often illustrated in the context of emptying a storage tank by pumping the fluid out the top. This situation is different than our previous examples for the forces involved are constant. After all, the force required to move one cubic foot of water (about 62.4 lb) is the same regardless of its location in the tank. What is variable is the distance that cubic foot of water has to travel; water closer to the top travels less distance than water at the bottom, producing less work.

Table 7.5.7 Weight and Mass densities

Fluid	lb/ft^3	kg/m^3
Concrete	150	2400
Fuel Oil	55.46	890.13
Gasoline	45.93	737.22
Iodine	307	4927
Methanol	49.3	791.3
Mercury	844	13546
Milk	63.6–65.4	1020–1050
Water	62.4	1000

We demonstrate how to compute the total work done in pumping a fluid out of the top of a tank in the next two examples.

Example 7.5.8 Computing work performed: pumping fluids.

A cylindrical storage tank with a radius of 10 ft and a height of 30 ft is filled with water, which weighs approximately $62.4\,\frac{\text{lb}}{\text{ft}^3}$. Compute the amount of work performed by pumping the water up to a point 5 feet above the top of the tank.

Solution. We will refer often to Figure 7.5.9 which illustrates the salient aspects of this problem.

We start as we often do: we partition an interval into subintervals. We orient our tank vertically since this makes intuitive sense with the base of the tank at $y=0$. Hence the top of the water is at $y=30$, meaning we are interested in subdividing the y-interval $[0,30]$ into n subintervals as

$$0 = y_0 < y_1 < \cdots < y_n = 30.$$

Consider the work W_i of pumping only the water residing in the ith subinterval, illustrated in Figure 7.5.9. The force required to move this water is equal to its weight which we calculate as volume × density. The volume of water in this subinterval is $V_i = 10^2\pi\Delta y_i$; its density is $62.4\,\frac{\text{lb}}{\text{ft}^3}$. Thus the required force is $6240\pi\Delta y_i$ lb.

We approximate the distance the force is applied by using any y-value contained in the ith subinterval; for simplicity, we arbitrarily use y_i for now (it will not matter later on). The water will be pumped to a point 5 feet above the top of the tank, that is, to the height of $y=35$ ft. Thus the distance the water at height y_i travels is $35-y_i$ ft.

In all, the approximate work W_i performed in moving the water in the ith subinterval to a point 5 feet above the tank is

$$W_i \approx 6240\pi\Delta y_i(35-y_i).$$

To approximate the total work performed in pumping out all the water from the tank, we sum all the work W_i performed in pumping the water from each of the n subintervals of $[0,30]$:

$$W \approx \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} 6240\pi\Delta y_i(35-y_i).$$

This is a Riemann sum. Taking the limit as the subinterval length goes to 0 gives

$$\begin{aligned} W &= \int_0^{30} 6240\pi(35-y)\,dy \\ &= 6240\pi\left(35y - 1/2y^2\right)\Big|_0^{30} \\ &= 11{,}762{,}123 \text{ ft–lb} \\ &\approx 1.176\times 10^7 \text{ ft–lb.} \end{aligned}$$

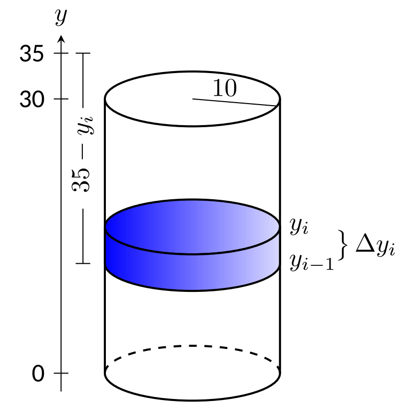


Figure 7.5.9 Illustrating a water tank in order to compute the work required to empty it in Example 7.5.8

We can “streamline” the above process a bit as we may now recognize what the important features of the problem are. Figure 7.5.10 shows the tank from Example 7.5.8 without the ith subinterval identified.

Instead, we just draw one differential element. This helps establish the height

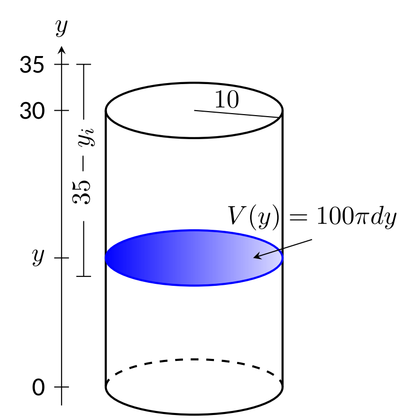


Figure 7.5.10 A simplified illustration for computing work

a small amount of water must travel along with the force required to move it (where the force is volume × density).

We demonstrate the concepts again in the next examples.

Example 7.5.11 Computing work performed: pumping fluids.

A conical water tank has its top at ground level and its base 10 feet below ground. The radius of the cone at ground level is 2 ft. It is filled with water weighing $62.4\,\frac{\text{lb}}{\text{ft}^3}$ and is to be emptied by pumping the water to a spigot 3 feet above ground level. Find the total amount of work performed in emptying the tank.

Solution. The conical tank is sketched in Figure 7.5.12. We can orient the tank in a variety of ways; we could let $y = 0$ represent the base of the tank and $y = 10$ represent the top of the tank, but we choose to keep the convention of the wording given in the problem and let $y = 0$ represent ground level and hence $y = -10$ represents the bottom of the tank. The actual "height" of the water does not matter; rather, we are concerned with the distance the water travels.

The figure also sketches a differential element, a cross-sectional circle. The radius of this circle is variable, depending on y. When $y = -10$, the circle has radius 0; when $y = 0$, the circle has radius 2. These two points, $(-10, 0)$ and $(0, 2)$, allow us to find the equation of the line that gives the radius of the cross-sectional circle, which is $r(y) = 1/5y + 2$. Hence the volume of water at this height is $V(y) = \pi(1/5y + 2)^2 dy$, where dy represents a very small height of the differential element. The force required to move the water at height y is $F(y) = 62.4 \times V(y)$.

The distance the water at height y travels is given by $h(y) = 3 - y$. Thus the total work done in pumping the water from the tank is

$$\begin{aligned} W &= \int_{-10}^{0} 62.4\pi(1/5y + 2)^2(3 - y)\,dy \\ &= 62.4\pi \int_{-10}^{0} \left(-\frac{1}{25}y^3 - \frac{17}{25}y^2 - \frac{8}{5}y + 12\right)\,dy \\ &= 62.2\pi \cdot \frac{220}{3} \approx 14,376\text{ft–lb}. \end{aligned}$$

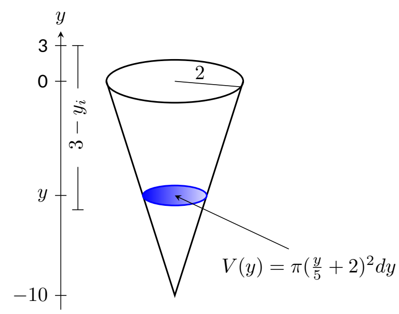


Figure 7.5.12 A graph of the conical water tank in Example 7.5.11

Example 7.5.13 Computing work performed: pumping fluids.

A rectangular swimming pool is 20 ft wide and has a 3 ft "shallow end" and a 6 ft "deep end." It is to have its water pumped out to a point 2 ft above the current top of the water. The cross-sectional dimensions of the water in the pool are given in Figure 7.5.14; note that the dimensions are for the water, not the pool itself. Compute the amount of work performed in draining the pool.

Solution. For the purposes of this problem we choose to set $y = 0$ to represent the bottom of the pool, meaning the top of the water is at $y = 6$.

Figure 7.5.15 shows the pool oriented with this y-axis, along with 2 differential elements as the pool must be split into two different regions.

The top region lies in the y-interval of $[3, 6]$, where the length of the differential element is 25 ft as shown. As the pool is 20 ft wide, this differential element represents a thin slice of water with volume $V(y) =$

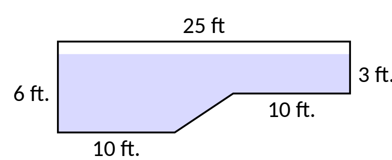


Figure 7.5.14 The cross-section of a swimming pool filled with water in Example 7.5.13

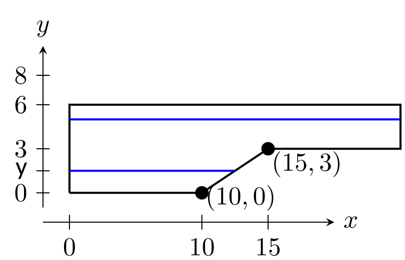


Figure 7.5.15 Orienting the pool and showing differential elements for Example 7.5.13

$20 \cdot 25 \cdot dy$. The water is to be pumped to a height of $y = 8$, so the height function is $h(y) = 8 - y$. The work done in pumping this top region of water is

$$W_t = 62.4 \int_3^6 500(8 - y)\, dy = 327,600 \text{ ft–lb.}$$

The bottom region lies in the y-interval of $[0, 3]$; we need to compute the length of the differential element in this interval.
One end of the differential element is at $x = 0$ and the other is along the line segment joining the points $(10, 0)$ and $(15, 3)$. The equation of this line is $y = 3/5(x - 10)$; as we will be integrating with respect to y, we rewrite this equation as $x = 5/3y + 10$. So the length of the differential element is a difference of x-values: $x = 0$ and $x = 5/3y + 10$, giving a length of $x = 5/3y + 10$.
Again, as the pool is 20 ft wide, this differential element represents a thin slice of water with volume $V(y) = 20 \cdot (5/3y + 10) \cdot dy$; the height function is the same as before at $h(y) = 8 - y$. The work performed in emptying this part of the pool is

$$W_b = 62.4 \int_0^3 20(5/3y + 10)(8 - y)\, dy = 299,520 \text{ ft–lb.}$$

The total work in empyting the pool is

$$W = W_b + W_t = 327,600 + 299,520 = 627,120 \text{ ft–lb.}$$

Notice how the emptying of the bottom of the pool performs almost as much work as emptying the top. The top portion travels a shorter distance but has more water. In the end, this extra water produces more work.

The next section introduces one final application of the definite integral, the calculation of fluid force on a plate.

7.5.4 Exercises

Terms and Concepts

1. What are the typical units of work?

2. If a man has a mass of 80 kg on Earth, will his mass on the moon be bigger, smaller, or the same?

3. If a woman weighs 130 lb on Earth, will her weight on the moon be bigger, smaller, or the same?

4. Fill in the blanks:
Some integrals in this section are set up by multiplying a variable __________ by a constant distance; others are set up by multiplying a constant force by a variable __________.

Problems

5. A 100 ft rope, weighing $0.1\ \frac{\text{lb}}{\text{ft}}$, hangs over the edge of a tall building.

(a) How much work is done pulling the entire rope to the top of the building?

(b) How much rope is pulled in when half of the total work is done?

6. A 50 m rope, with a mass density of $0.2\ \frac{\text{kg}}{\text{m}}$, hangs over the edge of a tall building.

(a) How much work is done pulling the entire rope to the top of the building?

(b) How much work is done pulling in the first 20 m?

7. A rope of length ℓ ft hangs over the edge of tall cliff. (Assume the cliff is taller than the length of the rope.) The rope has a weight density of $d\ \frac{\text{lb}}{\text{ft}}$.

(a) How much work is done pulling the entire rope to the top of the cliff?

(b) What percentage of the total work is done pulling in the first half of the rope?

(c) How much rope is pulled in when half of the total work is done?

8. A 20 m rope with mass density of $0.5\ \frac{\text{kg}}{\text{m}}$ hangs over the edge of a 10 m building. How much work is done pulling the rope to the top?

9. A crane lifts a 2000 lb load vertically 30 ft with a 1 in cable weighing $1.68\ \frac{\text{lb}}{\text{ft}}$.

(a) How much work is done lifting the cable alone?

(b) How much work is done lifting the load alone?

(c) Could one conclude that the work done lifting the cable is negligible compared to the work done lifting the load?

10. A100 lb bag of sand is lifted uniformly 120 ft in one minute. Sand leaks from the bag at a rate of $1/4\ \frac{\text{lb}}{\text{s}}$. What is the total work done in lifting the bag?

11. A box weighing 2 lb lifts 10 lb of sand vertically 50 ft. A crack in the box allows the sand to leak out such that 9 lb of sand is in the box at the end of the trip. Assume the sand leaked out at a uniform rate. What is the total work done in lifting the box and sand?

12. A force of 1000 lb compresses a spring 3 in. How much work is performed in compressing the spring?

13. A force of 2 N stretches a spring 5 cm. How much work is performed in stretching the spring?

14. A force of 50 lb compresses a spring from a natural length of 18 in to 12 in. How much work is performed in compressing the spring?

15. A force of 20 lb stretches a spring from a natural length of 6 in to 8 in. How much work is performed in stretching the spring?

16. A force of 7 N stretches a spring from a natural length of 11 cm to 21 cm. How much work is performed in stretching the spring from a length of 16 cm to 21 cm?

17. A force of f N stretches a spring d m from its natural length. How much work is performed in stretching the spring?

18. A 20 lb weight is attached to a spring. The weight rests on the spring, compressing the spring from a natural length of 1 ft to 6 in.
How much work is done in lifting the box 1.5 ft (i.e, the spring will be stretched 1 ft beyond its natural length)?

19. A 20 lb weight is attached to a spring. The weight rests on the spring, compressing the spring from a natural length of 1 ft to 6 in.
How much work is done in lifting the box 6 in (i.e, bringing the spring back to its natural length)?

20. A 5 m tall cylindrical tank with radius of 2 m is filled with 3 m of gasoline, with a mass density of $737.22\ \frac{\text{kg}}{\text{m}^3}$. Compute the total work performed in pumping all the gasoline to the top of the tank.

21. A 6 ft cylindrical tank with a radius of 3 ft is filled with water, which has a weight density of $62.4\ \frac{\text{lb}}{\text{ft}^3}$. The water is to be pumped to a point 2 ft above the top of the tank.

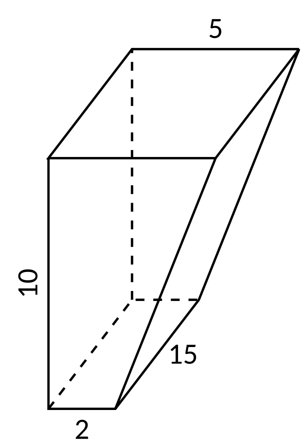
(a) How much work is performed in pumping all the water from the tank?

(b) How much work is performed in pumping 3 ft of water from the tank?

(c) At what point is 1/2 of the total work done?

22. A gasoline tanker is filled with gasoline with a weight density of $45.93\ \frac{\text{lb}}{\text{ft}^3}$. The dispensing valve at the base is jammed shut, forcing the operator to empty the tank via pumping the gas to a point 1 ft above the top of the tank. Assume the tank is a perfect cylinder, 20 ft long with a diameter of 7.5 ft. How much work is performed in pumping all the gasoline from the tank?

23. A fuel oil storage tank is 10 ft deep with trapezoidal sides, 5 ft at the top and 2 ft at the bottom, and is 15 ft wide (see diagram below). Given that fuel oil weighs $55.46\ \frac{\text{lb}}{\text{ft}^3}$, find the work performed in pumping all the oil from the tank to a point 3 ft above the top of the tank.



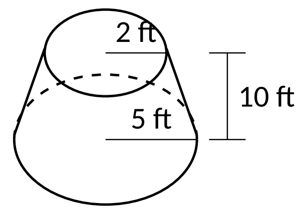
24. A conical water tank is 5 m deep with a top radius of 3 m. (This is similar to Example 7.5.11.) The tank is filled with pure water, with a mass density of $1000\ \frac{\text{kg}}{\text{m}^3}$.

(a) Find the work performed in pumping all the water to the top of the tank.

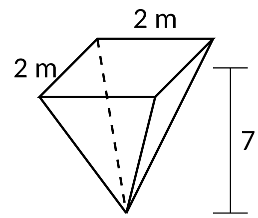
(b) Find the work performed in pumping the top 2.5 m of water to the top of the tank.

(c) Find the work performed in pumping the top half of the water, by volume, to the top of the tank.

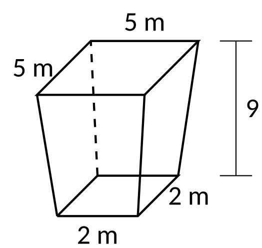
25. A water tank has the shape of a truncated cone, with dimensions given below, and is filled with water with a weight density of $62.4\ \frac{\text{lb}}{\text{ft}^3}$. Find the work performed in pumping all water to a point 1 ft above the top of the tank.



26. A water tank has the shape of an inverted pyramid, with dimensions given below, and is filled with water with a mass density of $1000\,\frac{\text{kg}}{\text{m}^3}$. Find the work performed in pumping all water to a point 5 m above the top of the tank.



27. A water tank has the shape of a truncated, inverted pyramid, with dimensions given below, and is filled with water with a mass density of $1000\,\frac{\text{kg}}{\text{m}^3}$. Find the work performed in pumping all water to a point 1 m above the top of the tank.



7.6 Fluid Forces

In the unfortunate situation of a car driving into a body of water, the conventional wisdom is that the water pressure on the doors will quickly be so great that they will be effectively unopenable. (Survival techniques suggest immediately opening the door, rolling down or breaking the window, or waiting until the water fills up the interior at which point the pressure is equalized and the door will open. See Mythbusters episode #72 to watch Adam Savage test these options.)

How can this be true? How much force does it take to open the door of a submerged car? In this section we will find the answer to this question by examining the forces exerted by fluids.

We start with *pressure*, which is related to *force* by the following equations:

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}} \Leftrightarrow \text{Force} = \text{Pressure} \times \text{Area}.$$

In the context of fluids, we have the following definition.

Definition 7.6.1 Fluid Pressure.

Let w be the weight-density of a fluid. The **pressure** p exerted on an object at depth d in the fluid is $p = w \cdot d$.

We use this definition to find the *force* exerted on a horizontal sheet by considering the sheet's area.

Example 7.6.2 Computing fluid force.

1. A cylindrical storage tank has a radius of 2 ft and holds 10 ft of a fluid with a weight-density of $50\ \frac{\text{lb}}{\text{ft}^3}$. (See Figure 7.6.3.) What is the force exerted on the base of the cylinder by the fluid?

2. A rectangular tank whose base is a 5 ft square has a circular hatch at the bottom with a radius of 2 ft. The tank holds 10 ft of a fluid with a weight-density of $50\ \frac{\text{lb}}{\text{ft}^3}$. (See Figure 7.6.4.) What is the force exerted on the hatch by the fluid?

Solution.

1. Using Definition 7.6.1, we calculate that the pressure exerted on the cylinder's base is $w \cdot d = 50\ \frac{\text{lb}}{\text{ft}^3} \times 10\,\text{ft} = 500\ \frac{\text{lb}}{\text{ft}^2}$. The area of the base is $\pi \cdot 2^2 = 4\pi$ ft^2. So the force exerted by the fluid is

$$F = 500 \times 4\pi = 6283\,\text{lb}.$$

Note that we effectively just computed the *weight* of the fluid in the tank.

2. The dimensions of the tank in this problem are irrelevant. All we are concerned with are the dimensions of the hatch and the depth of the fluid. Since the dimensions of the hatch are the same as the base of the tank in the previous part of this example, as is the depth, we see that the fluid force is the same. That is, $F = 6283$ lb. A key concept to understand here is that we are effectively measuring the weight of a 10 ft column of water above the hatch. The size of the tank holding the fluid does not matter.

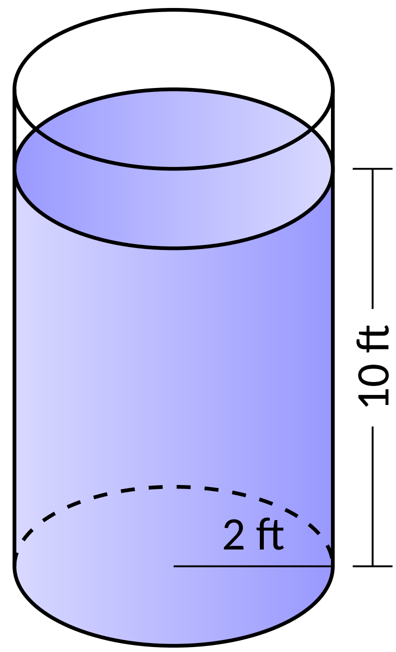


Figure 7.6.3 A cylindrical tank in Example 7.6.2

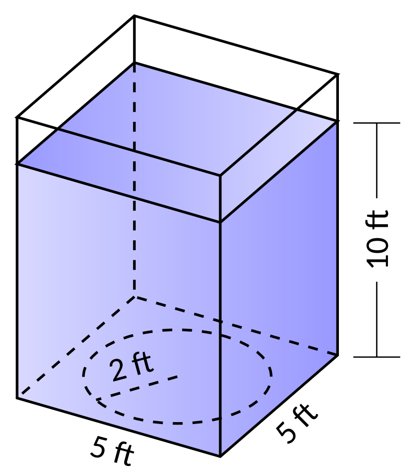


Figure 7.6.4 A rectangular tank in Example 7.6.2

The previous example demonstrates that computing the force exerted on a horizontally oriented plate is relatively easy to compute. What about a vertically oriented plate? For instance, suppose we have a circular porthole located on the side of a submarine. How do we compute the fluid force exerted on it?

Pascal's Principle states that the pressure exerted by a fluid at a depth is equal in all directions. Thus the pressure on any portion of a plate that is 1 ft below the surface of water is the same no matter how the plate is oriented. (Thus a hollow cube submerged at a great depth will not simply be "crushed" from above, but the sides will also crumple in. The fluid will exert force on *all* sides of the cube.)

So consider a vertically oriented plate as shown in Figure 7.6.5 submerged in a fluid with weight-density w. What is the total fluid force exerted on this plate? We find this force by first approximating the force on small horizontal strips.

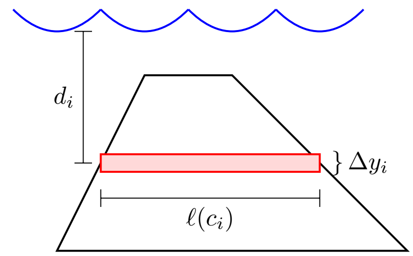


Figure 7.6.5 A thin, vertically oriented plate submerged in a fluid with weight-density w

Let the top of the plate be at depth b and let the bottom be at depth a. (For now we assume that surface of the fluid is at depth 0, so if the bottom of the plate is 3 ft under the surface, we have $a = -3$. We will come back to this later.) We partition the interval $[a, b]$ into n subintervals

$$a = y_0 < y_1 < \cdots < y_n = b,$$

with the ith subinterval having length Δy_i. The force F_i exerted on the plate in the ith subinterval is $F_i = \text{Pressure} \times \text{Area}$.

The pressure is depth times the weight density w. We approximate the depth of this thin strip by choosing any value d_i in $[y_{i-1}, y_i]$; the depth is approximately $-d_i$. (Our convention has d_i being a negative number, so $-d_i$ is positive.) For convenience, we let d_i be an endpoint of the subinterval; we let $d_i = y_i$.

The area of the thin strip is approximately length × width. The width is Δy_i. The length is a function of some y-value c_i in the ith subinterval. We state the length is $\ell(c_i)$. Thus

$$\begin{aligned} F_i &= \text{Pressure} \times \text{Area} \\ &= -y_i \cdot w \times \ell(c_i) \cdot \Delta y_i. \end{aligned}$$

To approximate the total force, we add up the approximate forces on each of the n thin strips:

$$F = \sum_{i=1}^{n} F_i \approx \sum_{i=1}^{n} -w \cdot y_i \cdot \ell(c_i) \cdot \Delta y_i.$$

This is, of course, another Riemann Sum. We can find the exact force by taking a limit as the subinterval lengths go to 0; we evaluate this limit with a definite integral.

Key Idea 7.6.6 Fluid Force on a Vertically Oriented Plate.

Let a vertically oriented plate be submerged in a fluid with weight-density w, where the top of the plate is at $y = b$ and the bottom is at $y = a$. Let $\ell(y)$ be the length of the plate at y.

1. If $y = 0$ corresponds to the surface of the fluid, then the force exerted on the plate by the fluid is

$$F = \int_a^b w \cdot (-y) \cdot \ell(y)\, dy.$$

2. In general, let $d(y)$ represent the distance between the surface of the fluid and the plate at y. Then the force exerted on the plate by the fluid is
$$F = \int_a^b w \cdot d(y) \cdot \ell(y)\, dy.$$

Example 7.6.7 Finding fluid force.

Consider a thin plate in the shape of an isosceles triangle as shown in Figure 7.6.8, submerged in water with a weight-density of $62.4\,\frac{\text{lb}}{\text{ft}^3}$. If the bottom of the plate is 10 ft below the surface of the water, what is the total fluid force exerted on this plate?

Solution. We approach this problem in two different ways to illustrate the different ways Key Idea 7.6.6 can be implemented. First we will let $y = 0$ represent the surface of the water, then we will consider an alternate convention.

1. We let $y = 0$ represent the surface of the water; therefore the bottom of the plate is at $y = -10$. We center the triangle on the y-axis as shown in Figure 7.6.9. The depth of the plate at y is $-y$ as indicated by the Key Idea. We now consider the length of the plate at y. We need to find equations of the left and right edges of the plate. The right hand side is a line that connects the points $(0, -10)$ and $(2, -6)$: that line has equation $x = 1/2(y + 10)$. (Find the equation in the familiar $y = mx + b$ format and solve for x.) Likewise, the left hand side is described by the line $x = -1/2(y+10)$. The total length is the distance between these two lines: $\ell(y) = 1/2(y + 10) - (-1/2(y + 10)) = y + 10$.

   The total fluid force is then:
$$\begin{aligned} F &= \int_{-10}^{-6} 62.4(-y)(y + 10)\, dy \\ &= 62.4 \cdot \frac{176}{3} \approx 3660.8\,\text{lb}. \end{aligned}$$

2. Sometimes it seems easier to orient the thin plate nearer the origin. For instance, consider the convention that the bottom of the triangular plate is at $(0, 0)$, as shown in Figure 7.6.10. The equations of the left and right hand sides are easy to find. They are $y = 2x$ and $y = -2x$, respectively, which we rewrite as $x = 1/2y$ and $x = -1/2y$. Thus the length function is $\ell(y) = 1/2y - (-1/2y) = y$.

   As the surface of the water is 10 ft above the base of the plate, we have that the surface of the water is at $y = 10$. Thus the depth function is the distance between $y = 10$ and y; $d(y) = 10 - y$. We compute the total fluid force as:
$$\begin{aligned} F &= \int_0^4 62.4(10 - y)(y)\, dy \\ &\approx 3660.8\,\text{lb}. \end{aligned}$$

The correct answer is, of course, independent of the placement of the plate in the coordinate plane as long as we are consistent.

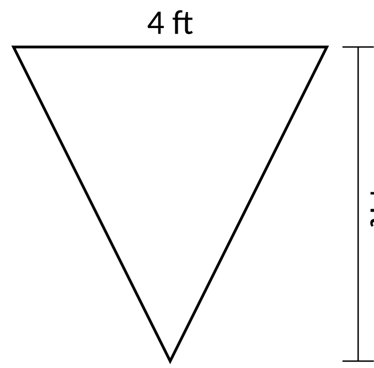


Figure 7.6.8 A thin plate in the shape of an isosceles triangle in Example 7.6.7

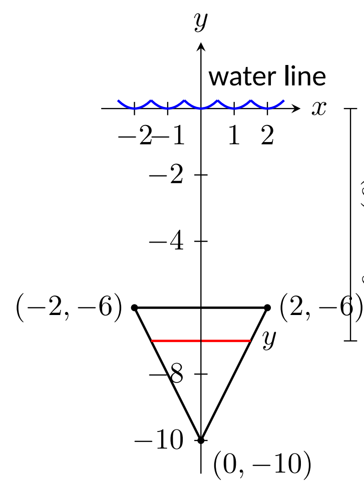


Figure 7.6.9 Sketching the triangular plate in Example 7.6.7 with the convention that the water level is at $y = 0$

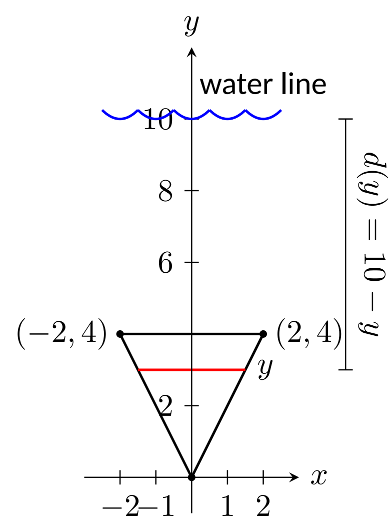


Figure 7.6.10 Sketching the triangular plate in Example 7.6.7 with the convention that the base of the triangle is at $(0, 0)$

Example 7.6.11 Finding fluid force.

Find the total fluid force on a car door submerged up to the bottom of its window in water, where the car door is a rectangle 40 in long and 27 in high (based on the dimensions of a 2005 Fiat Grande Punto.)

Solution. The car door, as a rectangle, is drawn in Figure 7.6.12. Its length is $10/3$ ft and its height is 2.25 ft. We adopt the convention that the top of the door is at the surface of the water, both of which are at $y = 0$. Using the weight-density of water of $62.4\,\frac{\text{lb}}{\text{ft}^3}$, we have the total force as

$$\begin{aligned} F &= \int_{-2.25}^{0} 62.4(-y)10/3\,dy \\ &= \int_{-2.25}^{0} -208y\,dy \\ &= -104y^2\Big|_{-2.25}^{0} \\ &= 526.5\text{ lb.} \end{aligned}$$

Most adults would find it very difficult to apply over 500 lb of force to a car door while seated inside, making the door effectively impossible to open. This is counter-intuitive as most assume that the door would be relatively easy to open. The truth is that it is not, hence the survival tips mentioned at the beginning of this section.

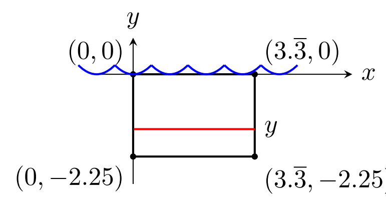


Figure 7.6.12 Sketching a submerged car door in Example 7.6.11

Example 7.6.13 Finding fluid force.

An underwater observation tower is being built with circular viewing portholes enabling visitors to see underwater life. Each vertically oriented porthole is to have a 3 ft diameter whose center is to be located 50 ft underwater. Find the total fluid force exerted on each porthole. Also, compute the fluid force on a horizontally oriented porthole that is under 50 ft of water.

Solution. We place the center of the porthole at the origin, meaning the surface of the water is at $y = 50$ and the depth function will be $d(y) = 50 - y$; see Figure 7.6.14

The equation of a circle with a radius of 1.5 is $x^2 + y^2 = 2.25$; solving for x we have $x = \pm\sqrt{2.25 - y^2}$, where the positive square root corresponds to the right side of the circle and the negative square root corresponds to the left side of the circle. Thus the length function at depth y is $\ell(y) = 2\sqrt{2.25 - y^2}$. Integrating on $[-1.5, 1.5]$ we have:

$$\begin{aligned} F &= 62.4\int_{-1.5}^{1.5} 2(50 - y)\sqrt{2.25 - y^2}\,dy \\ &= 62.4\int_{-1.5}^{1.5} \left(100\sqrt{2.25 - y^2} - 2y\sqrt{2.25 - y^2}\right)dy \\ &= 6240\int_{-1.5}^{1.5} \left(\sqrt{2.25 - y^2}\right)dy - 62.4\int_{-1.5}^{1.5} \left(2y\sqrt{2.25 - y^2}\right)dy. \end{aligned}$$

The second integral above can be evaluated using substitution. Let $u = 2.25 - y^2$ with $du = -2y\,dy$. The new bounds are: $u(-1.5) = 0$ and

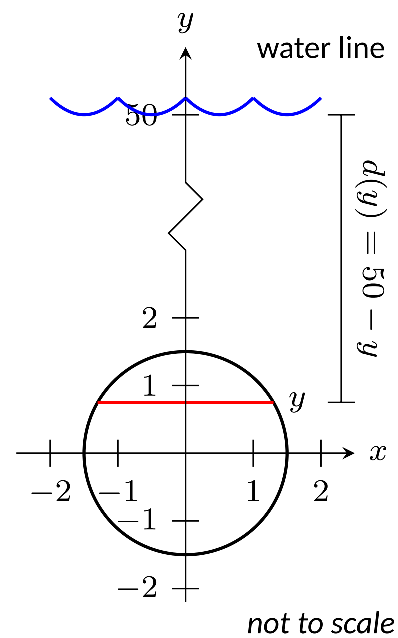


Figure 7.6.14 Measuring the fluid force on an underwater porthole in Example 7.6.13

$u(1.5) = 0$; the new integral will integrate from $u = 0$ to $u = 0$, hence the integral is 0.

The first integral above finds the area of half a circle of radius 1.5, thus the first integral evaluates to $6240 \cdot \pi \cdot 1.5^2/2 = 22,054$. Thus the total fluid force on a vertically oriented porthole is $22,054$ lb.

Finding the force on a horizontally oriented porthole is more straightforward:

$$F = \text{Pressure} \times \text{Area} = 62.4 \cdot 50 \times \pi \cdot 1.5^2 = 22,054 \text{ lb.}$$

That these two forces are equal is not coincidental; it turns out that the fluid force applied to a vertically oriented circle whose center is at depth d is the same as force applied to a horizontally oriented circle at depth d.

We end this chapter with a reminder of the true skills meant to be developed here. We are not truly concerned with an ability to find fluid forces or the volumes of solids of revolution. Work done by a variable force is important, though measuring the work done in pulling a rope up a cliff is probably not.

What we are actually concerned with is the ability to solve certain problems by first approximating the solution, then refining the approximation, then recognizing if/when this refining process results in a definite integral through a limit. Knowing the formulas found inside the special boxes within this chapter is beneficial as it helps solve problems found in the exercises, and other mathematical skills are strengthened by properly applying these formulas. However, more importantly, understand how each of these formulas was constructed. Each is the result of a summation of approximations; each summation was a Riemann sum, allowing us to take a limit and find the exact answer through a definite integral.

7.6.1 Exercises

Terms and Concepts

1. State in your own words Pascal's Principle.
2. State in your own words how pressure is different from force.

Problems

Exercise Group. In the following exercises, find the fluid force exerted on the given plate, submerged in water with a weight density of $62.4\,\frac{\text{lb}}{\text{ft}^3}$.

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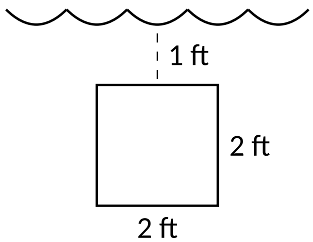

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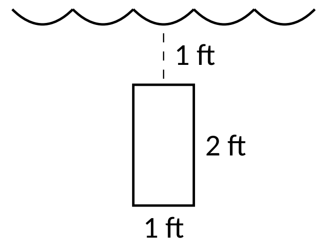

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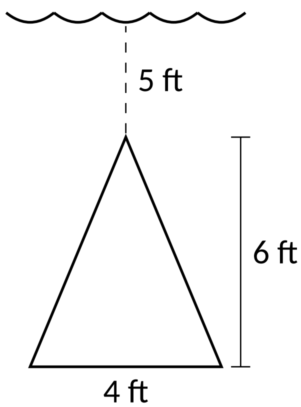

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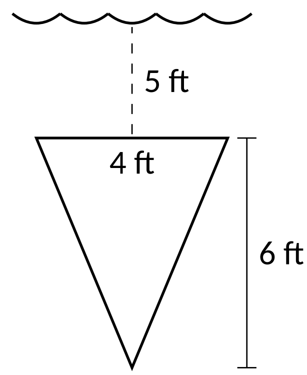

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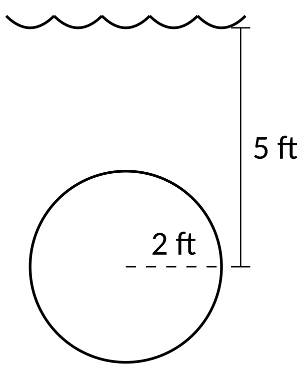

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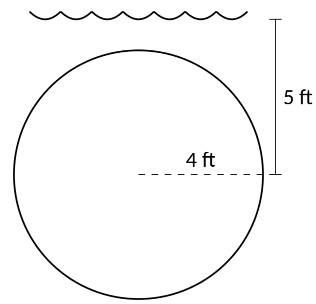

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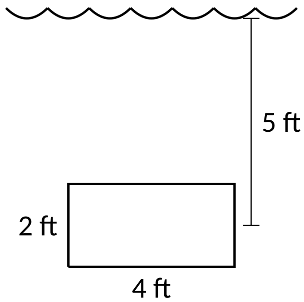

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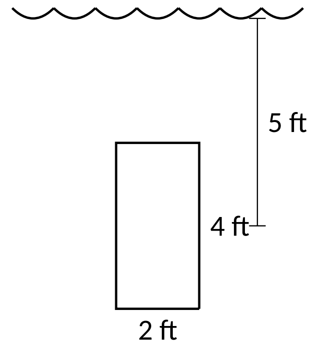

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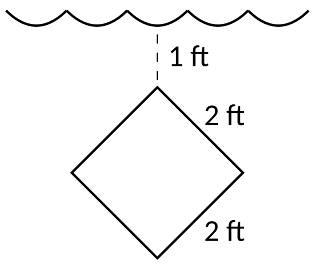

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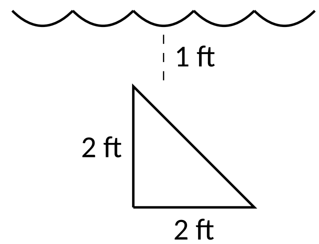
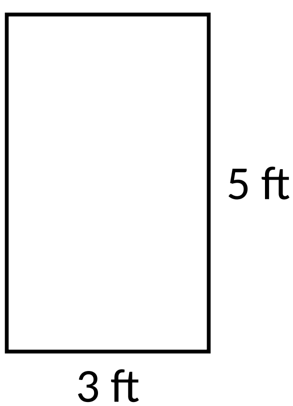


Exercise Group. In the following exercises, the side of a container is pictured. Find the fluid force exerted on this plate when the container is full of:

(a) water, with a weight density of $62.4\,\frac{\text{lb}}{\text{ft}^3}$, and

(b) concrete, with a weight density of $150\,\frac{\text{lb}}{\text{ft}^3}$.

13.



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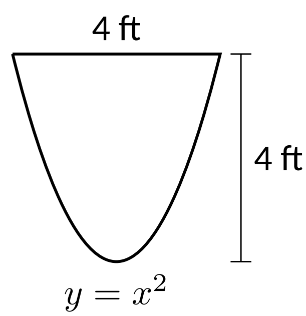

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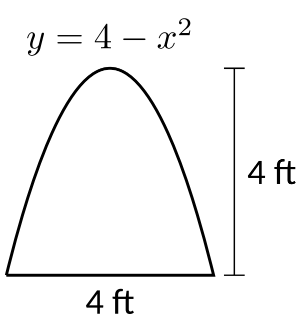

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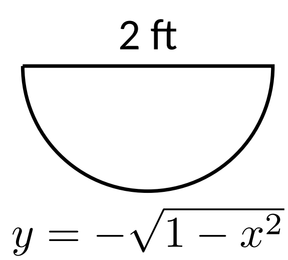

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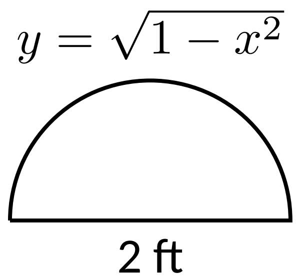

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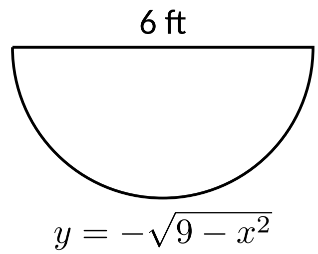


19. How deep must the center of a vertically oriented circular plate with a radius of 1 ft be submerged in water, with a weight density of $62.4\,\frac{\text{lb}}{\text{ft}^3}$, for the fluid force on the plate to reach $1,000$ lb?

20. How deep must the center of a vertically oriented square plate with a side length of 2 ft be submerged in water, with a weight density of $62.4\,\frac{\text{lb}}{\text{ft}^3}$, for the fluid force on the plate to reach 1,000 lb?

Chapter 8

Differential Equations

One of the strengths of calculus is its ability to describe real-world phenomena. We have seen hints of this in our discussion of the applications of derivatives and integrals in the previous chapters. The process of formulating an equation or multiple equations to describe a physical phenomenon is called *mathematical modeling*. As a simple example, populations of bacteria are often described as "growing exponentially." Looking in a biology text, we might see $P(t) = P_0 e^{kt}$, where $P(t)$ is the bacteria population at time t, P_0 is the initial population at time $t = 0$, and the constant k describes how quickly the population grows. This equation for exponential growth arises from the assumption that the population of bacteria grows at a rate proportional to its size. Recalling that the derivative gives the rate of change of a function, we can describe the growth assumption precisely using the equation $P' = kP$. This equation is called a *differential equation*, and these equations are the subject of the current chapter.

8.1 Graphical and Numerical Solutions to Differential Equations

In Section 5.1, we were introduced to the idea of a differential equation. Given a function $y = f(x)$, we defined a *differential equation* as an equation involving y, x, and derivatives of y. We explored the simple differential equation $y' = 2x$, and saw that a *solution* to a differential equation is simply a function that satisfies the differential equation.

youtu.be/watch?v=aevFioTbghg

Figure 8.1.1 Video introduction to Section 8.1

8.1.1 Introduction and Terminology

Definition 8.1.2 Differential Equation.

Given a function $y = f(x)$, a **differential equation** is an equation relating x, y, and derivatives of y.

- The variable x is called the *independent variable*.
- The variable y is called the *dependent variable*.
- The *order* of the differential equation is the order of the highest derivative of y that appears in the equation.

Let us return to the simple differential equation

$$y' = 2x.$$

To find a solution, we must find a function whose derivative is $2x$. In other words, we seek an antiderivative of $2x$. The function

$$y = x^2$$

is an antiderivative of $2x$, and solves the differential equation. So do the functions

$$y = x^2 + 1$$

and

$$y = x^2 - 2346.$$

We call the function

$$y = x^2 + C,$$

with C an arbitrary constant of integration, the *general solution* to the differential equation.

In order to specify the value of the integration constant C, we require additional information. For example, if we know that $y(1) = 3$, it follows that $C = 2$. This additional information is called an *initial condition*.

Note: A general solution typically includes one or more arbitrary constants. Different values of the constant(s) specify different members in the family of solutions. The particular solution to an initial value problem is the specific member in the family of solutions that corresponds to the given initial condition(s).

Definition 8.1.3 Initial Value Problem.

A differential equation paired with an initial condition (or initial conditions) is called an **initial value problem.**
The solution to an initial value problem is called a **particular solution**. A particular solution does not include arbitrary constants.
The family of solutions to a differential equation that encompasses all possible solutions is called the **general solution** to the differential equation.

Example 8.1.4 A simple first-order differential equation.

Solve the differential equation $y' = 2y$.

Solution. The solution is a function y such that differentiation yields twice the original function. Unlike our starting example, finding the solution here does not involve computing an antiderivative. Notice that "integrating both sides" would yield the result $y = \int 2y\, dx$, which is not useful. Without knowledge of the function y, we can't compute the indefinite integral. Later sections will explore systematic ways to find analytic solutions to simple differential equations. For now, a bit of thought might let us guess the solution

$$y = e^{2x}.$$

Notice that application of the chain rule yields $y' = 2e^{2x} = 2y$. Another solution is given by

$$y = -3e^{2x}.$$

In fact,

$$y = Ce^{2x},$$

where C is any constant, is the *general solution* to the differential equation because $y' = 2Ce^{2x} = 2y$.

If we are provided with a single initial condition, say $y(0) = 3/2$, we can identify $C = 3/2$ so that

$$y = \frac{3}{2}e^{2x}$$

is the *particular solution* to the initial value problem

$$y' = 2y, \text{ with } y(0) = \frac{3}{2}.$$

Figure 8.1.5 shows various members of the general solution to the differential equation $y' = 2y$. Each C value yields a different member of the family, and a different function. We emphasize the particular solution corresponding to the initial condition $y(0) = 3/2$.

Video solution

youtu.be/watch?v=PAn_TrwF27M

Example 8.1.6 A second-order differential equation.

Solve the differential equation $y'' + 9y = 0$.

Solution. We seek a function whose second derivative is negative 9 multiplied by the original function. Both $\sin(3x)$ and $\cos(3x)$ have this feature. The general solution to the differential equation is given by

$$y = C_1 \sin(3x) + C_2 \cos(3x),$$

where C_1 and C_2 are arbitrary constants. To fully specify a particular solution, we require two additional conditions. For example, the initial conditions $y(0) = 1$ and $y'(0) = 3$ yield $C_1 = C_2 = 1$.

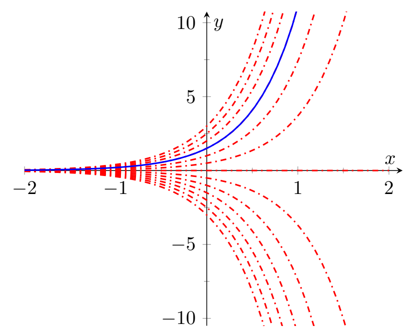


Figure 8.1.5 A representation of some of the members of general solution to the differential equation $y' = 2y$, including the particular solution to the initial value problem with $y(0) = 3/2$, from Example 8.1.4

The differential equation in Example 8.1.6 is *second order*, because the equation involves a second derivative. In general, the number of initial conditions required to specify a particular solution depends on the order of the differential equation. For the remainder of the chapter, we restrict our attention to first order differential equations and first order initial value problems.

Example 8.1.7 Verifying a solution to the differential equation.

Which of the following is a solution to the differential equation

$$y' + \frac{y}{x} - \sqrt{y} = 0?$$

(a) $y = C\left(1 + \ln(x)\right)^2$

(b) $y = \left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)^2$

(c) $y = Ce^{-3x} + \sqrt{\sin(x)}$

Solution. Verifying a solution to a differential equation is simply an exercise in differentiation and simplification. We substitute each potential solution into the differential equation to see if it satisfies the equation.

(a) Testing the potential solution $y = C\left(1 + \ln(x)\right)^2$:

Differentiating, we have $y' = \dfrac{2C(1 + \ln(x))}{x}$. Substituting into the differential equation,

$$\begin{aligned}&\frac{2C(1+\ln(x))}{x}+\frac{C(1+\ln(x))^2}{x}-\sqrt{C}(1+\ln(x))\\&=(1+\ln(x))\left(\frac{2C}{x}+\frac{C(1+\ln(x))}{x}-\sqrt{C}\right)\\&\neq 0.\end{aligned}$$

Since it doesn't satisfy the differential equation, $y = C(1+\ln(x))^2$ is *not* a solution.

(b) Testing the potential solution $y = \left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)^2$:

Differentiating, we have $y' = 2\left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)\left(\frac{1}{3} - \frac{C}{2x^{3/2}}\right)$. Substituting into the differential equation,

$$\begin{aligned}&2\left(\frac{1}{3}x+\frac{C}{\sqrt{x}}\right)\left(\frac{1}{3}-\frac{C}{2x^{3/2}}\right)+\frac{1}{x}\left(\frac{1}{3}x+\frac{C}{\sqrt{x}}\right)^2-\left(\frac{1}{3}x+\frac{C}{\sqrt{x}}\right)\\&=\left(\frac{1}{3}x+\frac{C}{\sqrt{x}}\right)\left(\frac{2}{3}-\frac{C}{x^{3/2}}+\frac{1}{3}+\frac{C}{x^{3/2}}-1\right)\\&=0.\end{aligned}$$

(Note how the second parenthetical grouping above reduces to 0.)

Thus $y = \left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)^2$ *is* a solution to the differential equation.

(c) Testing the potential solution $y = Ce^{-3x} + \sqrt{\sin(x)}$:

Differentiating, $y' = -3Ce^{-3x} + \frac{\cos(x)}{2\sqrt{\sin(x)}}$. Substituting into the differential equation,

$$-3Ce^{-3x}+\frac{\cos(x)}{2\sqrt{\sin(x)}}+\frac{Ce^{-3x}+\sqrt{\sin(x)}}{x}-\sqrt{Ce^{-3x}+\sqrt{\sin(x)}}\neq 0.$$

The function $y = Ce^{-3x} + \sqrt{\sin(x)}$ is *not* a solution to the differential equation.

Video solution

youtu.be/watch?v=bf_WyPauK0Y

Example 8.1.8 Verifying a solution to a differential equation.

Verify that $x^2 + y^2 = Cy$ is a solution to $y' = \frac{2xy}{x^2 - y^2}$.

Solution. The solution in this example is called an *implicit solution*. That means the dependent variable y is a function of x, but has not been explicitly solved for. Verifying the solution still involves differentiation, but we must take the derivatives implicitly. Differentiating, we have

$$2x + 2yy' = Cy'.$$

Solving for y', we have

$$y' = \frac{2x}{C - 2y}.$$

From the solution, we know that $C = \dfrac{x^2 + y^2}{y}$. Then

$$\begin{aligned} y' &= \frac{2x}{\dfrac{x^2+y^2}{y} - 2y} \\ &= \frac{2xy}{x^2 + y^2 - 2y^2} \\ &= \frac{2xy}{x^2 - y^2}. \end{aligned}$$

We have verified that $x^2 + y^2 = Cy$ is a solution to $y' = \dfrac{2xy}{x^2 - y^2}$.

Video solution

youtu.be/watch?v=B0gxkvJf9oY

8.1.2 Graphical Solutions to Differential Equations

In the examples we have explored so far, we have found exact forms for the functions that solve the differential equations. Solutions of this type are called *analytic solutions*. Many times a differential equation has a solution, but it is difficult or impossible to find the solution analytically. This is analogous to algebraic equations. The algebraic equation $x^2+3x-1 = 0$ has two real solutions that can be found analytically by using the quadratic formula. The equation $\cos(x) = x$ has one real solution, but we can't find it analytically. As shown in Figure 8.1.9, we can find an approximate solution graphically by plotting $\cos(x)$ and x and observing the x-value of the intersection. We can similarly use graphical tools to understand the qualitative behavior of solutions to a first order-differential equation.

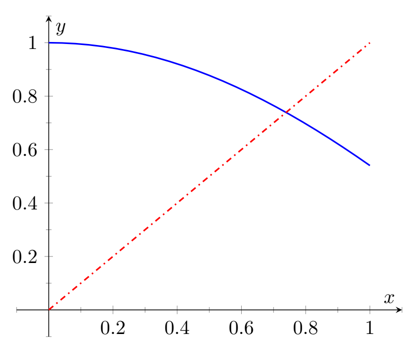


Figure 8.1.9 Graphically finding an approximate solution to $\cos(x) = x$

Consider the first-order differential equation

$$y' = f(x, y).$$

The function f could be any function of the two variables x and y. Written in this way, we can think of the function f as providing a formula to find the slope of a solution at a given point in the xy-plane. In other words, suppose a solution to the differential equation passes through the point (x_0, y_0). At the point (x_0, y_0), the slope of the solution curve will be $f(x_0, y_0)$. Since this calculation of the slope is possible at any point (x, y) where the function $f(x, y)$ is defined, we can produce a plot called a *slope field* (or direction field) that shows the slope of a solution at any point in the xy-plane where the solution is defined. Further, this process can be done purely by working with the differential equation itself. In other words, we can draw a slope field and use it to determine the qualitative behavior of solutions to a differential equation without having to solve the differential equation.

Definition 8.1.10 Slope Field.

A **slope field** for a first-order differential equation $y' = f(x, y)$ is a plot in the xy-plane made up of short line segments or arrows. At each point (x_0, y_0) where $f(x, y)$ is defined, the slope of the line segment is given by $f(x_0, y_0)$. Plots of solutions to a differential equation are tangent to the line segments in the slope field.

Example 8.1.11 Sketching a slope field.

Find a slope field for the differential equation $y' = x + y$.

Solution. Because the function $f(x, y) = x + y$ is defined for all points (x, y), every point in the xy-plane has an associated line segment. It is not practical to draw an entire slope field by hand, but many tools exist for drawing slope fields on a computer. Here, we explicitly calculate a few of the line segments in the slope field.

- The slope of the line segment at $(0, 0)$ is $f(0, 0) = 0 + 0 = 0$.
- The slope of the line segment at $(1, 1)$ is $f(1, 1) = 1 + 1 = 2$.
- The slope of the line segment at $(1, -1)$ is $f(1, -1) = 1 - 1 = 0$.
- The slope of the line segment at $(-2, -1)$ is $f(-2, -1) = -2 - 1 = -3$.

Though it is possible to continue this process to sketch a slope field, we usually use a computer to make the drawing. Most popular computer algebra systems can draw slope fields. There are also various online tools that can make the drawings. The slope field for $y' = x + y$ is shown in Figure 8.1.12.

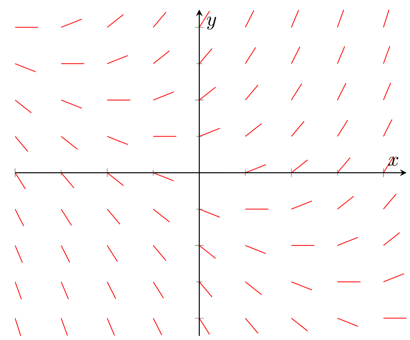


Figure 8.1.12 Slope field for $y' = x + y$ from Example 8.1.11

Example 8.1.13 A graphical solution to an initial value problem.

Approximate, with a sketch, the solution to the initial value problem $y' = x + y$, with $y(1) = -1$.

Solution. The solution to the initial value problem should be a continuous smooth curve. Using the slope field, we can draw of a sketch of the solution using the following two criteria:

1. The solution must pass through the point $(1, -1)$.
2. When the solution passes through a point (x_0, y_0) it must be tangent to the line segment at (x_0, y_0).

Essentially, we sketch a solution to the initial value problem by starting at the point $(1, -1)$ and "following the lines" in either direction. A sketch of the solution is shown in Figure 8.1.14.

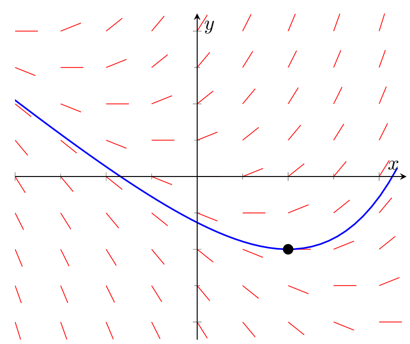


Figure 8.1.14 Solution to the initial value problem $y' = x + y$, with $y(1) = -1$ from Example 8.1.13

Example 8.1.15 Using a slope field to predict long term behavior.

Use the slope field for the differential equation $y' = y(1 - y)$, shown in Figure 8.1.16, to predict long term behavior of solutions to the equation.

Solution. This differential equation, called the *logistic differential equation*, often appears in population biology to describe the size of a population. For that reason, we use t (time) as the independent variable instead of x. We also often restrict attention to non-negative y-values because negative values correspond to a negative population.

Looking at the slope field in Figure 8.1.16, we can predict long term be-

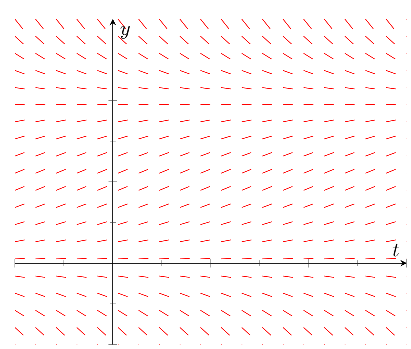


Figure 8.1.16 Slope field for the logistic differential equation $y' = y(1 - y)$ from Example 8.1.15

havior for a given initial condition.

- If the initial y-value is negative ($y(0) < 0$), the solution curve must pass though the point $(0, y(0))$ and follow the slope field. We expect the solution y to become more and more negative as time increases. Note that this result is not physically relevant when considering a population.
- If the initial y-value is greater than 0 but less than 1, we expect the solution y to increase and level off at $y = 1$.
- If the initial y-value is greater than 1, we expect the solution y to decrease and level off at $y = 1$.

The slope field for the logistic differential equation, along with representative solution curves, is shown in Figure 8.1.17. Notice that any solution curve with positive initial value will tend towards the value $y = 1$. We call this the *carrying capacity*.

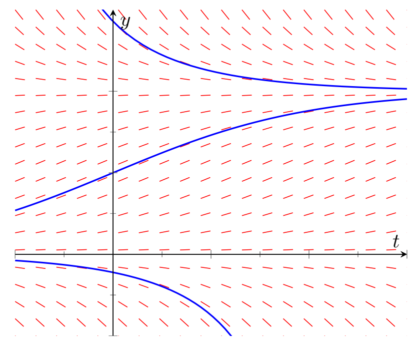


Figure 8.1.17 Slope field for the logistic differential equation $y' = y(1-y)$ from Example 8.1.15 with a few representative solution curves

8.1.3 Numerical Solutions to Differential Equations: Euler's Method

While the slope field is an effective way to understand the qualitative behavior of solutions to a differential equation, it is difficult to use a slope field to make quantitative predictions. For example, if we have the slope field for the differential equation $y' = x + y$ from Example 8.1.11 along with the initial condition $y(0) = 1$, we can understand the qualitative behavior of the solution to the initial value problem, but will struggle to predict a specific value, $y(2)$ for example, with any degree of confidence. The most straightforward way to predict $y(2)$ is to find the analytic solution to the the initial value problem and evaluate it at $x = 2$. Unfortunately, we have already mentioned that it is impossible to find analytic solutions to many differential equations. In the absence of an analytic solution, a *numerical solution* can serve as an effective tool to make quantitative predictions about the solution to an initial value problem.

There are many techniques for computing numerical solutions to initial value problems. A course in numerical analysis will discuss various techniques along with their strengths and weaknesses. The simplest technique is called *Euler's Method*.

Euler's Method is named for Leonhard Euler, a prolific Swiss mathematician during the 1700's. His last name is properly pronounced "oil-er", not "you-ler."

Consider the first-order initial value problem

$$y' = f(x, y), \text{ with } y(x_0) = y_0.$$

Using the definition of the derivative,

$$y'(x) = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h}.$$

This notation can be confusing at first, but "$y(x)$" simply means "the y-value of the solution when the x-value is x", and "$y(x + h)$" means "the y-value of the solution when the x-value is $x + h$".

If we remove the limit but restrict h to be "small," we have

$$y'(x) \approx \frac{y(x+h) - y(x)}{h},$$

so that

$$f(x, y) \approx \frac{y(x+h) - y(x)}{h},$$

because $y' = f(x, y)$ according to the differential equation. Rearranging terms,

$$y(x + h) \approx y(x) + h\,f(x, y).$$

This statement says that if we know the solution (y-value) to the initial value problem for some given x-value, we can find an approximation for the solution at the value $x+h$ by taking our y-value and adding h times the function f evaluated at the x and y values. Euler's method uses the initial condition of an initial value problem as the starting point, and then uses the above idea to find approximate values for the solution y at later x-values. The algorithm is summarized in Key Idea 8.1.18.

Key Idea 8.1.18 Euler's Method.

Consider the initial value problem

$$y' = f(x, y) \text{ with } y(x_0) = y_0.$$

Let h be a small positive number and N be an integer.

1. For $i = 0, 1, 2, \ldots, N$, define

$$x_i = x_0 + ih.$$

2. The value y_0 is given by the initial condition. For $i = 0, 1, 2, \ldots, N - 1$, define

$$y_{i+1} = y_i + hf(x_i, y_i).$$

This process yields a sequence of $N+1$ points (x_i, y_i) for $i = 0, 1, 2, \ldots, N$, where (x_i, y_i) is an approximation for $(x_i, y(x_i))$.

Let's practice Euler's Method using a few concrete examples.

Example 8.1.19 Using Euler's Method 1.

Find an approximation at $x = 2$ for the solution to $y' = x + y$ with $y(1) = -1$ using Euler's Method with $h = 0.5$.

Solution. Our initial condition yields the starting values $x_0 = 1$ and $y_0 = -1$. With $h = 0.5$, it takes $N = 2$ steps to get to $x = 2$. Using steps 1 and 2 from the Euler's Method algorithm,

$$\begin{aligned} x_0 &= 1 & y_0 &= -1\\ x_1 &= x_0 + h & y_1 &= y_0 + hf(x_0, y_0)\\ &= 1 + 0.5 & &= -1 + 0.5(1 - 1)\\ &= 1.5 & &= -1\\ x_2 &= x_0 + 2h & y_2 &= y_1 + hf(x_1, y_1)\\ &= 1 + 2(0.5) & &= -1 + 0.5(1.5 - 1)\\ &= 2 & &= -0.75. \end{aligned}$$

Using Euler's method, we find the approximate $y(2) \approx -0.75$.
To help visualize the Euler's method approximation, these three points (connected by line segments) are plotted along with the analytical solution to the initial value problem in Figure 8.1.20.

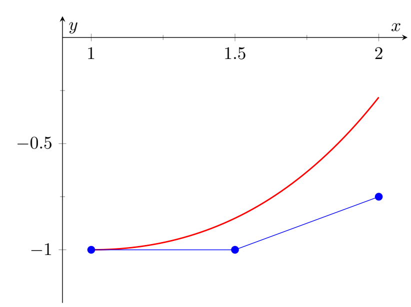


Figure 8.1.20 Euler's Method approximation to $y' = x + y$ with $y(1) = -1$ from Example 8.1.19, along with the analytical solution to the initial value problem

This approximation doesn't appear terrific, though it is better than merely guessing. Let's repeat the previous example using a smaller h-value.

Example 8.1.21 Using Euler's Method 2.

Find an approximation on the interval $[1, 2]$ for the solution to $y' = x+y$ with $y(1) = -1$ using Euler's Method with $h = 0.25$.

Solution. Our initial condition yields the starting values $x_0 = 1$ and $y_0 = -1$. With $h = 0.25$, we need $N = 4$ steps on the interval $[1, 2]$ Using steps 1 and 2 from the Euler's Method algorithm (and rounding to 4 decimal points), we have

$$\begin{aligned}
x_0 &= 1 & y_0 &= -1\\
x_1 &= 1.25 & y_1 &= -1 + 0.25(1-1)\\
& & &= -1\\
x_2 &= 1.5 & y_2 &= -1 + 0.25(1.25-1)\\
& & &= -0.9375\\
x_3 &= 1.75 & y_3 &= -0.9375 + 0.25(1.5-0.9375)\\
& & &= -0.7969\\
x_4 &= 2 & y_4 &= -0.7969 + 0.25(1.75-0.7969)\\
& & &= -0.5586.
\end{aligned}$$

Using Euler's method, we find $y(2) \approx -0.5586$.
These five points, along with the points from Example 8.1.19 and the analytic solution, are plotted in Figure 8.1.22.

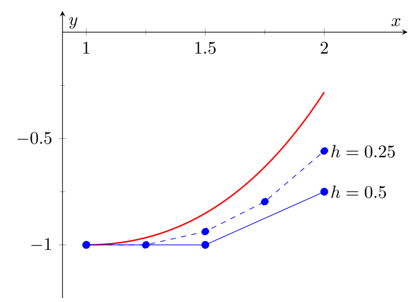


Figure 8.1.22 Euler's Method approximations to $y' = x + y$ with $y(1) = -1$ from Examples 8.1.19 and 8.1.21, along with the analytical solution

Using the results from Examples 8.1.19 and 8.1.21, we can make a few observations about Euler's method. First, the Euler approximation generally gets worse as we get farther from the initial condition. This is because Euler's method involves two sources of error. The first comes from the fact that we're using a positive h-value in the derivative approximation instead of using a limit as h approaches zero. Essentially, we're using a linear approximation to the solution y (similar to the process described in Section 4.4 on Differentials.) This error is often called the *local truncation error.* The second source of error comes from the fact that every step in Euler's method uses the result of the previous step. That means we're using an approximate y-value to approximate the next y-value. Doing this repeatedly causes the errors to build on each other. This second type of error is often called the *propagated* or *accumulated error.*

A second observation is that the Euler approximation is more accurate for smaller h-values. This accuracy comes at a cost, though. Example 8.1.21 is more accurate than Example 8.1.19, but takes twice as many computations. In general, numerical algorithms (even when performed by a computer program) require striking a balance between a desired level of accuracy and the amount of computational effort we are willing to undertake.

Let's do one final example of Euler's Method.

Example 8.1.23 Using Euler's Method 3.

Find an approximation for the solution to the logistic differential equation
$y' = y(1-y)$ with $y(0) = 0.25$, for $0 \le y \le 4$. Use $N = 10$ steps.

Solution. The logistic differential equation is what is called an *autonomous equation*. An autonomous differential equation has no explicit dependence on the independent variable (t in this case). This has no real effect on the application of Euler's method other than the fact that the function $f(t, y)$ is really just a function of y. To take steps in the y variable, we use

$$y_{i+1} = y_i + hf(t_i, y_i) = y_i + hy_i(1 - y_i).$$

Using $N = 10$ steps requires $h = \dfrac{4-0}{10} = 0.4$. Implementing Euler's Method, we have

$$\begin{aligned}
x_0 &= 0 & y_0 &= 0.25\\
x_1 &= 0.4 & y_1 &= 0.25 + 0.4(0.25)(1 - 0.25)\\
& & &= 0.325\\
x_2 &= 0.8 & y_2 &= 0.325 + 0.4(0.325)(1 - 0.325)\\
& & &= 0.41275\\
x_3 &= 1.2 & y_3 &= 0.41275 + 0.4(0.41275)(1 - 0.41275)\\
& & &= 0.50970\\
x_4 &= 1.6 & y_4 &= 0.50970 + 0.4(0.50970)(1 - 0.50970)\\
& & &= 0.60966\\
x_5 &= 2.0 & y_5 &= 0.60966 + 0.4(0.60966)(1 - 0.60966)\\
& & &= 0.70485\\
x_6 &= 2.4 & y_6 &= 0.70485 + 0.4(0.70485)(1 - 0.70485)\\
& & &= 0.78806\\
x_7 &= 2.8 & y_7 &= 0.78806 + 0.4(0.78806)(1 - 0.78806)\\
& & &= 0.85487\\
x_8 &= 3.2 & y_8 &= 0.85487 + 0.4(0.85487)(1 - 0.85487)\\
& & &= 0.90450\\
x_9 &= 3.6 & y_9 &= 0.90450 + 0.4(0.90450)(1 - 0.90450)\\
& & &= 0.93905\\
x_{10} &= 4.0 & y_{10} &= 0.93905 + 0.4(0.93905)(1 - 0.93905)\\
& & &= 0.96194.
\end{aligned}$$

These 11 points, along with the the analytic solution, are plotted in Figure 8.1.24. Notice how well they seem to match the true solution.

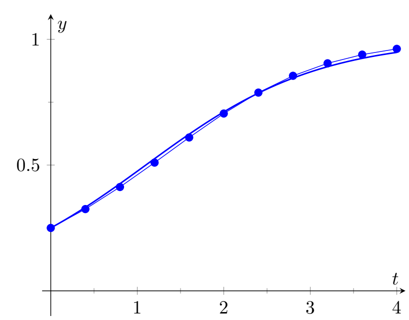


Figure 8.1.24 Euler's Method approximation to $y' = y(1 - y)$ with $y(0) = 0.25$ from Example 8.1.23, along with the analytical solution

The study of differential equations is a natural extension of the study of derivatives and integrals. The equations themselves involve derivatives, and methods to find analytic solutions often involve finding antiderivatives. In this section, we focus on graphical and numerical techniques to understand solutions to differential equations. We restrict our examples to relatively simple initial value problems that permit analytic solutions to the equations, but we should remember that this is only for comparison purposes. In reality, many differential equations, even some that appear straightforward, do not have solutions we can find analytically. Even so, we can use the techniques presented in this section to understand the behavior of solutions. In the next two sections, we explore two techniques to find analytic solutions to two different classes of differential equations.

8.1.4 Exercises

Terms and Concepts

1. In your own words, what is an initial value problem, and how is it different than a differential equation?

2. In your own words, describe what it means for a function to be a solution to a differential equation.

3. How can we verify that a function is a solution to a differential equation?

4. Describe the difference between a particular solution and a general solution.

5. Why might we use a graphical or numerical technique to study solutions to a differential equation instead of simply solving the differential equation to find an analytic solution?

6. Describe the considerations that should be made when choosing an h value to use in a numerical method like Euler's Method.

Problems

Exercise Group. In the following exercises, verify that the given function is a solution to the differential equation or initial value problem.

7. $y = Ce^{-6x^2}$; $y' = -12xy$.

8. $y = x\sin(x)$; $y' - x\cos(x) = (x^2+1)\sin(x) - xy$, with $y(\pi) = 0$.

9. $2x^2 - y^2 = C$; $yy' - 2x = 0$

10. $y = xe^x$; $y'' - 2y' + y = 0$

Exercise Group. In the following exercises, verify that the given function is a solution to the differential equation and find the C value required to make the function satisfy the initial condition.

11. $y = 4e^{3x}\sin(x) + Ce^{3x}$; $y' - 3y = 4e^{3x}\cos(x)$, with $y(0) = 2$

12. $y(x^2+y) = C$; $2xy + (x^2+2y)y' = 0$, with $y(1) = 2$

Exercise Group. In the following exercises, sketch a slope field for the given differential equation. Let x and y range between -2 and 2.

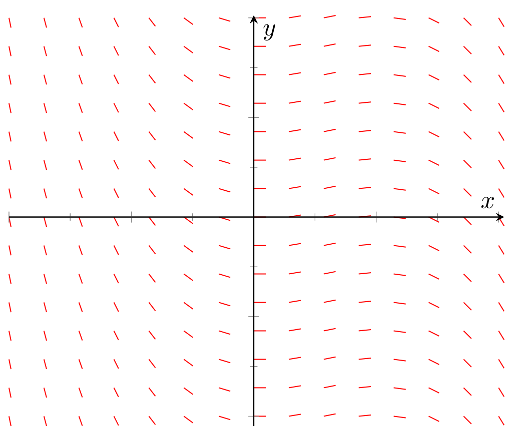
13. $y' = y - x$

14. $y' = \dfrac{x}{2y}$

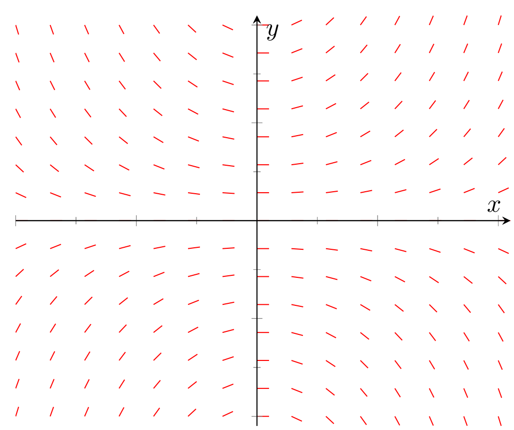
15. $y' = \sin(\pi y)$

16. $y' = \frac{y}{4}$

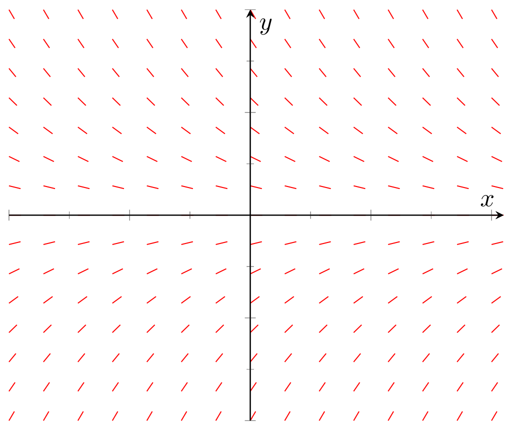
Exercise Group. Match each slope field below with the appropriate differential equation.



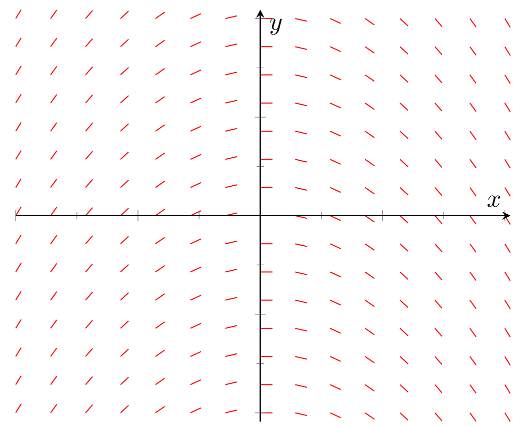
(a)



(b)



(c)



(d)

17. $y' = xy$

18. $y' = -y$

19. $y' = -x$

20. $y' = x(1-x)$

Exercise Group. In the following exercises, sketch the slope field for the differential equation, and use it to draw a sketch of the solution to the initial value problem.

21. $y' = \dfrac{y}{x} - y$, with $y(0.5) = 1$.

22. $y' = y\,\text{sin}(x)$, with $y(0) = 1$.

23. $y' = y^2 - 3y + 2$, with $y(0) = 2$.

24. $y' = -\dfrac{xy}{1+x^2}$, with $y(0) = 1$.

Exercise Group. In the following exercises, use Euler's Method to make a table of values that approximates the solution to the initial value problem on the given interval. Use the specified h or N value.

25. $y' = x + 2y$
$y(0) = 1$
interval: $[0, 1]$
$h = 0.25$

26. $y' = xe^{-y}$
$y(0) = 1$
interval: $[0, 0.5]$
$N = 5$

27. $y' = y + \text{sin}(x)$
$y(0) = 2$
interval: $[0, 1]$
$h = 0.2$

28. $y' = e^{x-y}$
$y(0) = 0$
interval: $[0, 2]$
$h = 0.5$

Exercise Group. In the following exercises, use the provided solution $y(x)$ and Euler's Method with the $h = 0.2$ and $h = 0.1$ to complete the following table.

x	0.0	0.2	0.4	0.6	0.8	1.0
$y(x)$						
$h = 0.2$						
$h = 0.1$						

29. $y' = xy^2$
$y(0) = 1$
Solution: $y(x) = \dfrac{2}{1-x^2}$

30. $y' = xe^{x^2} + \dfrac{1}{2}xy$
$y(0) = \dfrac{1}{2}$
Solution: $y(x) = \dfrac{1}{2}(x^2+1)e^{x^2}$

8.2 Separable Differential Equations

There are specific techniques that can be used to solve specific types of differential equations. This is similar to solving algebraic equations. In algebra, we can use the quadratic formula to solve a quadratic equation, but not a linear or cubic equation. In the same way, techniques that can be used for a specific type of differential equation are often ineffective for a differential equation of a different type. In this section, we describe and practice a technique to solve a class of differential equations called *separable equations*.

youtu.be/watch?v=tIkZsA3kK6o

Figure 8.2.1 Video introduction to Section 8.2

Definition 8.2.2 Separable Differential Equation.

A **separable differential equation** is one that can be written in the form

$$n(y)\frac{dy}{dx} = m(x),$$

where n is a function that depends only on the dependent variable y, and m is a function that depends only on the independent variable x.

Below, we show a few examples of separable differential equations, along with similar looking equations that are not separable.

1. $\dfrac{dy}{dx} = x^2 y$
2. $y\sqrt{y^2-5}\dfrac{dy}{dx} - \sin(x)\cos(y) = 0$
3. $\dfrac{dy}{dx} = \dfrac{(x^2+1)e^y}{y}$

List 8.2.3 Separable

1. $\dfrac{dy}{dx} = x^2 + y$
2. $y\sqrt{y^2-5}\dfrac{dy}{dx} - \sin(x)\cos(y) = 1$
3. $\dfrac{dy}{dx} = \dfrac{(xy+1)e^y}{y}$

List 8.2.4 Not Separable

Notice that a separable equation requires that the functions of the dependent and independent variables be multiplied, not added (like Item 1 in List 8.2.4). An alternate definition of a separable differential equation states that an equation is separable if it can be written in the form

$$\frac{dy}{dx} = f(x)g(y),$$

for some functions f and g.

8.2.1 Separation of Variables

Let's find a formal solution to the separable equation

$$n(y)\frac{dy}{dx} = m(x).$$

Since the functions on the left and right hand sides of the equation are equal, their antiderivatives should be equal up to an arbitrary constant of integration. That is

$$\int n(y)\frac{dy}{dx}\,dx = \int m(x)\,dx + C.$$

Though the integral on the left may look a bit strange, recall that y itself is a function of x. Consider the substitution $u = y(x)$. The differential is $du =$

$\frac{dy}{dx}\,dx$. Using this substitution, the above equation becomes

$$\int n(u)\,du = \int m(x)\,dx + C.$$

Let $N(u)$ and $M(x)$ be antiderivatives of $n(u)$ and $m(x)$, respectively. Then

$$N(u) = M(x) + C.$$

Since $u = y(x)$, this is

$$N(y) = M(x) + C.$$

This relationship between y and x is an implicit form of the solution to the differential equation. Sometimes (but not always) it is possible to solve for y to find an explicit version of the solution.

Though the technique outlined above is formally correct, what we did essentially amounts to integrating the function n with respect to its variable and integrating the function m with respect to its variable. The informal way to solve a separable equation is to treat the derivative $\frac{dy}{dx}$ as if it were a fraction. The separated form of the equation is

$$n(y)\,dy = m(x)\,dx.$$

To solve, we integrate the left hand side with respect to y and the right hand side with respect to x and add a constant of integration. As long as we are able to find the antiderivatives, we can find an implicit form for the solution. Sometimes we are able to solve for y in the implicit solution to find an explicit form of the solution to the differential equation. We practice the technique by solving the three differential equations listed in the separable column above, and conclude by revisiting and finding the general solution to the logistic differential equation from Section 8.1.

Example 8.2.5 Solving a Separable Differential Equation.

Find the general solution to the differential equation $y' = x^2y$.

Solution. Using the informal solution method outlined above, we treat $\frac{dy}{dx}$ as a fraction, and write the separated form of the differential equation as

$$\frac{dy}{y} = x^2dx.$$

Integrating the left hand side of the equation with respect to y and the right hand side of the equation with respect to x yields

$$\ln|y| = \frac{1}{3}x^3 + C.$$

This is an implicit form of the solution to the differential equation. Solving for y yields an explicit form for the solution. Exponentiating both sides, we have

$$|y| = e^{x^3/3+C} = e^{x^3/3}e^C.$$

This solution is a bit problematic. First, the absolute value makes the solution difficult to understand. The second issue comes from our desire to find the *general solution*. Recall that a general solution includes all possible solutions to the differential equation. In other words, for any

The indefinite integrals $\int \frac{dy}{y}$ and $\int x^2\,dx$ both produce arbitrary constants. Since both constants are arbitrary, we combine them into a single constant of integration.

given initial condition, the general solution must include the solution to that specific initial value problem. We can often satisfy any given initial condition by choosing an appropriate C value. When solving separable equations, though, it is possible to lose solutions that have the form $y =$ constant. Notice that $y = 0$ solves the differential equation, but it is not possible to choose a finite C to make our solution look like $y = 0$. Our solution cannot solve the initial value problem $\frac{dy}{dx} = x^2 y$, with $y(a) = 0$ (where a is any value). Thus, we haven't actually found a general solution to the problem. We can clean up the solution and recover the missing solution with a bit of clever thought.

Recall the formal definition of the absolute value: $|y| = y$ if $y \geq 0$ and $|y| = -y$ if $y < 0$. Our solution is either $y = e^C e^{\frac{x^3}{3}}$ or $y = -e^C e^{\frac{x^3}{3}}$. Further, note that C is constant, so e^C is also constant. If we write our solution as $y = Ae^{\frac{x^3}{3}}$, and allow the constant A to take on either positive or negative values, we incorporate both cases of the absolute value. Finally, if we allow A to be zero, we recover the missing solution discussed above. The best way to express the general solution to our differential equation is

$$y = Ae^{\frac{x^3}{3}}.$$

Video solution

youtu.be/watch?v=pDXfO52xNVw

Missing constant solutions can't always be recovered by cleverly redefining the arbitrary constant. The differential equation $y' = y^2 - 1$ is an example of this fact. Both $y = 1$ and $y = -1$ are constant solutions to this differential equation. Separation of variables yields a solution where $y = 1$ can be attained by choosing an appropriate C value, but $y = -1$ can't. The general solution is the set containing the solution produced by separation of variables *and* the missing solution $y = -1$. We should always be careful to look for missing constant solutions when seeking the general solution to a separable differential equation.

Example 8.2.6 Solving a Separable Initial Value Problem.

Solve the initial value problem $(y\sqrt{y^2-5})y' - \sin(x)\cos(x) = 0$, with $y(0) = -3$.

Solution. We first put the differential equation in separated form

$$y\sqrt{y^2-5}\,dy = \sin(x)\cos(x)\,dx.$$

The indefinite integral $\int y\sqrt{y^2-5}\,dy$ requires the substitution $u = y^2 - 5$. Using this substitute yields the antiderivative $\frac{1}{3}(y^2-5)^{3/2}$. The indefinite integral $\int \sin(x)\cos(x)\,dx$ requires the substitution $u = \sin(x)$. Using this substitution yields the antiderivative $\frac{1}{2}\sin^2 x$. Thus, we have an implicit form of the solution to the differential equation given by

$$\frac{1}{3}(y^2-5)^{3/2} = \frac{1}{2}\sin^2 x + C.$$

The initial condition says that y should be -3 when x is 0, or

$$\frac{1}{3}((-3)^2-5)^{3/2} = \frac{1}{2}\sin^2 0 + C.$$

Evaluating the line above, we find $C = 8/3$, yielding the particular solution to the initial value problem

$$\frac{1}{3}(y^2-5)^{3/2} = \frac{1}{2}\sin^2 x + \frac{8}{3}.$$

Video solution

youtu.be/watch?v=Bl3ugfR-Guw

Example 8.2.7 Solving a Separable Differential Equation.

Find the general solution to the differential equation $\dfrac{dy}{dx} = \dfrac{(x^2+1)e^y}{y}$.

Solution. We start by observing that there are no constant solutions to this differential equation because there are no constant y values that make the right hand side of the equation identically zero. Thus, we need not worry about losing solutions during the separation of variables process. The separated form of the equation is given by

$$ye^{-y}\,dy = (x^2+1)\,dx.$$

The antiderivative of the left hand side requires Integration by Parts. Evaluating both indefinite integrals yields the implicit solution

$$-(y+1)e^{-y} = \frac{1}{3}x^3 + x + C.$$

Since we cannot solve for y, we cannot find an explicit form of the solution.

Video solution

youtu.be/watch?v=OharserepNU

Example 8.2.8 Solving the Logistic Differential Equation.

Solve the logistic differential equation $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{M}\right)$

Solution. We looked at a slope field for this equation in Section 8.1 in the specific case of $k = M = 1$. Here, we use separation of variables to find an analytic solution to the more general equation. Notice that the independent variable t does not explicitly appear in the differential equation. We mentioned that an equation of this type is called *autonomous*. All autonomous first order differential equations are separable.

We start by making the observation that both $y = 0$ and $y = M$ are constant solutions to the differential equation. We must check that these solutions are not lost during the separation of variables process. The separated form of the equation is

$$\frac{1}{y\left(1 - \dfrac{y}{M}\right)}\,dy = k\,dt.$$

The antiderivative of the left hand side of the equation can be found by making use of partial fractions. Using the techniques discussed in Section 6.4, we write

$$\frac{1}{y\left(1 - \frac{y}{M}\right)} = \frac{1}{y} + \frac{1}{M-y}.$$

Then an implicit form of the solution is given by

$$\ln|y| - \ln|M-y| = kt + C.$$

Combining the logarithms,

$$\ln\left|\frac{y}{M-y}\right| = kt + C.$$

Similarly to Example 8.2.5, we can write

$$\frac{y}{M-y} = Ae^{kt}.$$

Letting A take on positive values or negative values incorporates both cases of the absolute value. This is another implicit form of the solution. Solving for y gives the explicit form

$$y = \frac{M}{1+be^{-kt}},$$

where b is an arbitrary constant. Notice that $b=0$ recovers the constant solution $y = M$. The constant solution $y = 0$ cannot be produced with a finite b value, and has been lost. The general solution the logistic differential equation is the set containing $y = \dfrac{M}{1+be^{-kt}}$ and $y = 0$.

Video solution

youtu.be/watch?v=nLItalqug6A

Solving for y initially yields the explicit solution $y = \dfrac{AMe^{kt}}{1+Ae^{kt}}$. Dividing numerator and denominator by Ae^{kt} and defining $b = 1/A$ yields the commonly presented form of the solution given in Example 8.2.8.

8.2.2 Exercises

Problems

Exercise Group. In the following exercises, decide whether the differential equation is separable or not separable. If the equation is separable, write it in separated form.

1. $y' = y^2 - y$

2. $xy' + x^2y = \dfrac{\sin(x)}{x-y}$

3. $(y+3)y' + (\ln(x))y' - x\sin y = (y+3)\ln(x)$

4. $y' - x^2\cos y + y = \cos y - x^2 y$

Exercise Group. In the following exercises, find the general solution to the separable differential equation. Be sure to check for missing constant solutions.

5. $y' + 1 - y^2 = 0$

6. $y' = y - 2$

7. $xy' = 4y$

8. $yy' = 4x$

9. $e^x yy' = e^{-y} + e^{-2x-y}$

10. $(x^2+1)y' = \dfrac{x}{y-1}$

11. $y' = \dfrac{x\sqrt{1-4y^2}}{x^4+2x^2+2}$

12. $(e^x + e^{-x})y' = y^2$

Exercise Group. In the following exercises, find the particular solution to the separable initial value problem.

13. $y' = \dfrac{\sin(x)}{\cos y}$, with $y(0) = \dfrac{\pi}{2}$

14. $y' = \dfrac{x^2}{1-y^2}$, with $y(0) = -2$

15. $y' = \dfrac{2x}{y+x^2y}$, with $y(0) = -4$

16. $x + ye^{-x}y' = 0$, with $y(0) = -2$

17. $y' = \dfrac{x\ln(x^2+1)}{y-1}$, with $y(0) = 2$

18. $\sqrt{1-x^2}\,y' - \dfrac{\arcsin x}{y\cos(y^2)} = 0$, with $y(0) = \sqrt{\dfrac{7\pi}{6}}$

19. $y' = (\cos^2 x)(\cos^2 2y)$, with $y(0) = 0$

20. $y' = \dfrac{y^2\sqrt{1-y^2}}{x}$, with $y(0) = 1$

8.3 First Order Linear Differential Equations

In the previous section, we explored a specific techique to solve a specific type of differential equation called a separable differential equation. In this section, we develop and practice a technique to solve a type of differential equation called a *first order linear* differential equation.

Recall than a linear algebraic equation in one variable is one that can be written $ax + b = 0$, where a and b are real numbers. Notice that the variable x appears to the first power. The equations $\sqrt{x} + 1 = 0$ and $\sin(x) - 3x = 0$ are both nonlinear. A linear differential equation is one in which the dependent variable and its derivatives appear only to the first power. We focus on first order equations, which involve first (but not higher order) derivatives of the dependent variable.

youtu.be/watch?v=aGk_H6jc5BE

Figure 8.3.1 Introduction to Section 8.3, and presentation of Example 8.3.3

8.3.1 Solving First Order Linear Equations

Definition 8.3.2 First Order Linear Differential Equation.

A **first order linear differential equation** is a differential equation that can be written in the form

$$\frac{dy}{dx} + p(x)y = q(x),$$

where p and q are arbitrary functions of the independent variable x.

Example 8.3.3 Classifying Differential Equations.

Classify each differential equation as first order linear, separable, both, or neither.

(a) $y' = xy$

(b) $y' = e^y + 3x$

(c) $y' - (\cos(x))y = \cos(x)$

(d) $yy' - 3xy = 4\ln(x)$

Solution.

(a) Both. We identify $p(x) = -x$ and $q(x) = 0$. The separated form of the equation is $\frac{dy}{y} = x\,dx$.

(b) Neither. The e^y term makes the equation nonlinear. Because of the addition, it is not possible to write the equation in separated form.

(c) First order linear. We identify $p(x) = -\cos(x)$ and $q(x) = \cos(x)$. The equation cannot be written in separated form.

(d) Neither. Notice that dividing by y results in the nonlinear term $\frac{4\ln(x)}{y}$. It is not possible to write the equation in separated form.

Notice that linearity depends on the dependent variable y, not the independent variable x. The functions $p(x)$ and $q(x)$ need not be linear, as demonstrated in part (c) of Example 8.3.3. Neither $\cos(x)$ nor $\sin(x)$ are linear functions of x, but the differential equation is still linear.

Before working out a general technique for solving first order linear differential equations, we look at a specific example. Consider the differential equation

$$\frac{d}{dx}(xy) = \sin(x)\cos(x).$$

This is an easy differential equation to solve. On the left, the antiderivative of the derivative is simply the function xy. Using the substitution $u = \sin(x)$ on the right and integrating results in the implicit solution

$$xy = \frac{1}{2}\sin^2 x + C.$$

Solving for y yields the explicit solution

$$y = \frac{\sin^2 x}{2x} + \frac{C}{x}.$$

Though not obvious, the differential equation above is actually a linear differential equation. Using the product rule and implicit differentiation, we can write $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$. Our original differential equation can be written

$$x\frac{dy}{dx} + y = \sin(x)\cos(x).$$

If we divide by x, we have

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{\sin(x)\cos(x)}{x},$$

which matches the form in Definition 8.3.2. Reversing our steps would lead us back to the original form our our differential equation.

In the examples in the previous section, we performed operations on the arbitrary constant C, but still called the result C. The justification is that the result after the operation is *still* an arbitrary contant. Here, we divide C by x, so the result depends explicitly on the independent variable x. Since C/x is *not* contant, we can't just call it C.

As motivated by the problem we just explored, the basic idea behind solving first order linear differential equations is to multiply both sides of the differential equation by a function, called an *integrating factor*, that makes the left hand side of the equation look like an expanded Product Rule. We then condense the left hand side into the derivative of a product and integrate both sides. An obvious question is, "How do you find this integrating factor?"

Consider the first order linear equation

$$\frac{dy}{dx} + p(x)y = q(x).$$

youtu.be/watch?v=f-Bea35N11g

Figure 8.3.4 Using an integrating factor to solve a linear differential equation

Let's call the integrating factor $\mu(x)$. We multiply both sides of the differential equation by $\mu(x)$ to get

$$\mu(x)\left(\frac{dy}{dx} + p(x)y\right) = \mu(x)q(x).$$

Our goal is to choose $\mu(x)$ so that the left hand side of the differential equation looks like the result of a Product Rule. The left hand side of the equation is

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y.$$

Though we use $\mu(x)$ for our integrating factor, the symbol is unimportant. The notation $\mu(x)$ is a common choice, but other texts my use $\alpha(x), I(x)$, or some other symbol to designate the integrating factor.

Using the Product Rule and Implicit Differentiation,

$$\frac{d}{dx}(\mu(x)y) = \frac{d\mu}{dx}y + \mu(x)\frac{dy}{dx}.$$

Equating $\frac{d}{dx}\big(\mu(x)y\big)$ and $\mu(x)\left(\frac{dy}{dx}+p(x)y\right)$ gives

$$\frac{d\mu}{dx}y+\mu(x)\frac{dy}{dx}=\mu(x)\frac{dy}{dx}+\mu(x)p(x)y,$$

which is equivalent to

$$\frac{d\mu}{dx}=\mu(x)p(x).$$

In order for the integrating factor $\mu(x)$ to perform its job, it must solve the differential equation above. But that differential equation is separable, so we can solve it. The separated form is

$$\frac{d\mu}{\mu}=p(x)\,dx.$$

Integrating,

$$\ln\mu=\int p(x)\,dx,$$

or

$$\mu(x)=e^{\int p(x)\,dx}.$$

Following the steps outlined in the previous section, we should technically end up with $\mu(x)=Ce^{\int p(x)\,dx}$, where C is an arbitrary constant. Because we multiply both sides of the differential equation by $\mu(x)$, the arbitrary constant cancels, and we omit it when finding the integrating factor.

If $\mu(x)$ is chosen this way, after multiplying by $\mu(x)$, we can always write the differential equation in the form

$$\frac{d}{dx}\big(\mu(x)y\big)=\mu(x)q(x).$$

Integrating and solving for y, the explicit solution is

$$y=\frac{1}{\mu(x)}\int\big(\mu(x)q(x)\big)\,dx.$$

Though this formula can be used to write down the solution to a first order linear equation, we shy away from simply memorizing a formula. The process is lost, and it's easy to forget the formula. Rather, we always always follow the steps outlined in Key Idea 8.3.5 when solving equations of this type.

Key Idea 8.3.5 Solving First Order Linear Equations.

1. Write the differential equation in the form

$$\frac{dy}{dx}+p(x)y=q(x).$$

2. Compute the integrating factor

$$\mu(x)=e^{\int p(x)\,dx}.$$

3. Multiply both sides of the differential equation by $\mu(x)$, and condense the left hand side to get

$$\frac{d}{dx}\big(\mu(x)y\big)=\mu(x)q(x).$$

4. Integrate both sides of the differential equation with respect to x, taking care to remember the arbitrary constant.
5. Solve for y to find the explicit solution to the differential equation.

Let's practice the process by solving the two first order linear differential equations from Example 8.3.3.

Example 8.3.6 Solving a First Order Linear Equation.

Find the general solution to $y' = xy$.

Solution. We solve by following the steps in Key Idea 8.3.5. Unlike the process for solving separable equations, we need not worry about losing constant solutions. The answer we find *will* be the general solution to the differential equation. We first write the equation in the form

$$\frac{dy}{dx} - xy = 0.$$

By identifying $p(x) = -x$, we can compute the integrating factor

$$\mu(x) = e^{\int -x\,dx} = e^{-\frac{1}{2}x^2}.$$

Multiplying both side of the differential equation by $\mu(x)$, we have

$$e^{-\frac{1}{2}x^2}\left(\frac{dy}{dx} - xy\right) = 0.$$

The left hand side of the differential equation condenses to yield

$$\frac{d}{dx}\left(e^{-\frac{1}{2}x^2}y\right) = 0.$$

We integrate both sides with respect to x to find the implicit solution

$$e^{-\frac{1}{2}x^2}y = C,$$

or the explicit solution

$$y = Ce^{\frac{1}{2}x^2}.$$

Video solution

youtu.be/watch?v=fgeo61eY3qo

The step where the left hand side of the differential equation condenses to the derivative of a product can feel a bit magical. The reality is that we choose $\mu(x)$ so that we can get exactly this condensing behavior. It's not magic, it's math! If you're still skeptical, try using the Product Rule and Implicit Differentiation to evaluate $\frac{d}{dx}\left(e^{-\frac{1}{2}x^2}y\right)$, and verify that it becomes $e^{-\frac{1}{2}x^2}\left(\frac{dy}{dx} - xy\right)$.

Example 8.3.7 Solving a First Order Linear Equation.

Find the general solution to $y' - (\cos(x))y = \cos(x)$.

Solution. The differential equation is already in the correct form. The integrating factor is given by

$$\mu(x) = e^{-\int \cos(x)\,dx} = e^{-\sin(x)}.$$

Multiplying both sides of the equation by the integrating factor and condensing,

$$\frac{d}{dx}\left(e^{-\sin(x)}y\right) = (\cos(x))e^{-\sin(x)}$$

Using the substitution $u = -\sin(x)$, we can integrate to find the implicit solution

$$e^{-\sin(x)}y = -e^{-\sin(x)} + C.$$

The explicit form of the general solution is

$$y = -1 + Ce^{\sin(x)}.$$

Video solution

youtu.be/watch?v=qB_aSFCQfcE

We continue our practice by finding the particular solution to an initial value problem.

Example 8.3.8 Solving a First Order Linear Initial Value Problem.

Solve the initial value problem $xy' - y = x^3 \ln(x)$, with $y(1) = 0$.

Solution. We first divide by x to get

$$\frac{dy}{dx} - \frac{1}{x}y = x^2 \ln(x).$$

The integrating factor is given by

$$\begin{aligned}\mu(x) &= e^{\int -\frac{1}{x}\,dx} \\ &= e^{-\ln(x)} \\ &= e^{\ln(x)^{-1}} \\ &= x^{-1}.\end{aligned}$$

Multiplying both sides of the differential equation by the integrating factor and condensing the left hand side, we have

$$\frac{d}{dx}\left(\frac{y}{x}\right) = x\ln(x).$$

Using Integrating by Parts to find the antiderivative of $x\ln(x)$, we find the implicit solution

$$\frac{y}{x} = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C.$$

Solving for y, the explicit solution is

$$y = \frac{1}{2}x^3\ln(x) - \frac{1}{4}x^3 + Cx.$$

The initial condition $y(1) = 0$ yields $C = 1/4$. The solution to the initial value problem is

$$y = \frac{1}{2}x^3\ln(x) - \frac{1}{4}x^3 + \frac{1}{4}x.$$

Video solution

youtu.be/watch?v=XsFRAzdk7WI

Differential equations are a valuable tool for exploring various physical problems. This process of using equations to describe real world situations is called mathematical modeling, and is the topic of the next section. The last two examples in this section begin our discussion of mathematical modeling.

Example 8.3.9 A Falling Object Without Air Resistance.

Suppose an object with mass m is dropped from an airplane. Find and solve a differential equation describing the vertical velocity of the object assuming no air resistance.

Solution. The basic physical law at play is Newton's second law,

$$\text{mass} \times \text{acceleration} = \text{the sum of the forces}.$$

Using the fact that acceleration is the derivative of velocity, mass × acceleration can be writting mv'. In the absence of air resistance, the only force of interest is the force due to gravity. This force is approximately constant, and is given by mg, where g is the gravitational constant. The word equation above can be written as the differential equation

$$m\frac{dv}{dt} = mg.$$

Because g is constant, this differential equation is simply an integration problem, and we find

$$v = gt + C.$$

Since $v = C$ with $t = 0$, we see that the arbitrary constant here corresponds to the initial vertical velocity of the object.

The process of mathematical modeling does not stop simply because we have found an answer. We must examine the answer to see how well it can describe real world observations. In the previous example, the answer may be somewhat useful for short times, but intuition tells us that something is missing. Our answer says that a falling object's velocity will increase linearly as a function of time, but we know that a falling object does not speed up indefinitely. In order to more fully describe real world behavior, our mathematical model must be revised.

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Figure 8.3.10 Video presentation of Examples 8.3.9–8.3.11

Example 8.3.11 A Falling Object with Air Resistance.

Suppose an object with mass m is dropped from an airplane. Find and solve a differential equation describing the vertical velocity of the object, taking air resistance into account.

Solution. We still begin with Newon's second law, but now we assume that the forces in the object come both from gravity and from air resistance. The gravitational force is still given by mg. For air resistance, we assume the force is related to the velocity of the object. A simple way to describe this assumption might be kv^p, where k is a proportionality constant and p is a positive real number. The value k depends on various factors such as the density of the object, surface area of the object, and density of the air. The value p affects how changes in the velocity affect the force. Taken together, a function of the form kv^p is often called a *power law*. The differential equation for the velocity is given by

$$m\frac{dv}{dt} = mg - kv^p.$$

(Notice that the force from air resistance opposes motion, and points in the opposite direction as the force from gravity.) This differential equation is separable, and can be written in the separated form

$$\frac{m}{mg - kv^p}\,dv = dt.$$

For arbitrary positive p, the integration is difficult, making this problem hard to solve analytically. In the case that $p = 1$, the differential equation becomes linear, and is easy to solve either using either separation of variables or integrating factor techniques. We assume $p = 1$, and proceed with an integrating factor so we can continue practicing the process. Writing

$$\frac{dv}{dt} + \frac{k}{m}v = g,$$

we identify the integrating factor

$$\mu(t) = e^{\int \frac{k}{m}\,dt} = e^{\frac{k}{m}t}.$$

Then

$$\frac{d}{dt}\left(e^{\frac{k}{m}t}v\right) = ge^{\frac{k}{m}t},$$

so

$$e^{\frac{k}{m}t}v = \frac{mg}{k}e^{\frac{k}{m}t} + C,$$

or

$$v = \frac{mg}{k} + Ce^{-\frac{k}{m}t}.$$

In the solution above, the exponential term decays as time increases, causing the velocity to approach the constant value mg/k in the limit as t approaches infinity. This value is called the *terminal velocity*. If we assume a zero initial velocity (the object is dropped, not thrown from the plane), the velocities from Examples 8.3.9 and 8.3.11 are given by $v = gt$ and $v = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$, respectively. These two functions are shown in Figure 8.3.12, with $g = 9.8, m = 1$, and $k = 1$. Notice that the two curves agree well for short times, but have dramatically different behaviors as t increases. Part of the art in mathematical modeling is deciding on the level of detail required to answer the question of interest. If we are only interested in the initial behavior of the falling object, the simple model in Example 8.3.9 may be sufficient. If we are interested in the longer term behavior of the object, the simple model is not sufficient, and we should consider a more complicated model.

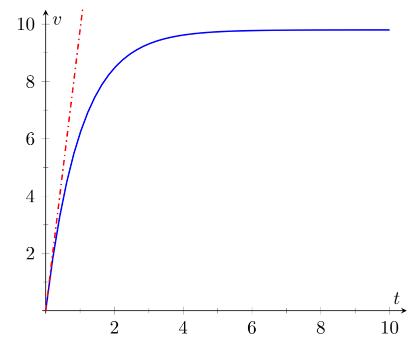


Figure 8.3.12 The velocity functions from Examples 8.3.9 (dashed) and 8.3.11 (solid) under the assumption that $v(0) = 0$, with $g = 9.8, m = 1$, and $k = 1$

8.3.2 Exercises

Problems

Exercise Group. In the following exercises, Find the general solution to the first order linear differential equation.

1. $y' = 2y - 3$

2. $x^2y' + xy = 1$

3. $x^2y' - xy = 1$

4. $xy' + 4y = x^3 - x$

5. $(\cos^2 x \sin(x))y' + (\cos^3 x)y = 1$

6. $\dfrac{y'}{x} = 1 - 2y$

7. $x^3y' - 3x^3y = x^4e^{2x}$

8. $y' + y = 5\sin(2x)$

Exercise Group. In the following exercises, Find the particular solution to the initial value problem.

9. $y' = y + 2xe^x$, $y(0) = 2$

10. $xy' + 2y = x^2 - x + 1$, $y(1) = 1$

11. $xy' + (x+2)y = x$, $y(1) = 0$

12. $y' + 2y = 0$, $y(0) = 3$

13. $(x+1)y' + (x+2)y = 2xe^{-x}$, $y(0) = 1$

14. $(\cos(x))y' + (\sin(x))y = 1$, $y(0) = -3$

15. $(x^2-1)y' + 2y = (x+1)^2$, $y(0) = 2$

16. $xy' - 2y = \dfrac{x^3}{1+x^2}$, $y(1) = 0$

Exercise Group. In the following exercises, classify the differential equation as separable, first order linear, or both, and solve the initial value problem using an appropriate method.

17. $y' = y + yx^2$, $y(0) = -5$

18. $xe^yy' = x^2\sin(x)$, $y(0) = 0$

19. $(x-1)y' + y = x^2 - 1$, $y(0) = 2$

20. $y' = y^2 + y - 2$, $y(0) = 1$

Exercise Group. In the following exercises, draw a slope field for the differential equation. Use the slope field to predict the behavior of the solution to the initial value problem for large x values. Solve the initial value problem, and verify your prediction.

21. $y' = x - y$, $y(0) = 0$

22. $(X+1)y' + y = \dfrac{1}{x+1}$, $y(0) = 2$

8.4 Modeling with Differential Equations

In the first three sections of this chapter, we focused on the basic ideas behind differential equations and the mechanics of solving certain types of differential equations. We have only hinted at their practical use. In this section, we use differential equations for mathematical modeling, the process of using equations to describe real world processes. We explore a few different mathematical models with the goal of gaining an introduction to this large field of applied mathematics.

8.4.1 Models Involving Proportional Change

Some of the simplest differential equation models involve one quantity that changes at a rate proportional to another quantity. In the introduction to this chapter, we considered a population that grows at a rate proportional to the current population. The words in this assumption can be directly translated into a differential equation as shown below.

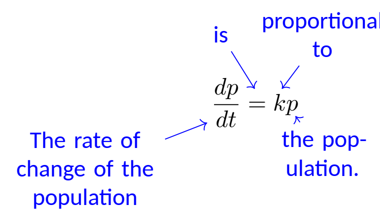


Figure 8.4.1 Translating words into a differential equation

There are some key ideas that can be helpful when translating words into a differential equation. Any time we see something about rates or changes, we should think about derivatives. The word "is" usually corresponds to an equal sign in the equation. The words "proportional to" mean we have a constant multiplied by something.

The differential equation in Figure 8.4.1 is easily solved using separation of variables. We find

$$p = Ce^{kt}.$$

Notice that we need values for both C and k before we can use this formula to predict population size. We require information about the population at two different times in order to fully determine the population model.

Example 8.4.2 Bacterial Growth.

Suppose a population of *e-coli* bacteria grows at a rate proportional to the current population. If an initial popluation of 200 bacteria has grown to 1600 three hours later, find a function for the size of the population at time t, and use it to predict when the population size will reach 10,000.

Solution. We already know that the population at time t is given by $p = Ce^{kt}$ for some C and k. The information about the initial size of the population means that $p(0) = 200$. Thus $C = 200$. Our knowledge of the population size after three hours allows us to solve for k via the equation

$$1600 = 200e^{3k}.$$

Solving this exponential equation yields $k = \ln(8)/3 \approx 0.6931$. The popluation at time t is given by

$$p = 200e^{(\ln(8)/3)t}.$$

Solving

$$10000 = 200e^{(\ln(8)/3)t}$$

yields $t = (3\ln(50))/\ln(8) \approx 5.644$. The population is predicted to reach 10,000 bacteria in slightly more than five and a half hours.

Video solution

youtu.be/watch?v=EC18tbH7SQw

Another example of porportional change is *Newton's Law of Cooling*. The laws of thermodynamics state that heat flows from areas of higher temperature to areas of lower temperature. A simple example is a hot object that cools down when placed in a cool room. Newton's Law of Cooling is the simple assumption that the temperature of the object changes at a rate proportional to the difference between the temperature of the object and the ambient temperature of the room. If T is the temperature of the object and A is the constant ambient temperature, Newton's Law of Cooling can be expressed as the differential equation

$$\frac{dT}{dt} = k(A - T).$$

This differential equation is both linear and separable. The separated form is

$$\frac{1}{A - T}\, dT = k\, dt.$$

Then an implicit definition of the temperature is given by

$$-\ln|A - T| = kt + C.$$

If we solve for T, we find the explicit temperature

$$T = A - Ce^{-kt}.$$

Though we didn't show the steps, the explicit solution involves the typical process of renaming the constant $\pm e^{-C}$ as C, and allowing C to be positive, negative, or zero to account for both cases of the absolution value and to catch the constant solution $T = A$. Notice that the temperature of the object approaches the ambient temperature in the limit as $t \to \infty$.

The equation $\frac{dT}{dt} = k(T - A)$ is also a valid representation of Newton's Law of Cooling. Intuition tells us that T will increase if T is less than A and decrease if T is greater then A. The form we use in the text follows this intuition with a positive k value. The form above will require that k take on a negative value. In the end, both forms result in the same function.

Example 8.4.3 Hot Coffee.

A freshly brewed cup of coffee is set on the counter and has a temperature of 200° Fahrenheit. After 3 minutes, it has cooled to 190°, but is still too hot to drink. If the room is 72° and the coffee cools according to Newton's Law of Cooling, how long will the impatient coffee drinker have to wait until the coffee has cooled to 165°?

Solution. Since we have already solved the differential equation for Newton's Law of Cooling, we can immediately use the function

$$T = A - Ce^{-kt}.$$

Since the room is 72°, we know $A = 72$. The initial temperature is 200°, which means $C = -128$. At this point, we have

$$T = 72 + 128e^{-kt}$$

The information about the coffee cooling to 190° in 3 minutes leads to the equation

$$190 = 72 + 128e^{-3k}.$$

Solving the exponential equation for k, we have

$$k = -\frac{1}{3}\ln\left(\frac{59}{64}\right) \approx 0.0271.$$

Finally, we finish the problem by solving the exponential equation

$$165 = 72 + 128e^{\frac{1}{3}\ln\left(\frac{59}{64}\right)t}.$$

The coffee drinker must wait $t = \dfrac{3\ln\left(\frac{93}{128}\right)}{\ln\left(\frac{59}{64}\right)} \approx 11.78$ minutes.

Video solution

youtu.be/watch?v=tEMLcz1yvFI

We finish our discussion of models of proportional change by exploring three different models of disease spread through a population. In all of the models, we let y denote the proportion of the population that is sick ($0 \leq y \leq 1$). We assume a proportion of 0.05 is initially sick and that a proportion of 0.1 is sick 1 week later.

Example 8.4.4 Disease Spread 1.

Suppose a disease spreads through a population at a rate proportional to the number of individuals who are sick. If 5% of the population is sick initially and 10% of the population is sick one week later, find a formula for the proportion of the popoulation that is sick at time t.

Solution. The assumption here seems to have some merit because it matches our intuition that a disease should spread more rapidly when more individuals are sick. The differential equation is simply

$$\frac{dy}{dt} = ky,$$

with solution

$$y = Ce^{kt}.$$

The conditions $y(0) = 0.05$ and $y(1) = 0.1$ lead to $C = 0.05$ a and $k = \ln(2)$, so the function is

$$y = 0.05e^{(\ln(2)t}.$$

We should point out a glaring problem with this model. The variable y is a proportion and should take on values between 0 and 1, but the function $y = 0.05e^{2t}$ grows without bound. After $t \approx 4.32$ weeks, y exceeds 1, and the model ceases to make physical sense.

youtu.be/watch?v=UjRpT852su4

Figure 8.4.5 Video presentation of Examples 8.4.4–8.4.6

Example 8.4.6 Disease Spread 2.

Suppose a disease spreads through a population at a rate proportional to the number of individuals who are not sick. If 5% of the population is sick initially and 10% of the population is sick one week later, find a formula for the proportion of the popoulation that is sick at time t.

Solution. The intuition behind the assumption here is that a disease can only spread if there are individuals who are susceptible to the infection. As fewer and fewer people are able to be infected, the disease spread should slow down. Since y is proportion of the population that is sick, $1 - y$ is the proportion who are not sick, and the differential equation is

$$\frac{dy}{dt} = k(1 - y).$$

Though the context is quite different, the differential equation is identical to the differential equation for Newton's Law of Cooling, with $A = 1$. The solution is

$$y = 1 - Ce^{-kt}.$$

The conditions $y(0) = 0.05$ and $y(1) = 0.1$ yield $C = 0.95$ and $k = -\ln\left(\frac{18}{19}\right) \approx 0.0541$, so the final function is

$$y = 1 - .95e^{\ln\left(\frac{18}{19}\right)t}.$$

Notice that this function approaches $y = 1$ in the limit as $t \to \infty$, and does not suffer from the non-physical behavior described in Example 8.4.4.

In Example 8.4.4, we assumed disease spread depends on the number of infected individuals. In Example 8.4.6, we assumed disease spread depends on the number of susceptible individuals who are able to become infected. In reality, we would expect many diseases to require the interaction of both infected and susceptible individuals in order to spread. One of the simplest ways to model this required interaction is to assume disease spread depends on the product of the proportions of infected and uninfected individuals. This assumption (regularly seen in the context of chemical reactions) is often called the *law of mass action*.

Example 8.4.7 Disease Spread 3.

Suppose a disease spreads through a population at a rate proportional to the product of the number of infected and uninfected individuals. If 5% of the population is sick initially and 10% of the population is sick one week later, find a formula for the proportion of the population that is sick at time t.

Solution. The differential equation is

$$\frac{dy}{dt} = ky(1-y).$$

This is exactly the logistic equation with $M = 1$. We solved this differential equation in Example 8.2.8, and found

$$y = \frac{1}{1+be^{-kt}}.$$

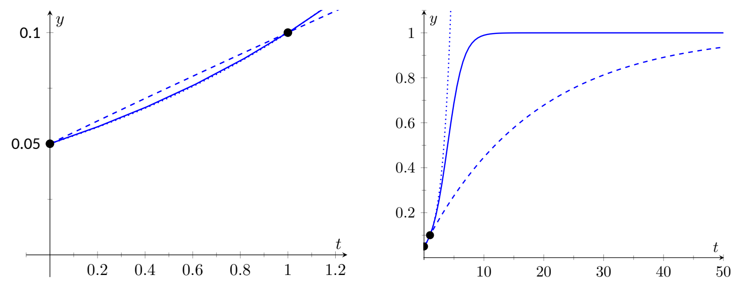
The conditions $y(0) = 0.05$ and $y(1) = 0.1$ yield $b = 19$ and $k = -\ln\left(\frac{9}{19}\right) \approx 0.7472$. The final function is

$$y = \frac{1}{1+19e^{\ln\left(\frac{9}{19}\right)t}}.$$

Based on the three different assumptions about the rate of disease spread explored in the last three examples, we now have three different functions giving the proportion of a population that is sick at time t. Each of the three functions meets the conditions $y(0) = 0.05$ and $y(1) = 0.1$. The three functions are shown in Figure 8.4.8.

Notice that the logistic function mimics specific parts of the functions from Examples 8.4.4 and 8.4.6. We see in Figure 8.4.8(a) that the logistic and exponential functions are virtually indistinguishable for small t values. When there are few infected individuals and lots of susceptible individuals, the spread of a disease is largely determined by the number of sick people. The logistic curve captures this feature, and is "almost exponential" early on.

In Figure 8.4.8(b), we see that the logistic curve leaves the exponential curve from Example 8.4.4 and approaches the curve from Example 8.4.6. This result implies that when most of the population is sick, the spread of the disease is largely dependent on the number of susceptible individuals. Though there are much more sophisticated mathematical models describing the spread of infections, we could argue that the logistic model presented in this example is the "best" of the three.



(a) **(b)**

Figure 8.4.8 Plots of the functions from Example 8.4.4 (dotted), Example 8.4.6 (dashed), and Example 8.4.7 (solid)

Video solution

youtu.be/watch?v=vV8Jy231gLk

8.4.2 Rate-in Rate-out Problems

One of the classic ways to build a mathematical model involves tracking the way the amount of something can change. We sometimes say these models are based on *conservation laws*. Consider a box with some amount of a specific type of material inside. (Some type of chemical, for example.) The amount of material of the specific type in the box can only change in four ways; we can add more to the box, we can remove some from the box, some of the material can change into material of a different type, or some other type of material can turn into the type we're tracking. In the examples that follow, we assume material doesn't change type, so we only need to keep track of material coming into the box and material leaving the box. To derive a differential equation, we track rates:

$$\text{rate of change of some quantity} = \text{rate in} - \text{rate out}\,.$$

Though we stick to relatively simple examples, this basic idea can be used to derive some very important differential equations in mathematics and physics.

The examples to follow involve tracking the amount of a chemical in solution. We assume liquid containing some chemical flows into a container at some rate. That liquid mixes instantaneously with the liquid already in the container. Then the liquid from the container flows out at some (potentially different) rate.

youtu.be/watch?v=G3nvU0Jc5pw

Figure 8.4.9 Introduction to Rate-in Rate-out problems

The assumption about instantaneous mixing, though not physically accurate, leads to a differential equation we have hope of solving. In reality, the amount of chemical at a specific location in the container depends both on the location and how long we have been waiting. This dependence on both space and time leads to a type of differential equation called a *partial differential equation*. Differential equations of this type are more interesting, but significantly harder to study. Instantaneous mixing removes any spatial dependence from the problem, and leaves us with an *ordinary differential equation*.

Example 8.4.10 Equal Flow Rates.

Suppose a 10 liter tank has 5 liters of salt solution in it. The initial concentration of the salt solution is 1 gram per liter. A salt solution with concentration $3\,\frac{\text{g}}{\text{L}}$ flows into the tank at a rate of $2\,\frac{\text{L}}{\text{min}}$. Suppose the salt solution mixes instantaneously with the solution already in the tank, and that the mixed solution from the tank flows out at a rate of $2\,\frac{\text{L}}{\text{min}}$. Find a function that gives the amount of salt in the tank at time t.

Solution. We use the rate in - rate out setup described above. The quantity here is the amount (in grams) of salt in the tank at time t. Let y denote the amount of salt. In words, the differential equation is given by

$$\frac{dy}{dt} = \text{rate in} - \text{rate out}\,.$$

Thinking in terms of units can help fill in the details of the differential equation. Since y has units of grams, the left hand side of the equation has units g/min. Both terms on the right hand side must have these same units. Notice that the product of a concentration (with units g/L) and a flow rate (with units L/min) results in a quantity with units g/min. Both terms on the right hand side of the equation will include a concentration multiplied by a flow rate.

For the rate in, we multiply the inflow concentration by the rate that fluid is flowing into the bucket. This is $\left(3\frac{\text{g}}{\text{L}}\right)\left(2\frac{\text{L}}{\text{min}}\right) = 6$ g/min.

The rate out is more complicated. The flow rate is still $2\,\frac{\text{L}}{\text{min}}$, meaning that the overall volume of the fluid in the bucket is the constant 5 L. The salt concentration in the bucket is not constant though, meaning that the outflow concentration is not constant. In particular, the outflow concentration is *not* the constant $1\,\frac{\text{L}}{\text{min}}$. This is simply the initial concentration. To find the concentration at any time, we need the amount of salt in the bucket at that time and the volume of liquid in the bucket at that time. The volume of liquid is the constant 5 L, and the amount of salt is given by the dependent variable y. Thus, the outflow concentration is $\frac{y}{5}$ g/L, yielding a rate out given by

$$\left(\frac{y}{5}\frac{\text{g}}{\text{L}}\right)\left(2\frac{\text{L}}{\text{min}}\right) = \frac{2y}{5}\text{ g/min}\,.$$

The differential equation we wish to solve is given by

$$\frac{dy}{dt} = 6 - \frac{2y}{5}.$$

To furnish an initial condition, we must convert the initial salt concentration into an initial amount of salt. This is $\left(1\frac{\text{g}}{\text{L}}\right)(5\text{ L}) = 5$ g, so $y(0) = 5$ is our initial condition.

Our differential equation is both separable and linear. We solve using separation of variables. The separated form of the differential equation is

$$\frac{5}{30-2y}\,dy = dt.$$

Integration yields the implicit solution

$$-\frac{5}{2}\ln|30-2y| = t + C.$$

Solving for y (and redefining the arbitrary constant C as necessary) yields the explicit solution

$$y = 15 + Ce^{-\frac{2}{5}t}.$$

The initial condition $y(0) = 5$ means that $C = -10$ so that

$$y = 15 - 10e^{-\frac{2}{5}t}$$

is the particular solution to our initial value problem.

This function is plotted in Figure 8.4.11. Notice that in the limit as $t \to \infty$, y approaches 15. This corresponds to a bucket concentration of $15/5 = 3$ g/L. It should not be surprising that salt concentration inside the tank will move to match the inflow salt concentration.

Video solution

youtu.be/watch?v=Au4n_QP73Ko

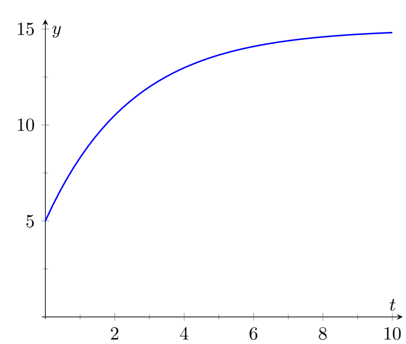


Figure 8.4.11 Salt concentration at time t, from Example 8.4.10

Example 8.4.12 Unequal Flow Rates.

Suppose the setup is identical to the setup in Example 8.4.10 except that now liquid flows out of the bucket at a rate of 1 L/min. Find a function that gives the amount of salt in the bucket at time t. What is the salt concentration when the solution ceases to be valid?

Solution. Because the inflow and outflow rates no longer match, the volume of liquid in the bucket is not the constant 5 L. In general, we can find the volume of liquid via the equation

$$\text{volume} = \text{initial volume} + \text{(inflow rate - outflow rate)}\, t.$$

In this example, the volume at time t is $5+t$ liters. Because the total volume of the bucket is only 10 L, it follows that our solution will only be valid for $0 \le t \le 5$. At that point it is no longer possible to have liquid flow into a the bucket at a rate of 2 L/min and out of the bucket at a rate of 1 L/min.

To update the differential equation, we must modify the rate out. Since the volume is $5+t$, the concentration at time t is given by $\frac{y}{5+t}$ g/L. Thus for rate out, we must use $\left(\frac{y}{5+t}\right)(1)$ g/min. The initial value problem is

$$\frac{dy}{dt} = 6 - \frac{y}{5+t}, \text{ with } y(0) = 5.$$

Unlike Example 8.4.10, where we had equal flow rates, this differential equation is no longer separable. We must proceed with an integrating factor. Writing the differential equation in the form

$$\frac{dy}{dt} + \frac{1}{5+t}y = 6,$$

we identify the integrating factor

$$\mu(t) = e^{\int \frac{1}{5+t}\,dt} = e^{\ln(5+t)} = 5+t.$$

Then

$$\frac{d}{dt}\big((5+t)y\big) = 6(5+t),$$

yielding the implicit solution

$$(5+t)y = 30t + 3t^2 + C.$$

The initial condition $y(0) = 5$ implies $C = 25$, so the explicit solution to our initial value problem is given by

$$y = \frac{3t^2 + 30t + 25}{5+t}.$$

This solution ceases to be valid at $t = 5$. At that time, there are 25 g of salt in the tank. The volume of liquid is 10 L, resulting in a salt concentration of 2.5 g/L.

Video solution

youtu.be/watch?v=P3fkhn-TQEk

Differential equations are powerful tools that can be used to help describe the world around us. Though relatively simple in concept, the ideas of proportional change and matching rates can serve as building blocks in the development of more sophisticated mathematical models. As we saw in this section, some simple mathematical models can be solved analytically using the techniques developed in this chapter. Most sophisticated mathematical models don't allow for analytic solutions. Even so, there are an array of graphical and numerical techniques that can be used to analyze the model to make predictions and infer information about real world phenomena.

8.4.3 Exercises

Problems

Exercise Group. In the following exercises, use the tools in the section to answer the questions presented.

1. Suppose the rate of change of y with respect to x is proportional to $10 - y$. Write down and solve a differential equation for y.

2. A rumor is spreading through a middle school with 250 students. Suppose the rumor spreads at a rate proportional to the number of students who haven't heard the rumor yet. If 1 person starts the rumor, and 75 students have heard the rumor 3 days later, how many days will it take until 80% of the students in the school have heard the rumor?

3. A rumor is spreading through a middle school with 250 students. Suppose the rumor spreads at a rate proportional to the product of number of students who have heard the rumor and the number who haven't heard the rumor. If 1 person starts the rumor, and 75 students have heard the rumor 3 days later, how many days will it take until 80% of the students in the school have heard the rumor?

4. A feature of radioactive decay is that the amount of a radioactive substance decreases at a rate proportional to the current amount of the substance. The *half life* of a substance is the amount of time it takes for half of a given amount of substance to decay. The half life of carbon-14 is approximately 5730 years. If an ancient object has a carbon-14 amount that is 20% of the original amount, how old is the object?

5. Consider a chemical reaction where molecules of type A combine with molecules of type B to form molecules of type C. Suppose one molecule of type A combines with one molecule of type B to form one molecule of type C, and that type C is produced at a rate proportional the product of the remaining number of molecules of types A and B. Let x denote moles of molecules of type C. Find a function giving the number of moles of type C at time t if there are originally a moles of type A, b moles of type B, and zero moles of type C.

6. Suppose an object with a temperature of 100° is introduced into a room with an ambient temperature of 70°. Suppose the temperature of the object changes at a rate proportional to the difference between the temperature of the object and the temperature of the room (Newton's Law of Cooling). If the object has cooled to 92° in 10 minutes, how long until the object has cooled to 84°?

7. Suppose an object with a temperature of 100° is introduced into a room with an ambient temperature given by $60 + 20e^{-\frac{1}{4}t}$ degrees. Suppose the temperature of the object changes at a rate proportional to the difference between the temperature of the object and the temperature of the room (Newton's Law of Cooling). If the object is 80° after 20 minutes, find a formula giving the temperature of the object at time t. (Note: This problem requires a numerical technique to solve for the unknown constants.)

8. A tank contains 5 gallons of salt solution with concentration 0.5 g/gal. Pure water flows into the tank at a rate of 1 gallon per minute. Salt solution flows out of the tank at a rate of 1 gallon per minute. (Assume instantaneous mixing.) Find the concentration of the salt solution at 10 minutes.

9. Dead leaves accumulate on the ground at a rate of 4 grams per square centimeter per year. The dead leaves on the ground decompose at a rate of 50% per year. Find a formula giving grams per square centimeter on the ground if there are no leaves on the ground at time $t = 0$.

10. A pond initially contains 10 million gallons of fresh water. Water containing an undesirable chemical flows into the pond at a rate of 5 million gallons per year, and fluid from the pond flows out at the same rate. (Assume instantaneous mixing.) If the concentration (in grams per million gallons) of the incoming chemical varies periodically according to the expression $2 + \sin(2t)$, find a formula giving the amount of chemical in the pond at time t.

11. A large tank contains 1 gallon of a salt solution with concentration 2 g/gal. A salt solution with concentration 1 g/gal flows into the tank at a rate of 4 gal/min. Salt solution flows out of the tank at a rate of 3 gal/min. (Assume instantaneous mixing.) Find the amount of salt in the tank at 10 minutes.

12. A stream flows into a pond containing 2 million gallons of fresh water at a rate of 1 million gallons per day. The stream flows out of the first pond and into a second pond containing 3 million gallons of fresh water. The stream then flows out of the second pond. Suppose the inflow and outflow rates are the same so that both ponds maintain their volumes. A factory upstream of the first pond starts polluting the stream. Directly below the factory, pollutant has a concentration of 55 grams per million gallons, and this concentration starts to flow into the first pond. Find the concentration of pollutant in the first and second ponds at 5 days.

Chapter 9

Sequences and Series

This chapter introduces *sequences* and *series*, important mathematical constructions that are useful when solving a large variety of mathematical problems. The content of this chapter is considerably different from the content of the chapters before it. While the material we learn here definitely falls under the scope of "calculus," we will make very little use of derivatives or integrals. Limits are extremely important, though, especially limits that involve infinity.

One of the problems addressed by this chapter is this: suppose we know information about a function and its derivatives at a point, such as $f(1) = 3$, $f'(1) = 1$, $f''(1) = -2$, $f'''(1) = 7$, and so on. What can I say about $f(x)$ itself? Is there any reasonable approximation of the value of $f(2)$? The topic of Taylor Series addresses this problem, and allows us to make excellent approximations of functions when limited knowledge of the function is available.

9.1 Sequences

We commonly refer to a set of events that occur one after the other as a *sequence* of events. In mathematics, we use the word *sequence* to refer to an ordered set of numbers, i.e., a set of numbers that "occur one after the other."

For instance, the numbers 2, 4, 6, 8, ..., form a sequence. The order is important; the first number is 2, the second is 4, etc. It seems natural to seek a formula that describes a given sequence, and often this can be done. For instance, the sequence above could be described by the function $a(n) = 2n$, for the values of $n = 1, 2, \ldots$ To find the 10th term in the sequence, we would compute $a(10)$. This leads us to the following, formal definition of a sequence.

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Figure 9.1.1 Video introduction to Section 9.1

Definition 9.1.2 Sequence.

A **sequence** is a function $a(n)$ whose domain is $\mathbb{N}$. The **range** of a sequence is the set of all distinct values of $a(n)$.
The **terms** of a sequence are the values $a(1)$, $a(2)$, ..., which are usually denoted with subscripts as a_1, a_2, ...
A sequence $a(n)$ is often denoted as $\{a_n\}$.

Notation: We use $\mathbb{N}$ to describe the set of natural numbers, that is, the positive integers $1, 2, 3, \ldots$

Definition 9.1.3

A **factorial** refers to the product of a descending sequence of natural numbers. For example, the expression $4!$ (read as 4 factorial) refers to the number $4 \cdot 3 \cdot 2 \cdot 1 = 24$.

In general, $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$, where n is a natural number. We define $0! = 1$. While this does not immediately make sense, it makes many mathematical formulas work properly.

Example 9.1.4 Listing terms of a sequence.

List the first four terms of the following sequences.

1. $\{a_n\} = \left\{\dfrac{3^n}{n!}\right\}$
2. $\{a_n\} = \{4 + (-1)^n\}$
3. $\{a_n\} = \left\{\dfrac{(-1)^{n(n+1)/2}}{n^2}\right\}$

Solution.

1. $a_1 = \dfrac{3^1}{1!} = 3;\quad a_2 = \dfrac{3^2}{2!} = \dfrac{9}{2};\quad a_3 = \dfrac{3^3}{3!} = \dfrac{9}{2};\quad a_4 = \dfrac{3^4}{4!} = \dfrac{27}{8}$

   We can plot the terms of a sequence with a scatter plot. The horizontal axis is used for the values of n, and the values of the terms are plotted on the vertical axis. To visualize this sequence, see Figure 9.1.5.

2. $a_1 = 4+(-1)^1 = 3;\quad a_2 = 4+(-1)^2 = 5;\quad a_3 = 4+(-1)^3 = 3;\quad a_4 = 4+(-1)^4 = 5\quad .$

   Note that the range of this sequence is finite, consisting of only the values 3 and 5. This sequence is plotted in Figure 9.1.6.

3. $a_1 = \dfrac{(-1)^{1(2)/2}}{1^2} = -1;\quad a_2 = \dfrac{(-1)^{2(3)/2}}{2^2} = -\dfrac{1}{4};$ $a_3 = \dfrac{(-1)^{3(4)/2}}{3^2} = \dfrac{1}{9};\quad a_4 = \dfrac{(-1)^{4(5)/2}}{4^2} = \dfrac{1}{16};\quad ;\ a_5 = \dfrac{(-1)^{5(6)/2}}{5^2} = -\dfrac{1}{25}\quad .$

   We gave one extra term to begin to show the pattern of signs is "$-,-,+,+,-,-,\ldots$", due to the fact that the exponent of -1 is a special quadratic. This sequence is plotted in Figure 9.1.7.

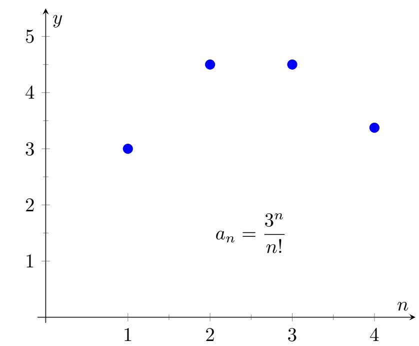

Figure 9.1.5 Plotting the sequence in Item 1

Video solution

youtu.be/watch?v=j-iqXR_bANY

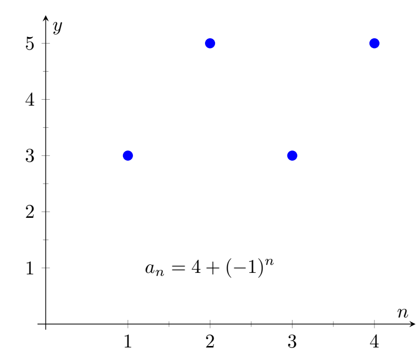

Figure 9.1.6 Plotting the sequence in Item 2

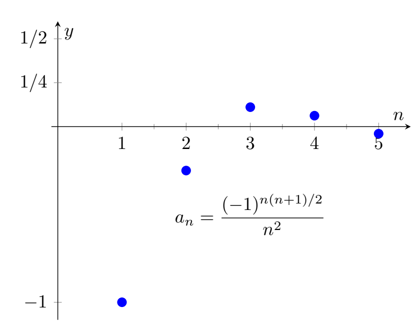

Figure 9.1.7 Plotting the sequence in Item 3

Example 9.1.8 Determining a formula for a sequence.

Find the nth term of the following sequences, i.e., find a function that describes each of the given sequences.

1. $\{a_n\} = \{2, 5, 8, 11, 14, \ldots\}$
2. $\{b_n\} = \{2, -5, 10, -17, 26, -37, \ldots\}$
3. $\{c_n\} = \{1, 1, 2, 6, 24, 120, 720, \ldots\}$

4. $\{d_n\} = \left\{\frac{5}{2}, \frac{5}{2}, \frac{15}{8}, \frac{5}{4}, \frac{25}{32}, \dots\right\}$

Solution. We should first note that there is never exactly one function that describes a finite set of numbers as a sequence. There are many sequences that start with 2, then 5, as our first example does. We are looking for a simple formula that describes the terms given, knowing there is possibly more than one answer.

1. Note how each term is 3 more than the previous one. This implies a linear function would be appropriate: $a(n) = a_n = 3n + b$ for some appropriate value of b. If we were to think in terms of ordered pairs, they would be of the form $(n, a(n))$. So one such ordered pair would be $(1, 2)$. As we want $a_1 = 2$, we set $b = -1$. Thus $a_n = 3n - 1$.

2. First notice how the sign changes from term to term. This is most commonly accomplished by multiplying the terms by either $(-1)^n$ or $(-1)^{n+1}$. Using $(-1)^n$ multiplies the odd indexed terms by (-1). Thus the first term would be negative and the second term would be positive. Multiplying by $(-1)^{n+1}$ multiplies the even indexed terms by (-1). Thus the second term would be negative and the first term would be positive. As this sequence has negative even indexed terms, we will multiply by $(-1)^{n+1}$.

   After this, we might feel a bit stuck as to how to proceed. At this point, we are just looking for a pattern of some sort: what do the numbers 2, 5, 10, 17, etc., have in common? There are many correct answers, but the one that we'll use here is that each is one more than a perfect square. That is, $2 = 1^2 + 1$, $5 = 2^2 + 1$, $10 = 3^2 + 1$, etc. Thus our formula is $b_n = (-1)^{n+1}(n^2 + 1)$.

3. One who is familiar with the factorial function will readily recognize these numbers. They are $0!$, $1!$, $2!$, $3!$, etc. Since our sequences start with $n = 1$, we cannot write $c_n = n!$, for this misses the $0!$ term. Instead, we shift by 1, and write $c_n = (n - 1)!$.

4. This one may appear difficult, especially as the first two terms are the same, but a little "sleuthing" will help. Notice how the terms in the numerator are always multiples of 5, and the terms in the denominator are always powers of 2. Does something as simple as $d_n = \frac{5n}{2^n}$ work?

   When $n = 1$, we see that we indeed get $5/2$ as desired. When $n = 2$, we get $10/4 = 5/2$. Further checking shows that this formula indeed matches the other terms of the sequence.

Video solution

youtu.be/watch?v=-Tu12lkQtTs

A common mathematical endeavor is to create a new mathematical object (for instance, a sequence) and then apply previously known mathematics to the new object. We do so here. The fundamental concept of calculus is the limit, so we will investigate what it means to find the limit of a sequence.

Definition 9.1.9 Limit of a Sequence, Convergent, Divergent.

Let $\{a_n\}$ be a sequence and let L be a real number. Given any $\varepsilon > 0$, if an N can be found such that $|a_n - L| < \varepsilon$ for all $n > N$, then we say the **limit** of $\{a_n\}$, as n **approaches infinity**, is L, denoted

$$\lim_{n\to\infty} a_n = L.$$

If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges**; otherwise, the sequence **diverges**.

This definition states, informally, that if the limit of a sequence is L, then if you go far enough out along the sequence, all subsequent terms will be *really close* to L. Of course, the terms "far enough" and "really close" are subjective terms, but hopefully the intent is clear.

This definition is reminiscent of the ε-δ proofs of Chapter 1. In that chapter we developed other tools to evaluate limits apart from the formal definition; we do so here as well.

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Figure 9.1.10 Video presentation of Definition 9.1.9

Definition 9.1.11 Limit of Infinity, Divergent Sequence.

Let $\{a_n\}$ be a sequence. We say $\lim_{n\to\infty} a_n = \infty$ if for all $M > 0$, there exists a number N such that if $n \geq N$, then $a_n > M$. In this case, we say the sequence diverges to ∞.

This definition states, informally, that if the limit of a_n is ∞, then you can guarantee that the terms of a_n will get arbitrarily large (larger than any value of M that you think of), by going out far enough in the sequence.

Theorem 9.1.12 Limit of a Sequence.

Let $\{a_n\}$ be a sequence, let L be a real number, and let $f(x)$ be a function whose domain contains the positive real numbers where $f(n) = a_n$ for all n in $\mathbb{N}$.

1. *If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.*
2. *If $\lim_{x\to\infty} f(x) = \infty$, then $\lim_{n\to\infty} a_n = \infty$.*

Theorem 9.1.12 allows us, in certain cases, to apply the tools developed in Chapter 1 to limits of sequences. Note two things *not* stated by the theorem:

1. If $\lim_{x\to\infty} f(x)$ does not exist, we cannot conclude that $\lim_{n\to\infty} a_n$ does not exist. It may, or may not, exist. For instance, we can define a sequence $\{a_n\} = \{\cos(2\pi n)\}$. Let $f(x) = \cos(2\pi x)$. Since the cosine function oscillates over the real numbers, the limit $\lim_{x\to\infty} f(x)$ does not exist. However, for every positive integer n, $\cos(2\pi n) = 1$, so $\lim_{n\to\infty} a_n = 1$.

2. If we cannot find a function $f(x)$ whose domain contains the positive real numbers where $f(n) = a_n$ for all n in $\mathbb{N}$, we cannot conclude $\lim_{n\to\infty} a_n$ does not exist. It may, or may not, exist.

Example 9.1.13 Determining convergence/divergence of a sequence.

Determine the convergence or divergence of the following sequences.

1. $\{a_n\} = \left\{\dfrac{3n^2-2n+1}{n^2-1000}\right\}$
2. $\{b_n\} = \{\cos(n)\}$
3. $\{c_n\} = \left\{\dfrac{(-1)^n}{n}\right\}$

Solution.

1. Using Theorem 1.6.21, we can state that $\lim\limits_{x\to\infty} \frac{3x^2-2x+1}{x^2-1000} = 3$. (We could have also directly applied L'Hospital's Rule.) Thus the sequence $\{a_n\}$ converges, and its limit is 3. A scatter plot of every 5 values of a_n is given in Figure 9.1.14. The values of a_n vary widely near $n = 30$, ranging from about -73 to 125, but as n grows, the values approach 3.

2. The limit $\lim\limits_{x\to\infty} \cos(x)$ does not exist, as $\cos(x)$ oscillates (and takes on every value in $[-1, 1]$ infinitely many times). Thus we cannot apply Theorem 9.1.12. The fact that the cosine function oscillates strongly hints that $\cos(n)$, when n is restricted to $\mathbb{N}$, will also oscillate. Figure 9.1.15, where the sequence is plotted, shows that this is true. Because only discrete values of cosine are plotted, it does not bear strong resemblance to the familiar cosine wave. The proof of the following statement is beyond the scope of this text, but it is true: there are infinitely many integers n that are arbitrarily (i.e., *very*) close to an even multiple of π, so that $\cos n \approx 1$. Similarly, there are infinitely many integers m that are arbitrarily close to an odd multiple of π, so that $\cos m \approx -1$. As the sequence takes on values near 1 and -1 infinitely many times, we conclude that $\lim\limits_{n\to\infty} a_n$ does not exist.

3. We cannot actually apply Theorem 9.1.12 here, as the function $f(x) = (-1)^x/x$ is not well defined. (What does $(-1)^{\sqrt{2}}$ mean? In actuality, there is an answer, but it involves *complex analysis*, beyond the scope of this text.) Instead, we invoke the definition of the limit of a sequence. By looking at the plot in Figure 9.1.16, we would like to conclude that the sequence converges to $L = 0$. Let $\epsilon > 0$ be given. We can find a natural number m such that $1/m < \varepsilon$. Let $n > m$, and consider $|a_n - L|$:

$$\begin{aligned} |a_n - L| &= \left|\frac{(-1)^n}{n} - 0\right| \\ &= \frac{1}{n} \\ &< \frac{1}{m} \text{ (since } n > m\text{)} \\ &< \varepsilon. \end{aligned}$$

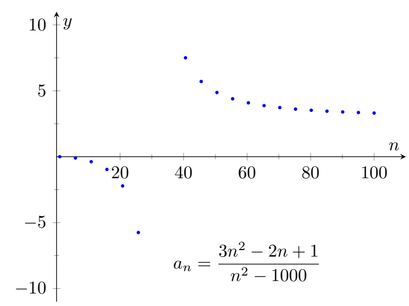

Figure 9.1.14 Scatter plot for the sequence in Item 1

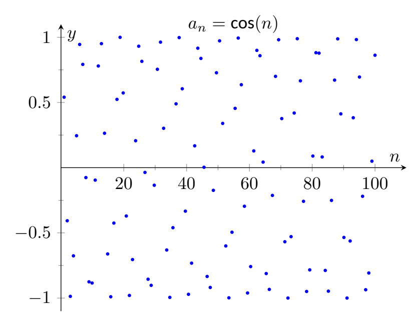

Figure 9.1.15 Scatter plot for the sequence in Item 2

We have shown that by picking m large enough, we can ensure that a_n is arbitrarily close to our limit, $L = 0$, hence by the definition of the limit of a sequence, we can say $\lim\limits_{n\to\infty} a_n = 0$.

Video solution

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In the previous example we used the definition of the limit of a sequence to determine the convergence of a sequence as we could not apply Theorem 9.1.12. In general, we like to avoid invoking the definition of a limit, and the following theorem gives us tool that we could use in that example instead.

Theorem 9.1.17 Absolute Value Theorem.

Let $\{a_n\}$ be a sequence. If $\lim\limits_{n\to\infty} |a_n| = 0$, *then* $\lim\limits_{n\to\infty} a_n = 0$

Proof. Let $\lim\limits_{n\to\infty} |a_n| = 0$. We start by noting that $-|a_n| \le a_n \le |a_n|$. If we apply limits to this inequality:

$$\lim_{n\to\infty} (-|a_n|) \le \lim_{n\to\infty} a_n \le \lim_{n\to\infty} |a_n|$$
$$-\lim_{n\to\infty} |a_n| \le \lim_{n\to\infty} a_n \le \lim_{n\to\infty} |a_n|$$

Using the fact that $\lim\limits_{n\to\infty} |a_n| = 0$:

$$0 \le \lim_{n\to\infty} a_n \le 0$$

We conclude that the only possible answer for $\lim\limits_{n\to\infty} a_n$ is 0. ■

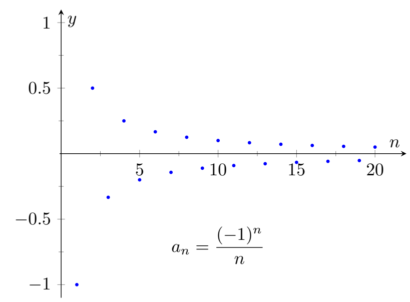

Figure 9.1.16 Scatter plot for the sequence in Item 3

Example 9.1.18 Determining the convergence/divergence of a sequence.

Determine the convergence or divergence of the following sequences.

1. $\{a_n\} = \left\{\dfrac{(-1)^n}{n}\right\}$
2. $\{a_n\} = \left\{\dfrac{(-1)^n(n+1)}{n}\right\}$

Solution.

1. This appeared in Example 9.1.13. We want to apply Theorem 9.1.17, so consider the limit of $\{|a_n|\}$:

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right|$$
$$= \lim_{n\to\infty} \frac{1}{n}$$
$$= 0.$$

   Since this limit is 0, we can apply Theorem 9.1.17 and state that $\lim\limits_{n\to\infty} a_n = 0$.

2. Because of the alternating nature of this sequence (i.e., every other term is multiplied by -1), we cannot simply look at the limit $\lim\limits_{x\to\infty} \frac{(-1)^x(x+1)}{x}$. We can try to apply the techniques of The-

orem 9.1.17:

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left|\frac{(-1)^n(n+1)}{n}\right| = \lim_{n\to\infty} \frac{n+1}{n} = 1.$$

We have concluded that when we ignore the alternating sign, the sequence approaches 1. This means we cannot apply Theorem 9.1.17; it states the the limit must be 0 in order to conclude anything.

Since we know that the signs of the terms alternate *and* we know that the limit of $|a_n|$ is 1, we know that as n approaches infinity, the terms will alternate between values close to 1 and -1, meaning the sequence diverges. A plot of this sequence is given in Figure 9.1.19.

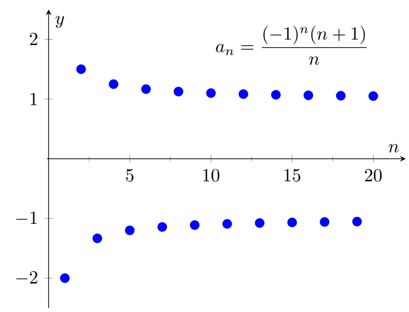

Figure 9.1.19 A plot of a sequence in Example 9.1.18, part 2

We continue our study of the limits of sequences by considering some of the properties of these limits.

Theorem 9.1.20 Properties of the Limits of Sequences.

Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = K$, and let c be a real number.

1. $\lim_{n\to\infty} (a_n \pm b_n) = L \pm K$
2. $\lim_{n\to\infty} (a_n \cdot b_n) = L \cdot K$
3. $\lim_{n\to\infty} (a_n/b_n) = L/K$, $K \neq 0$
4. $\lim_{n\to\infty} c \cdot a_n = c \cdot L$

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Figure 9.1.21 Video presentation of Theorem 9.1.17 and Theorem 9.1.20

Example 9.1.22 Applying properties of limits of sequences.

Let the following sequences, and their limits, be given:

- $\{a_n\} = \left\{\frac{n+1}{n^2}\right\}$, and $\lim_{n\to\infty} a_n = 0$;
- $\{b_n\} = \left\{\left(1+\frac{1}{n}\right)^n\right\}$, and $\lim_{n\to\infty} b_n = e$; and
- $\{c_n\} = \{n \cdot \sin(5/n)\}$, and $\lim_{n\to\infty} c_n = 5$.

Evaluate the following limits.

1. $\lim_{n\to\infty} (a_n + b_n)$
2. $\lim_{n\to\infty} (b_n \cdot c_n)$
3. $\lim_{n\to\infty} (1000 \cdot a_n)$

Solution. We will use Theorem 9.1.20 to answer each of these.

1. Since $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = e$, we conclude that $\lim_{n\to\infty} (a_n + b_n) = 0 + e = e$. So even though we are adding something to each term of the sequence b_n, we are adding something so small that the final limit is the same as before.

Video solution

youtu.be/watch?v=2PVI6iUVYcI

2. Since $\lim_{n\to\infty} b_n = e$ and $\lim_{n\to\infty} c_n = 5$, we conclude that $\lim_{n\to\infty}(b_n \cdot c_n) = e\cdot 5 = 5e$.

3. Since $\lim_{n\to\infty} a_n = 0$, we have $\lim_{n\to\infty} 1000a_n = 1000\cdot 0 = 0$. It does not matter that we multiply each term by 1000; the sequence still approaches 0. (It just takes longer to get close to 0.)

There is more to learn about sequences than just their limits. We will also study their range and the relationships terms have with the terms that follow. We start with some definitions describing properties of the range.

Definition 9.1.23 Bounded and Unbounded Sequences.

A sequence $\{a_n\}$ is said to be **bounded** if there exist real numbers m and M such that $m \leq a_n \leq M$ for all n in $\mathbb{N}$. The number m is called a **lower bound** for the sequence, and the number M is called an **upper bound** for the sequence.
A sequence $\{a_n\}$ is said to be **unbounded** if it is not bounded.
A sequence $\{a_n\}$ is said to be **bounded above** if there exists an M such that $a_n < M$ for all n in $\mathbb{N}$; it is **bounded below** if there exists an m such that $m < a_n$ for all n in $\mathbb{N}$.

It follows from this definition that an unbounded sequence may be bounded above or bounded below; a sequence that is both bounded above and below is simply a bounded sequence.

Example 9.1.24 Determining boundedness of sequences.

Determine the boundedness of the following sequences.

1. $\{a_n\} = \left\{\dfrac{1}{n}\right\}$
2. $\{a_n\} = \{2^n\}$

Solution.

1. The terms of this sequence are always positive but are decreasing, so we have $0 < a_n < 2$ for all n. Thus this sequence is bounded. Figure 9.1.25(a) illustrates this.

2. The terms of this sequence obviously grow without bound. However, it is also true that these terms are all positive, meaning $0 < a_n$. Thus we can say the sequence is unbounded, but also bounded below. Figure 9.1.25(b) illustrates this.

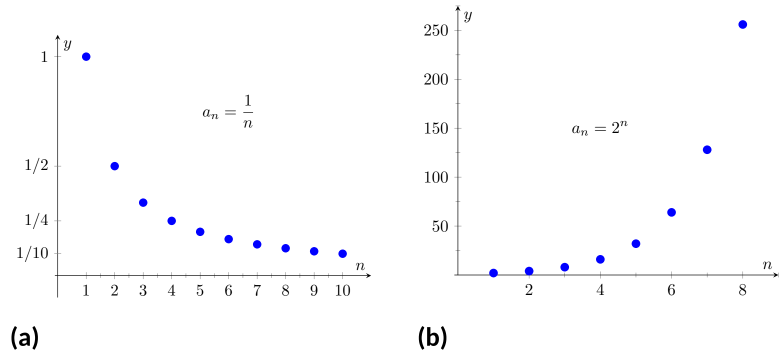


Figure 9.1.25 A plot of $\{a_n\} = \{1/n\}$ and $\{a_n\} = \{2^n\}$ from Example 9.1.24

The previous example produces some interesting concepts. First, we can recognize that the sequence $\{1/n\}$ converges to 0. This says, informally, that "most" of the terms of the sequence are "really close" to 0. This implies that the sequence is bounded, using the following logic. First, "most" terms are near 0, so we could find some sort of bound on these terms (using Definition 9.1.9, the bound is ε). That leaves a "few" terms that are not near 0 (i.e., a *finite* number of terms). A finite list of numbers is always bounded.

This logic implies that if a sequence converges, it must be bounded. This is indeed true, as stated by the following theorem.

Theorem 9.1.26 Convergent Sequences are Bounded.

Let $\{a_n\}$ be a convergent sequence. Then $\{a_n\}$ is bounded.

In Example 9.1.22 we saw the sequence $\{b_n\} = \{(1+1/n)^n\}$, where it was stated that $\lim\limits_{n\to\infty} b_n = e$. (Note that this is simply restating part of Theorem 1.3.17. The limit can also be found using logarithms and L'Hospital's rule.) Even though it may be difficult to intuitively grasp the behavior of this sequence, we know immediately that it is bounded.

Another interesting concept to come out of Example 9.1.24 again involves the sequence $\{1/n\}$. We stated, without proof, that the terms of the sequence were decreasing. That is, that $a_{n+1} < a_n$ for all n. (This is easy to show. Clearly $n < n+1$. Taking reciprocals flips the inequality: $1/n > 1/(n+1)$. This is the same as $a_n > a_{n+1}$.) Sequences that either steadily increase or decrease are important, so we give this property a name.

Definition 9.1.28 Monotonic Sequences.

1. A sequence $\{a_n\}$ is *monotonically increasing* if $a_n \le a_{n+1}$ for all n, i.e.,
$$a_1 \le a_2 \le a_3 \le \cdots a_n \le a_{n+1} \cdots$$
2. A sequence $\{a_n\}$ is *monotonically decreasing* if $a_n \ge a_{n+1}$ for all n, i.e.,
$$a_1 \ge a_2 \ge a_3 \ge \cdots a_n \ge a_{n+1} \cdots$$
3. A sequence is *monotonic* if it is monotonically increasing or monotonically decreasing.

Video solution

youtu.be/watch?v=d_C_rw2LXwk

youtu.be/watch?v=Ljrqs5azcCI

Figure 9.1.27 Video presentation of Theorem 9.1.26

Keep in mind what Theorem 9.1.26 does *not* say. It does not say that bounded sequences must converge, nor does it say that if a sequence does not converge, it is not bounded.

It is sometimes useful to call a monotonically increasing sequence *strictly increasing* if $a_n < a_{n+1}$ for all n; i.e, we remove the possibility that subsequent terms are equal.

A similar statement holds for *strictly decreasing*.

youtu.be/watch?v=ZxaUovyGMBI

Figure 9.1.29 Video presentation of Definition 9.1.28 and Theorem 9.1.32

Example 9.1.30 Determining monotonicity.

Determine the monotonicity of the following sequences.

1. $\{a_n\} = \left\{\dfrac{n+1}{n}\right\}$
2. $\{a_n\} = \left\{\dfrac{n^2+1}{n+1}\right\}$
3. $\{a_n\} = \left\{\dfrac{n^2-9}{n^2-10n+26}\right\}$
4. $\{a_n\} = \left\{\dfrac{n^2}{n!}\right\}$

Solution. In each of the following, we will examine $a_{n+1} - a_n$. If $a_{n+1} - a_n \geq 0$, we conclude that $a_n \leq a_{n+1}$ and hence the sequence is increasing. If $a_{n+1} - a_n \leq 0$, we conclude that $a_n \geq a_{n+1}$ and the sequence is decreasing. Of course, a sequence need not be monotonic and perhaps neither of the above will apply.
We also give a scatter plot of each sequence. These are useful as they suggest a pattern of monotonicity, but analytic work should be done to confirm a graphical trend.

1. $$\begin{aligned} a_{n+1} - a_n &= \frac{n+2}{n+1} - \frac{n+1}{n} \\ &= \frac{(n+2)(n) - (n+1)^2}{(n+1)n} \\ &= \frac{-1}{n(n+1)} \\ &< 0 \text{ for all } n. \end{aligned}$$

   Since $a_{n+1} - a_n < 0$ for all n, we conclude that the sequence is decreasing.

2. $$\begin{aligned} a_{n+1} - a_n &= \frac{(n+1)^2+1}{n+2} - \frac{n^2+1}{n+1} \\ &= \frac{\big((n+1)^2+1\big)(n+1) - (n^2+1)(n+2)}{(n+1)(n+2)} \\ &= \frac{n^2+3n}{(n+1)(n+2)} \\ &> 0 \text{ for all } n. \end{aligned}$$

   Since $a_{n+1} - a_n > 0$ for all n, we conclude the sequence is increasing.

3. We can clearly see in Figure 9.1.31(c), where the sequence is plotted, that it is not monotonic. However, it does seem that after the first 4 terms it is decreasing. To understand why, perform the same analysis as done before:

   $$\begin{aligned} a_{n+1} - a_n &= \frac{(n+1)^2-9}{(n+1)^2-10(n+1)+26} - \frac{n^2-9}{n^2-10n+26} \\ &= \frac{n^2+2n-8}{n^2-8n+17} - \frac{n^2-9}{n^2-10n+26} \end{aligned}$$

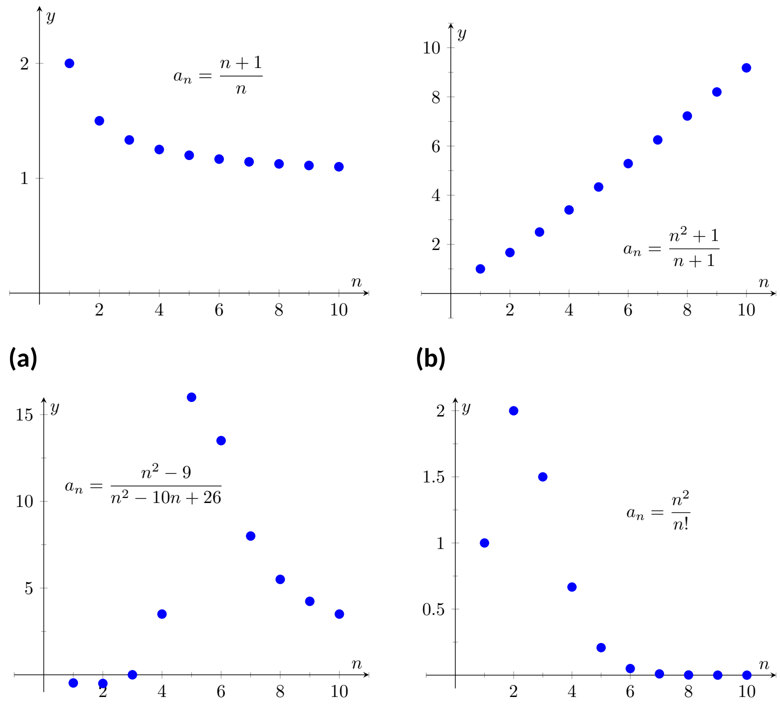
$$= \frac{(n^2+2n-8)(n^2-10n+26)-(n^2-9)(n^2-8n+17)}{(n^2-8n+17)(n^2-10n+26)}$$
$$= \frac{-10n^2+60n-55}{(n^2-8n+17)(n^2-10n+26)}.$$

We want to know when this is greater than, or less than, 0. The denominator is always positive, therefore we are only concerned with the numerator. For small values of n, the numerator is positive. As n grows large, the numerator is dominated by $-10n^2$, meaning the entire fraction will be negative; i.e., for large enough n, $a_{n+1}-a_n<0$. Using the quadratic formula we can determine that the numerator is negative for $n\geq 5$. In short, the sequence is simply not monotonic, though it is useful to note that for $n\geq 5$, the sequence is monotonically decreasing.

4. Again, the plot in Figure 9.1.31(d) shows that the sequence is not monotonic, but it suggests that it is monotonically decreasing after the first term. We perform the usual analysis to confirm this.

$$a_{n+1}-a_n = \frac{(n+1)^2}{(n+1)!}-\frac{n^2}{n!}$$
$$= \frac{(n+1)^2-n^2(n+1)}{(n+1)!}$$
$$= \frac{-n^3+2n+1}{(n+1)!}$$

When $n=1$, the above expression is >0; for $n\geq 2$, the above expression is <0. Thus this sequence is not monotonic, but it is monotonically decreasing after the first term.



(a) (b) (c) (d)

Figure 9.1.31 Plots of sequences in Example 9.1.30

Knowing that a sequence is monotonic can be useful. Consider, for example, a sequence that is monotonically decreasing and is bounded below. We know the sequence is always getting smaller, but that there is a bound to how small it can become. This is enough to prove that the sequence will converge, as stated in the following theorem.

Theorem 9.1.32 Bounded Monotonic Sequences are Convergent.

1. *Let $\{a_n\}$ be a monotonically increasing sequence that is bounded above. Then $\{a_n\}$ converges.*

2. *Let $\{a_n\}$ be a monotonically decreasing sequence that is bounded below. Then $\{a_n\}$ converges.*

Consider once again the sequence $\{a_n\} = \{1/n\}$. It is easy to show it is monotonically decreasing and that it is always positive (i.e., bounded below by 0). Therefore we can conclude by Theorem 9.1.32 that the sequence converges. We already knew this by other means, but in the following section this theorem will become very useful.

We can replace Theorem 9.1.32 with the statement "Let $\{a_n\}$ be a bounded, monotonic sequence. Then $\{a_n\}$ converges; i.e., $\lim\limits_{n\to\infty} a_n$ exists." We leave it to the reader in the exercises to show the theorem and the above statement are equivalent.

Sequences are a great source of mathematical inquiry. The On-Line Encyclopedia of Integer Sequences (oeis.org) contains thousands of sequences and their formulae. (As of this writing, there are 328,977 sequences in the database.) Perusing this database quickly demonstrates that a single sequence can represent several different "real life" phenomena.

Video solution

youtu.be/watch?v=F8mDR0ZnMCM

youtu.be/watch?v=fRMKbUCIb_4

Figure 9.1.33 Finding the limit of a bounded, monotonic sequence

Interesting as this is, our interest actually lies elsewhere. We are more interested in the *sum* of a sequence. That is, given a sequence $\{a_n\}$, we are very interested in $a_1 + a_2 + a_3 + \cdots$. Of course, one might immediately counter with "Doesn't this just add up to 'infinity'?" Many times, yes, but there are many important cases where the answer is no. This is the topic of *series*, which we begin to investigate in Section 9.2.

9.1.1 Exercises

Terms and Concepts

1. Use your own words to define a *sequence*.

2. The domain of a sequence is the __________ numbers.

3. Use your own words to describe the *range* of a sequence.

4. Describe what it means for a sequence to be *bounded*.

Problems

Exercise Group. In the following exercises, give the first five terms of the given sequence.

5. $\{a_n\} = \left\{\dfrac{4^n}{(n+1)!}\right\}$

6. $\{b_n\} = \left\{\left(-\dfrac{3}{2}\right)^n\right\}$

7. $\{c_n\} = \left\{-\dfrac{n^{n+1}}{n+2}\right\}$

8. $\{d_n\} = \left\{\dfrac{1}{\sqrt{5}}\left(\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right)\right\}$

Exercise Group. In the following exercises, determine the nth term of the given sequence.

9. 4, 7, 10, 13, 16, ...

10. $3, -\dfrac{3}{2}, \dfrac{3}{4}, -\dfrac{3}{8}, \ldots$

11. $10, 20, 40, 80, 160, \ldots$

12. $1, 1, \dfrac{1}{2}, \dfrac{1}{6}, \dfrac{1}{24}, \dfrac{1}{120}, \ldots$

Exercise Group. In the following exercises, use the following information to determine the limit of the given sequences.

- $\{a_n\} = \left\{\dfrac{2^n - 20}{2^n}\right\}$; $\lim\limits_{n\to\infty} a_n = 1$
- $\{b_n\} = \left\{\left(1 + \dfrac{2}{n}\right)^n\right\}$; $\lim\limits_{n\to\infty} b_n = e^2$
- $\{c_n\} = \{\sin(3/n)\}$; $\lim\limits_{n\to\infty} c_n = 0$

13. $\{a_n\} = \left\{\dfrac{2^n - 20}{7 \cdot 2^n}\right\}$

14. $\{a_n\} = \{3b_n - a_n\}$

15. $\{a_n\} = \left\{\sin(3/n)\left(1 + \dfrac{2}{n}\right)^n\right\}$

16. $\{a_n\} = \left\{\left(1 + \dfrac{2}{n}\right)^{2n}\right\}$

Exercise Group. In the following exercises, determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.

17. $\{a_n\} = \left\{(-1)^n \dfrac{n}{n+1}\right\}$

18. $\{a_n\} = \left\{\dfrac{4n^2 - n + 5}{3n^2 + 1}\right\}$

19. $\{a_n\} = \left\{\dfrac{4^n}{5^n}\right\}$

20. $\{a_n\} = \left\{\dfrac{n-1}{n} - \dfrac{n}{n-1}\right\}, n \geq 2$

21. $\{a_n\} = \{\ln(n)\}$

22. $\{a_n\} = \left\{\dfrac{3n}{\sqrt{n^2+1}}\right\}$

23. $\{a_n\} = \left\{\left(1 + \dfrac{1}{n}\right)^n\right\}$

24. $\{a_n\} = \left\{5 - \dfrac{1}{n}\right\}$

25. $\{a_n\} = \left\{\dfrac{(-1)^{n+1}}{n}\right\}$

26. $\{a_n\} = \left\{\dfrac{1.1^n}{n}\right\}$

27. $\{a_n\} = \left\{\dfrac{2n}{n+1}\right\}$

28. $\{a_n\} = \left\{(-1)^n\dfrac{n^2}{2^n-1}\right\}$

Exercise Group. In the following exercises, determine whether the sequence is bounded, bounded above, bounded below, or none of the above.

29. $\{a_n\} = \{\sin(n)\}$

30. $\{a_n\} = \{\tan(n)\}$

31. $\{a_n\} = \left\{(-1)^n\dfrac{3n-1}{n}\right\}$

32. $\{a_n\} = \left\{\dfrac{3n^2-1}{n}\right\}$

33. $\{a_n\} = \{n\cos(n)\}$

34. $\{a_n\} = \{2^n - n!\}$

Exercise Group. In the following exercises, determine whether the sequence is monotonically increasing or decreasing. If it is not, determine if there is an m such that it is monotonic for all $n \geq m$.

35. $\{a_n\} = \left\{\dfrac{n}{n+2}\right\}$

36. $\{a_n\} = \left\{\dfrac{n^2-6n+9}{n}\right\}$

37. $\{a_n\} = \left\{(-1)^n\dfrac{1}{n^3}\right\}$

38. $\{a_n\} = \left\{\dfrac{n^2}{2^n}\right\}$

Exercise Group. The following exercises explore further the theory of sequences.

39. Prove Theorem 9.1.17; that is, use the definition of the limit of a sequence to show that if $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

40. Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = K$.

(a) Show that if $a_n < b_n$ for all n, then $L \leq K$.

(b) Give an example where $L = K$.

41. Prove the Squeeze Theorem for sequences: Let $\{a_n\}$ and $\{b_n\}$ be such that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = L$, and let $\{c_n\}$ be such that $a_n \leq c_n \leq b_n$ for all n. Then $\lim\limits_{n\to\infty} c_n = L$

42. Prove the statement "Let $\{a_n\}$ be a bounded, monotonic sequence. Then $\{a_n\}$ converges; i.e., $\lim\limits_{n\to\infty} a_n$ exists." is equivalent to Theorem 9.1.32. That is,

(a) Show that if Theorem 9.1.32 is true, then above statement is true, and

(b) Show that if the above statement is true, then Theorem 9.1.32 is true.

9.2 Infinite Series

Given the sequence $\{a_n\} = \{1/2^n\} = 1/2,\ 1/4,\ 1/8, \ldots$, consider the following sums:

$$\begin{aligned} a_1 &= 1/2 &&= 1/2 \\ a_1 + a_2 &= 1/2 + 1/4 &&= 3/4 \\ a_1 + a_2 + a_3 &= 1/2 + 1/4 + 1/8 &&= 7/8 \\ a_1 + a_2 + a_3 + a_4 &= 1/2 + 1/4 + 1/8 + 1/16 &&= 15/16 \end{aligned}$$

In general, we can show that

$$a_1 + a_2 + a_3 + \cdots + a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}.$$

Let S_n be the sum of the first n terms of the sequence $\{1/2^n\}$. From the above, we see that $S_1 = 1/2$, $S_2 = 3/4$, etc. Our formula at the end shows that $S_n = 1 - 1/2^n$.

Now consider the following limit: $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - 1/2^n\right) = 1$. This limit can be interpreted as saying something amazing: *the sum of* all *the terms of the sequence* $\{1/2^n\}$ *is 1.*

This example illustrates some interesting concepts that we explore in this section. We begin this exploration with some definitions.

youtu.be/watch?v=RV9LNPv20TA

Figure 9.2.1 Video introduction to Section 9.2

9.2.1 Convergence of sequences

Definition 9.2.2 Infinite Series, nth Partial Sums, Convergence, Divergence.

Let $\{a_n\}$ be a sequence, beginning at some index value $n = k$.

1. The sum $\displaystyle\sum_{n=k}^{\infty} a_n$ is called an **infinite series** (or, simply **series**).
2. Let S_n denote the sum of the first n terms in the sequence $\{a_n\}$, known as the nth **partial sum** of the sequence. We can then define the sequence $\{S_n\}$ of partial sums of $\{a_n\}$.
3. If the sequence $\{S_n\}$ converges to L, we say the series $\displaystyle\sum_{n=k}^{\infty} a_n$ **converges** to L, and we write $\displaystyle\sum_{n=k}^{\infty} a_n = L$.
4. If the sequence $\{S_n\}$ diverges, the series $\displaystyle\sum_{n=k}^{\infty} a_n$ **diverges**.

Using our new terminology, we can state that the series $\displaystyle\sum_{n=1}^{\infty} 1/2^n$ converges, and $\displaystyle\sum_{n=1}^{\infty} 1/2^n = 1$.

Note that in the definition above, we do not necessarily assume that our sum begins with $n = 1$. In fact, it is quite common to have a series beginning at $n = 0$, and in some cases we may need to consider other values as well. The

nth partial sum S_n will always denote the sum of the first n terms: For example, $\sum_{n=1}^{\infty} 1/n$ has

$$S_n = \overbrace{1 + \frac{1}{2} + \cdots + \frac{1}{n}}^{n \text{ terms}},$$

while $\sum_{n=0}^{\infty} 3^{-n}$ has

$$S_n = \overbrace{1 + \frac{1}{3} + \cdots + \frac{1}{3^{n-1}}}^{n \text{ terms}},$$

and $\sum_{n=3}^{\infty} \frac{1}{n^2-2n}$ has

$$S_n = \overbrace{\frac{1}{3} + \frac{1}{8} + \cdots + \frac{1}{(n+2)^2 - 2(n+2)}}^{n \text{ terms}}.$$

In general, for the series $\displaystyle\sum_{n=k}^{\infty} a_n$, the nth partial sum will be $S_n = \displaystyle\sum_{i=k}^{k+n-1} a_i$.

We will explore a variety of series in this section. We start with two series that diverge, showing how we might discern divergence.

Example 9.2.3 Showing series diverge.

1. Let $\{a_n\} = \{n^2\}$. Show $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

2. Let $\{b_n\} = \{(-1)^{n+1}\}$. Show $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

Solution.

1. Consider S_n, the nth partial sum.

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &= 1^2 + 2^2 + 3^2 \cdots + n^2. \end{aligned}$$

By Theorem 5.3.9, this is

$$= \frac{n(n+1)(2n+1)}{6}.$$

Since $\displaystyle\lim_{n\to\infty} S_n = \infty$, we conclude that the series $\displaystyle\sum_{n=1}^{\infty} n^2$ diverges.

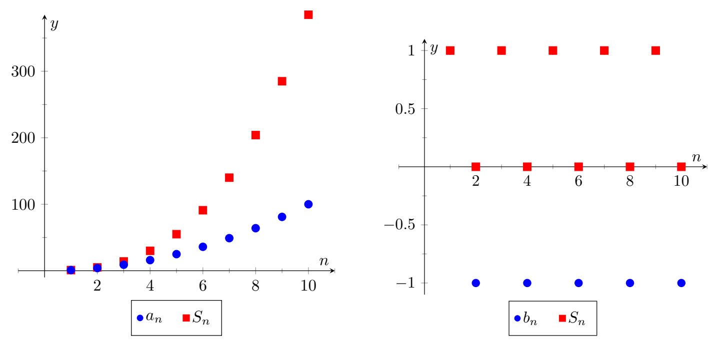
It is instructive to write $\displaystyle\sum_{n=1}^{\infty} n^2 = \infty$ for this tells us *how* the series diverges: it grows without bound. A scatter plot of the sequences $\{a_n\}$ and $\{S_n\}$ is given in Figure 9.2.4(a). The terms of $\{a_n\}$ are growing, so the terms of the partial sums $\{S_n\}$ are growing even faster, illustrating that the series diverges.

2. The sequence $\{b_n\}$ starts with 1, -1, 1, -1, $\ldots$. Consider some of the partial sums S_n of $\{b_n\}$:

$$S_1 = 1$$

$$S_2 = 0$$
$$S_3 = 1$$
$$S_4 = 0$$

This pattern repeats; we find that $S_n = \begin{cases} 1 & n \text{ is odd} \\ ,0 & n \text{ is even} \end{cases}$. As $\{S_n\}$ oscillates, repeating 1, 0, 1, 0, . . ., we conclude that $\lim_{n\to\infty} S_n$ does not exist, hence $\sum_{n=1}^{\infty}(-1)^{n+1}$ diverges. A scatter plot of the sequence $\{b_n\}$ and the partial sums $\{S_n\}$ is given in Figure 9.2.4(b). When n is odd, $b_n = S_n$ so the marks for b_n are drawn oversized to show they coincide.



(a) **(b)**

Figure 9.2.4 Scatter plots relating to Example 9.2.3

Video solution

youtu.be/watch?v=NLXq_m8S2tw

While it is important to recognize when a series diverges, we are generally more interested in the series that converge. In this section we will demonstrate a few general techniques for determining convergence; later sections will delve deeper into this topic.

9.2.2 Geometric Series

One important type of series is a *geometric series*.

Definition 9.2.5 Geometric Series.

A **geometric series** is a series of the form

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots + r^n + \cdots$$

Note that the index starts at $n = 0$, not $n = 1$.

We started this section with a geometric series, although we dropped the first term of 1. One reason geometric series are important is that they have nice convergence properties.

youtu.be/watch?v=Js5qK6AecSM

Figure 9.2.6 Video presentation of Definition 9.2.5 and Theorem 9.2.7

Theorem 9.2.7 Geometric Series Test.

Consider the geometric series $\displaystyle\sum_{n=0}^{\infty} r^n$.

1. *For* $r \neq 1$, *the* n*th partial sum is:*

$$S_n = 1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}.$$

   When $r = 1$, $S_n = n$.

2. *The series converges if, and only if,* $|r| < 1$. *When* $|r| < 1$,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}.$$

Proof. We begin by proving the formula for the simplied form for the partial sums. Consider the nth partial sum of the geometric series, $S_n = \sum_{i=0}^{n} r^i$:

$$S_n = 1 + r + r^2 + \cdots + r^{n-2} + r^{n-1}$$

Multiply both sides by r:

$$r \cdot S_n = r + r^2 + r^3 + \cdots + r^{n-1} + r^n$$

Now subtract the second line from the first and solve for S_n:

$$\begin{aligned} S_n - r \cdot S_n &= 1 - r^n \\ S_n(1 - r) &= 1 - r^n \\ S_n &= \frac{1 - r^n}{1 - r}. \end{aligned}$$

We have shown Part 1 of Geometric Series Test.
Now, examining the partial sums, we consider five cases to determine when S_n converges:

1. If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$, so we have $\lim\limits_{n\to\infty} S_n = \frac{1-0}{1-r} = \frac{1}{1-r}$, a convergent sequence of partial sums.

2. If $r > 1$, then $r^n \to \infty$ as $n \to \infty$, so

$$S_n = \frac{1 - r^n}{1 - r} = \frac{r^n}{r - 1} - \frac{1}{r - 1}$$

   diverges to infinity. (Note that $r - 1$ is a positive constant.)

3. If $r < -1$, then r^n will oscillate between large positive and large negative values as n increases. The same will be true of S_n, so $\lim\limits_{n\to\infty} S_n$ does not exist.

4. If $r = 1$, then $S_n = \frac{1-1^{n+1}}{1-1}$ is undefined. However, examining $S_n = 1 + r + r^2 + \cdots + r^n$ for $r = 1$, we can see that the partial sums simplify to $S_n = n$, and this sequence diverges to ∞.

5. If $r = -1$, then $S_n = \frac{1-(-1)^n}{2}$. For even values of n, the partial sums are always 0. For odd values of n, the partial sums are always 1. So the sequence of partial sums diverges.

Therefore, a geometric series converges if and only if $|r| < 1$. ■

According to Theorem 9.2.7, the series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^2 = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

converges as $r = 1/2 < 1$, and $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-1/2} = 2$. This concurs with our introductory example; while there we got a sum of 1, we skipped the first term of 1.

Example 9.2.8 Exploring geometric series.

Check the convergence of the following series. If the series converges, find its sum.

1. $\displaystyle\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n$
2. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n$
3. $\displaystyle\sum_{n=0}^{\infty} 3^n$

Solution.

1. Since $r = 3/4 < 1$, this series converges. By Theorem 9.2.7, we have that

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{1-3/4} = 4.$$

However, note the subscript of the summation in the given series: we are to start with $n = 2$. Therefore we subtract off the first two terms, giving:

$$\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n = 4 - 1 - \frac{3}{4} = \frac{9}{4}.$$

This is illustrated in Figure 9.2.9.

2. Since $|r| = 1/2 < 1$, this series converges, and by Theorem 9.2.7,

$$\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n = \frac{1}{1-(-1/2)} = \frac{2}{3}.$$

The partial sums of this series are plotted in Figure 9.2.10. Note how the partial sums are not purely increasing as some of the terms of the sequence $\{(-1/2)^n\}$ are negative.

3. Since $r > 1$, the series diverges. (This makes "common sense"; we expect the sum

$$1 + 3 + 9 + 27 + 81 + 243 + \cdots$$

to diverge.) This is illustrated in Figure 9.2.11.

9.2.3 p-Series

Another important type of series is the *p-series*.

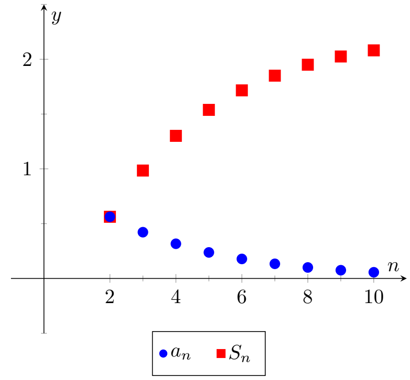


Figure 9.2.9 Scatter plots for the series in Item 1

Video solution

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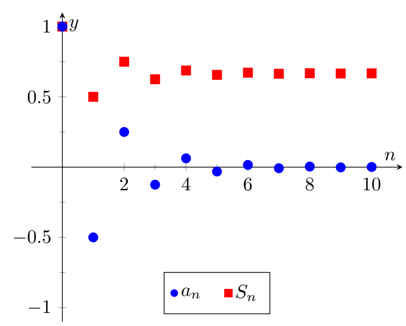


Figure 9.2.10 Scatter plots for the series in Item 2

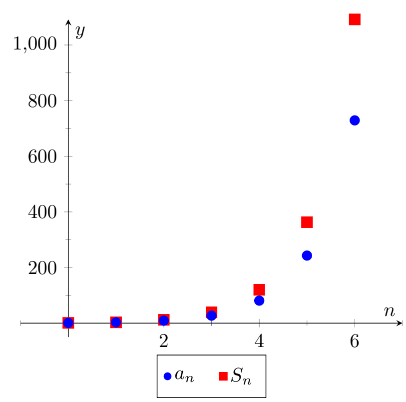


Figure 9.2.11 Scatter plots for the series in Item 3

Definition 9.2.12 p-Series, General p-Series.

1. A *p-series* is a series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \qquad \text{where } p > 0.$$

2. A *general p-series* is a series of the form

$$\sum_{n=1}^{\infty} \frac{1}{(an+b)^p},$$

where $p > 0$ and a, b are real numbers such that $a \neq 0$ and $an + b > 0$ for all $n \geq 1$.

Like geometric series, one of the nice things about p-series is that they have easy to determine convergence properties.

Theorem 9.2.13 p-Series Test.

A general p-series $\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$ *will converge if, and only if,* $p > 1$.

We will be able to prove Theorem 9.2.13 in Section 9.3. This theorem assumes that $an+b > 0$ for all n; if $an+b < 0$, $(an+b)^p$ won't be defined when p is not an integer, and if $an+b = 0$ for some n, then of course the series does not converge regardless of p as not all of the terms of the sequence are defined. These requirements actually force us to have $a > 0$, since if $a < 0$, we'll have $an+b < 0$ for sufficiently large n.

Example 9.2.14 Determining convergence of series.

Determine the convergence of the following series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
5. $\displaystyle\sum_{n=11}^{\infty} \frac{1}{(\frac{1}{2}n-5)^3}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

Solution.

1. This is a p-series with $p = 1$. By Theorem 9.2.13, this series diverges. This series is a famous series, called the *Harmonic Series*, so named because of its relationship to *harmonics* in the study of music and sound.

2. This is a p-series with $p = 2$. By Theorem 9.2.13, it converges. Note that the theorem does not give a formula by which we can determine *what* the series converges to; we just know it converges. A famous, unexpected result is that this series converges to $\pi^2/6$.

3. This is a p-series with $p = 1/2$; the theorem states that it diverges.

4. This is not a p-series; the definition does not allow for alternating signs. Therefore we cannot apply Theorem 9.2.13. (Another famous result states that this series, the *Alternating Harmonic Series*, converges to ln(2).)

5. This is a general p-series with $p = 3$, therefore it converges.

6. This is not a p-series, but a geometric series with $r = 1/2$. It converges.

Later sections will provide tests by which we can determine whether or not a given series converges. This, in general, is much easier than determining *what* a given series converges to. There are many cases, though, where the sum can be determined.

Example 9.2.15 Telescoping series.

Evaluate the sum $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$.

Solution. It will help to write down some of the first few partial sums of this series.

$$\begin{aligned}
S_1 &= \frac{1}{1}-\frac{1}{2} = 1-\frac{1}{2}\\
S_2 &= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right) = 1-\frac{1}{3}\\
S_3 &= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right) = 1-\frac{1}{4}\\
S_4 &= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right) = 1-\frac{1}{5}
\end{aligned}$$

Note how most of the terms in each partial sum are canceled out! In general, we see that $S_n = 1-\dfrac{1}{n+1}$. The sequence $\{S_n\}$ converges, as $\displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\tfrac{1}{n+1}\right) = 1$, and so we conclude that $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1$. Partial sums of the series are plotted in Figure 9.2.16.

Video solution

youtu.be/watch?v=XYW0z0LMaJc

Video solution

youtu.be/watch?v=ckj4xm6ZHgU

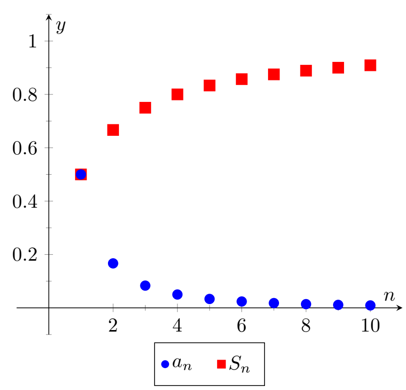


Figure 9.2.16 Scatter plots relating to the series of Example 9.2.15

The series in Example 9.2.15 is an example of a *telescoping series*. Informally, a telescoping series is one in which most terms cancel with preceding or following terms, reducing the number of terms in each partial sum. The partial sum S_n did not contain n terms, but rather just two: 1 and $1/(n+1)$.

When possible, seek a way to write an explicit formula for the nth partial sum S_n. This makes evaluating the limit $\displaystyle\lim_{n\to\infty} S_n$ much more approachable. We do so in the next example.

Example 9.2.17 Evaluating series.

Evaluate each of the following infinite series.

1. $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n^2+2n}$

2. $\displaystyle\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right)$

Solution.

1. We can decompose the fraction $2/(n^2+2n)$ as

$$\frac{2}{n^2+2n} = \frac{1}{n}-\frac{1}{n+2}.$$

(See Section 6.4, Partial Fraction Decomposition, to recall how this is done, if necessary.) Expressing the terms of $\{S_n\}$ is now more instructive:

$$\begin{aligned}
S_1 &= 1 - \frac{1}{3} \\
S_2 &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) \\
&= 1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} \\
S_3 &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) \\
&= 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} \\
S_4 &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) \\
&= 1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6} \\
S_5 &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) \\
&= 1 + \frac{1}{2} - \frac{1}{6} - \frac{1}{7}
\end{aligned}$$

We again have a telescoping series. In each partial sum, most of the terms cancel and we obtain the formula $S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$. Taking limits allows us to determine the convergence of the series:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{2},$$

so $\sum_{n=1}^{\infty} \frac{1}{n^2+2n} = \frac{3}{2}$. This is illustrated in Figure 9.2.18(a).

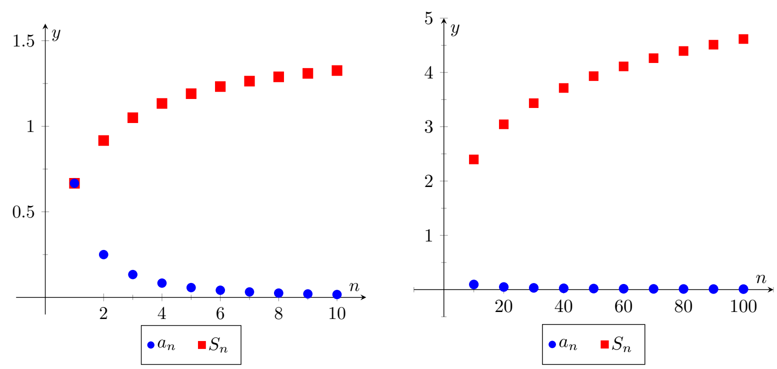
2. We begin by writing the first few partial sums of the series:

$$\begin{aligned}
S_1 &= \ln(2) \\
S_2 &= \ln(2) + \ln\left(\frac{3}{2}\right) \\
S_3 &= \ln(2) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) \\
S_4 &= \ln(2) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \ln\left(\frac{5}{4}\right)
\end{aligned}$$

At first, this does not seem helpful, but recall the logarithmic identity: $\ln(x) + \ln(y) = \ln(xy)$. Applying this to S_4 gives:

$$\begin{aligned}
S_4 &= \ln(2) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \ln\left(\frac{5}{4}\right) \\
&= \ln\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\right) = \ln(5).
\end{aligned}$$

We can conclude that $\{S_n\} = \big\{\ln(n+1)\big\}$. This sequence does not converge, as $\lim\limits_{n\to\infty} S_n = \infty$. Therefore $\sum\limits_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \infty$; the series diverges. Note in Figure 9.2.18(b) how the sequence of partial sums grows slowly; after 100 terms, it is not yet over 5. Graphically we may be fooled into thinking the series converges, but our analysis above shows that it does not.



(a) **(b)**

Figure 9.2.18 Scatter plots relating to the series in Example 9.2.17

Video solution

youtu.be/watch?v=BMn2QTtTdNQ

We are learning about a new mathematical object, the series. As done before, we apply "old" mathematics to this new topic.

Theorem 9.2.19 Properties of Infinite Series.

Let $\sum\limits_{n=1}^{\infty} a_n = L, \sum\limits_{n=1}^{\infty} b_n = K$, *and let* c *be a constant.*

1. *Constant Multiple Rule:* $\sum\limits_{n=1}^{\infty} c \cdot a_n = c \cdot \sum\limits_{n=1}^{\infty} a_n = c \cdot L.$
2. *Sum/Difference Rule:* $\sum\limits_{n=1}^{\infty} (a_n \pm b_n) = \sum\limits_{n=1}^{\infty} a_n \pm \sum\limits_{n=1}^{\infty} b_n = L \pm K.$

Before using this theorem, we provide a few "famous" series.

Key Idea 9.2.20 Important Series.

1. $\sum\limits_{n=0}^{\infty} \frac{1}{n!} = e$. (Note that the index starts with $n = 0$.)
2. $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.
3. $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$.

4. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (This is called the *Harmonic Series*.)

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$. (This is called the *Alternating Harmonic Series*.)

Example 9.2.21 Evaluating series.

Evaluate the given series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n^2-n)}{n^3}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1000}{n!}$

3. $\displaystyle\frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \cdots$

Solution.

1. We start by using algebra to break the series apart:

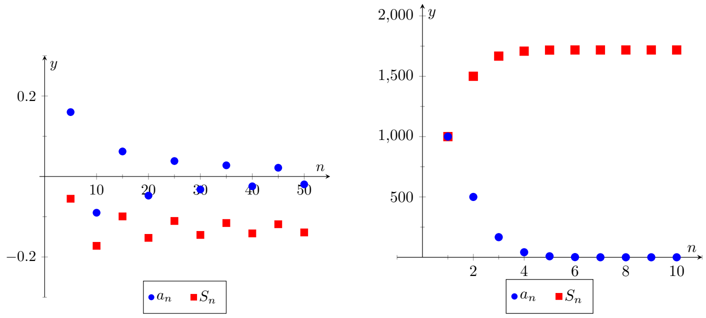
$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n^2-n)}{n^3} &= \sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}n^2}{n^3} - \frac{(-1)^{n+1}n}{n^3}\right)\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} - \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\\ &= \ln(2) - \frac{\pi^2}{12} \approx -0.1293.\end{aligned}$$

   This is illustrated in Figure 9.2.22(a).

2. This looks very similar to the series that involves e in Key Idea 9.2.20. Note, however, that the series given in this example starts with $n = 1$ and not $n = 0$. The first term of the series in the Key Idea is $1/0! = 1$, so we will subtract this from our result below:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1000}{n!} &= 1000 \cdot \sum_{n=1}^{\infty}\frac{1}{n!}\\ &= 1000 \cdot (e-1) \approx 1718.28.\end{aligned}$$

   This is illustrated in Figure 9.2.22(b). The graph shows how this particular series converges very rapidly.



(a) **(b)**

Figure 9.2.22 Scatter plots relating to the series in Example 9.2.21

3. The denominators in each term are perfect squares; we are adding $\displaystyle\sum_{n=4}^{\infty}\frac{1}{n^2}$ (note we start with $n = 4$, not $n = 1$). This series will converge. Using the formula from Key Idea 9.2.20, we have the following:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{3}\frac{1}{n^2} + \sum_{n=4}^{\infty}\frac{1}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{3}\frac{1}{n^2} = \sum_{n=4}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{6} - \left(\frac{1}{1}+\frac{1}{4}+\frac{1}{9}\right) = \sum_{n=4}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{6} - \frac{49}{36} = \sum_{n=4}^{\infty}\frac{1}{n^2}$$

$$0.2838 \approx \sum_{n=4}^{\infty}\frac{1}{n^2}$$

Video solution

youtu.be/watch?v=BMn2QTtTdNQ

It may take a while before one is comfortable with this statement, whose truth lies at the heart of the study of infinite series: *it is possible that the sum of an infinite list of nonzero numbers is finite*. We have seen this repeatedly in this section, yet it still may "take some getting used to."

As one contemplates the behavior of series, a few facts become clear.

1. In order to add an infinite list of nonzero numbers and get a finite result, "most" of those numbers must be "very near" 0.

2. If a series diverges, it means that the sum of an infinite list of numbers is not finite (it may approach $\pm\infty$ or it may oscillate), and:

   (a) The series will still diverge if the first term is removed.

   (b) The series will still diverge if the first 10 terms are removed.

   (c) The series will still diverge if the first $1,000,000$ terms are removed.

   (d) The series will still diverge if any finite number of terms from anywhere in the series are removed.

These concepts are very important and lie at the heart of the next two theorems.

Theorem 9.2.23 nth-Term Test for Divergence.

Consider the series $\displaystyle\sum_{n=1}^{\infty} a_n$. *If* $\displaystyle\lim_{n\to\infty} a_n \neq 0$, *then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges.*

Important! This theorem *does not state* that if $\displaystyle\lim_{n\to\infty} a_n = 0$ then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. The standard example of this is the Harmonic Series, as given in Key Idea 9.2.20. The Harmonic Sequence, $\{1/n\}$, converges to 0; the Harmonic Series, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$, diverges.

Looking back, we can apply this theorem to the series in Example 9.2.3. In that example, the nth terms of both sequences do not converge to 0, therefore we can quickly conclude that each series diverges.

One can rewrite Theorem 9.2.23 to state "If a series converges, then the underlying sequence converges to 0." While it is important to understand the truth of this statement, in practice it is rarely used. It is generally far easier to prove the convergence of a sequence than the convergence of a series.

Theorem 9.2.24 Infinite Nature of Series.

The convergence or divergence of an infinite series remains unchanged by the addition or subtraction of any finite number of terms. That is:

1. *A divergent series will remain divergent with the addition or subtraction of any finite number of terms.*
2. *A convergent series will remain convergent with the addition or subtraction of any finite number of terms. (Of course, the sum will likely change.)*

Consider once more the Harmonic Series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges; that is, the sequence of partial sums $\{S_n\}$ grows (very, very slowly) without bound. One might think that by removing the "large" terms of the sequence that perhaps the series will converge. This is simply not the case. For instance, the sum of the first 10 million terms of the Harmonic Series is about 16.7. Removing the first 10 million terms from the Harmonic Series changes the nth partial sums, effectively subtracting 16.7 from the sum. However, a sequence that is growing without bound will still grow without bound when 16.7 is subtracted from it.

The equations below illustrate this. The first line shows the infinite sum of the Harmonic Series split into the sum of the first 10 million terms plus the sum of "everything else." The next equation shows us subtracting these first 10 million terms from both sides. The final equation employs a bit of "psuedo-math": subtracting 16.7 from "infinity" still leaves one with "infinity."

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n} &= \sum_{n=1}^{10,000,000}\frac{1}{n} + \sum_{n=10,000,001}^{\infty}\frac{1}{n}\\ \sum_{n=1}^{\infty}\frac{1}{n} - \sum_{n=1}^{10,000,000}\frac{1}{n} &= \sum_{n=10,000,001}^{\infty}\frac{1}{n}\\ \infty - 16.7 &= \infty.\end{aligned}$$

Just for fun, we can show that the Harmonic Series diverges algebraically (without the use of p-Series Test).

Divergence of the harmonic series. If you just consider the partial sums

$$S_1, S_2, S_3, \ldots, S_{1000}, S_{1001}, \ldots,$$

it is not apparent that the partial sums diverge. Indeed they do diverge, but very, very slowly. (If you graph them on a logarithmic scale however, you can clearly see the divergence of the partial sums.) Instead, we will consider the partial sums, indexed by powers of 2. That is, we will consider $S_2, S_4, S_8, S_{16}, \ldots$.

$$S_2 = 1 + \frac{1}{2}$$
$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$
$$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8}$$

Next, we consider grouping together terms in each partial sum. We will use these groupings to set up inequalities.

$$S_2 = 1 + \frac{1}{2}$$
$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right)$$
$$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

In the partial sum S_4, we note that since $1/3 > 1/4$, we can say

$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{1/2} = 1 + \frac{2}{2}.$$

Do the same in S_8 and also note that every term in the group $\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$ is larger than $1/8$. So

$$\begin{aligned} S_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\ &> 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{1/2} + \underbrace{\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)}_{1/2} = 1 + \frac{3}{2} \end{aligned}$$

Generally, we can see that $S_{2^n} > 1 + \frac{n}{2}$. (In order to *really* show this, we should employ proof by induction.) Since the sequence of partial sums clearly diverges, so does the series $\sum_{n=1}^{\infty} 1/n$. ■

This section introduced us to series and defined a few special types of series whose convergence properties are well known: we know when a p-series or a geometric series converges or diverges. Most series that we encounter are not one of these types, but we are still interested in knowing whether or not they converge. The next three sections introduce tests that help us determine whether or not a given series converges.

9.2.4 Exercises

Terms and Concepts

1. Use your own words to describe how sequences and series are related.

2. Use your own words to define a *partial sum*.

3. Given a series $\sum_{n=1}^{\infty} a_n$, describe the two sequences related to the series that are important.

4. Use your own words to explain what a geometric series is.

5. T/F: If $\{a_n\}$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

6. T/F: If $\{a_n\}$ converges to 0, then $\sum_{n=0}^{\infty} a_n$ converges.

Problems

Exercise Group. In the following exercises, a series $\sum_{n=1}^{\infty} a_n$ is given.

(a) Give the first 5 partial sums of the series.

(b) Give a graph of the first 5 terms of a_n and S_n on the same axes.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

9. $\displaystyle\sum_{n=1}^{\infty} \cos(\pi n)$

10. $\displaystyle\sum_{n=1}^{\infty} n$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$

13. $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{9}{10}\right)^n$

14. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$

Exercise Group. In the following exercises, use Theorem 9.2.23 to show the given series diverges.

15. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{n(n+2)}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n - n^5}{5^n + n^5}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 1}{2^{n+1}}$

20. $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$

Exercise Group. In the following exercises, state whether the given series converges or diverges.

21. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$

22. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{5^n}$

23. $\displaystyle\sum_{n=0}^{\infty} \frac{6^n}{5^n}$

24. $\displaystyle\sum_{n=1}^{\infty} n^{-4}$

25. $\displaystyle\sum_{n=1}^{\infty} \sqrt{n}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{10}{n!}$

27. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n!} + \frac{1}{n}\right)$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{(2n+8)^2}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$

Exercise Group. In the following exercises, a series is given.

(a) Find a formula for S_n, the nth partial sum of the series.

(b) Determine whether the series converges or diverges. If it converges, state what it converges to.

31. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n}$

32. $\displaystyle\sum_{n=1}^{\infty} 2$

33. $1^3 + 2^3 + 3^3 + 4^3 + \cdots$

34. $\displaystyle\sum_{n=1}^{\infty} (-1)^n n$

35. $\displaystyle\sum_{n=0}^{\infty} \frac{5}{2^n}$

36. $\displaystyle\sum_{n=0}^{\infty} e^{-n}$

37. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} + \cdots$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+2)}$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$

41. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

43. $\frac{1}{1\cdot 4} + \frac{1}{2\cdot 5} + \frac{1}{3\cdot 6} + \frac{1}{4\cdot 7} + \cdots$

44. $2 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{9}\right) + \left(\frac{1}{8} + \frac{1}{27}\right) + \cdots$

45. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$

46. $\displaystyle\sum_{n=0}^{\infty} \big(\sin(1)\big)^n$

47. Break the Harmonic Series into the sum of the odd and even terms:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{2n-1} + \sum_{n=1}^{\infty} \frac{1}{2n}.$$

The goal is to show that each of the series on the right diverge.

(a) Show why $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1} > \sum_{n=1}^{\infty} \frac{1}{2n}$.

(Compare each nth partial sum.)

(b) Show why $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1} < 1 + \sum_{n=1}^{\infty} \frac{1}{2n}$

(c) Explain why (a) and (b) demonstrate that the series of odd terms is convergent, if, and only if, the series of even terms is also convergent. (That is, show both converge or both diverge.)

(d) Explain why knowing the Harmonic Series is divergent determines that the even and odd series are also divergent.

48. Show the series $\sum_{n=1}^{\infty} \frac{n}{(2n-1)(2n+1)}$ diverges.

9.3 Integral and Comparison Tests

Knowing whether or not a series converges is very important, especially when we discuss Power Series in Section 9.6. Theorems 9.2.7 and 9.2.13 give criteria for when Geometric and p-series converge, and Theorem 9.2.23 gives a quick test to determine if a series diverges. There are many important series whose convergence cannot be determined by these theorems, though, so we introduce a set of tests that allow us to handle a broad range of series. We start with the Integral Test.

9.3.1 Integral Test

We stated in Section 9.1 that a sequence $\{a_n\}$ is a function $a(n)$ whose domain is $\mathbb{N}$, the set of natural numbers. If we can extend $a(n)$ to $\mathbb{R}$, the real numbers, and it is both positive and decreasing on $[1,\infty)$, then the convergence of $\displaystyle\sum_{n=1}^{\infty} a_n$ is the same as $\displaystyle\int_1^{\infty} a(x)\,dx$.

Theorem 9.3.1 Integral Test.

Let a sequence $\{a_n\}$ be defined by $a_n = a(n)$, where $a(n)$ is continuous, positive and decreasing on $[1,\infty)$. Then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, if, and only if, $\displaystyle\int_1^{\infty} a(x)\,dx$ converges.

Theorem 9.3.1 does not state that the integral and the summation have the same value.

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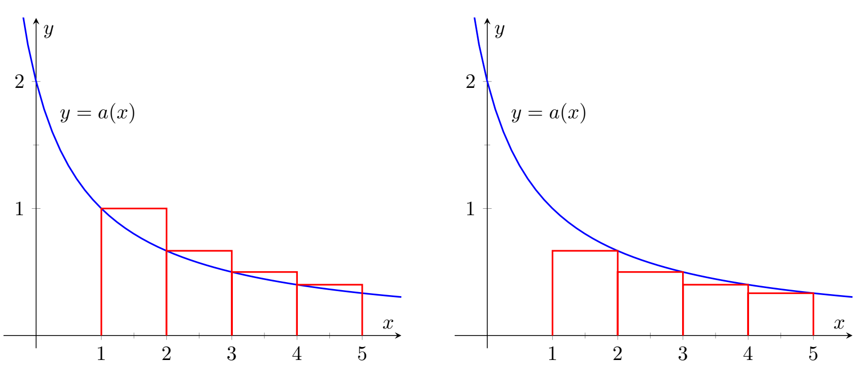
Figure 9.3.2 Video presentation of Theorem 9.3.1

We can demonstrate the truth of the Integral Test with two simple graphs. In Figure 9.3.3(a), the height of each rectangle is $a(n) = a_n$ for $n = 1, 2, \ldots$, and clearly the rectangles enclose more area than the area under $y = a(x)$. Therefore we can conclude that

$$\int_1^{\infty} a(x)\,dx < \sum_{n=1}^{\infty} a_n. \tag{9.3.1}$$



(a) **(b)**

Figure 9.3.3 Illustrating the truth of the Integral Test

In Figure 9.3.3(b), we draw rectangles under $y = a(x)$ with the Right-Hand rule, starting with $n = 2$. This time, the area of the rectangles is less than the area under $y = a(x)$, so $\displaystyle\sum_{n=2}^{\infty} a_n < \int_1^{\infty} a(x)\,dx$. Note how this summation starts with $n = 2$; adding a_1 to both sides lets us rewrite the summation starting with $n = 1$:

$$\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\,dx. \tag{9.3.2}$$

Combining Equations (9.3.1) and (9.3.2), we have

$$\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\,dx < a_1 + \sum_{n=1}^{\infty} a_n. \tag{9.3.3}$$

From Equation (9.3.3) we can make the following two statements:

1. If $\sum_{n=1}^{\infty} a_n$ diverges, so does $\int_1^{\infty} a(x)\,dx$ (because $\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\,dx$)
2. If $\sum_{n=1}^{\infty} a_n$ converges, so does $\int_1^{\infty} a(x)\,dx$ (because $\int_1^{\infty} a(x)\,dx < \sum_{n=1}^{\infty} a_n$.)

Therefore the series and integral either both converge or both diverge. Theorem 9.2.24 allows us to extend this theorem to series where $a(n)$ is positive and decreasing on $[b, \infty)$ for some $b > 1$. A formal proof of the Integral Test is shown below.

Proof of the Integral Test. Let $a(x) = a_x$ be a postive, continuous, decreasing function on $[1, \infty)$. We will consider how the partial sums of $\sum_{n=1}^{\infty} a_n$ compare to the integral $\int_0^{\infty} a(x)\,dx$. We first consider the case where $\int_1^{\infty} a(x)\,dx$ diverges.

1. Suppose that $\int_1^{\infty} a(x)\,dx$ diverges. Using Figure 9.3.3(a), we can say that $S_n = \sum_{i=1}^{n} a_i > \int_1^{n+1} a(x)\,dx$. If we let $n \to \infty$ in this inequality, we know that $\int_1^{n+1} a(x)\,dx$ will get arbitrarily large as $n \to \infty$ (since $a(x) > 0$ and $\int_1^{\infty} a(x)\,dx$ diverges). Therefore we conclude that $S_n = \sum_{i=1}^{n} a_i$ will also get arbitrarily large as $n \to \infty$, and thus $\sum_{n=1}^{\infty} a_n$ diverges.
2. Now suppose that $\int_1^{\infty} a(x)\,dx$ converges to M, where M is some positive, finite number. Using Figure 9.3.3(b), we can say that $0 < S_n = \sum_{i=1}^{n} a_i < \int_1^{\infty} a(x)\,dx = M$. Therefore our sequence of partial sums, S_n is bounded. Furthermore, S_n is a monotonically increasing sequence since all of the terms a_n are positive. Since S_n is both bounded and monotonic, S_n converges by Convergent Sequences are Bounded and by Definition 9.2.2, the series $\sum_{n=1}^{\infty} a_n$ converges as well.

■

Example 9.3.4 Using the Integral Test.

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$. (The terms of the sequence $\{a_n\} = \{\ln(n)/n^2\}$ and the nth partial sums are given in Figure 9.3.5.)

Solution. Figure 9.3.5 implies that $a(n) = (\ln(n))/n^2$ is positive and decreasing on $[2, \infty)$. We can determine this analytically, too. We know $a(n)$ is positive as both $\ln(n)$ and n^2 are positive on $[2, \infty)$. Treating $a(n)$ as a continuous function of n defined on $[1, \infty)$, consider $a'(n) = (1 - 2\ln(n))/n^3$, which is negative for $n \geq 2$. Since $a'(n)$ is negative,

$a(n)$ is decreasing for $n \geq 2$. We can still use the integral test since a finite number of terms will not affect convergence of the series.
Applying the Integral Test, we test the convergence of $\int_1^\infty \frac{\ln(x)}{x^2}\,dx$. Integrating this improper integral requires the use of Integration by Parts, with $u = \ln(x)$ and $dv = 1/x^2\,dx$.

$$\begin{aligned}
\int_1^\infty \frac{\ln(x)}{x^2}\,dx &= \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x^2}\,dx \\
&= \lim_{b\to\infty} -\frac{1}{x}\ln(x)\Big|_1^b + \int_1^b \frac{1}{x^2}\,dx \\
&= \lim_{b\to\infty} -\frac{1}{x}\ln(x) - \frac{1}{x}\Big|_1^b \\
&= \lim_{b\to\infty} 1 - \frac{1}{b} - \frac{\ln(b)}{b}. \text{ Apply L'Hospital's Rule:} \\
&= 1.
\end{aligned}$$

Since $\int_1^\infty \frac{\ln(x)}{x^2}\,dx$ converges, so does $\sum_{n=1}^\infty \frac{\ln(n)}{n^2}$.

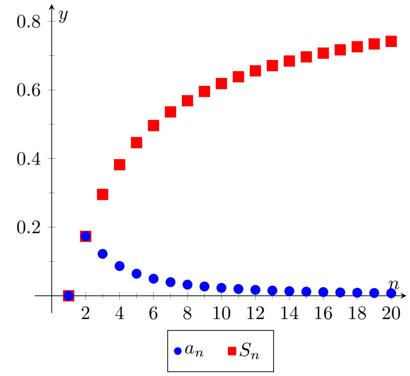


Figure 9.3.5 Plotting the sequence and series in Example 9.3.4

Theorem 9.2.13 was given without justification, stating that the general p-series $\sum_{n=1}^\infty \frac{1}{(an+b)^p}$ converges if, and only if, $p > 1$. In the following example, we prove this to be true by applying the Integral Test.

Video solution

youtu.be/watch?v=YuAg9zOh2Hk

Example 9.3.6 Using the Integral Test to establish Theorem 9.2.13.

Let a, b be real numbers such that $a \neq 0$ and $an + b > 0$ for all $n \geq 1$. Use the Integral Test to prove that $\sum_{n=1}^\infty \frac{1}{(an+b)^p}$ converges if, and only if, $p > 1$.

Solution. Consider the integral $\int_1^\infty \frac{1}{(ax+b)^p}\,dx$; assuming $p \neq 1$ and $a \neq 0$,

$$\begin{aligned}
\int_1^\infty \frac{1}{(ax+b)^p}\,dx &= \lim_{c\to\infty} \int_1^c \frac{1}{(ax+b)^p}\,dx \\
&= \lim_{c\to\infty} \frac{1}{a(1-p)}(ax+b)^{1-p}\Big|_1^c \\
&= \lim_{c\to\infty} \frac{1}{a(1-p)}\big((ac+b)^{1-p} - (a+b)^{1-p}\big).
\end{aligned}$$

This limit converges if, and only if, $p > 1$ so that $1 - p < 0$. It is easy to show that the integral also diverges in the case of $p = 1$. (This result is similar to the work preceding Key Idea 6.5.17.)

Therefore $\sum_{n=1}^\infty \frac{1}{(an+b)^p}$ converges if, and only if, $p > 1$.

Video solution

youtu.be/watch?v=fBQkA2ntBuM

We consider two more convergence tests in this section, both *comparison* tests. That is, we determine the convergence of one series by comparing it to another series with known convergence.

9.3.2 Direct Comparison Test

Theorem 9.3.7 Direct Comparison Test.

Let $\{a_n\}$ and $\{b_n\}$ be positive sequences where $a_n \leq b_n$ for all $n \geq N$, for some $N \geq 1$.

1. *If $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.*
2. *If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.*

Proof. Let $0 < a_n \leq b_n$ for all $n \geq N \geq 1$. Note that both partial sums for both series are positive and increasing since the terms of both sequences are positive.

1. Suppose that $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, so $\displaystyle\sum_{n=1}^{\infty} b_n = S$, where S is a finite, positive number. (S must be positive since $b_n > 0$.)

   Comparing the partial sums, we must have $\displaystyle\sum_{i=N}^{n} a_i \leq \sum_{i=N}^{n} b_i$ since $a_n \leq b_n$ for all $n \geq N$. Furthermore since $\displaystyle\sum_{n=1}^{\infty} b_n$ converges to S, our partial sums for a_n are bounded (note that the partial sums started at $i = N$, but a finite number of terms will not affect the boundedness of the partial sums).

   $$0 < \sum_{i=N}^{n} a_i \leq \sum_{i=N}^{n} b_i < S.$$

   Since the sequence of partial sums, $s_n = \sum_{i=1}^{n} a_i$ is both monotonically increasing and bounded, we can say that s_n converges (by Convergent Sequences are Bounded), and therefore so does $\sum_{n=1}^{\infty} a_n$.

2. Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, so $\displaystyle\sum_{i=1}^{n} a_n = \infty$. (We can say that the series diverges to ∞ since the terms of the series are always positive). Comparing the partial sums, we have

   $$\sum_{i=N}^{n} a_i \leq \sum_{i=N}^{n} b_i$$

   Then applying limits, we get

   $$\lim_{n\to\infty} \sum_{i=N}^{n} a_i \leq \lim_{n\to\infty} \sum_{i=N}^{n} b_i.$$

   Since the limit on the left side diverges to ∞, we can say that $\lim_{n\to\infty} \sum_{i=N}^{n} b_i$ also diverges to ∞.

■

youtu.be/watch?v=KhPpyQVNR5Y

Figure 9.3.8 Video presentation of Theorem 9.3.7

A sequence $\{a_n\}$ is a *positive sequence* if $a_n > 0$ for all n.

Because of Theorem 9.2.24, any theorem that relies on a positive sequence still holds true when $a_n > 0$ for all but a finite number of values of n.

Example 9.3.9 Applying the Direct Comparison Test.

Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{3^n + n^2}$.

Solution. This series is neither a geometric or p-series, but seems related. We predict it will converge, so we look for a series with larger terms that converges. (Note too that the Integral Test seems difficult to apply here.)

Since $3^n < 3^n + n^2$, $\frac{1}{3^n} > \frac{1}{3^n + n^2}$ for all $n \geq 1$. The series $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent geometric series; by Theorem 9.3.7, $\sum_{n=1}^{\infty} \frac{1}{3^n + n^2}$ converges.

Video solution

youtu.be/watch?v=DAfdDWo948U

Example 9.3.10 Applying the Direct Comparison Test.

Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{n - \ln(n)}$.

Solution. We know the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and it seems that the given series is closely related to it, hence we predict it will diverge.

Since $n \geq n - \ln(n)$ for all $n \geq 1$, $\frac{1}{n} \leq \frac{1}{n - \ln(n)}$ for all $n \geq 1$.

The Harmonic Series diverges, so we conclude that $\sum_{n=1}^{\infty} \frac{1}{n - \ln(n)}$ diverges as well.

Video solution

youtu.be/watch?v=G1j5JNagVmU

The concept of direct comparison is powerful and often relatively easy to apply. Practice helps one develop the necessary intuition to quickly pick a proper series with which to compare. However, it is easy to construct a series for which it is difficult to apply the Direct Comparison Test.

Consider $\sum_{n=1}^{\infty} \frac{1}{n + \ln(n)}$. It is very similar to the divergent series given in Example 9.3.10. We suspect that it also diverges, as $\frac{1}{n} \approx \frac{1}{n + \ln(n)}$ for large n. However, the inequality that we naturally want to use "goes the wrong way": since $n \leq n + \ln(n)$ for all $n \geq 1$, $\frac{1}{n} \geq \frac{1}{n + \ln(n)}$ for all $n \geq 1$. The given series has terms *less than* the terms of a divergent series, and we cannot conclude anything from this.

Fortunately, we can apply another test to the given series to determine its convergence.

youtu.be/watch?v=bH7U2fgSWXs

Figure 9.3.11 Motivating Theorem 9.3.12

9.3.3 Limit Comparison Test

Theorem 9.3.12 Limit Comparison Test.

Let $\{a_n\}$ and $\{b_n\}$ be positive sequences.

1. *If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where L is a positive real number, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.*
2. *If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then if $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.*
3. *If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then if $\sum_{n=1}^{\infty} b_n$ diverges, then so does $\sum_{n=1}^{\infty} a_n$.*

Theorem 9.3.12 is most useful when the convergence of the series from $\{b_n\}$ is known and we are trying to determine the convergence of the series from $\{a_n\}$.

We use the Limit Comparison Test in the next example to examine the series $\sum_{n=1}^{\infty} \frac{1}{n+\ln(n)}$ which motivated this new test.

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Figure 9.3.13 Video presentation of Theorem 9.3.12

Example 9.3.14 Applying the Limit Comparison Test.

Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{n+\ln(n)}$ using the Limit Comparison Test.

Solution. We compare the terms of $\sum_{n=1}^{\infty} \frac{1}{n+\ln(n)}$ to the terms of the Harmonic Sequence $\sum_{n=1}^{\infty} \frac{1}{n}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{1/(n+\ln(n))}{1/n} &= \lim_{n\to\infty} \frac{n}{n+\ln(n)}\\ &= 1 \text{ (after applying L'Hospital's Rule)}.\end{aligned}$$

Since the Harmonic Series diverges, we conclude that $\sum_{n=1}^{\infty} \frac{1}{n+\ln(n)}$ diverges as well.

Video solution

youtu.be/watch?v=RBeu0Tgsj_c

Example 9.3.15 Applying the Limit Comparison Test.

Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{3^n - n^2}$

Solution. This series is similar to the one in Example 9.3.9, but now we are considering "$3^n - n^2$" instead of "$3^n + n^2$." This difference makes applying the Direct Comparison Test difficult.

Instead, we use the Limit Comparison Test and compare with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$:

$$\lim_{n\to\infty} \frac{1/(3^n - n^2)}{1/3^n} = \lim_{n\to\infty} \frac{3^n}{3^n - n^2}$$
$$= 1 \text{ (after applying L'Hospital's Rule twice)} .$$

We know $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent geometric series, hence $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - n^2}$ converges as well.

Video solution

youtu.be/watch?v=1qaiCHhP3GE

As mentioned before, practice helps one develop the intuition to quickly choose a series with which to compare. A general rule of thumb is to pick a series based on the dominant term in the expression of $\{a_n\}$. It is also helpful to note that factorials dominate exponentials, which dominate algebraic functions (e.g., polynomials), which dominate logarithms. In the previous example, the dominant term of $\dfrac{1}{3^n - n^2}$ was 3^n, so we compared the series to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$. It is hard to apply the Limit Comparison Test to series containing factorials, though, as we have not learned how to apply L'Hospital's Rule to $n!$.

Example 9.3.16 Applying the Limit Comparison Test.

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^2 - n + 1}$.

Solution. We naïvely attempt to apply the rule of thumb given above and note that the dominant term in the expression of the series is $1/n^2$. Knowing that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we attempt to apply the Limit Comparison Test:

$$\lim_{n\to\infty} \frac{(\sqrt{n}+3)/(n^2 - n + 1)}{1/n^2} = \lim_{n\to\infty} \frac{n^2(\sqrt{n}+3)}{n^2 - n + 1}$$
$$= \infty \text{ (Apply L'Hospital's Rule)} .$$

Theorem 9.3.12 part (3) only applies when $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges; in our case, it converges. Ultimately, our test has not revealed anything about the convergence of our series.

The problem is that we chose a poor series with which to compare. Since the numerator and denominator of the terms of the series are both algebraic functions, we should have compared our series to the dominant term of the numerator divided by the dominant term of the denominator.

The dominant term of the numerator is $n^{1/2}$ and the dominant term of the denominator is n^2. Thus we should compare the terms of the given series to $n^{1/2}/n^2 = 1/n^{3/2}$:

$$\lim_{n\to\infty} \frac{(\sqrt{n}+3)/(n^2 - n + 1)}{1/n^{3/2}} = \lim_{n\to\infty} \frac{n^{3/2}(\sqrt{n}+3)}{n^2 - n + 1}$$

$= 1$ (Apply L'Hospital's Rule) .

Since the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^2-n+1}$ converges as well.

We mentioned earlier that the Integral Test did not work well with series containing factorial terms. The next section introduces the Ratio Test, which does handle such series well. We also introduce the Root Test, which is good for series where each term is raised to a power.

Video solution

youtu.be/watch?v=D-OsPkY8khE

9.3.4 Exercises

Terms and Concepts

1. In order to apply the Integral Test to a sequence $\{a_n\}$, the function $a(n) = a_n$ must be ________, ________ and ________.

2. T/F: The Integral Test can be used to determine the sum of a convergent series.

3. What test(s) in this section do not work well with factorials?

4. Suppose $\sum_{n=0}^{\infty} a_n$ is convergent, and there are sequences $\{b_n\}$ and $\{c_n\}$ such that $b_n \leq a_n \leq c_n$ for all n. What can be said about the series $\sum_{n=0}^{\infty} b_n$ and $\sum_{n=0}^{\infty} c_n$?

Problems

Exercise Group. In the following exercises, use the Integral Test to determine the convergence of the given series.

5. $\sum_{n=1}^{\infty} \frac{1}{2^n}$

6. $\sum_{n=1}^{\infty} \frac{1}{n^4}$

7. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

8. $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$

9. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

10. $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$

11. $\sum_{n=1}^{\infty} \frac{n}{2^n}$

12. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$

Exercise Group. In the following exercises, use the Direct Comparison Test to determine the convergence of the given series; state what series is used for comparison.

13. $\sum_{n=1}^{\infty} \frac{1}{n^2+3n-5}$

14. $\sum_{n=1}^{\infty} \frac{1}{4^n+n^2-n}$

15. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

16. $\sum_{n=1}^{\infty} \frac{1}{n!+n}$

17. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$

18. $\sum_{n=5}^{\infty} \frac{1}{\sqrt{n}-2}$

19. $\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^3-5}$

20. $\sum_{n=1}^{\infty} \frac{2^n}{5^n+10}$

21. $\sum_{n=2}^{\infty} \frac{n}{n^2-1}$

22. $\sum_{n=2}^{\infty} \frac{1}{n^2 \ln(n)}$

Exercise Group. In the following exercises, use the Limit Comparison Test to determine the convergence of the given series; state what series is used for comparison.

23. $\sum_{n=1}^{\infty} \frac{1}{n^2-3n+5}$

24. $\sum_{n=1}^{\infty} \frac{1}{4^n-n^2}$

25. $\displaystyle\sum_{n=4}^{\infty} \frac{\ln(n)}{n-3}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{n-10}{n^2+10n+10}$

29. $\displaystyle\sum_{n=1}^{\infty} \sin\left(1/n\right)$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{n^3-5}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^2+17}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+100}$

Exercise Group. In the following exercises, determine the convergence of the given series. State the test used; more than one test may be appropriate.

33. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+5)^3}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n!}$

37. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n+n}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{10n+5}$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(1/n)}{\sqrt{n}}$

41. Given that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, state which of the following series converges, may converge, or does not converge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n}$

(b) $\displaystyle\sum_{n=1}^{\infty} a_n a_{n+1}$

(c) $\displaystyle\sum_{n=1}^{\infty} (a_n)^2$

(d) $\displaystyle\sum_{n=1}^{\infty} n a_n$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$

42. In this exercise, we explore an approximation method for series to which the Integral Test applies.

(a) Let $a(x)$ be a function to which the Integral Test applies, and for which the series $\sum_{n=1}^{\infty} a_n$ converges. Let $R_n = \sum_{n+1}^{\infty} a_n$ denote the **remainder**; that is, the difference between $\sum_{n=1}^{\infty} a_n$ and the nth partial sum. (Note that R_n is the size of the error that results if we approximate the series by the nth partial sum.) Explain why we must have the following inequality:

$$\int_n^{\infty} a(x)\,dx \leq R_n \leq \int_{n+1}^{\infty} a(x)\,dx$$

(b) Estimate the error involved in using the first 12 terms to approximate the series $\sum_{n=1}^{\infty} 1/n^4$. What is the approximate value of the series?

(c) How many terms must we take to ensure that the nth partial sum approximation for $\sum_{n=1}^{\infty} 1/n^4$ is accurate to 5 decimal places?

9.4 Ratio and Root Tests

The nth-Term Test of Theorem 9.2.23 states that in order for a series $\sum_{n=1}^{\infty} a_n$ to converge, $\lim_{n\to\infty} a_n = 0$. That is, the terms of $\{a_n\}$ must get very small. Not only must the terms approach 0, they must approach 0 "fast enough": while $\lim_{n\to\infty} 1/n = 0$, the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges as the terms of $\{1/n\}$ do not approach 0 "fast enough."

The comparison tests of the previous section determine convergence by comparing terms of a series to terms of another series whose convergence is known. This section introduces the Ratio and Root Tests, which determine convergence by analyzing the terms of a series to see if they approach 0 "fast enough."

9.4.1 Ratio Test

Theorem 9.4.1 Ratio Test.

Let $\{a_n\}$ be a positive sequence and consider $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

1. *If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.*
2. *If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$ or $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.*
3. *If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, the Ratio Test is inconclusive.*

Theorem 9.2.24 allows us to apply the Ratio Test to series where $\{a_n\}$ is positive for all but a finite number of terms.

youtu.be/watch?v=DlrdbRa-t84

Figure 9.4.2 Video presentation of Theorem 9.4.1

The principle of the Ratio Test is this: if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$, then for large n, each term of $\{a_n\}$ is significantly smaller than its previous term which is enough to ensure convergence.

Example 9.4.3 Applying the Ratio Test.

Use the Ratio Test to determine the convergence of the following series:

1. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$
2. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$
3. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Solution.

1. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!}\\ &= \lim_{n\to\infty} \frac{2^{n+1}n!}{2^n(n+1)!}\\ &= \lim_{n\to\infty} \frac{2}{n+1}\\ &= 0.\end{aligned}$$

Since the limit is $0 < 1$, by the Ratio Test $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges. The fact that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$ can be interpreted to mean that in the long run, the term a_{n+1} is roughly 0 times as large as a_n. In other words, not only is a_n decreasing to 0, it is decreasing *very quickly*. That is, the terms of a_n decrease to 0 sufficiently fast enough to guarantee the convergence of $\sum_{n=1}^{\infty} a_n$.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{3^{n+1}/(n+1)^3}{3^n/n^3} \\ &= \lim_{n\to\infty} \frac{3^{n+1}n^3}{3^n(n+1)^3} \\ &= \lim_{n\to\infty} \frac{3n^3}{(n+1)^3} \\ &= 3.\end{aligned}$$

Since the limit is $3 > 1$, by the Ratio Test $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ diverges. The fact that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 3$ can be interpreted to mean that in the long run, the term a_{n+1} is roughly 3 times as large as a_n, so a_n is *increasing* by roughly a factor of 3 in the long run. We could also use Theorem 9.2.23 to determine that this series diverges. The exponential will dominate the polynomial in the long run, so $\lim_{n\to\infty} 3^n/n^3 = \infty$.

3. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{1/\big((n+1)^2+1\big)}{1/(n^2+1)} \\ &= \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1} \\ &= 1.\end{aligned}$$

Since the limit is 1, the Ratio Test is inconclusive. We can easily show this series converges using the Integral Test. We can also use Direct Comparison Test or Limit Comparison Test, with each comparing to the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Video solution

youtu.be/watch?v=Zpn9qvIGlG0

The Ratio Test is not effective when the terms of a series *only* contain algebraic functions (e.g., polynomials). It is most effective when the terms contain some factorials or exponentials. The previous example also reinforces our developing intuition: factorials dominate exponentials, which dominate algebraic functions, which dominate logarithmic functions. In Part 1 of the example, the factorial in the denominator dominated the exponential in the numerator, causing the series to converge. In Part 2, the exponential in the numerator dominated the algebraic function in the denominator, causing the series to diverge.

While we have used factorials in previous sections, we have not explored them closely and one is likely to not yet have a strong intuitive sense for how they behave. The following example gives more practice with factorials.

Example 9.4.4 Applying the Ratio Test.

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty}\frac{n!n!}{(2n)!}$.

Solution. Before we begin, be sure to note the difference between $(2n)!$ and $2n!$. When $n=4$, the former is $8! = 8\cdot 7\cdot \ldots \cdot 2\cdot 1 = 40,320$, whereas the latter is $2(4\cdot 3\cdot 2\cdot 1) = 48$.
Applying the Ratio Test:

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{(n+1)!(n+1)!/\big(2(n+1)\big)!}{n!n!/(2n)!}\\ &= \lim_{n\to\infty}\frac{(n+1)!(n+1)!(2n)!}{n!n!(2n+2)!}\end{aligned}$$

Noting that $(n+1)! = (n+1)\cdot n!$ and $(2n+2)! = (2n+2)\cdot(2n+1)\cdot(2n)!$, we have

$$\begin{aligned}&= \lim_{n\to\infty}\frac{(n+1)(n+1)}{(2n+2)(2n+1)}\\ &= 1/4.\end{aligned}$$

Since the limit is $1/4 < 1$, by the Ratio Test we conclude $\displaystyle\sum_{n=1}^{\infty}\frac{n!n!}{(2n)!}$ converges.
To find the limit in the second to last line, recall that we just need to examine the leading terms of the numerator and denominator, which are n^2 and $4n^2$ respectively.

Video solution

youtu.be/watch?v=JghEjy4pykA

9.4.2 Root Test

The final test we introduce is the Root Test, which works particularly well on series where each term is raised to a power, and does not work well with terms containing factorials.

Theorem 9.4.5 Root Test.

Let $\{a_n\}$ be a positive sequence, and consider $\displaystyle\lim_{n\to\infty}(a_n)^{1/n}$.

1. *If $\displaystyle\lim_{n\to\infty}(a_n)^{1/n} < 1$, then $\displaystyle\sum_{n=1}^{\infty}a_n$ converges.*
2. *If $\displaystyle\lim_{n\to\infty}(a_n)^{1/n} > 1$ or $\displaystyle\lim_{n\to\infty}(a_n)^{1/n} = \infty$, then $\displaystyle\sum_{n=1}^{\infty}a_n$ diverges.*
3. *If $\displaystyle\lim_{n\to\infty}(a_n)^{1/n} = 1$, the Root Test is inconclusive.*

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Figure 9.4.6 Video presentation of Theorem 9.4.5

Example 9.4.7 Applying the Root Test.

Determine the convergence of the following series using the Root Test:

1. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3n+1}{5n-2}\right)^n$ 2. $\displaystyle\sum_{n=1}^{\infty}\frac{n^4}{(\ln(n))^n}$ 3. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n^2}$

Solution.

1.

$$\begin{aligned}\lim_{n\to\infty}(a_n)^{1/n} &= \lim_{n\to\infty}\left(\left(\frac{3n+1}{5n-2}\right)^n\right)^{1/n}\\ &= \lim_{n\to\infty}\frac{3n+1}{5n-2}=\frac{3}{5}.\end{aligned}$$

Since the limit is less than 1, we conclude the series converges. Note: it is difficult to apply the Ratio Test to this series.

2.

$$\begin{aligned}\lim_{n\to\infty}(a_n)^{1/n} &= \lim_{n\to\infty}\left(\frac{n^4}{(\ln(n))^n}\right)^{1/n}\\ &= \lim_{n\to\infty}\frac{\left(n^{4/n}\right)}{\ln(n)}\end{aligned}$$

The limit of the numerator must be found using L'Hospital's Rule for indeterminate powers

$$\begin{aligned}\lim_{n\to\infty}\left(n^{4/n}\right) &= \lim_{n\to\infty}e^{\ln\left(n^{4/n}\right)}\\ &= \lim_{n\to\infty}e^{4\ln(n)/n}\end{aligned}$$

Now apply L'Hospital's to the expression in the exponent:

$$\begin{aligned}&\overset{\text{by LHR}}{=}\lim_{n\to\infty}e^{4/n}\\ &= e^0 = 1.\end{aligned}$$

Since the numerator approaches 1 (by L'Hospital's Rule) and the denominator grows to infinity, we have

$$\lim_{n\to\infty}\frac{\left(n^{4/n}\right)}{\ln(n)}=0.$$

Since the limit is less than 1, we conclude the series converges.

3. $\displaystyle\lim_{n\to\infty}\left(\frac{2^n}{n^2}\right)^{1/n}=\lim_{n\to\infty}\frac{2}{\left(n^{2/n}\right)}=2$. Since this is greater than 1, we conclude the series diverges. (Note: The Ratio Test is easy to apply to this series.)

(Also note: The limit in the denominator is found in a similar fashion as was illustrated in Part 2. In general $\displaystyle\lim_{n\to\infty}(n)^{b/n}=1$ for any real number b.)

Video solution

youtu.be/watch?v=Y7IaFXjMLlw

Each of the tests we have encountered so far has required that we analyze series from *positive* sequences. Section 9.5 relaxes this restriction by considering *alternating series*, where the underlying sequence has terms that alternate between being positive and negative.

Theorem 9.2.24 allows us to apply the Root Test to series where $\{a_n\}$ is positive for all but a finite number of terms.

9.4.3 Exercises

Terms and Concepts

1. The Ratio Test is not effective when the terms of a sequence only contain ___________ functions.

2. The Ratio Test is most effective when the terms of a sequence contains __________ and/or __________ functions.

3. What three convergence tests do not work well with terms containing factorials?

4. The Root Test works particularly well on series where each term is ___________ to a ___________.

Problems

Exercise Group. In the following exercises, determine the convergence of the given series using the Ratio Test. If the Ratio Test is inconclusive, state so and determine convergence with another test.

5. $\displaystyle\sum_{n=0}^{\infty} \frac{2n}{n!}$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{5^n - 3n}{4^n}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{n!10^n}{(2n)!}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n + n^4}{7^n + n^2}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^3 + 7}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{10 \cdot 5^n}{7^n - 3}$

12. $\displaystyle\sum_{n=1}^{\infty} n \cdot \left(\frac{3}{5}\right)^n$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots 2n}{3 \cdot 6 \cdot 9 \cdot 12 \cdots 3n}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{5 \cdot 10 \cdot 15 \cdots (5n)}$

Exercise Group. In the following exercises, determine the convergence of the given series using the Root Test. If the Root Test is inconclusive, state so and determine convergence with another test.

15. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+5}{3n+11}\right)^n$

16. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{0.9n^2 - n - 3}{n^2 + n + 3}\right)^n$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n^2}{3^n}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^2 2^{n+1}}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+7}}{7^n}$

21. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2 - n}{n^2 + n}\right)^n$

22. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\big(\ln(n)\big)^n}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{\big(\ln(n)\big)^n}$

Exercise Group. In the following exercises, determine the convergence of the given series. State the test used; more than one test may be appropriate.

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 4n - 2}{n^3 + 4n^2 - 3n + 7}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n^4 4^n}{n!}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n + n}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^n}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+4n+1}}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{n!n!n!}{(3n)!}$

31. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$

32. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+2}{n+1}\right)^n$

33. $\displaystyle\sum_{n=2}^{\infty} \frac{n^3}{\big(\ln(n)\big)^n}$

34. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$

9.5 Alternating Series and Absolute Convergence

All of the series convergence tests we have used require that the underlying sequence $\{a_n\}$ be a positive sequence. (We can relax this with Theorem 9.2.24 and state that there must be an $N > 0$ such that $a_n > 0$ for all $n > N$; that is, $\{a_n\}$ is positive for all but a finite number of values of n.)

In this section we explore series whose summation includes negative terms. We start with a very specific form of series, where the terms of the summation alternate between being positive and negative.

Definition 9.5.1 Alternating Series.

Let $\{a_n\}$ be a positive sequence. An **alternating series** is a series of either the form

$$\sum_{n=1}^{\infty}(-1)^n a_n \qquad \text{or} \qquad \sum_{n=1}^{\infty}(-1)^{n+1}a_n.$$

Recall the terms of Harmonic Series come from the Harmonic Sequence $\{a_n\} = \{1/n\}$. An important alternating series is the *Alternating Harmonic Series*:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Geometric Series can also be alternating series when $r < 0$. For instance, if $r = -1/2$, the geometric series is

$$\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots$$

Theorem 9.2.7 states that geometric series converge when $|r| < 1$ and gives the sum: $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$. When $r = -1/2$ as above, we find

$$\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n = \frac{1}{1-(-1/2)} = \frac{1}{3/2} = \frac{2}{3}.$$

A powerful convergence theorem exists for other alternating series that meet a few conditions.

Theorem 9.5.2 Alternating Series Test.

Let $\{a_n\}$ be a positive, decreasing sequence where $\lim_{n\to\infty} a_n = 0$. Then

$$\sum_{n=1}^{\infty}(-1)^n a_n \qquad \text{and} \qquad \sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

converge.

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Figure 9.5.3 Video presentation of Definition 9.5.1 and Theorem 9.5.2

The basic idea behind Theorem 9.5.2 is illustrated in Figure 9.5.4–9.5.5. A positive, decreasing sequence $\{a_n\}$ is shown along with the partial sums

$$S_n = \sum_{i=1}^{n}(-1)^{i+1}a_i = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1}a_n.$$

Because $\{a_n\}$ is decreasing, the amount by which S_n bounces up/down decreases. Moreover, the odd terms of S_n form a decreasing, bounded sequence, while the even terms of S_n form an increasing, bounded sequence. Since bounded, monotonic sequences converge (see Theorem 9.1.32) and the terms of $\{a_n\}$ approach 0, one can show the odd and even terms of S_n converge to the same common limit L, the sum of the series.

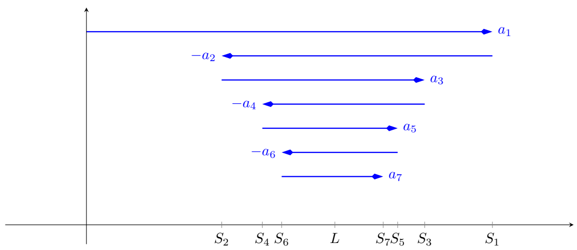

Figure 9.5.5 A visual representation of adding terms of an alternating series. The arrows represent the length and direction of each term of the sequence.

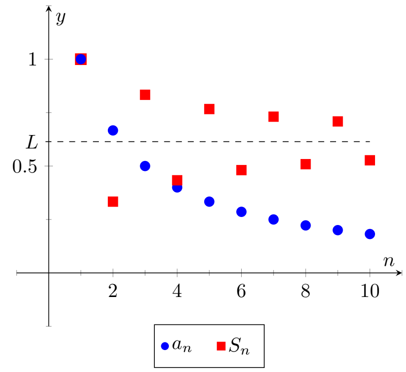

Figure 9.5.4 Illustrating convergence with the Alternating Series Test

Example 9.5.6 Applying the Alternating Series Test.

Determine if the Alternating Series Test applies to each of the following series.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ 2. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{\ln(n)}{n}$ 3. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{|\sin(n)|}{n^2}$

Solution.

1. This is the Alternating Harmonic Series as seen previously. The underlying sequence is $\{a_n\} = \{1/n\}$, which is positive, decreasing, and approaches 0 as $n \to \infty$. Therefore we can apply the Alternating Series Test and conclude this series converges. While the test does not state what the series converges to, we will see later that $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \ln(2)$.

2. The underlying sequence is $\{a_n\} = \{\ln(n)/n\}$. This is positive and approaches 0 as $n \to \infty$ (use L'Hospital's Rule). However, the sequence is not decreasing for all n. It is straightforward to compute $a_1 = 0$, $a_2 \approx 0.347$, $a_3 \approx 0.366$, and $a_4 \approx 0.347$: the sequence is increasing for at least the first 3 terms. We do not immediately conclude that we cannot apply the Alternating Series Test. Rather, consider the long-term behavior of $\{a_n\}$. Treating $a_n = a(n)$ as a continuous function of n defined on $[1, \infty)$, we can take its derivative:

$$a'(n) = \frac{1 - \ln(n)}{n^2}.$$

The derivative is negative for all $n \geq 3$ (actually, for all $n > e$), meaning $a(n) = a_n$ is decreasing on $[3, \infty)$. We can apply the Alternating Series Test to the series when we start with $n = 3$ and conclude that $\displaystyle\sum_{n=3}^{\infty}(-1)^{n}\frac{\ln(n)}{n}$ converges; adding the terms with $n = 1$ and $n = 2$ do not change the convergence (i.e., we apply

Theorem 9.2.24). The important lesson here is that as before, if a series fails to meet the criteria of the Alternating Series Test on only a finite number of terms, we can still apply the test.

3. The underlying sequence is $\{a_n\} = |\sin(n)|/n$. This sequence is positive and approaches 0 as $n \to \infty$. However, it is not a decreasing sequence; the value of $|\sin(n)|$ oscillates between 0 and 1 as $n \to \infty$. We cannot remove a finite number of terms to make $\{a_n\}$ decreasing, therefore we cannot apply the Alternating Series Test. Keep in mind that this does not mean we conclude the series diverges; in fact, it does converge. We are just unable to conclude this based on Theorem 9.5.2. We will be able to show that this series converges shortly.

Video solution

youtu.be/watch?v=WmsfSSlc-W0

Key Idea 9.2.20 gives the sum of some important series. Two of these are

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.64493 \text{ and } \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \approx 0.82247.$$

These two series converge to their sums at different rates. To be accurate to two places after the decimal, we need 202 terms of the first series though only 13 of the second. To get 3 places of accuracy, we need 1069 terms of the first series though only 33 of the second. Why is it that the second series converges so much faster than the first?

While there are many factors involved when studying rates of convergence, the alternating structure of an alternating series gives us a powerful tool when approximating the sum of a convergent series.

Theorem 9.5.7 The Alternating Series Approximation Theorem.

Let $\{a_n\}$ be a sequence that satisfies the hypotheses of the Alternating Series Test, and let S_n and L be the nth partial sums and sum, respectively, of either $\sum_{n=1}^{\infty}(-1)^n a_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$. Then

1. $E_n = |S_n - L| < a_{n+1}$, *and*
2. L *is between* S_n *and* S_{n+1}.

Part 1 of Theorem 9.5.7 states that the nth partial sum of a convergent alternating series will be within a_{n+1} of its total sum. You can see this visually in Figure 9.5.5. Look at the distance between S_6 and L. Clearly this distance is less than the length of the arrow corresponding to a_7.

Also consider the alternating series we looked at before the statement of the theorem, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$. Since $a_{14} = 1/14^2 \approx 0.0051$, we know that S_{13} is within 0.0051 of the total sum.

Moreover, Part 2 of the theorem states that since $S_{13} \approx 0.8252$ and $S_{14} \approx 0.8201$, we know the sum L lies between 0.8201 and 0.8252. One use of this is the knowledge that S_{14} is accurate to two places after the decimal.

Some alternating series converge slowly. In Example 9.5.6 we determined the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln(n)}{n}$ converged. With $n = 1001$, we find $\ln(n)/n \approx 0.0069$, meaning that $S_{1000} \approx 0.1633$ is accurate to one, maybe two, places

youtu.be/watch?v=xtsP2Kfy6Bk

Figure 9.5.8 Video presentation of Theorem 9.5.7

after the decimal. Since $S_{1001} \approx 0.1564$, we know the sum L is $0.1564 \leq L \leq 0.1633$.

Example 9.5.9 Approximating the sum of convergent alternating series.

Approximate the sum of the following series, accurate to within 0.001.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^3}$
2. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln(n)}{n}$

Solution.

1. Using Theorem 9.5.7, we want to find n where $1/n^3 \leq 0.001$. That is, we want to find the the first time a term in the sequence a_n is smaller than the desired level of error:

$$\begin{aligned}\frac{1}{n^3} &\leq 0.001 = \frac{1}{1000}\\ n^3 &\geq 1000\\ n &\geq \sqrt[3]{1000}\\ n &\geq 10.\end{aligned}$$

Let L be the sum of this series. By Part 1 of the theorem, $|S_9 - L| < a_{10} = 1/1000$. (We found $a_{10} = a_{n+1} < 0.0001$, so $n = 9$). We can compute $S_9 = 0.902116$, which our theorem states is within 0.001 of the total sum. We can use Part 2 of the theorem to obtain an even more accurate result. As we know the 10th term of the series is $(-1)^n/10^3 = -1/1000$, we can easily compute $S_{10} = 0.901116$. Part 2 of the theorem states that L is between S_9 and S_{10}, so $0.901116 < L < 0.902116$.

2. We want to find n where $\ln(n)/n < 0.001$. We start by solving $\ln(n)/n = 0.001$ for n. This cannot be solved algebraically, so we will use Newton's Method to approximate a solution. (Note: we can also use a "Brute Force" technique. That is, we can guess and check numerically until we find a solution.) Let $f(x) = \ln(x)/x - 0.001$; we want to know where $f(x) = 0$. We make a guess that x must be "large," so our initial guess will be $x_1 = 1000$. Recall how Newton's Method works: given an approximate solution x_n, our next approximation x_{n+1} is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

We find $f'(x) = \big(1 - \ln(x)\big)/x^2$. This gives

$$\begin{aligned}x_2 &= 1000 - \frac{\ln(1000)/1000 - 0.001}{\big(1-\ln(1000)\big)/1000^2}\\ &= 2000.\end{aligned}$$

Using a computer, we find that Newton's Method seems to converge to a solution $x = 9118.01$ after 8 iterations. Taking the next integer higher, we have $n = 9119$, where $\ln(9119)/9119 =$

$0.000999903 < 0.001$. Again using a computer, we find $S_{9118} = -0.160369$. Part 1 of the theorem states that this is within 0.001 of the actual sum L. Already knowing the $9{,}119$th term, we can compute $S_{9119} = -0.159369$, meaning $-0.159369 < L < -0.160369$.

Notice how the first series converged quite quickly, where we needed only 10 terms to reach the desired accuracy, whereas the second series took over 9,000 terms.

Video solution

youtu.be/watch?v=f_iiMYNpqXE

One of the famous results of mathematics is that the Harmonic Series, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, yet the Alternating Harmonic Series, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$, converges. The notion that alternating the signs of the terms in a series can make a series converge leads us to the following definitions.

Definition 9.5.10 Absolute and Conditional Convergence.

1. A series $\sum_{n=1}^{\infty} a_n$ *converges absolutely* if $\sum_{n=1}^{\infty} |a_n|$ converges.
2. A series $\sum_{n=1}^{\infty} a_n$ *converges conditionally* if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

In Definition 9.5.10, $\sum_{n=1}^{\infty} a_n$ is not necessarily an alternating series; it just may have some negative terms.

Thus we say the Alternating Harmonic Series converges conditionally.

Example 9.5.11 Determining absolute and conditional convergence.

Determine if the following series converge absolutely, conditionally, or diverge.

1. $\sum_{n=1}^{\infty}(-1)^n\frac{n+3}{n^2+2n+5}$
2. $\sum_{n=1}^{\infty}(-1)^n\frac{n^2+2n+5}{2^n}$
3. $\sum_{n=3}^{\infty}(-1)^n\frac{3n-3}{5n-10}$

Solution.

1. We can show the series
$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{n+3}{n^2+2n+5}\right| = \sum_{n=1}^{\infty}\frac{n+3}{n^2+2n+5}$$
diverges using the Limit Comparison Test, comparing with $1/n$.
The series $\sum_{n=1}^{\infty}(-1)^n\frac{n+3}{n^2+2n+5}$ converges using the Alternating Series Test; we conclude it converges conditionally.

2. We can show the series

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{n^2+2n+5}{2^n}\right| = \sum_{n=1}^{\infty}\frac{n^2+2n+5}{2^n}$$

converges using the Ratio Test. Therefore we conclude $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2+2n+5}{2^n}$ converges absolutely.

3. The series

$$\sum_{n=3}^{\infty}\left|(-1)^n\frac{3n-3}{5n-10}\right| = \sum_{n=3}^{\infty}\frac{3n-3}{5n-10}$$

diverges using the nth Term Test, so it does not converge absolutely. The series $\displaystyle\sum_{n=3}^{\infty}(-1)^n\frac{3n-3}{5n-10}$ fails the conditions of the Alternating Series Test as $(3n-3)/(5n-10)$ does not approach 0 as $n \to \infty$. We can state further that this series diverges; as $n \to \infty$, the series effectively adds and subtracts $3/5$ over and over. This causes the sequence of partial sums to oscillate and not converge. Therefore the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{3n-3}{5n-10}$ diverges.

Video solution

youtu.be/watch?v=I02aGt0Ce5M

Knowing that a series converges absolutely allows us to make two important statements, given in Theorem 9.5.13 below. The first is that absolute convergence is “stronger” than regular convergence. That is, just because $\sum_{n=1}^{\infty} a_n$ converges, we cannot conclude that $\sum_{n=1}^{\infty} |a_n|$ will converge, but knowing a series converges absolutely tells us that $\sum_{n=1}^{\infty} a_n$ will converge.

One reason this is important is that our convergence tests all require that the underlying sequence of terms be positive. By taking the absolute value of the terms of a series where not all terms are positive, we are often able to apply an appropriate test and determine absolute convergence. This, in turn, determines that the series we are given also converges.

The second statement relates to *rearrangements* of series. When dealing with a finite set of numbers, the sum of the numbers does not depend on the order which they are added. (So $1+2+3 = 3+1+2$.) One may be surprised to find out that when dealing with an infinite set of numbers, the same statement does not always hold true: some infinite lists of numbers may be rearranged in different orders to achieve different sums. The theorem states that the terms of an absolutely convergent series can be rearranged in any way without affecting the sum.

youtu.be/watch?v=d0enMDgDON8

Figure 9.5.12 Video presentation of Definition 9.5.10 and Theorem 9.5.13

Theorem 9.5.13 Absolute Convergence Theorem.

Let $\displaystyle\sum_{n=1}^{\infty} a_n$ *be a series that converges absolutely.*

1. $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges.*

2. *Let $\{b_n\}$ be any rearrangement of the sequence $\{a_n\}$. Then*

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

Proof. We will provide a proof for Part 1 of Absolute Convergence Theorem. Suppose that $\sum_{n=1}^{\infty} |a_n|$ converges. We start by noting that for any sequence a_n, we have

$$-|a_n| \leq a_n \leq |a_n|$$

If we add $|a_n|$ to all three sides:

$$0 \leq a_n + |a_n| \leq 2|a_n|.$$

We are now in a position to apply the Direct Comparison Test to the series $\sum_{n=1}^{\infty} (a_n + |a_n|)$. Since $\sum_{n=1}^{\infty} |a_n|$ converges by our supposition, so does $\sum_{n=1}^{\infty} 2|a_n|$ (the scalar multiple of a convergent series also converges by Theorem 9.2.19). Therefore $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges by the Direct Comparison Test.

Now we turn our attention to $\sum_{n=1}^{\infty} a_n$. We can say

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} (a_n + |a_n| - |a_n|) \\ &= \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|.\end{aligned}$$

The last line is the difference between two convergent series, which is also convergent by Theorem 9.2.19. Therefore $\sum_{n=1}^{\infty} a_n$ converges. ■

In Example 9.5.11, we determined the series in Part 2 converges absolutely. Theorem 9.5.13 tells us the series converges (which we could also determine using the Alternating Series Test).

The theorem states that rearranging the terms of an absolutely convergent series does not affect its sum. This implies that perhaps the sum of a conditionally convergent series can change based on the arrangement of terms. Indeed, it can. The Riemann Rearrangement Theorem (named after Bernhard Riemann) states that any conditionally convergent series can have its terms rearranged so that the sum is any desired value, including ∞!

As an example, consider the Alternating Harmonic Series once more. We have stated that

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} \cdots = \ln(2),$$

(see Key Idea 9.2.20 or Example 9.5.6).

Consider the rearrangement where every positive term is followed by two negative terms:

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} \cdots$$

(Convince yourself that these are exactly the same numbers as appear in the Alternating Harmonic Series, just in a different order.) Now group some terms and simplify:

$$\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\cdots=$$
$$\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\cdots=$$
$$\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots\right)=\frac{1}{2}\ln(2).$$

By rearranging the terms of the series, we have arrived at a different sum! (One could *try* to argue that the Alternating Harmonic Series does not actually converge to $\ln(2)$, because rearranging the terms of the series *shouldn't* change the sum. However, the Alternating Series Test proves this series converges to L, for some number L, and if the rearrangement does not change the sum, then $L = L/2$, implying $L = 0$. But the Alternating Series Approximation Theorem quickly shows that $L > 0$. The only conclusion is that the rearrangement *did* change the sum.) This is an incredible result.

We end here our study of tests to determine convergence. The end of this text contains a table summarizing the tests that one may find useful.

While series are worthy of study in and of themselves, our ultimate goal within calculus is the study of Power Series, which we will consider in the next section. We will use power series to create functions where the output is the result of an infinite summation.

9.5.1 Exercises

Terms and Concepts

1. Why is $\sum_{n=1}^{\infty} \sin(n)$ not an alternating series?

2. A series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges when $\{a_n\}$ is ___________, ___________ and $\lim_{n\to\infty} a_n$ =___________.

3. Give an example of a series where $\sum_{n=0}^{\infty} a_n$ converges but $\sum_{n=0}^{\infty} |a_n|$ does not.

4. The sum of a ___________ convergent series can be changed by rearranging the order of its terms.

Problems

Exercise Group. In the following exercises, an alternating series $\sum_{n=i}^{\infty} a_n$ is given.

(a) Determine if the series converges or diverges.

(b) Determine if $\sum_{n=0}^{\infty} |a_n|$ converges or diverges.

(c) If $\sum_{n=0}^{\infty} a_n$ converges, determine if the convergence is conditional or absolute.

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

6. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n!}}$

7. $\sum_{n=0}^{\infty} (-1)^n \frac{n+5}{3n-5}$

8. $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2}$

9. $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{3n+5}{n^2-3n+1}$

10. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n)+1}$

11. $\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln(n)}$

12. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1+3+5+\cdots+(2n-1)}$

13. $\sum_{n=1}^{\infty} \cos(\pi n)$

14. $\sum_{n=2}^{\infty} \frac{\sin\left((n+1/2)\pi\right)}{n\ln(n)}$

15. $\sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n$

16. $\sum_{n=0}^{\infty} (-e)^{-n}$

17. $\sum_{n=0}^{\infty} \frac{(-1)^n n^2}{n!}$

18. $\sum_{n=0}^{\infty} (-1)^n 2^{-n^2}$

19. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

20. $\sum_{n=1}^{\infty} \frac{(-1000)^n}{n!}$

Exercise Group. Let S_n be the n^{th} partial sum of a series. In the following exercises a convergent alternating series is given and a value of n. Compute S_n and S_{n+1} and use these values to find bounds on the sum of the series.

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}, n = 5$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}, n = 4$

23. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}, n = 6$

24. $\displaystyle\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n, n = 9$

Exercise Group. In the following exercises, a convergent alternating series is given along with its sum and a value of ε. Use Theorem 9.5.7 to find n such that the nth partial sum of the series is within ε of the sum of the series.

25. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}, \varepsilon = 0.001$

26. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}, \varepsilon = 0.0001$

27. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}, \varepsilon = 0.001$

28. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} = \cos(1), \varepsilon = 10^{-8}$

9.6 Power Series

So far, our study of series has examined the question of "Is the sum of these infinite terms finite?," i.e., "Does the series converge?" We now approach series from a different perspective: as a function. Given a value of x, we evaluate $f(x)$ by finding the sum of a particular series that depends on x (assuming the series converges). We start this new approach to series with a definition.

youtu.be/watch?v=y12Zn3QZpbE

Figure 9.6.1 Video introduction to Section 9.6

Definition 9.6.2 Power Series.

Let $\{a_n\}$ be a sequence, let x be a variable, and let c be a real number.

1. The *power series in* x is the series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

2. The *power series in* x *centered at* c is the series

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots$$

Example 9.6.3 Examples of power series.

Write out the first five terms of the following power series:

1. $\displaystyle\sum_{n=0}^{\infty} x^n$

2. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x+1)^n}{n}$

3. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\pi)^{2n}}{(2n)!}$

Solution.

1. One of the conventions we adopt is that $x^0 = 1$ regardless of the value of x. Therefore

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots$$

This is a geometric series in x with $r = x$.

2. This series is centered at $c = -1$. Note how this series starts with $n = 1$. We could rewrite this series starting at $n = 0$ with the understanding that $a_0 = 0$, and hence the first term is 0.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x+1)^n}{n}$$
$$= (x+1) - \frac{(x+1)^2}{2} + \frac{(x+1)^3}{3} - \frac{(x+1)^4}{4} + \frac{(x+1)^5}{5} \ldots$$

3. This series is centered at $c = \pi$. Recall that $0! = 1$.

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-\pi)^{2n}}{(2n)!}$$
$$= -1 + \frac{(x-\pi)^2}{2} - \frac{(x-\pi)^4}{24} + \frac{(x-\pi)^6}{6!} - \frac{(x-\pi)^8}{8!}\dots$$

Video solution

youtu.be/watch?v=NK8i9T-4hSg

We introduced power series as a type of function, where a value of x is given and the sum of a series is returned. Of course, not every series converges. For instance, in part 1 of Example 9.6.3, we recognized the series $\sum_{n=0}^{\infty} x^n$ as a geometric series in x. Theorem 9.2.7 states that this series converges only when $|x| < 1$.

This raises the question: "For what values of x will a given power series converge?," which leads us to a theorem and definition.

Theorem 9.6.4 Convergence of Power Series.

Let a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ *be given. Then one of the following is true:*

1. *The series converges only at* $x = c$.
2. *There is an* $R > 0$ *such that the series converges for all* x *in* $(c - R, c + R)$ *and diverges for all* $x < c - R$ *and* $x > c + R$.
3. *The series converges for all* x.

youtu.be/watch?v=RRzviD89Phg

Figure 9.6.5 Video presentation of Theorem 9.6.4 and Definition 9.6.6

The value of R is important when understanding a power series, hence it is given a name in the following definition. Also, note that part 2 of Theorem 9.6.4 makes a statement about the interval $(c - R, c + R)$, but the not the endpoints of that interval. A series may/may not converge at these endpoints.

Definition 9.6.6 Radius and Interval of Convergence.

1. The number R given in Theorem 9.6.4 is the *radius of convergence* of a given series. When a series converges for only $x = c$, we say the radius of convergence is 0, i.e., $R = 0$. When a series converges for all x, we say the series has an infinite radius of convergence, i.e., $R = \infty$.
2. The *interval of convergence* is the set of all values of x for which the series converges.

To find the interval of convergence, we start by using the ratio test to find the radius of convergence R. If $0 < R < \infty$, we know the series converges on $(c - R, c + R)$, and it remains to check for convergence at the endpoints.

Given $\sum_{n=0}^{\infty} a_n(x-c)^n$ we apply the ratio test to $\sum_{n=0}^{\infty} |a_n(x-c)^n|$ since the ratio test requires positive terms. We find

$$\lim_{n\to\infty} \frac{|a_{n+1}(x-c)^{n+1}|}{|a_n(x-c)^n|} = L\,|x-c|\,,$$

where $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. It follows that the series converges absolutely (and therefore converges) for any x such that $L\,|x-c|<1$; that is, for x in $\left(c-\frac{1}{L}, c+\frac{1}{L}\right)$.

On the other hand, suppose for some x that $L\,|x-c|>1$. Then, for sufficiently large n, $|a_{n+1}|>|a_n|$. This means that the terms of $\sum_{n=0}^{\infty} a_n(x-c)^n$ are growing in absolute value, and therefore cannot converge to zero. This means that the series diverges, by Theorem 9.2.23.

From the above observations, it follows that $R=\frac{1}{L}$ must be the radius of convergence.

Key Idea 9.6.7 Determining the Radius and Interval of Convergence.

Given the power series $\displaystyle\sum_{n=0}^{\infty} a_n(x-c)^n$, apply the ratio test to the series $\displaystyle\sum_{n=0}^{\infty} |a_n(x_c)^n|$. The result will be $L\,|x-c|$, where $L=\displaystyle\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}$.

1. If $L=0$, then the power series converges for every x by the ratio test, since $L\,|x-c|=0<1$.
2. If $L=\infty$, then power series converges only when $x=c$.
3. If $0<L<\infty$, then $R=1/L$ is the radius of convergence: by the ratio test, the series converges when $|x-c|<R$.

   To determine the interval of convergence, plug the endpoints ($x=c-R$ and $x=c+R$) into the power series, and test the resulting series for convergence. If the series converges, we include the endpoint. If it diverges, we exclude the endpoint.

Key Idea 9.6.7 allows us to find the radius of convergence R of a series by applying the Ratio Test (or any applicable test) to the absolute value of the terms of the series. We practice this in the following example.

Example 9.6.8 Determining the radius and interval of convergence.

Find the radius and interval of convergence for each of the following series:

1. $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$
2. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$
3. $\displaystyle\sum_{n=0}^{\infty}2^n(x-3)^n$
4. $\displaystyle\sum_{n=0}^{\infty}n!x^n$

Solution.

1. We apply the Ratio Test to the series $\displaystyle\sum_{n=1}^{\infty}\left|\frac{x^n}{n!}\right|$:

$$\begin{aligned}\lim_{n\to\infty}\frac{|x^{n+1}/(n+1)!|}{|x^n/n!|} &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\cdot\frac{n!}{(n+1)!}\right| \\ &= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| \\ &= 0 \text{ for all } x.\end{aligned}$$

The Ratio Test shows us that regardless of the choice of x, the series converges. Therefore the radius of convergence is $R = \infty$, and the interval of convergence is $(-\infty, \infty)$.

2. We apply the Ratio Test to the series $\displaystyle\sum_{n=1}^{\infty} \left|(-1)^{n+1}\frac{x^n}{n}\right| = \sum_{n=1}^{\infty} \left|\frac{x^n}{n}\right|$:

$$\begin{aligned}\lim_{n\to\infty} \frac{\left|x^{n+1}/(n+1)\right|}{|x^n/n|} &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\cdot\frac{n}{n+1}\right| \\ &= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)|x| \\ &= |x|\,.\end{aligned}$$

The Ratio Test states a series converges if the limit of $|a_{n+1}/a_n| = L < 1$. We found the limit above to be $|x|$; therefore, the power series converges when $|x| < 1$, or when x is in $(-1, 1)$. Thus the radius of convergence is $R = 1$. To determine the interval of convergence, we need to check the endpoints of $(-1, 1)$. When $x = -1$, we have the opposite of the Harmonic Series:

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^n}{n} &= \sum_{n=1}^{\infty}\frac{-1}{n} \\ &= -\infty.\end{aligned}$$

The series diverges when $x = -1$. When $x = 1$, we have the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1)^n}{n}$, which is the Alternating Harmonic Series, which converges. Therefore the interval of convergence is $(-1, 1]$.

3. We apply the Ratio Test to the series $\displaystyle\sum_{n=0}^{\infty}|2^n(x-3)^n|$:

$$\begin{aligned}\lim_{n\to\infty}\frac{\left|2^{n+1}(x-3)^{n+1}\right|}{|2^n(x-3)^n|} &= \lim_{n\to\infty}\left|\frac{2^{n+1}}{2^n}\cdot\frac{(x-3)^{n+1}}{(x-3)^n}\right| \\ &= \lim_{n\to\infty}|2(x-3)|\,.\end{aligned}$$

According to the Ratio Test, the series converges when $|2(x-3)| < 1 \implies |x-3| < 1/2$. The series is centered at 3, and x must be within $1/2$ of 3 in order for the series to converge. Therefore the radius of convergence is $R = 1/2$, and we know that the series converges absolutely for all x in $(3 - 1/2, 3 + 1/2) = (2.5, 3.5)$. We check for convergence at the endpoints to find the interval of convergence. When $x = 2.5$, we have:

$$\begin{aligned}\sum_{n=0}^{\infty}2^n(2.5-3)^n &= \sum_{n=0}^{\infty}2^n(-1/2)^n \\ &= \sum_{n=0}^{\infty}(-1)^n,\end{aligned}$$

which diverges. A similar process shows that the series also diverges at $x = 3.5$. Therefore the interval of convergence is $(2.5, 3.5)$.

4. We apply the Ratio Test to $\sum_{n=0}^{\infty} |n!x^n|$:

$$\lim_{n\to\infty} \frac{\left|(n+1)!x^{n+1}\right|}{|n!x^n|} = \lim_{n\to\infty} |(n+1)x|$$
$$= \infty \text{ for all } x, \text{ except } x = 0.$$

The Ratio Test shows that the series diverges for all x except $x = 0$. Therefore the radius of convergence is $R = 0$.

Video solution

youtu.be/watch?v=1YzPDWYUWO8

We can use a power series to define a function:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

where the domain of f is a subset of the interval of convergence of the power series. One can apply calculus techniques to such functions; in particular, we can find derivatives and antiderivatives.

Theorem 9.6.9 Derivatives and Indefinite Integrals of Power Series Functions.

Let $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ be a function defined by a power series, with radius of convergence R.

1. *$f(x)$ is continuous and differentiable on $(c - R, c + R)$.*
2. *$f'(x) = \sum_{n=1}^{\infty} a_n \cdot n \cdot (x-c)^{n-1}$, with radius of convergence R.*
3. *$\int f(x)\,dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$, with radius of convergence R.*

youtu.be/watch?v=qErrT8xRKts

Figure 9.6.10 Video presentation of Theorem 9.6.9

A few notes about Theorem 9.6.9:

1. The theorem states that differentiation and integration do not change the radius of convergence. It does not state anything about the *interval* of convergence. They are not always the same.
2. Notice how the summation for $f'(x)$ starts with $n = 1$. This is because the constant term a_0 of $f(x)$ becomes 0 through differentiation.
3. Differentiation and integration are simply calculated term-by-term using the Power Rules.

Example 9.6.11 Derivatives and indefinite integrals of power series.

Let $f(x) = \sum_{n=0}^{\infty} x^n$. Find $f'(x)$ and $F(x) = \int f(x)\,dx$, along with their respective intervals of convergence.

Solution. We find the derivative and indefinite integral of $f(x)$, following Theorem 9.6.9.

1.
$$f'(x) = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$
$$= \sum_{n=0}^{\infty} (n+1)x^n.$$

In Example 9.6.3, we recognized that $\sum_{n=0}^{\infty} x^n$ is a geometric series in x. We know that such a geometric series converges when $|x| < 1$; that is, the interval of convergence is $(-1, 1)$. To determine the interval of convergence of $f'(x)$, we consider the endpoints of $(-1, 1)$:

$$f'(-1) = 1 - 2 + 3 - 4 + \cdots, \text{ which diverges.}$$

$$f'(1) = 1 + 2 + 3 + 4 + \cdots, \text{ which diverges.}$$

Therefore, the interval of convergence of $f'(x)$ is $(-1, 1)$.

2. $F(x) = \int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

To find the interval of convergence of $F(x)$, we again consider the endpoints of $(-1, 1)$:

$$F(-1) = C - 1 + 1/2 - 1/3 + 1/4 + \cdots$$

The value of C is irrelevant; notice that the rest of the series is an Alternating Series that whose terms converge to 0. By the Alternating Series Test, this series converges. (In fact, we can recognize that the terms of the series after C are the opposite of the Alternating Harmonic Series. We can thus say that $F(-1) = C - \ln(2)$.)

$$F(1) = C + 1 + 1/2 + 1/3 + 1/4 + \cdots$$

Notice that this summation is $C +$ the Harmonic Series, which diverges. Since F converges for $x = -1$ and diverges for $x = 1$, the interval of convergence of $F(x)$ is $[-1, 1)$.

Video solution

youtu.be/watch?v=XE6m9CGME5Q

The previous example showed how to take the derivative and indefinite integral of a power series without motivation for why we care about such operations. We may care for the sheer mathematical enjoyment "that we can", which is motivation enough for many. However, we would be remiss to not recognize that we can learn a great deal from taking derivatives and indefinite integrals.

Recall that $f(x) = \sum_{n=0}^{\infty} x^n$ in Example 9.6.11 is a geometric series. According to Theorem 9.2.7, this series converges to $1/(1-x)$ when $|x| < 1$. Thus we can say

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \text{ on } (-1, 1).$$

Integrating the power series, (as done in Example 9.6.11,) we find

$$F(x) = C_1 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \tag{9.6.1}$$

while integrating the function $f(x) = 1/(1-x)$ gives

$$F(x) = -\ln|1-x| + C_2. \tag{9.6.2}$$

Equating Equations (9.6.1) and (9.6.2), we have

$$F(x) = C_1 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln|1-x| + C_2.$$

Letting $x = 0$, we have $F(0) = C_1 = C_2$. This implies that we can drop the constants and conclude

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln|1-x|.$$

We established in Example 9.6.11 that the series on the left converges at $x = -1$; substituting $x = -1$ on both sides of the above equality gives

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots = -\ln(2).$$

On the left we have the opposite of the Alternating Harmonic Series; on the right, we have $-\ln(2)$. We conclude that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln(2).$$

Important: We stated in Key Idea 9.2.20 (in Section 9.2) that the Alternating Harmonic Series converges to $\ln(2)$, and referred to this fact again in Example 9.5.6 of Section 9.5. However, we never gave an argument for why this was the case. The work above finally shows how we conclude that the Alternating Harmonic Series converges to $\ln(2)$.

We use this type of analysis in the next example.

Example 9.6.12 Analyzing power series functions.

Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Find $f'(x)$ and $\int f(x)\,dx$, and use these to analyze the behavior of $f(x)$.

Solution. We start by making two notes: first, in Example 9.6.8, we found the interval of convergence of this power series is $(-\infty, \infty)$. Second, we will find it useful later to have a few terms of the series written out:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \qquad (9.6.3)$$

We now find the derivative:

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} n\frac{x^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = 1 + x + \frac{x^2}{2!} + \cdots . \end{aligned}$$

Since the series starts at $n = 1$ and each term refers to $(n-1)$, we can re-index the series starting with $n = 0$:

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= f(x). \end{aligned}$$

We found the derivative of $f(x)$ is $f(x)$. The only functions for which this is true are of the form $y = ce^x$ for some constant c. As $f(0) = 1$ (see Equation (9.6.3)), c must be 1. Therefore we conclude that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

for all x.

We can also find $\displaystyle\int f(x)\,dx$:

$$\begin{aligned} \int f(x)\,dx &= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!(n+1)} \\ &= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \end{aligned}$$

We write out a few terms of this last series:

$$C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} = C + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

The integral of $f(x)$ differs from $f(x)$ only by a constant, again indicating that $f(x) = e^x$.

Video solution

youtu.be/watch?v=SQm1BC7bwEw

Example 9.6.12 and the work following Example 9.6.11 established relationships between a power series function and "regular" functions that we have dealt with in the past. In general, given a power series function, it is difficult (if not impossible) to express the function in terms of elementary functions. We chose examples where things worked out nicely.

In this section's last example, we show how to solve a simple differential equation with a power series.

Example 9.6.13 Solving a differential equation with a power series.

Give the first 4 terms of the power series solution to $y' = 2y$, where $y(0) = 1$.

Solution. The differential equation $y' = 2y$ describes a function $y = f(x)$ where the derivative of y is twice y and $y(0) = 1$. This is a rather simple differential equation; with a bit of thought one should realize that if $y = Ce^{2x}$, then $y' = 2Ce^{2x}$, and hence $y' = 2y$. By letting $C = 1$ we satisfy the initial condition of $y(0) = 1$.

Let's ignore the fact that we already know the solution and find a power series function that satisfies the equation. The solution we seek will have the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

for unknown coefficients a_n. We can find $f'(x)$ using Theorem 9.6.9:

$$f'(x) = \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \cdots .$$

Since $f'(x) = 2f(x)$, we have

$$\begin{aligned} a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \cdots &= 2\big(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\big) \\ &= 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + \cdots \end{aligned}$$

The coefficients of like powers of x must be equal, so we find that

$$a_1 = 2a_0, 2a_2 = 2a_1, 3a_3 = 2a_2, 4a_4 = 2a_3, \text{ etc.}$$

The initial condition $y(0) = f(0) = 1$ indicates that $a_0 = 1$; with this, we can find the values of the other coefficients:

$$\begin{aligned} a_0 = 1 \text{ and } a_1 = 2a_0 &\Rightarrow a_1 = 2; \\ a_1 = 2 \text{ and } 2a_2 = 2a_1 &\Rightarrow a_2 = 4/2 = 2; \\ a_2 = 2 \text{ and } 3a_3 = 2a_2 &\Rightarrow a_3 = 8/(2 \cdot 3) = 4/3; \\ a_3 = 4/3 \text{ and } 4a_4 = 2a_3 &\Rightarrow a_4 = 16/(2 \cdot 3 \cdot 4) = 2/3. \end{aligned}$$

Thus the first 5 terms of the power series solution to the differential equation $y' = 2y$ is

$$f(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots$$

In Section 9.7, as we study Taylor Series, we will learn how to recognize this series as describing $y = e^{2x}$.

Our last example illustrates that it can be difficult to recognize an elementary function by its power series expansion. It is far easier to start with a known function, expressed in terms of elementary functions, and represent it as a power series function. One may wonder why we would bother doing so, as the latter function probably seems more complicated. In the next two sections, we show both *how* to do this and *why* such a process can be beneficial.

9.6.1 Exercises

Terms and Concepts

1. We adopt the convention that $x^0 =$___________, regardless of the value of x.

2. What is the difference between the radius of convergence and the interval of convergence?

3. If the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is 5, what is the radius of convergence of $\sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$?

4. If the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is 5, what is the radius of convergence of $\sum_{n=0}^{\infty} (-1)^n a_n x^n$?

Problems

Exercise Group. In the following exercises, write out the sum of the first 5 terms of the given power series.

5. $\sum_{n=0}^{\infty} 2^n x^n$

6. $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$

7. $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$

8. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

Exercise Group. In the following exercises, a power series is given.

(a) Find the radius of convergence.

(b) Find the interval of convergence.

9. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} x^n$

10. $\sum_{n=0}^{\infty} n x^n$

11. $\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{n}$

12. $\sum_{n=0}^{\infty} \frac{(x+4)^n}{n!}$

13. $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

14. $\sum_{n=0}^{\infty} \frac{(-1)^n (x-5)^n}{10^n}$

15. $\sum_{n=0}^{\infty} 5^n (x-1)^n$

16. $\sum_{n=0}^{\infty} (-2)^n x^n$

17. $\sum_{n=0}^{\infty} \sqrt{n} x^n$

18. $\sum_{n=0}^{\infty} \frac{n}{3^n} x^n$

19. $\sum_{n=0}^{\infty} \frac{3^n}{n!} (x-5)^n$

20. $\sum_{n=0}^{\infty} (-1)^n n! (x-10)^n$

21. $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

22. $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n^3}$

23. $\sum_{n=0}^{\infty} n! \left(\frac{x}{10}\right)^n$

24. $\sum_{n=0}^{\infty} n^2 \left(\frac{x+4}{4}\right)^n$

Exercise Group. In the following exercises, a function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is given.

(a) Give a power series for $f'(x)$ and its interval of convergence.

(b) Give a power series for $\int f(x)\,dx$ and its interval of convergence.

25. $\displaystyle\sum_{n=0}^{\infty} nx^n$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

27. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$

28. $\displaystyle\sum_{n=0}^{\infty} (-3x)^n$

29. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

30. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

Exercise Group. In the following exercises, give the first 5 terms of the series that is a solution to the given differential equation.

31. $y' = 3y, y(0) = 1$

32. $y' = 5y, y(0) = 5$

33. $y' = y^2, y(0) = 1$

34. $y' = y + 1, y(0) = 1$

35. $y'' = -y, y(0) = 0, y'(0) = 1$

36. $y'' = 2y, y(0) = 1, y'(0) = 1$

9.7 Taylor Series

In Section 9.6, we showed how certain functions can be represented by a power series function. In Section 4.5, we showed how we can approximate functions with polynomials, given that enough derivative information is available. In this section we combine these concepts: if a function $f(x)$ is infinitely differentiable, we show how to represent it with a power series function.

youtu.be/watch?v=4RC4SLmEjro

Figure 9.7.1 Video introduction to Section 9.7

Definition 9.7.2 Taylor and Maclaurin Series.

Let $f(x)$ have derivatives of all orders at $x = c$.

1. The *Taylor Series of* $f(x)$, *centered at* c is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

2. Setting $c = 0$ gives the *Maclaurin Series of* $f(x)$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

If $p_n(x)$ is the nth degree Taylor polynomial for $f(x)$ centered at $x = c$, we saw how $f(x)$ is *approximately equal* to $p_n(x)$ near $x = c$. We also saw how increasing the degree of the polynomial generally reduced the error.

We are now considering *series*, where we sum an infinite set of terms. Our ultimate hope is to see the error vanish and claim a function is *equal* to its Taylor series.

When creating the Taylor polynomial of degree n for a function $f(x)$ at $x = c$, we needed to evaluate f, and the first n derivatives of f, at $x = c$. When creating the Taylor series of f, it helps to find a pattern that describes the nth derivative of f at $x = c$. We demonstrate this in the next two examples.

Example 9.7.3 The Maclaurin series of $f(x) = \cos(x)$.

Find the Maclaurin series of $f(x) = \cos(x)$.

Solution. In Example 4.5.19 we found the 8th degree Maclaurin polynomial of $\cos(x)$. In doing so, we created the table shown in Figure 9.7.4. Notice how $f^{(n)}(0) = 0$ when n is odd, $f^{(n)}(0) = 1$ when n is divisible by 4, and $f^{(n)}(0) = -1$ when n is even but not divisible by 4. Thus the Maclaurin series of $\cos(x)$ is

$$1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

We can go further and write this as a summation. The coefficients alternate between positive and negative. Since we only need the terms where the power of x is even, we write the power series in terms of x^{2n}:

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}.$$

This Maclaurin series is a special type of power series. As such, we should determine its interval of convergence. Applying the Ratio Test,

$f(x) = \cos(x)$	$f(0) = 1$
$f'(x) = -\sin(x)$	$f'(0) = 0$
$f''(x) = -\cos(x)$	$f''(0) = -1$
$f'''(x) = \sin(x)$	$f'''(0) = 0$
$f^{(4)}(x) = \cos(x)$	$f^{(4)}(0) = 1$
$f^{(5)}(x) = -\sin(x)$	$f^{(5)}(0) = 0$
$f^{(6)}(x) = -\cos(x)$	$f^{(6)}(0) = -1$
$f^{(7)}(x) = \sin(x)$	$f^{(7)}(0) = 0$
$f^{(8)}(x) = \cos(x)$	$f^{(8)}(0) = 1$
$f^{(9)}(x) = -\sin(x)$	$f^{(9)}(0) = 0$

Figure 9.7.4 Derivatives of $f(x) = \cos(x)$ evaluated at $x = 0$

Video solution

youtu.be/watch?v=97x6p1616z0

we have

$$\lim_{n\to\infty} \frac{\left|(-1)^{n+1}\dfrac{x^{2(n+1)}}{(2(n+1))!}\right|}{\left|(-1)^n \frac{x^{2n}}{(2n)!}\right|} = \lim_{n\to\infty} \left|\frac{x^{2n+2}}{x^{2n}}\right| \frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} \frac{|x|^2}{(2n+2)(2n+1)}.$$

For any fixed x, this limit is 0. Therefore this power series has an infinite radius of convergence, converging for all x. It is important to note what we have, and have not, determined: we have determined the Maclaurin series for $\cos(x)$ along with its interval of convergence. We *have not* shown that $\cos(x)$ is *equal* to this power series.

Example 9.7.3 found the Taylor Series representation of $\cos(x)$. We can easily find the Taylor Series representation of $\sin(x)$ by recognizing that $\int \cos(x)\,dx = \sin(x)$ and apply Theorem 9.6.9.

Example 9.7.5 The Taylor series of $f(x) = \ln(x)$ at $x = 1$.

Find the Taylor series of $f(x) = \ln(x)$ centered at $x = 1$.

Solution. Figure 9.7.6 shows the nth derivative of $\ln(x)$ evaluated at $x = 1$ for $n = 0, \ldots, 5$, along with an expression for the nth term:

$$f^{(n)}(1) = (-1)^{n+1}(n-1)! \text{ for } n \geq 1.$$

Remember that this is what distinguishes Taylor series from Taylor polynomials; we are very interested in finding a pattern for the nth term, not just finding a finite set of coefficients for a polynomial.
Since $f(1) = \ln(1) = 0$, we skip the first term and start the summation with $n = 1$, giving the Taylor series for $\ln(x)$, centered at $x = 1$, as

$$\sum_{n=1}^{\infty}(-1)^{n+1}(n-1)!\frac{1}{n!}(x-1)^n = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}.$$

$f(x) = \ln(x)$	$f(1) = 0$
$f'(x) = 1/x$	$f'(1) = 1$
$f''(x) = -1/x^2$	$f''(1) = -1$
$f'''(x) = 2/x^3$	$f'''(1) = 2$
$f^{(4)}(x) = -6/x^4$	$f^{(4)}(1) = -6$
$f^{(5)}(x) = 24/x^5$	$f^{(5)}(1) = 24$
$\vdots$	$\vdots$
$f^{(n)}(x) = \dfrac{(-1)^{n+1}(n-1)!}{x^n}$	$f^{(n)}(1) = (-1)^{n+1}(n-1)!$

Figure 9.7.6 Derivatives of $\ln(x)$ evaluated at $x = 1$

We now determine the interval of convergence, using the Ratio Test.

$$\lim_{n\to\infty} \frac{\left|(-1)^{n+2}\dfrac{(x-1)^{n+1}}{n+1}\right|}{\left|(-1)^{n+1}\dfrac{(x-1)^n}{n}\right|} = \lim_{n\to\infty} \left|\frac{(x-1)^{n+1}}{(x-1)^n}\right| \frac{n}{n+1}$$

$$= |x-1|.$$

By the Ratio Test, we have convergence when $|x-1| < 1$: the radius of convergence is 1, and we have convergence on $(0, 2)$. We now check the endpoints.
At $x = 0$, the series is

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^n}{n} = -\sum_{n=1}^{\infty}\frac{1}{n},$$

which diverges (it is the Harmonic Series times (-1).)

At $x=2$, the series is

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1)^n}{n}=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n},$$

the Alternating Harmonic Series, which converges.

We have found the Taylor series of $\ln x$ centered at $x=1$, and have determined the series converges on $(0,2]$. We cannot (yet) say that $\ln x$ is equal to this Taylor series on $(0,2]$.

Video solution

youtu.be/watch?v=Bdk4lGCkz7o

It is important to note that Definition 9.7.2 defines a Taylor series given a function $f(x)$, but makes no claim about their equality. We will find that "most of the time" they are equal, but we need to consider the conditions that allow us to conclude this.

It can be shown that $\ln(x)$ is equal to this Taylor series on $(0,2]$. From the work in Example 9.7.5, this justifies our previous declaration that the Alternating Harmonic Series converges to $\ln(2)$.

Theorem 4.5.16 states that the error between a function $f(x)$ and its nth-degree Taylor polynomial $p_n(x)$ is $R_n(x)$, where

$$|R_n(x)|\leq\frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|(x-c)^{(n+1)}\right|.$$

If $R_n(x)$ goes to 0 for each x in an interval I as n approaches infinity, we conclude that the function is equal to its Taylor series expansion.

Theorem 9.7.7 Function and Taylor Series Equality.

Let $f(x)$ have derivatives of all orders at $x=c$, let $R_n(x)$ be as stated in Theorem 4.5.16, and let I be an interval on which the Taylor series of $f(x)$ converges. If $\lim_{n\to\infty} R_n(x)=0$ for all x in I, then

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n \text{ on } I.$$

We demonstrate the use of this theorem in an example.

youtu.be/watch?v=FF3m792UGiA

Figure 9.7.8 Video presentation of Theorem 9.7.7

Example 9.7.9 Establishing equality of a function and its Taylor series.

Show that $f(x)=\cos(x)$ is equal to its Maclaurin series, as found in Example 9.7.3, for all x.

Solution. Given a value x, the magnitude of the error term $R_n(x)$ is bounded by

$$|R_n(x)|\leq\frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|x^{n+1}\right|.$$

Since all derivatives of $\cos(x)$ are $\pm\sin(x)$ or $\pm\cos(x)$, whose magnitudes are bounded by 1, we can state

$$|R_n(x)|\leq\frac{1}{(n+1)!}\left|x^{n+1}\right|$$

which implies

$$-\frac{\left|x^{n+1}\right|}{(n+1)!}\leq R_n(x)\leq\frac{\left|x^{n+1}\right|}{(n+1)!}. \tag{9.7.1}$$

For any x, $\lim_{n\to\infty}\frac{x^{n+1}}{(n+1)!}=0$. Applying the Squeeze Theorem to Equation (9.7.1), we conclude that $\lim_{n\to\infty} R_n(x) = 0$ for all x, and hence

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \text{ for all } x\,.$$

It is natural to assume that a function is equal to its Taylor series on the series' interval of convergence, but this is not always the case. In order to properly establish equality, one must use Theorem 9.7.7. This is a bit disappointing, as we developed beautiful techniques for determining the interval of convergence of a power series, and proving that $R_n(x) \to 0$ can be difficult. For instance, it is not a simple task to show that ln x equals its Taylor series on $(0,2]$ as found in Example 9.7.5; in the Exercises, the reader is only asked to show equality on $(1,2)$, which is simpler.

There is good news. A function $f(x)$ that is equal to its Taylor series, centered at any point the domain of $f(x)$, is said to be an *analytic function*, and most, if not all, functions that we encounter within this course are analytic functions. Generally speaking, any function that one creates with elementary functions (polynomials, exponentials, trigonometric functions, etc.) that is not piecewise defined is probably analytic. For most functions, we assume the function is equal to its Taylor series on the series' interval of convergence and only use Theorem 9.7.7 when we suspect something may not work as expected.

We develop the Taylor series for one more important function, then give a table of the Taylor series for a number of common functions.

Example 9.7.10 The Binomial Series.

Find the Maclaurin series of $f(x) = (1+x)^k$, $k \neq 0$.

Solution. When k is a positive integer, the Maclaurin series is finite. For instance, when $k = 4$, we have

$$f(x) = (1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4.$$

The coefficients of x when k is a positive integer are known as the *binomial coefficients*, giving the series we are developing its name.

When $k = 1/2$, we have $f(x) = \sqrt{1+x}$. Knowing a series representation of this function would give a useful way of approximating $\sqrt{1.3}$, for instance.

To develop the Maclaurin series for $f(x) = (1+x)^k$ for any value of $k \neq 0$, we consider the derivatives of f evaluated at $x = 0$:

$$\begin{aligned} f(x) &= (1+x)^k & f(0) &= 1 \\ f'(x) &= k(1+x)^{k-1} & f'(0) &= k \\ f''(x) &= k(k-1)(1+x)^{k-2} & f''(0) &= k(k-1) \\ f'''(x) &= k(k-1)(k-2)(1+x)^{k-3} & f'''(0) &= k(k-1)(k-2) \\ &\vdots & &\vdots \end{aligned}$$

For a general n,

$$f^{(n)}(x) = k(k-1)\cdots\big(k-(n-1)\big)(1+x)^{k-n},$$

giving $f^{(n)}(0) = k(k-1)\cdots\big(k-(n-1)\big)$.

Thus the Maclaurin series for $f(x) = (1+x)^k$ is

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$$
$$\dots + \frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!}(x-c)^n + \dots$$

It is important to determine the interval of convergence of this series. With

$$a_n = \frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!}x^n,$$

we apply the Ratio Test:

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{\left|\frac{k(k-1)\cdots(k-(n-1))(k-n)}{(n+1)!}x^{n+1}\right|}{\left|\frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!}x^n\right|}$$
$$= \lim_{n\to\infty}\left|\frac{k-n}{n+1}x\right|$$
$$= |x|\,.$$

The series converges absolutely when the limit of the Ratio Test is less than 1; therefore, we have absolute convergence when $|x| < 1$.
While outside the scope of this text, the interval of convergence depends on the value of k. When $k > 0$, the interval of convergence is $[-1,1]$. When $-1 < k < 0$, the interval of convergence is $[-1,1)$. If $k \leq -1$, the interval of convergence is $(-1,1)$.

Video solution

youtu.be/watch?v=uQouiDtMuDY

We learned that Taylor polynomials offer a way of approximating a "difficult to compute" function with a polynomial. Taylor series offer a way of exactly representing a function with a series. One probably can see the use of a good approximation; is there any use of representing a function exactly as a series?

While we should not overlook the mathematical beauty of Taylor series (which is reason enough to study them), there are practical uses as well. They provide a valuable tool for solving a variety of problems, including problems relating to integration and differential equations.

In Key Idea 9.7.11 (on the following page) we give a table of the Taylor series of a number of common functions. We then give a theorem about the "algebra of power series," that is, how we can combine power series to create power series of new functions. This allows us to find the Taylor series of functions like $f(x) = e^x\cos(x)$ by knowing the Taylor series of e^x and $\cos(x)$.

Before we investigate combining functions, consider the Taylor series for the arctangent function (see Key Idea 9.7.11). Knowing that $\tan^{-1}(1) = \pi/4$, we can use this series to approximate the value of π:

$$\frac{\pi}{4} = \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$
$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)$$

Unfortunately, this particular expansion of π converges very slowly. The first 100 terms approximate π as 3.13159, which is not particularly good.

Key Idea 9.7.11 Important Taylor Series Expansions.

Function and Series	First Few Terms	Interval of Convergence
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$(-\infty, \infty)$
$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$(-\infty, \infty)$
$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$(-\infty, \infty)$
$\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$	$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$	$(0, 2]$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + \cdots$	$(-1, 1)$
$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$[-1, 1]$
$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$	$1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots$	$(-1, 1)$

Note that for $(1+x)^k$, the interval of convergence may contain one or both endpoints, depending on the value of k, and we are using the generalized binomial coefficients

$$\binom{k}{n} = \frac{k(k-1)\cdots(k-(n-1))}{n!}.$$

Theorem 9.7.12 Algebra of Power Series.

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $g(x) = \sum_{n=0}^{\infty} b_n x^n$ *converge absolutely for* $|x| < R$, *and let* $h(x)$ *be a polynomial function.*

1. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$ for $|x| < R$.
2. $f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} \big(a_0 b_n + a_1 b_{n-1} + \ldots a_n b_0\big) x^n$ for $|x| < R$.
3. $f\big(h(x)\big) = \sum_{n=0}^{\infty} a_n \big(h(x)\big)^n$ for $|h(x)| < R$.

Note that we require $h(x)$ to be a polynomial function in Theorem 9.7.12. If we plug a function that is not polynomial into a power series, the result will no longer be a power series. If one is *very* careful about the centre and radius of convergence, it is technically possible to substitute the Taylor series for a general function $h(x)$ into the Taylor series for $f(x)$, and the result will be the Taylor series for $f(h(x))$.

In practice, $h(x)$ is typically a monomial function of the form $h(x) = ax^n$. For anything more complicated, rearranging the power series into a standard form becomes a nightmare.

Example 9.7.13 Combining Taylor series.

Write out the first 3 terms of the Taylor Series for $f(x) = e^x \cos(x)$ using Key Idea 9.7.11 and Theorem 9.7.12.

Solution. Key Idea 9.7.11 informs us that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \text{ and } \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots.$$

Applying Theorem 9.7.12, we find that

$$e^x \cos(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right).$$

Distribute the right hand expression across the left:

$$\begin{aligned} e^x \cos(x) &= 1\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \\ &\quad + \frac{x^2}{2!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{x^3}{3!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \\ &\quad + \frac{x^4}{4!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \cdots \end{aligned}$$

If we distribute again and collect like terms, we find

$$e^x \cos(x) = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \frac{x^7}{630} + \cdots.$$

While this process is a bit tedious, it is much faster than evaluating all the necessary derivatives of $e^x \cos(x)$ and computing the Taylor series directly.

Because the series for e^x and $\cos(x)$ both converge on $(-\infty, \infty)$, so does the series expansion for $e^x \cos(x)$.

youtu.be/watch?v=v-y5IH796gY

Figure 9.7.14 Deriving the Taylor series for $\arctan(x)$ in Key Idea 9.7.11

Video solution

youtu.be/watch?v=rUBF6BC201g

Example 9.7.15 Creating new Taylor series.

Use Theorem 9.7.12 to create series for $y = \sin(x^2)$ and $y = x^3/(3 + x^4)$.

Solution. Given that

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

we simply substitute x^2 for x in the series, giving

$$\begin{aligned} \sin(x^2) &= \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} \\ &\sum_{n=0}^{\infty} (-1)^n \frac{(x^{4n+2}}{(2n+1)!} \\ &= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} \cdots. \end{aligned}$$

Since the Taylor series for $\sin(x)$ has an infinite radius of convergence, so does the Taylor series for $\sin(x^2)$.

For $y = x^3/(3 + x^4)$, we begin with the geometric series expansion

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Note that we can write

$$\frac{1}{3+x^4} = \frac{1}{3} \cdot \frac{1}{1+x^4/3} = \frac{1}{3} \cdot \frac{1}{1-(-x^4/3)}.$$

Substituting $-x^4/3$ into the geometric series expansion, we get

$$\frac{1}{3+x^4} = \sum_{n=0}^{\infty}(-x^4/3)^n = \sum_{n=0}^{\infty} \frac{(-1^n)x^{4n}}{3^n}.$$

Finally, we can multiply both sides of the above equation by x^3 to obtain

$$\frac{x^3}{3+x^4} = x^3 \sum_{n=0}^{\infty} \frac{(-1^n)x^{4n}}{3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{3^n}.$$

Video solution

youtu.be/watch?v=j3eHOO9taNQ

Example 9.7.16 A (somewhat foolish) combination of Taylor series.

Discuss possible methods for obtaining a Taylor series expansion for $f(x) = \ln(\sqrt{x})$.

Solution. Since $f(x)$ is a composition, our first instict might be to apply Theorem 9.7.12 to the problem. However, $\sqrt{x}$ is not a polynomial function, and neither $\ln(x)$ nor $\sqrt{x}$ have Maclaurin series expansions.
You might already see a simple way to proceed, but let us first consider the following: $\sqrt{x} = (1+(x-1))^{1/2}$ can be expanded as a binomial series centered at $x = 1$. We also know the Taylor series for $\ln(x)$ at $x = 1$, and note that $\sqrt{1} = 1$, so when x is near 1, so is $\sqrt{x}$.
What happens if we take the Taylor series

$$\ln(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$$

and substitute in

$$\sqrt{x} = \sum_{n=0}^{\infty}\binom{1/2}{n}(x-1)^n = 1 + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \cdots,$$

where $\binom{1/2}{n} = \frac{1/2(1/2-1)\cdots(1/2-(n-1))}{n!}$ denotes the binomial coefficient?
Short answer: a mess. We have to replace each occurence of $x-1$ in the power series for $\ln(x)$ with $\sqrt{x}-1 = \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{16}(x-1)^3 + \cdots$, and then expand, and collect terms. If we do this, keeping only terms up to $(x-1)^3$, we find:

$$\begin{aligned}
\ln(\sqrt{x}) &= \left(\frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{16}(x-1)^3 + \cdots\right) \\
&\quad - \frac{1}{2}\left(\frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{16}(x-1)^3 + \cdots\right)^2 \\
&\quad + \frac{1}{3}\left(\frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{16}(x-1)^3 + \cdots\right) + \cdots \\
&= \frac{1}{2}(x-1) - \frac{1}{4}(x-2)^2 + \frac{1}{6}(x-1)^3 - \cdots.
\end{aligned}$$

But of course, there was a better way all along:

$$\ln(\sqrt{x}) = \ln(x^{1/2}) = \frac{1}{2}\ln(x)$$

using properties of the logarithm, and indeed, the result above is the same as the one we would have obtained by simply multiplying the Taylor series for $\ln(x)$ by $\frac{1}{2}$. Power series manipulation is a powerful technique, but one should not apply it blindly.

Example 9.7.17 Using Taylor series to approximate a composition.

Use Taylor series to determine a degree 5 Taylor polynomial approximation to $f(x) = e^{\sin(x)}$.

Solution. Here we want to apply Theorem 9.7.12, but $h(x) = \sin(x)$ is not a polynomial. However, we are interested in approximation, so we replace $\sin(x)$ by the Maclaurin polynomial

$$q(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

The Maclaurin series for $f(x) = e^x$ is given by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Next, we substitute $q(x)$ into the series for e^x. The algebra gets *very* messy, but we can simplify things: since we want the degree 5 approximation, there is no need to write down terms involving x^6 or higher powers.

$$\begin{aligned}
e^{\sin(x)} &\approx 1 + q(x) + \frac{1}{2!}q(x)^2 + \frac{1}{3!}q(x)^3 + \frac{1}{4!}q(x)^4 + \frac{1}{5!}q(x)^5\\
&= 1 + \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right) + \frac{1}{2}\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\\
&\quad + \frac{1}{6}\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)^3 + \frac{1}{24}\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)^4\\
&\quad + \frac{1}{120}\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)^5\\
&= 1 + x - \frac{x^3}{6} + \frac{x^5}{120} + \frac{1}{2}\left(x^2 - \frac{1}{3}x^4 + \cdots\right) + \frac{1}{6}\left(x^3 - \frac{1}{2}x^5 + \cdots\right)\\
&\quad + \frac{1}{24}\left(x^4 + \cdots\right) + \frac{1}{120}\left(x^5 + \cdots\right)\\
&= 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{15}x^5 + \cdots.
\end{aligned}$$

While the algebra is a bit of a mess, it is often less work than computing the Taylor polynomial directly, as the derivatives of a composite function quickly get complicated. The function $f(x) = e^{\sin(x)}$ and its approximation are plotted in Figure 9.7.18 below. Note that our polynomial approximation is very good on $[-1, 1]$.

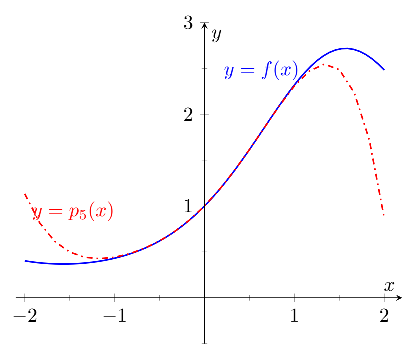

Figure 9.7.18 A graph of $f(x)$ and its degree 5 Maclaurin polynomial

In the previous example, the reader might be left wondering why we would bother with all that algebra, when the computer could have given us the result in seconds. One reason is simply that it lets us see how these different pieces fit together. Computing a Taylor polynomial by combining existing results will give the same polynomial as computing derivatives. Also we see that we can compute an approximation by replacing both parts of a composition with approximations. In the last couple of examples in this chapter, we see another reason: often we have to define functions in terms of power series derived through integration, or the solution of a differential equation, where there is no known function we can simply plug into the computer.

Example 9.7.19 Using Taylor series to evaluate definite integrals.

Use the Taylor series of e^{-x^2} to evaluate $\displaystyle\int_0^1 e^{-x^2}\,dx$.

Solution. We learned, when studying Numerical Integration, that e^{-x^2} does not have an antiderivative expressible in terms of elementary functions. This means any definite integral of this function must have its value approximated, and not computed exactly.

We can quickly write out the Taylor series for e^{-x^2} using the Taylor series of e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and so

$$\begin{aligned} e^{-x^2} &= \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots. \end{aligned}$$

We use Theorem 9.6.9 to integrate:

$$\int e^{-x^2}\,dx = C + x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots$$

This *is* the antiderivative of e^{-x^2}; while we can write it out as a series, we cannot write it out in terms of elementary functions. We can evaluate the definite integral $\displaystyle\int_0^1 e^{-x^2}\,dx$ using this antiderivative; substituting 1 and 0 for x and subtracting gives

$$\int_0^1 e^{-x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} \cdots.$$

Summing the 5 terms shown above give the approximation of 0.74749. Since this is an alternating series, we can use the Alternating Series Approximation Theorem, (Theorem 9.5.7), to determine how accurate this approximation is. The next term of the series is $1/(11\cdot 5!) \approx 0.00075758$. Thus we know our approximation is within 0.00075758 of the actual value of the integral. This is arguably much less work than using Simpson's Rule to approximate the value of the integral.

Video solution

youtu.be/watch?v=WhpEci26gUA

Example 9.7.20 Using Taylor series to solve differential equations.

Solve the differential equation $y' = 2y$ in terms of a power series, and use the theory of Taylor series to recognize the solution in terms of an elementary function.

Solution. We found the first 5 terms of the power series solution to this differential equation in Example 9.6.13 in Section 9.6. These are:

$$a_0 = 1, a_1 = 2, a_2 = \frac{4}{2} = 2, a_3 = \frac{8}{2\cdot 3} = \frac{4}{3}, a_4 = \frac{16}{2\cdot 3\cdot 4} = \frac{2}{3}.$$

We include the "unsimplified" expressions for the coefficients found in Example 9.6.13 as we are looking for a pattern. It can be shown that $a_n = 2^n/n!$. Thus the solution, written as a power series, is

$$y = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}.$$

Using Key Idea 9.7.11 and Theorem 9.7.12, we recognize $f(x) = e^{2x}$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \Rightarrow \qquad e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}.$$

Video solution

youtu.be/watch?v=ojjEGO5H8qQ

Finding a pattern in the coefficients that match the series expansion of a known function, such as those shown in Key Idea 9.7.11, can be difficult. What if the coefficients in the previous example were given in their reduced form; how could we still recover the function $y = e^{2x}$?

Suppose that all we know is that

$$a_0 = 1, a_1 = 2, a_2 = 2, a_3 = \frac{4}{3}, a_4 = \frac{2}{3}.$$

Definition 9.7.2 states that each term of the Taylor expansion of a function includes an $n!$. This allows us to say that

$$a_2 = 2 = \frac{b_2}{2!}, a_3 = \frac{4}{3} = \frac{b_3}{3!}, \text{ and } a_4 = \frac{2}{3} = \frac{b_4}{4!}$$

for some values b_2, b_3 and b_4. Solving for these values, we see that $b_2 = 4$, $b_3 = 8$ and $b_4 = 16$. That is, we are recovering the pattern we had previously seen, allowing us to write

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n \\ &= 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4 + \cdots \end{aligned}$$

From here it is easier to recognize that the series is describing an exponential function.

There are simpler, more direct ways of solving the differential equation $y' = 2y$, as discussed in Chapter 8. We applied power series techniques to this equation to demonstrate its utility, and went on to show how *sometimes* we are able to recover the solution in terms of elementary functions using the theory of Taylor series. Most differential equations faced in real scientific and engineering situations are much more complicated than this one, but power series can offer a valuable tool in finding, or at least approximating, the solution.

This chapter introduced sequences, which are ordered lists of numbers, followed by series, wherein we add up the terms of a sequence. We quickly saw that such sums do not always add up to "infinity," but rather converge. We studied tests for convergence, then ended the chapter with a formal way of defining functions based on series. Such "series-defined functions" are a valuable tool in solving a number of different problems throughout science and engineering.

Coming in the next chapters are new ways of defining curves in the plane apart from using functions of the form $y = f(x)$. Curves created by these new methods can be beautiful, useful, and important.

9.7.1 Exercises

Terms and Concepts

1. What is the difference between a Taylor polynomial and a Taylor series?

2. What theorem must we use to show that a function is equal to its Taylor series?

Problems

Exercise Group. Key Idea 9.7.11 gives the nth term of the Taylor series of common functions. In the following exercises, verify the formula given in the Key Idea by finding the first few terms of the Taylor series of the given function and identifying a pattern.

3. $f(x) = e^x$;$c = 0$

4. $f(x) = \sin(x)$;$c = 0$

5. $f(x) = 1/(1-x)$;$c = 0$

6. $f(x) = \tan^{-1}(x)$;$c = 0$

Exercise Group. In the following exercises, find a formula for the nth term of the Taylor series of $f(x)$, centered at c, by finding the coefficients of the first few powers of x and looking for a pattern. (The formulas for several of these are found in Key Idea 9.7.11; show work verifying these formula.)

7. $f(x) = \cos(x)$;$c = \pi/2$

8. $f(x) = 1/x$;$c = 1$

9. $f(x) = e^{-x}$;$c = 0$

10. $f(x) = \ln(1+x)$;$c = 0$

11. $f(x) = x/(x+1)$;$c = 1$

12. $f(x) = \sin(x)$;$c = \pi/4$

Exercise Group. In the following exercises, show that the Taylor series for $f(x)$, as given in Key Idea 9.7.11, is equal to $f(x)$ by applying Theorem 9.7.7; that is, show $\lim\limits_{n\to\infty} R_n(x) = 0$.

13. $f(x) = e^x$

14. $f(x) = \sin(x)$

15. $f(x) = \ln(x)$ (show equality only on $(1, 2)$)

16. $f(x) = 1/(1-x)$ (show equality only on $(-1, 0)$)

Exercise Group. In the following exercises, use the Taylor series given in Key Idea 9.7.11 to verify the given identity.

17. $\cos(-x) = \cos(x)$

18. $\sin(-x) = -\sin(x)$

19. $\frac{d}{dx}\big(\sin(x)\big) = \cos(x)$

20. $\frac{d}{dx}\big(\cos(x)\big) = -\sin(x)$

Exercise Group. In the following exercises, write out the first 5 terms of the Binomial series with the given k-value.

21. $k = 1/2$

22. $k = -1/2$

23. $k = 1/3$

24. $k = 4$

Exercise Group. In the following exercises, use the Taylor series given in Key Idea 9.7.11 to create the Taylor series of the given functions.

25. $f(x) = \cos\left(x^2\right)$

26. $f(x) = e^{-x}$

27. $f(x) = \sin\left(2x+3\right)$

28. $f(x) = \tan^{-1}\left(x/2\right)$

29. $f(x) = e^x\sin(x)$(only find the first 4 terms)

30. $f(x) = (1+x)^{1/2}\cos(x)$(only find the first 4 terms)

Exercise Group. In the following exercises, approximate the value of the given definite integral by using the first 4 nonzero terms of the integrand's Taylor series.

31. $\displaystyle\int_0^{\sqrt{\pi}} \sin\left(x^2\right)\,dx$

32. $\displaystyle\int_0^{\sqrt[3]{\pi}} \cos\left(x^3\right)\,dx$

Chapter 10

Curves in the Plane

We have explored functions of the form $y = f(x)$ closely throughout this text. We have explored their limits, their derivatives and their antiderivatives; we have learned to identify key features of their graphs, such as relative maxima and minima, inflection points and asymptotes; we have found equations of their tangent lines, the areas between portions of their graphs and the x-axis, and the volumes of solids generated by revolving portions of their graphs about a horizontal or vertical axis.

Despite all this, the graphs created by functions of the form $y = f(x)$ are limited. Since each x-value can correspond to only 1 y-value, common shapes like circles cannot be fully described by a function in this form. Fittingly, the "vertical line test" excludes vertical lines from being functions of x, even though these lines are important in mathematics.

In this chapter we'll explore new ways of drawing curves in the plane. We'll still work within the framework of functions, as an input will still only correspond to one output. However, our new techniques of drawing curves will render the vertical line test pointless, and allow us to create important — and beautiful — new curves. Once these curves are defined, we'll apply the concepts of calculus to them, continuing to find equations of tangent lines and the areas of enclosed regions.

10.1 Conic Sections

The ancient Greeks recognized that interesting shapes can be formed by intersecting a plane with a *double napped* cone (i.e., two identical cones placed tip-to-tip as shown in the following figures). As these shapes are formed as sections of conics, they have earned the official name "conic sections."

The three "most interesting" conic sections are given in the top row of Figure 10.1.2. They are the parabola, the ellipse (which includes circles) and the hyperbola. In each of these cases, the plane does not intersect the tips of the cones (usually taken to be the origin).

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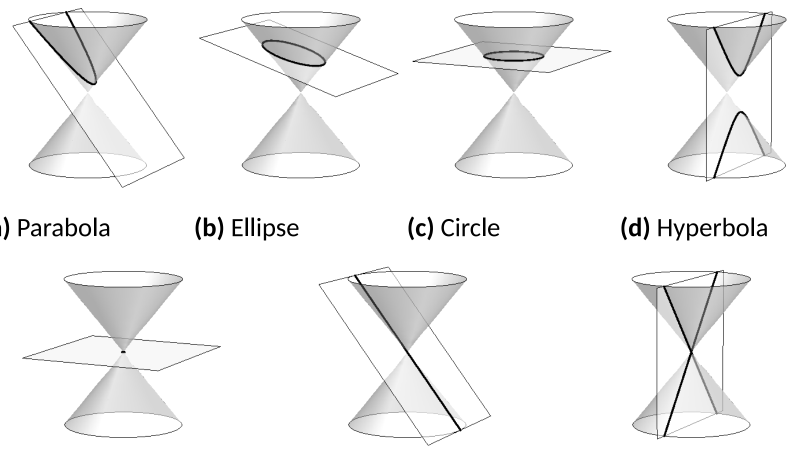
Figure 10.1.1 Video introduction to Section 10.1



(a) Parabola **(b)** Ellipse **(c)** Circle **(d)** Hyperbola

(e) Point **(f)** Line **(g)** Crossed Lines

Figure 10.1.2 Conic Sections

When the plane does contain the origin, three *degenerate* cones can be formed as shown the bottom row of Figure 10.1.2: a point, a line, and crossed lines. We focus here on the nondegenerate cases.

While the above geometric constructs define the conics in an intuitive, visual way, these constructs are not very helpful when trying to analyze the shapes algebraically or consider them as the graph of a function. It can be shown that all conics can be defined by the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

While this algebraic definition has its uses, most find another geometric perspective of the conics more beneficial.

Each nondegenerate conic can be defined as the *locus*, or set, of points that satisfy a certain distance property. These distance properties can be used to generate an algebraic formula, allowing us to study each conic as the graph of a function.

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Figure 10.1.3 Video introduction to the parabola

10.1.1 Parabolas

Definition 10.1.4 Parabola.

A **parabola** is the locus of all points equidistant from a point (called a **focus**) and a line (called the **directrix**) that does not contain the focus.

Figure 10.1.5 illustrates this definition. The point halfway between the focus and the directrix is the *vertex*. The line through the focus, perpendicular to the directrix, is the *axis of symmetry*, as the portion of the parabola on one side of this line is the mirror-image of the portion on the opposite side.

The definition leads us to an algebraic formula for the parabola. Let $P = (x, y)$ be a point on a parabola whose focus is at $F = (0, p)$ and whose directrix is at $y = -p$. (We'll assume for now that the focus lies on the y-axis; by placing the focus p units above the x-axis and the directrix p units below this axis, the vertex will be at $(0, 0)$.)

We use the Distance Formula to find the distance d_1 between F and P:

$$d_1 = \sqrt{(x-0)^2 + (y-p)^2}.$$

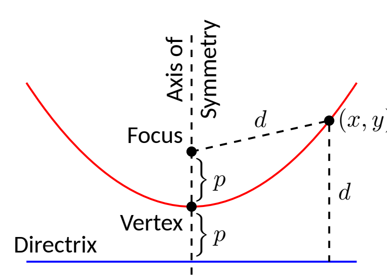

Figure 10.1.5 Illustrating the definition of the parabola and establishing an algebraic formula

The distance d_2 from P to the directrix is more straightforward:

$$d_2 = y - (-p) = y + p.$$

These two distances are equal. Setting $d_1 = d_2$, we can solve for y in terms of x:

$$\begin{aligned} d_1 &= d_2 \\ \sqrt{x^2 + (y-p)^2} &= y + p \end{aligned}$$

Now square both sides.

$$\begin{aligned} x^2 + (y-p)^2 &= (y+p)^2 \\ x^2 + y^2 - 2yp + p^2 &= y^2 + 2yp + p^2 \\ x^2 &= 4yp \\ y &= \frac{1}{4p}x^2. \end{aligned}$$

The geometric definition of the parabola has led us to the familiar quadratic function whose graph is a parabola with vertex at the origin. When we allow the vertex to not be at $(0, 0)$, we get the following standard form of the parabola.

Key Idea 10.1.6 General Equation of a Parabola.

1. *Vertical Axis of Symmetry:* The equation of the parabola with vertex at (h, k) and directrix $y = k - p$ in standard form is

$$y = \frac{1}{4p}(x-h)^2 + k.$$

The focus is at $(h, k + p)$.

2. *Horizontal Axis of Symmetry:* The equation of the parabola with vertex at (h, k) and directrix $x = h - p$ in standard form is

$$x = \frac{1}{4p}(y-k)^2 + h.$$

The focus is at $(h + p, k)$.

Note: p is not necessarily a positive number.

Example 10.1.7 Finding the equation of a parabola.

Give the equation of the parabola with focus at $(1, 2)$ and directrix at $y = 3$.

Solution. The vertex is located halfway between the focus and directrix, so $(h, k) = (1, 2.5)$. This gives $p = -0.5$. Using Key Idea 10.1.6 we have the equation of the parabola as

$$y = \frac{1}{4(-0.5)}(x-1)^2 + 2.5 = -\frac{1}{2}(x-1)^2 + 2.5.$$

The parabola is sketched in Figure 10.1.8.

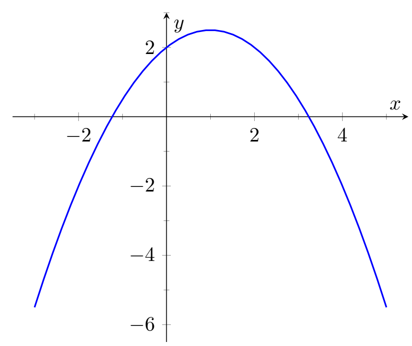

Figure 10.1.8 The parabola described in Example 10.1.7

Video solution

youtu.be/watch?v=dk8lrQac8Qg

Example 10.1.9 Finding the focus and directrix of a parabola.

Find the focus and directrix of the parabola $x = \frac{1}{8}y^2 - y + 1$. The point $(7, 12)$ lies on the graph of this parabola; verify that it is equidistant from the focus and directrix.

Solution. We need to put the equation of the parabola in its general form. This requires us to complete the square:

$$\begin{aligned} x &= \frac{1}{8}y^2 - y + 1 \\ &= \frac{1}{8}\left(y^2 - 8y + 8\right) \\ &= \frac{1}{8}\left(y^2 - 8y + 16 - 16 + 8\right) \\ &= \frac{1}{8}\left((y-4)^2 - 8\right) \\ &= \frac{1}{8}(y-4)^2 - 1. \end{aligned}$$

Hence the vertex is located at $(-1, 4)$. We have $\frac{1}{8} = \frac{1}{4p}$, so $p = 2$. We conclude that the focus is located at $(1, 4)$ and the directrix is $x = -3$. The parabola is graphed in Figure 10.1.10, along with its focus and directrix.

The point $(7, 12)$ lies on the graph and is $7 - (-3) = 10$ units from the directrix. The distance from $(7, 12)$ to the focus is:

$$\sqrt{(7-1)^2 + (12-4)^2} = \sqrt{100} = 10.$$

Indeed, the point on the parabola is equidistant from the focus and directrix.

Video solution

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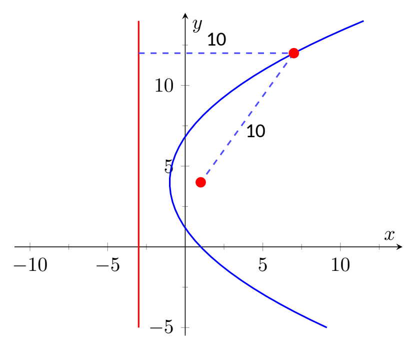


Figure 10.1.10 The parabola described in Example 10.1.9. The distances from a point on the parabola to the focus and directrix are given.

Reflective Property. One of the fascinating things about the nondegenerate conic sections is their reflective properties. Parabolas have the following reflective property:

> Any ray emanating from the focus that intersects the parabola reflects off along a line perpendicular to the directrix.

This is illustrated in Figure 10.1.11. The following theorem states this more rigorously.

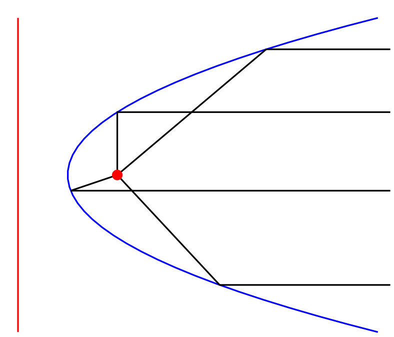


Figure 10.1.11 Illustrating the parabola's reflective property

Theorem 10.1.12 Reflective Property of the Parabola.

Let P be a point on a parabola. The tangent line to the parabola at P makes equal angles with the following two lines:

1. *The line containing P and the focus F, and*
2. *The line perpendicular to the directrix through P.*

Because of this reflective property, paraboloids (the 3D analogue of parabolas) make for useful flashlight reflectors as the light from the bulb, ideally located at the focus, is reflected along parallel rays. Satellite dishes also have paraboloid shapes. Signals coming from satellites effectively approach the dish along parallel rays. The dish then *focuses* these rays at the focus, where the sensor is located.

10.1.2 Ellipses

Definition 10.1.14 Ellipse.

An **ellipse** is the locus of all points whose sum of distances from two fixed points, each a **focus** of the ellipse, is constant.

youtu.be/watch?v=eAKfJphIwIE

Figure 10.1.13 Video introduction to the ellipse

An easy way to visualize this construction of an ellipse is to pin both ends of a string to a board. The pins become the foci. Holding a pencil tight against the string places the pencil on the ellipse; the sum of distances from the pencil to the pins is constant: the length of the string. See Figure 10.1.15.

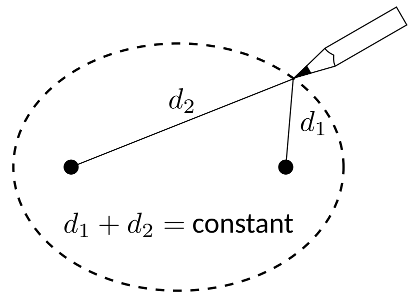


Figure 10.1.15 Illustrating the construction of an ellipse with pins, pencil and string

We can again find an algebraic equation for an ellipse using this geometric definition. Let the foci be located along the x-axis, c units from the origin. Let these foci be labeled as $F_1 = (-c, 0)$ and $F_2 = (c, 0)$. Let $P = (x, y)$ be a point on the ellipse. The sum of distances from F_1 to P (d_1) and from F_2 to P (d_2) is a constant d. That is, $d_1 + d_2 = d$. Using the Distance Formula, we have

$$\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = d.$$

Using a fair amount of algebra can produce the following equation of an ellipse (note that the equation is an implicitly defined function; it has to be, as an ellipse fails the Vertical Line Test):

$$\frac{x^2}{\left(\frac{d}{2}\right)^2} + \frac{y^2}{\left(\frac{d}{2}\right)^2 - c^2} = 1.$$

This is not particularly illuminating, but by making the substitution $a = d/2$ and $b = \sqrt{a^2 - c^2}$, we can rewrite the above equation as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This choice of a and b is not without reason; as shown in Figure 10.1.16, the values of a and b have geometric meaning in the graph of the ellipse.

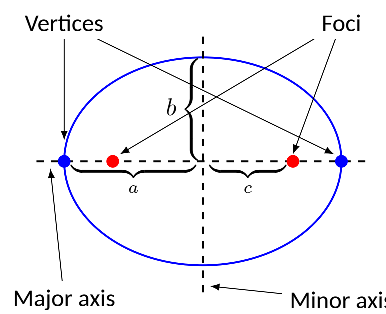


Figure 10.1.16 Labeling the significant features of an ellipse

In general, the two foci of an ellipse lie on the *major axis* of the ellipse, and the midpoint of the segment joining the two foci is the *center*. The major axis intersects the ellipse at two points, each of which is a *vertex*. The line segment through the center and perpendicular to the major axis is the *minor axis*. The "constant sum of distances" that defines the ellipse is the length of the major axis, i.e., $2a$.

Allowing for the shifting of the ellipse gives the following standard equations.

Key Idea 10.1.17 Standard Equation of the Ellipse.

The equation of an ellipse centered at (h, k) with major axis of length $2a$ and minor axis of length $2b$ in standard form is:

1. *Horizontal major axis:* $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1.$
2. *Vertical major axis:* $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1.$

The foci lie along the major axis, c units from the center, where $c^2 = a^2 - b^2$.

Example 10.1.18 Finding the equation of an ellipse.

Find the general equation of the ellipse graphed in Figure 10.1.19.

Solution. The center is located at $(-3, 1)$. The distance from the center to a vertex is 5 units, hence $a = 5$. The minor axis seems to have length 4, so $b = 2$. Thus the equation of the ellipse is

$$\frac{(x+3)^2}{4} + \frac{(y-1)^2}{25} = 1.$$

Video solution

youtu.be/watch?v=NVqlZCWDDnI

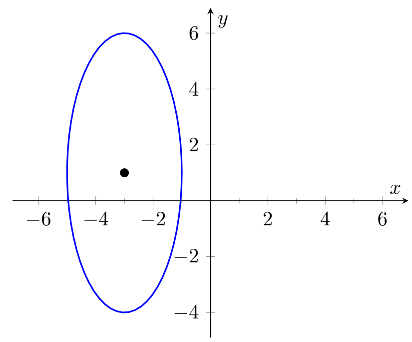


Figure 10.1.19 The ellipse used in Example 10.1.18

Example 10.1.20 Graphing an ellipse.

Graph the ellipse defined by $4x^2 + 9y^2 - 8x - 36y = -4$.

Solution. It is simple to graph an ellipse once it is in standard form. In order to put the given equation in standard form, we must complete the square with both the x and y terms. We first rewrite the equation by regrouping:

$$4x^2 + 9y^2 - 8x - 36y = -4 \Rightarrow (4x^2 - 8x) + (9y^2 - 36y) = -4.$$

Now we complete the squares.

$$\begin{aligned}
(4x^2 - 8x) + (9y^2 - 36y) &= -4 \\
4(x^2 - 2x) + 9(y^2 - 4y) &= -4 \\
4(x^2 - 2x + 1 - 1) + 9(y^2 - 4y + 4 - 4) &= -4 \\
4\big((x-1)^2 - 1\big) + 9\big((y-2)^2 - 4\big) &= -4 \\
4(x-1)^2 - 4 + 9(y-2)^2 - 36 &= -4 \\
4(x-1)^2 + 9(y-2)^2 &= 36 \\
\frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} &= 1.
\end{aligned}$$

We see the center of the ellipse is at $(1, 2)$. We have $a = 3$ and $b = 2$; the major axis is horizontal, so the vertices are located at $(-2, 2)$ and $(4, 2)$. We find $c = \sqrt{9-4} = \sqrt{5} \approx 2.24$. The foci are located along the major axis, approximately 2.24 units from the center, at $(1 \pm 2.24, 2)$. This is all graphed in Figure 10.1.21

Video solution

youtu.be/watch?v=U3uhopYrG8o

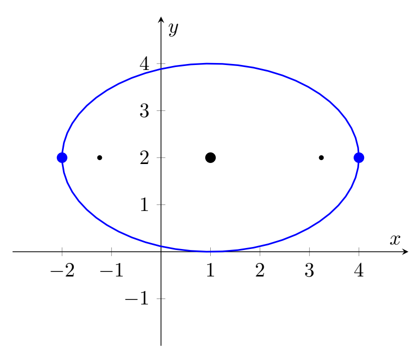


Figure 10.1.21 Graphing the ellipse in Example 10.1.20

Eccentricity. When $a = b$, we have a circle. The general equation becomes

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} = 1 \Rightarrow (x-h)^2 + (y-k)^2 = a^2,$$

the familiar equation of the circle centered at (h, k) with radius a. Since $a = b$, $c = \sqrt{a^2 - b^2} = 0$. The circle has "two" foci, but they lie on the same point, the center of the circle.

Consider Figure 10.1.22, where several ellipses are graphed with $a = 1$. In Figure 10.1.22(a), we have $c = 0$ and the ellipse is a circle. As c grows, the resulting ellipses look less and less circular. A measure of this "noncircularness" is *eccentricity*.

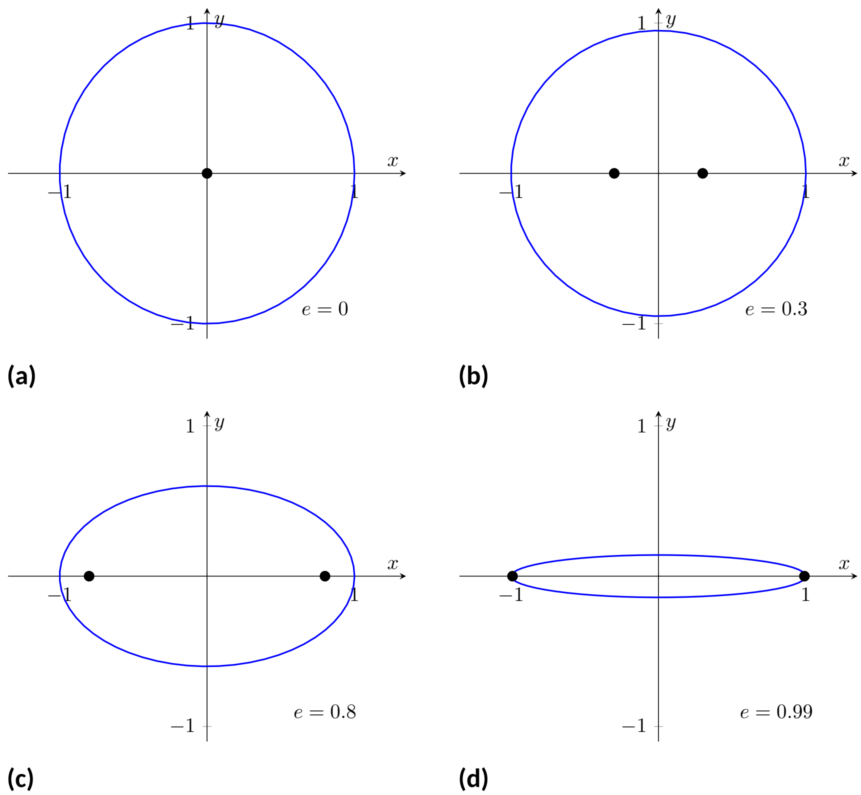


Figure 10.1.22 Understanding the eccentricity of an ellipse

Definition 10.1.23 Eccentricity of an Ellipse.

The eccentricity e of an ellipse is $e = \dfrac{c}{a}$.

The eccentricity of a circle is 0; that is, a circle has no “noncircularness.” As c approaches a, e approaches 1, giving rise to a very noncircular ellipse, as seen in Figure 10.1.22(d).

It was long assumed that planets had circular orbits. This is known to be incorrect; the orbits are elliptical. Earth has an eccentricity of 0.0167 — it has a nearly circular orbit. Mercury’s orbit is the most eccentric, with $e = 0.2056$. (Pluto’s eccentricity is greater, at $e = 0.248$, the greatest of all the currently known dwarf planets.) The planet with the most circular orbit is Venus, with $e = 0.0068$. The Earth’s moon has an eccentricity of $e = 0.0549$, also very circular.

Reflective Property. The ellipse also possesses an interesting reflective property. Any ray emanating from one focus of an ellipse reflects off the ellipse along a line through the other focus, as illustrated in Figure 10.1.24. This property is given formally in the following theorem.

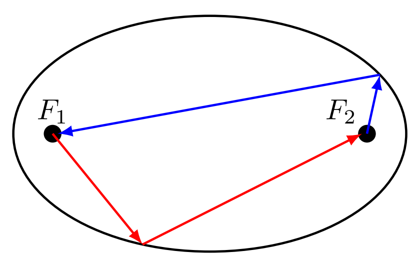


Figure 10.1.24 Illustrating the reflective property of an ellipse

Theorem 10.1.25 Reflective Property of an Ellipse.

Let P be a point on a ellipse with foci F_1 and F_2. The tangent line to the ellipse at P makes equal angles with the following two lines:

1. *The line through F_1 and P, and*

2. *The line through F_2 and P.*

This reflective property is useful in optics and is the basis of the phenomena experienced in whispering halls.

10.1.3 Hyperbolas

youtu.be/watch?v=9aLDYQrTaBo

Figure 10.1.26 Video introduction to hyperbolas

The definition of a hyperbola is very similar to the definition of an ellipse; we essentially just change the word "sum" to "difference."

Definition 10.1.27 Hyperbola.

A **hyperbola** is the locus of all points where the absolute value of difference of distances from two fixed points, each a focus of the hyperbola, is constant.

We do not have a convenient way of visualizing the construction of a hyperbola as we did for the ellipse. The geometric definition does allow us to find an algebraic expression that describes it. It will be useful to define some terms first.

The two foci lie on the *transverse axis* of the hyperbola; the midpoint of the line segment joining the foci is the *center* of the hyperbola. The transverse axis intersects the hyperbola at two points, each a *vertex* of the hyperbola. The line through the center and perpendicular to the transverse axis is the *conjugate axis*. This is illustrated in Figure 10.1.28. It is easy to show that the constant difference of distances used in the definition of the hyperbola is the distance between the vertices, i.e., $2a$.

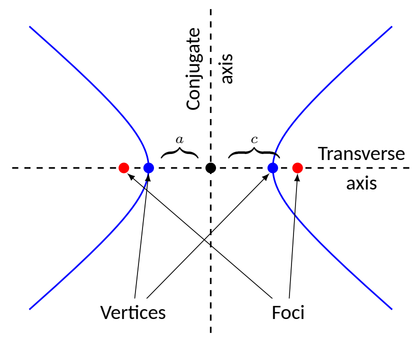


Figure 10.1.28 Labeling the significant features of a hyperbola

Key Idea 10.1.29 Standard Equation of a Hyperbola.

The equation of a hyperbola centered at (h, k) in standard form is:

1. *Horizontal Transverse Axis:* $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1.$

2. *Vertical Transverse Axis:* $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1.$

The vertices are located a units from the center and the foci are located c units from the center, where $c^2 = a^2 + b^2$.

Graphing Hyperbolas. Consider the hyperbola $\frac{x^2}{9} - \frac{y^2}{1} = 1$. Solving for y, we find $y = \pm\sqrt{x^2/9 - 1}$. As x grows large, the "-1" part of the equation for y becomes less significant and $y \approx \pm\sqrt{x^2/9} = \pm x/3$. That is, as x gets large, the graph of the hyperbola looks very much like the lines $y = \pm x/3$. These lines are asymptotes of the hyperbola, as shown in Figure 10.1.30.

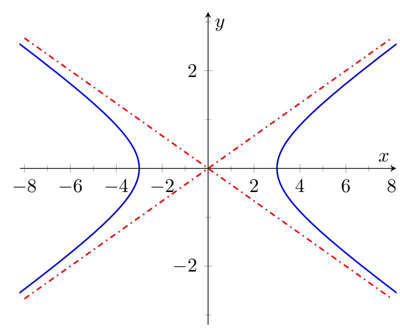


Figure 10.1.30 Graphing the hyperbola $\frac{x^2}{9} - \frac{y^2}{1} = 1$ along with its asymptotes, $y = \pm x/3$

This is a valuable tool in sketching. Given the equation of a hyperbola in general form, draw a rectangle centered at (h, k) with sides of length $2a$ parallel to the transverse axis and sides of length $2b$ parallel to the conjugate axis. (See Figure 10.1.31 for an example with a horizontal transverse axis.) The diagonals of the rectangle lie on the asymptotes.

These lines pass through (h, k). When the transverse axis is horizontal, the slopes are $\pm b/a$; when the transverse axis is vertical, their slopes are $\pm a/b$. This gives equations:

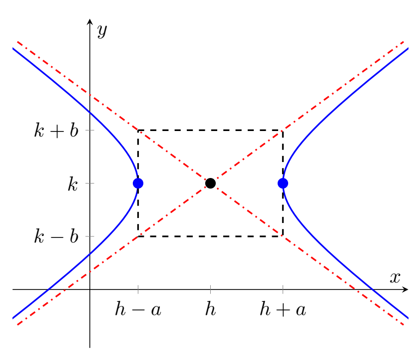


Figure 10.1.31 Using the asymptotes of a hyperbola as a graphing aid

Horizontal Transverse Axis	Vertical Transverse Axis
$y = \pm \frac{b}{a}(x-h)+k$	$y = \pm \frac{a}{b}(x-h)+k.$

Example 10.1.32 Graphing a hyperbola.

Sketch the hyperbola given by $\frac{(y-2)^2}{25} - \frac{(x-1)^2}{4} = 1.$

Solution. The hyperbola is centered at $(1,2)$; $a=5$ and $b=2$. In Figure 10.1.33 we draw the prescribed rectangle centered at $(1,2)$ along with the asymptotes defined by its diagonals. The hyperbola has a vertical transverse axis, so the vertices are located at $(1,7)$ and $(1,-3)$. This is enough to make a good sketch.

We also find the location of the foci: as $c^2 = a^2+b^2$, we have $c = \sqrt{29} \approx 5.4$. Thus the foci are located at $(1, 2 \pm 5.4)$ as shown in the figure.

Video solution

youtu.be/watch?v=0YVNci7ZOfo

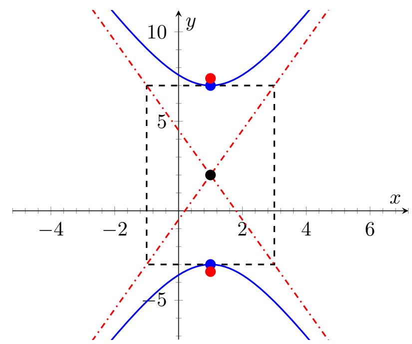


Figure 10.1.33 Graphing the hyperbola in Example 10.1.32

Example 10.1.34 Graphing a hyperbola.

Sketch the hyperbola given by $9x^2 - y^2 + 2y = 10.$

Solution. We must complete the square to put the equation in general form. (We recognize this as a hyperbola since it is a general quadratic equation and the x^2 and y^2 terms have opposite signs.)

$$\begin{aligned} 9x^2 - y^2 + 2y &= 10 \\ 9x^2 - (y^2 - 2y) &= 10 \\ 9x^2 - (y^2 - 2y + 1 - 1) &= 10 \\ 9x^2 - \big((y-1)^2 - 1\big) &= 10 \\ 9x^2 - (y-1)^2 &= 9 \\ x^2 - \frac{(y-1)^2}{9} &= 1 \end{aligned}$$

We see the hyperbola is centered at $(0,1)$, with a horizontal transverse axis, where $a=1$ and $b=3$. The appropriate rectangle is sketched in Figure 10.1.35 along with the asymptotes of the hyperbola. The vertices are located at $(\pm 1, 1)$. We have $c = \sqrt{10} \approx 3.2$, so the foci are located at $(\pm 3.2, 1)$ as shown in the figure.

Video solution

youtu.be/watch?v=b-1_3ATvn9A

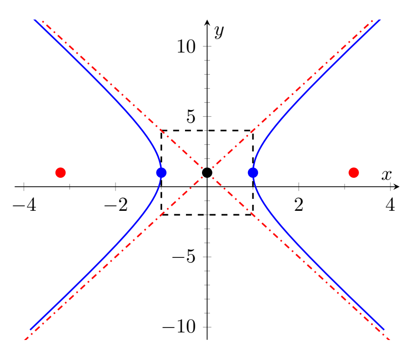


Figure 10.1.35 Graphing the hyperbola in Example 10.1.34

Eccentricity.

Definition 10.1.36 Eccentricity of a Hyperbola.

The eccentricity of a hyperbola is $e = \frac{c}{a}$.

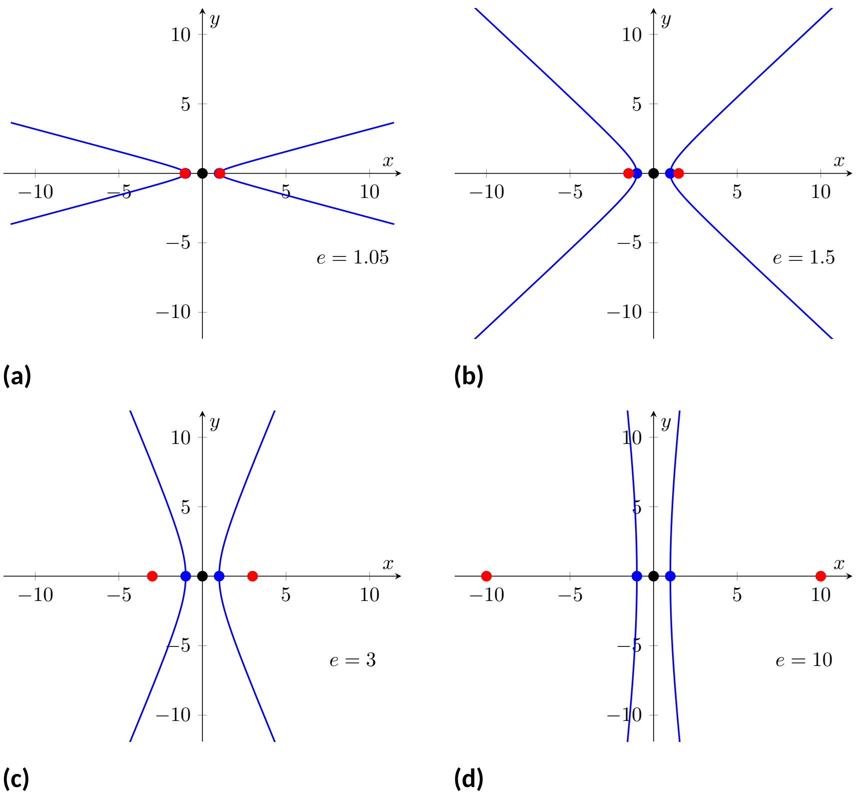


Figure 10.1.37 Understanding the eccentricity of a hyperbola

Note that this is the definition of eccentricity as used for the ellipse. When c is close in value to a (i.e., $e \approx 1$), the hyperbola is very narrow (looking almost like crossed lines). Figure 10.1.37 shows hyperbolas centered at the origin with $a = 1$. The graph in Figure 10.1.37(a) has $c = 1.05$, giving an eccentricity of $e = 1.05$, which is close to 1. As c grows larger, the hyperbola widens and begins to look like parallel lines, as shown in Figure 10.1.37(d).

Reflective Property. Hyperbolas share a similar reflective property with ellipses. However, in the case of a hyperbola, a ray emanating from a focus that intersects the hyperbola reflects along a line containing the other focus, but moving *away* from that focus. This is illustrated in Figure 10.1.39 (on the next page). Hyperbolic mirrors are commonly used in telescopes because of this reflective property. It is stated formally in the following theorem.

Theorem 10.1.38 Reflective Property of Hyperbolas.

Let P be a point on a hyperbola with foci F_1 and F_2. The tangent line to the hyperbola at P makes equal angles with the following two lines:

1. *The line through F_1 and P, and*
2. *The line through F_2 and P.*

Location Determination. Determining the location of a known event has many practical uses (locating the epicenter of an earthquake, an airplane crash site, the position of the person speaking in a large room, etc.).

To determine the location of an earthquake's epicenter, seismologists use *trilateration* (not to be confused with *triangulation*). A seismograph allows one to determine how far away the epicenter was; using three separate readings, the location of the epicenter can be approximated.

A key to this method is knowing distances. What if this information is not available? Consider three microphones at positions A, B and C which all record a noise (a person's voice, an explosion, etc.) created at unknown location D. The microphone does not "know" when the sound was *created*, only when the sound was *detected*. How can the location be determined in such a situation?

If each location has a clock set to the same time, hyperbolas can be used to determine the location. Suppose the microphone at position A records the sound at exactly 12:00, location B records the time exactly 1 second later, and location C records the noise exactly 2 seconds after that. We are interested in the *difference* of times. Since the speed of sound is approximately 340 m/s, we can conclude quickly that the sound was created 340 meters closer to position A than position B. If A and B are a known distance apart (as shown in Figure 10.1.40(a)), then we can determine a hyperbola on which D must lie.

The "difference of distances" is 340; this is also the distance between vertices of the hyperbola. So we know $2a = 340$. Positions A and B lie on the foci, so $2c = 1000$. From this we can find $b \approx 470$ and can sketch the hyperbola, given in Figure 10.1.40(b). We only care about the side closest to A. (Why?)

We can also find the hyperbola defined by positions B and C. In this case, $2a = 680$ as the sound traveled an extra 2 seconds to get to C. We still have $2c = 1000$, centering this hyperbola at $(-500, 500)$. We find $b \approx 367$. This hyperbola is sketched in Figure 10.1.40(c). The intersection point of the two graphs is the location of the sound, at approximately $(188, -222.5)$.

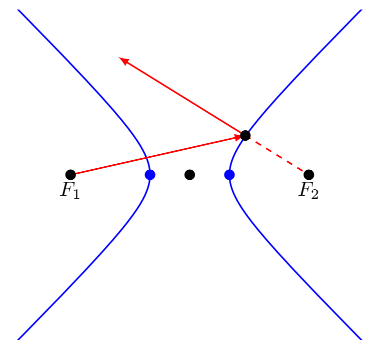
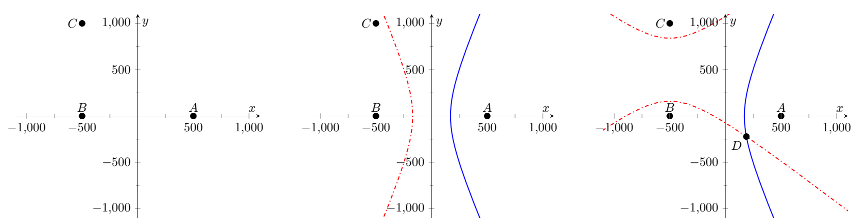


Figure 10.1.39 Illustrating the reflective property of a hyperbola



(a) **(b)** **(c)**

Figure 10.1.40

This chapter explores curves in the plane, in particular curves that cannot be described by functions of the form $y = f(x)$. In this section, we learned of ellipses and hyperbolas that are defined implicitly, not explicitly. In the following sections, we will learn completely new ways of describing curves in the plane, using *parametric equations* and *polar coordinates*, then study these curves using calculus techniques.

10.1.4 Exercises

Terms and Concepts

1. What is the difference between degenerate and nondegenerate conics?

2. Use your own words to explain what the eccentricity of an ellipse measures.

3. What has the largest eccentricity: an ellipse or a hyperbola?

4. Explain why the following is true: "If the coefficient of the x^2 term in the equation of an ellipse in standard form is smaller than the coefficient of the y^2 term, then the ellipse has a horizontal major axis."

5. Explain how one can quickly look at the equation of a hyperbola in standard form and determine whether the transverse axis is horizontal or vertical.

6. Fill in the blank: It can be said that ellipses and hyperbolas share the *same* reflective property: "A ray emanating from one focus will reflect off the conic along a ___________ that contains the other focus."

Problems

Exercise Group. In the following exercises, find the equation of the parabola defined by the given information. Sketch the parabola.

7. Focus: $(3, 2)$; directrix: $y = 1$

8. Focus: $(-1, -4)$; directrix: $y = 2$

9. Focus: $(1, 5)$; directrix: $x = 3$

10. Focus: $(1/4, 0)$; directrix: $x = -1/4$

11. Focus: $(1, 1)$; vertex: $(1, 2)$

12. Focus: $(-3, 0)$; vertex: $(0, 0)$

13. Vertex: $(0, 0)$; directrix: $y = -1/16$

14. Vertex: $(2, 3)$; directrix: $x = 4$

Exercise Group. In the following exercises, the equation of a parabola and a point on its graph are given. Find the focus and directrix of the parabola, and verify that the given point is equidistant from the focus and directrix.

15. $y = \frac{1}{4}x^2$, $P = (2, 1)$

16. $x = \frac{1}{8}(y - 2)^2 + 3$, $P = (11, 10)$

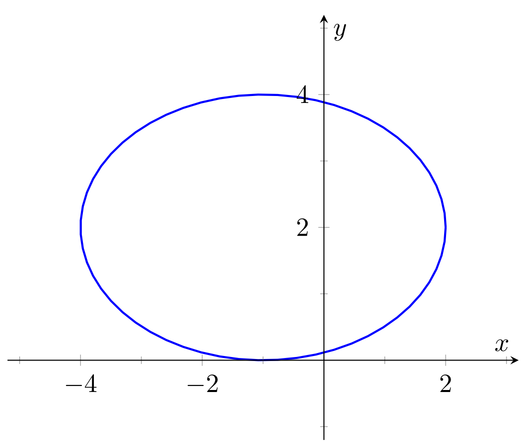
Exercise Group. In the following exercises, sketch the ellipse defined by the given equation. Label the center, foci and vertices.

17. $\dfrac{(x-1)^2}{3} + \dfrac{(y-2)^2}{5} = 1$

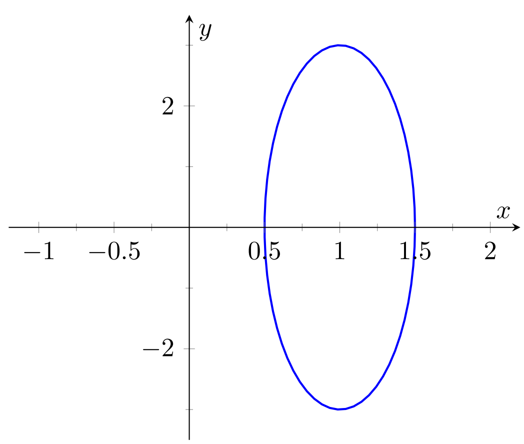
18. $\dfrac{1}{25}x^2 + \dfrac{1}{9}(y+3)^2 = 1$

Exercise Group. In the following exercises, find the equation of the ellipse shown in the graph. Give the location of the foci and the eccentricity of the ellipse.

19.



20.



Exercise Group. In the following exercises, find the equation of the ellipse defined by the given information. Sketch the elllipse.

21. Foci: $(\pm 2, 0)$; vertices: $(\pm 3, 0)$

22. Foci: $(-1, 3)$ and $(5, 3)$; vertices: $(-3, 3)$ and $(7, 3)$

23. Foci: $(2, \pm 2)$; vertices: $(2, \pm 7)$

24. Focus: $(-1, 5)$; vertex: $(-1, -4)$; center: $(-1, 1)$

Exercise Group. In the following exercises, write the equation of the given ellipse in standard form.

25. $x^2 - 2x + 2y^2 - 8y = -7$

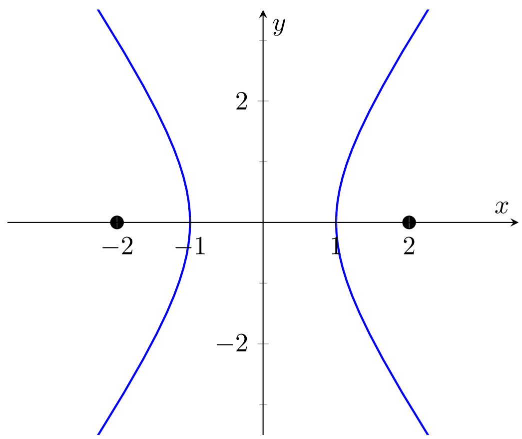
26. $5x^2 + 3y^2 = 15$

27. $3x^2 + 2y^2 - 12y + 6 = 0$

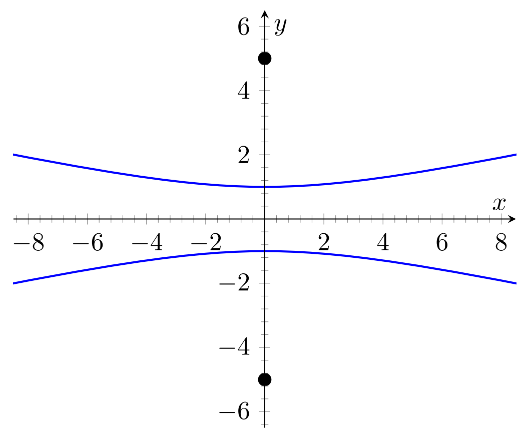
28. $x^2 + y^2 - 4x - 4y + 4 = 0$

Exercise Group. In the following exercises, find the equation of the hyperbola shown in the graph.

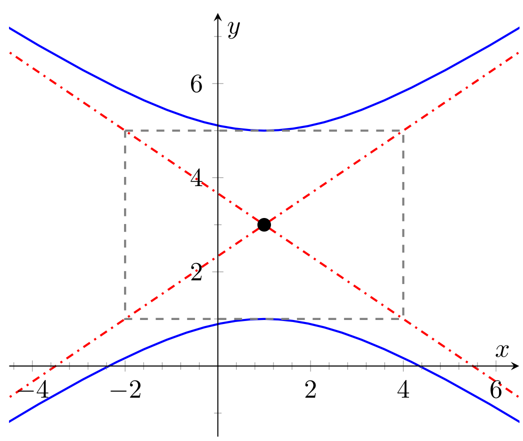
29.



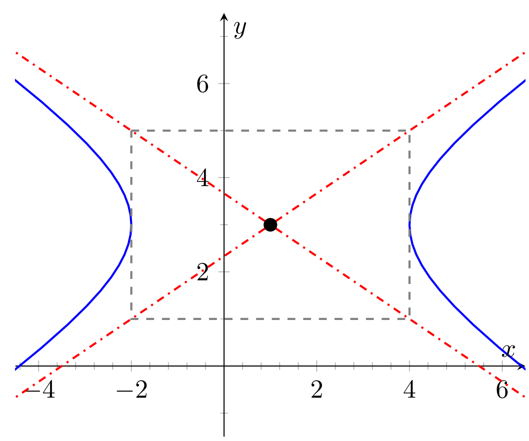
30.



31.



32.



Exercise Group. In the following exercises, sketch the hyperbola defined by the given equation. Label the center and foci.

33. $\dfrac{(x-1)^2}{16} - \dfrac{(y+2)^2}{9} = 1$

34. $(y-4)^2 - \dfrac{(x+1)^2}{25} = 1$

Exercise Group. In the following exercises, find the equation of the hyperbola defined by the given information. Sketch the hyperbola.

35. Foci: $(\pm 3, 0)$; vertices: $(\pm 2, 0)$

36. Foci: $(0, \pm 3)$; vertices: $(0, \pm 2)$

37. Foci: $(-2, 3)$ and $(8, 3)$; vertices: $(-1, 3)$ and $(7, 3)$

38. Foci: $(3, -2)$ and $(3, 8)$; vertices: $(3, 0)$ and $(3, 6)$

Exercise Group. In the following exercises, write the equation of the hyperbola in standard form.

39. $3x^2 - 4y^2 = 12$

40. $3x^2 - y^2 + 2y = 10$

41. $x^2 - 10y^2 + 40y = 30$

42. $(4y - x)(4y + x) = 4$

43. Consider the ellipse given by $\frac{(x-1)^2}{4} + \frac{(y-3)^2}{12} = 1$.

(a) Verify that the foci are located at $(1, 3 \pm 2\sqrt{2})$.

(b) The points $P_1 = (2, 6)$ and $P_2 = (1 + \sqrt{2}, 3 + \sqrt{6}) \approx (2.414, 5.449)$ lie on the ellipse. Verify that the sum of distances from each point to the foci is the same.

44. Johannes Kepler discovered that the planets of our solar system have elliptical orbits with the Sun at one focus. The Earth's elliptical orbit is used as a standard unit of distance; the distance from the center of Earth's elliptical orbit to one vertex is 1 Astronomical Unit, or A.U.

The following table gives information about the orbits of three planets.

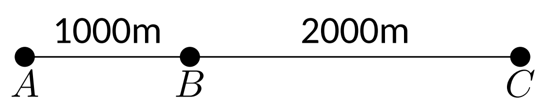
Planet	Distance from center to vertex	Orbit eccentricity
Mercury	0.387 A.U.	0.2056
Earth	1 A.U.	0.0167
Mars	1.524 A.U.	0.0934

(a) In an ellipse, knowing $c^2 = a^2 - b^2$ and $e = c/a$ allows us to find b in terms of a and e. Show $b = a\sqrt{1 - e^2}$.

(b) For each planet, find equations of their elliptical orbit of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (This places the center at $(0, 0)$, but the Sun is in a different location for each planet.)

(c) Shift the equations so that the Sun lies at the origin. Plot the three elliptical orbits.

45. A loud sound is recorded at three stations that lie on a line as shown in the figure below. Station A recorded the sound 1 second after Station B, and Station C recorded the sound 3 seconds after B. Using the speed of sound as 340m/s, determine the location of the sound's origination.



10.2 Parametric Equations

youtu.be/watch?v=ktYtHfYoWi4

Figure 10.2.1 Video introduction to Section 10.2

We are familiar with sketching shapes, such as parabolas, by following this basic procedure:

Choose x ⟶ Use a function f to find y $(y = f(x))$ ⟶ Plot point (x, y)

Figure 10.2.2 Plotting a graph $y = f(x)$

The *rectangular equation* $y = f(x)$ works well for some shapes like a parabola with a vertical axis of symmetry, but in the previous section we encountered several shapes that could not be sketched in this manner. (To plot an ellipse using the above procedure, we need to plot the "top" and "bottom" separately.)

In this section we introduce a new sketching procedure:

Choose t ⟶ Use a function f to find x $(x = f(t))$ ⟶ Plot point (x, y)

Choose t ⟶ Use a function g to find y $(y = g(t))$ ⟶ Plot point (x, y)

Figure 10.2.3 Plotting a curve $(x(t), y(t))$

Here, x and y are found separately but then plotted together: for each value of the input t, we plot the output - the point $(x(t), y(t))$.

10.2.1 Plotting parametric curves

The procedure outlined in Figure 10.2.3 leads us to a definition.

Definition 10.2.4 Parametric Equations and Curves.

Let f and g be continuous functions on an interval I. The set of all points $(x, y) = (f(t), g(t))$ in the Cartesian plane, as t varies over I, is the **graph** of the **parametric equations** $x = f(t)$ and $y = g(t)$, where t is the **parameter**. A **curve** is a graph along with the parametric equations that define it.

This is a formal definition of the word *curve*. When a curve lies in a plane (such as the Cartesian plane), it is often referred to as a *plane curve*. Examples will help us understand the concepts introduced in the definition.

Example 10.2.5 Plotting parametric functions.

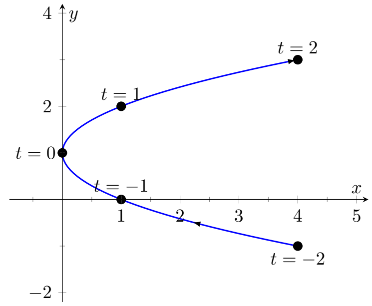
Plot the graph of the parametric equations $x = t^2$, $y = t + 1$ for t in $[-2, 2]$.

Solution. We plot the graphs of parametric equations in much the same manner as we plotted graphs of functions like $y = f(x)$: we make a table of values, plot points, then connect these points with a "reasonable" looking curve. Figure 10.2.6(a) shows such a table of values; note how we have 3 columns.

The points (x, y) from the table are plotted in Figure 10.2.6(b). The points have been connected with a smooth curve. Each point has been labeled with its corresponding t-value. These values, along with the two arrows along the curve, are used to indicate the *orientation* of the graph. This information helps us determine the direction in which the graph is "moving."

t	x	y
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3

(a)



(b)

Video solution

youtu.be/watch?v=7Ytw9RTrFtM

Figure 10.2.6 A table of values of the parametric equations in Example 10.2.5 along with a sketch of their graph

We often use the letter t as the parameter as we often regard t as representing *time*. Certainly there are many contexts in which the parameter is not time, but it can be helpful to think in terms of time as one makes sense of parametric plots and their orientation (for instance, "At time $t = 0$ the position is $(1, 2)$ and at time $t = 3$ the position is $(5, 1)$.").

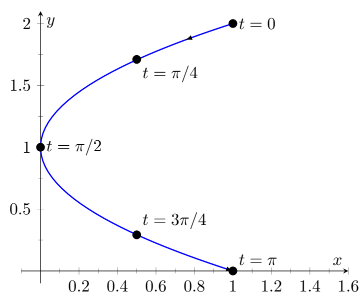
Example 10.2.7 Plotting parametric functions.

Sketch the graph of the parametric equations $x = \cos^2(t)$, $y = \cos(t)+1$ for t in $[0, \pi]$.

Solution. We again start by making a table of values in Figure 10.2.8(a), then plot the points (x, y) on the Cartesian plane in Figure 10.2.8(b).

t	x	y
0	1	2
$\pi/4$	$1/2$	$1+\sqrt{2}/2$
$\pi/2$	0	1
$3\pi/4$	$1/2$	$1-\sqrt{2}/2$
π	1	0

(a)



(b)

Figure 10.2.8 A table of values of the parametric equations in Example 10.2.7 along with a sketch of their graph

It is not difficult to show that the curves in Examples 10.2.5 and 10.2.7 are portions of the same parabola. While the *parabola* is the same, the *curves* are different. In Example 10.2.5, if we let t vary over all real numbers, we'd obtain the entire parabola. In this example, letting t vary over all real numbers would still produce the same graph; this portion of the parabola would be traced, and re-traced, infinitely many times. The orientation shown in Figure 10.2.8 shows the orientation on $[0, \pi]$, but this orientation is reversed on $[\pi, 2\pi]$.

These examples begin to illustrate the powerful nature of parametric equations. Their graphs are far more diverse than the graphs of functions produced by "$y = f(x)$" functions.

Technology Note: Most graphing utilities can graph functions given in parametric form. Often the word "parametric" is abbreviated as "PAR" or "PARAM" in the options. The user usually needs to determine the graphing window (i.e, the minimum and maximum x- and y-values), along with the values of t that are to be plotted. The user is often prompted to give a t minimum, a t maximum, and a "t-step" or "Δt." Graphing utilities effectively plot parametric functions just as we've shown here: they plots lots of points. A smaller t-step plots more points, making for a smoother graph (but may take longer). In Figure 10.2.6, the t-step is 1; in Figure 10.2.8, the t-step is $\pi/4$.

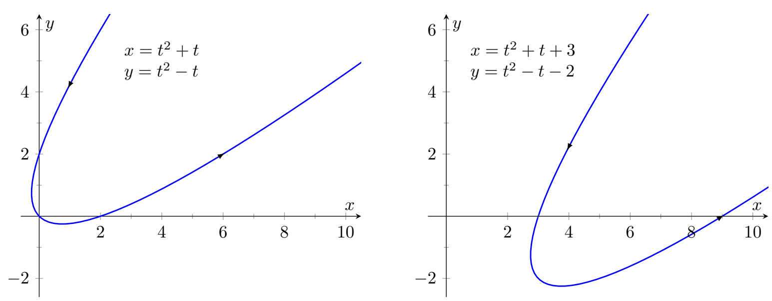
One nice feature of parametric equations is that their graphs are easy to shift. While this is not too difficult in the "$y = f(x)$" context, the resulting function can look rather messy. (Plus, to shift to the right by two, we replace x with $x - 2$, which is counter-intuitive.) The following example demonstrates this.

Example 10.2.9 Shifting the graph of parametric functions.

Sketch the graph of the parametric equations $x = t^2 + t$, $y = t^2 - t$. Find new parametric equations that shift this graph to the right 3 places and down 2.

Solution. The graph of the parametric equations is given in Figure 10.2.10(a). It is a parabola with a axis of symmetry along the line $y = x$; the vertex is at $(0, 0)$.

In order to shift the graph to the right 3 units, we need to increase the x-value by 3 for every point. The straightforward way to accomplish this is simply to add 3 to the function defining x: $x = t^2 + t + 3$. To shift the graph down by 2 units, we wish to decrease each y-value by 2, so we subtract 2 from the function defining y: $y = t^2 - t - 2$. Thus our parametric equations for the shifted graph are $x = t^2 + t + 3$, $y = t^2 - t - 2$. This is graphed in Figure 10.2.10(a). Notice how the vertex is now at $(3, -2)$.



(a) **(b)**

Figure 10.2.10 Illustrating how to shift graphs in Example 10.2.9

Because the x- and y-values of a graph are determined independently, the graphs of parametric functions often possess features not seen on "$y = f(x)$" type graphs. The next example demonstrates how such graphs can arrive at the same point more than once.

Example 10.2.11 Graphs that cross themselves.

Plot the parametric functions $x = t^3 - 5t^2 + 3t + 11$ and $y = t^2 - 2t + 3$ and determine the t-values where the graph crosses itself.

Solution. Using the methods developed in this section, we again plot points and graph the parametric equations as shown in Figure 10.2.12. It appears that the graph crosses itself at the point $(2, 6)$, but we'll need to analytically determine this.

We are looking for two different values, say, s and t, where $x(s) = x(t)$ and $y(s) = y(t)$. That is, the x-values are the same precisely when the y-values are the same. This gives us a system of 2 equations with 2 unknowns:

$$s^3 - 5s^2 + 3s + 11 = t^3 - 5t^2 + 3t + 11$$
$$s^2 - 2s + 3 = t^2 - 2t + 3$$

Solving this system is not trivial but involves only algebra. Using the quadratic formula, one can solve for t in the second equation and find that $t = 1 \pm \sqrt{s^2 - 2s + 1}$. This can be substituted into the first equation, revealing that the graph crosses itself at $t = -1$ and $t = 3$. We confirm our result by computing $x(-1) = x(3) = 2$ and $y(-1) = y(3) = 6$.

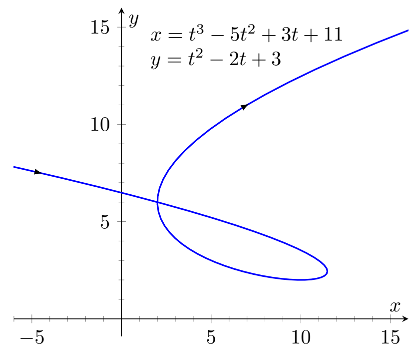


Figure 10.2.12 A graph of the parametric equations from Example 10.2.11

10.2.2 Converting between rectangular and parametric equations

It is sometimes useful to rewrite equations in rectangular form (i.e., $y = f(x)$) into parametric form, and vice-versa. Converting from rectangular to parametric can be very simple: given $y = f(x)$, the parametric equations $x = t$, $y = f(t)$ produce the same graph. As an example, given $y = x^2$, the parametric equations $x = t$, $y = t^2$ produce the familiar parabola. However, other parametrizations can be used. The following example demonstrates one possible alternative.

Example 10.2.13 Converting from rectangular to parametric.

Consider $y = x^2$. Find parametric equations $x = f(t)$, $y = g(t)$ for the parabola where $t = \frac{dy}{dx}$. That is, $t = a$ corresponds to the point on the graph whose tangent line has slope a.

Solution. We start by computing $\frac{dy}{dx}$: $y' = 2x$. Thus we set $t = 2x$. We can solve for x and find $x = t/2$. Knowing that $y = x^2$, we have $y = t^2/4$. Thus parametric equations for the parabola $y = x^2$ are

$$x = t/2 y = t^2/4.$$

To find the point where the tangent line has a slope of -2, we set $t = -2$. This gives the point $(-1, 1)$. We can verify that the slope of the line tangent to the curve at this point indeed has a slope of -2.

Video solution

youtu.be/watch?v=YsOEEcCXNb8

We sometimes choose the parameter to accurately model physical behavior.

Example 10.2.14 Converting from rectangular to parametric.

An object is fired from a height of 0 feet and lands 6 seconds later, 192 feet away. Assuming ideal projectile motion, the height, in feet, of the object can be described by $h(x) = -x^2/64 + 3x$, where x is the distance in feet from the initial location. (Thus $h(0) = h(192) = 0$ feet.) Find parametric equations $x = f(t)$, $y = g(t)$ for the path of the projectile where x is the horizontal distance the object has traveled at time t (in seconds) and y is the height at time t.

Solution. Physics tells us that the horizontal motion of the projectile is linear; that is, the horizontal speed of the projectile is constant. Since the object travels 192 ft in 6 s, we deduce that the object is moving horizontally at a rate of $32\,\frac{\text{ft}}{\text{s}}$, giving the equation $x = 32t$. As $y = -x^2/64+3x$, we find $y = -16t^2 + 96t$. We can quickly verify that $y'' = -32\,\frac{\text{ft}}{\text{ft}^2}$, the acceleration due to gravity, and that the projectile reaches its maximum at $t = 3$, halfway along its path.

These parametric equations make certain determinations about the object's location easy: 2 seconds into the flight the object is at the point $\big(x(2), y(2)\big) = \big(64, 128\big)$. That is, it has traveled horizontally 64 ft and is at a height of 128 ft, as shown in Figure 10.2.15.

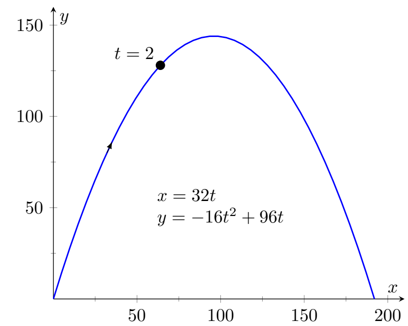


Figure 10.2.15 Graphing projectile motion in Example 10.2.14

It is sometimes necessary to convert given parametric equations into rectangular form. This can be decidedly more difficult, as some "simple" looking parametric equations can have very "complicated" rectangular equations. This conversion is often referred to as "eliminating the parameter," as we are looking for a relationship between x and y that does not involve the parameter t.

Example 10.2.16 Eliminating the parameter.

Find a rectangular equation for the curve described by

$$x = \frac{1}{t^2+1} \text{ and } y = \frac{t^2}{t^2+1}.$$

Solution. There is not a set way to eliminate a parameter. One method is to solve for t in one equation and then substitute that value in the second. We use that technique here, then show a second, simpler method. Starting with $x = 1/(t^2+1)$, solve for t: $t = \pm\sqrt{1/x-1}$. Substitute this value for t in the equation for y:

$$\begin{aligned} y &= \frac{t^2}{t^2+1} \\ &= \frac{1/x-1}{1/x-1+1} \\ &= \frac{1/x-1}{1/x} \\ &= \left(\frac{1}{x}-1\right)\cdot x \\ &= 1-x. \end{aligned}$$

Thus $y = 1-x$. One may have recognized this earlier by manipulating the equation for y:

$$y = \frac{t^2}{t^2+1} = 1 - \frac{1}{t^2+1} = 1-x.$$

This is a shortcut that is very specific to this problem; sometimes shortcuts exist and are worth looking for.

We should be careful to limit the domain of the function $y = 1-x$. The parametric equations limit x to values in $(0,1]$, thus to produce the same graph we should limit the domain of $y = 1-x$ to the same.

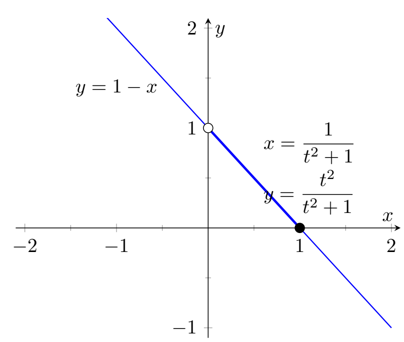


Figure 10.2.17 Graphing parametric and rectangular equations for a graph in Example 10.2.16

The graphs of these functions is given in Figure 10.2.17. The portion of the graph defined by the parametric equations is given in a thick line; the graph defined by $y = 1 - x$ with unrestricted domain is given in a thin line.

Video solution

youtu.be/watch?v=8Vgv74zCWFQ

Example 10.2.18 Eliminating the parameter.

Eliminate the parameter in $x = 4\cos(t) + 3$, $y = 2\sin(t) + 1$

Solution. We should not try to solve for t in this situation as the resulting algebra/trig would be messy. Rather, we solve for $\cos(t)$ and $\sin(t)$ in each equation, respectively. This gives

$$\cos(t) = \frac{x-3}{4} \text{ and } \sin(t) = \frac{y-1}{2}.$$

The Pythagorean Theorem gives $\cos^2(t) + \sin^2(t) = 1$, so:

$$\begin{aligned} \cos^2(t) + \sin^2(t) &= 1 \\ \left(\frac{x-3}{4}\right)^2 + \left(\frac{y-1}{2}\right)^2 &= 1 \\ \frac{(x-3)^2}{16} + \frac{(y-1)^2}{4} &= 1 \end{aligned}$$

This final equation should look familiar — it is the equation of an ellipse! Figure 10.2.19 plots the parametric equations, demonstrating that the graph is indeed of an ellipse with a horizontal major axis and center at $(3, 1)$.

Video solution

youtu.be/watch?v=RYkPHhpgnas

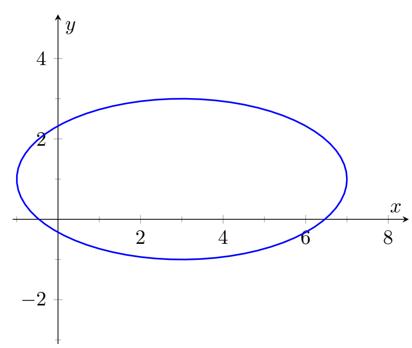


Figure 10.2.19 Graphing the parametric equations $x = 4\cos(t) + 3$, $y = 2\sin(t) + 1$ in Example 10.2.18

The Pythagorean Theorem can also be used to identify parametric equations for hyperbolas. We give the parametric equations for ellipses and hyperbolas in the following Key Idea.

Key Idea 10.2.20 Parametric Equations of Ellipses and Hyperbolas.

- The parametric equations

  $$x = a\cos(t) + h, y = b\sin(t) + k$$

  define an ellipse with horizontal axis of length $2a$ and vertical axis of length $2b$, centered at (h, k).

- The parametric equations

  $$x = a\tan(t) + h, y = \pm b\sec(t) + k$$

  define a hyperbola with vertical transverse axis centered at (h, k), and

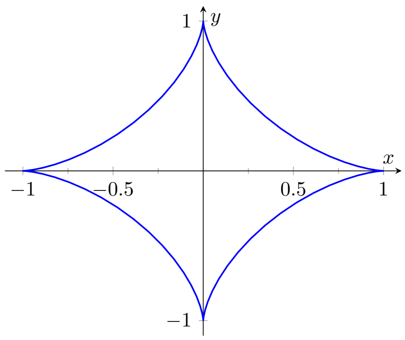
  $$x = \pm a\sec(t) + h, y = b\tan(t) + k$$

  defines a hyperbola with horizontal transverse axis. Each has asymptotes at $y = \pm b/a(x - h) + k$.

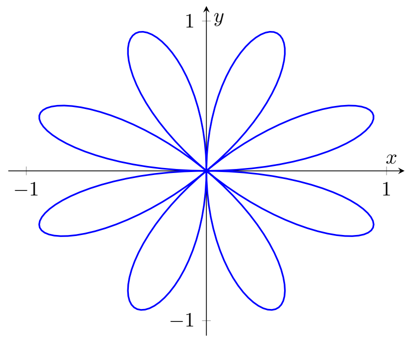
10.2.3 Special Curves

Figure 10.2.21 gives a small gallery of "interesting" and "famous" curves along with parametric equations that produce them. Interested readers can begin learning more about these curves through internet searches.

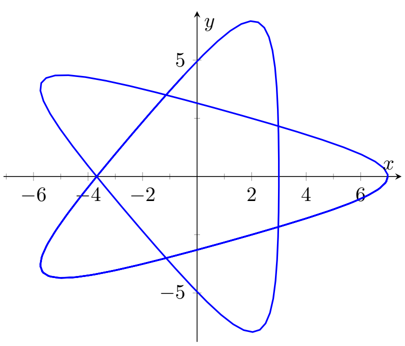
One might note a feature shared by two of these graphs: "sharp corners," or *cusps*. We have seen graphs with cusps before and determined that such functions are not differentiable at these points. This leads us to a definition.



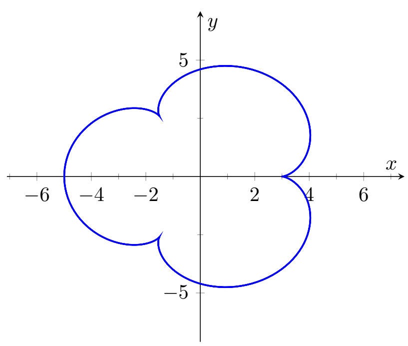
(a) Astroid where $x = \cos^3(t)$ and $y = \sin^3(t)$



(b) Rose Curve where $x = \cos(t)\sin(4t)$ and $y = \sin(t)\sin(4t)$



(c) Hypotrochoid where $x = 2\cos(t) + 5\cos(2t/3)$ and $y = 2\sin(t) - 5\sin(2t/3)$



(d) Epicycloid where $x = 4\cos(t) - \cos(4t)$ and $y = 4\sin(t) - \sin(4t)$

Figure 10.2.21 A gallery of interesting planar curves

Definition 10.2.22 Smooth.

A curve C defined by $x = f(t)$, $y = g(t)$ is **smooth** on an interval I if f' and g' are continuous on I and not simultaneously 0 (except possibly at the endpoints of I). A curve is **piecewise smooth** on I if I can be partitioned into subintervals where C is smooth on each subinterval.

Consider the astroid, given by $x = \cos^3(t)$, $y = \sin^3(t)$. Taking derivatives, we have:

$$x' = -3\cos^2(t)\sin(t) \text{ and } y' = 3\sin^2(t)\cos(t).$$

It is clear that each is 0 when $t = 0, \pi/2, \pi, \ldots$. Thus the astroid is not smooth at these points, corresponding to the cusps seen in the figure.

We demonstrate this once more.

Example 10.2.23 Determine where a curve is not smooth.

Let a curve C be defined by the parametric equations $x = t^3 - 12t + 17$ and $y = t^2 - 4t + 8$. Determine the points, if any, where it is not smooth.

Solution. We begin by taking derivatives.

$$x' = 3t^2 - 12, y' = 2t - 4.$$

We set each equal to 0:

$$x' = 0 \Rightarrow 3t^2 - 12 = 0 \Rightarrow t = \pm 2$$
$$y' = 0 \Rightarrow 2t - 4 = 0 \Rightarrow t = 2$$

We see at $t = 2$ both x' and y' are 0; thus C is not smooth at $t = 2$, corresponding to the point $(1, 4)$. The curve is graphed in Figure 10.2.24, illustrating the cusp at $(1, 4)$.

Video solution

youtu.be/watch?v=rqhh0McArDM

If a curve is not smooth at $t = t_0$, it means that $x'(t_0) = y'(t_0) = 0$ as defined. This, in turn, means that rate of change of x (and y) is 0; that is, at that instant, neither x nor y is changing. If the parametric equations describe the path of some object, this means the object is at rest at t_0. An object at rest can make a "sharp" change in direction, whereas moving objects tend to change direction in a "smooth" fashion.

One should be careful to note that a "sharp corner" does not have to occur when a curve is not smooth. For instance, one can verify that $x = t^3$, $y = t^6$ produce the familiar $y = x^2$ parabola. However, in this parametrization, the curve is not smooth. A particle traveling along the parabola according to the given parametric equations comes to rest at $t = 0$, though no sharp point is created.

Our previous experience with cusps taught us that a function was not differentiable at a cusp. This can lead us to wonder about derivatives in the context of parametric equations and the application of other calculus concepts. Given a curve defined parametrically, how do we find the slopes of tangent lines? Can we determine concavity? We explore these concepts and more in the next section.

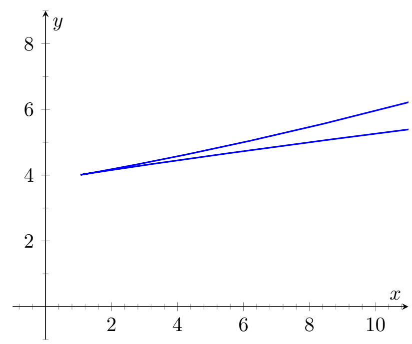


Figure 10.2.24 Graphing the curve in Example 10.2.23; note it is not smooth at $(1, 4)$

10.2.4 Exercises

Terms and Concepts

1. True or False? When sketching the graph of parametric equations, the x- and y-values are found separately, then plotted together. (☐ True ☐ False)

2. The direction in which a graph is "moving" is called the __________ of the graph.

3. An equation written as $y = f(x)$ is written in __________ form.

4. Create parametric equations $x = f(t)$, $y = g(t)$ and sketch their graph. Explain any interesting features of your graph based on the functions f and g.

Problems

Exercise Group. In the following exercises, sketch the graph of the given parametric equations *by hand*, making a table of points to plot. Be sure to indicate the orientation of the graph.

5. $x = t^2 + t, y = 1 - t^2, -3 \le t \le 3$

6. $x = 1, y = 5\sin(t), -\pi/2 \le t \le \pi/2$

7. $x = t^2, y = 2, -2 \le t \le 2$

8. $x = t^3 - t + 3, y = t^2 + 1, -2 \le t \le 2$

Exercise Group. In the following exercises, sketch the graph of the given parametric equations; using a graphing utility is advisable. Be sure to indicate the orientation of the graph.

9. $x = t^3 - 2t^2, y = t^2, -2 \le t \le 3$

10. $x = 1/t, y = \sin(t), 0 < t \le 10$

11. $x = 3\cos(t), y = 5\sin(t), 0 \le t \le 2\pi$

12. $x = 3\cos(t) + 2, y = 5\sin(t) + 3, 0 \le t \le 2\pi$

13. $x = \cos(t), y = \cos(2t), 0 \le t \le \pi$

14. $x = \cos(t), y = \sin(2t), 0 \le t \le 2\pi$

15. $x = 2\sec(t), y = 3\tan(t), -\pi/2 < t < \pi/2$

16. $x = \cosh(t), y = \sinh(t), -2 \le t \le 2$

17. $x = \cos(t) + \frac{1}{4}\cos(8t), y = \sin(t) + \frac{1}{4}\sin(8t), 0 \le t \le 2\pi$

18. $x = \cos(t) + \frac{1}{4}\sin(8t), y = \sin(t) + \frac{1}{4}\cos(8t), 0 \le t \le 2\pi$

Exercise Group. In the following exercises, four sets of parametric equations are given. Describe how their graphs are similar and different. Be sure to discuss orientation and ranges.

19.

- **(a)** $x = t\ y = t^2, -\infty < t < \infty$
- **(b)** $x = \sin(t)\ y = \sin^2(t), -\infty < t < \infty$
- **(c)** $x = e^t\ y = e^{2t}, -\infty < t < \infty$
- **(d)** $x = -t\ y = t^2, -\infty < t < \infty$

20.

- **(a)** $x = \cos(t)\ y = \sin(t), 0 \le t \le 2\pi$
- **(b)** $x = \cos(t^2)\ y = \sin(t^2), 0 \le t \le 2\pi$
- **(c)** $x = \cos(1/t)\ y = \sin(1/t), 0 < t < 1$
- **(d)** $x = \cos(\cos(t))\ y = \sin(\cos(t)), 0 \le t \le 2\pi$

Exercise Group. Eliminate the parameter in the given parametric equations.

21. $x = 2t + 5, y = -3t + 1$

22. $x = \sec(t), y = \tan(t)$

23. $x = 4\sin(t) + 1, y = 3\cos(t) - 2$

24. $x = t^2, y = t^3$

25. $x = \frac{1}{t+1}, y = \frac{3t+5}{t+1}$

26. $x = e^t, y = e^{3t} - 3$

27. $x = \ln(t), y = t^2 - 1$

28. $x = \cot(t), y = \csc(t)$

29. $x = \cosh(t), y = \sinh(t)$

30. $x = \cos(2t), y = \sin(t)$

Exercise Group. In the following exercises, eliminate the parameter in the given parametric equations. Describe the curve defined by the parametric equations based on its rectangular form.

31. $x = at + x_0, y = bt + y_0$

32. $x = r\cos(t), y = r\sin(t)$

33. $x = a\cos(t) + h, y = b\sin(t) + k$

34. $x = a\sec(t) + h, y = b\tan(t) + k$

Exercise Group. In the following exercises, find parametric equations for the given rectangular equation using the parameter $t = \frac{dy}{dx}$. Verify that at $t = 1$, the point on the graph has a tangent line with slope of 1.

35. $y = 3x^2 - 11x + 2$

36. $y = e^x$

37. $y = \sin(x)$

38. $y = \sqrt{x}$ on $[0, \infty)$

Exercise Group. In the following exercises, find the values of t where the graph of the parametric equations crosses itself.

39. $x = t^3 - t + 3, y = t^2 - 3$

40. $x = t^3 - 4t^2 + t + 7, y = t^2 - t$

41. $x = \cos(t), y = \sin(2t)$ on $[0, 2\pi]$

42. $x = \cos(t)\cos(3t), y = \sin(t)\cos(3t)$ on $[0, \pi]$

Exercise Group. In the following exercises, find the value(s) of t where the curve defined by the parametric equations is not smooth.

43. $x = t^3 + t^2 - t, y = t^2 + 2t + 3$

44. $x = t^2 - 4t, y = t^3 - 2t^2 - 4t$

45. $x = \cos(t), y = 2\cos(t)$

46. $x = 2\cos(t) - \cos(2t), y = 2\sin(t) - \sin(2t)$

Exercise Group. Find parametric equations that describe the given situation.

47. A projectile is fired from a height of 0 ft, landing 16 ft away in 4 s.

48. A projectile is fired from a height of 0 ft, landing 200 ft away in 4 s.

49. A projectile is fired from a height of 0 ft, landing 200 ft away in 20 s.

50. Find parametric equations that describe a circle of radius 2, centered at the origin, that is traced clockwise once at constant speed on $[0, 2\pi]$.

51. Find parametric equations that describe a circle of radius 3, centered at $(1, 1)$, that is traced once counter-clockwise at constant speed on $[0, 1]$.

52. Find parametric equations that describe an ellipse centered at $(1, 3)$, with vertical major axis of length 6 and minor axis of length 2.

53. An ellipse with foci at $(\pm 1, 0)$ and vertices at $(\pm 5, 0)$.

54. A hyperbola with foci at $(5, -3)$ and $(-1, -3)$, and with vertices at $(1, -3)$ and $(3, -3)$.

55. A hyperbola with vertices at $(0, \pm 6)$ and asymptotes $y = \pm 3x$.

10.3 Calculus and Parametric Equations

The previous section defined curves based on parametric equations. In this section we'll employ the techniques of calculus to study these curves.

We are still interested in lines tangent to points on a curve. They describe how the y-values are changing with respect to the x-values, they are useful in making approximations, and they indicate instantaneous direction of travel.

The slope of the tangent line is still $\frac{dy}{dx}$, and the Chain Rule allows us to calculate this in the context of parametric equations. If $x = f(t)$ and $y = g(t)$, the Chain Rule states that

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt} = \frac{g'(t)}{f'(t)},$$

provided that $f'(t) \neq 0$. This is important so we label it a Key Idea.

youtu.be/watch?v=FIvX66HAaj8

Figure 10.3.1 Video introduction to Section 10.3

Key Idea 10.3.2 Finding $\frac{dy}{dx}$ with Parametric Equations.

Let $x = f(t)$ and $y = g(t)$, where f and g are differentiable on some open interval I and $f'(t) \neq 0$ on I. Then

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}.$$

We use this to define the tangent line.

Definition 10.3.3 Tangent and Normal Lines.

Let a curve C be parametrized by $x = f(t)$ and $y = g(t)$, where f and g are differentiable functions on some interval I containing $t = t_0$. The **tangent line** to C at $t = t_0$ is the line through $\big(f(t_0), g(t_0)\big)$ with slope $m = g'(t_0)/f'(t_0)$, provided $f'(t_0) \neq 0$.
The **normal line** to C at $t = t_0$ is the line through $\big(f(t_0), g(t_0)\big)$ with slope $m = -f'(t_0)/g'(t_0)$, provided $g'(t_0) \neq 0$.

The definition leaves two special cases to consider. When the tangent line is horizontal, the normal line is undefined by the above definition as $g'(t_0) = 0$. Likewise, when the normal line is horizontal, the tangent line is undefined. It seems reasonable that these lines be defined (one can draw a line tangent to the "right side" of a circle, for instance), so we add the following to the above definition.

1. If the tangent line at $t = t_0$ has a slope of 0, the normal line to C at $t = t_0$ is the line $x = f(t_0)$.
2. If the normal line at $t = t_0$ has a slope of 0, the tangent line to C at $t = t_0$ is the line $x = f(t_0)$.

Example 10.3.4 Tangent and Normal Lines to Curves.

Let $x = 5t^2 - 6t + 4$ and $y = t^2 + 6t - 1$, and let C be the curve defined by these equations.

1. Find the equations of the tangent and normal lines to C at $t = 3$.

2. Find where C has vertical and horizontal tangent lines.

Solution.

1. We start by computing $f'(t) = 10t - 6$ and $g'(t) = 2t + 6$. Thus

$$\frac{dy}{dx} = \frac{2t+6}{10t-6}.$$

Make note of something that might seem unusual: $\frac{dy}{dx}$ is a function of t, not x. Just as points on the curve are found in terms of t, so are the slopes of the tangent lines. The point on C at $t = 3$ is $(31, 26)$. The slope of the tangent line is $m = 1/2$ and the slope of the normal line is $m = -2$. Thus,

- the equation of the tangent line is $y = \frac{1}{2}(x - 31) + 26$, and
- the equation of the normal line is $y = -2(x - 31) + 26$.

This is illustrated in Figure 10.3.5.

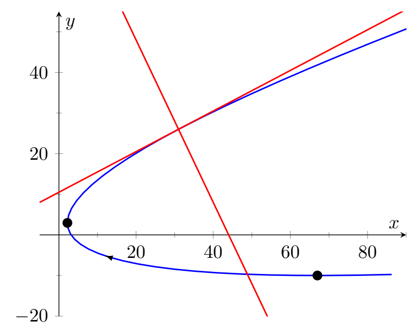


Figure 10.3.5 Graphing tangent and normal lines in Example 10.3.4

2. To find where C has a horizontal tangent line, we set $\frac{dy}{dx} = 0$ and solve for t. In this case, this amounts to setting $g'(t) = 0$ and solving for t (and making sure that $f'(t) \neq 0$).

$$g'(t) = 0 \Rightarrow 2t + 6 = 0 \Rightarrow t = -3.$$

The point on C corresponding to $t = -3$ is $(67, -10)$; the tangent line at that point is horizontal (hence with equation $y = -10$). To find where C has a vertical tangent line, we find where it has a horizontal normal line, and set $-\frac{f'(t)}{g'(t)} = 0$. This amounts to setting $f'(t) = 0$ and solving for t (and making sure that $g'(t) \neq 0$).

$$f'(t) = 0 \Rightarrow 10t - 6 = 0 \Rightarrow t = 0.6.$$

The point on C corresponding to $t = 0.6$ is $(2.2, 2.96)$. The tangent line at that point is $x = 2.2$. The points where the tangent lines are vertical and horizontal are indicated on the graph in Figure 10.3.5.

Video solution

youtu.be/watch?v=A62eUDxiOw4

Example 10.3.6 Tangent and Normal Lines to a Circle.

1. Find where the unit circle, defined by $x = \cos(t)$ and $y = \sin(t)$ on $[0, 2\pi]$, has vertical and horizontal tangent lines.
2. Find the equation of the normal line at $t = t_0$.

Solution.

1. We compute the derivative following Key Idea 10.3.2:

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} = -\frac{\cos(t)}{\sin(t)}.$$

The derivative is 0 when $\cos(t) = 0$; that is, when $t = \pi/2, 3\pi/2$. These are the points $(0, 1)$ and $(0, -1)$ on the circle. The normal line is horizontal (and hence, the tangent line is vertical) when $\sin(t) = 0$; that is, when $t = 0, \pi, 2\pi$, corresponding to the points $(-1,0)$ and $(0,1)$ on the circle. These results should make intuitive sense.

2. The slope of the normal line at $t = t_0$ is $m = \dfrac{\sin(t_0)}{\cos(t_0)} = \tan(t_0)$. This normal line goes through the point $(\cos(t_0), \sin(t_0))$, giving the line

$$\begin{aligned} y &= \frac{\sin(t_0)}{\cos(t_0)}(x - \cos(t_0)) + \sin(t_0) \\ &= (\tan(t_0))x, \end{aligned}$$

as long as $\cos(t_0) \neq 0$. It is an important fact to recognize that the normal lines to a circle pass through its center, as illustrated in Figure 10.3.7. Stated in another way, any line that passes through the center of a circle intersects the circle at right angles.

Video solution

youtu.be/watch?v=PpmsaMVJAZI

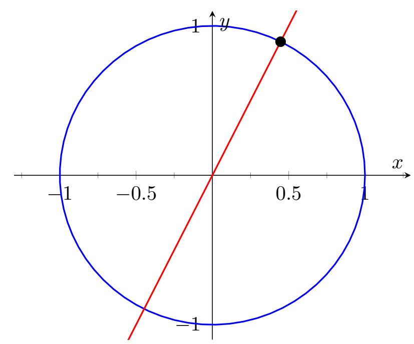


Figure 10.3.7 Illustrating how a circle's normal lines pass through its center

Example 10.3.8 Tangent lines when $\frac{dy}{dx}$ is not defined.

Find the equation of the tangent line to the astroid $x = \cos^3(t)$, $y = \sin^3(t)$ at $t = 0$, shown in Figure 10.3.9.

Solution. We start by finding $x'(t)$ and $y'(t)$:

$$x'(t) = -3\sin(t)\cos^2(t), \qquad y'(t) = 3\cos(t)\sin^2(t).$$

Note that both of these are 0 at $t = 0$; the curve is not smooth at $t = 0$ forming a cusp on the graph. Evaluating $\frac{dy}{dx}$ at this point returns the indeterminate form of "0/0".
We can, however, examine the slopes of tangent lines near $t = 0$, and take the limit as $t \to 0$.

$$\begin{aligned} \lim_{t\to 0} \frac{y'(t)}{x'(t)} &= \lim_{t\to 0} \frac{3\cos(t)\sin^2(t)}{-3\sin(t)\cos^2(t)} \quad \text{(We can cancel as } t \neq 0.\text{)} \\ &= \lim_{t\to 0} -\frac{\sin(t)}{\cos(t)} \\ &= 0. \end{aligned}$$

We have accomplished something significant. When the derivative $\frac{dy}{dx}$ returns an indeterminate form at $t = t_0$, we can define its value by setting it to be $\lim_{t\to t_0} \frac{dy}{dx}$, if that limit exists. This allows us to find slopes of tangent lines at cusps, which can be very beneficial.
We found the slope of the tangent line at $t = 0$ to be 0; therefore the tangent line is $y = 0$, the x-axis.

Video solution

youtu.be/watch?v=rb4wEkhcUtE

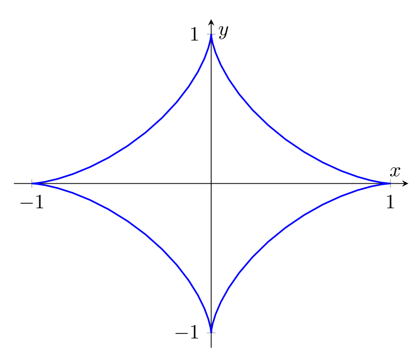


Figure 10.3.9 A graph of an astroid

10.3.1 Concavity

We continue to analyze curves in the plane by considering their concavity; that is, we are interested in $\frac{d^2y}{dx^2}$, "the second derivative of y with respect to x." To find this, we need to find the derivative of $\frac{dy}{dx}$ with respect to x; that is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right],$$

but recall that $\frac{dy}{dx}$ is a function of t, not x, making this computation not straightforward.

To make the upcoming notation a bit simpler, let $h(t) = \frac{dy}{dx}$. We want $\frac{d}{dx}[h(t)]$; that is, we want $\frac{dh}{dx}$. We again appeal to the Chain Rule. Note:

$$\frac{dh}{dt} = \frac{dh}{dx}\cdot\frac{dx}{dt} \Rightarrow \frac{dh}{dx} = \frac{dh}{dt}\Big/\frac{dx}{dt}.$$

In words, to find $\frac{d^2y}{dx^2}$, we first take the derivative of $\frac{dy}{dx}$ *with respect to* t, then divide by $x'(t)$. We restate this as a Key Idea.

Key Idea 10.3.10 Finding $\frac{d^2y}{dx^2}$ with Parametric Equations.

Let $x = f(t)$ and $y = g(t)$ be twice differentiable functions on an open interval I, where $f'(t) \neq 0$ on I. Then

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{dy}{dx}\right]\Big/\frac{dx}{dt} = \frac{d}{dt}\left[\frac{dy}{dx}\right]\Big/ f'(t).$$

Examples will help us understand this Key Idea.

Example 10.3.11 Concavity of Plane Curves.

Let $x = 5t^2 - 6t + 4$ and $y = t^2 + 6t - 1$ as in Example 10.3.4. Determine the t-intervals on which the graph is concave up/down.

Solution (a). Concavity is determined by the second derivative of y with respect to x, $\frac{d^2y}{dx^2}$, so we compute that here following Key Idea 10.3.10.

In Example 10.3.4, we found $\dfrac{dy}{dx} = \dfrac{2t+6}{10t-6}$ and $f'(t) = 10t - 6$. So:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dt}\left[\frac{2t+6}{10t-6}\right]\Big/(10t-6)\\ &= -\frac{72}{(10t-6)^2}\Big/(10t-6)\\ &= -\frac{72}{(10t-6)^3}\\ &= -\frac{9}{(5t-3)^3}\end{aligned}$$

The graph of the parametric functions is concave up when $\frac{d^2y}{dx^2} > 0$ and concave down when $\frac{d^2y}{dx^2} < 0$. We determine the intervals when the second derivative is greater/less than 0 by first finding when it is 0 or undefined.

As the numerator of $-\dfrac{9}{(5t-3)^3}$ is never 0, $\frac{d^2y}{dx^2} \neq 0$ for all t. It is undefined when $5t - 3 = 0$; that is, when $t = 3/5$. Following the work established in Section 3.4, we look at values of t greater/less than $3/5$ on a number line:

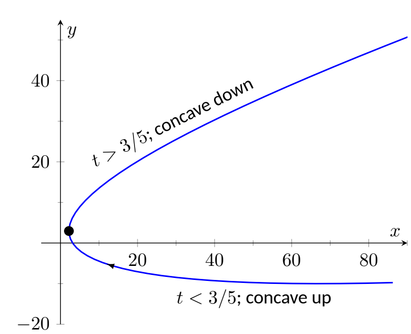
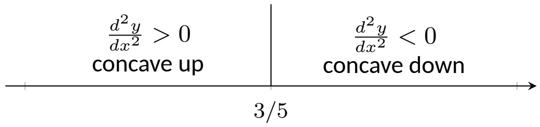


Figure 10.3.12 Graphing the parametric equations in Example 10.3.11 to demonstrate concavity



Reviewing Example 10.3.4, we see that when $t = 3/5 = 0.6$, the graph of the parametric equations has a vertical tangent line. This point is also a point of inflection for the graph, illustrated in Figure 10.3.12.
The video in Figure 10.3.13 shows how this information can be used to sketch the curve by hand.

Video solution

youtu.be/watch?v=ZUN_apodMWw

youtu.be/watch?v=HSBSVFSVqms

Figure 10.3.13 Sketching the curve in Example 10.3.11

Example 10.3.14 Concavity of Plane Curves.

Find the points of inflection of the graph of the parametric equations $x = \sqrt{t}$, $y = \sin(t)$, for $0 \le t \le 16$.

Solution. We need to compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

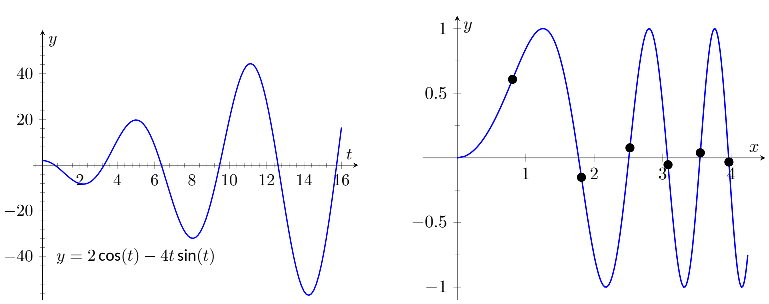
$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\cos(t)}{1/(2\sqrt{t})} = 2\sqrt{t}\cos(t).$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{x'(t)} = \frac{\cos(t)/\sqrt{t} - 2\sqrt{t}\sin(t)}{1/(2\sqrt{t})} = 2\cos(t) - 4t\sin(t).$$

The points of inflection are found by setting $\frac{d^2y}{dx^2} = 0$. This is not trivial, as equations that mix polynomials and trigonometric functions generally do not have "nice" solutions.
In Figure 10.3.15(a) we see a plot of the second derivative. It shows that it has zeros at approximately $t = 0.5, 3.5, 6.5, 9.5, 12.5$ and 16. These approximations are not very good, made only by looking at the graph. Newton's Method provides more accurate approximations. Accurate to 2 decimal places, we have:

$$t = 0.65,\ 3.29,\ 6.36,\ 9.48,\ 12.61 \text{ and } 15.74.$$

The corresponding points have been plotted on the graph of the parametric equations in Figure 10.3.15(b). Note how most occur near the x-axis, but not exactly on the axis.



(a) **(b)**

Figure 10.3.15 In (a), a graph of $\frac{d^2y}{dx^2}$, showing where it is approximately 0. In (b), graph of the parametric equations in Example 10.3.14 along with the points of inflection

10.3.2 Arc Length

We continue our study of the features of the graphs of parametric equations by computing their arc length.

Recall in Section 7.4 we found the arc length of the graph of a function, from $x = a$ to $x = b$, to be

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

We can use this equation and convert it to the parametric equation context. Letting $x = f(t)$ and $y = g(t)$, we know that $\frac{dy}{dx} = g'(t)/f'(t)$. It will also be useful to calculate the differential of x:

$$dx = f'(t)dt \qquad \Rightarrow \qquad dt = \frac{1}{f'(t)} \cdot dx.$$

Starting with the arc length formula above, consider:

$$\begin{aligned} L &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_a^b \sqrt{1 + \frac{g'(t)^2}{f'(t)^2}}\, dx. \end{aligned}$$

Factor out the $f'(t)^2$:

$$\begin{aligned} &= \int_a^b \sqrt{f'(t)^2 + g'(t)^2} \cdot \underbrace{\frac{1}{f'(t)}\, dx}_{=dt} \\ &= \int_{t_1}^{t_2} \sqrt{f'(t)^2 + g'(t)^2}\, dt. \end{aligned}$$

Note the new bounds (no longer "x" bounds, but "t" bounds). They are found by finding t_1 and t_2 such that $a = f(t_1)$ and $b = f(t_2)$. This formula is important, so we restate it as a theorem.

youtu.be/watch?v=57F7ZspP0lU

Figure 10.3.16 Video introduction to arc length for parametric curves

Theorem 10.3.17 Arc Length of Parametric Curves.

Let $x = f(t)$ and $y = g(t)$ be parametric equations with f' and g' continuous on $[t_1, t_2]$, on which the graph traces itself only once. The arc length of the graph, from $t = t_1$ to $t = t_2$, is

$$L = \int_{t_1}^{t_2} \sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

Note: Theorem 10.3.17 makes use of differentiability on closed intervals, just as was done in Section 7.4.

As before, these integrals are often not easy to compute. We start with a simple example, then give another where we approximate the solution.

Example 10.3.18 Arc Length of a Circle.

Find the arc length of the circle parametrized by $x = 3\cos(t)$, $y = 3\sin(t)$ on $[0, 3\pi/2]$.

Video solution

youtu.be/watch?v=It8vHw3RHEw

Solution. By direct application of Theorem 10.3.17, we have

$$L = \int_0^{3\pi/2} \sqrt{(-3\sin(t))^2 + (3\cos(t))^2}\, dt.$$

Apply the Pythagorean Theorem.

$$= \int_0^{3\pi/2} 3\, dt$$
$$= 3t\Big|_0^{3\pi/2} = 9\pi/2.$$

This should make sense; we know from geometry that the circumference of a circle with radius 3 is 6π; since we are finding the arc length of $3/4$ of a circle, the arc length is $3/4 \cdot 6\pi = 9\pi/2$.

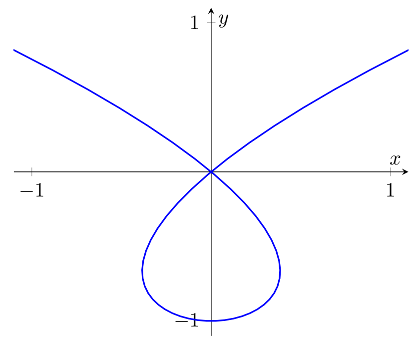


Figure 10.3.20 A graph of the parametric equations in Example 10.3.19, where the arc length of the teardrop is calculated

Example 10.3.19 Arc Length of a Parametric Curve.

The graph of the parametric equations $x = t(t^2-1)$, $y = t^2-1$ crosses itself as shown in Figure 10.3.20, forming a "teardrop." Find the arc length of the teardrop.

Solution. We can see by the parametrizations of x and y that when $t = \pm 1$, $x = 0$ and $y = 0$. This means we'll integrate from $t = -1$ to $t = 1$. Applying Theorem 10.3.17, we have

$$L = \int_{-1}^{1} \sqrt{(3t^2-1)^2 + (2t)^2}\, dt$$
$$= \int_{-1}^{1} \sqrt{9t^4 - 2t^2 + 1}\, dt.$$

Unfortunately, the integrand does not have an antiderivative expressible by elementary functions. We turn to numerical integration to approximate its value. Using 4 subintervals, Simpson's Rule approximates the value of the integral as 2.65051. Using a computer, more subintervals are easy to employ, and $n = 20$ gives a value of 2.71559. Increasing n shows that this value is stable and a good approximation of the actual value.

Video solution

youtu.be/watch?v=G5U9BVhB3PE

10.3.3 Surface Area of a Solid of Revolution

Related to the formula for finding arc length is the formula for finding surface area. We can adapt the formula found in Theorem 7.4.13 from Section 7.4 in a similar way as done to produce the formula for arc length done before.

Theorem 10.3.21 Surface Area of a Solid of Revolution.

Consider the graph of the parametric equations $x = f(t)$ and $y = g(t)$, where f' and g' are continuous on an open interval I containing t_1 and t_2 on which the graph does not cross itself.

1. *The surface area of the solid formed by revolving the graph about*

the x-axis is (where $g(t) \geq 0$ on $[t_1, t_2]$):

$$\text{Surface Area } = 2\pi \int_{t_1}^{t_2} g(t)\sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

2. *The surface area of the solid formed by revolving the graph about the y-axis is (where $f(t) \geq 0$ on $[t_1, t_2]$):*

$$\text{Surface Area } = 2\pi \int_{t_1}^{t_2} f(t)\sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

Example 10.3.22 Surface Area of a Solid of Revolution.

Consider the teardrop shape formed by the parametric equations $x = t(t^2-1)$, $y = t^2-1$ as seen in Example 10.3.19. Find the surface area if this shape is rotated about the x-axis, as shown in Figure 10.3.23.

Solution. The teardrop shape is formed between $t = -1$ and $t = 1$. Using Theorem 10.3.21, we see we need for $g(t) \geq 0$ on $[-1, 1]$, and this is not the case. To fix this, we simplify replace $g(t)$ with $-g(t)$, which flips the whole graph about the x-axis (and does not change the surface area of the resulting solid). The surface area is:

$$\begin{aligned}\text{Area } S &= 2\pi \int_{-1}^{1} (1-t^2)\sqrt{(3t^2-1)^2 + (2t)^2}\, dt \\ &= 2\pi \int_{-1}^{1} (1-t^2)\sqrt{9t^4 - 2t^2 + 1}\, dt.\end{aligned}$$

Once again we arrive at an integral that we cannot compute in terms of elementary functions. Using Simpson's Rule with $n = 20$, we find the area to be $S = 9.44$. Using larger values of n shows this is accurate to 2 places after the decimal.

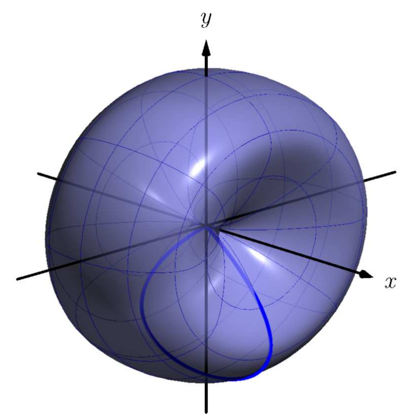


Figure 10.3.23 Rotating a teardrop shape about the x-axis in Example 10.3.22

After defining a new way of creating curves in the plane, in this section we have applied calculus techniques to the parametric equation defining these curves to study their properties. In the next section, we define another way of forming curves in the plane. To do so, we create a new coordinate system, called *polar coordinates*, that identifies points in the plane in a manner different than from measuring distances from the y- and x- axes.

10.3.4 Exercises

Terms and Concepts

1. True or False? Given parametric equations $x = f(t)$ and $y = g(t)$, $\frac{dy}{dx} = f'(t)/g'(t)$, as long as $g'(t) \neq 0$. (☐ True ☐ False)

2. Given parametric equations $x = f(t)$ and $y = g(t)$, the derivative $\frac{dy}{dx}$ as given in Key Idea 10.3.2 is a function of ___________?

3. True or False? Given parametric equations $x = f(t)$ and $y = g(t)$, to find $\frac{d^2y}{dx^2}$, one simply computes $\frac{d}{dt}\left(\frac{dy}{dx}\right)$. (☐ True ☐ False)

4. True or False? If $\frac{dy}{dx} = 0$ at $t = t_0$, then the normal line to the curve at $t = t_0$ is a vertical line. (☐ True ☐ False)

Problems

Exercise Group. In the following exercises, parametric equations for a curve are given.

(a) Find $\dfrac{dy}{dx}$.

(b) Find the equations of the tangent and normal line(s) at the point(s) given.

(c) Sketch the graph of the parametric functions along with the found tangent and normal lines.

5. $x = t$, $y = t^2$;$t = 1$

6. $x = \sqrt{t}$, $y = 5t + 2$;$t = 4$

7. $x = t^2 - t$, $y = t^2 + t$;$t = 1$

8. $x = t^2 - 1$, $y = t^3 - t$;$t = 0$ and $t = 1$

9. $x = \sec(t)$, $y = \tan(t)$ on $(-\pi/2, \pi/2)$;$t = \pi/4$

10. $x = \cos(t)$, $y = \sin(2t)$ on $[0, 2\pi]$;$t = \pi/4$

11. $x = \cos(t)\sin(2t)$, $y = \sin(t)\sin(2t)$ on $[0, 2\pi]$; $t = 3\pi/4$

12. $x = e^{t/10}\cos(t)$, $y = e^{t/10}\sin(t)$; $t = \pi/2$

Exercise Group. Find the t-values where the curve defined by the given parametric equations has a horizontal tangent line. Note: these are the same equations as in Exercises 5–12.

13. $x = t$, $y = t^2$

14. $x = \sqrt{t}$, $y = 5t + 2$

15. $x = t^2 - t$, $y = t^2 + t$

16. $x = t^2 - 1$, $y = t^3 - t$

17. $x = \sec(t)$, $y = \tan(t)$ on $(-\pi/2, \pi/2)$

18. $x = \cos(t)$, $y = \sin(2t)$, on $[0, 2\pi)$

19. $x = \cos(t)\sin(2t)$, $y = \sin(t)\sin(2t)$ on $[0, 2\pi]$

20. $x = e^{t/10}\cos(t)$, $y = e^{t/10}\sin(t)$

Exercise Group. Find the point $t = t_0$ where the graph of the given parametric equations is not smooth, then find $\lim\limits_{t\to t_0} \frac{dy}{dx}$.

21. $x = \frac{1}{t^2+1}$, $y = t^3$

22. $x = -t^3 + 7t^2 - 16t + 13$, $y = t^3 - 5t^2 + 8t - 2$

23. $x = t^3 - 3t^2 + 3t - 1$,$y = t^2 - 2t + 1$

24. $x = \cos^2(t)$,$y = 1 - \sin^2(t)$

Exercise Group. For the given parametric equations for a curve, find $\frac{d^2y}{dx^2}$, then determine the intervals on which the graph of the curve is concave up/down. Note: these are the same equations as in Exercises 5–12.

25. $x = t$,$y = t^2$

26. $x = \sqrt{t}$,$y = 5t + 2$

27. $x = t^2 - t$, $y = t^2 + t$

28. $x = t^2 - 1$,$y = t^3 - t$

29. $x = \sec(t)$,$y = \tan(t)$ on $(-\pi/2, \pi/2)$

30. $x = \cos(t)$, $y = \sin(2t)$, on $[0, 2\pi)$

31. $x = \cos(t)\sin(2t)$,$y = \sin(t)\sin(2t)$ on $[-\pi/2, \pi/2]$

32. $x = e^{t/10}\cos(t)$,$y = e^{t/10}\sin(t)$

Exercise Group. Find the arc length of the graph of the parametric equations on the given interval(s).

33. $x = -3\sin(2t)$, $y = 3\cos(2t)$ on $[0, \pi]$

34. $x = e^{t/10}\cos(t)$, $y = e^{t/10}\sin(t)$ on $[0, 2\pi]$ and $[2\pi, 4\pi]$.

35. $x = 5t + 2$, $y = 1 - 3t$ on $[-1, 1]$

36. $x = 2t^{3/2}$,$y = 3t$ on $[0, 1]$

Exercise Group. In the following exercises, numerically approximate the given arc length.

37. Approximate the arc length of one petal of the rose curve $x = \cos(t)\cos(2t)$,$y = \sin(t)\cos(2t)$ using Simpson's Rule and $n = 4$.

38. Approximate the arc length of the "bow tie curve" $x = \cos(t)$,$y = \sin(2t)$ using Simpson's Rule and $n = 6$.

39. Approximate the arc length of the parabola $x = t^2 - t$,$y = t^2 + t$ on $[-1, 1]$ using Simpson's Rule and $n = 4$.

40. A common approximate of the circumference of an ellipse given by $x = a\cos(t)$,$y = b\sin(t)$ is $C \approx 2\pi\sqrt{\frac{a^2+b^2}{2}}$. Use this formula to approximate the circumference of $x = 5\cos(t)$, $y = 3\sin(t)$ and compare this to the approximation given by Simpson's Rule and $n = 6$.

Exercise Group. In the following exercises, a solid of revolution is described. Find or approximate its surface area as specified.

41. Find the surface area of the sphere formed by rotating the circle $x = 2\cos(t)$,$y = 2\sin(t)$ about:

- **(a)** The x-axis.
- **(b)** The y-axis.

42. Find the surface area of the torus (or "donut") formed by rotating the circle $x = \cos(t) + 2$,$y = \sin(t)$ about the y-axis.

43. Approximate the surface area of the solid formed by rotating the "upper right half" of the bow tie curve $x = \cos(t)$,$y = \sin(2t)$ on $[0, \pi/2]$ about the x-axis, using Simpson's Rule and $n = 4$.

44. Approximate the surface area of the solid formed by rotating the one petal of the rose curve $x = \cos(t)\cos(2t)$,$y = \sin(t)\cos(2t)$ on $[0, \pi/4]$ about the x-axis, using Simpson's Rule and $n = 4$.

10.4 Introduction to Polar Coordinates

youtu.be/watch?v=pyS8sweaJ9w

Figure 10.4.1 Video introduction to Section 10.4

We are generally introduced to the idea of graphing curves by relating x-values to y-values through a function f. That is, we set $y = f(x)$, and plot lots of point pairs (x, y) to get a good notion of how the curve looks. This method is useful but has limitations, not least of which is that curves that "fail the vertical line test" cannot be graphed without using multiple functions.

The previous two sections introduced and studied a new way of plotting points in the x, y-plane. Using parametric equations, x and y values are computed independently and then plotted together. This method allows us to graph an extraordinary range of curves. This section introduces yet another way to plot points in the plane: using *polar coordinates*.

10.4.1 Polar Coordinates

Start with a point O in the plane called the *pole* (we will always identify this point with the origin). From the pole, draw a ray, called the *initial ray* (we will always draw this ray horizontally, identifying it with the positive x-axis). A point P in the plane is determined by the distance r that P is from O, and the angle θ formed between the initial ray and the segment $\overline{OP}$ (measured counter-clockwise). We record the distance and angle as an ordered pair (r, θ). To avoid confusion with rectangular coordinates, we will denote polar coordinates with the letter P, as in $P(r, \theta)$. This is illustrated in Figure 10.4.2

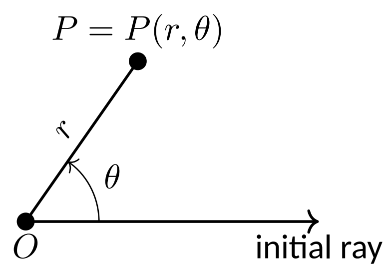


Figure 10.4.2 Illustrating polar coordinates

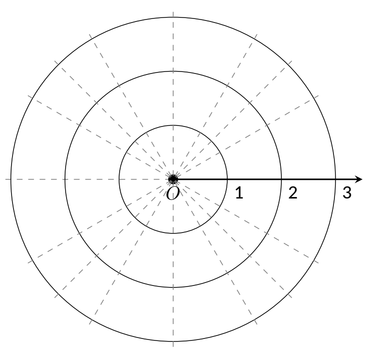
Practice will make this process more clear.

Example 10.4.3 Plotting Polar Coordinates.

Plot the following polar coordinates:

$$A = P(1, \pi/4) B = P(1.5, \pi) C = P(2, -\pi/3) D = P(-1, \pi/4)$$

Solution. To aid in the drawing, a polar grid is provided below. To place the point A, go out 1 unit along the initial ray (putting you on the inner circle shown on the grid), then rotate counter-clockwise $\pi/4$ radians (or $45°$). Alternately, one can consider the rotation first: think about the ray from O that forms an angle of $\pi/4$ with the initial ray, then move out 1 unit along this ray (again placing you on the inner circle of the grid).



To plot B, go out 1.5 units along the initial ray and rotate π radians $(180°)$.
To plot C, go out 2 units along the initial ray then rotate *clockwise* $\pi/3$ radians, as the angle given is negative.
To plot D, move along the initial ray "-1" units — in other words, "back up" 1 unit, then rotate counter-clockwise by $\pi/4$. The results are given in Figure 10.4.4.

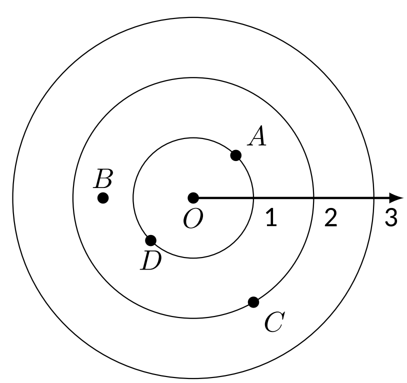


Figure 10.4.4 Plotting polar points in Example 10.4.3

Consider the following two points: $A = P(1, \pi)$ and $B = P(-1, 0)$. To locate A, go out 1 unit on the initial ray then rotate π radians; to locate B, go out -1 units on the initial ray and don't rotate. One should see that A and B are located at the same point in the plane. We can also consider $C = P(1, 3\pi)$, or $D = P(1, -\pi)$; all four of these points share the same location.

This ability to identify a point in the plane with multiple polar coordinates is both a "blessing" and a "curse." We will see that it is beneficial as we can plot beautiful functions that intersect themselves (much like we saw with parametric functions). The unfortunate part of this is that it can be difficult to determine when this happens. We'll explore this more later in this section.

10.4.2 Polar to Rectangular Conversion

It is useful to recognize both the rectangular (or, Cartesian) coordinates of a point in the plane and its polar coordinates. Figure 10.4.5 shows a point P in the plane with rectangular coordinates (x, y) and polar coordinates $P(r, \theta)$. Using trigonometry, we can make the identities given in the following Key Idea.

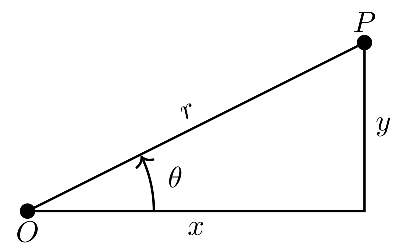


Figure 10.4.5 Converting between rectangular and polar coordinates

Key Idea 10.4.6 Converting Between Rectangular and Polar Coordinates.

Given the polar point $P(r, \theta)$, the rectangular coordinates are determined by

$$x = r\cos(\theta) \qquad y = r\sin(\theta).$$

Given the rectangular coordinates (x, y), the polar coordinates are determined by

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}.$$

Example 10.4.7 Converting Between Polar and Rectangular Coordinates.

1. Convert the polar coordinates $P(2, 2\pi/3)$ and $P(-1, 5\pi/4)$ to rectangular coordinates.

2. Convert the rectangular coordinates $(1, 2)$ and $(-1, 1)$ to polar coordinates.

Solution.

1. (a) We start with $P(2, 2\pi/3)$. Using Key Idea 10.4.6, we have

$$x = 2\cos(2\pi/3) = -1 \qquad y = 2\sin(2\pi/3) = \sqrt{3}.$$

So the rectangular coordinates are $(-1, \sqrt{3}) \approx (-1, 1.732)$.

(b) The polar point $P(-1, 5\pi/4)$ is converted to rectangular with:

$$x = -1\cos(5\pi/4) = \sqrt{2}/2 \qquad y = -1\sin(5\pi/4) = \sqrt{2}/2.$$

So the rectangular coordinates are $(\sqrt{2}/2, \sqrt{2}/2) \approx (0.707, 0.707)$.

These points are plotted in Figure 10.4.8(a). The rectangular coordinate system is drawn lightly under the polar coordinate system so that the relationship between the two can be seen.

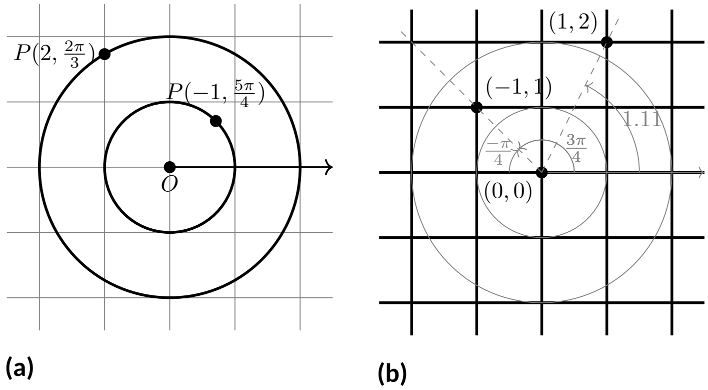


Figure 10.4.8 Plotting rectangular and polar points in Example 10.4.7

2. (a) To convert the rectangular point $(1, 2)$ to polar coordinates, we use the Key Idea to form the following two equations:

$$1^2 + 2^2 = r^2 \qquad \tan(\theta) = \frac{2}{1}.$$

The first equation tells us that $r = \sqrt{5}$. Using the inverse tangent function, we find

$$\tan(\theta) = 2 \Rightarrow \theta = \tan^{-1}(2) \approx 1.11 \approx 63.43°.$$

Thus polar coordinates of $(1, 2)$ are $P(\sqrt{5}, 1.11)$.

(b) To convert $(-1, 1)$ to polar coordinates, we form the equations

$$(-1)^2 + 1^2 = r^2 \qquad \tan(\theta) = \frac{1}{-1}.$$

Thus $r = \sqrt{2}$. We need to be careful in computing θ: using the inverse tangent function, we have

$$\tan(\theta) = -1 \Rightarrow \theta = \tan^{-1}(-1) = -\pi/4 = -45°.$$

This is not the angle we desire. The range of $\tan^{-1}(x)$ is $(-\pi/2, \pi/2)$; that is, it returns angles that lie in the 1st and 4th quadrants. To find locations in the 2nd and 3rd quadrants, add π to the result of $\tan^{-1}(x)$. So $\pi + (-\pi/4)$ puts the angle at $3\pi/4$. Thus the polar point is $P(\sqrt{2}, 3\pi/4)$. An alternate method is to use the angle θ given by arctangent, but change the sign of r. Thus we could also refer to $(-1, 1)$ as $P(-\sqrt{2}, -\pi/4)$.

These points are plotted in Figure 10.4.8(b). The polar system is drawn lightly under the rectangular grid with rays to demonstrate the angles used.

10.4.3 Polar Functions and Polar Graphs

Defining a new coordinate system allows us to create a new kind of function, a *polar function*. Rectangular coordinates lent themselves well to creating functions that related x and y, such as $y = x^2$. Polar coordinates allow us to create functions that relate r and θ. Normally these functions look like $r = f(\theta)$, although we can create functions of the form $\theta = f(r)$. The following examples introduce us to this concept.

Example 10.4.9 Introduction to Graphing Polar Functions.

Describe the graphs of the following polar functions.

1. $r = 1.5$
2. $\theta = \pi/4$

Solution.

1. The equation $r = 1.5$ describes all points that are 1.5 units from the pole; as the angle is not specified, any θ is allowable. All points 1.5 units from the pole describes a circle of radius 1.5. We can consider the rectangular equivalent of this equation; using $r^2 = x^2 + y^2$, we see that $1.5^2 = x^2 + y^2$, which we recognize as the equation of a circle centered at $(0, 0)$ with radius 1.5. This is sketched in Figure 10.4.10.

2. The equation $\theta = \pi/4$ describes all points such that the line through them and the pole make an angle of $\pi/4$ with the initial ray. As the radius r is not specified, it can be any value (even negative). Thus $\theta = \pi/4$ describes the line through the pole that makes an angle of $\pi/4 = 45°$ with the initial ray. We can again consider the rectangular equivalent of this equation. Combine $\tan(\theta) = y/x$ and $\theta = \pi/4$:

$$\tan(\pi)/4 = y/x \Rightarrow x\tan(\pi)/4 = y \Rightarrow y = x.$$

This graph is also plotted in Figure 10.4.10.

Video solution

youtu.be/watch?v=HTCbzFnW9KU

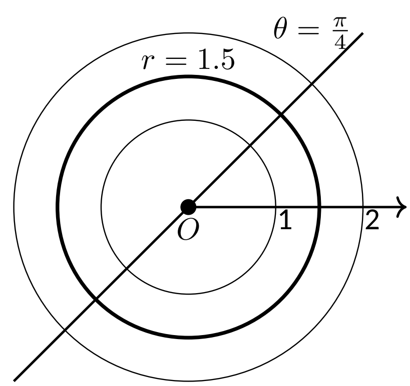


Figure 10.4.10 Plotting standard polar plots

The basic rectangular equations of the form $x = h$ and $y = k$ create vertical and horizontal lines, respectively; the basic polar equations $r = h$ and $\theta = \alpha$ create circles and lines through the pole, respectively. With this as a foundation, we can create more complicated polar functions of the form $r = f(\theta)$. The input is an angle; the output is a length, how far in the direction of the angle to go out.

We sketch these functions much like we sketch rectangular and parametric functions: we plot lots of points and "connect the dots" with curves. We demonstrate this in the following example.

Example 10.4.11 Sketching Polar Functions.

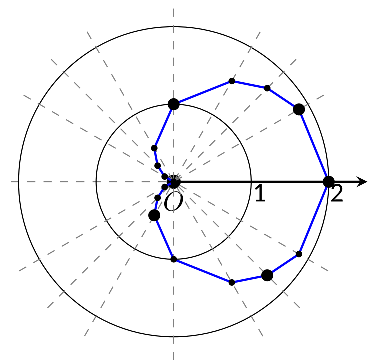
Sketch the polar function $r = 1 + \cos(\theta)$ on $[0, 2\pi]$ by plotting points.

Solution. A common question when sketching curves by plotting points is "Which points should I plot?" With rectangular equations, we often choose "easy" values — integers, then add more if needed. When plotting polar equations, start with the "common" angles — multiples of $\pi/6$ and $\pi/4$. Figure 10.4.12 gives a table of just a few values of θ in $[0, \pi]$. Consider the point $P(2, 0)$ determined by the first line of the table. The angle is 0 radians — we do not rotate from the initial ray — then we go out 2 units from the pole. When $\theta = \pi/6$, $r = 1.866$ (actually, it is

$1+\sqrt{3}/2$); so rotate by $\pi/6$ radians and go out 1.866 units.
The graph shown uses more points, connected with straight lines. (The points on the graph that correspond to points in the table are signified with larger dots.) Such a sketch is likely good enough to give one an idea of what the graph looks like.

θ	$r = 1+\cos(\theta)$
0	2
$\pi/6$	1.86603
$\pi/2$	1
$4\pi/3$	0.5
$7\pi/4$	1.70711

(a)



(b)

Figure 10.4.12 Graphing a polar function in Example 10.4.11 by plotting points

Video solution

youtu.be/watch?v=1omaozpN7wI

Technology Note: Plotting functions in this way can be tedious, just as it was with rectangular functions. To obtain very accurate graphs, technology is a great aid. Most graphing calculators can plot polar functions; in the menu, set the plotting mode to something like `polar` or `POL`, depending on one's calculator. As with plotting parametric functions, the viewing "window" no longer determines the x-values that are plotted, so additional information needs to be provided. Often with the "window" settings are the settings for the beginning and ending θ values (often called θ_{min} and θ_{max}) as well as the θ_{step} — that is, how far apart the θ values are spaced. The smaller the θ_{step} value, the more accurate the graph (which also increases plotting time). Using technology, we graphed the polar function $r = 1+\cos(\theta)$ from Example 10.4.11 in Figure 10.4.13.

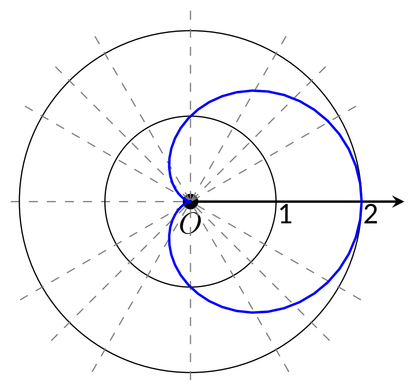


Figure 10.4.13 Using technology to graph a polar function

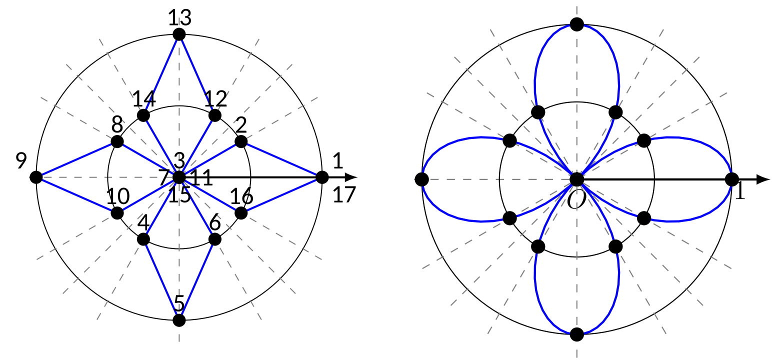
Example 10.4.14 Sketching Polar Functions.

Sketch the polar function $r = \cos(2\theta)$ on $[0, 2\pi]$ by plotting points.

Solution. We start by making a table of $\cos(2\theta)$ evaluated at common angles θ, as shown in Figure 10.4.15. These points are then plotted in Figure 10.4.16(a). This particular graph "moves" around quite a bit and one can easily forget which points should be connected to each other. To help us with this, we numbered each point in the table and on the graph.
Using more points (and the aid of technology) a smoother plot can be made as shown in Figure 10.4.16(b). This plot is an example of a *rose curve*.

Pt.	θ	$\cos(2\theta)$
1	0	1
2	$\pi/6$	0.5
3	$\pi/4$	0
4	$\pi/3$	-0.5
5	$\pi/2$	-1
6	$2\pi/3$	-0.5
7	$3\pi/4$	0
8	$5\pi/6$	0.5
9	π	1
10	$7\pi/6$	0.5
11	$5\pi/4$	0
12	$4\pi/3$	-0.5
13	$3\pi/2$	-1
14	$5\pi/3$	-0.5
15	$7\pi/4$	0
16	$11\pi/6$	0.5
17	2π	1

Figure 10.4.15 Table of points for plotting a polar curve in Example 10.4.14



(a) **(b)**

Figure 10.4.16 Polar plots from Example 10.4.14

Video solution

youtu.be/watch?v=DSmu6HQXiS4

It is sometimes desirable to refer to a graph via a polar equation, and other times by a rectangular equation. Therefore it is necessary to be able to convert between polar and rectangular functions, which we practice in the following example. We will make frequent use of the identities found in Key Idea 10.4.6.

Example 10.4.17 Converting between rectangular and polar equations.

Convert from rectangular to polar.

1. $y = x^2$
2. $xy = 1$

Convert from polar to rectangular.

1. $r = \dfrac{2}{\sin(\theta) - \cos(\theta)}$
2. $r = 2\cos(\theta)$

Solution.

1. Replace y with $r\sin(\theta)$ and replace x with $r\cos(\theta)$, giving:

$$\begin{aligned} y &= x^2 \\ r\sin(\theta) &= r^2\cos^2(\theta) \\ \frac{\sin(\theta)}{\cos^2(\theta)} &= r \end{aligned}$$

   We have found that $r = \sin(\theta)/\cos^2(\theta) = \tan(\theta)\sec(\theta)$. The domain of this polar function is $(-\pi/2, \pi/2)$; plot a few points to see how the familiar parabola is traced out by the polar equation.

2. We again replace x and y using the standard identities and work to solve for r:

$$\begin{aligned} xy &= 1 \\ r\cos(\theta)\cdot r\sin(\theta) &= 1 \\ r^2 &= \frac{1}{\cos(\theta)\sin(\theta)} \\ r &= \frac{1}{\sqrt{\cos(\theta)\sin(\theta)}} \end{aligned}$$

   This function is valid only when the product of $\cos(\theta)\sin(\theta)$ is positive. This occurs in the first and third quadrants, meaning the domain of this polar function is $(0, \pi/2) \cup (\pi, 3\pi/2)$. We can rewrite the original rectangular equation $xy = 1$ as $y = 1/x$. This is graphed in Figure 10.4.18; note how it only exists in the first and third quadrants.

3. There is no set way to convert from polar to rectangular; in general, we look to form the products $r\cos(\theta)$ and $r\sin(\theta)$, and then replace these with x and y, respectively. We start in this problem by multiplying both sides by $\sin(\theta) - \cos(\theta)$:

$$\begin{aligned} r &= \frac{2}{\sin(\theta)-\cos(\theta)} \\ r(\sin(\theta)-\cos(\theta)) &= 2 \\ r\sin(\theta) - r\cos(\theta) &= 2. \qquad \text{Now replace with } y \text{ and } x\text{:} \\ y - x &= 2 \\ y &= x + 2. \end{aligned}$$

The original polar equation, $r = 2/(\sin(\theta) - \cos(\theta))$ does not easily reveal that its graph is simply a line. However, our conversion shows that it is. The upcoming gallery of polar curves gives the general equations of lines in polar form.

4. By multiplying both sides by r, we obtain both an r^2 term and an $r\cos(\theta)$ term, which we replace with $x^2 + y^2$ and x, respectively.

$$\begin{aligned} r &= 2\cos(\theta) \\ r^2 &= 2r\cos(\theta) \\ x^2 + y^2 &= 2x. \end{aligned}$$

We recognize this as a circle; by completing the square we can find its radius and center.

$$\begin{aligned} x^2 - 2x + y^2 &= 0 \\ (x-1)^2 + y^2 &= 1. \end{aligned}$$

The circle is centered at $(1, 0)$ and has radius 1. The upcoming gallery of polar curves gives the equations of *some* circles in polar form; circles with arbitrary centers have a complicated polar equation that we do not consider here.

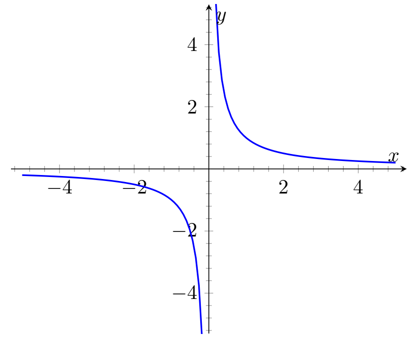


Figure 10.4.18 Graphing $xy = 1$ from Example 10.4.17

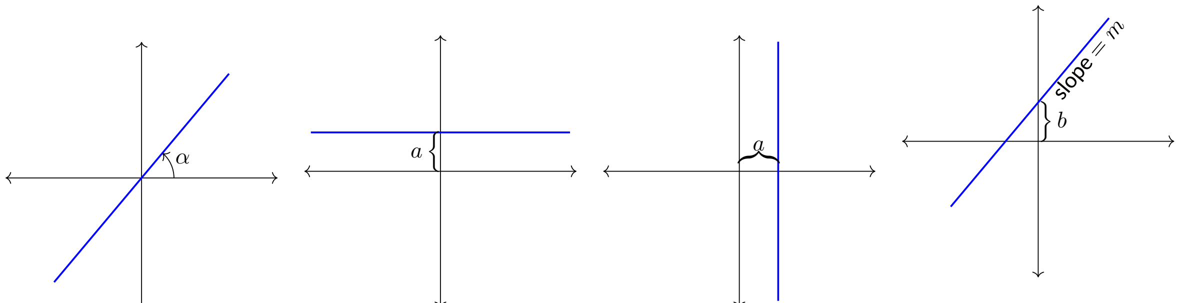
Video solution

youtu.be/watch?v=kWnHXtXTzSw

Some curves have very simple polar equations but rather complicated rectangular ones. For instance, the equation $r = 1 + \cos(\theta)$ describes a *cardioid* (a shape important the sensitivity of microphones, among other things; one is graphed in the gallery in the Limaçon section). It's rectangular form is not nearly as simple; it is the implicit equation $x^4 + y^4 + 2x^2y^2 - 2xy^2 - 2x^3 - y^2 = 0$. The conversion is not "hard," but takes several steps, and is left as a problem in the Exercise section.

Gallery of Polar Curves

There are a number of basic and "classic" polar curves, famous for their beauty and/or applicability to the sciences. This section ends with a small gallery of some of these graphs. We encourage the reader to understand how these graphs are formed, and to investigate with technology other types of polar functions.



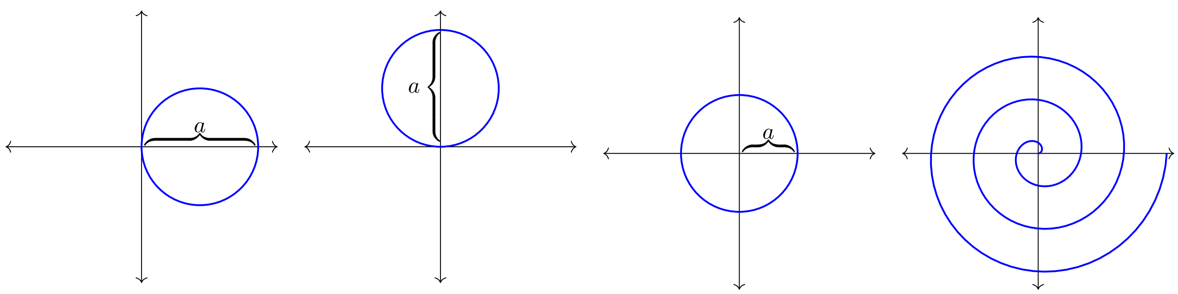
(a) Through the origin: $\theta = \alpha$

(b) Horizontal line: $r = a\csc(\theta)$

(c) Vertical line: $r = a\sec(\theta)$

(d) Not through origin: $r = \dfrac{b}{\sin(\theta) - m\cos(\theta)}$

Figure 10.4.19 Lines in polar coordinates



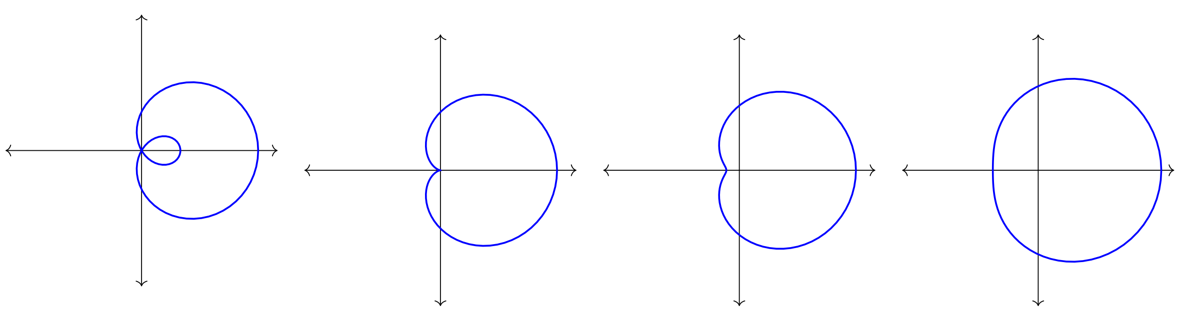
(a) Centered on x-axis: $r = a\cos(\theta)$

(b) Centered on y-axis: $r = a\sin(\theta)$

(c) Centered on origin: $r = a$

(d) Archimedean spiral: $r = \theta$

Figure 10.4.20 Circles and Spirals



(a) With inner loop: $\dfrac{a}{b} < 1$

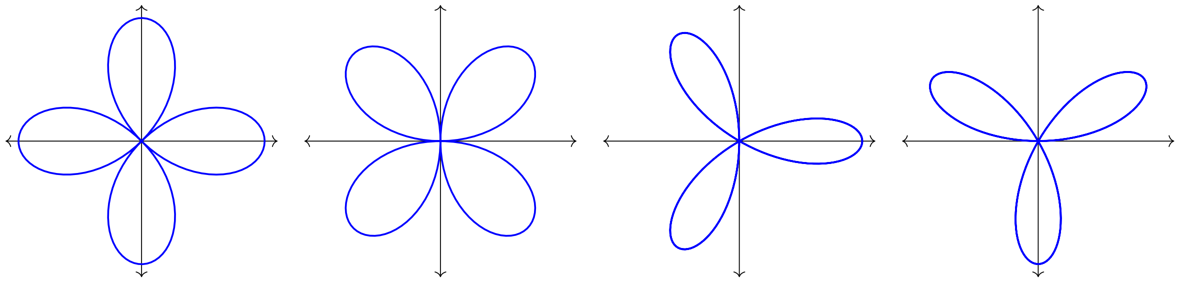
(b) Cardioid: $\dfrac{a}{b} = 1$

(c) Dimpled: $1 < \dfrac{a}{b} < 2$

(d) Convex: $\dfrac{a}{b} > 2$

Figure 10.4.21 Limaçons

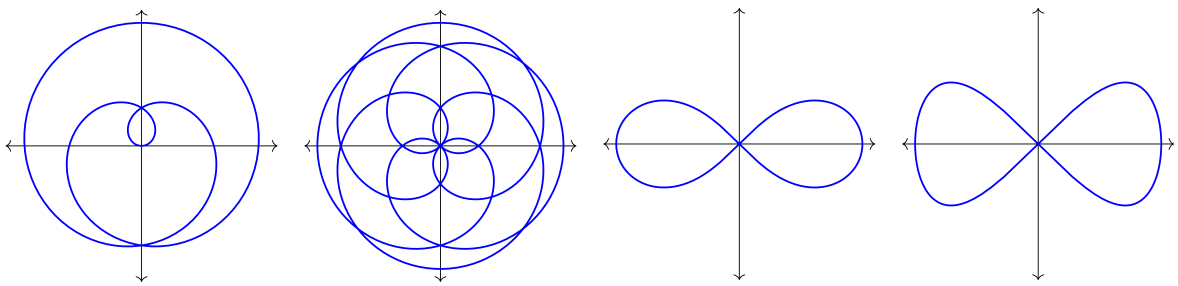
Symmetric about x-axis: $r = a \pm b\cos(\theta)$
Symmetric about y-axis: $r = a \pm b\sin(\theta)$; $a, b > 0$



(a) $r = a\cos(2\theta)$ **(b)** $r = a\sin(2\theta)$ **(c)** $r = a\cos(3\theta)$ **(d)** $r = a\sin(3\theta)$

Figure 10.4.22 Rose curves

Symmetric about x-axis: $r = a\cos(n\theta)$
Symmetric about y-axis: $r = a\sin(n\theta)$
Curve contains $2n$ petals when n is even and n petals when n is odd.



(a) Rose curve: $r = a\sin(\theta/5)$ **(b)** Rose curve: $r = a\sin(2\theta/5)$ **(c)** Lemniscate: $r^2 = a^2\cos(2\theta)$ **(d)** Eight Curve: $r^2 = a^2\sec^4(\theta)\cos(2\theta)$

Figure 10.4.23 Special Curves

Earlier we discussed how each point in the plane does not have a unique representation in polar form. This can be a "good" thing, as it allows for the beautiful and interesting curves seen in the preceding gallery. However, it can also be a "bad" thing, as it can be difficult to determine where two curves intersect.

Example 10.4.24 Finding points of intersection with polar curves.

Determine where the graphs of the polar equations $r = 1 + 3\cos(\theta)$ and $r = \cos(\theta)$ intersect.

Solution. As technology is generally readily available, it is usually a good idea to start with a graph. We have graphed the two functions in Figure 10.4.25(a); to better discern the intersection points, Figure 10.4.25(b) zooms in around the origin.

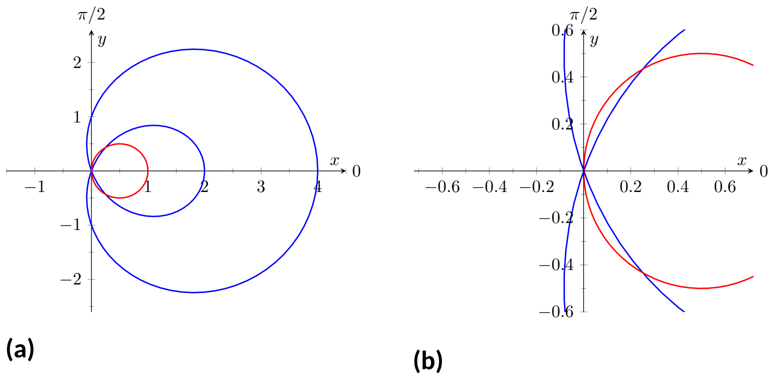


Figure 10.4.25 Graphs to help determine the points of intersection of the polar functions given in Example 10.4.24

We start by setting the two functions equal to each other and solving for θ:

$$\begin{aligned} 1 + 3\cos(\theta) &= \cos(\theta) \\ 2\cos(\theta) &= -1 \\ \cos(\theta) &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3}, \frac{4\pi}{3}. \end{aligned}$$

(There are, of course, infinite solutions to the equation $\cos(\theta) = -1/2$; as the limaçon is traced out once on $[0, 2\pi]$, we restrict our solutions to this interval.)

We need to analyze this solution. When $\theta = 2\pi/3$ we obtain the point of intersection that lies in the 4th quadrant. When $\theta = 4\pi/3$, we get the point of intersection that lies in the second quadrant. There is more to say about this second intersection point, however. The circle defined by $r = \cos(\theta)$ is traced out once on $[0, \pi]$, meaning that this point of intersection occurs while tracing out the circle a second time. It seems strange to pass by the point once and then recognize it as a point of intersection only when arriving there a "second time." The first time the circle arrives at this point is when $\theta = \pi/3$. It is key to understand that these two points are the same: $(\cos(\pi/3), \pi/3)$ and $(\cos(4\pi/3), 4\pi/3)$.

To summarize what we have done so far, we have found two points of intersection: when $\theta = 2\pi/3$ and when $\theta = 4\pi/3$. When referencing the circle $r = \cos(\theta)$, the latter point is better referenced as when $\theta = \pi/3$.

There is yet another point of intersection: the pole (or, the origin). We did not recognize this intersection point using our work above as each graph arrives at the pole at a different θ value.

A graph intersects the pole when $r = 0$. Considering the circle $r = \cos(\theta)$, $r = 0$ when $\theta = \pi/2$ (and odd multiples thereof, as the circle is repeatedly traced). The limaçon intersects the pole when $1 + 3\cos(\theta) = 0$; this occurs when $\cos(\theta) = -1/3$, or for $\theta = \cos^{-1}(-1/3)$. This is a nonstandard angle, approximately $\theta = 1.9106 = 109.47°$. The limaçon intersects the pole twice in $[0, 2\pi]$; the other angle at which the limaçon is at the pole is the reflection of the first angle across the x-axis. That is, $\theta = 4.3726 = 250.53°$.

Video solution

youtu.be/watch?v=mI8vfQxub9g

If all one is concerned with is the (x, y) coordinates at which the graphs intersect, much of the above work is extraneous. We know they intersect at $(0, 0)$; we might not care at what θ value. Likewise, using $\theta = 2\pi/3$ and $\theta = 4\pi/3$ can give us the needed rectangular coordinates. However, in the next section we apply calculus concepts to polar functions. When computing the area of a region bounded by polar curves, understanding the nuances of the points of intersection becomes important.

10.4.4 Exercises

Terms and Concepts

1. In your own words, describe how to plot the polar point $P(r,\theta)$.

2. True or False? When plotting a point with polar coordinate $P(r,\theta)$, r must be positive. (☐ True ☐ False)

3. True or False? Every point in the Cartesian plane can be represented by a polar coordinate. (☐ True ☐ False)

4. True or False? Every point in the Cartesian plane can be represented uniquely by a polar coordinate. (☐ True ☐ False)

Problems

5. Plot the points with the given polar coordinates.

(a) $A = P(2,0)$

(b) $B = P(1,\pi)$

(c) $C = P(-2,\pi/2)$

(d) $D = P(1,\pi/4)$

6. Plot the points with the given polar coordinates.

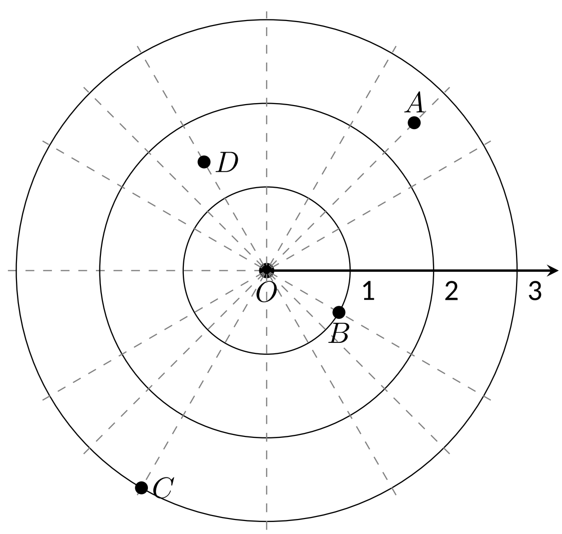
(a) $A = P(2,3\pi)$

(b) $B = P(1,-\pi)$

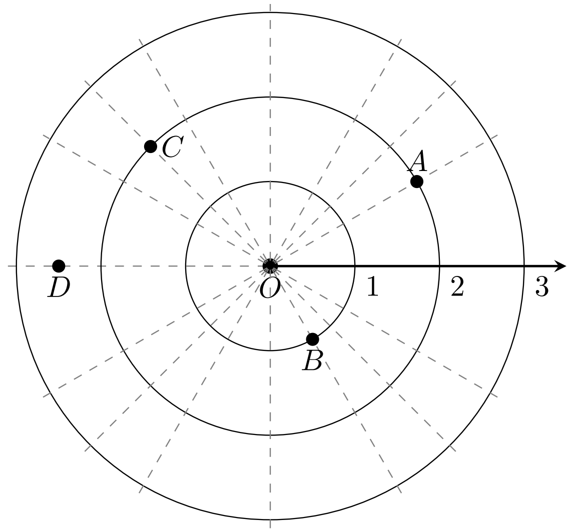
(c) $C = P(1,2)$

(d) $D = P(1/2,5\pi/6)$

7. For each of the given points give two sets of polar coordinates that identify it, where $0 \le \theta \le 2\pi$.



8. For each of the given points give two sets of polar coordinates that identify it, where $-\pi < \theta \le \pi$.



9. Convert each of the following polar coordinates to rectangular, and each of the following rectangular coordinates to polar.

a. $A = P(2, \pi/4)$
$(x, y) =$________________

b. $B = P(2, -\pi/4)$
$(x, y) =$________________

c. $C = (2, -1)$
$P(r, \theta) = P$________________

d. $D = (-2, 1)$
$P(r, \theta) = P$________________

10. Convert each of the following polar coordinates to rectangular, and each of the followingrectangular coordinates to polar.

a. $A = P(3, \pi)$
$(x, y) =$________________

b. $B = P(1, 2\pi/3)$
$(x, y) =$________________

c. $C = (0, 4)$
$P(r, \theta) = P$________________

d. $D = (1, -\sqrt{3})$
$P(r, \theta) = P$________________

Exercise Group. In the following exercises, graph the polar function on the given interval.

11. $r = 2, 0 \leq \theta \leq \pi/2$

12. $\theta = \pi/6, -1 \leq r \leq 2$

13. $r = 1 - \cos(\theta), [0, 2\pi]$

14. $r = 2 + \sin(\theta), [0, 2\pi]$

15. $r = 2 - \sin(\theta), [0, 2\pi]$

16. $r = 1 - 2\sin(\theta), [0, 2\pi]$

17. $r = 1 + 2\sin(\theta), [0, 2\pi]$

18. $r = \cos(2\theta), [0, 2\pi]$

19. $r = \sin(3\theta), [0, \pi]$

20. $r = \cos(\theta/3), [0, 3\pi]$

21. $r = \cos(2\theta/3), [0, 6\pi]$

22. $r = \theta/2, [0, 4\pi]$

23. $r = 3\sin(\theta), [0, \pi]$

24. $r = 2\cos(\theta), [0, \pi/2]$

25. $r = \cos(\theta)\sin(\theta)$,$[0, 2\pi]$

26. $r = \theta^2 - (\pi/2)^2$,$[-\pi, \pi]$

27. $r = \dfrac{3}{5\sin(\theta) - \cos(\theta)}$,$[0, 2\pi]$

28. $r = \dfrac{-2}{3\cos(\theta) - 2\sin(\theta)}$,$[0, 2\pi]$

29. $r = 3\sec(\theta)$,$(-\pi/2, \pi/2)$

30. $r = 3\csc(\theta)$,$(0, \pi)$

Exercise Group. In the following exercises, convert the polar equation to a rectangular equation.

31. Convert the polar equation to a rectangular equation.
$r = 6\cos(\theta)$

32. Convert the polar equation to a rectangular equation.
$r = -4\sin(\theta)$

33. Convert the polar equation to a rectangular equation.
$r = \cos(\theta) + \sin(\theta)$

34. Convert the polar equation to a rectangular equation.
$r = \dfrac{7}{5\sin(\theta) - 2\cos(\theta)}$

35. Convert the polar equation to a rectangular equation.
$r = \dfrac{3}{\cos(\theta)}$

36. Convert the polar equation to a rectangular equation.
$r = \dfrac{4}{\sin(\theta)}$

37. $r = \tan(\theta)$

38. $r = \cot\theta$

39. Convert the polar equation to a rectangular equation.
$r = 2$

40. Convert the polar equation to a rectangular equation.
$\theta = \pi/6$

Exercise Group. In the following exercises, convert the rectangular equation to a polar equation.

41. Convert the rectangular equation to a polar equation. Type ‘`theta`’ for θ.
$y = x$

42. Convert the rectangular equation to a polar equation. Type ‘`theta`’ for θ.
$y = 4x + 7$

43. Convert the rectangular equation to a polar equation. Type ‘`theta`’ for θ.
$x = 5$

44. Convert the rectangular equation to a polar equation. Type ‘`theta`’ for θ.
$y = 5$

45. Convert the rectangular equation to a polar equation. Type ‘`theta`’ for θ.
$x = y^2$

46. $x^2 y = 1$

47. Convert the rectangular equation to a polar equation. Type ‘`theta`’ for θ.
$x^2 + y^2 = 7$

48. $(x + 1)^2 + y^2 = 1$

Exercise Group. In the following exercises, find the points of intersection of the polar graphs.

49. Find the points where $r = \sin(2\theta)$ intersects $r = \cos(\theta)$ on $[0, \pi]$, expressed in polar coordinates with notation $P(r, \theta)$.

50. $r = \cos(2\theta)$ and $r = \cos(\theta)$ on $[0, \pi]$

51. Find the points where $r = 2\cos(\theta)$ intersects $r = 2\sin(\theta)$ on $[0, \pi]$, expressed in polar coordinates with notation $P(r, \theta)$.

52. $r = \sin(\theta)$ and $r = \sqrt{3} + 3\sin(\theta)$ on $[0, 2\pi]$

53. $r = \sin(3\theta)$ and $r = \cos(3\theta)$ on $[0, \pi]$

54. Find the points where $r = 3\cos(\theta)$ intersects $r = 1 + \cos(\theta)$ on $[-\pi, \pi]$, expressed in polar coordinates with notation $P(r, \theta)$.

55. $r = 1$ and $r = 2\sin(2\theta)$ on $[0, 2\pi]$

56. $r = 1 - \cos(\theta)$ and $r = 1 + \sin(\theta)$ on $[0, 2\pi]$

57. Pick a integer value for n, where $n \neq 2, 3$, and use technology to plot $r = \sin\left(\frac{m}{n}\theta\right)$ for three different integer values of m. Sketch these and determine a minimal interval on which the entire graph is shown.

58. Create your own polar function, $r = f(\theta)$ and sketch it. Describe why the graph looks as it does.

10.5 Calculus and Polar Functions

The previous section defined polar coordinates, leading to polar functions. We investigated plotting these functions and solving a fundamental question about their graphs, namely, where do two polar graphs intersect?

We now turn our attention to answering other questions, whose solutions require the use of calculus. A basis for much of what is done in this section is the ability to turn a polar function $r = f(\theta)$ into a set of parametric equations. Using the identities $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we can create the parametric equations $x = f(\theta)\cos(\theta)$, $y = f(\theta)\sin(\theta)$ and apply the concepts of Section 10.3.

youtu.be/watch?v=rj1KoMXvhKw

Figure 10.5.1 Video introduction to Section 10.5

10.5.1 Polar Functions and dy/dx

We are interested in the lines tangent to a given graph, regardless of whether that graph is produced by rectangular, parametric, or polar equations. In each of these contexts, the slope of the tangent line is $\frac{dy}{dx}$. Given $r = f(\theta)$, we are generally *not* concerned with $r' = f'(\theta)$; that describes how fast r changes with respect to θ. Instead, we will use $x = f(\theta)\cos(\theta)$, $y = f(\theta)\sin(\theta)$ to compute $\frac{dy}{dx}$.

Using Key Idea 10.3.2 we have

$$\frac{dy}{dx} = \frac{dy}{d\theta}\bigg/\frac{dx}{d\theta}.$$

Each of the two derivatives on the right hand side of the equality requires the use of the Product Rule. We state the important result as a Key Idea.

Key Idea 10.5.2 Finding $\frac{dy}{dx}$ with Polar Functions.

Let $r = f(\theta)$ be a polar function. With $x = f(\theta)\cos(\theta)$ and $y = f(\theta)\sin(\theta)$,

$$\frac{dy}{dx} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}.$$

Example 10.5.3 Finding $\frac{dy}{dx}$ with polar functions.

Consider the limaçon $r = 1 + 2\sin(\theta)$ on $[0, 2\pi]$.

1. Find the equations of the tangent and normal lines to the graph at $\theta = \pi/4$.
2. Find where the graph has vertical and horizontal tangent lines.

Solution.

1. We start by computing $\frac{dy}{dx}$. With $f'(\theta) = 2\cos(\theta)$, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{2\cos(\theta)\sin(\theta) + \cos(\theta)(1 + 2\sin(\theta))}{2\cos^2(\theta) - \sin(\theta)(1 + 2\sin(\theta))}\\ &= \frac{\cos(\theta)(4\sin(\theta) + 1)}{2(\cos^2(\theta) - \sin^2(\theta)) - \sin(\theta)}.\end{aligned}$$

When $\theta = \pi/4$, $\frac{dy}{dx} = -2\sqrt{2} - 1$ (this requires a bit of simplification). In rectangular coordinates, the point on the graph at

$\theta = \pi/4$ is $(1 + \sqrt{2}/2, 1 + \sqrt{2}/2)$. Thus the rectangular equation of the line tangent to the limaçon at $\theta = \pi/4$ is

$$y = (-2\sqrt{2} - 1)\big(x - (1 + \sqrt{2}/2)\big) + 1 + \sqrt{2}/2 \approx -3.83x + 8.24.$$

The limaçon and the tangent line are graphed in Figure 10.5.4. The normal line has the opposite-reciprocal slope as the tangent line, so its equation is

$$y \approx \frac{1}{3.83}x + 1.26.$$

2. To find the horizontal lines of tangency, we find where $\frac{dy}{dx} = 0$; thus we find where the numerator of our equation for $\frac{dy}{dx}$ is 0.

$$\cos(\theta)(4\sin(\theta) + 1) = 0 \Rightarrow \cos(\theta) = 0 \text{ or } 4\sin(\theta) + 1 = 0.$$

On $[0, 2\pi]$, $\cos(\theta) = 0$ when $\theta = \pi/2, 3\pi/2$. Setting $4\sin(\theta) + 1 = 0$ gives $\theta = \sin^{-1}(-1/4) \approx -0.2527 = -14.48°$. We want the results in $[0, 2\pi]$; we also recognize there are two solutions, one in the third quadrant and one in the fourth. Using reference angles, we have our two solutions as $\theta = 3.39$ and 6.03 radians. The four points we obtained where the limaçon has a horizontal tangent line are given in Figure 10.5.4 with black-filled dots. To find the vertical lines of tangency, we set the denominator of $\frac{dy}{dx} = 0$.

$$2(\cos^2(\theta) - \sin^2(\theta)) - \sin(\theta) = 0.$$

Convert the $\cos^2(\theta)$ term to $1 - \sin^2(\theta)$:

$$\begin{aligned} 2(1 - \sin^2(\theta) - \sin^2(\theta)) - \sin(\theta) &= 0 \\ 4\sin^2(\theta) + \sin(\theta) - 2 &= 0. \end{aligned}$$

Recognize this as a quadratic in the variable $\sin(\theta)$. Using the quadratic formula, we have

$$\sin(\theta) = \frac{-1 \pm \sqrt{33}}{8}.$$

We solve $\sin(\theta) = \frac{-1+\sqrt{33}}{8}$ and $\sin(\theta) = \frac{-1-\sqrt{33}}{8}$:

$$\begin{aligned} \sin(\theta) &= \frac{-1 + \sqrt{33}}{8} \\ \theta &= \sin^{-1}\left(\frac{-1 + \sqrt{33}}{8}\right) \\ \theta &= 0.6349 \end{aligned} \qquad \begin{aligned} \sin(\theta) &= \frac{-1 - \sqrt{33}}{8} \\ \theta &= \sin^{-1}\left(\frac{-1 - \sqrt{33}}{8}\right) \\ \theta &= -1.0030 \end{aligned}$$

In each of the solutions above, we only get one of the possible two solutions as $\sin^{-1}(x)$ only returns solutions in $[-\pi/2, \pi/2]$, the 4th and 1st quadrants. Again using reference angles, we have:

$$\sin\theta = \frac{-1 + \sqrt{33}}{8} \Rightarrow \theta = 0.6349,\ 2.5067 \text{ radians}$$

and

$$\sin(\theta) = \frac{-1 - \sqrt{33}}{8} \Rightarrow \theta = 4.1446,\ 5.2802 \text{ radians}.$$

These points are also shown in Figure 10.5.4 with white-filled dots.

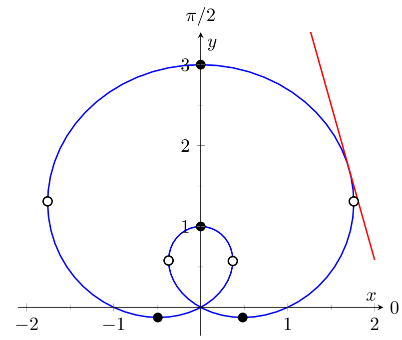


Figure 10.5.4 The limaçon in Example 10.5.3 with its tangent line at $\theta = \pi/4$ and points of vertical and horizontal tangency

Video solution

youtu.be/watch?v=QLsLabLb6I4

When the graph of the polar function $r = f(\theta)$ intersects the pole, it means that $f(\alpha) = 0$ for some angle α. Thus the formula for $\frac{dy}{dx}$ in such instances is very simple, reducing simply to

$$\frac{dy}{dx} = \tan\alpha.$$

This equation makes an interesting point. It tells us the slope of the tangent line at the pole is $\tan\alpha$; some of our previous work (see, for instance, Example 10.4.9) shows us that the line through the pole with slope $\tan\alpha$ has polar equation $\theta = \alpha$. Thus when a polar graph touches the pole at $\theta = \alpha$, the equation of the tangent line at the pole is $\theta = \alpha$.

Example 10.5.5 Finding tangent lines at the pole.

Let $r = 1 + 2\sin(\theta)$, a limaçon. Find the equations of the lines tangent to the graph at the pole.

Solution. We need to know when $r = 0$.

$$\begin{aligned} 1 + 2\sin(\theta) &= 0 \\ \sin(\theta) &= -1/2 \\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6}. \end{aligned}$$

Thus the equations of the tangent lines, in polar, are $\theta = 7\pi/6$ and $\theta = 11\pi/6$. In rectangular form, the tangent lines are $y = \tan(7\pi/6)x$ and $y = \tan(11\pi/6)x$. The full limaçon can be seen in Figure 10.5.4; we zoom in on the tangent lines in Figure 10.5.6.

Video solution

youtu.be/watch?v=1bAf6kE9F1Y

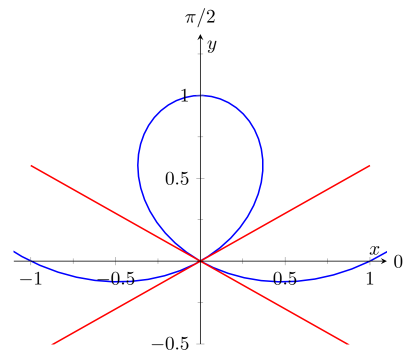


Figure 10.5.6 Graphing the tangent lines at the pole in Example 10.5.5

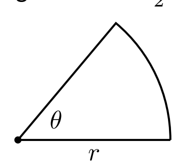
10.5.2 Area

When using rectangular coordinates, the equations $x = h$ and $y = k$ defined vertical and horizontal lines, respectively, and combinations of these lines create rectangles (hence the name "rectangular coordinates"). It is then somewhat natural to use rectangles to approximate area as we did when learning about the definite integral.

When using polar coordinates, the equations $\theta = \alpha$ and $r = c$ form lines through the origin and circles centered at the origin, respectively, and combinations of these curves form sectors of circles. It is then somewhat natural to calculate the area of regions defined by polar functions by first approximating with sectors of circles.

Consider Figure 10.5.7(a) where a region defined by $r = f(\theta)$ on $[\alpha, \beta]$ is given. (Note how the "sides" of the region are the lines $\theta = \alpha$ and $\theta = \beta$, whereas in rectangular coordinates the "sides" of regions were often the vertical lines $x = a$ and $x = b$.)

Recall that the area of a sector of a circle with radius r subtended by an angle θ is $A = \frac{1}{2}\theta r^2$.



Partition the interval $[\alpha, \beta]$ into n equally spaced subintervals as $\alpha = \theta_0 < \theta_1 < \cdots < \theta_n = \beta$. The length of each subinterval is $\Delta\theta = (\beta - \alpha)/n$, representing a small change in angle. The area of the region defined by the ith subinterval $[\theta_{i-1}, \theta_i]$ can be approximated with a sector of a circle with radius $f(c_i)$, for some c_i in $[\theta_{i-1}, \theta_i]$. The area of this sector is $\frac{1}{2}f(c_i)^2\Delta\theta$. This is shown in Figure 10.5.7(b), where $[\alpha, \beta]$ has been divided into 4 subintervals. We approximate the area of the whole region by summing the areas of all sectors:

$$\text{Area} \approx \sum_{i=1}^{n} \frac{1}{2} f(c_i)^2 \Delta\theta.$$

This is a Riemann sum. By taking the limit of the sum as $n \to \infty$, we find the exact area of the region in the form of a definite integral.

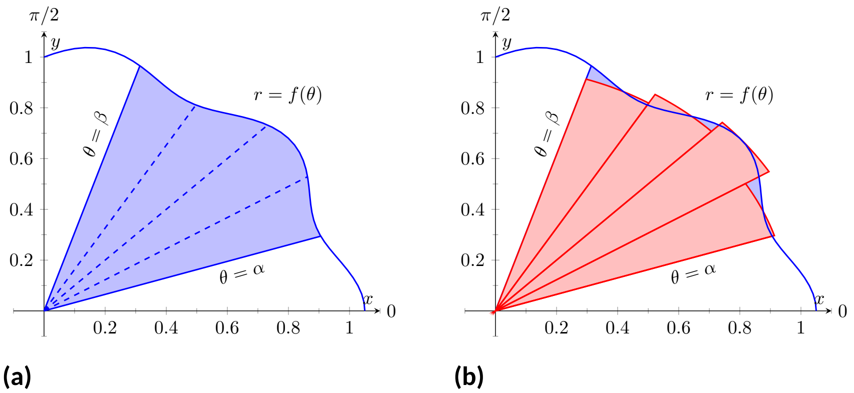


Figure 10.5.7 Computing the area of a polar region

Theorem 10.5.9 Area of a Polar Region.

Let f be continuous and non-negative on $[\alpha, \beta]$, where $0 \leq \beta - \alpha \leq 2\pi$. The area A of the region bounded by the curve $r = f(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_\alpha^\beta f(\theta)^2 \, d\theta = \frac{1}{2}\int_\alpha^\beta r^2 \, d\theta$$

The theorem states that $0 \leq \beta - \alpha \leq 2\pi$. This ensures that region does not overlap itself, which would give a result that does not correspond directly to the area.

Example 10.5.10 Area of a polar region.

Find the area of the circle defined by $r = \cos(\theta)$. (Recall this circle has radius $1/2$.)

Solution. This is a direct application of Theorem 10.5.9. The circle is traced out on $[0, \pi]$, leading to the integral

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\int_0^\pi \cos^2(\theta)\, d\theta \\
&= \frac{1}{2}\int_0^\pi \frac{1+\cos(2\theta)}{2}\, d\theta \\
&= \frac{1}{4}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_0^\pi \\
&= \frac{1}{4}\pi.
\end{aligned}$$

Of course, we already knew the area of a circle with radius $1/2$. We did this example to demonstrate that the area formula is correct.

youtu.be/watch?v=dYEMKbxRxpY

Figure 10.5.8 Video presentation of Theorem 10.5.9

Example 10.5.10 requires the use of the integral $\int \cos^2(\theta)\, d\theta$. This is handled well by using the power reducing formula as found in Subsection B.3.2 of the Quick Reference Appendix. Due to the nature of the area formula, integrating $\cos^2(\theta)$ and $\sin^2(\theta)$ is required often. We offer here these indefinite integrals as a time-saving measure.

$$\int \cos^2\theta \, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C$$

$$\int \sin^2\theta \, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) + C$$

Video solution

youtu.be/watch?v=T_z8FNS3Whs

Example 10.5.11 Area of a polar region.

Find the area of the cardioid $r = 1 + \cos(\theta)$ bound between $\theta = \pi/6$ and $\theta = \pi/3$, as shown in Figure 10.5.12.

Solution. This is again a direct application of Theorem 10.5.9.

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_{\pi/6}^{\pi/3} (1+\cos(\theta))^2\, d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{\pi/3} (1+2\cos(\theta)+\cos^2(\theta))\, d\theta \\ &= \frac{1}{2}\left(\theta + 2\sin(\theta) + \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right)\Bigg|_{\pi/6}^{\pi/3} \\ &= \frac{1}{8}\left(\pi + 4\sqrt{3} - 4\right) \approx 0.7587.\end{aligned}$$

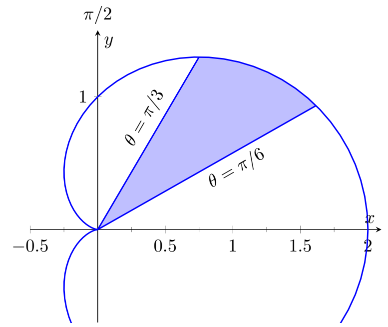


Figure 10.5.12 Finding the area of the shaded region of a cardioid in Example 10.5.11

Video solution

youtu.be/watch?v=yZpBCQnB7r8

Area Between Curves. Our study of area in the context of rectangular functions led naturally to finding area bounded between curves. We consider the same in the context of polar functions.

Consider the shaded region shown in Figure 10.5.13. We can find the area of this region by computing the area bounded by $r_2 = f_2(\theta)$ and subtracting the area bounded by $r_1 = f_1(\theta)$ on $[\alpha, \beta]$. Thus

$$\text{Area} = \frac{1}{2}\int_\alpha^\beta r_2^2\, d\theta - \frac{1}{2}\int_\alpha^\beta r_1^2\, d\theta = \frac{1}{2}\int_\alpha^\beta \left(r_2^2 - r_1^2\right) d\theta.$$

Key Idea 10.5.14 Area Between Polar Curves.

The area A of the region bounded by $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$, $\theta = \alpha$ and $\theta = \beta$, where $f_1(\theta) \leq f_2(\theta)$ on $[\alpha, \beta]$, is

$$A = \frac{1}{2}\int_\alpha^\beta \left(r_2^2 - r_1^2\right) d\theta.$$

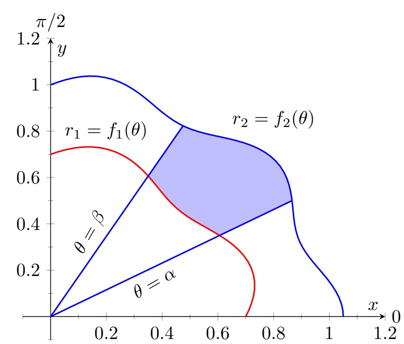


Figure 10.5.13 Illustrating area bound between two polar curves

Example 10.5.15 Area between polar curves.

Find the area bounded between the curves $r = 1 + \cos(\theta)$ and $r = 3\cos(\theta)$, as shown in Figure 10.5.16.

Solution. We need to find the points of intersection between these two functions. Setting them equal to each other, we find:

$$\begin{aligned}1 + \cos(\theta) &= 3\cos(\theta) \\ \cos(\theta) &= 1/2 \\ \theta &= \pm\pi/3\end{aligned}$$

Thus we integrate $\frac{1}{2}\left((3\cos(\theta))^2 - (1+\cos(\theta))^2\right)$ on $[-\pi/3, \pi/3]$.

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left((3\cos(\theta))^2 - (1+\cos(\theta))^2\right) d\theta \\ &= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(8\cos^2(\theta) - 2\cos(\theta) - 1\right) d\theta\end{aligned}$$

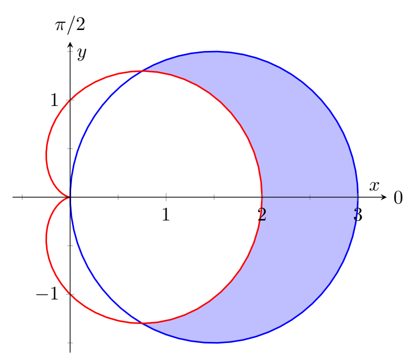


Figure 10.5.16 Finding the area between polar curves in Example 10.5.15

$$= \frac{1}{2}\big(2\sin(2\theta) - 2\sin(\theta) + 3\theta\big)\Big|_{-\pi/3}^{\pi/3}$$
$$= \pi.$$

Amazingly enough, the area between these curves has a “nice” value.

Video solution

youtu.be/watch?v=CM7XsZRRqSI

Example 10.5.17 Area defined by polar curves.

Find the area bounded between the polar curves $r = 1$ and $r = 2\cos(2\theta)$, as shown in Figure 10.5.18.

Solution. We need to find the point of intersection between the two curves. Setting the two functions equal to each other, we have

$$2\cos(2\theta) = 1 \Rightarrow \cos(2\theta) = \frac{1}{2} \Rightarrow 2\theta = \pi/3 \Rightarrow \theta = \pi/6.$$

In Figure 10.5.19, we zoom in on the region and note that it is not really bounded *between* two polar curves, but rather *by* two polar curves, along with $\theta = 0$. The dashed line breaks the region into its component parts. Below the dashed line, the region is defined by $r = 1$, $\theta = 0$ and $\theta = \pi/6$. (Note: the dashed line lies on the line $\theta = \pi/6$.) Above the dashed line the region is bounded by $r = 2\cos(2\theta)$ and $\theta = \pi/6$. Since we have two separate regions, we find the area using two separate integrals.

Call the area below the dashed line A_1 and the area above the dashed line A_2. They are determined by the following integrals:

$$A_1 = \frac{1}{2}\int_0^{\pi/6} (1)^2\, d\theta \qquad A_2 = \frac{1}{2}\int_{\pi/6}^{\pi/4} \big(2\cos(2\theta)\big)^2\, d\theta.$$

(The upper bound of the integral computing A_2 is $\pi/4$ as $r = 2\cos(2\theta)$ is at the pole when $\theta = \pi/4$.)

We omit the integration details and let the reader verify that $A_1 = \pi/12$ and $A_2 = \pi/12 - \sqrt{3}/8$; the total area is $A = \pi/6 - \sqrt{3}/8$.

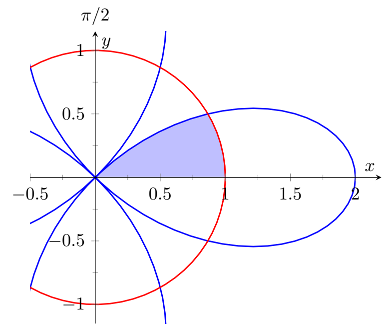


Figure 10.5.18 The region bounded by the functions in Example 10.5.17

Video solution

youtu.be/watch?v=5MHcrQVTjjU

10.5.3 Arc Length

As we have already considered the arc length of curves defined by rectangular and parametric equations, we now consider it in the context of polar equations. Recall that the arc length L of the graph defined by the parametric equations $x = f(t)$, $y = g(t)$ on $[a, b]$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt. \qquad (10.5.1)$$

Now consider the polar function $r = f(\theta)$. We again use the identities $x = f(\theta)\cos(\theta)$ and $y = f(\theta)\sin(\theta)$ to create parametric equations based on the polar function. We compute $x'(\theta)$ and $y'(\theta)$ as done before when computing $\frac{dy}{dx}$, then apply Equation (10.5.1).

The expression $x'(\theta)^2 + y'(\theta)^2$ can be simplified a great deal; we leave this as an exercise and state that

$$x'(\theta)^2 + y'(\theta)^2 = f'(\theta)^2 + f(\theta)^2.$$

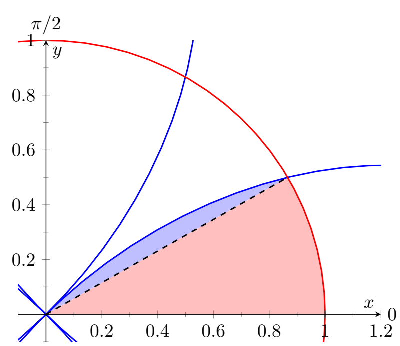


Figure 10.5.19 Breaking the region bounded by the functions in Example 10.5.17 into its component parts

This leads us to the arc length formula.

Theorem 10.5.20 Arc Length of Polar Curves.

Let $r = f(\theta)$ be a polar function with f' continuous on $[\alpha, \beta]$, on which the graph traces itself only once. The arc length L of the graph on $[\alpha, \beta]$ is

$$L = \int_\alpha^\beta \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta = \int_\alpha^\beta \sqrt{(r\,')^2 + r^2}\, d\theta.$$

Example 10.5.21 Arc length of a limaçon.

Find the arc length of the limaçon $r = 1 + 2\sin(\theta)$.

Solution. With $r = 1 + 2\sin(\theta)$, we have $r\,' = 2\cos(\theta)$. The limaçon is traced out once on $[0, 2\pi]$, giving us our bounds of integration. Applying Theorem 10.5.20, we have

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(2\cos\theta)^2 + (1 + 2\sin\theta)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4\cos^2\theta + 4\sin^2\theta + 4\sin\theta + 1}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4\sin\theta + 5}\, d\theta \\ &\approx 13.3649. \end{aligned}$$

The final integral cannot be solved in terms of elementary functions, so we resorted to a numerical approximation. (Simpson's Rule, with $n = 4$, approximates the value with 13.0608. Using $n = 22$ gives the value above, which is accurate to 4 places after the decimal.)

Video solution

youtu.be/watch?v=o-cetriP4Ms

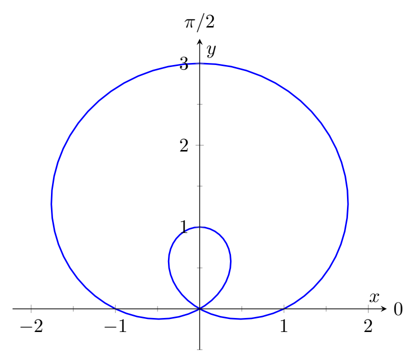


Figure 10.5.22 The limaçon in Example 10.5.21 whose arc length is measured

10.5.4 Surface Area

The formula for arc length leads us to a formula for surface area. The following Theorem is based on Theorem 10.3.21.

Theorem 10.5.23 Surface Area of a Solid of Revolution.

Consider the graph of the polar equation $r = f(\theta)$, where f' is continuous on $[\alpha, \beta]$, on which the graph does not cross itself.

1. *The surface area of the solid formed by revolving the graph about the initial ray ($\theta = 0$) is:*

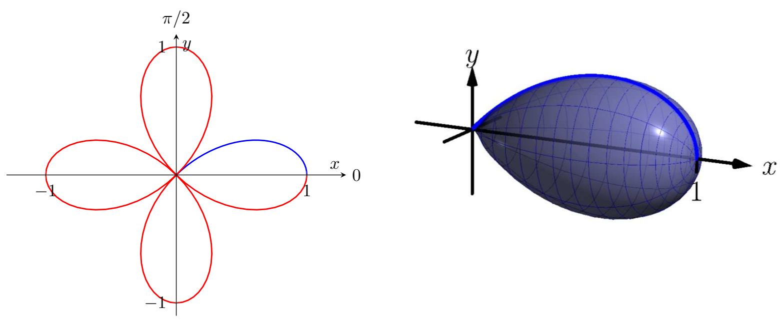
$$\text{Surface Area} = 2\pi \int_\alpha^\beta f(\theta)\sin(\theta)\sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta.$$

2. *The surface area of the solid formed by revolving the graph about the line $\theta = \pi/2$ is:*

$$\text{Surface Area} = 2\pi \int_\alpha^\beta f(\theta)\cos(\theta)\sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta.$$

Example 10.5.24 Surface area determined by a polar curve.

Find the surface area formed by revolving one petal of the rose curve $r = \cos(2\theta)$ about its central axis, as shown in Figure 10.5.25.



(a) **(b)**

Figure 10.5.25 Finding the surface area of a rose-curve petal that is revolved around its central axis

Solution. We choose, as implied by the figure, to revolve the portion of the curve that lies on $[0, \pi/4]$ about the initial ray. Using Theorem 10.5.23 and the fact that $f'(\theta) = -2\sin(2\theta)$, we have

$$\begin{aligned}\text{Surface Area } &= 2\pi\int_0^{\pi/4} \cos(2\theta)\sin(\theta)\sqrt{\big(-2\sin(2\theta)\big)^2 + \big(\cos(2\theta)\big)^2}\,d\theta\\ &\approx 1.36707.\end{aligned}$$

The integral is another that cannot be evaluated in terms of elementary functions. Simpson's Rule, with $n = 4$, approximates the value at 1.36751.

This chapter has been about curves in the plane. While there is great mathematics to be discovered in the two dimensions of a plane, we live in a three dimensional world and hence we should also look to do mathematics in 3D — that is, in *space*. The next chapter begins our exploration into space by introducing the topic of *vectors*, which are incredibly useful and powerful mathematical objects.

10.5.5 Exercises

Terms and Concepts

1. Given polar equation $r = f(\theta)$, how can one create parametric equations of the same curve?

2. With rectangular coordinates, it is natural to approximate area with ___________; with polar coordinates, it is natural to approximate area with ___________.

Problems

Exercise Group. Find $\frac{dy}{dx}$ (in terms of θ). Then find the equations of the tangent and normal lines to the curve at the indicated θ-value.

3. $r = 1, \theta = \pi/4$

4. $r = \cos(\theta), \theta = \pi/4$

5. $r = 1 + \sin(\theta), \theta = \pi/6$

6. $r = 1 - 3\cos(\theta), \theta = 3\pi/4$

7. $r = \theta, \theta = \pi/2$

8. $r = \cos(3\theta), \theta = \pi/6$

9. $r = \sin(4\theta), \theta = \pi/3$

10. $r = \dfrac{1}{\sin(\theta) - \cos(\theta)}; \theta = \pi$

Exercise Group. Find the values of θ in the given interval where the graph of the polar function has horizontal and vertical tangent lines.

11. $r = 3$; $[0, 2\pi]$

12. $r = 2\sin(\theta)$; $[0, \pi]$

13. $r = \cos(2\theta)$; $[0, 2\pi]$

14. $r = 1 + \cos(\theta)$; $[0, 2\pi)$

Exercise Group. Find the equation of the lines tangent to the graph at the pole.

15. $r = \sin(\theta)$;$[0, \pi]$

16. $r = \sin(3\theta)$;$[0, \pi]$

Exercise Group. Find the area of the described region.

17. Enclosed by the circle: $r = 4\sin(\theta)$

18. Enclosed by the circle $r = 5$

19. Find the area enclosed by one petal of $r = \sin(3\theta)$.

20. Enclosed by one petal of the rose curve $r = \cos(n\,\theta)$, where n is a positive integer.

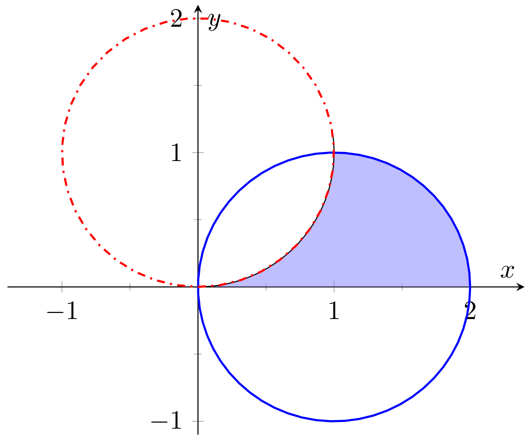
21. Find the area enclosed by the cardioid $r = 1 - \sin(\theta)$.

22. Enclosed by the inner loop of the limaçon $r = 1 + 2\cos(\theta)$

23. Find the area enclosed by the outer loop of the limaçon $r = 1 + 2\cos(\theta)$ (including area enclosed by the inner loop).

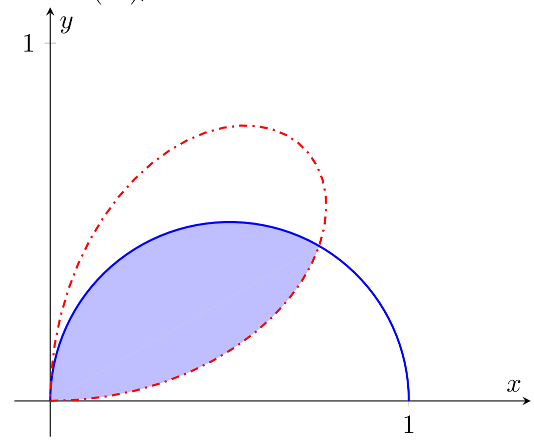
24. Find the area enclosed between the inner and outer loop of the limaçon $r = 1 + 2\cos(\theta)$.

25. Find the area enclosed by $r = 2\cos(\theta)$, $r = 2\sin(\theta)$, and the x-axis, as shown:



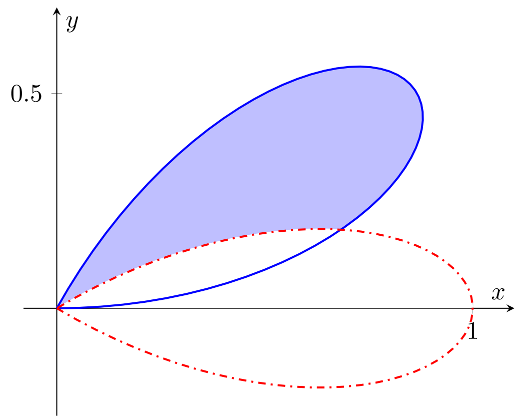
The area is ________________.

26. Find the area enclosed by $r = \cos(\theta)$ and $r = \sin(2\theta)$, as shown:

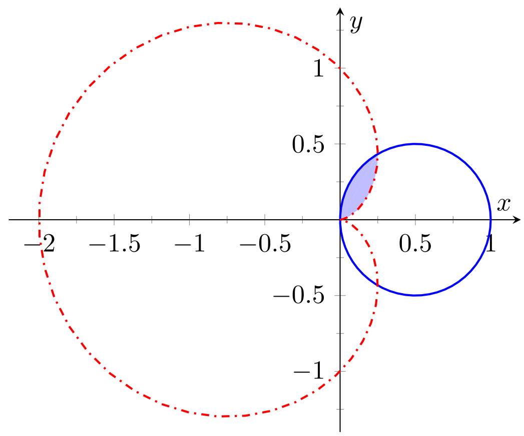


The area is ________________.

27. Enclosed by $r = \cos(3\theta)$ and $r = \sin(3\theta)$, as shown:



28. Enclosed by $r = \cos(\theta)$ and $r = 1 - \cos(\theta)$, as shown:



Exercise Group. In the following exercises, answer the questions involving arc length.

29. Use the arc length formula to compute the arc length of the circle $r = 2$.

30. Use the arc length formula to compute the arc length of the circle $r = 4\sin(\theta)$.

31. Use the arc length formula to compute the arc length of $r = \cos\theta + \sin\theta$.

32. Use the arc length formula to compute the arc length of the cardioid $r = 1 + \cos\theta$. (Hint: apply the formula, simplify, then use a Power-Reducing Formula to convert $1 + \cos\theta$ into a square.)

33. Approximate the arc length of one petal of the rose curve $r = \sin(3\theta)$ with Simpson's Rule and $n = 4$.

34. Let $x(\theta) = f(\theta)\cos(\theta)$ and $y(\theta) = f(\theta)\sin(\theta)$. Show, as suggested by the text, that

$$x'(\theta)^2 + y'(\theta)^2 = f'(\theta)^2 + f(\theta)^2.$$

Exercise Group. In the following exercises, answer the questions involving surface area.

35. Use Theorem 10.5.23 to find the surface area of the sphere formed by revolving the circle $r = 2$ about the initial ray.

36. Use Theorem 10.5.23 to find the surface area of the sphere formed by revolving the circle $r = 2\cos(\theta)$ about the initial ray.

37. Find the surface area of the solid formed by revolving the cardioid $r = 1 + \cos(\theta)$ about the initial ray.

38. Find the surface area of the solid formed by revolving the circle $r = 2\cos(\theta)$ about the line $\theta = \pi/2$.

39. Find the surface area of the solid formed by revolving the line $r = 3\sec(\theta)$, $-\pi/4 \leq \theta \leq \pi/4$, about the line $\theta = \pi/2$.

40. Find the surface area of the solid formed by revolving the line $r = 3\sec\theta$, $0 \leq \theta \leq \pi/4$, about the initial ray.

Chapter 11

Introduction to Functions of Several Variables

The videos in the last three chapters were actually the first to be recorded. At the time, the intent was simply to provide video content for a course: Math 2580, Calculus IV, at the University of Lethbridge. The project to create videos directly aligned to APEX Calculus came later.

As a result, the videos included here do not align perfectly with the textbook. In particular, examples done in the videos are not the same as examples done in the book.

A function of the form $y = f(x)$ is a function of a single variable; given a value of x, we can find a value y. Even the vector-valued functions of Chapter 13 are single-variable functions; the input is a single variable though the output is a vector.

There are many situations where a desired quantity is a function of two or more variables. For instance, wind chill is measured by knowing the temperature and wind speed; the volume of a gas can be computed knowing the pressure and temperature of the gas; to compute a baseball player's batting average, one needs to know the number of hits and the number of at-bats.

This chapter introduces *multivariable* functions, that is, functions with more than one input. A more detailed study of differential calculus for multivariable functions continues in Chapter 14.

11.1 Introduction to Multivariable Functions

Definition 11.1.1 Function of Two Variables.

Let D be a subset of $\mathbb{R}^2$. A function f **of two variables** is a rule that assigns each pair (x, y) in D a value $z = f(x, y)$ in $\mathbb{R}$. D is the **domain** of f; the set of all outputs of f is the **range**.

youtu.be/watch?v=w4K2L9-OPk8

Figure 11.1.2 Video introduction to multivariable function notation

Example 11.1.3 Understanding a function of two variables.

Let $z = f(x, y) = x^2 - y$. Evaluate $f(1, 2)$, $f(2, 1)$, and $f(-2, 4)$; find the domain and range of f.

Solution. Using the definition $f(x, y) = x^2 - y$, we have:

$$\begin{aligned} f(1,2) &= 1^2 - 2 = -1 \\ f(2,1) &= 2^2 - 1 = 3 \\ f(-2,4) &= (-2)^2 - 4 = 0 \end{aligned}$$

The domain is not specified, so we take it to be all possible pairs in $\mathbb{R}^2$ for which f is defined. In this example, f is defined for *all* pairs (x, y), so the domain D of f is $\mathbb{R}^2$.

The output of f can be made as large or small as possible; any real number r can be the output. (In fact, given any real number r, $f(0,-r) = r$.) So the range R of f is $\mathbb{R}$.

Example 11.1.4 Understanding a function of two variables.

Let $f(x,y) = \sqrt{1-\frac{x^2}{9}-\frac{y^2}{4}}$. Find the domain and range of f.

Solution. The domain is all pairs (x,y) allowable as input in f. Because of the square root, we need (x,y) such that $0 \le 1-\frac{x^2}{9}-\frac{y^2}{4}$:

$$0 \le 1-\frac{x^2}{9}-\frac{y^2}{4}$$

$$\frac{x^2}{9}+\frac{y^2}{4} \le 1$$

The above equation describes an ellipse and its interior as shown in Figure 11.1.5. We can represent the domain D graphically with the figure; in set notation, we can write $D = \{(x,y) | \frac{x^2}{9}+\frac{y^2}{4} \le 1\}$.
The range is the set of all possible output values. The square root ensures that all output is ≥ 0. Since the x and y terms are squared, then subtracted, inside the square root, the largest output value comes at $x=0, y=0$: $f(0,0)=1$. Thus the range R is the interval $[0,1]$.

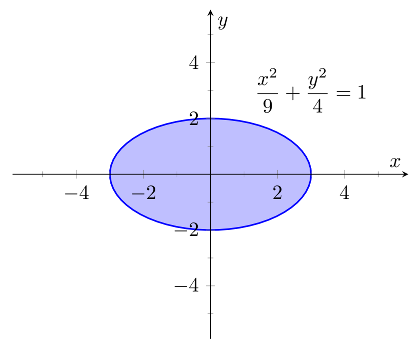
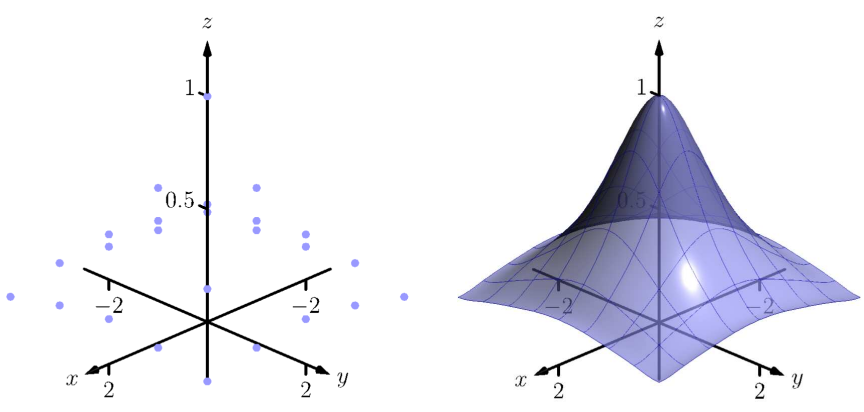


Figure 11.1.5 Illustrating the domain of $f(x,y)$ in Example 11.1.4

11.1.1 Graphing Functions of Two Variables

The *graph* of a function f of two variables is the set of all points $(x,y,f(x,y))$ where (x,y) is in the domain of f. This creates a *surface* in space.



(a) **(b)**

Figure 11.1.6 Graphing a function of two variables

One can begin sketching a graph by plotting points, but this has limitations. Consider Figure 11.1.6(a) where 25 points have been plotted of $f(x,y) = \frac{1}{x^2+y^2+1}$. More points have been plotted than one would reasonably want to do by hand, yet it is not clear at all what the graph of the function looks like. Technology allows us to plot lots of points, connect adjacent points with lines and add shading to create a graph like Figure 11.1.6(b) which does a far better job of illustrating the behavior of f.

While technology is readily available to help us graph functions of two variables, there is still a paper-and-pencil approach that is useful to understand and master as it, combined with high-quality graphics, gives one great insight into the behavior of a function. This technique is known as sketching *level curves*.

11.1.2 Level Curves

It may be surprising to find that the problem of representing a three dimensional surface on paper is familiar to most people (they just don't realize it). Topographical maps, like the one shown in Figure 11.1.7, represent the surface of Earth by indicating points with the same elevation with *contour lines*. The elevations marked are equally spaced; in this example, each thin line indicates an elevation change in 50ft increments and each thick line indicates a change of 200ft. When lines are drawn close together, elevation changes rapidly (as one does not have to travel far to rise 50ft). When lines are far apart, such as near "Aspen Campground," elevation changes more gradually as one has to walk farther to rise 50ft.

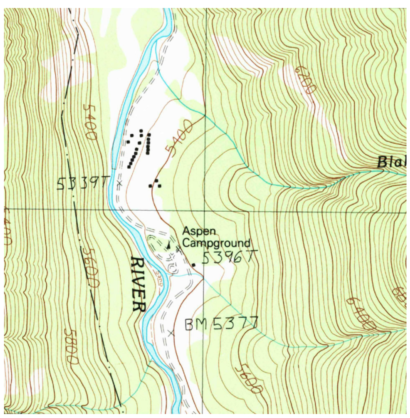


Figure 11.1.7 A topographical map displays elevation by drawing contour lines, along with the elevation is constant. USGS 1:24000-scale Quadrangle for Chrome Mountain, MT 1987.

Given a function $f(x,y)$, we can draw a "topographical map" of the graph $z=f(x,y)$ by drawing *level curves* (or, contour lines). A level curve at $z=c$ is a curve in the xy-plane such that for all points (x,y) on the curve, $f(x,y)=c$.

When drawing level curves, it is important that the c values are spaced equally apart as that gives the best insight to how quickly the "elevation" is changing. Examples will help one understand this concept.

Example 11.1.8 Drawing Level Curves.

Let $f(x,y)=\sqrt{1-\frac{x^2}{9}-\frac{y^2}{4}}$. Find the level curves of f for $c=0, 0.2$, $0.4, 0.6, 0.8$ and 1.

Solution. Consider first $c=0$. The level curve for $c=0$ is the set of all points (x,y) such that $0=\sqrt{1-\frac{x^2}{9}-\frac{y^2}{4}}$. Squaring both sides gives us

$$\frac{x^2}{9}+\frac{y^2}{4}=1,$$

an ellipse centered at $(0,0)$ with horizontal major axis of length 6 and minor axis of length 4. Thus for any point (x,y) on this curve, $f(x,y)=0$.

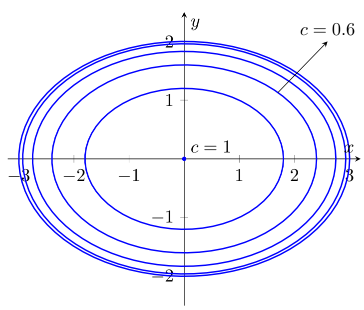
Now consider the level curve for $c = 0.2$

$$\begin{aligned} 0.2 &= \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \\ 0.04 &= 1 - \frac{x^2}{9} - \frac{y^2}{4} \\ \frac{x^2}{9} + \frac{y^2}{4} &= 0.96 \\ \frac{x^2}{8.64} + \frac{y^2}{3.84} &= 1. \end{aligned}$$

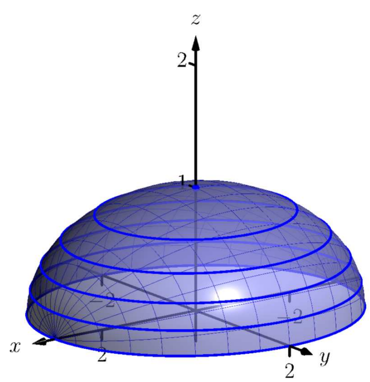
This is also an ellipse, where $a = \sqrt{8.64} \approx 2.94$ and $b = \sqrt{3.84} \approx 1.96$.
In general, for $z = c$, the level curve is:

$$\begin{aligned} c &= \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \\ c^2 &= 1 - \frac{x^2}{9} - \frac{y^2}{4} \\ \frac{x^2}{9} + \frac{y^2}{4} &= 1 - c^2 \\ \frac{x^2}{9(1-c^2)} + \frac{y^2}{4(1-c^2)} &= 1, \end{aligned}$$

ellipses that are decreasing in size as c increases. A special case is when $c = 1$; there the ellipse is just the point $(0, 0)$.
The level curves are shown in Figure 11.1.9(a). Note how the level curves for $c = 0$ and $c = 0.2$ are very, very close together: this indicates that f is growing rapidly along those curves.



(a)



(b)

Figure 11.1.9 Graphing the level curves in Example 11.1.8

In Figure 11.1.9(b), the curves are drawn on a graph of f in space. Note how the elevations are evenly spaced. Near the level curves of $c = 0$ and $c = 0.2$ we can see that f indeed is growing quickly.

Example 11.1.10 Analyzing Level Curves.

Let $f(x, y) = \frac{x+y}{x^2+y^2+1}$. Find the level curves for $z = c$.
Solution. We begin by setting $f(x, y) = c$ for an arbitrary c and seeing

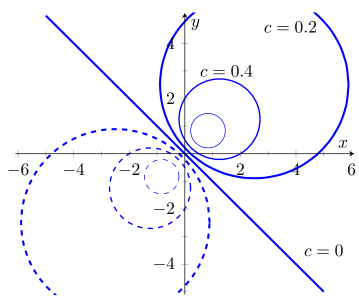
if algebraic manipulation of the equation reveals anything significant.

$$\frac{x+y}{x^2+y^2+1} = c$$
$$x+y = c(x^2+y^2+1).$$

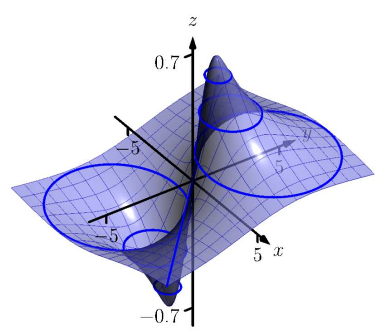
We recognize this as a circle, though the center and radius are not yet clear. By completing the square, we can obtain:

$$\left(x-\frac{1}{2c}\right)^2 + \left(y-\frac{1}{2c}\right)^2 = \frac{1}{2c^2} - 1,$$

a circle centered at $\big(1/(2c), 1/(2c)\big)$ with radius $\sqrt{1/(2c^2)-1}$, where $|c| < 1/\sqrt{2}$. The level curves for $c = \pm 0.2,\ \pm 0.4$ and ± 0.6 are sketched in Figure 11.1.11(a). To help illustrate "elevation," we use thicker lines for c values near 0, and dashed lines indicate where $c < 0$.
There is one special level curve, when $c = 0$. The level curve in this situation is $x + y = 0$, the line $y = -x$.
In Figure 11.1.11(b) we see a graph of the surface. Note how the y-axis is pointing away from the viewer to more closely resemble the orientation of the level curves in Figure 11.1.11(a).



(a)



(b)

Figure 11.1.11 Graphing the level curves in Example 11.1.10

Seeing the level curves helps us understand the graph. For instance, the graph does not make it clear that one can "walk" along the line $y = -x$ without elevation change, though the level curve does.

11.1.3 Functions of Three Variables

We extend our study of multivariable functions to functions of three variables. (One can make a function of as many variables as one likes; we limit our study to three variables.)

Definition 11.1.12 Function of Three Variables.

Let D be a subset of $\mathbb{R}^3$. A function f **of three variables** is a rule that assigns each triple (x, y, z) in D a value $w = f(x, y, z)$ in $\mathbb{R}$. D is the **domain** of f; the set of all outputs of f is the **range**.

Note how this definition closely resembles that of Definition 11.1.1.

Example 11.1.13 Understanding a function of three variables.

Let $f(x,y,z) = \frac{x^2+z+3\sin(y)}{x+2y-z}$. Evaluate f at the point $(3,0,2)$ and find the domain and range of f.

Solution. To evaluate the function we simply set $x = 3$, $y = 0$, and $z = 3$ in the definition of f:

$$f(3,0,2) = \frac{3^2+2+3\sin(0)}{3+2(0)-2} = 11.$$

As the domain of f is not specified, we take it to be the set of all triples (x,y,z) for which $f(x,y,z)$ is defined. As we cannot divide by 0, we find the domain D is

$$D = \{(x,y,z) \,|\, x+2y-z \neq 0\}.$$

We recognize that the set of all points in $\mathbb{R}^3$ that *are not* in D form a plane in space that passes through the origin (with normal vector $\langle 1,2,-1\rangle$).
We determine the range R is $\mathbb{R}$; that is, all real numbers are possible outputs of f. There is no set way of establishing this. Rather, to get numbers near 0 we can let $y = 0$ and choose $z \approx -x^2$. To get numbers of arbitrarily large magnitude, we can let $z \approx x + 2y$.

11.1.4 Level Surfaces

It is very difficult to produce a meaningful graph of a function of three variables. A function of *one* variable is a *curve* drawn in *2* dimensions; a function of *two* variables is a *surface* drawn in *3* dimensions; a function of *three* variables is a *hypersurface* drawn in *4* dimensions.

There are a few techniques one can employ to try to "picture" a graph of three variables. One is an analogue of level curves: *level surfaces*. Given $w = f(x,y,z)$, the level surface at $w = c$ is the surface in space formed by all points (x,y,z) where $f(x,y,z) = c$.

Example 11.1.14 Finding level surfaces.

If a point source S is radiating energy, the intensity I at a given point P in space is inversely proportional to the square of the distance between S and P. That is, when $S = (0,0,0)$, $I(x,y,z) = \frac{k}{x^2+y^2+z^2}$ for some constant k.
Let $k = 1$; find the level surfaces of I.

Solution. We can (mostly) answer this question using "common sense." If energy (say, in the form of light) is emanating from the origin, its intensity will be the same at all points equidistant from the origin. That is, at any point on the surface of a sphere centered at the origin, the intensity should be the same. Therefore, the level surfaces are spheres.
We now find this mathematically. The level surface at $I = c$ is defined by

$$c = \frac{1}{x^2+y^2+z^2}.$$

A small amount of algebra reveals

$$x^2 + y^2 + z^2 = \frac{1}{c}.$$

Given an intensity c, the level surface $I = c$ is a sphere of radius $1/\sqrt{c}$, centered at the origin.

Figure 11.1.15 gives a table of the radii of the spheres for given c values. Normally one would use equally spaced c values, but these values have been chosen purposefully. At a distance of 0.25 from the point source, the intensity is 16; to move to a point of half that intensity, one just moves out 0.1 to 0.35 — not much at all. To again halve the intensity, one moves 0.15, a little more than before.

Note how each time the intensity if halved, the distance required to move away grows. We conclude that the closer one is to the source, the more rapidly the intensity changes.

c	r
16	0.25
8	0.35
4	0.5
2	0.71
1	1
0.5	1.41
0.25	2
0.125	2.83
0.0625	4

Figure 11.1.15 A table of c values and the corresponding radius r of the spheres of constant value in Example 11.1.14

In the next section we apply the concepts of limits to functions of two or more variables.

11.1.5 Exercises

Terms and Concepts

1. Give two examples (other than those given in the text) of "real world" functions that require more than one input.

2. The graph of a function of two variables is a ___________.

3. Most people are familiar with the concept of level curves in the context of ___________ maps.

4. T/F: Along a level curve, the output of a function does not change.

5. The analogue of a level curve for functions of three variables is a level ___________.

6. What does it mean when level curves are close together? Far apart?

Problems

Exercise Group. In the following exercises, give the domain and range of the multivariable function.

7. $f(x, y) = x^2 + y^2 + 2$

8. $f(x, y) = x + 2y$

9. $f(x, y) = x - 2y$

10. $f(x, y) = \dfrac{1}{x + 2y}$

11. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

12. $f(x, y) = \sin(x)\cos(y)$

13. $f(x, y) = \sqrt{9 - x^2 - y^2}$

14. $f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2 - 9}}$

Exercise Group. In the following exercises, describe in words and sketch the level curves for the function and given c values.

15. $f(x, y) = 3x - 2y; c = -2, 0, 2$

16. $f(x, y) = x^2 - y^2; c = -1, 0, 1$

17. $f(x, y) = x - y^2; c = -2, 0, 2$

18. $f(x, y) = \dfrac{1 - x^2 - y^2}{2y - 2x}; c = -2, 0, 2$

19. $f(x, y) = \dfrac{2x - 2y}{x^2 + y^2 + 1}; c = -1, 0, 1$

20. $f(x, y) = \dfrac{y - x^3 - 1}{x}; c = -3, -1, 0, 1, 3$

21. $f(x, y) = \sqrt{x^2 + 4y^2}; c = 1, 2, 3, 4$

22. $f(x, y) = x^2 + 4y^2; c = 1, 2, 3, 4$

Exercise Group. In the following exercises, give the domain and range of the functions of three variables.

23. $f(x, y, z) = \dfrac{x}{x + 2y - 4z}$

24. $f(x, y, z) = \dfrac{1}{1 - x^2 - y^2 - z^2}$

25. $f(x, y, z) = \sqrt{z - x^2 + y^2}$

26. $f(x, y, z) = z^2 \sin(x)\cos(y)$

Exercise Group. In the following exercises, describe the level surfaces of the given functions of three variables.

27. $f(x, y, z) = x^2 + y^2 + z^2$

28. $f(x, y, z) = z - x^2 + y^2$

29. $f(x, y, z) = \dfrac{x^2 + y^2}{z}$

30. $f(x, y, z) = \dfrac{z}{x - y}$

31. Compare the level curves of Exercises 21 and 22. How are they similar, and how are they different? Each surface is a quadric surface; describe how the level curves are consistent with what we know about each surface.

11.2 Limits and Continuity of Multivariable Functions

We continue with the pattern we have established in this text: after defining a new kind of function, we apply calculus ideas to it. The previous section defined functions of two and three variables; this section investigates what it means for these functions to be "continuous."

We begin with a series of definitions. We are used to "open intervals" such as $(1, 3)$, which represents the set of all x such that $1 < x < 3$, and "closed intervals" such as $[1, 3]$, which represents the set of all x such that $1 \leq x \leq 3$. We need analogous definitions for open and closed sets in the xy-plane.

youtu.be/watch?v=GIamhgb3Ilk

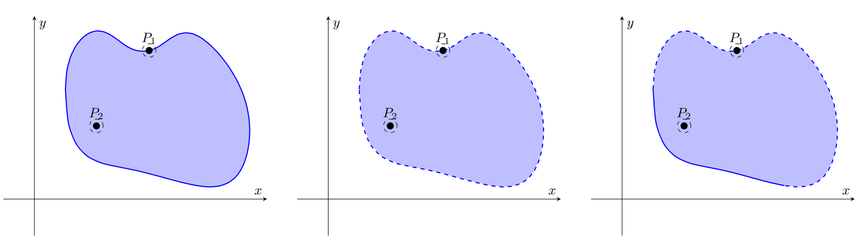
Figure 11.2.1 Introducing limits and continuity for functions of several variables

11.2.1 Open and Closed Subsets in Higher Dimensions

Definition 11.2.2 Open Disk, Boundary and Interior Points, Open and Closed Sets, Bounded Sets.

An **open disk** B in $\mathbb{R}^2$ centered at (x_0, y_0) with radius r is the set of all points (x, y) such that $\sqrt{(x - x_0)^2 + (y - y_0)^2} < r$.
Let S be a set of points in $\mathbb{R}^2$. A point P in $\mathbb{R}^2$ is a **boundary point** of S if all open disks centered at P contain both points in S and points not in S.
A point P in S is an **interior point** of S if there is an open disk centered at P that contains only points in S.
A set S is **open** if every point in S is an interior point.
A set S is **closed** if it contains all of its boundary points.
A set S is **bounded** if there is an $M > 0$ such that the open disk, centered at the origin with radius M, contains S. A set that is not bounded is **unbounded**.

Figure 11.2.3 shows several sets in the xy-plane. In each set, point P_1 lies on the boundary of the set as all open disks centered there contain both points in, and not in, the set. In contrast, point P_2 is an interior point for there is an open disk centered there that lies entirely within the set.



(a) **(b)** **(c)**

Figure 11.2.3 Illustrating open and closed sets in the xy-plane

The set depicted in Figure 11.2.3(a) is a closed set as it contains all of its boundary points. The set in Figure 11.2.3(b) is open, for all of its points are interior points (or, equivalently, it does not contain any of its boundary points). The set in Figure 11.2.3(c) is neither open nor closed as it contains some of its boundary points.

Example 11.2.4 Determining open/closed, bounded/unbounded.

Determine if the domain of the function $f(x, y) = \sqrt{1 - x^2/9 - y^2/4}$ is open, closed, or neither, and if it is bounded.

Solution. This domain of this function was found in Example 11.1.4 to be $D = \{(x,y)\,|\, \frac{x^2}{9} + \frac{y^2}{4} \leq 1\}$, the region *bounded* by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Since the region includes the boundary (indicated by the use of "$\leq$"), the set contains all of its boundary points and hence is closed. The region is bounded as a disk of radius 4, centered at the origin, contains D.

Example 11.2.5 Determining open/closed, bounded/unbounded.

Determine if the domain of $f(x,y) = \frac{1}{x-y}$ is open, closed, or neither.

Solution. As we cannot divide by 0, we find the domain to be $D = \{(x,y)\,|\, x - y \neq 0\}$. In other words, the domain is the set of all points (x,y) *not* on the line $y = x$.

The domain is sketched in Figure 11.2.6. Note how we can draw an open disk around any point in the domain that lies entirely inside the domain, and also note how the only boundary points of the domain are the points on the line $y = x$. We conclude the domain is an open set. The set is unbounded.

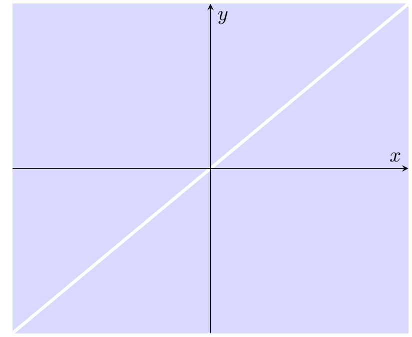


Figure 11.2.6 Sketching the domain of the function in Example 11.2.5

While our first limit definition was defined over an open interval, we now define limits over a set S in the plane (where S does not have to be open). As planar sets can be far more complicated than intervals, our definition adds the restriction "... where every open disk centered at P contains points in S other than P." In this text, all sets we'll consider will satisfy this condition and we won't bother to check; it is included in the definition for completeness.

11.2.2 Limits

Recall a pseudo-definition of the limit of a function of one variable:

"$\lim\limits_{x\to c} f(x) = L$"

means that if x is "really close" to c, then $f(x)$ is "really close" to L. A similar pseudo-definition holds for functions of two variables. We'll say that

"$\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$"

means "if the point (x,y) is really close to the point (x_0,y_0), then $f(x,y)$ is really close to L." The formal definition is given below.

Definition 11.2.7 Limit of a Function of Two Variables.

Let S be a set containing $P = (x_0,y_0)$ where every open disk centered at P contains points in S other than P, let f be a function of two variables defined on S, except possibly at P, and let L be a real number. The **limit** of $f(x,y)$ as (x,y) approaches (x_0,y_0) is L, denoted

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L,$$

means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all (x,y) in S, where $(x,y) \neq (x_0,y_0)$, if (x,y) is in the open disk centered at (x_0,y_0) with radius δ, then $|f(x,y) - L| < \varepsilon$.

The concept behind Definition 11.2.7 is sketched in Figure 11.2.8. Given $\varepsilon > 0$, find $\delta > 0$ such that if (x,y) is any point in the open disk centered at (x_0,y_0) in the xy-plane with radius δ, then $f(x,y)$ should be within ε of L.

Computing limits using this definition is rather cumbersome. The following theorem allows us to evaluate limits much more easily.

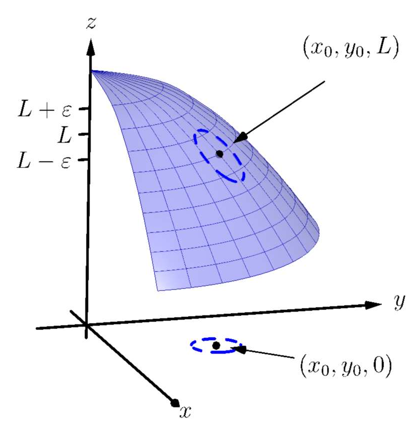


Figure 11.2.8 *Illustrating the definition of a limit.* The open disk in the xy-plane has radius δ. Let (x,y) be any point in this disk; $f(x,y)$ is within ε of L.

Theorem 11.2.9 Basic Limit Properties of Functions of Two Variables.

Let b, x_0, y_0, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \text{ and } \lim_{(x,y)\to(x_0,y_0)} g(x,y) = K.$$

The following limits hold.

1. *Constants:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} b = b$
2. *Identity* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} x = x_0;\ \lim_{(x,y)\to(x_0,y_0)} y = y_0$
3. *Sums/Differences:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \big(f(x,y) \pm g(x,y)\big) = L \pm K$
4. *Scalar Multiples:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} b\cdot f(x,y) = bL$
5. *Products:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)\cdot g(x,y) = LK$
6. *Quotients:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)/g(x,y) = L/K$, $(K \neq 0)$
7. *Powers:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)^n = L^n$

This theorem, combined with Theorems 1.3.4 and 1.3.7 of Section 1.3, allows us to evaluate many limits.

Example 11.2.10 Evaluating a limit.

Evaluate the following limits:

1. $\displaystyle\lim_{(x,y)\to(1,\pi)} \left(\frac{y}{x} + \cos(xy)\right)$
2. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}$

Solution.

1. The aforementioned theorems allow us to simply evaluate $y/x + \cos(xy)$ when $x = 1$ and $y = \pi$. If an indeterminate form is returned, we must do more work to evaluate the limit; otherwise, the result is the limit. Therefore

$$\begin{aligned}\lim_{(x,y)\to(1,\pi)} \frac{y}{x} + \cos(xy) &= \frac{\pi}{1} + \cos(\pi)\\ &= \pi - 1.\end{aligned}$$

2. We attempt to evaluate the limit by substituting 0 in for x and y, but the result is the indeterminate form "$0/0$." To evaluate this limit, we must "do more work," but we have not yet learned what "kind" of work to do. Therefore we cannot yet evaluate this limit.

When dealing with functions of a single variable we also considered onesided limits and stated

$$\lim_{x\to c} f(x) = L \text{ if, and only if, } \lim_{x\to c^+} f(x) = L \textbf{ and } \lim_{x\to c^-} f(x) = L.$$

That is, the limit is L if and only if $f(x)$ approaches L when x approaches c from *either* direction, the left or the right.

In the plane, there are infinitely many directions from which (x, y) might approach (x_0, y_0). In fact, we do not have to restrict ourselves to approaching (x_0, y_0) from a particular direction, but rather we can approach that point along a path that is not a straight line. It is possible to arrive at different limiting values by approaching (x_0, y_0) along different paths. If this happens, we say that $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist (this is analogous to the left and right hand limits of single variable functions not being equal).

Our theorems tell us that we can evaluate most limits quite simply, without worrying about paths. When indeterminate forms arise, the limit may or may not exist. If it does exist, it can be difficult to prove this as we need to show the same limiting value is obtained regardless of the path chosen. The case where the limit does not exist is often easier to deal with, for we can often pick two paths along which the limit is different.

Example 11.2.11 Showing limits do not exist.

1. Show $\lim\limits_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}$ does not exist by finding the limits along the lines $y = mx$.
2. Show $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$ does not exist by finding the limit along the path $y = -\sin(x)$.

Solution.

1. Evaluating $\lim\limits_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}$ along the lines $y = mx$ means replace all y's with mx and evaluating the resulting limit:

$$\begin{aligned}\lim_{(x,mx)\to(0,0)} \frac{3x(mx)}{x^2+(mx)^2} &= \lim_{x\to 0} \frac{3mx^2}{x^2(m^2+1)}\\ &= \lim_{x\to 0} \frac{3m}{m^2+1}\\ &= \frac{3m}{m^2+1}.\end{aligned}$$

   While the limit exists for each choice of m, we get a *different* limit for each choice of m. That is, along different lines we get differing limiting values, meaning *the* limit does not exist.

2. Let $f(x, y) = \frac{\sin(xy)}{x+y}$. We are to show that $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist by finding the limit along the path $y = -\sin(x)$. First, however, consider the limits found along the lines $y = mx$ as done above.

$$\begin{aligned}\lim_{(x,mx)\to(0,0)} \frac{\sin\big(x(mx)\big)}{x+mx} &= \lim_{x\to 0} \frac{\sin(mx^2)}{x(m+1)}\\ &= \lim_{x\to 0} \frac{\sin(mx^2)}{x}\cdot\frac{1}{m+1}.\end{aligned}$$

By applying L'Hospital's Rule, we can show this limit is 0 *except* when $m = -1$, that is, along the line $y = -x$. This line is not in the domain of f, so we have found the following fact: along every line $y = mx$ in the domain of f, $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0$. Now consider the limit along the path $y = -\sin(x)$:

$$\lim_{(x,-\sin(x))\to(0,0)} \frac{\sin\big(-x\sin(x)\big)}{x-\sin(x)} = \lim_{x\to 0} \frac{\sin\big(-x\sin(x)\big)}{x-\sin(x)}$$

Now apply L'Hospital's Rule twice:

$$= \lim_{x\to 0} \frac{\cos\big(-x\sin(x)\big)(-\sin(x)-x\cos(x))}{1-\cos(x)} \qquad (0/0)$$

$$= \lim_{x\to 0} \frac{-\sin\big(-x\sin(x)\big)(-\sin(x)-x\cos(x))^2+\cos\big(-x\sin(x)\big)(-2\cos(x)+x\sin(x))}{\sin(x)}.$$

This last limit is of the form "$2/0$", which suggests that the limit does not exist. Step back and consider what we have just discovered. Along any line $y = mx$ in the domain of the $f(x, y)$, the limit is 0. However, along the path $y = -\sin(x)$, which lies in the domain of $f(x, y)$ for all $x \neq 0$, the limit does not exist. Since the limit is not the same along every path to $(0, 0)$, we say $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$ does not exist.

Example 11.2.12 Finding a limit.

Let $f(x, y) = \frac{5x^2y^2}{x^2+y^2}$. Find $\lim\limits_{(x,y)\to(0,0)} f(x, y)$.

Solution. It is relatively easy to show that along any line $y = mx$, the limit is 0. This is not enough to prove that the limit exists, as demonstrated in the previous example, but it tells us that if the limit does exist then it must be 0.

To prove the limit is 0, we apply Definition 11.2.7. Let $\varepsilon > 0$ be given. We want to find $\delta > 0$ such that if $\sqrt{(x-0)^2+(y-0)^2} < \delta$, then $|f(x, y) - 0| < \varepsilon$.

Set $\delta < \sqrt{\varepsilon/5}$. Note that $\left|\frac{5y^2}{x^2+y^2}\right| < 5$ for all $(x, y) \neq (0, 0)$, and that if $\sqrt{x^2+y^2} < \delta$, then $x^2 < \delta^2$.

Let $\sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2} < \delta$. Consider $|f(x, y) - 0|$:

$$\begin{aligned}
|f(x,y)-0| &= \left|\frac{5x^2y^2}{x^2+y^2} - 0\right| \\
&= \left|x^2 \cdot \frac{5y^2}{x^2+y^2}\right| \\
&< \delta^2 \cdot 5 \\
&< \frac{\varepsilon}{5} \cdot 5 \\
&= \varepsilon.
\end{aligned}$$

Thus if $\sqrt{(x-0)^2+(y-0)^2} < \delta$ then $|f(x, y) - 0| < \varepsilon$, which is what we wanted to show. Thus $\lim\limits_{(x,y)\to(0,0)} \frac{5x^2y^2}{x^2+y^2} = 0$.

11.2.3 Continuity

Definition 1.5.1 defines what it means for a function of one variable to be continuous. In brief, it meant that the graph of the function did not have breaks, holes, jumps, etc. We define continuity for functions of two variables in a similar way as we did for functions of one variable.

Definition 11.2.13 Continuous.

Let a function $f(x,y)$ be defined on a set S containing the point (x_0,y_0).

1. f is *continuous* at (x_0,y_0) if $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.
2. f is *continuous on* S if f is continuous at all points in S. If f is continuous at all points in $\mathbb{R}^2$, we say that f is *continuous everywhere*.

Example 11.2.14 Continuity of a function of two variables.

Let $f(x,y) = \begin{cases} \frac{\cos(y)\sin(x)}{x} & x \neq 0 \\ \cos(y) & x = 0 \end{cases}$. Is f continuous at $(0,0)$? Is f continuous everywhere?

Solution. To determine if f is continuous at $(0,0)$, we need to compare $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ to $f(0,0)$.

Applying the definition of f, we see that $f(0,0) = \cos(0) = 1$.

We now consider the limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$. Substituting 0 for x and y in $(\cos(y)\sin(x))/x$ returns the indeterminate form "0/0", so we need to do more work to evaluate this limit.

Consider two related limits: $\lim\limits_{(x,y)\to(0,0)} \cos(y)$ and $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x)}{x}$. The first limit does not contain x, and since $\cos(y)$ is continuous,

$$\lim_{(x,y)\to(0,0)} \cos(y) = \lim_{y\to 0} \cos(y) = \cos(0) = 1.$$

The second limit does not contain y. By Theorem 1.3.17 we can say

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

Finally, Theorem 11.2.9 of this section states that we can combine these two limits as follows:

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{\cos(y)\sin(x)}{x} &= \lim_{(x,y)\to(0,0)} (\cos(y)) \left(\frac{\sin(x)}{x}\right) \\ &= \left(\lim_{(x,y)\to(0,0)} \cos(y)\right)\left(\lim_{(x,y)\to(0,0)} \frac{\sin(x)}{x}\right) \\ &= (1)(1) \\ &= 1. \end{aligned}$$

We have found that $\lim_{(x,y)\to(0,0)} \frac{\cos(y)\sin(x)}{x} = f(0,0)$, so f is continuous at $(0,0)$.

A similar analysis shows that f is continuous at all points in $\mathbb{R}^2$. As long as $x \neq 0$, we can evaluate the limit directly; when $x = 0$, a similar analysis shows that the limit is $\cos(y)$. Thus we can say that f is continuous everywhere. A graph of f is given in Figure 11.2.15. Notice how it has no breaks, jumps, etc.

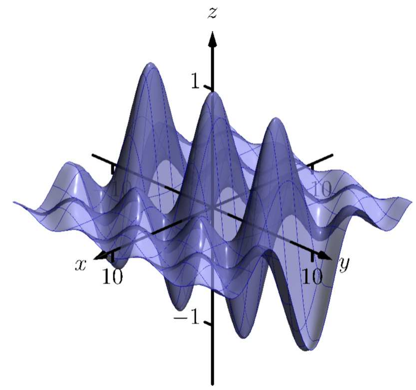


Figure 11.2.15 A graph of $f(x,y)$ in Example 11.2.14

The following theorem is very similar to Theorem 1.5.10, giving us ways to combine continuous functions to create other continuous functions.

Theorem 11.2.16 Properties of Continuous Functions.

Let f and g be continuous on a set S, let c be a real number, and let n be a positive integer. The following functions are continuous on S.

1. *Sums/Differences:* $f \pm g$
2. *Constant Multiples:* $c \cdot f$
3. *Products:* $f \cdot g$
4. *Quotients:* f/g *{ (as longs as $g \neq 0$ on S)}*
5. *Powers:* f^n
6. *Roots:* $\sqrt[n]{f}$ *(if n is even then $f \geq 0$ on S; if n is odd, then true for all values of f on S.)*
7. *Compositions:Adjust the definitions of f and g to: Let f be continuous on S, where the range of f on S is J, and let g be a single variable function that is continuous on J. Then $g \circ f$, i.e., $g(f(x,y))$, is continuous on S.*

Example 11.2.17 Establishing continuity of a function.

Let $f(x,y) = \sin(x^2\cos(y))$. Show f is continuous everywhere.

Solution. We will apply both Theorems 1.5.10 and 11.2.16. Let $f_1(x,y) = x^2$. Since y is not actually used in the function, and polynomials are continuous (by Theorem 1.5.10), we conclude f_1 is continuous everywhere. A similar statement can be made about $f_2(x,y) = \cos(y)$. Part 3 of Theorem 11.2.16 states that $f_3 = f_1 \cdot f_2$ is continuous everywhere, and Part 7 of the theorem states the composition of sine with f_3 is continuous: that is, $\sin(f_3) = \sin(x^2\cos(y))$ is continuous everywhere.

11.2.4 Functions of Three Variables

The definitions and theorems given in this section can be extended in a natural way to definitions and theorems about functions of three (or more) variables. We cover the key concepts here; some terms from Definitions 11.2.2 and 11.2.13 are not redefined but their analogous meanings should be clear to the reader.

Definition 11.2.18 Open Balls, Limit, Continuous.

1. An *open ball* in $\mathbb{R}^3$ centered at (x_0, y_0, z_0) with radius r is the set of all points (x, y, z) such that

$$\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}=r.$$

2. Let D be a set in $\mathbb{R}^3$ containing (x_0, y_0, z_0) where every open ball centered at (x_0, y_0, z_0) contains points of D other than (x_0, y_0, z_0), and let $f(x, y, z)$ be a function of three variables defined on D, except possibly at (x_0, y_0, z_0). The *limit* of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0) is L, denoted

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z)=L,$$

means that given any $\varepsilon > 0$, there is a $\delta > 0$ such that for all (x, y, z) in D, $(x, y, z) \neq (x_0, y_0, z_0)$, if (x, y, z) is in the open ball centered at (x_0, y_0, z_0) with radius δ, then $|f(x, y, z) - L| < \varepsilon$.

3. Let $f(x, y, z)$ be defined on a set D containing (x_0, y_0, z_0). We say f is *continuous* at (x_0, y_0, z_0) if

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z)=f(x_0,y_0,z_0).$$

If f is continuous at all points in D, we say f is *continuous on* D.

These definitions can also be extended naturally to apply to functions of four or more variables. Theorem 11.2.16 also applies to function of three or more variables, allowing us to say that the function

$$f(x,y,z)=\frac{e^{x^2+y}\sqrt{y^2+z^2+3}}{\sin(xyz)+5}$$

is continuous everywhere.

When considering single variable functions, we studied limits, then continuity, then the derivative. In our current study of multivariable functions, we have studied limits and continuity. In the next section we study derivation, which takes on a slight twist as we are in a multivariable context.

11.2.5 Exercises

Terms and Concepts

1. Describe in your own words the difference between the boundary and interior points of a set.

2. Use your own words to describe (informally) what $\lim\limits_{(x,y)\to(1,2)} f(x,y) = 17$ means.

3. Give an example of a closed, bounded set.

4. Give an example of a closed, unbounded set.

5. Give an example of a open, bounded set.

6. Give an example of a open, unbounded set.

Problems

Exercise Group. A set S is given.

(a) Give one boundary point and one interior point, when possible, of S.

(b) State whether S is open, closed, or neither.

(c) State whether S is bounded or unbounded.

7. $S = \left\{(x,y) \,\middle|\, \dfrac{(x-1)^2}{4} + \dfrac{(y-3)^2}{9} \le 1\right\}$

8. $S = \{(x,y) \mid y \neq x^2\}$

9. $S = \{(x,y) \mid x^2 + y^2 = 1\}$

10. $S = \{(x,y) \mid y > \sin(x)\}$.

Exercise Group. In the following exercises:

(a) Find the domain D of the given function.

(b) State whether D is an open or closed set.

(c) State whether D is bounded or unbounded.

11. $f(x,y) = \sqrt{9 - x^2 - y^2}$

12. $f(x,y) = \sqrt{y - x^2}$

13. $f(x,y) = \dfrac{1}{\sqrt{y - x^2}}$

14. $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

Exercise Group. In the following exercises, a limit is given. Evaluate the limit along the paths given, then state why these results show the given limit does not exist.

15. $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

(a) Along the path $y = 0$.

(b) Along the path $x = 0$.

16. $\lim\limits_{(x,y)\to(0,0)} \frac{x+y}{x-y}$

Along the path $y = mx$.

17. $\lim\limits_{(x,y)\to(0,0)} \frac{xy-y^2}{y^2+x}$

(a) Along the path $y = mx$.

(b) Along the path $x = 0$.

18. $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x^2)}{y}$

(a) Along the path $y = mx$.

(b) Along the path $y = x^2$.

19. $\lim\limits_{(x,y)\to(1,2)} \frac{x+y-3}{x^2-1}$

(a) Along the path $y = 2$.

(b) Along the path $y = x + 1$.

20. $\lim\limits_{(x,y)\to(\pi,\pi/2)} \frac{\sin(x)}{\cos(y)}$

(a) Along the path $x = \pi$.

(b) Along the path $y = x - \pi/2$.

11.3 Partial Derivatives

Let y be a function of x. We have studied in great detail the derivative of y with respect to x, that is, $\frac{dy}{dx}$, which measures the rate at which y changes with respect to x. Consider now $z = f(x, y)$. It makes sense to want to know how z changes with respect to x and/or y. This section begins our investigation into these rates of change.

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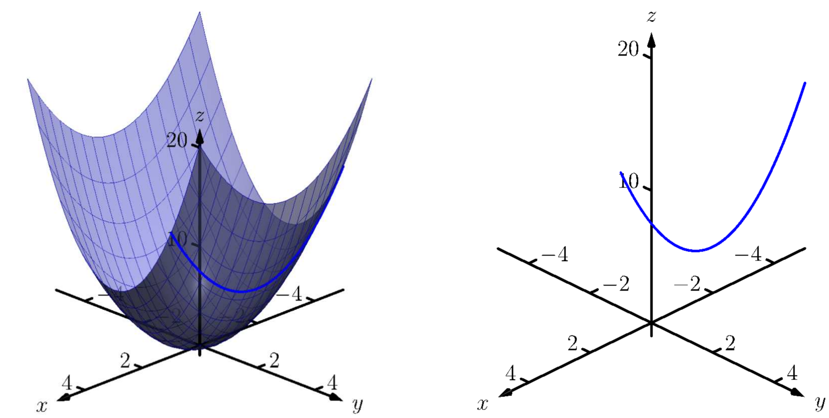
Figure 11.3.1 Motivating the concept of the partial derivative

11.3.1 First-order partial derivatives

Consider the function $f(x, y) = x^2 + 2y^2$, as graphed in Figure 11.3.2(a). By fixing $y = 2$, we focus our attention to all points on the surface where the y-value is 2, shown in both Figure 11.3.2(a) and Figure 11.3.2(b). These points form a curve in the plane $y = 2$: $z = f(x, 2) = x^2 + 8$ which defines z as a function of just one variable. We can take the derivative of z with respect to x along this curve and find equations of tangent lines, etc.



(a) **(b)**

Figure 11.3.2 By fixing $y = 2$, the surface $z = f(x, y) = x^2 + 2y^2$ is a curve in space

The key notion to extract from this example is: by treating y as constant (it does not vary) we can consider how z changes with respect to x. In a similar fashion, we can hold x constant and consider how z changes with respect to y. This is the underlying principle of *partial derivatives*. We state the formal, limit-based definition first, then show how to compute these partial derivatives without directly taking limits.

Definition 11.3.3 Partial Derivative.

Let $z = f(x, y)$ be a continuous function on a set S in $\mathbb{R}^2$.

1. The *partial derivative of f with respect to x* is:

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

2. The *partial derivative of f with respect to y* is:

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

Alternate notations for $f_x(x, y)$ include:

$$\frac{\partial}{\partial x} f(x, y), \quad \frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x}, \quad \text{and } z_x,$$

with similar notations for $f_y(x, y)$. For ease of notation, $f_x(x, y)$ is often abbreviated f_x.

Example 11.3.4 Computing partial derivatives with the limit definition.

Let $f(x,y) = x^2y + 2x + y^3$. Find $f_x(x,y)$ using the limit definition.

Solution. Using Definition 11.3.3, we have:

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2y + 2(x+h) + y^3 - (x^2y + 2x + y^3)}{h} \\ &= \lim_{h\to 0} \frac{x^2y + 2xhy + h^2y + 2x + 2h + y^3 - (x^2y + 2x + y^3)}{h} \\ &= \lim_{h\to 0} \frac{2xhy + h^2y + 2h}{h} \\ &= \lim_{h\to 0} 2xy + hy + 2 \\ &= 2xy + 2. \end{aligned}$$

We have found $f_x(x,y) = 2xy + 2$.

Example 11.3.4 found a partial derivative using the formal, limit-based definition. Using limits is not necessary, though, as we can rely on our previous knowledge of derivatives to compute partial derivatives easily. When computing $f_x(x,y)$, we hold y fixed — it does not vary. Therefore we can compute the derivative with respect to x by treating y as a constant or coefficient.

Just as $\frac{d}{dx}\left(5x^2\right) = 10x$, we compute $\frac{\partial}{\partial x}\left(x^2y\right) = 2xy$. Here we are treating y as a coefficient.

Just as $\frac{d}{dx}\left(5^3\right) = 0$, we compute $\frac{\partial}{\partial x}\left(y^3\right) = 0$. Here we are treating y as a constant. More examples will help make this clear.

Example 11.3.5 Finding partial derivatives.

Find $f_x(x,y)$ and $f_y(x,y)$ in each of the following.

1. $f(x,y) = x^3y^2 + 5y^2 - x + 7$
2. $f(x,y) = \cos(xy^2) + \sin(x)$
3. $f(x,y) = e^{x^2y^3}\sqrt{x^2+1}$

Solution.

1. We have $f(x,y) = x^3y^2 + 5y^2 - x + 7$. Begin with $f_x(x,y)$. Keep y fixed, treating it as a constant or coefficient, as appropriate:

   $$f_x(x,y) = 3x^2y^2 - 1.$$

   Note how the $5y^2$ and 7 terms go to zero. To compute $f_y(x,y)$, we hold x fixed:

   $$f_y(x,y) = 2x^3y + 10y.$$

   Note how the $-x$ and 7 terms go to zero.

2. We have $f(x,y) = \cos(xy^2) + \sin(x)$. Begin with $f_x(x,y)$. We need to apply the Chain Rule with the cosine term; y^2 is the coefficient of the x-term inside the cosine function.

   $$f_x(x,y) = -\sin(xy^2)(y^2) + \cos(x) = -y^2\sin(xy^2) + \cos(x).$$

To find $f_y(x, y)$, note that x is the coefficient of the y^2 term inside of the cosine term; also note that since x is fixed, $\sin(x)$ is also fixed, and we treat it as a constant.

$$f_y(x, y) = -\sin(xy^2)(2xy) = -2xy\sin(xy^2).$$

3. We have $f(x, y) = e^{x^2y^3}\sqrt{x^2+1}$. Beginning with $f_x(x, y)$, note how we need to apply the Product Rule.

$$\begin{aligned} f_x(x, y) &= e^{x^2y^3}(2xy^3)\sqrt{x^2+1} + e^{x^2y^3}\frac{1}{2}\left(x^2+1\right)^{-1/2}(2x) \\ &= 2xy^3e^{x^2y^3}\sqrt{x^2+1} + \frac{xe^{x^2y^3}}{\sqrt{x^2+1}}. \end{aligned}$$

Note that when finding $f_y(x, y)$ we do not have to apply the Product Rule; since $\sqrt{x^2+1}$ does not contain y, we treat it as fixed and hence becomes a coefficient of the $e^{x^2y^3}$ term.

$$f_y(x, y) = e^{x^2y^3}(3x^2y^2)\sqrt{x^2+1} = 3x^2y^2e^{x^2y^3}\sqrt{x^2+1}.$$

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Figure 11.3.6 Additional partial derivative computation examples

We have shown *how* to compute a partial derivative, but it may still not be clear what a partial derivative *means*. Given $z = f(x, y)$, $f_x(x, y)$ measures the rate at which z changes as only x varies: y is held constant.

Imagine standing in a rolling meadow, then beginning to walk due east. Depending on your location, you might walk up, sharply down, or perhaps not change elevation at all. This is similar to measuring z_x: you are moving only east (in the "x"-direction) and not north/south at all. Going back to your original location, imagine now walking due north (in the "y"-direction). Perhaps walking due north does not change your elevation at all. This is analogous to $z_y = 0$: z does not change with respect to y. We can see that z_x and z_y do not have to be the same, or even similar, as it is easy to imagine circumstances where walking east means you walk downhill, though walking north makes you walk uphill.

The following example helps us visualize this more.

Example 11.3.8 Evaluating partial derivatives.

Let $z = f(x, y) = -x^2 - \frac{1}{2}y^2 + xy + 10$. Find $f_x(2, 1)$ and $f_y(2, 1)$ and interpret their meaning.

Solution. We begin by computing $f_x(x, y) = -2x + y$ and $f_y(x, y) = -y + x$. Thus

$$f_x(2, 1) = -3 \text{ and } f_y(2, 1) = 1.$$

It is also useful to note that $f(2, 1) = 7.5$. What does each of these numbers mean?

Consider $f_x(2, 1) = -3$, along with Figure 11.3.9(a). If one "stands" on the surface at the point $(2, 1, 7.5)$ and moves parallel to the x-axis (i.e., only the x-value changes, not the y-value), then the instantaneous rate of change is -3. Increasing the x-value will decrease the z-value; decreasing the x-value will increase the z-value.

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Figure 11.3.7 Interpreting partial derivatives

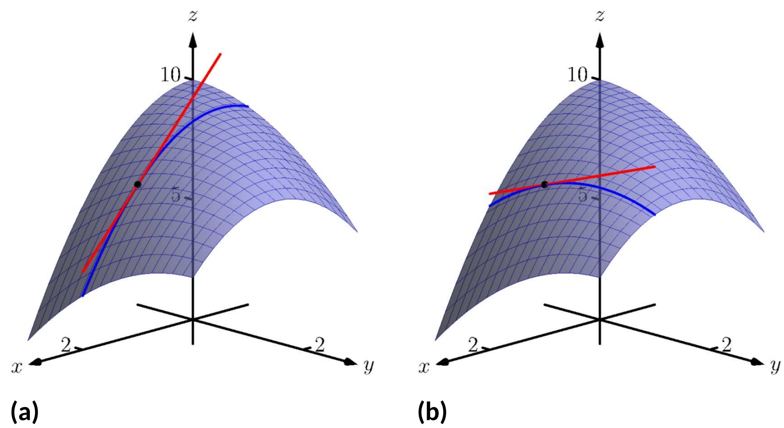


Figure 11.3.9 Illustrating the meaning of partial derivatives

Now consider $f_y(2,1) = 1$, illustrated in Figure 11.3.9(b). Moving along the curve drawn on the surface, i.e., parallel to the y-axis and not changing the x-values, increases the z-value instantaneously at a rate of 1. Increasing the y-value by 1 would increase the z-value by approximately 1.

Since the magnitude of f_x is greater than the magnitude of f_y at $(2,1)$, it is "steeper" in the x-direction than in the y-direction.

11.3.2 Tangent Planes

Another way to interpret partial derivatives is in terms of the *tangent plane*. Consider the graph of a function $f(x, y)$, such as the one in Figure 11.3.2. Setting $x = a$, $y = b$ defines a point $(a, b, f(a, b))$ on the graph. Through the point (a, b), we have the lines $x = a + s, y = b$, and $x = a, y = b + t$, parallel to the x and y axes, respectively (where s, t are parameters).

Using the function $f(x, y)$ we define two vector-valued functions:

$$\vec{r}_1(s) = \langle a+s, b, f(a+s,b)\rangle$$
$$\vec{r}_2(t) = \langle a, b+t, f(a,b+t)\rangle .$$

Both vector-valued functions define space curves that lie on the surface $z = f(x, y)$, and these curves intersect at the point $(a, b, f(a, b))$, when $s = t = 0$.

Now consider computing $\vec{r}_1'(s)$. The first two components of this derivative are found in a straightforward manner: they are 1 and 0, respectively. To find the third component of the derivative, notice that in $\vec{r}_1(s)$ we vary the x-component of f while holding the y-component constant. Using the Chain Rule and Definition 11.3.3, we find that the third component is $f_x(a + s, b)$. Altogether, we have

$$\vec{r}_1'(s) = \langle 1, 0, f_x(a+s,b)\rangle .$$

Evaluating this at $s = 0$ gives

$$\vec{v} = \vec{r}_1'(0) = \langle 1, 0, f_x(a,b)\rangle .$$

We can perform a similar process with $\vec{r}_2(t)$, ultimately leading to

$$\vec{w} = \vec{r}_2'(0) = \langle 0, 1, f_y(a,b)\rangle .$$

From Section 13.2, we know that $\vec{r}_1'(0)$ defines a tangent vector to the curve $\vec{r}_1(s)$ when $s = 0$, and similarly, $\vec{r}_2'(0)$ defines a tangent vector to the curve $\vec{r}_2(t)$ when $t = 0$.

It seems reasonable that any vector that is tangent to these curves, which lie on our surface, should also be considered tangent to that surface. The vectors $\vec{v}$ and $\vec{w}$ are therefore tangent to $z = f(x,y)$ at $(a,b,f(a,b))$, and they are definitely not parallel. From Section 12.6 we know that any two non-parallel vectors at a point define a plane through that point. We also know that taking the cross product of these two vectors gives us a normal vector: the cross product gives us

$$\vec{n} = \vec{v} \times \vec{w} = \langle -f_x(a,b), -f_y(a,b), 1 \rangle .$$

The equation of the plane through $(a,b,f(a,b))$ with normal vector $\vec{n} = \langle -f_x(a,b), -f_y(a,b), 1\rangle$ is

$$-f_x(a,b)(x-a) - f_y(a,b)(y-b) + (z - f(a,b)) = 0.$$

It is customary to solve for z in this equation and make the following definition.

Definition 11.3.10

Let $f(x,y)$ be a function whose first-order partial derivatives exist at (a,b). The **tangent plane** to the surface $z = f(x,y)$ at the point $(a,b,f(a,b))$ is the plane defined by the equation

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Example 11.3.11 Finding a tangent plane equation.

Find the equation of tangent plane to the surface $z = x^2 + 3y^2$ at $(x,y) = (1,-1)$.

Solution. Our function is $f(x,y) = x^2 + 3y^2$, and we have $f(1,-1) = 4$, so the point on the surface is $(1,-1,4)$. The partial derivatives are $f_x(x,y) = 2x$ and $f_y(x,y) = 6y$, so $f_x(1,-1) = 2$, $f_y(1,-1) = -6$. Using Definition 11.3.10, our plane is given by

$$z = 4 + 2(x-1) - 6(y+1).$$

Notice the similarity between the tangent plane equation in Definition 11.3.10 and the single variable tangent line equation $y = f(c) + f'(c)(x-c)$. As with functions of one variable, this suggests a connection between derivatives and linear approximation. We explore this connection in Section 14.1, where we'll see that Definition 11.3.10 should be strengthed to require that the partial derivatives of f be *continuous*.

11.3.3 Second-order partial derivatives

Let $z = f(x,y)$. We have learned to find the partial derivatives $f_x(x,y)$ and $f_y(x,y)$, which are each functions of x and y. Therefore we can take partial derivatives of them, each with respect to x and y. We define these "second partials" along with the notation, give examples, then discuss their meaning.

Definition 11.3.12 Second Partial Derivative, Mixed Partial Derivative.

Let $z = f(x,y)$ be continuous on a set S.

1. The *second partial derivative of f with respect to x then x* is

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = (f_x)_x = f_{xx}$$

2. The *second partial derivative of f with respect to x then y* is

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy}$$

Similar definitions hold for $\frac{\partial^2 f}{\partial y^2} = f_{yy}$ and $\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$.
The second partial derivatives f_{xy} and f_{yx} are **mixed partial derivatives**.

The notation of second partial derivatives gives some insight into the notation of the second derivative of a function of a single variable. If $y = f(x)$, then $f''(x) = \frac{d^2y}{dx^2}$. The "$d^2y$" portion means "take the derivative of y twice," while "dx^2" means "with respect to x both times." When we only know of functions of a single variable, this latter phrase seems silly: there is only one variable to take the derivative with respect to. Now that we understand functions of multiple variables, we see the importance of specifying which variables we are referring to.

The terms in Definition 11.3.12 all depend on limits, so each definition comes with the caveat "where the limit exists."

Example 11.3.13 Second partial derivatives.

For each of the following, find all six first and second partial derivatives. That is, find

$$f_x, f_y, f_{xx}, f_{yy}, f_{xy} \text{ and } f_{yx}.$$

1. $f(x,y) = x^3y^2 + 2xy^3 + \cos(x)$

2. $f(x,y) = \dfrac{x^3}{y^2}$

3. $f(x,y) = e^x \sin(x^2 y)$

Solution. In each, we give f_x and f_y immediately and then spend time deriving the second partial derivatives.

1.

$$\begin{aligned}
f(x,y) &= x^3y^2 + 2xy^3 + \cos(x)\\
f_x(x,y) &= 3x^2y^2 + 2y^3 - \sin(x)\\
f_y(x,y) &= 2x^3y + 6xy^2\\
f_{xx}(x,y) &= \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\big(3x^2y^2 + 2y^3 - \sin(x)\big)\\
&= 6xy^2 - \cos(x)\\
f_{yy}(x,y) &= \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\big(2x^3y + 6xy^2\big)\\
&= 2x^3 + 12xy\\
f_{xy}(x,y) &= \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\big(3x^2y^2 + 2y^3 - \sin(x)\big)
\end{aligned}$$

$$\begin{aligned}
&= 6x^2y + 6y^2 \\
f_{yx}(x,y) &= \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(2x^3y + 6xy^2) \\
&= 6x^2y + 6y^2
\end{aligned}$$

2.

$$\begin{aligned}
f(x,y) &= \frac{x^3}{y^2} = x^3y^{-2} \\
f_x(x,y) &= \frac{3x^2}{y^2} \\
f_y(x,y) &= -\frac{2x^3}{y^3} \\
f_{xx}(x,y) &= \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(\frac{3x^2}{y^2}) \\
&= \frac{6x}{y^2} \\
f_{yy}(x,y) &= \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(-\frac{2x^3}{y^3}) \\
&= \frac{6x^3}{y^4} \\
f_{xy}(x,y) &= \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(\frac{3x^2}{y^2}) \\
&= -\frac{6x^2}{y^3} \\
f_{yx}(x,y) &= \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(-\frac{2x^3}{y^3}) \\
&= -\frac{6x^2}{y^3}
\end{aligned}$$

3. $f(x,y) = e^x \sin(x^2y)$ Because the following partial derivatives get rather long, we omit the extra notation and just give the results. In several cases, multiple applications of the Product and Chain Rules will be necessary, followed by some basic combination of like terms.

$$\begin{aligned}
f_x(x,y) &= e^x \sin(x^2y) + 2xye^x \cos(x^2y) \\
f_y(x,y) &= x^2e^x \cos(x^2y) \\
f_{xx}(x,y) &= e^x \sin(x^2y) + 4xye^x \cos(x^2y) + 2ye^x \cos(x^2y) - 4x^2y^2e^x \sin(x^2y) \\
f_{yy}(x,y) &= -x^4e^x \sin(x^2y) \\
f_{xy}(x,y) &= x^2e^x \cos(x^2y) + 2xe^x \cos(x^2y) - 2x^3ye^x \sin(x^2y) \\
f_{yx}(x,y) &= x^2e^x \cos(x^2y) + 2xe^x \cos(x^2y) - 2x^3ye^x \sin(x^2y)
\end{aligned}$$

youtu.be/watch?v=H81s72zLfI8

Figure 11.3.14 Second order partial derivatives and differentiability classes

Notice how in each of the three functions in Example 11.3.13, $f_{xy} = f_{yx}$. Due to the complexity of the examples, this likely is not a coincidence. The following theorem states that it is not.

Theorem 11.3.15 Mixed Partial Derivatives.

Let f be defined such that f_{xy} and f_{yx} are continuous on a set S. Then for each point (x, y) in S, $f_{xy}(x, y) = f_{yx}(x, y)$.

Finding f_{xy} and f_{yx} independently and comparing the results provides a convenient way of checking our work.

11.3.4 Understanding Second Partial Derivatives

Now that we know *how* to find second partials, we investigate *what* they tell us.

Again we refer back to a function $y = f(x)$ of a single variable. The second derivative of f is "the derivative of the derivative," or "the rate of change of the rate of change." The second derivative measures how much the derivative is changing. If $f''(x) < 0$, then the derivative is getting smaller (so the graph of f is concave down); if $f''(x) > 0$, then the derivative is growing, making the graph of f concave up.

Now consider $z = f(x, y)$. Similar statements can be made about f_{xx} and f_{yy} as could be made about $f''(x)$ above. When taking derivatives with respect to x twice, we measure how much f_x changes with respect to x. If $f_{xx}(x, y) < 0$, it means that as x increases, f_x decreases, and the graph of f will be concave down *in the x-direction*. Using the analogy of standing in the rolling meadow used earlier in this section, f_{xx} measures whether one's path is concave up/down when walking due east.

Similarly, f_{yy} measures the concavity in the y-direction. If $f_{yy}(x, y) > 0$, then f_y is increasing with respect to y and the graph of f will be concave up in the y-direction. Appealing to the rolling meadow analogy again, f_{yy} measures whether one's path is concave up/down when walking due north.

We now consider the mixed partials f_{xy} and f_{yx}. The mixed partial f_{xy} measures how much f_x changes with respect to y. Once again using the rolling meadow analogy, f_x measures the slope if one walks due east. Looking east, begin walking *north* (side-stepping). Is the path towards the east getting steeper? If so, $f_{xy} > 0$. Is the path towards the east not changing in steepness? If so, then $f_{xy} = 0$. A similar thing can be said about f_{yx}: consider the steepness of paths heading north while side-stepping to the east.

The following example examines these ideas with concrete numbers and graphs.

Example 11.3.16 Understanding second partial derivatives.

Let $z = x^2 - y^2 + xy$. Evaluate the 6 first and second partial derivatives at $(-1/2, 1/2)$ and interpret what each of these numbers mean.

Solution. We find that:
$f_x(x, y) = 2x + y$,$f_y(x, y) = -2y + x$,$f_{xx}(x, y) = 2$, $f_{yy}(x, y) = -2$ and $f_{xy}(x, y) = f_{yx}(x, y) = 1$. Thus at $(-1/2, 1/2)$ we have

$$f_x(-1/2, 1/2) = -1/2, \qquad f_y(-1/2, 1/2) = -3/2.$$

The slope of the tangent line at $(-1/2, 1/2, -1/4)$ in the direction of x is $-1/2$: if one moves from that point parallel to the x-axis, the instantaneous rate of change will be $-1/2$. The slope of the tangent line at this point in the direction of y is $-3/2$: if one moves from this point parallel to the y-axis, the instantaneous rate of change will be $-3/2$. These tangents lines are graphed in Figure 11.3.17(a) and Figure 11.3.17(b), respectively, where the tangent lines are drawn in a solid line.

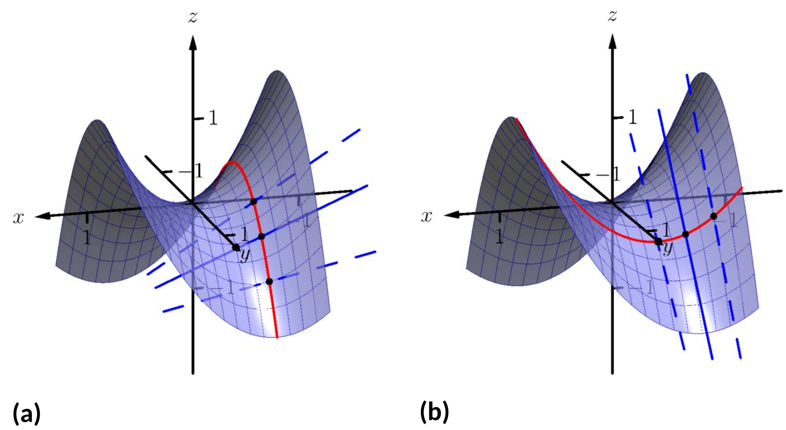


Figure 11.3.17 Understanding the second partial derivatives in Example 11.3.16

Now consider only Figure 11.3.17(a). Three directed tangent lines are drawn (two are dashed), each in the direction of x; that is, each has a slope determined by f_x. Note how as y increases, the slope of these lines get closer to 0. Since the slopes are all negative, getting closer to 0 means the *slopes are increasing*. The slopes given by f_x are increasing as y increases, meaning f_{xy} must be positive.

Since $f_{xy} = f_{yx}$, we also expect f_y to increase as x increases. Consider Figure 11.3.17(b) where again three directed tangent lines are drawn, this time each in the direction of y with slopes determined by f_y. As x increases, the slopes become less steep (closer to 0). Since these are negative slopes, this means the slopes are increasing.

Thus far we have a visual understanding of f_x, f_y, and $f_{xy} = f_{yx}$. We now interpret f_{xx} and f_{yy}. In Figure 11.3.17(a), we see a curve drawn where x is held constant at $x = -1/2$: only y varies. This curve is clearly concave down, corresponding to the fact that $f_{yy} < 0$. In part Figure 11.3.17(b) of the figure, we see a similar curve where y is constant and only x varies. This curve is concave up, corresponding to the fact that $f_{xx} > 0$.

11.3.5 Partial Derivatives and Functions of Three Variables

The concepts underlying partial derivatives can be easily extend to more than two variables. We give some definitions and examples in the case of three variables and trust the reader can extend these definitions to more variables if needed.

Definition 11.3.18 Partial Derivatives with Three Variables.

Let $w = f(x, y, z)$ be a continuous function on a set D in $\mathbb{R}^3$.
The **partial derivative of f with respect to x** is:

$$f_x(x, y, z) = \lim_{h\to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}.$$

Similar definitions hold for $f_y(x, y, z)$ and $f_z(x, y, z)$.

By taking partial derivatives of partial derivatives, we can find second partial derivatives of f with respect to z then y, for instance, just as before.

Example 11.3.19 Partial derivatives of functions of three variables.

For each of the following, find $f_x,\ f_y,\ f_z,\ f_{xz},\ f_{yz}$, and f_{zz}.

1. $f(x, y, z) = x^2y^3z^4 + x^2y^2 + x^3z^3 + y^4z^4$
2. $f(x, y, z) = x \sin(yz)$

Solution.

1.

$$\begin{aligned} f_x(x, y, z) &= 2xy^3z^4 + 2xy^2 + 3x^2z^3 \\ f_y(x, y, z) &= 3x^2y^2z^4 + 2x^2y + 4y^3z^4 \\ f_z(x, y, z) &= 4x^2y^3z^3 + 3x^3z^2 + 4y^4z^3 \\ f_{xz}(x, y, z) &= 8xy^3z^3 + 9x^2z^2 \\ f_{yz}(x, y, z) &= 12x^2y^2z^3 + 16y^3z^3 \\ f_{zz}(x, y, z) &= 12x^2y^3z^2 + 6x^3z + 12y^4z^2 \end{aligned}$$

2. $f_x = \sin(yz)$; $f_y = xz\cos(yz)$; $f_z = xy\cos(yz)$, and

$$\begin{aligned} f_{xz}(x, y, z) &= y\cos(yz) \\ f_{yz}(x, y, z) &= x\cos(yz) - xyz\sin(yz) \\ f_{zz}(x, y, z) &= -xy^2\sin(yz) \end{aligned}$$

11.3.6 Higher Order Partial Derivatives

We can continue taking partial derivatives of partial derivatives of partial derivatives of ...; we do not have to stop with second partial derivatives. These higher order partial derivatives do not have a tidy graphical interpretation; nevertheless they are not hard to compute and worthy of some practice.

We do not formally define each higher order derivative, but rather give just a few examples of the notation.

$$f_{xyx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) \text{ and}$$
$$f_{xyz}(x, y, z) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right).$$

Example 11.3.20 Higher order partial derivatives.

1. Let $f(x, y) = x^2y^2 + \sin(xy)$. Find f_{xxy} and f_{yxx}.
2. Let $f(x, y, z) = x^3e^{xy} + \cos(z)$. Find f_{xyz}.

Solution.

1. To find f_{xxy}, we first find f_x, then f_{xx}, then f_{xxy}:

$$\begin{aligned} f_x(x, y) &= 2xy^2 + y\cos(xy) \\ f_{xx}(x, y) &= 2y^2 - y^2\sin(xy) \end{aligned}$$

$$f_{xxy}(x,y) = 4y - 2y\,\text{sin}(xy) - xy^2\,\text{cos}(xy).$$

To find f_{yxx}, we first find f_y, then f_{yx}, then f_{yxx}:

$$\begin{aligned}
f_y(x,y) &= 2x^2y + x\,\text{cos}(xy)\\
f_{yx}(x,y) &= 4xy + \text{cos}(xy) - xy\,\text{sin}(xy)\\
f_{yxx}(x,y) &= 4y - y\,\text{sin}(xy) - \big(y\,\text{sin}(xy) + xy^2\,\text{cos}(xy)\big)\\
&= 4y - 2y\,\text{sin}(xy) - xy^2\,\text{cos}(xy).
\end{aligned}$$

Note how $f_{xxy} = f_{yxx}$.

2. To find f_{xyz}, we find f_x, then f_{xy}, then f_{xyz}:

$$\begin{aligned}
f_x(x,y,z) &= 3x^2e^{xy} + x^3ye^{xy}\\
f_{xy}(x,y,z) &= 3x^3e^{xy} + x^3e^{xy} + x^4ye^{xy}\\
&= 4x^3e^{xy} + x^4ye^{xy}\\
f_{xyz}(x,y,z) &= 0.
\end{aligned}$$

In the previous example we saw that $f_{xxy} = f_{yxx}$; this is not a coincidence. While we do not state this as a formal theorem, as long as each partial derivative is continuous, it does not matter the order in which the partial derivatives are taken. For instance, $f_{xxy} = f_{xyx} = f_{yxx}$.

This can be useful at times. Had we known this, the second part of Example 11.3.20 would have been much simpler to compute. Instead of computing f_{xyz} in the x, y then z orders, we could have applied the z, then x then y order (as $f_{xyz} = f_{zxy}$). It is easy to see that $f_z = -\text{sin}(z)$; then f_{zx} and f_{zxy} are clearly 0 as f_z does not contain an x or y.

A brief review of this section: partial derivatives measure the instantaneous rate of change of a multivariable function with respect to one variable. With $z = f(x,y)$, the partial derivatives f_x and f_y measure the instantaneous rate of change of z when moving parallel to the x- and y-axes, respectively. How do we measure the rate of change at a point when we do not move parallel to one of these axes? What if we move in the direction given by the vector $\langle 2,1\rangle$? Can we measure that rate of change? The answer is, of course, yes, we can. This is the topic of Section 14.3. First, we need to define what it means for a function of two variables to be *differentiable*.

11.3.7 Exercises

Terms and Concepts

1. What is the difference between a constant and a coefficient?

2. Given a function $f(x, y)$, explain in your own words how to compute f_x.

3. In the mixed partial fraction f_{xy}, which is computed first, f_x or f_y?

- f_x
- f_y

4. In the mixed partial fraction $\frac{\partial^2 f}{\partial x \partial y}$, which is computed first, f_x or f_y?

- f_x
- f_y

Problems

Exercise Group. In the following exercises, evaluate $f_x(x, y)$ and $f_y(x, y)$ at the indicated point.

5. $f(x, y) = x^2y - x + 2y + 3$ at $(1, 2)$

6. $f(x, y) = x^3 - 3x + y^2 - 6y$ at $(-1, 3)$.

7. $f(x, y) = \sin(y)\cos(x)$ at $(\pi/3, \pi/3)$

8. $f(x, y) = \ln(xy)$ at $(-2, -3)$ Find:

Exercise Group. In the following exercises, find f_x, f_y, f_{xx}, f_{yy}, f_{xy} and f_{yx}.

9. $f(x, y) = x^2y + 3x^2 + 4y - 5$

10. $f(x, y) = y^3 + 3xy^2 + 3x^2y + x^3$

11. $f(x, y) = \dfrac{x}{y}$

12. $f(x, y) = \frac{4}{xy}$

13. $f(x, y) = e^{x^2+y^2}$

14. $f(x, y) = e^{x+2y}$

15. $f(x, y) = \sin(x)\cos(y)$

16. $f(x, y) = (x + y)^3$

17. $f(x, y) = \cos(5xy^3)$

18. $f(x, y) = \sin\left(5x^2 + 2y^3\right)$

19. $f(x, y) = \sqrt{4xy^2 + 1}$

20. $f(x, y) = (2x + 5y)\sqrt{y}$

21. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

22. $f(x, y) = 5x - 17y$

23. $f(x, y) = 3x^2 + 1$

24. $f(x, y) = \ln(x^2 + y)$

25. $f(x, y) = \dfrac{\ln(x)}{4y}$

26. $f(x, y) = 5e^x \sin(y) + 9$

Exercise Group. In the following exercises, form a function $f(x, y)$ such that f_x and f_y match those given.

27. $f_x = \sin(y) + 1$,$f_y = x\cos(y)$

28. $f_x = x + y$ and $f_y = x + y$

29. $f_x = 6xy - 4y^2$,$f_y = 3x^2 - 8xy + 2$

30. $f_x = \frac{2x}{x^2+y^2}$ and $f_y = \frac{2y}{x^2+y^2}$

Exercise Group. In the following exercises, find f_x, f_y, f_z, f_{yz} and f_{zy}.

31. $f(x, y, z) = x^2e^{2y-3z}$

32. $f(x, y, z) = x^3y^2 + x^3z + y^2z$

33. $f(x, y, z) = \dfrac{3x}{7y^2z}$

34. $f(x, y, z) = \ln(xyz)$

Appendices

Appendix A

Answers to Selected Exercises

II · Math 2565: Accelerated Calculus II
6 · Techniques of Antidifferentiation
6.1 · Integration by Parts
6.1 · Exercises

Terms and Concepts

6.1.1. True
6.1.2. False
6.1.4. False

Problems

6.1.5. $\sin(x) - x\cos(x) + C$
6.1.6. $-e^{-x}(x+1) + C$
6.1.7. $-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$
6.1.8. $-x^3\cos(x) + 3x^2\sin(x) + 6x\cos(x) - 6\sin(x) + C$
6.1.9. $\frac{1}{2}e^{x^2} + C$
6.1.10. $e^x\left(x^3 - 3x^2 + 6x - 6\right) + C$
6.1.11. $-\frac{1}{2}xe^{-2x} - \frac{e^{-2x}}{4} + C$
6.1.12. $\frac{1}{2}e^x(\sin(x) - \cos(x)) + C$
6.1.13. $\frac{1}{5}e^{2x}(\sin(x) + 2\cos(x)) + C$
6.1.14. $\left(\frac{1}{58}\right)e^{7x}(7\sin(3x) - 3\cos(3x)) + C$
6.1.15. $\left(\frac{1}{16}\right)e^{8x}(\sin(8x) + \cos(8x)) + C$
6.1.16. $0.5\sin^2(x) + C$
6.1.17. $\sqrt{1-x^2} + x\sin^{-1}(x) + C$
6.1.18. $x\tan^{-1}(3x) - 0.166667\ln\left(9x^2+1\right) + C$
6.1.19. $0.5x^2\tan^{-1}(x) - \frac{x}{2} + 0.5\tan^{-1}(x) + C$
6.1.20. $-\sqrt{1-x^2} + x\cos^{-1}(x) + C$
6.1.21. $0.5x^2\ln(x) - \frac{x^2}{4} + C$
6.1.22. $\frac{1}{2}x^2\ln(x) - \frac{x^2}{4} + 2x\ln(x) - 2x + C$
6.1.23. $\frac{1}{2}x^2\ln(x-4) - \frac{1}{4}(x-4)^2 - 4x - 8\ln(x-4) + C$
6.1.24. $0.5x^2\ln\left(x^2\right) - \frac{x^2}{2} + C$
6.1.25. $0.333333x^3\ln(x) - \frac{x^3}{9} + C$
6.1.26. $2x + x\ln^2(x) - 2x\ln(x) + C$
6.1.27. $2(x-7) + (x-7)\left(\ln(x-7)\right)^2 - 2(x-7)\ln(x-7) + C$
6.1.28. $x\tan(x) + \ln(|\cos(x)|) + C$
6.1.29. $\ln(|\sin(x)|) - x\cot(x) + C$
6.1.30. $\left(\frac{2}{5}(x+2)^2 - \left(\frac{4}{3}\right)(x+2)\right)\sqrt{x+2} + C$
6.1.31. $\frac{1}{3}\left(x^2-6\right)^{\left(\frac{3}{2}\right)} + C$
6.1.32. $\sec(x) + C$
6.1.33. $x\sec(x) - \ln(|\sec(x) + \tan(x)|) + C$
6.1.34. $-x\csc(x) - \ln(|\csc(x) + \cot(x)|) + C$
6.1.35. $\frac{x}{2}(\sin(\ln(x)) - \cos(\ln(x))) + C$
6.1.36. $\sin(e^x) - e^x\cos(e^x) + C$
6.1.37. $2\sin(\sqrt{x}) - 2\sqrt{x}\cos(\sqrt{x}) + C$
6.1.38. $x\ln(\sqrt{x}) - \frac{x}{2} + C$
6.1.39. $2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$
6.1.40. $\frac{x^2}{2} + C$
6.1.41. -1
6.1.42. $-3\frac{1}{e^2}$
6.1.43. 0
6.1.44. $\frac{\pi^2}{12} - \frac{1.73205\pi^3}{216} + 1.73205\pi - 6$
6.1.45. 1
6.1.46. 0.563436
6.1.47. $\left(-\frac{9}{4}\right)e^{-8} - \left(-\frac{5}{4}\right)e^{-4}$
6.1.48. $0.5e^{\pi} + 0.5$
6.1.49. $0.2\left(-e^{3\pi} - e^{-3\pi}\right)$

6.2 · Trigonometric Integrals

6.2 · Exercises

Terms and Concepts

6.2.1. False
6.2.2. False
6.2.3. False
6.2.4. False

Problems

6.2.5. $-0.2\cos^5(x)+C$

6.2.6. $0.25\sin^4(x)+C$

6.2.7. $\frac{1}{7}(\cos(x))^7-\frac{1}{5}(\cos(x))^5+C$

6.2.8. $\frac{1}{8}(\cos(x))^8-\frac{1}{6}(\cos(x))^6+C$

6.2.9. $\frac{1}{11}(\sin(x))^{11}-\frac{2}{9}(\sin(x))^9+\frac{1}{7}(\sin(x))^7+C$

6.2.10. $-0.111111\sin^9(x)+0.428571\sin^7(x)-0.6\sin^5(x)+0.333333\sin^3(x)+C$

6.2.11. $\frac{x}{8}-0.03125\sin(4x)+C$

6.2.12. $0.5(-0.125\cos(8x)-0.5\cos(2x))+C$

6.2.13. $C-\left(\left(\frac{1}{10}\right)\cos(5x)+\left(\frac{1}{14}\right)\cos(7x)\right)$

6.2.14. $\left(\frac{1}{2}\right)\sin(x)-\left(\frac{1}{14}\right)\sin(7x)+C$

6.2.15. $\frac{1}{14\pi}\sin(7\pi x)-\frac{1}{18\pi}\sin(9\pi x)+C$

6.2.16. $0.5(\sin(x)+0.333333\sin(3x))+C$

6.2.17. $\frac{3}{\pi}\cos\left(\frac{\pi}{6}\pi x\right)+\frac{1}{\pi}\cos\left(\frac{\pi}{2}\pi x\right)+C$

6.2.18. $\frac{\tan^5(x)}{5}+C$

6.2.19. $\frac{\tan^5(x)}{5}+\frac{\tan^3(x)}{3}+C$

6.2.20. $\frac{1}{11}(\tan(x))^{11}+\frac{1}{9}(\tan(x))^9+C$

6.2.21. $\frac{1}{10}(\tan(x))^{10}+C$

6.2.22. $\frac{1}{4}(\sec(x))^4-\frac{1}{2}(\sec(x))^2+C$

6.2.23. $\frac{1}{7}(\sec(x))^7-\frac{2}{5}(\sec(x))^5+\frac{1}{3}(\sec(x))^3+C$

6.2.24. $\frac{\tan^3(x)}{3}-\tan(x)+x+C$

6.2.25. $0.25\tan(x)\sec^3(x)+0.375(\sec(x)\tan(x)+\ln(|\sec(x)+\tan(x)|))+C$

6.2.26. $0.5(\sec(x)\tan(x)-\ln(|\sec(x)+\tan(x)|))+C$

6.2.27. $0.25\tan(x)\sec^3(x)-0.125(\sec(x)\tan(x)+\ln(|\sec(x)+\tan(x)|))+C$

6.2.28. $\frac{1}{5}$

6.2.29. 0

6.2.30. 0

6.2.31. $\frac{21.3333333333333}{85.3333333333333}$

6.2.32. $\frac{2}{3}$

6.2.33. $\frac{1}{5}$

6.2.34. $\frac{8}{15}$

6.3 · Trigonometric Substitution
6.3 · Exercises

Terms and Concepts

6.3.1. backward
6.3.2. $6\sin(\theta)$ or $6\cos(\theta)$
6.3.3. **(a).** $\tan^2(\theta)+1=\sec^2(\theta)$
(b). $7\sec^2(\theta)$

Problems

6.3.5. $\frac{1}{2}\left(x\sqrt{x^2+1}+\ln\left(\sqrt{x^2+1}+x\right)\right)+C$

6.3.6. $\frac{x}{2}\sqrt{x^2+4}+2\ln\left(\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right)+C$

6.3.7. $\frac{1}{2}\sin^{-1}(x)+\frac{x}{2}\sqrt{1-x^2}+C$

6.3.8. $\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)+\frac{x}{2}\sqrt{9-x^2}+C$

6.3.9. $\frac{1}{2}x\sqrt{x^2-1}-\frac{1}{2}\ln\left(\left|x+\sqrt{x^2-1}\right|\right)+C$

6.3.10. $\frac{1}{2}x\sqrt{x^2-16}-8\ln\left(\left|\frac{x}{4}+\frac{\sqrt{x^2-16}}{4}\right|\right)+C$

6.3.11. $\frac{x}{2}\sqrt{36x^2+1}+\frac{1}{12}\ln\left(6x+\sqrt{36x^2+1}\right)+C$

6.3.12. $\frac{x}{2}\sqrt{1-49x^2}+\frac{1}{14}\sin^{-1}(7x)+C$

6.3.13. $\frac{x}{2}\sqrt{64x^2-1}-\frac{1}{16}\ln\left(\left|8x+\sqrt{64x^2-1}\right|\right)+C$

6.3.14. $9\ln\left(\frac{x}{2.44949}+\sqrt{\frac{x^2}{6}+1}\right)+C$

6.3.15. $2\sin^{-1}\left(\frac{x}{3.87298}\right)+C$

6.3.16. $3\ln\left(\left|\frac{x}{3.31662}+\sqrt{\frac{x^2}{11}-1}\right|\right)+C$

6.3.17. $\sqrt{x^2-5}-2.23607\sec^{-1}\left(\frac{x}{2.23607}\right)+C$

6.3.18. $\frac{1}{2}\tan^{-1}(x)+\frac{x}{2(x^2+1)}+C$

6.3.19. $\sqrt{x^2-7}+C$

6.3.20. $\frac{1}{8}\sin^{-1}(x)+\frac{x}{8}\sqrt{1-x^2}\left(2x^2-1\right)+C$

6.3.21. $C-\frac{1}{\sqrt{x^2+49}}$

6.3.22. $4x\sqrt{x^2-10}+40\ln\left(\left|\frac{x}{3.16228}+\sqrt{\frac{x^2}{10}-1}\right|\right)+C$

6.3.23. $\left(\frac{1}{8}\right)\frac{x-8}{x^2-16x+68}+\left(\frac{1}{16}\right)\tan^{-1}\left(\frac{x-8}{2}\right)+C$

6.3.24. $\frac{x}{\sqrt{1-x^2}}-\sin^{-1}(x)+C$

6.3.25. $C-\left(\frac{\sqrt{10-x^2}}{3x}+\frac{1}{3}\sin^{-1}\left(\frac{x}{3.16228}\right)\right)$

6.3.26. $\frac{x}{2}\sqrt{x^2+5}-\left(\frac{5}{2}\right)\ln\left(\frac{x}{1.73205}+\sqrt{\frac{x^2}{5}+1}\right)+C$

6.3.27. $\frac{\pi}{2}$

6.3.28. $\left(\frac{7}{2}\right)\sqrt{33}-8\ln\left(\left|\left(\frac{7}{4}\right)+\left(\frac{1}{4}\right)\sqrt{33}\right|\right)$

6.3.29. $6\sqrt{10}+2\ln\left(3+1\sqrt{10}\right)$

6.3.30. $\tan^{-1}(7)+\left(\frac{7}{50}\right)$

6.3.31. $9\sin^{-1}\left(\left(\frac{2}{3}\right)\right)+2\sqrt{5}$

6.3.32. $\frac{\pi}{8}$

6.4 · Partial Fraction Decomposition

6.4 · Exercises

Terms and Concepts

6.4.1. rational

6.4.2. True

6.4.3. $\frac{A}{x}+\frac{B}{x-5}$

6.4.4. $\frac{A}{x-4}+\frac{B}{x+4}$

6.4.5. $\frac{A}{x-\sqrt{7}}+\frac{B}{x+\sqrt{7}}$

6.4.6. $\frac{A}{x}+\frac{Bx+C}{x^2+6}$

Problems

6.4.7. $2\ln(|x-2|)+3\ln(|x-4|)+C$

6.4.8. $3\ln(|x|)-4\ln(|x-5|)+C$

6.4.9. $\left(\frac{9}{5}\right)\ln(|x-1|)-\left(\frac{9}{5}\right)\ln(|x+1|)+C$

6.4.10. $\ln(|x+9|)+\ln(|3x+1|)+C$

6.4.11. $3\ln(|x+7|)-\frac{8}{x+7}+C$

6.4.12. $9\ln(|x+5|)-\frac{3}{x+5}+C$

6.4.13. $6\ln(|x|)+7\ln(|x+3|)+\frac{4}{x+3}+C$

6.4.14. $C-(7\ln(|-(8x+5)|)+5\ln(|x+1|)+6\ln(|x+6|))$

6.4.15. $C - \left(\left(\frac{1}{2}\right) \ln(|6x+2|) + \left(\frac{1}{9}\right) \ln(|9x-1|) + \frac{\left(\frac{2}{5}\right)}{5x-4} \right)$

6.4.16. $x + 3\ln(|x-4|) + \ln(|x+1|) + C$

6.4.17. $\frac{1}{2}x^2 - x + \left(\frac{216}{13}\right) \ln(|x-6|) + \left(\frac{343}{13}\right) \ln(|x+7|) + C$

6.4.18. $4x + C$

6.4.19. $\left(\frac{1}{19}\right) \ln(|x|) - \left(\frac{1}{38}\right) \ln\left(x^2+8x+19\right) - 0.121547 \tan^{-1}\left(\frac{x+4}{1.73205}\right) + C$

6.4.20. $x + \ln\left(x^2+6x+11\right) - 3.53553 \tan^{-1}\left(\frac{x+3}{1.41421}\right) + C$

6.4.21. $\ln\left(\left|3x^2+5x-9\right|\right) - 5\ln(|x+4|) + C$

6.4.22. $3\ln(|x+1|) + 2\ln\left(x^2+2x+2\right) - 5\tan^{-1}(x+1) + C$

6.4.23. $\left(\frac{463}{73}\right) \ln(|x+8|) + \left(\frac{121}{146}\right) \ln\left(x^2+9\right) - \left(\frac{201}{73}\right) \tan^{-1}\left(\frac{x}{3}\right) + C$

6.4.24. $3\ln\left(x^2-2x+26\right) - 5\ln(|x+4|) - 2\tan^{-1}\left(\frac{x-1}{5}\right) + C$

6.4.25. $5\ln\left(x^2-4x+10\right) - 4\ln(|x-4|) + 0.816497 \tan^{-1}\left(\frac{x-2}{2.44949}\right) + C$

6.4.26. $4\ln(|x+7|) + \left(\frac{5}{2}\right) \ln\left(x^2-10x+27\right) - 7.77817 \tan^{-1}\left(\frac{x-5}{1.41421}\right) + C$

6.4.27. $\ln\left(\left(\frac{6144}{117649}\right)\right)$

6.4.28. 0.536267

6.4.29. $\ln\left(\left(\frac{9}{11}\right)\right) + \tan^{-1}(4) - \tan^{-1}(2)$

6.4.30. $\frac{1}{8}$

6.5 · Improper Integration

6.5 · Exercises

Terms and Concepts

6.5.4. $p > 1$

6.5.5. $p > 1$

6.5.6. $p < 1$

Problems

6.5.7. $\frac{e^5}{2}$

6.5.8. $\frac{1}{2}$

6.5.9. $\frac{1}{3}$

6.5.10. $\frac{\pi}{3}$

6.5.11. $\frac{1}{\ln(2)}$

6.5.12. ∞

6.5.13. ∞

6.5.14. ∞

6.5.15. 1

6.5.16. ∞

6.5.17. ∞

6.5.18. ∞

6.5.19. ∞

6.5.20. ∞

6.5.21. ∞

6.5.22. $2 + 2\sqrt{2}$

6.5.23. 1

6.5.24. $\frac{1}{2}$

6.5.25. 0

6.5.26. $\frac{\pi}{2}$

6.5.27. $\frac{-1}{4}$

6.5.28. $\frac{-1}{9}$

6.5.29. ∞

6.5.30. -1

6.5.31. 1

6.5.32. ∞

6.5.33. $\frac{1}{2}$

6.5.34. $\frac{1}{2}$

6.5.35. **(a).** Limit Comparison Test
(b). diverges
(c). $\frac{1}{x}$

6.5.36. **(a).** Limit Comparison Test
(b). converges
(c). $\frac{1}{x^{1.5}}$

6.5.37. **(a).** Limit Comparison Test
(b). diverges
(c). $\frac{1}{x}$

6.5.38. **(a).** Direct Comparison Test
(b). converges
(c). xe^{-x}

6.5.39. **(a).** Direct Comparison Test
(b). converges
(c). e^{-x}

6.5.40. **(a).** Direct Comparison Test
(b). converges
(c). xe^{-x}

6.5.41. **(a).** Direct Comparison Test
(b). converges
(c). $\frac{1}{x^2-1}$

6.5.42. **(a).** Direct Comparison Test
(b). diverges
(c). $\frac{x}{x^2+1}$

6.5.43. **(a).** Direct Comparison Test
(b). converges
(c). $\frac{1}{e^x}$

6.5.44. **(a).** Limit Comparison Test
(b). converges
(c). $\frac{1}{e^x}$

7 • Applications of Integration
7.1 • Area Between Curves
7.1 • Exercises

Terms and Concepts

7.1.1. True
7.1.2. True

Problems

7.1.5. 22.436
7.1.6. 5.33333
7.1.7. 3.14159
7.1.8. 3.14159
7.1.9. 0.5
7.1.10. 2.82843
7.1.11. 0.721354
7.1.12. $4/3$
7.1.13. 4.5
7.1.14. 1.33333
7.1.15. 0.429204
7.1.16. 8
7.1.17. 0.166667
7.1.18. 3.08333

7.1.19. All enclosed regions have the same area, with regions being the reflection of adjacent regions. One region is formed on $[\pi/4, 5\pi/4]$, with area $2\sqrt{2}$.

7.1.20. 3.89711
7.1.21. 1
7.1.22. 1.66667
7.1.23. 4.5
7.1.24. 2.25
7.1.25. 0.514298
7.1.26. $4/3$
7.1.27. 1
7.1.28. 5
7.1.29. 4
7.1.30. 10.5

7.1.31. 262800 ft^2
7.1.32. 623333 ft^2

7.2 • Volume by Cross-Sectional Area; Disk and Washer Methods
7.2 • Exercises

Terms and Concepts

7.2.1. T
7.2.2. Answers will vary.

Problems

7.2.4. $48\pi\sqrt{3}/5$ units^3
7.2.5. $175\pi/3$ units^3
7.2.6. $\pi^2/4$ units^3
7.2.7. $\pi/6$ units^3
7.2.8. $9\pi/2$ units^3
7.2.9. $35\pi/3$ units^3
7.2.10. $\pi^2 - 2\pi$ units^3
7.2.11. $2\pi/15$ units^3

7.2.12.

(a) $\pi/2$

(b) $5\pi/6$

(c) $4\pi/5$

(d) $8\pi/15$

7.2.13.

(a) $512\pi/15$

(b) $256\pi/5$

(c) $832\pi/15$

(d) $128\pi/3$

7.2.14.

(a) $4\pi/3$

(b) $2\pi/3$

(c) $4\pi/3$

(d) $\pi/3$

7.2.15.

(a) $104\pi/15$

(b) $64\pi/15$

(c) $32\pi/5$

7.2.16.

(a) $\pi^2/2$

(b) $\pi^2/2 - 4\pi \sinh^{-1}(1)$

(c) $\pi^2/2 + 4\pi \sinh^{-1}(1)$

7.2.17.

(a) 8π

(b) 8π

(c) $16\pi/3$

(d) $8\pi/3$

7.2.18. $250\pi/3$
7.2.19. $250\pi/3$
7.2.20. $80/3$
7.2.21. 187.5

7.3 · The Shell Method
7.3 · Exercises

Terms and Concepts

7.3.1. T
7.3.2. F
7.3.3. F
7.3.4. T

Problems

7.3.5. $9\pi/2$ units^3

7.3.6. $70\pi/3$ units^3

7.3.7. $\pi^2 - 2\pi$ units^3

7.3.8. $2\pi/15$ units^3

7.3.9. $48\pi\sqrt{3}/5$ units^3

7.3.10. $350\pi/3$ units^3

7.3.11. $\pi^2/4$ units^3

7.3.12. $\pi/6$ units^3

7.3.13.

- **(a)** $4\pi/5$
- **(b)** $8\pi/15$
- **(c)** $\pi/2$
- **(d)** $5\pi/6$

7.3.14.

- **(a)** $128\pi/3$
- **(b)** $128\pi/3$
- **(c)** $512\pi/15$
- **(d)** $256\pi/5$

7.3.15.

- **(a)** $4\pi/3$
- **(b)** $\pi/3$
- **(c)** $4\pi/3$
- **(d)** $2\pi/3$

7.3.16.

- **(a)** $16\pi/3$
- **(b)** $8\pi/3$
- **(c)** 8π

7.3.17.

- **(a)** $2\pi(\sqrt{2} - 1)$
- **(b)** $2\pi(1 - \sqrt{2} + \sinh^{-1}(1))$

7.3.18.

- **(a)** $16\pi/3$
- **(b)** $8\pi/3$
- **(c)** 8π
- **(d)** 8π

7.4 • Arc Length and Surface Area

7.4 • Exercises

Problems

7.4.3. $\sqrt{2}$

7.4.4. 6

7.4.5. $\frac{10}{3}$

7.4.6. 6

7.4.7. $\frac{157}{3}$

7.4.8. $\frac{3}{2}$

7.4.9. $\frac{12}{5}$

7.4.10. $\frac{7.99533\times 10^7}{400000}$

7.4.11. $-\ln(2 - \sqrt{3}) \approx 1.31696$

7.4.12. $\sinh^{-1}(1)$

7.4.13. $\int_0^1 \sqrt{1 + 4x^2}\, dx$

7.4.14. $\int_0^1 \sqrt{1 + 100x^{18}}\, dx$

7.4.15. $\int_1^e \sqrt{1 + \frac{1}{x^2}}\, dx$

7.4.16. $\int_1^2 \sqrt{1 + \frac{1}{x^4}}\, dx$

7.4.17. $\int_0^{\pi/2} \sqrt{1 + \sin^2(x)}\, dx$

7.4.18. $\int_{-\pi/4}^{\pi/4} \sqrt{1 + \sec^2(x)\tan^2(x)}\, dx$

7.4.19. 1.4790

7.4.20. 1.8377

7.4.21. 2.1300

7.4.22. 1.3254

7.4.23. 1.00013

7.4.24. 1.7625

7.4.25. $2\pi \int_0^1 2x\sqrt{5}\,dx = 2\pi\sqrt{5}$

7.4.26. $2\pi \int_0^1 x\sqrt{5}\,dx = \pi\sqrt{5}$

7.4.27. $2\pi \int_0^1 x\sqrt{1+4x^2}\,dx = \pi/6(5\sqrt{5}-1)$

7.4.28. $2\pi \int_0^1 x^3\sqrt{1+9x^4}\,dx = \pi/27(10\sqrt{10}-1)$

7.4.29. $\int_0^1 \sqrt{1+\frac{1}{4x}}\,dx$

7.4.30. $\int_{-1}^1 \sqrt{1+\frac{x^2}{1-x^2}}\,dx$

7.4.31. $\int_{-3}^3 \sqrt{1+\frac{x^2}{81-9x^2}}\,dx$

7.4.32. $2\pi \int_0^1 \sqrt{x}\sqrt{1+1/(4x)}\,dx = \pi/6(5\sqrt{5}-1)$

7.4.33. $2\pi \int_0^1 \sqrt{1-x^2}\sqrt{1+x/(1-x^2)}\,dx = 4\pi$

7.5 · Work

7.5 · Exercises

Terms and Concepts

7.5.1. In SI units, it is one joule, i.e., one newton–meter, or $\frac{\text{kg}\cdot\text{m}}{\text{s}^2}$m In Imperial Units, it is ft–lb.

7.5.2. The same.

7.5.3. Smaller.

7.5.4. force; distance

Problems

7.5.5.

- **(a)** 500 ft–lb
- **(b)** $100 - 50\sqrt{2} \approx 29.29$ ft

7.5.6.

- **(a)** 2450 J
- **(b)** 1568 J

7.5.7.

- **(a)** $\frac{1}{2}\cdot d\cdot l^2$ ft–lb
- **(b)** 75 %
- **(c)** $\ell(1-\sqrt{2}/2) \approx 0.2929\ell$

7.5.8. 735 J

7.5.9.

- **(a)** 756 ft–lb
- **(b)** 60,000 ft–lb
- **(c)** Yes, for the cable accounts for about 1% of the total work.

7.5.10. 11,100 ft–lb

7.5.11. 575 ft–lb

7.5.12. 125 ft–lb

7.5.13. 0.05 J
7.5.14. 12.5 ft–lb
7.5.15. 5/3 ft–lb
7.5.16. 0.2625 = 21/80 J
7.5.17. $f \cdot d/2$ J
7.5.18. 45 ft–lb
7.5.19. 5 ft–lb
7.5.20. $953, 284$ J
7.5.21.

- **(a)** 52,929.6 ft–lb
- **(b)** 18,525.3 ft–lb
- **(c)** When 3.83 ft of water have been pumped from the tank, leaving about 2.17 ft in the tank.

7.5.22. 192,767 ft–lb. Note that the tank is oriented horizontally. Let the origin be the center of one of the circular ends of the tank. Since the radius is 3.75 ft, the fluid is being pumped to $y = 4.75$; thus the distance the gas travels is $h(y) = 4.75 - y$. A differential element of water is a rectangle, with length 20 and width $2\sqrt{3.75^2 - y^2}$. Thus the force required to move that slab of gas is $F(y) = 40 \cdot 45.93 \cdot \sqrt{3.75^2 - y^2}dy$. Total work is $\int_{-3.75}^{3.75} 40 \cdot 45.93 \cdot (4.75 - y)\sqrt{3.75^2 - y^2}\,dy$. This can be evaluated without actual integration; split the integral into $\int_{-3.75}^{3.75} 40 \cdot 45.93 \cdot (4.75)\sqrt{3.75^2 - y^2}\,dy + \int_{-3.75}^{3.75} 40 \cdot 45.93 \cdot (-y)\sqrt{3.75^2 - y^2}\,dy$. The first integral can be evaluated as measuring half the area of a circle; the latter integral can be shown to be 0 without much difficulty. (Use substitution and realize the bounds are both 0.)
7.5.23. 212,135 ft–lb
7.5.24.

- **(a)** approx. 577,000 J
- **(b)** approx. 399,000 J
- **(c)** approx 110,000 J (By volume, half of the water is between the base of the cone and a height of 3.9685 m. If one rounds this to 4 m, the work is approx 104,000 J.)

7.5.25. 187,214 ft–lb
7.5.26. 617,400 J
7.5.27. 4,917,150 J

7.6 • Fluid Forces
7.6 • Exercises

Terms and Concepts

7.6.1. Answers will vary.
7.6.2. Answers will vary.

Problems

7.6.3. 499.2 lb
7.6.4. 249.6 lb
7.6.5. 6739.2 lb
7.6.6. 5241.6 lb
7.6.7. 3920.7 lb
7.6.8. 15682.8 lb
7.6.9. 2496 lb
7.6.10. 2496 lb
7.6.11. 602.59 lb
7.6.12. 291.2 lb

7.6.13.

(a) 2340 lb

(b) 5625 lb

7.6.14.

(a) 1064.96 lb

(b) 2560 lb

7.6.15.

(a) 1597.44 lb

(b) 3840 lb

7.6.16.

(a) 41.6 lb

(b) 100 lb

7.6.17.

(a) 56.42 lb

(b) 135.62 lb

7.6.18.

(a) 1123.2 lb

(b) 2700 lb

7.6.19. 5.1 ft

7.6.20. 4.1 ft

8 · Differential Equations
8.1 · Graphical and Numerical Solutions to Differential Equations
8.1 · Exercises

Terms and Concepts

8.1.1. An initial value problems is a differential equation that is paired with one or more initial conditions. A differential equation is simply the equation without the initial conditions.

8.1.2. Answers will vary.

8.1.3. Substitute the proposed function into the differential equation, and show the the statement is satisfied.

8.1.4. A particular solution is one specifica member of a family of solutions, and has no arbitrary constants. A general solution is a family of solutions, includes all possible solutions to the differential equation, and typically includes one or more arbitrary constants.

8.1.5. Many differential equations are impossible to solve analytically.

8.1.6. A smaller h value leads to a numerical solution that is closer to the true solution, but decreasing the h value leads to more computational effort.

Problems

8.1.7. Answers will vary.

8.1.8. Answers will vary.

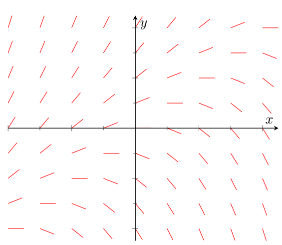
8.1.9. Answers will vary.

8.1.10. Answers will vary.

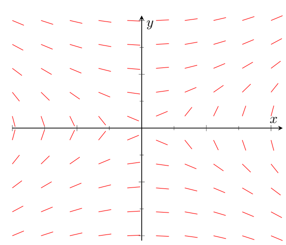
8.1.11. $C = 2$

8.1.12. $C = 6$

8.1.13.



8.1.14.



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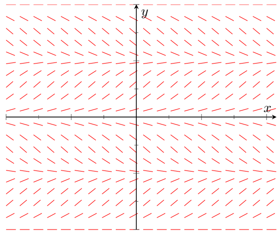

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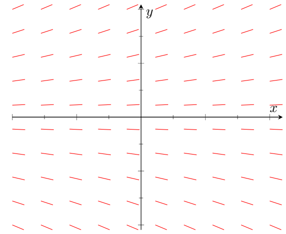


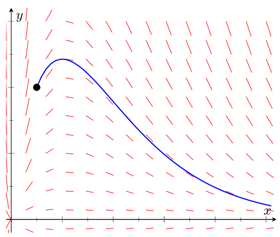
8.1.17. b

8.1.18. c

8.1.19. d

8.1.20. a

8.1.21.



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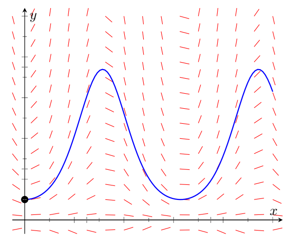
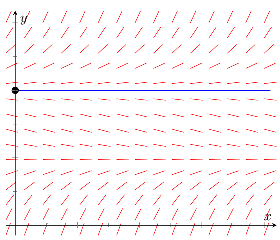
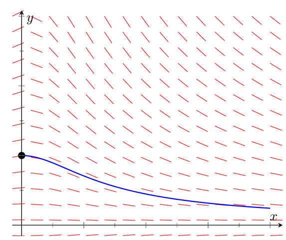


8.1.23.



8.1.24.



8.1.25.

x_i	y_i
0.00	1.0000
0.25	1.5000
0.50	2.3125
0.75	3.5938
1.00	5.5781

8.1.26.

x_i	y_i
0.0	1.0000
0.1	1.0000
0.2	1.0037
0.3	1.0110
0.4	1.0219
0.5	1.0363

8.1.27.

x_i	y_i
0.0	2.0000
0.2	2.4000
0.4	2.9197
0.6	3.5816
0.8	4.4108
1.0	5.4364

8.1.28.

x_i	y_i
0.0	0.0000
0.5	0.5000
1.0	1.8591
1.5	10.5824
2.0	88378.1190

8.1.29.

x	0.0	0.2	0.4	0.6	0.8	1.0
$y(x)$	1.0000	1.0204	1.0870	1.2195	1.4706	2.0000
$h=0.2$	1.0000	1.0000	1.0400	1.1265	1.2788	1.5405
$h=0.1$	1.0000	1.0100	1.0623	1.1687	1.3601	1.7129

8.1.30.

x	0.0	0.2	0.4	0.6	0.8	1.0
$y(x)$	0.5000	0.5412	0.6806	0.9747	1.5551	2.7183
$h=0.2$	0.5000	0.5000	0.5816	0.7686	1.1250	1.7885
$h=0.1$	0.5000	0.5201	0.6282	0.8622	1.3132	2.1788

8.2 · Separable Differential Equations

8.2 · Exercises

Problems

8.2.1. Separable. $\frac{1}{y^2-y}\,dy = dx$

8.2.2. Not separable.

8.2.3. Not separable.

8.2.4. Separable. $\frac{1}{\cos y - y}\,dy = (x^2+1)\,dx$

8.2.5. $\left\{y = \frac{1+Ce^{2x}}{1-Ce^{2x}}, y=-1\right\}$

8.2.6. $y = 2 + Ce^x$

8.2.7. $y = Cx^4$

8.2.8. $y^2 - 4x^2 = C$

8.2.9. $(y-1)e^y = -e^{-x} - \frac{1}{3}e^{-3x} + C$

8.2.10. $(y-1)^2 = \ln(x^2+1) + C$

8.2.11. $\left\{\arcsin 2y - \arctan(x^2+1) = C, y = \pm\frac{1}{2}\right\}$

8.2.12. $\left\{y = \frac{1}{C - \arctan x}, y = 0\right\}$

8.2.13. $\sin y + \cos(x) = 2$

8.2.14. $-x^3 + 3y - y^3 = 2$

8.2.15. $\frac{1}{2}y^2 - \ln(1+x^2) = 8$

8.2.16. $y^2 + 2xe^x - 2e^x = 2$

8.2.17. $\frac{1}{2}y^2 - y = \frac{1}{2}\big((x^2+1)\ln(x^2+1) - (x^2+1)\big) + \frac{1}{2}$

8.2.18. $\sin(y^2) - (\arcsin x)^2 = -\frac{1}{2}$

8.2.19. $2\tan 2y = 2x + \sin 2x$

8.2.20. $x = exp\left(-\frac{\sqrt{1-y^2}}{y}\right)$

8.3 · First Order Linear Differential Equations

8.3 · Exercises

Problems

8.3.1. $y = \frac{3}{2} + Ce^{2x}$

8.3.2. $y = \frac{\ln|x| + C}{x}$

8.3.3. $y = -\frac{1}{2x} + Cx$

8.3.4. $y = \frac{x^3}{7} - \frac{x}{5} + \frac{C}{x^4}$

8.3.5. $y = \sec x + C(\csc x)$

8.3.6. $y = \frac{1}{2} + Ce^{-x^2}$

8.3.7. $y = Ce^{3x} - (x+1)e^{2x}$

8.3.8. $y = sin(2x) - 2\cos(2x) + Ce^{-x}$

8.3.9. $y = (x^2+2)e^x$

8.3.10. $y = \frac{1}{4}x^2 - \frac{1}{3}x + \frac{1}{2} + \frac{7}{12x^2}$

8.3.11. $y = 1 - \frac{2}{x} + \frac{2-e^{1-x}}{x^2}$

8.3.12. $y = 3e^{-2x}$

8.3.13. $y = \frac{x^2+1}{x+1}e^{-x}$

8.3.14. $y = \sin(x) - 3\cos(x)$

8.3.15. $y = \frac{(x-2)(x+1)}{x-1}$

8.3.16. $y = x^2\left(\arctan x - \frac{\pi}{4}\right)$

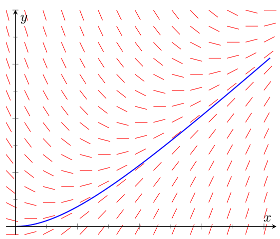
8.3.17. Both; $y = -5e^{x+\frac{1}{3}x^3}$

8.3.18. separable; $e^y = \sin(x) - x\cos(x) + 1$

8.3.19. linear; $y = \frac{x^3-3x-6}{3(x-1)}$

8.3.20. separable; $y = 1$

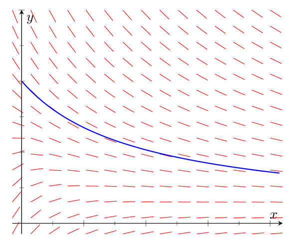
8.3.21.



The solution will increase and begin to follow the line $y = x - 1$.

$y = x - 1 + e^{-x}$

8.3.22.



The solution will decrease and approach $y = 0$.

$y = \frac{2+\ln(x+1)}{x+1}$

8.4 · Modeling with Differential Equations

8.4 · Exercises

Problems

8.4.1. $y = 10 + Ce^{-kx}$

8.4.2. 13.66 days

8.4.3. 4.43 days

8.4.4. 13,304.65 years old

8.4.5. $x = \begin{cases} \frac{ab(1-e^{(a-b)kt})}{b-ae^{(a-b)kt}} & \text{if } a \neq b \\ \frac{a^2kt}{1+akt} & \text{if } a = b \end{cases}$

8.4.6. 24.57 minutes

8.4.7. $y = 60 - 3.69858e^{-\frac{1}{4}t} + 43.69858e^{-0.0390169t}$

8.4.8. 0.06767 g/gal

8.4.9. $y = 8(1 - e^{-\frac{1}{2}t})$ g/cm^2

8.4.10. $y = 20 - \frac{10}{17}(4\cos(2t) - \sin(2t)) - \frac{300}{17}e^{-\frac{1}{2}t}$ g

8.4.11. 11.00075 g

8.4.12. pond 1: 50.4853 grams per million gallons
pond 2: 32.8649 grams per million gallons

9 · Sequences and Series

9.1 · Sequences

9.1 · Exercises

Terms and Concepts

9.1.1. Answers will vary.

9.1.2. natural

9.1.3. Answers will vary.

9.1.4. Answers will vary.

Problems

9.1.5. $2, \frac{8}{3}, \frac{8}{3}, \frac{32}{15}, \frac{64}{45}$

9.1.6. $-\frac{3}{2}, \frac{9}{4}, -\frac{27}{8}, \frac{81}{16}, -\frac{243}{32}$

9.1.7. $-\frac{1}{3}, -2, -\frac{81}{5}, -\frac{512}{3}, -\frac{15625}{7}$

9.1.8. $1, 1, 2, 3, 5$

9.1.9. $a_n = 3n + 1$

9.1.10. $a_n = (-1)^{n+1}\frac{3}{2^{n-1}}$

9.1.11. $a_n = 10 \cdot 2^{n-1}$

9.1.12. $a_n = 1/(n-1)!$

9.1.13. $1/7$

9.1.14. $3e^2 - 1$

9.1.15. 0

9.1.16. e^4

9.1.17. diverges

9.1.18. converges to $4/3$

9.1.19. converges to 0

9.1.20. converges to 0

9.1.21. diverges

9.1.22. converges to 3

9.1.23. converges to e

9.1.24. converges to 5

9.1.25. converges to 0

9.1.26. diverges

9.1.27. converges to 2

9.1.28. converges to 0

9.1.29. bounded

9.1.30. neither bounded above or below

9.1.31. bounded

9.1.32. bounded below

9.1.33. neither bounded above or below

9.1.34. bounded above

9.1.35. monotonically increasing

9.1.36. monotonically increasing for $n \geq 3$

9.1.37. never monotonic

9.1.38. monotonically decreasing for $n \geq 3$

9.1.40.

(b) $a_n = 1/3^n$ and $b_n = 1/2^n$

9.2 • Infinite Series

9.2 • Exercises

Terms and Concepts

9.2.1. Answers will vary.

9.2.2. Answers will vary.

9.2.4. Answers will vary.

9.2.5. F

9.2.6. F

9.3 • Integral and Comparison Tests

9.3 • Exercises

Terms and Concepts

9.3.1. continuous, positive and decreasing

9.3.2. F

Problems

9.3.5. Converges
9.3.6. Converges
9.3.7. Diverges
9.3.8. Diverges
9.3.9. Converges
9.3.10. Converges
9.3.11. Converges
9.3.12. Converges

9.4 · Ratio and Root Tests

9.4 · Exercises

Terms and Concepts

9.4.1. algebraic, or polynomial.

9.4.2. factorial and/or exponential

9.4.3. Integral Test, Limit Comparison Test, and Root Test

9.4.4. raised to a power

Problems

9.4.5. Converges
9.4.6. Diverges
9.4.7. Converges
9.4.8. Converges
9.4.9. The Ratio Test is inconclusive; the p-Series Test states it diverges.
9.4.10. The Ratio Test is inconclusive; the Direct Comparison Test with $1/n^3$ shows it converges.
9.4.11. Converges
9.4.12. Converges
9.4.13. Converges; note the summation can be rewritten as $\sum_{n=1}^{\infty} \frac{2^n n!}{3^n n!}$, from which the Ratio Test or Geometric Series Test can be applied.
9.4.14. Converges; rewrite the summation as $\sum_{n=1}^{\infty} \frac{n!}{5^n n!}$ then apply the Ratio Test or Geometric Series Test.
9.4.15. Converges
9.4.16. Converges
9.4.17. Converges
9.4.18. Converges
9.4.19. Diverges
9.4.20. Converges
9.4.21. Diverges. The Root Test is inconclusive, but the nth-Term Test shows divergence. (The terms of the sequence approach e^{-2}, not 0, as $n \to \infty$.)
9.4.22. Converges
9.4.23. Converges
9.4.24. Converges

9.5 · Alternating Series and Absolute Convergence

9.5 · Exercises

Terms and Concepts

9.5.2. postive, decreasing, 0

9.5.3. Many examples exist; one common example is $a_n = (-1)^n/n$.

9.5.4. conditionally

10 · Curves in the Plane
10.1 · Conic Sections
10.1 · Exercises

Terms and Concepts

10.1.6. line

Problems

10.1.19. $\frac{(x+1)^2}{9} + \frac{(y-2)^2}{4} = 1$; foci at $(-1 \pm \sqrt{5}, 2)$; $e = \sqrt{5}/3$

10.1.20. $\frac{(x-1)^2}{1/4} + \frac{y^2}{9} = 1$; foci at $(1, \pm\sqrt{8.75})$; $e = \sqrt{8.75}/3 \approx 0.99$

10.1.29. $x^2 - \frac{y^2}{3} = 1$

10.1.30. $y^2 - \frac{x^2}{24} = 1$

10.1.31. $\frac{(y-3)^2}{4} - \frac{(x-1)^2}{9} = 1$

10.1.32. $\frac{(x-1)^2}{9} - \frac{(y-3)^2}{4} = 1$

10.1.45. The sound originated from a point approximately 31m to the right of B and 1390m above or below it. (Since the three points are collinear, we cannot distinguish whether the sound originated above/below the line containing the points.)

10.2 · Parametric Equations
10.2 · Exercises

Terms and Concepts

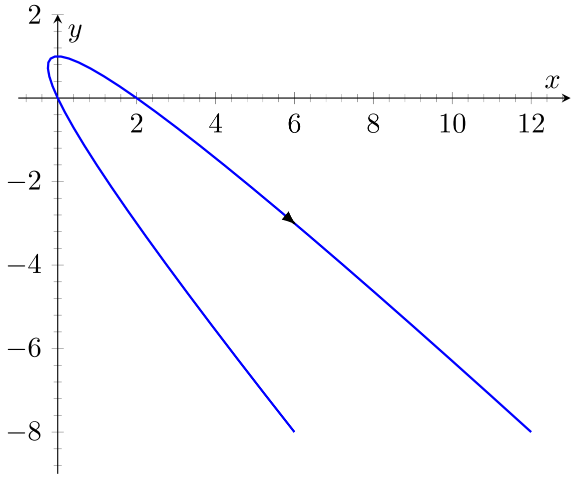
10.2.1. True

10.2.2. orientation

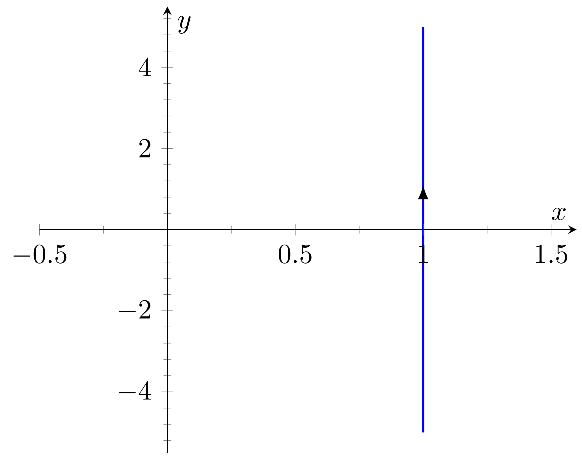
10.2.3. rectangular

Problems

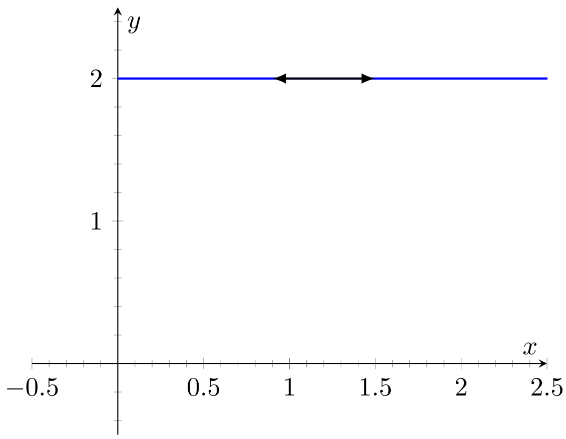
10.2.5.



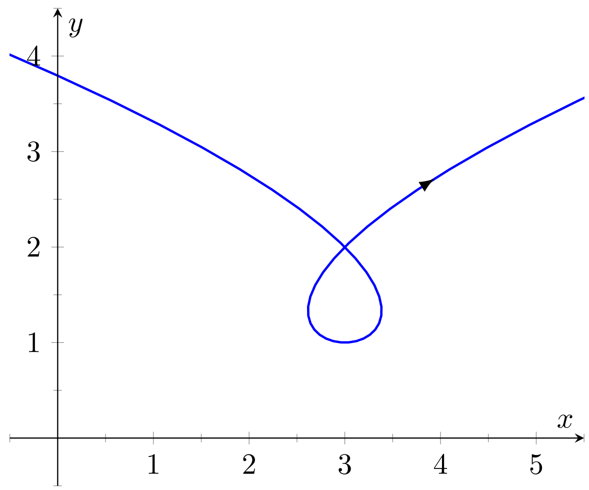
10.2.6.



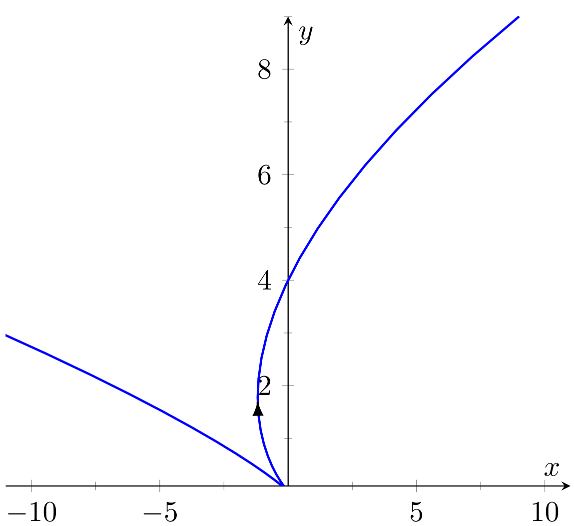
10.2.7.



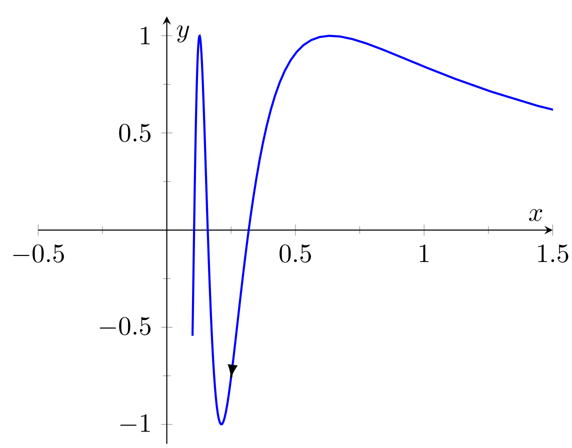
10.2.8.



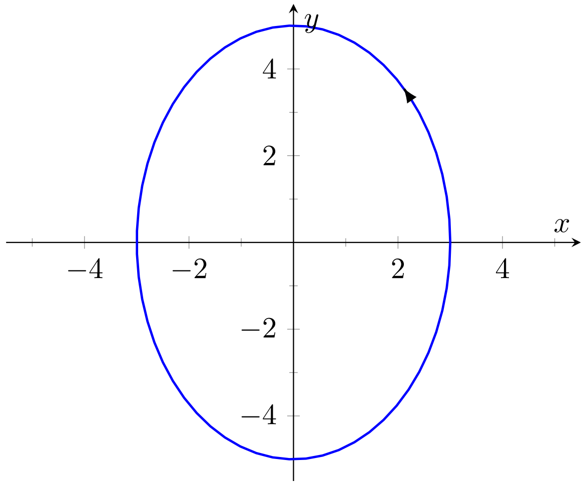
10.2.9.



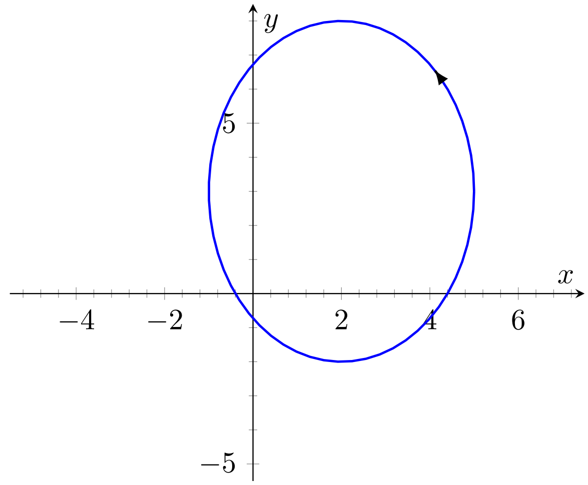
10.2.10.



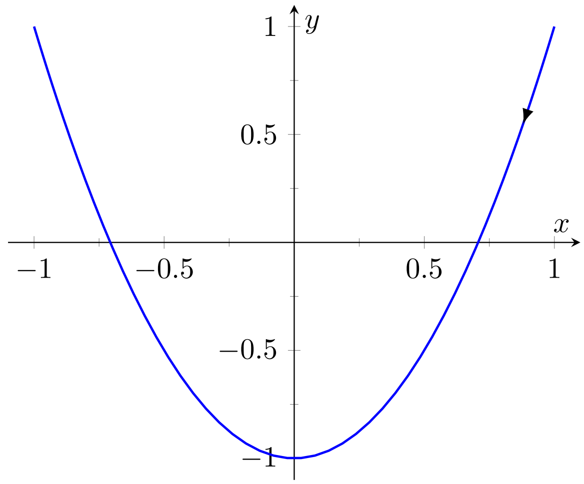
10.2.11.



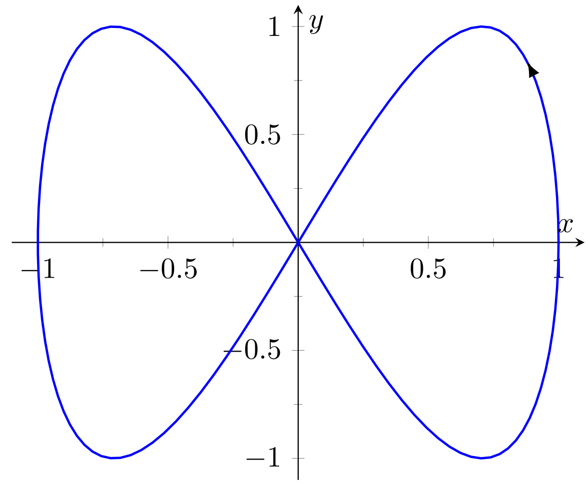
10.2.12.



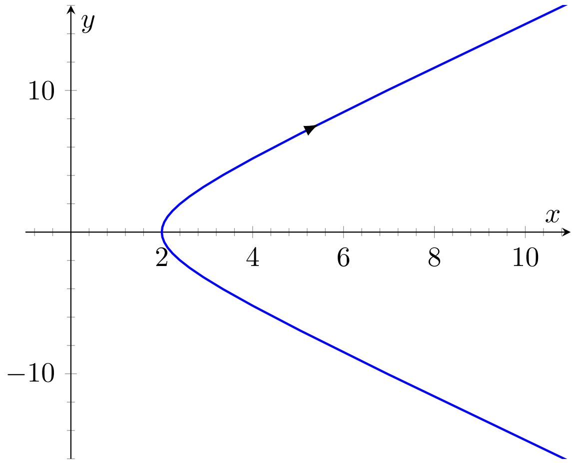
10.2.13.



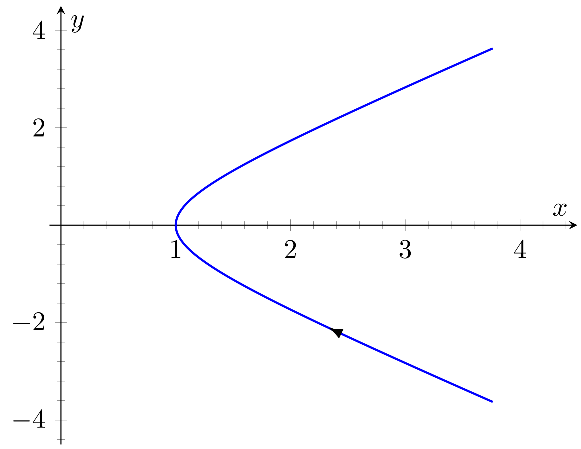
10.2.14.



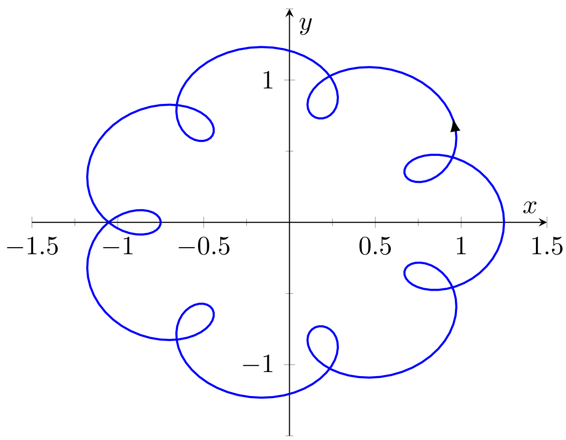
10.2.15.



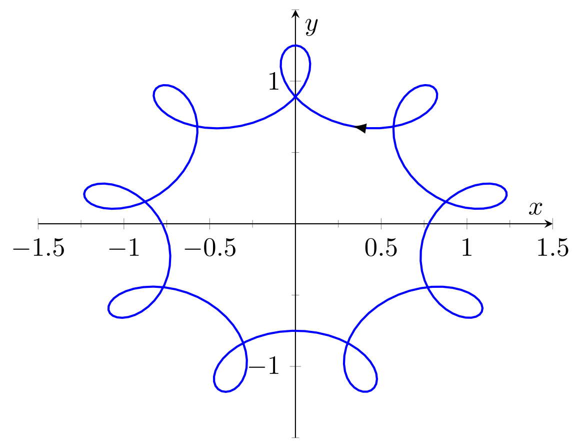
10.2.16.



10.2.17.



10.2.18.



10.2.19.

(a) Traces the parabola $y = x^2$, moves from left to right.

(b) Traces the parabola $y = x^2$, but only from $-1 \leq x \leq 1$; traces this portion back and forth infinitely.

(c) Traces the parabola $y = x^2$, but only for $0 < x$. Moves left to right.

(d) Traces the parabola $y = x^2$, moves from right to left.

10.2.20.

(a) Traces a circle of radius 1 counterclockwise once.

(b) Traces a circle of radius 1 counterclockwise over 6 times.

(c) Traces a circle of radius 1 clockwise infinite times.

(d) Traces an arc of a circle of radius 1, from an angle of -1 radians to 1 radian, twice.

10.2.21. $3x + 2y = 17$

10.2.30. $x = 1 - 2y^2$

10.2.25. $y - 2x = 3$

10.2.35. (a). $\frac{t+11}{6}$

(b). $\frac{t^2-97}{12}$

(c). $(2, -8)$

(d). $6x - 11$

(e). 1

10.2.37. (a). $\cos^{-1}(t)$

(b). $\sqrt{1-t^2}$

(c). $(0, 0)$

(d). $\cos(x)$

(e). 1

10.2.36. (a). $\ln(t)$

(b). t

(c). $(0, 1)$

(d). e^x

(e). 1

10.2.39. (a). $-1, 1$

(b). $(3, -2)$

10.2.44. (a). 2

(b). $(-4, -8)$

10.2.46. (a). 0

(b). $(1, 0)$

10.2.50. $2\cos(t)\,;\ -2\sin(t)$

10.2.52. $3\cos(2\pi t) + 1;\ 3\sin(2\pi t) + 1$

10.2.51. $3\cos(2\pi t) + 1;\ 3\sin(2\pi t) + 1$

10.3 · Calculus and Parametric Equations

10.3 · Exercises

Terms and Concepts

10.3.1. False
10.3.3. False
10.3.4. True

Problems

10.3.15. **(a).** -0.5
(b). $(0.75, -0.25)$

10.3.18. **(a).** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
(b). $\left(\frac{\sqrt{2}}{2}, 1\right), \left(\frac{-\sqrt{2}}{2}, -1\right), \left(\frac{-\sqrt{2}}{2}, 1\right), \left(\frac{\sqrt{2}}{2}, -1\right)$

10.3.21. **(a).** 0
(b). 0

10.3.22. **(a).** 2
(b). 1

10.3.27. **(a).** $-\frac{4}{(2t-1)^3}$
(b). $(-\infty, 0.5]$
(c). $[0.5, \infty)$

10.3.30. **(a).** $\frac{2(\sin(t)(-2)\sin(2t)-\cos(2t)\cos(t))}{\sin^3(t)}$
(b). $\left[\frac{\pi}{2}, \pi\right], \left[\frac{3\pi}{2}, 2\pi\right]$
(c). $\left[0, \frac{\pi}{2}\right], \left[\pi, \frac{3\pi}{2}\right]$

10.3.33. 6π

10.3.34. **(a).** $\sqrt{101}\left(e^{\frac{\pi}{5}} - 1\right)$
(b). $\sqrt{101}\left(e^{\frac{2\pi}{5}} - e^{\frac{\pi}{5}}\right)$

10.3.35. $2\sqrt{34}$

10.4 · Introduction to Polar Coordinates

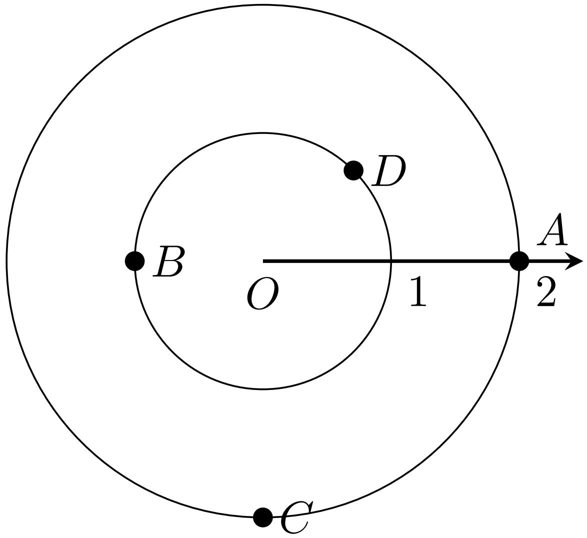
10.4 · Exercises

Terms and Concepts

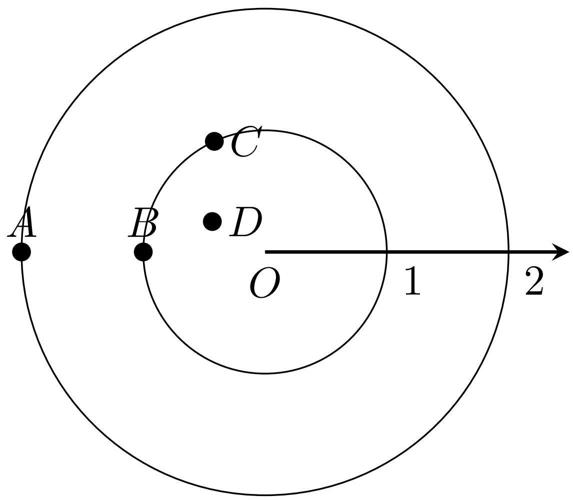
10.4.1. Answers will vary.
10.4.2. False
10.4.3. True
10.4.4. False

Problems

10.4.5.



10.4.6.



10.4.7. $A = P(2.5, \pi/4)$ and $P(-2.5, 5\pi/4)$;
$B = P(-1, 5\pi/6)$ and $P(1, 11\pi/6)$;
$C = P(3, 4\pi/3)$ and $P(-3, \pi/3)$;
$D = P(1.5, 2\pi/3)$ and $P(-1.5, 5\pi/3)$;

10.4.8. (a). $(2, 0.523599), (-2, -2.61799)$

(b). $(1, -1.0472), (-1, 2.0944)$

(c). $(2, 2.35619), (-2, -0.785398)$

(d). $(2.5, 3.14159), (2.5, -3.14159)$

10.4.9. (a). $\left(\sqrt{2}, \sqrt{2}\right)$

(b). $\left(\sqrt{2}, -\sqrt{2}\right)$

(c). $\left(\sqrt{5}, \tan^{-1}\left(\frac{-1}{2}\right)\right)$

(d). $\left(\sqrt{5}, \pi + \tan^{-1}\left(\frac{-1}{2}\right)\right)$

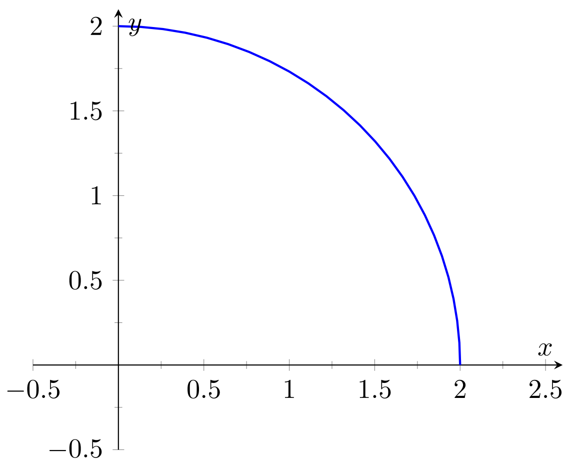
10.4.10. (a). $(-3, 0)$

(b). $\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$

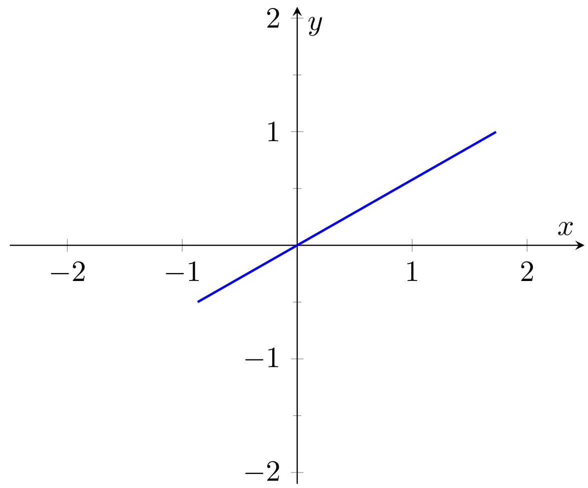
(c). $\left(4, \frac{\pi}{2}\right)$

(d). $\left(2, \frac{-\pi}{3}\right)$

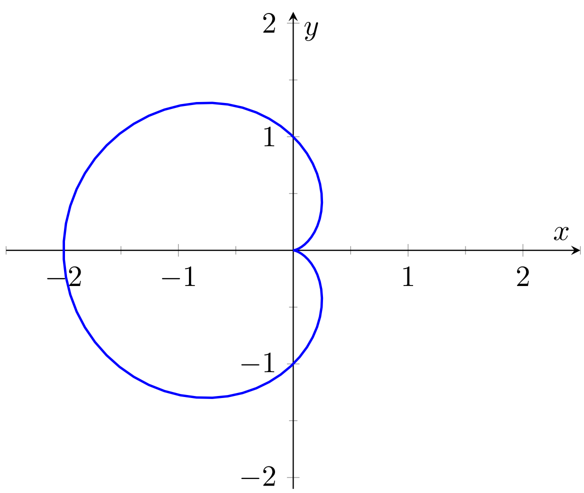
10.4.11.



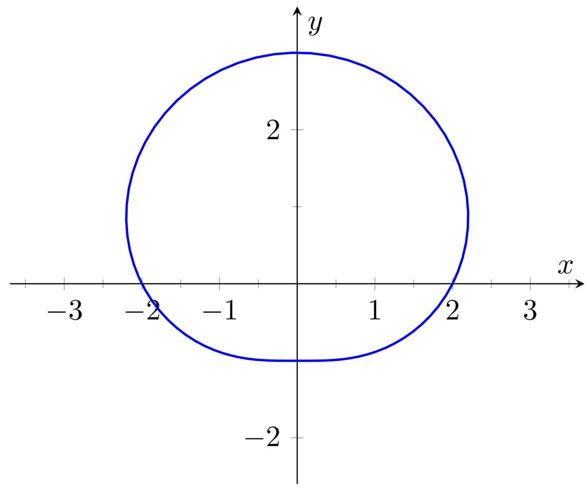
10.4.12.



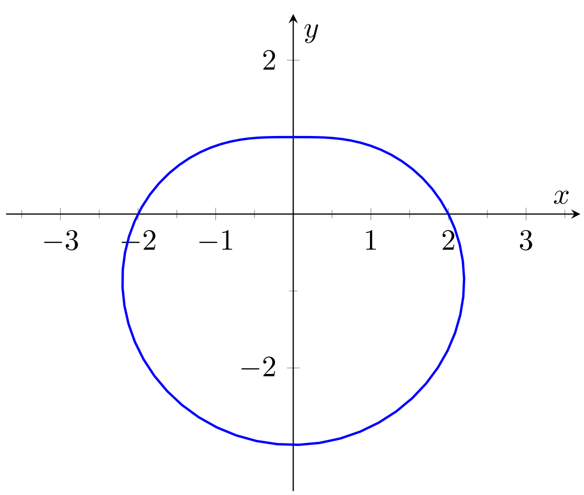
10.4.13.



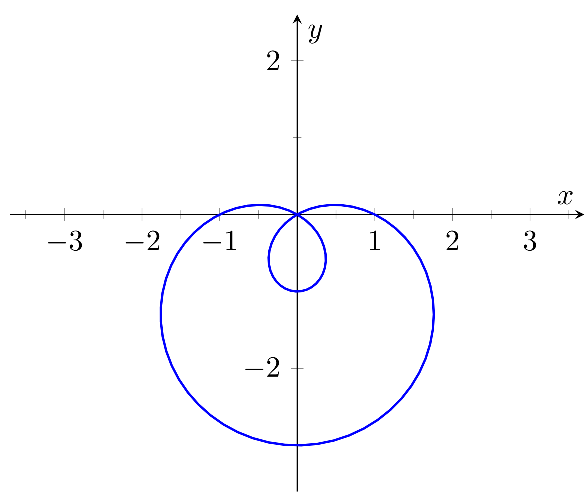
10.4.14.



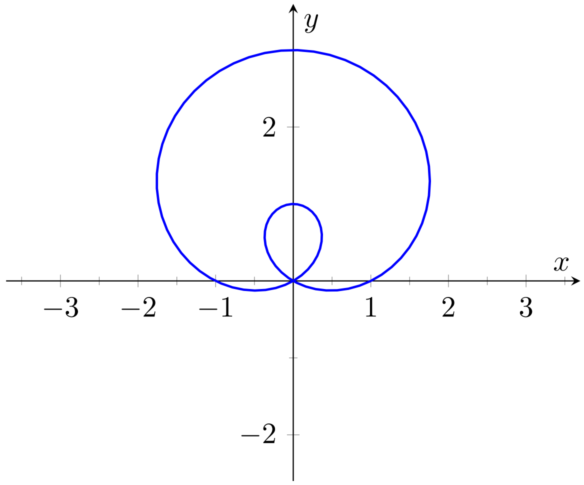
10.4.15.



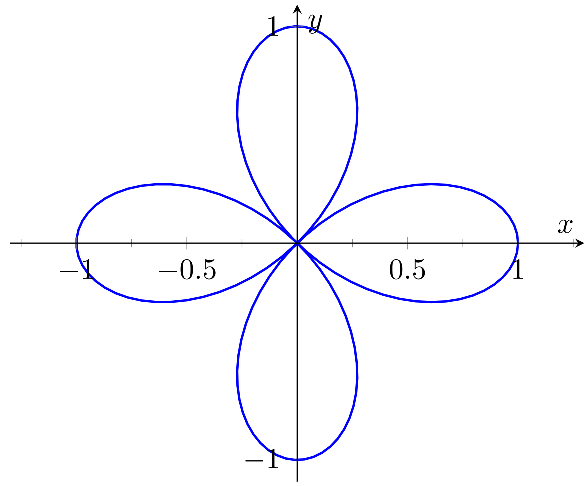
10.4.16.



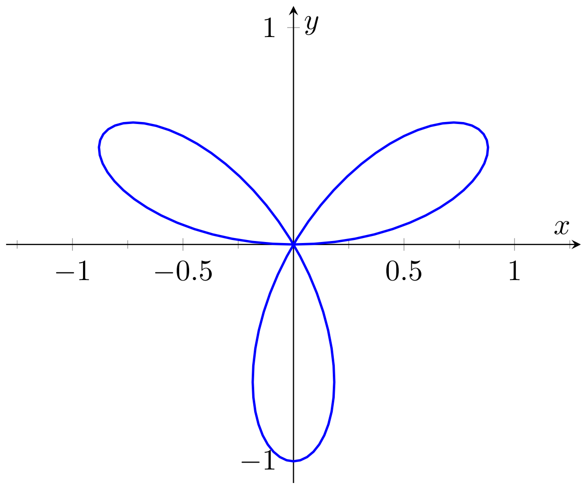
10.4.17.



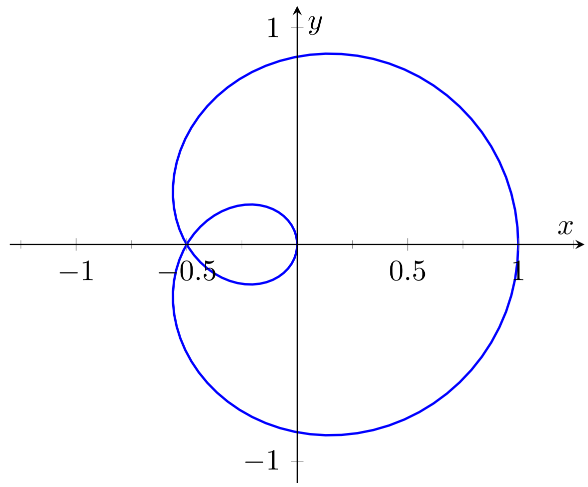
10.4.18.



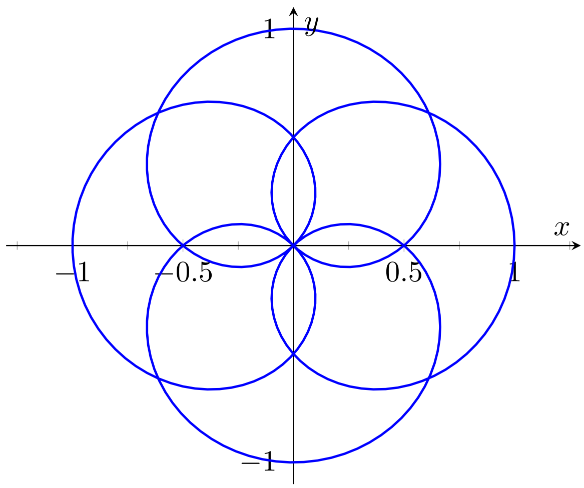
10.4.19.



10.4.20.



10.4.21.



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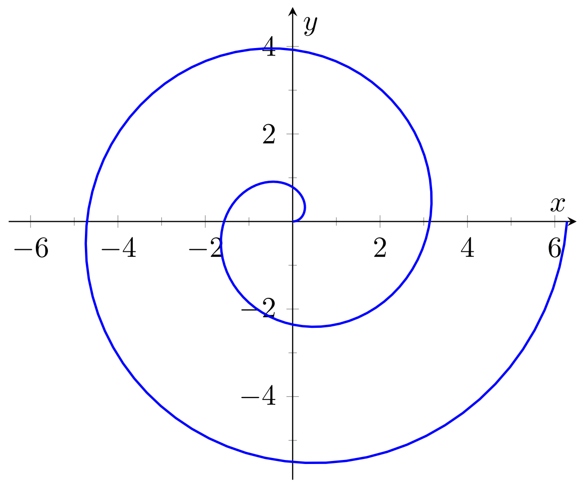

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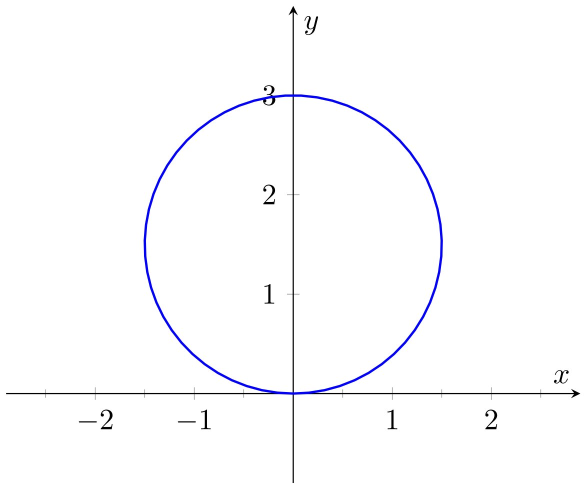

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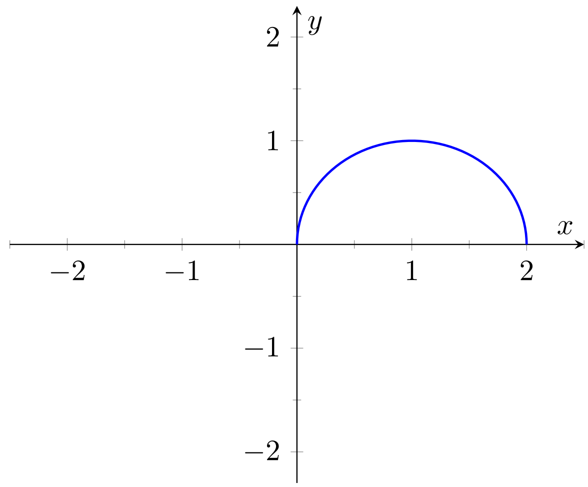

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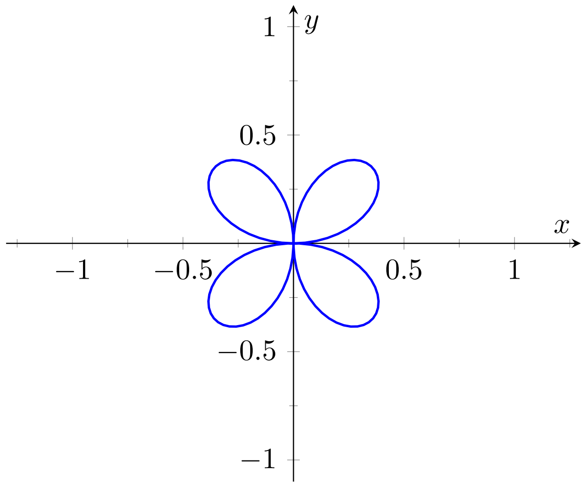

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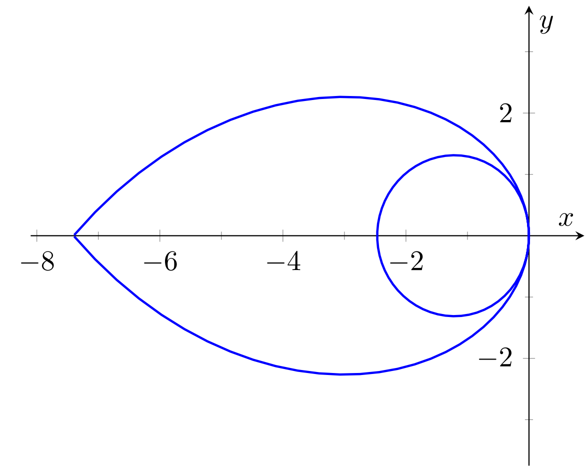

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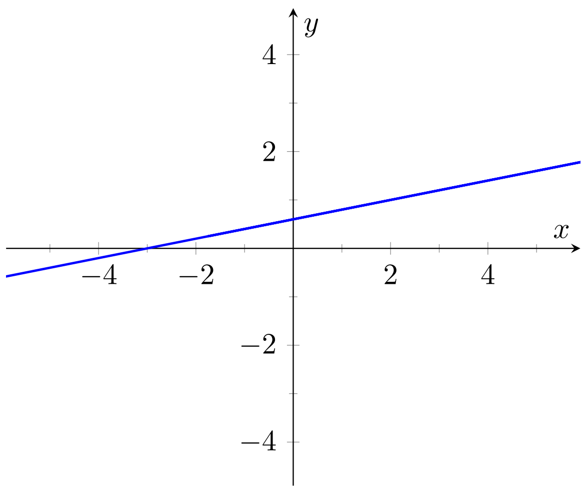

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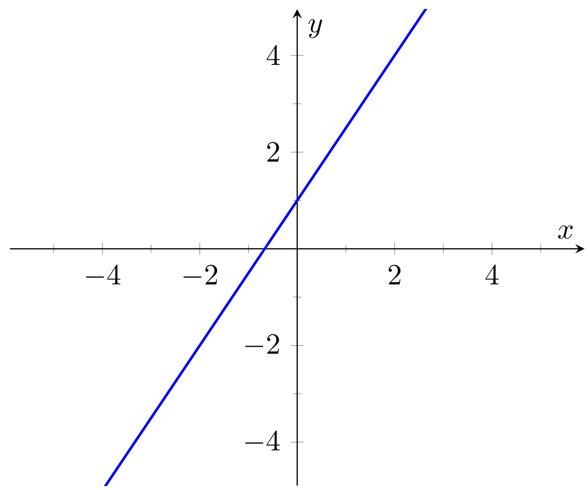

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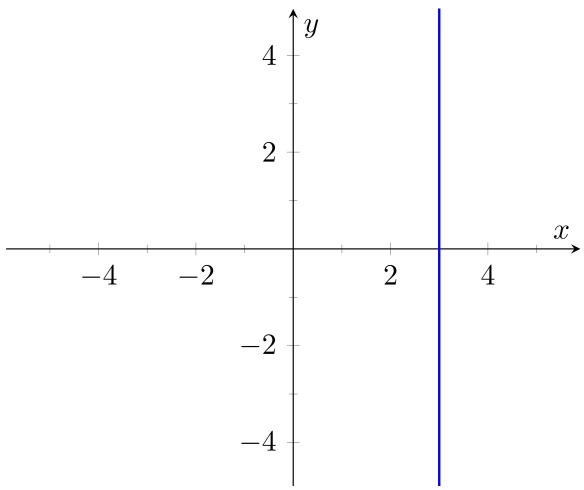

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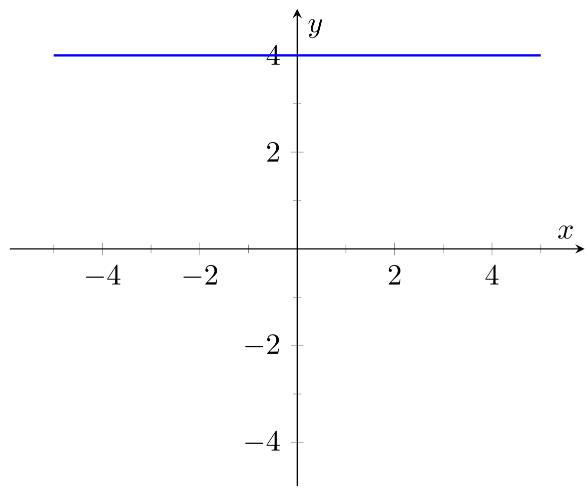


10.4.31. $(x-3)^2+y^2=9$

10.4.32. $x^2+(y+2)^2=4$

10.4.33. $(x-0.5)^2+(y-0.5)^2=0.5$

10.4.34. $y=0.4x+1.4$

10.4.35. $x=3$

10.4.36. $y=4$

10.4.38. $y^4+x^2y^2-x^2=0$

10.4.39. $x^2+y^2=4$

10.4.40. $y=\frac{x}{1.73205}$

10.4.41. $\theta=\frac{\pi}{4}$

10.4.42. $r=\frac{7}{\sin(\theta)-4\cos(\theta)}$

10.4.43. $r=5\sec(\theta)$

10.4.44. $r=5\csc(\theta)$

10.4.45. $r=\frac{\cos(\theta)}{\sin^2(\theta)}$

10.4.47. $r=\sqrt{7}$

10.4.49. $P\left(\frac{\sqrt{3}}{2},\frac{\pi}{6}\right),P\left(0,\frac{\pi}{2}\right),P\left(\frac{-\sqrt{3}}{2},\frac{5\pi}{6}\right)$

10.4.51. $P(0,0),P\left(\sqrt{2},\frac{\pi}{4}\right)$

10.4.54. $P\left(\frac{3}{2},\frac{\pi}{3}\right),P\left(\frac{3}{2},\frac{-\pi}{3}\right),P(0,\pi)$

10.5 · Calculus and Polar Functions

10.5 · Exercises

Problems

10.5.3. **(a).** $-\cot(\theta)$

(b). $y=-\left(x-\frac{\sqrt{2}}{2}\right)+\frac{\sqrt{2}}{2}$

(c). $y=x$

10.5.4. **(a).** $0.5(\tan(\theta)-\cot(\theta))$

(b). $y=\frac{1}{2}$

(c). $x=\frac{1}{2}$

10.5.7. **(a).** $\frac{\theta\cos(\theta)+\sin(\theta)}{\cos(\theta)-\theta\sin(\theta)}$

(b). $y=\frac{-2}{\pi}x+\frac{\pi}{2}$

(c). $y=\frac{\pi}{2}x+\frac{\pi}{2}$

10.5.8. **(a).** $\frac{\cos(\theta)\cos(3\theta)-3\sin(\theta)\sin(3\theta)}{-\cos(3\theta)\sin(\theta)-3\cos(\theta)\sin(3\theta)}$

(b). $y=\frac{x}{\sqrt{3}}$

(c). $y=-\sqrt{3}x$

10.5.9. **(a).** $\frac{4\sin(\theta)\cos(4\theta)+\sin(4\theta)\cos(\theta)}{4\cos(\theta)\cos(4\theta)-\sin(\theta)\sin(4\theta)}$

(b). $y=5\sqrt{3}\left(x+\frac{\sqrt{3}}{4}\right)-\frac{3}{4}$

(c). $y=\frac{-1}{5\sqrt{3}}\left(x+\frac{\sqrt{3}}{4}\right)-\frac{3}{4}$

10.5.14. **(a).** $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$
(b). $0, \frac{2\pi}{3}, \frac{4\pi}{3}$

10.5.19. $\frac{\pi}{12}$
10.5.20. area = $\pi/(4n)$
10.5.21. $\frac{3\pi}{2}$
10.5.23. $2\pi + \frac{3\cdot 1.73205}{2}$
10.5.24. $\pi + 3 \cdot 1.73205$
10.5.25. 1
10.5.26. $\frac{1}{32}(4\pi - 3 \cdot 1.73205)$

10.5.29. 4π
10.5.30. 4π
10.5.31. $\sqrt{2}\pi$
10.5.32. 8
10.5.33. 2.2592 or 2.22748

10.5.40. $SA = 9\pi$

11 · Introduction to Functions of Several Variables
11.2 · Limits and Continuity of Multivariable Functions
11.2 · Exercises

Problems

11.2.7.

(a) Answers will vary. interior point: $(1,3)$ boundary point: $(3,3)$

(b) S is a closed set

(c) S is bounded

11.2.8.

(a) Answers will vary. Interior point: $(1,0)$ (any point with $y \neq x^2$ will do). Boundary point: $(1,1)$ (any point with $y = x^2$ will do).

(b) S is an open set.

(c) S is unbounded.

11.2.11.

(a) $D = \left\{(x,y)\,|\,9 - x^2 - y^2 \geq 0\right\}$.

(b) D is a closed set.

(c) D is bounded.

11.2.12.

(a) $D = \left\{(x,y)\,|\,y \geq x^2\right\}$.

(b) D is a closed set.

(c) D is unbounded.

11.2.13.

(a) $D = \left\{(x,y)\,|\,y > x^2\right\}$.

(b) D is an open set.

(c) D is unbounded.

11.2.14.

(a) $D = \{(x,y)\,|\,(x,y) \neq (0,0)\}$.

(b) D is an open set.

(c) D is unbounded.

11.3 · Partial Derivatives
11.3 · Exercises

Terms and Concepts

11.3.3. `f_x`
11.3.4. `f_y`

Problems

11.3.6. **(a).** 0
(b). 0

11.3.8. **(a).** $-\frac{1}{2}$
(b). $-\frac{1}{3}$

11.3.10. **(a).** $3x^2+6xy+3y^2$
(b). $3x^2+6xy+3y^2$
(c). $6x+6y$
(d). $6x+6y$
(e). $6x+6y$
(f). $6x+6y$

11.3.12. **(a).** $\frac{-4}{x^2y}$
(b). $\frac{-4}{xy^2}$
(c). $\frac{8}{x^3y}$
(d). $\frac{4}{x^2y^2}$
(e). $\frac{4}{x^2y^2}$
(f). $\frac{8}{xy^3}$

11.3.14. **(a).** e^{x+2y}
(b). $2e^{x+2y}$
(c). e^{x+2y}
(d). $2e^{x+2y}$
(e). $2e^{x+2y}$
(f). $4e^{x+2y}$

11.3.16. **(a).** $3(x+y)^2$
(b). $3(x+y)^2$
(c). $6(x+y)$
(d). $6(x+y)$
(e). $6(x+y)$
(f). $6(x+y)$

11.3.18. **(a).** $10x\cos\big(5x^2+2y^3\big)$
(b). $6y^2\cos\big(5x^2+2y^3\big)$
(c). $10\cos\big(5x^2+2y^3\big)-100x^2\sin\big(5x^2+2y^3\big)$
(d). $-60xy^2\sin\big(5x^2+2y^3\big)$
(e). $-60xy^2\sin\big(5x^2+2y^3\big)$
(f). $12y\cos\big(5x^2+2y^3\big)-36y^4\sin\big(5x^2+2y^3\big)$

11.3.19. **(a).** $\frac{2y^2}{\sqrt{4xy^2+1}}$
(b). $\frac{4xy}{\sqrt{4xy^2+1}}$
(c). $\frac{-4y^4}{\left(\sqrt{4xy^2+1}\right)^3}$
(d). $\frac{-8xy^3}{\left(\sqrt{4xy^2+1}\right)^3}+\frac{4y}{\sqrt{4xy^2+1}}$
(e). $\frac{-8xy^3}{\left(\sqrt{4xy^2+1}\right)^3}+\frac{4y}{\sqrt{4xy^2+1}}$
(f). $\frac{-16x^2y^2}{\left(\sqrt{4xy^2+1}\right)^3}+\frac{4x}{\sqrt{4xy^2+1}}$

11.3.22. **(a).** 5
(b). -17
(c). 0
(d). 0
(e). 0
(f). 0

11.3.24. **(a).** $\frac{2x}{x^2+y}$
(b). $\frac{1}{x^2+y}$
(c). $\frac{-4x^2}{(x^2+y)^2}+\frac{2}{x^2+y}$
(d). $\frac{-2x}{(x^2+y)^2}$
(e). $\frac{-2x}{(x^2+y)^2}$
(f). $\frac{-1}{(x^2+y)^2}$

11.3.26. **(a).** $5e^x\sin(y)$
(b). $5e^x\cos(y)$
(c). $5e^x\sin(y)$
(d). $5e^x\cos(y)$
(e). $5e^x\cos(y)$
(f). $-5e^x\sin(y)$

11.3.28. $\frac{1}{2}x^2+xy+\frac{1}{2}y^2$

11.3.30. $\ln\big(x^2+y^2\big)$

11.3.32. (a). $3x^2y^2 + 3x^2z$

(b). $2x^3y + 2yz$

(c). $x^3 + y^2$

(d). $2y$

(e). $2y$

11.3.34. (a). $\frac{1}{x}$

(b). $\frac{1}{y}$

(c). $\frac{1}{z}$

(d). 0

(e). 0

Appendix B

Quick Reference

B.1 Differentiation Formulas

List B.1.1 Derivative Rules

1. $\frac{d}{dx}(cx) = c$
2. $\frac{d}{dx}(u \pm v) = u' \pm v'$
3. $\frac{d}{dx}(u \cdot v) = uv' + u'v$
4. $\frac{d}{dx}(\frac{u}{v}) = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}(u(v)) = u'(v)v'$
6. $\frac{d}{dx}(c) = 0$
7. $\frac{d}{dx}(x) = 1$

List B.1.2 Derivatives of Elementary Functions

1. $\frac{d}{dx}(x^n) = nx^{n-1}$
2. $\frac{d}{dx}(e^x) = e^x$
3. $\frac{d}{dx}(a^x) = \ln a \cdot a^x$
4. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
5. $\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$
6. $\frac{d}{dx}(\sin x) = \cos x$
7. $\frac{d}{dx}(\cos x) = -\sin x$
8. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
9. $\frac{d}{dx}(\sec x) = \sec x \tan x$
10. $\frac{d}{dx}(\tan x) = \sec^2 x$
11. $\frac{d}{dx}(\cot x) = -\csc^2 x$
12. $\frac{d}{dx}(\cosh x) = \sinh x$
13. $\frac{d}{dx}(\sinh x) = \cosh x$
14. $\frac{d}{dx}(\text{sech}\, x) = -\text{sech}\, x \tanh x$
15. $\frac{d}{dx}(\tanh x) = \text{sech}^2\, x$
16. $\frac{d}{dx}(\text{csch}\, x) = -\text{csch}\, x \coth x$
17. $\frac{d}{dx}(\coth x) = -\text{csch}^2\, x$

List B.1.3 Derivatives of Inverse Functions

1. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
2. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
3. $\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$
4. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
5. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
6. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
7. $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$
8. $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$
9. $\frac{d}{dx}(\text{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$
10. $\frac{d}{dx}(\text{csch}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}$
11. $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$
12. $\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$

B.2 Integration Formulas

List B.2.1 Basic Rules

1. $\int c \cdot f(x)\,dx = c\int f(x)\,dx$
2. $\int \big(f(x) \pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
3. $\int 0\,dx = C$
4. $\int 1\,dx = x + C$

List B.2.2 Integrals of Elementary (non-Trig) Functions

1. $\int e^x\,dx = e^x + C$
2. $\int \ln x\,dx = x\ln x - x + C$
3. $\int a^x\,dx = \frac{1}{\ln a} \cdot a^x + C$
4. $\int \frac{1}{x}\,dx = \ln|x| + C$
5. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, n \neq -1$

List B.2.3 Integrals Involving Trigonometric Functions

1. $\int \cos x\,dx = \sin x + C$
2. $\int \sin x\,dx = -\cos x + C$
3. $\int \tan x\,dx = -\ln|\cos x| + C$
4. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
5. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

6. $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$

7. $\displaystyle\int \sec^2 x\,dx = \tan x + C$

8. $\displaystyle\int \csc^2 x\,dx = -\cot x + C$

9. $\displaystyle\int \sec x \tan x\,dx = \sec x + C$

10. $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$

11. $\displaystyle\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$

12. $\displaystyle\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$

13. $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

14. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

15. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$

List B.2.4 Integrals Involving Hyperbolic Functions

1. $\displaystyle\int \cosh x\,dx = \sinh x + C$

2. $\displaystyle\int \sinh x\,dx = \cosh x + C$

3. $\displaystyle\int \tanh x\,dx = \ln(\cosh x) + C$

4. $\displaystyle\int \coth x\,dx = \ln|\sinh x| + C$

5. $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|x + \sqrt{x^2-a^2}\right| + C$

6. $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left|x + \sqrt{x^2+a^2}\right| + C$

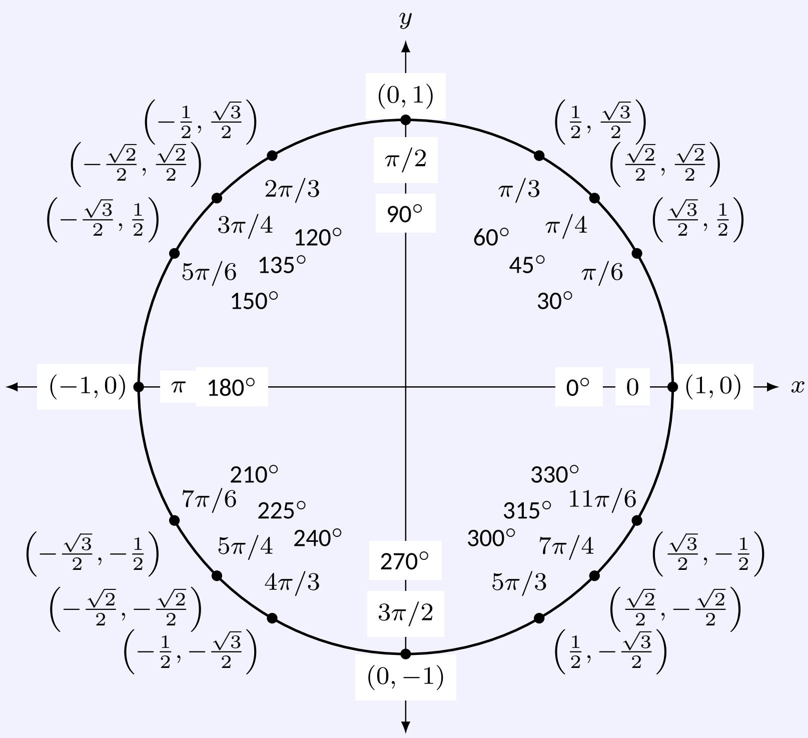
7. $\displaystyle\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

8. $\displaystyle\int \frac{1}{x\sqrt{a^2-x^2}}\,dx = \frac{1}{a}\ln\left(\frac{x}{a+\sqrt{a^2-x^2}}\right) + C$

9. $\displaystyle\int \frac{1}{x\sqrt{x^2+a^2}} = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{x^2+a^2}}\right| + C$

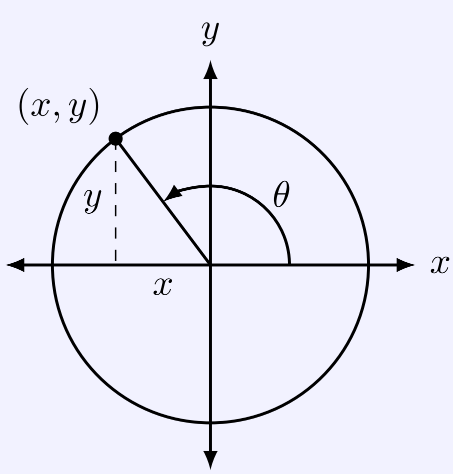
B.3 Trigonometry Reference

The Unit Circle.



B.3.1 Definitions of the Trigonometric Functions

Unit Circle Definition.

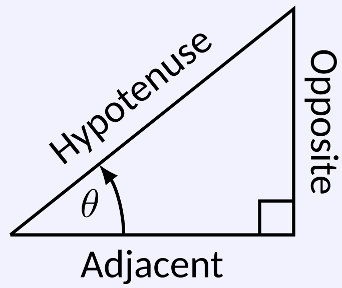


$\sin\theta = y \qquad \cos\theta = x$

$\csc\theta = \dfrac{1}{y} \qquad \sec\theta = \dfrac{1}{x}$

$\tan\theta = \dfrac{y}{x} \qquad \cot\theta = \dfrac{x}{y}$

Right Triangle Definition.



$\sin\theta = \frac{\text{O}}{\text{H}}$ $\quad\csc\theta = \frac{\text{H}}{\text{O}}$

$\cos\theta = \frac{\text{A}}{\text{H}}$ $\quad\sec\theta = \frac{\text{H}}{\text{A}}$

$\tan\theta = \frac{\text{O}}{\text{A}}$ $\quad\cot\theta = \frac{\text{A}}{\text{O}}$

B.3.2 Common Trigonometric Identities

1. $\sin^2 x + \cos^2 x = 1$
2. $\tan^2 x + 1 = \sec^2 x$
3. $1 + \cot^2 x = \csc^2 x$

List B.3.1 Pythagorean Identities

1. $\sin 2x = 2\sin x\cos x$
2. $$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x\\ &= 2\cos^2 x - 1\\ &= 1 - 2\sin^2 x\end{aligned}$$
3. $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

List B.3.2 Double Angle Formulas

1. $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
2. $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
3. $\tan\left(\frac{\pi}{2} - x\right) = \cot x$
4. $\csc\left(\frac{\pi}{2} - x\right) = \sec x$
5. $\sec\left(\frac{\pi}{2} - x\right) = \csc x$
6. $\cot\left(\frac{\pi}{2} - x\right) = \tan x$

List B.3.3 Cofunction Identities

1. $\sin(-x) = -\sin x$
2. $\cos(-x) = \cos x$
3. $\tan(-x) = -\tan x$
4. $\csc(-x) = -\csc x$
5. $\sec(-x) = \sec x$
6. $\cot(-x) = -\cot x$

List B.3.4 Even/Odd Identities

1. $\sin^2 x = \dfrac{1 - \cos 2x}{2}$
2. $\cos^2 x = \dfrac{1 + \cos 2x}{2}$
3. $\tan^2 x = \dfrac{1 - \cos 2x}{1 + \cos 2x}$

List B.3.5 Power-Reducing Formulas

1. $\sin x + \sin y = 2 \sin\left(\dfrac{x+y}{2}\right) \cos\left(\dfrac{x-y}{2}\right)$
2. $\sin x - \sin y = 2 \sin\left(\dfrac{x-y}{2}\right) \cos\left(\dfrac{x+y}{2}\right)$
3. $\cos x + \cos y = 2 \cos\left(\dfrac{x+y}{2}\right) \cos\left(\dfrac{x-y}{2}\right)$
4. $\cos x - \cos y = -2 \sin\left(\dfrac{x+y}{2}\right) \sin\left(\dfrac{x-y}{2}\right)$

List B.3.6 Sum to Product Formulas

List B.3.7 Product to Sum Formulas

1. $\sin x \sin y = \dfrac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$
2. $\cos x \cos y = \dfrac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$
3. $\sin x \cos y = \dfrac{1}{2}\big(\sin(x+y) + \sin(x-y)\big)$

List B.3.8 Angle Sum/Difference Formulas

1. $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
2. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
3. $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

B.4 Areas and Volumes

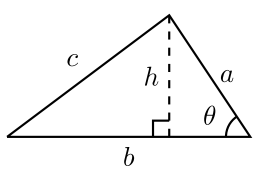
Triangles

$h = a \sin\theta$

Area = $\frac{1}{2}bh$

Law of Cosines:

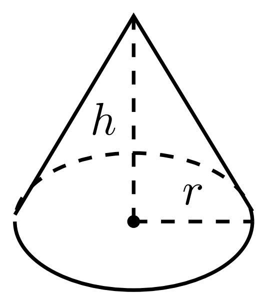
$c^2 = a^2 + b^2 - 2ab\cos\theta$



Right Circular Cone

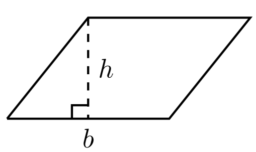
Volume = $\frac{1}{3}\pi r^2 h$

Surface Area = $\pi r\sqrt{r^2 + h^2} + \pi r^2$



Parallelograms

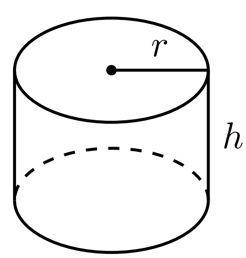
Area = bh



Right Circular Cylinder

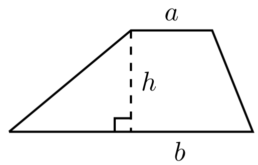
Volume = $\pi r^2 h$

Surface Area = $2\pi rh + 2\pi r^2$



Trapezoids

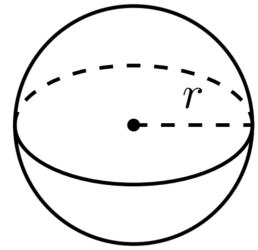
Area = $\frac{1}{2}(a+b)h$



Sphere

Volume = $\frac{4}{3}\pi r^3$

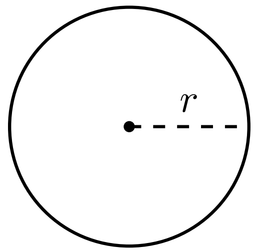
Surface Area =$4\pi r^2$



Circles

Area = πr^2

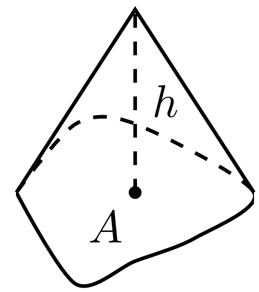
Circumference = $2\pi r$



General Cone

Area of Base = A

Volume = $\frac{1}{3}Ah$

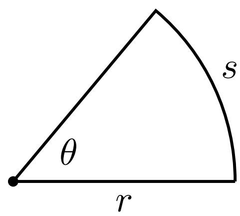


Sectors of Circles

θ in radians

Area = $\frac{1}{2}\theta r^2$

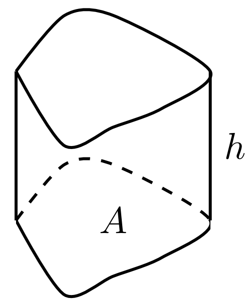
$s = r\theta$



General Right Cylinder

Area of Base = A

Volume = Ah



B.5 Algebra

Factors and Zeros of Polynomials.

Let $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial. If $p(a) = 0$, then a is a $zero$ of the polynomial and a solution of the equation $p(x) = 0$. Furthermore, $(x-a)$ is a $factor$ of the polynomial.

Fundamental Theorem of Algebra.

An nth degree polynomial has n (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

Quadratic Formula.

If $p(x) = ax^2 + bx + c$, and $0 \leq b^2 - 4ac$, then the real zeros of p are $x = (-b \pm \sqrt{b^2-4ac})/2a$

Special Factors.

$$x^2 - a^2 = (x-a)(x+a)$$
$$x^3 - a^3 = (x-a)(x^2+ax+a^2)$$
$$x^3 + a^3 = (x+a)(x^2-ax+a^2)$$
$$x^4 - a^4 = (x^2-a^2)(x^2+a^2)$$

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n$$
$$(x-y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \cdots \pm nxy^{n-1} \mp y^n$$

Binomial Theorem.

$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x-y)^2 = x^2 - 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$

Rational Zero Theorem.

If $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ has integer coefficients, then every $rational\ zero$ of p is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Factoring by Grouping.

$$acx^3 + adx^2 + bcx + bd = ax^2(cx+d) + b(cx+d) = (ax^2+b)(cx+d)$$

Arithmetic Operations.

$$ab + ac = a(b+c) \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \qquad \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc} \qquad \frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc} \qquad \frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c} \qquad \frac{a-b}{c-d} = \frac{b-a}{d-c} \qquad \frac{ab+ac}{a} = b+c$$

Exponents and Radicals.

$$a^0 = 1,\ a \neq 0 \qquad (ab)^x = a^xb^x \qquad a^xa^y = a^{x+y} \qquad \sqrt{a} = a^{1/2}$$

$$\frac{a^x}{a^y} = a^{x-y} \qquad \sqrt[n]{a} = a^{1/n} \qquad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \sqrt[n]{a^m} = a^{m/n}$$

$$a^{-x} = \frac{1}{a^x} \qquad \sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b} \qquad (a^x)^y = a^{xy} \qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

B.6 Additional Formulas

Summation Formulas:.

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Trapezoidal Rule:.

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\big[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\big]$$

with Error $\leq \dfrac{(b-a)^3}{12n^2}\big[\,\text{max}\,|f\,''(x)|\,\big]$

Simpson's Rule:.

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\big]$$

with Error $\leq \dfrac{(b-a)^5}{180n^4}\left[\,\text{max}\left|f^{\,(4)}(x)\right|\,\right]$

Arc Length:.

$$L = \int_a^b \sqrt{1 + f\,'(x)^2}\,dx$$

Surface of Revolution:.

$$2\pi \int_a^b f(x)\sqrt{1 + f\,'(x)^2}dx$$

(where $f(x) \geq 0$)

$$S = 2\pi \int_a^b x\sqrt{1 + f\,'(x)^2}dx$$

(where $a, b \geq 0$)

Work Done by a Variable Force:.

$$W = \int_a^b F(x)dx$$

Force Exerted by a Fluid:.

$$F = \int_a^b w\, d(y)\, \ell(y) dy$$

Taylor Series Expansion for $f(x)$:.

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

Maclaurin Series Expansion for $f(x)$, where $c = 0$:.

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

B.7 Summary of Tests for Series

Table B.7.1

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	Cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} r^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum $= \frac{1}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+a})$	$\lim_{n\to\infty} b_n = L$		Sum $= \left(\sum_{n=1}^{a} b_n\right) - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$	$p > 1$	$p \leq 1$	
Integral Test	$\sum_{n=0}^{\infty} a_n$	$\int_1^{\infty} a(n)\, dn$ converges	$\int_1^{\infty} a(n)\, dn$ diverges	$a_n = a(n)$ must be continuous
Direct Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$	$\sum_{n=0}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$	
Limit Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n\to\infty} \frac{a_n}{b_n} \geq 0$	$\sum_{n=0}^{\infty} b_n$ diverges and $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$	Also diverges if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$
Ratio Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$ Also diverges if	$\{a_n\}$ must be positive $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$
Root Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} (a_n)^{1/n} < 1$	$\lim_{n\to\infty} (a_n)^{1/n} > 1$ Also diverges if	$\{a_n\}$ must be positive $\lim_{n\to\infty} (a_n)^{1/n} = \infty$

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