MATH 1010
INTRODUCTION TO CALCULUS

Fall 2018 Abridged Edition, University of Lethbridge

Editor: Sean Fitzpatrick

Department of Mathematics and Computer Science

University of Lethbridge

Contributing Textbooks

Precalculus, Version $\lfloor \pi \rfloor = 3$

Carl Stitz and Jeff Zeager

www.stitz-zeager.com

APEX *Calculus*

Gregory Hartman et al

apexcalculus.com

Contents

Table of Contents **iii**

Preface **v**

1 The Real Numbers **1**
- 1.1 Some Basic Set Theory Notions 1
  - 1.1.1 Sets of Real Numbers 3
- 1.2 The Cartesian Coordinate Plane 10
  - 1.2.1 Distance in the Plane 13

2 Functions **17**
- 2.1 Function Notation 17
- 2.2 Operations on Functions 23
  - 2.2.1 Arithmetic with Functions 23
  - 2.2.2 Function Composition 26
  - 2.2.3 Inverse Functions 30

3 Essential Functions **41**
- 3.1 Linear and Quadratic Functions 41
  - 3.1.1 Linear Functions 41
  - 3.1.2 Absolute Value Functions 46
  - 3.1.3 Quadratic Functions 49
- 3.2 Polynomial Functions 56
  - 3.2.1 Graphs of Polynomial Functions 56
  - 3.2.2 Polynomial Arithmetic 64
  - 3.2.3 Polynomial Addition, Subtraction and Multiplication. . . . 65
  - 3.2.4 Polynomial Long Division. 67
- 3.3 Rational Functions 73
  - 3.3.1 Introduction to Rational Functions 73
- 3.4 Exponential and Logarithmic Functions 85
  - 3.4.1 Introduction to Exponential and Logarithmic Functions . 85
  - 3.4.2 Properties of Logarithms 94

4 Foundations of Trigonometry **105**
- 4.1 The Unit Circle: Sine and Cosine 105
- 4.2 The Six Circular Functions and Fundamental Identities 115
- 4.3 Trigonometric Identities 123
- 4.4 Graphs of the Trigonometric Functions 138
  - 4.4.1 Graphs of the Cosine and Sine Functions 138
  - 4.4.2 Graphs of the Secant and Cosecant Functions 143
  - 4.4.3 Graphs of the Tangent and Cotangent Functions 146
- 4.5 Inverse Trigonometric Functions 153

4.5.1 Inverses of Secant and Cosecant: Trigonometry Friendly Approach . . . 159
4.5.2 Inverses of Secant and Cosecant: Calculus Friendly Approach . . . 162

5 Limits **171**
5.1 An Introduction To Limits . . . 171
5.2 Finding Limits Analytically . . . 178
5.3 One Sided Limits . . . 188
5.4 Limits Involving Infinity . . . 194
5.5 Continuity . . . 204

6 Derivatives **213**
6.1 Instantaneous Rates of Change: The Derivative . . . 213
6.2 Interpretations of the Derivative . . . 225
6.3 Basic Differentiation Rules . . . 231
6.4 The Product and Quotient Rules . . . 238
6.5 The Chain Rule . . . 247

7 The Graphical Behaviour of Functions **257**
7.1 Extreme Values . . . 257
7.2 Increasing and Decreasing Functions . . . 264
7.3 Concavity and the Second Derivative . . . 271
7.4 Curve Sketching . . . 278
7.5 Antiderivatives and Indefinite Integration . . . 286

A Answers To Selected Problems **A.1**

Index **A.23**

Preface

One of the challenges with a course like Math 1010 is finding a suitable textbook. The course covers material from two topics – Precalculus and Calculus – that are usually offered as separate courses, with separate texts. Before the initial offering of Math 1010, I reviewed a number of commercially available options, but these all had two things in common: they did not quite meet our needs, and they were all very expensive (some were as much as $400).

Since writing a new textbook from scratch is a huge undertaking, requiring resources (like time) we simply did not have, I chose to explore non-commercial options. This took a bit of searching, since non-commercial texts, while inexpensive (or free), are of varying quality. Fortunately, there are some decent texts out there. Unfortunately, I couldn't find a single text that covered all of the material we need for Math 1010.

To get around this problem, I have selected two textbooks as our primary sources for the course. The first is *Precalculus*, version 3, by Carl Stitz and Jeff Zeager. The second is *APEX Calculus I*, version 3.0, by Hartman et al. (As of June, 2018, we have updated to version 4.0!) Both texts have two very useful advantages. First, they're both free (as in beer): you can download either text in PDF format from the authors' web pages. Second, they're also *open source* texts (that is, free, as in speech). Both books are written using the LaTeXmarkup language, as is typical in mathematics publishing. What is not typical is that the authors of both texts make their source code freely available, allowing others (such as myself) to edit and customize the books as they see fit.

In the first iteration of this project (Fall 2015), I was only able to edit each text individually for length and content, resulting in two separate textbooks for Math 1010. For Fall 2016, I had enough time to take the content of the Precalculus textbook and adapt its source code to be compatible with the formatting of the Calculus textbook, allowing me to produce a single textbook for all of Math 1010.

For Fall 2017, I produced this much shortened, abridged version of the "Complete (and Current) Edition" produced the previous year. That version has more material than an instructor can reasonably expect to cover in one semester. The unabridged version is still available for a student who wants a more complete treatment of the precalculus material in the text.

The book is very much a work in progress, and I will be editing it regularly. Feedback is always welcome.

Acknowledgements

First and foremost, I need to thank the authors of the two textbooks that provide the source material for this text. Without their hard work, and willingness to make their books (and the source code) freely available, it would not have been possible to create an affordable textbook for this course. You can find the original textbooks at their websites:

www.stitz-zeager.com, for the *Precalculus* textbook, by Stitz and Zeager, and

apexcalculus.com, for the APEX *Calculus* textbook, by Hartman et al.

I'd also like to thank Dave Morris for help with converting the graphics in the Precalculus textbook to work with the formatting code of the APEX text, Howard Cheng for providing some C++ code to convert the exercises, and the other faculty members involved with this course — Alia Hamieh, David Kaminski, and Nicole Wilson — for their input on the content of the text.

Sean Fitzpatrick
Department of Mathematics and Computer Science
University of Lethbridge
August, 2018

1: The Real Numbers

1.1 Some Basic Set Theory Notions

While the authors would like nothing more than to delve quickly and deeply into the sheer excitement that is *Precalculus*, experience has taught us that a brief refresher on some basic notions is welcome, if not completely necessary, at this stage. To that end, we present a brief summary of 'set theory' and some of the associated vocabulary and notations we use in the text. Like all good Math books, we begin with a definition.

> **Definition 1.1.1 Set**
>
> A **set** is a well-defined collection of objects which are called the 'elements' of the set. Here, 'well-defined' means that it is possible to determine if something belongs to the collection or not, without prejudice.

For example, the collection of letters that make up the word "pronghorns" is well-defined and is a set, but the collection of the worst math teachers in the world is **not** well-defined, and so is **not** a set. In general, there are three ways to describe sets. They are

One thing that student evaluations teach us is that any given Mathematics instructor can be simultaneously the best and worst teacher ever, depending on who is completing the evaluation.

> **Key Idea 1.1.1 Ways to Describe Sets**
>
> 1. **The Verbal Method:** Use a sentence to define a set.
> 2. **The Roster Method:** Begin with a left brace '$\{$', list each element of the set *only once* and then end with a right brace '$\}$'.
> 3. **The Set-Builder Method:** A combination of the verbal and roster methods using a "dummy variable" such as x.

For example, let S be the set described *verbally* as the set of letters that make up the word "pronghorns". A **roster** description of S would be $\{p, r, o, n, g, h, s\}$. Note that we listed 'r', 'o', and 'n' only once, even though they appear twice in "pronghorns." Also, the *order* of the elements doesn't matter, so $\{o, n, p, r, g, s, h\}$ is also a roster description of S. A **set-builder** description of S is:

$$\{x \mid x \text{ is a letter in the word "pronghorns".}\}$$

The way to read this is: 'The set of elements x such that x is a letter in the word "pronghorns."' In each of the above cases, we may use the familiar equals sign '$=$' and write $S = \{p, r, o, n, g, h, s\}$ or $S = \{x \mid x \text{ is a letter in the word "pronghorns".}\}$. Clearly r is in S and q is not in S. We express these sentiments mathematically by writing $r \in S$ and $q \notin S$.

More precisely, we have the following.

Definition 1.1.2 Notation for set inclusion

Let A be a set.

- If x is an element of A then we write $x \in A$ which is read 'x is in A'.
- If x is *not* an element of A then we write $x \notin A$ which is read 'x is not in A'.

Now let's consider the set $C = \{x \mid x \text{ is a consonant in the word "pronghorns"}\}$. A roster description of C is $C = \{p, r, n, g, h, s\}$. Note that by construction, every element of C is also in S. We express this relationship by stating that the set C is a **subset** of the set S, which is written in symbols as $C \subseteq S$. The more formal definition is given below.

Definition 1.1.3 Subset

Given sets A and B, we say that the set A is a **subset** of the set B and write '$A \subseteq B$' if every element in A is also an element of B.

Note that in our example above $C \subseteq S$, but not vice-versa, since $o \in S$ but $o \notin C$. Additionally, the set of vowels $V = \{a, e, i, o, u\}$, while it does have an element in common with S, is not a subset of S. (As an added note, S is not a subset of V, either.) We could, however, *build* a set which contains both S and V as subsets by gathering all of the elements in both S and V together into a single set, say $U = \{p, r, o, n, g, h, s, a, e, i, u\}$. Then $S \subseteq U$ and $V \subseteq U$. The set U we have built is called the **union** of the sets S and V and is denoted $S \cup V$. Furthermore, S and V aren't completely *different* sets since they both contain the letter 'o.' (Since the word 'different' could be ambiguous, mathematicians use the word *disjoint* to refer to two sets that have no elements in common.) The **intersection** of two sets is the set of elements (if any) the two sets have in common. In this case, the intersection of S and V is $\{o\}$, written $S \cap V = \{o\}$. We formalize these ideas below.

Definition 1.1.4 Intersection and Union

Suppose A and B are sets.

- The **intersection** of A and B is $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- The **union** of A and B is $A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ (or both)}\}$

The key words in Definition 1.1.4 to focus on are the conjunctions: 'intersection' corresponds to 'and' meaning the elements have to be in *both* sets to be in the intersection, whereas 'union' corresponds to 'or' meaning the elements have to be in one set, or the other set (or both). In other words, to belong to the union of two sets an element must belong to *at least one* of them.

Returning to the sets C and V above, $C \cup V = \{p, r, n, g, h, s, a, e, i, o, u\}$. When it comes to their intersection, however, we run into a bit of notational awkwardness since C and V have no elements in common. While we could write $C \cap V = \{\}$, this sort of thing happens often enough that we give the set with no elements a name.

Definition 1.1.5 Empty set

The **Empty Set** $\emptyset$ is the set which contains no elements. That is,

$$\emptyset = \{\} = \{x \,|\, x \neq x\}.$$

As promised, the empty set is the set containing no elements since no matter what 'x' is, '$x = x$.' Like the number '0,' the empty set plays a vital role in mathematics. We introduce it here more as a symbol of convenience as opposed to a contrivance. Using this new bit of notation, we have for the sets C and V above that $C \cap V = \emptyset$. A nice way to visualize relationships between sets and set operations is to draw a **Venn Diagram**. A Venn Diagram for the sets S, C and V is drawn in Figure 1.1.1.

In Figure 1.1.1 we have three circles - one for each of the sets C, S and V. We visualize the area enclosed by each of these circles as the elements of each set. Here, we've spelled out the elements for definitiveness. Notice that the circle representing the set C is completely inside the circle representing S. This is a geometric way of showing that $C \subseteq S$. Also, notice that the circles representing S and V overlap on the letter 'o'. This common region is how we visualize $S \cap V$. Notice that since $C \cap V = \emptyset$, the circles which represent C and V have no overlap whatsoever.

All of these circles lie in a rectangle labelled U (for 'universal' set). A universal set contains all of the elements under discussion, so it could always be taken as the union of all of the sets in question, or an even larger set. In this case, we could take $U = S \cup V$ or U as the set of letters in the entire alphabet. The usual triptych of Venn Diagrams indicating generic sets A and B along with $A \cap B$ and $A \cup B$ is given below.

(The reader may well wonder if there is an ultimate universal set which contains *everything*. The short answer is 'no'. Our definition of a set turns out to be overly simplistic, but correcting this takes us well beyond the confines of this course. If you want the longer answer, you can begin by reading about Russell's Paradox on Wikipedia.)

1.1.1 Sets of Real Numbers

The playground for most of this text is the set of **Real Numbers**. Many quantities in the 'real world' can be quantified using real numbers: the temperature at a given time, the revenue generated by selling a certain number of products and the maximum population of Sasquatch which can inhabit a particular region are just three basic examples. A succinct, but nonetheless incomplete definition of a real number is given below.

Definition 1.1.6 The real numbers

A **real number** is any number which possesses a decimal representation. The set of real numbers is denoted by the character $\mathbb{R}$.

The full extent of the empty set's role will not be explored in this text, but it is of fundamental importance in Set Theory. In fact, the empty set can be used to generate numbers - mathematicians can create something from nothing! If you're interested, read about the von Neumann construction of the natural numbers or consider signing up for Math 2000.

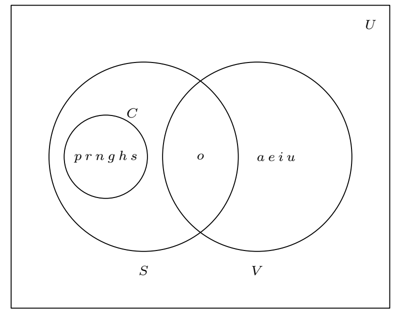
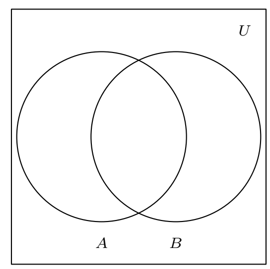
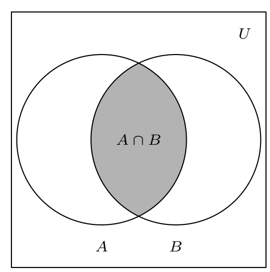


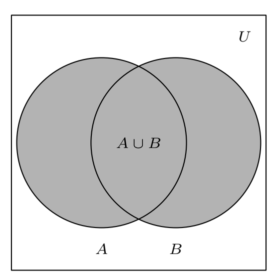
Figure 1.1.1: A Venn diagram for C, S, and V



Sets A and B.



$A \cap B$ is shaded.



$A \cup B$ is shaded.

Figure 1.1.2: Venn diagrams for intersection and union

Certain subsets of the real numbers are worthy of note and are listed below. In more advanced courses like Analysis, you learn that the real numbers can be *constructed* from the rational numbers, which in turn can be constructed from the integers (which themselves come from the natural numbers, which in turn can be defined as sets...).

Definition 1.1.7 Sets of Numbers

1. The **Empty Set**: $\emptyset = \{\} = \{x \mid x \neq x\}$. This is the set with no elements. Like the number '0,' it plays a vital role in mathematics.
2. The **Natural Numbers**: $\mathbb{N} = \{1, 2, 3, \ldots\}$ The periods of ellipsis here indicate that the natural numbers contain 1, 2, 3, 'and so forth'.
3. The **Integers**: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
4. The **Rational Numbers**: $\mathbb{Q} = \left\{\frac{a}{b} \mid a \in \mathbb{Z} \text{ and } b \in \mathbb{Z}\right\}$. Rational numbers are the ratios of integers (provided the denominator is not zero!) It turns out that another way to describe the rational numbers is:

   $\mathbb{Q} = \{x \mid x \text{ possesses a repeating or terminating decimal representation.}\}$
5. The **Real Numbers**: $\mathbb{R} = \{x \mid x \text{ possesses a decimal representation.}\}$
6. The **Irrational Numbers**: Real numbers that are not rational are called **irrational**. As a set, we have $\{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$. (There is no standard symbol for this set.) Every irrational number has a decimal expansion which neither repeats nor terminates.
7. The **Complex Numbers**: $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\}$ (We will not deal with complex numbers in Math 1010, although they usually make an appearance in Math 1410.)

An example of a number with a repeating decimal expansion is $a = 2.13234234234\ldots$. This is rational since $100a = 213.2342342342\ldots$, and $100000a = 213234.234234\ldots$ so $99900a = 100000a - 100a = 213021$. This gives us the rational expression $a = \dfrac{213021}{99900}$.

The classic example of an irrational number is the number π, but numbers like $\sqrt{2}$ and $0.101001000100001\ldots$ are other fine representatives.

It is important to note that every natural number is a whole number is an integer. Each integer is a rational number (take $b = 1$ in the above definition for $\mathbb{Q}$) and the rational numbers are all real numbers, since they possess decimal representations (via long division!). If we take $b = 0$ in the above definition of $\mathbb{C}$, we see that every real number is a complex number. In this sense, the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are 'nested' like Matryoshka dolls. More formally, these sets form a subset chain: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$. The reader is encouraged to sketch a Venn Diagram depicting $\mathbb{R}$ and all of the subsets mentioned above.

As you may recall, we often visualize the set of real numbers $\mathbb{R}$ as a line where each point on the line corresponds to one and only one real number. Given two different real numbers a and b, we write $a < b$ if a is located to the left of b on the number line, as shown in Figure 1.1.3.

While this notion seems innocuous, it is worth pointing out that this convention is rooted in two deep properties of real numbers. The first property is that $\mathbb{R}$ is complete. This means that there are no 'holes' or 'gaps' in the real number line. (This intuitive feel for what it means to be 'complete' is as good as it gets at this level. Completeness does get a much more precise meaning later in courses like Analysis and Topology.) Another way to think about this is that if you choose any two distinct (different) real numbers, and look between them, you'll find a solid line segment (or interval) consisting of infinitely many real numbers.

The next result tells us what types of numbers we can expect to find.

Theorem 1.1.1 Density Property of $\mathbb{Q}$ in $\mathbb{R}$

Between any two distinct real numbers, there is at least one rational number and irrational number. It then follows that between any two distinct real numbers there will be infinitely many rational and irrational numbers.

The root word 'dense' here communicates the idea that rationals and irrationals are 'thoroughly mixed' into $\mathbb{R}$. The reader is encouraged to think about how one would find both a rational and an irrational number between, say, 0.9999 and 1. Once you've done that, ask yourself whether there is any difference between the numbers $0.\overline{9}$ and 1.

The second property $\mathbb{R}$ possesses that lets us view it as a line is that the set is totally ordered. This means that given any two real numbers a and b, either $a < b$, $a > b$ or $a = b$ which allows us to arrange the numbers from least (left) to greatest (right). You may have heard this property given as the 'Law of Trichotomy'.

Definition 1.1.8 Law of Trichotomy

If a and b are real numbers then **exactly one** of the following statements is true:

$a < b$ $\qquad$ $a > b$ $\qquad$ $a = b$

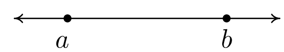


Figure 1.1.3: The real number line with two numbers a and b, where $a < b$.

The reader is probably familiar with the relations $a < b$ and $a > b$ in the context of *solving inequalities*. The **order properties** of the real number system can be summarized as a collection of rules for manipulating inequalities, as follows:

The Law of Trichotomy, strictly speaking, is an *axiom* of the real numbers: a basic requirement that we assume to be true. However, in any *construction* of the real numbers, such as the method of Dedekind cuts, it is necessary to *prove* that the Law of Trichotomy is satisfied.

Key Idea 1.1.2 Rules for inequalities

Let a, b, and c be any real numbers. Then:

- If $a < b$, then $a + c < b + c$.
- If $a < b$, then $a - c < b - c$.
- If $a < b$ **and** $c > 0$, then $ac < bc$.
- If $a < b$ and $c < 0$, then $ac > bc$. (In particular, $-a > -b$.)
- If $0 < a < b$, then $\frac{1}{b} < \frac{1}{a}$.

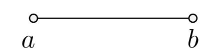
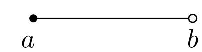
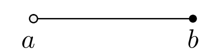
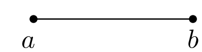
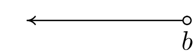
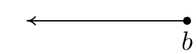
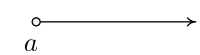
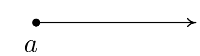
Note the emphasis in rule #3 above: caution must always be exercised when manipulating inequalities: multiplying by a negative number reverses the sign.

This is especially important to remember when dealing with inequalities involving variable quantities, for example, with rational inequalities (see Example 3.3.5).

Segments of the real number line are called **intervals** of numbers. Below is a summary of the so-called **interval notation** associated with given sets of numbers. For intervals with finite endpoints, we list the left endpoint, then the right endpoint. We use square brackets, '[' or ']', if the endpoint is included in the interval and use a filled-in or 'closed' dot to indicate membership in the interval. Otherwise, we use parentheses, '(' or ')' and an 'open' circle to indicate that the endpoint is not part of the set. If the interval does not have finite endpoints, we use the symbols $-\infty$ to indicate that the interval extends indefinitely to the left and ∞ to indicate that the interval extends indefinitely to the right. Since infinity is a concept, and not a number, we always use parentheses when using these symbols in interval notation, and use an appropriate arrow to indicate that the interval extends indefinitely in one (or both) directions.

Definition 1.1.9 Interval Notation

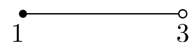
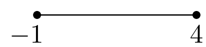
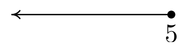
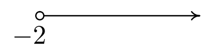
Let a and b be real numbers with $a < b$.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid a < x < b\}$	(a, b)	
$\{x \mid a \leq x < b\}$	$[a, b)$	
$\{x \mid a < x \leq b\}$	$(a, b]$	
$\{x \mid a \leq x \leq b\}$	$[a, b]$	
$\{x \mid x < b\}$	$(-\infty, b)$	
$\{x \mid x \leq b\}$	$(-\infty, b]$	
$\{x \mid x > a\}$	(a, ∞)	
$\{x \mid x \geq a\}$	$[a, \infty)$	
$\mathbb{R}$	$(-\infty, \infty)$	

The importance of understanding interval notation in Calculus cannot be overstated. If you don't find yourself getting the hang of it through repeated use, you may need to take the time to just memorize this chart.

As you can glean from the table, for intervals with finite endpoints we start by writing 'left endpoint, right endpoint'. We use square brackets, '[' or ']', if the endpoint is included in the interval. This corresponds to a 'filled-in' or 'closed' dot on the number line to indicate that the number is included in the set. Otherwise, we use parentheses, '(' or ')' that correspond to an 'open' circle which indicates that the endpoint is not part of the set. If the interval does not have finite endpoints, we use the symbol $-\infty$ to indicate that the interval extends indefinitely to the left and the symbol ∞ to indicate that the interval extends indefinitely to the right. Since infinity is a concept, and not a number, we always use parentheses when using these symbols in interval notation, and use the appropriate arrow to indicate that the interval extends indefinitely in one or both directions.

Let's do a few examples to make sure we have the hang of the notation:

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid 1 \leq x < 3\}$	$[1, 3)$	
$\{x \mid -1 \leq x \leq 4\}$	$[-1, 4]$	
$\{x \mid x \leq 5\}$	$(-\infty, 5]$	
$\{x \mid x > -2\}$	$(-2, \infty)$	

We defined the intersection and union of arbitrary sets in Definition 1.1.4. Recall that the union of two sets consists of the totality of the elements in each of the sets, collected together. For example, if $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$, then $A \cap B = \{2\}$ and $A \cup B = \{1, 2, 3, 4, 6\}$. If $A = [-5, 3)$ and $B = (1, \infty)$, then we can find $A \cap B$ and $A \cup B$ graphically. To find $A \cap B$, we shade the overlap of the two and obtain $A \cap B = (1, 3)$. To find $A \cup B$, we shade each of A and B and describe the resulting shaded region to find $A \cup B = [-5, \infty)$.

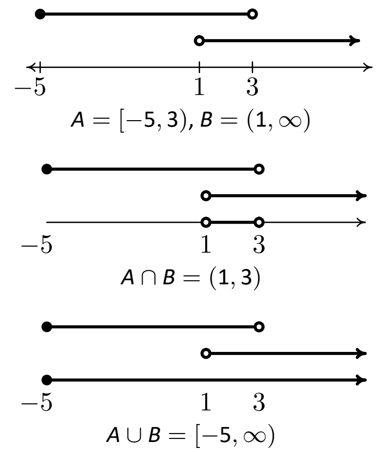


Figure 1.1.4: Union and intersection of intervals

While both intersection and union are important, we have more occasion to use union in this text than intersection, simply because most of the sets of real numbers we will be working with are either intervals or are unions of intervals, as the following example illustrates.

Example 1.1.1 Expressing sets as unions of intervals
Express the following sets of numbers using interval notation.

1. $\{x \mid x \leq -2 \text{ or } x \geq 2\}$
2. $\{x \mid x \neq 3\}$
3. $\{x \mid x \neq \pm 3\}$
4. $\{x \mid -1 < x \leq 3 \text{ or } x = 5\}$

Solution

1. The best way to proceed here is to graph the set of numbers on the number line and glean the answer from it. The inequality $x \leq -2$ corresponds to the interval $(-\infty, -2]$ and the inequality $x \geq 2$ corresponds to the interval $[2, \infty)$. Since we are looking to describe the real numbers x in one of these *or* the other, we have $\{x \mid x \leq -2 \text{ or } x \geq 2\} = (-\infty, -2] \cup [2, \infty)$.

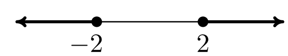


Figure 1.1.5: The set $(-\infty, -2] \cup [2, \infty)$

2. For the set $\{x \mid x \neq 3\}$, we shade the entire real number line except $x = 3$, where we leave an open circle. This divides the real number line into two intervals, $(-\infty, 3)$ and $(3, \infty)$. Since the values of x could be in either one of these intervals *or* the other, we have that $\{x \mid x \neq 3\} = (-\infty, 3) \cup (3, \infty)$

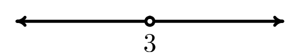


Figure 1.1.6: The set $(-\infty, 3) \cup (3, \infty)$

3. For the set $\{x \mid x \neq \pm 3\}$, we proceed as before and exclude both $x = 3$ and $x = -3$ from our set. This breaks the number line into *three* intervals, $(-\infty, -3)$, $(-3, 3)$ and $(3, \infty)$. Since the set describes real numbers which come from the first, second *or* third interval, we have $\{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

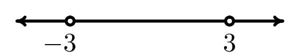


Figure 1.1.7: The set $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

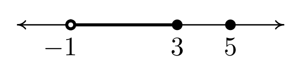


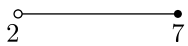
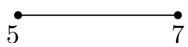
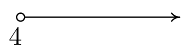
Figure 1.1.8: The set $(-1, 3] \cup \{5\}$

4. Graphing the set $\{x \mid -1 < x \leq 3 \text{ or } x = 5\}$, we get one interval, $(-1, 3]$ along with a single number, or point, $\{5\}$. While we *could* express the latter as $[5, 5]$ (Can you see why?), we choose to write our answer as $\{x \mid -1 < x \leq 3 \text{ or } x = 5\} = (-1, 3] \cup \{5\}$.

Exercises 1.1

Problems

1. Fill in the chart below:

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \leq x < 5\}$		
	$[0, 3)$	
		
$\{x \mid -5 < x \leq 0\}$		
	$(-3, 3)$	
		
$\{x \mid x \leq 3\}$		
	$(-\infty, 9)$	
		
$\{x \mid x \geq -3\}$		

In Exercises 2 – 7, find the indicated intersection or union and simplify if possible. Express your answers in interval notation.

2. $(-1, 5] \cap [0, 8)$

3. $(-1, 1) \cup [0, 6]$

4. $(-\infty, 4] \cap (0, \infty)$

5. $(-\infty, 0) \cap [1, 5]$

6. $(-\infty, 0) \cup [1, 5]$

7. $(-\infty, 5] \cap [5, 8)$

In Exercises 8 – 19, write the set using interval notation.

8. $\{x \mid x \neq 5\}$

9. $\{x \mid x \neq -1\}$

10. $\{x \mid x \neq -3,\ 4\}$

11. $\{x \mid x \neq 0,\ 2\}$

12. $\{x \mid x \neq 2,\ -2\}$

13. $\{x \mid x \neq 0,\ \pm 4\}$

14. $\{x \mid x \leq -1 \text{ or } x \geq 1\}$

15. $\{x \mid x < 3 \text{ or } x \geq 2\}$

16. $\{x \mid x \leq -3 \text{ or } x > 0\}$

17. $\{x \mid x \leq 5 \text{ or } x = 6\}$

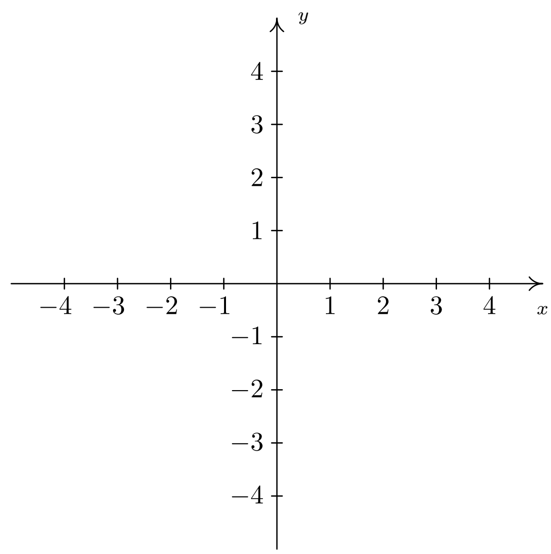
18. $\{x \mid x > 2 \text{ or } x = \pm 1\}$

19. $\{x \mid -3 < x < 3 \text{ or } x = 4\}$

1.2 The Cartesian Coordinate Plane

The Cartesian Plane is named in honour of René Descartes.

In order to visualize the pure excitement that is Precalculus, we need to unite Algebra and Geometry. Simply put, we must find a way to draw algebraic things. Let's start with possibly the greatest mathematical achievement of all time: the **Cartesian Coordinate Plane**. Imagine two real number lines crossing at a right angle at 0 as drawn below.

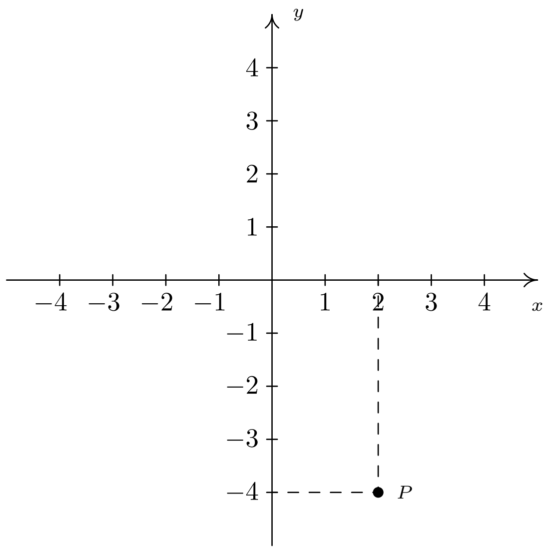


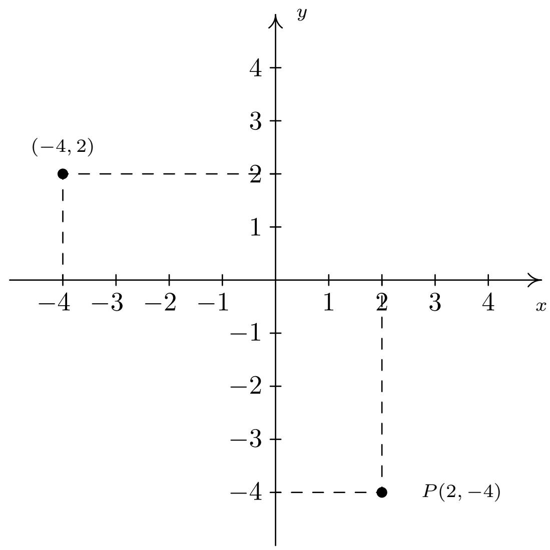
Usually extending off towards infinity is indicated by arrows, but here, the arrows are used to indicate the *direction* of increasing values of *x* and *y*.

The horizontal number line is usually called the ***x*-axis** while the vertical number line is usually called the ***y*-axis**. As with the usual number line, we imagine these axes extending off indefinitely in both directions. Having two number lines allows us to locate the positions of points off of the number lines as well as points on the lines themselves.

The names of the coordinates can vary depending on the context of the application. If, for example, the horizontal axis represented time we might choose to call it the *t*-axis. The first number in the ordered pair would then be the *t*-coordinate.

For example, consider the point *P* on the next page. To use the numbers on the axes to label this point, we imagine dropping a vertical line from the *x*-axis to *P* and extending a horizontal line from the *y*-axis to *P*. This process is sometimes called 'projecting' the point *P* to the *x*- (respectively *y*-) axis. We then describe the point *P* using the **ordered pair** $(2, -4)$. The first number in the ordered pair is called the **abscissa** or ***x*-coordinate** and the second is called the **ordinate** or ***y*-coordinate**. Taken together, the ordered pair $(2, -4)$ comprise the **Cartesian coordinates** of the point *P*. In practice, the distinction between a point and its coordinates is blurred; for example, we often speak of 'the point $(2, -4)$.' We can think of $(2, -4)$ as instructions on how to reach *P* from the **origin** $(0, 0)$ by moving 2 units to the right and 4 units downwards. Notice that the order in the ordered pair is important — if we wish to plot the point $(-4, 2)$, we would move to the left 4 units from the origin and then move upwards 2 units, as below on the right.





When we speak of the Cartesian Coordinate Plane, we mean the set of all possible ordered pairs (x, y) as x and y take values from the real numbers. Below is a summary of important facts about Cartesian coordinates.

Key Idea 1.2.1 Important Facts about the Cartesian Coordinate Plane

- (a, b) and (c, d) represent the same point in the plane if and only if $a = c$ and $b = d$.
- (x, y) lies on the x-axis if and only if $y = 0$.
- (x, y) lies on the y-axis if and only if $x = 0$.
- The origin is the point $(0, 0)$. It is the only point common to both axes.

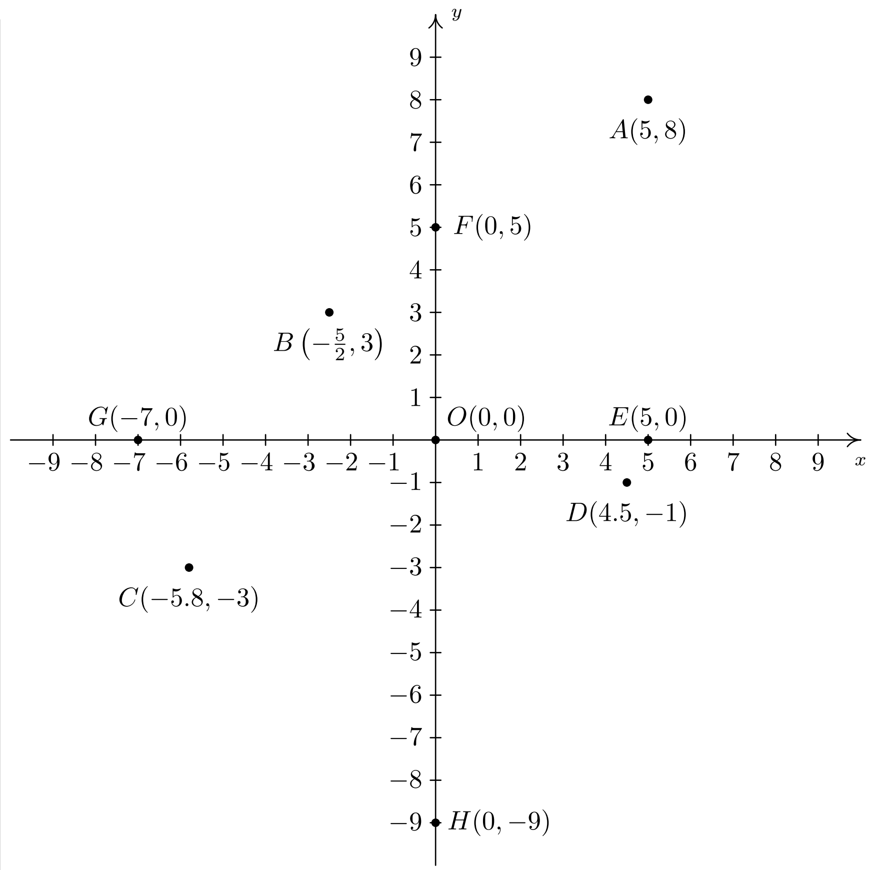
Cartesian coordinates are sometimes referred to as *rectangular coordinates*, to distinguish them from other coordinate systems such as *polar coordinates*.

Example 1.2.1 Plotting points in the Cartesian Plane

Plot the following points: $A(5, 8)$, $B\left(-\frac{5}{2}, 3\right)$, $C(-5.8, -3)$, $D(4.5, -1)$, $E(5, 0)$, $F(0, 5)$, $G(-7, 0)$, $H(0, -9)$, $O(0, 0)$.

The letter O is almost always reserved for the origin.

Solution To plot these points, we start at the origin and move to the right if the x-coordinate is positive; to the left if it is negative. Next, we move up if the y-coordinate is positive or down if it is negative. If the x-coordinate is 0, we start at the origin and move along the y-axis only. If the y-coordinate is 0 we move along the x-axis only.



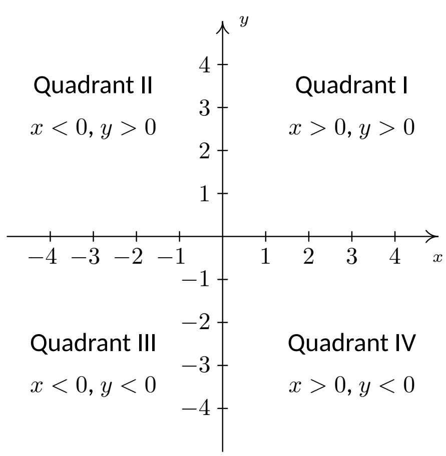


Figure 1.2.1: The four quadrants of the Cartesian plane

The axes divide the plane into four regions called **quadrants**. They are labelled with Roman numerals and proceed counterclockwise around the plane: see Figure 1.2.1.

For example, $(1,2)$ lies in Quadrant I, $(-1,2)$ in Quadrant II, $(-1,-2)$ in Quadrant III and $(1,-2)$ in Quadrant IV. If a point other than the origin happens to lie on the axes, we typically refer to that point as lying on the positive or negative x-axis (if $y = 0$) or on the positive or negative y-axis (if $x = 0$). For example, $(0,4)$ lies on the positive y-axis whereas $(-117,0)$ lies on the negative x-axis. Such points do not belong to any of the four quadrants.

One of the most important concepts in all of Mathematics is **symmetry**. There are many types of symmetry in Mathematics, but three of them can be discussed easily using Cartesian Coordinates.

Definition 1.2.1 Symmetry in the Cartesian Plane

Two points (a,b) and (c,d) in the plane are said to be

- **symmetric about the x-axis** if $a = c$ and $b = -d$
- **symmetric about the y-axis** if $a = -c$ and $b = d$
- **symmetric about the origin** if $a = -c$ and $b = -d$

In Figure 1.2.2, P and S are symmetric about the x-axis, as are Q and R; P and Q are symmetric about the y-axis, as are R and S; and P and R are symmetric about the origin, as are Q and S.

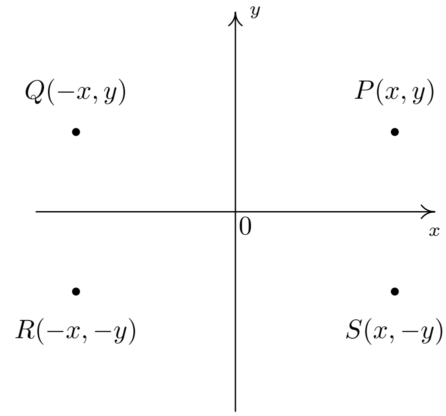


Figure 1.2.2: The three types of symmetry in the plane

Example 1.2.2 Finding points exhibiting symmetry

Let P be the point $(-2, 3)$. Find the points which are symmetric to P about the:

1. x-axis
2. y-axis
3. origin

Check your answer by plotting the points.

Solution The figure after Definition 1.2.1 gives us a good way to think about finding symmetric points in terms of taking the opposites of the x- and/or y-coordinates of $P(-2, 3)$.

1. To find the point symmetric about the x-axis, we replace the y-coordinate with its opposite to get $(-2, -3)$.
2. To find the point symmetric about the y-axis, we replace the x-coordinate with its opposite to get $(2, 3)$.
3. To find the point symmetric about the origin, we replace the x- and y-coordinates with their opposites to get $(2, -3)$.

   The points are plotted in Figure 1.2.3.

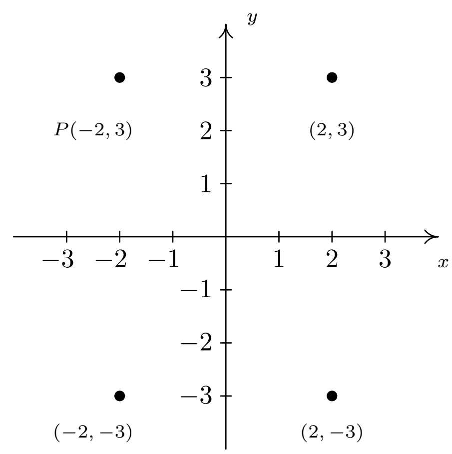


Figure 1.2.3: The point $P(-2, 3)$ and its three reflections

One way to visualize the processes in the previous example is with the concept of a **reflection**. If we start with our point $(-2, 3)$ and pretend that the x-axis is a mirror, then the reflection of $(-2, 3)$ across the x-axis would lie at $(-2, -3)$. If we pretend that the y-axis is a mirror, the reflection of $(-2, 3)$ across that axis would be $(2, 3)$. If we reflect across the x-axis and then the y-axis, we would go from $(-2, 3)$ to $(-2, -3)$ then to $(2, -3)$, and so we would end up at the point symmetric to $(-2, 3)$ about the origin. We summarize and generalize this process below.

Key Idea 1.2.2 Reflections in the Cartesian Plane

To reflect a point (x, y) about the:

- x-axis, replace y with $-y$.
- y-axis, replace x with $-x$.
- origin, replace x with $-x$ and y with $-y$.

1.2.1 Distance in the Plane

Another important concept in Geometry is the notion of length. If we are going to unite Algebra and Geometry using the Cartesian Plane, then we need to develop an algebraic understanding of what distance in the plane means. Suppose we have two points, $P(x_0, y_0)$ and $Q(x_1, y_1)$, in the plane. By the **distance** d between P and Q, we mean the length of the line segment joining P with Q. (Remember, given any two distinct points in the plane, there is a unique line containing both points.) Our goal now is to create an algebraic formula to compute the distance between these two points. Consider the generic situation in Figure 1.2.4.

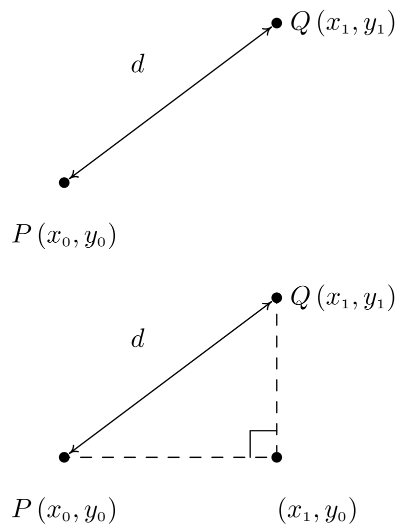


Figure 1.2.4: Distance between P and Q

With a little more imagination, we can envision a right triangle whose hypotenuse has length d as drawn above on the right. From the latter figure, we see that the lengths of the legs of the triangle are $|x_1 - x_0|$ and $|y_1 - y_0|$ so the Pythagorean Theorem gives us

$$|x_1 - x_0|^2 + |y_1 - y_0|^2 = d^2$$

$$(x_1 - x_0)^2 + (y_1 - y_0)^2 = d^2$$

(Do you remember why we can replace the absolute value notation with parentheses?) By extracting the square root of both sides of the second equation and using the fact that distance is never negative, we get

Key Idea 1.2.3 The Distance Formula

The distance d between the points $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$ is:

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

It is not always the case that the points P and Q lend themselves to constructing such a triangle. If the points P and Q are arranged vertically or horizontally, or describe the exact same point, we cannot use the above geometric argument to derive the distance formula. It is left to the reader in Exercise 16 to verify Equation 1.2.3 for these cases.

Example 1.2.3 Distance between two points

Find and simplify the distance between $P(-2, 3)$ and $Q(1, -3)$.

SOLUTION

$$\begin{aligned} d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\ &= \sqrt{(1 - (-2))^2 + (-3 - 3)^2} \\ &= \sqrt{9 + 36} \\ &= 3\sqrt{5} \end{aligned}$$

So the distance is $3\sqrt{5}$.

Example 1.2.4 Finding points at a given distance

Find all of the points with x-coordinate 1 which are 4 units from the point $(3, 2)$.

SOLUTION We shall soon see that the points we wish to find are on the line $x = 1$, but for now we'll just view them as points of the form $(1, y)$.

We require that the distance from $(3, 2)$ to $(1, y)$ be 4. The Distance Formula, Equation 1.2.3, yields

$$
\begin{aligned}
d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\
4 &= \sqrt{(1-3)^2 + (y-2)^2} \\
4 &= \sqrt{4 + (y-2)^2} \\
4^2 &= \left(\sqrt{4+(y-2)^2}\right)^2 && \text{squaring both sides} \\
16 &= 4 + (y-2)^2 \\
12 &= (y-2)^2 \\
(y-2)^2 &= 12 \\
y - 2 &= \pm\sqrt{12} && \text{extracting the square root} \\
y - 2 &= \pm 2\sqrt{3} \\
y &= 2 \pm 2\sqrt{3}
\end{aligned}
$$

We obtain two answers: $(1, 2+2\sqrt{3})$ and $(1, 2-2\sqrt{3})$. The reader is encouraged to think about why there are two answers.

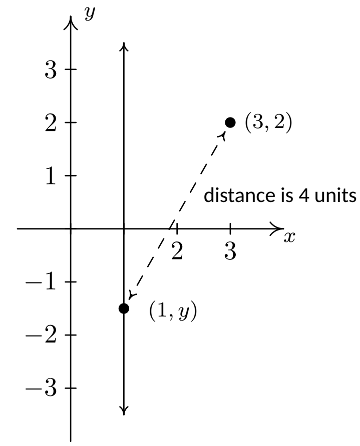


Figure 1.2.5: Diagram for Example 1.2.4

Related to finding the distance between two points is the problem of finding the **midpoint** of the line segment connecting two points. Given two points, $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$, the **midpoint** M of P and Q is defined to be the point on the line segment connecting P and Q whose distance from P is equal to its distance from Q.

Key Idea 1.2.4 The Midpoint Formula

The midpoint M of the line segment connecting $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$ is:

$$M = \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right)$$

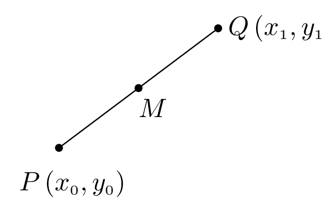


Figure 1.2.6: The midpoint of a line segment

If we let d denote the distance between P and Q, we leave it as Exercise 17 to show that the distance between P and M is $d/2$ which is the same as the distance between M and Q. This suffices to show that Key Idea 1.2.4 gives the coordinates of the midpoint.

Example 1.2.5 Finding the midpoint of a line segment

Find the midpoint of the line segment connecting $P(-2, 3)$ and $Q(1, -3)$.

Solution

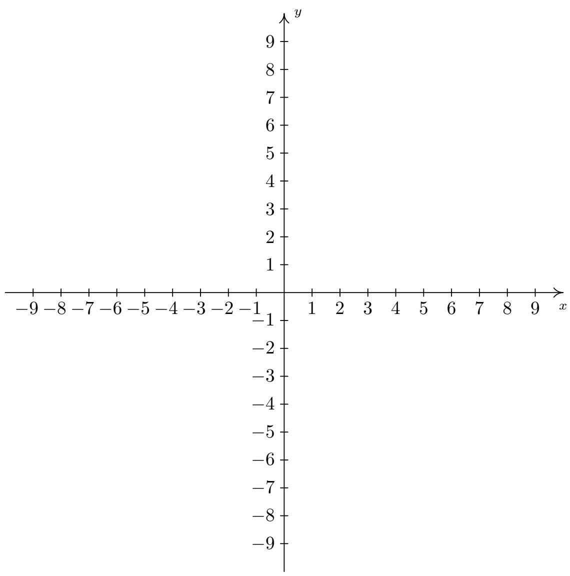
$$
\begin{aligned}
M &= \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right) \\
&= \left(\frac{(-2)+1}{2}, \frac{3+(-3)}{2}\right) = \left(-\frac{1}{2}, \frac{0}{2}\right) \\
&= \left(-\frac{1}{2}, 0\right)
\end{aligned}
$$

The midpoint is $\left(-\frac{1}{2}, 0\right)$.

Exercises 1.2

Problems

1. Plot and label the points $A(-3,-7)$, $B(1.3,-2)$, $C(\pi,\sqrt{10})$, $D(0,8)$, $E(-5.5,0)$, $F(-8,4)$, $G(9.2,-7.8)$ and $H(7,5)$ in the Cartesian Coordinate Plane given below.



2. For each point given in Exercise 1 above
   - Identify the quadrant or axis in/on which the point lies.
   - Find the point symmetric to the given point about the *x*-axis.
   - Find the point symmetric to the given point about the *y*-axis.
   - Find the point symmetric to the given point about the origin.

In Exercises 3 – 10, find the distance *d* between the points and the midpoint *M* of the line segment which connects them.

3. $(1,2)$, $(-3,5)$

4. $(3,-10)$, $(-1,2)$

5. $\left(\frac{1}{2},4\right)$, $\left(\frac{3}{2},-1\right)$

6. $\left(-\frac{2}{3},\frac{3}{2}\right)$, $\left(\frac{7}{3},2\right)$

7. $\left(\frac{24}{5},\frac{6}{5}\right)$, $\left(-\frac{11}{5},-\frac{19}{5}\right)$.

8. $\left(\sqrt{2},\sqrt{3}\right)$, $\left(-\sqrt{8},-\sqrt{12}\right)$

9. $\left(2\sqrt{45},\sqrt{12}\right)$, $\left(\sqrt{20},\sqrt{27}\right)$.

10. $(0,0)$, (x,y)

11. Find all of the points of the form $(x,-1)$ which are 4 units from the point $(3,2)$.

12. Find all of the points on the *y*-axis which are 5 units from the point $(-5,3)$.

13. Find all of the points on the *x*-axis which are 2 units from the point $(-1,1)$.

14. Find all of the points of the form $(x,-x)$ which are 1 unit from the origin.

15. Let's assume for a moment that we are standing at the origin and the positive *y*-axis points due North while the positive *x*-axis points due East. Our Sasquatch-o-meter tells us that Sasquatch is 3 miles West and 4 miles South of our current position. What are the coordinates of his position? How far away is he from us? If he runs 7 miles due East what would his new position be?

16. Verify the Distance Formula 1.2.3 for the cases when:
    (a) The points are arranged vertically. (Hint: Use $P(a,y_0)$ and $Q(a,y_1)$.)
    (b) The points are arranged horizontally. (Hint: Use $P(x_0,b)$ and $Q(x_1,b)$.)
    (c) The points are actually the same point. (You shouldn't need a hint for this one.)

17. Verify the Midpoint Formula by showing the distance between $P(x_1,y_1)$ and M and the distance between M and $Q(x_2,y_2)$ are both half of the distance between P and Q.

18. Show that the points A, B and C below are the vertices of a right triangle.
    (a) $A(-3,2)$, $B(-6,4)$, and $C(1,8)$
    (b) $A(-3,1)$, $B(4,0)$ and $C(0,-3)$

19. Find a point $D(x,y)$ such that the points $A(-3,1)$, $B(4,0)$, $C(0,-3)$ and D are the corners of a square. Justify your answer.

20. Discuss with your classmates how many numbers are in the interval $(0,1)$.

21. The world is not flat. (There are those who disagree with this statement. Look them up on the Internet some time when you're bored.) Thus the Cartesian Plane cannot possibly be the end of the story. Discuss with your classmates how you would extend Cartesian Coordinates to represent the three dimensional world. What would the Distance and Midpoint formulas look like, assuming those concepts make sense at all?

2: FUNCTIONS

2.1 Function Notation

Definition 2.1.1 Function

A **function** f from a set A to a set B is a rule that assigns each element $x \in A$ to a *unique* element $y \in B$. We express the fact that the function f relates the element x to the element y by writing $y = f(x)$.
The set A is called the **domain** of the function, and the set B is called the **codomain** of the function.

Informally, we view a function as a **process** by which each x in its domain is matched with some y in the codomain. If we think of the domain of a function as a set of **inputs** and the range as a set of **outputs**, we can think of a function f as a process by which each input x is matched with only one output y. Since the output is completely determined by the input x and the process f, we symbolize the output with **function notation**: '$f(x)$', read 'f **of** x.' In other words, $f(x)$ is the output which results by applying the process f to the input x. In this case, the parentheses here do not indicate multiplication, as they do elsewhere in Algebra. This can cause confusion if the context is not clear, so you must read carefully. This relationship is typically visualized using a diagram similar to the one in Figure 2.1.1.

The value of y is completely dependent on the choice of x. For this reason, x is often called the **independent variable**, or **argument** of f, whereas y is often called the **dependent variable**.

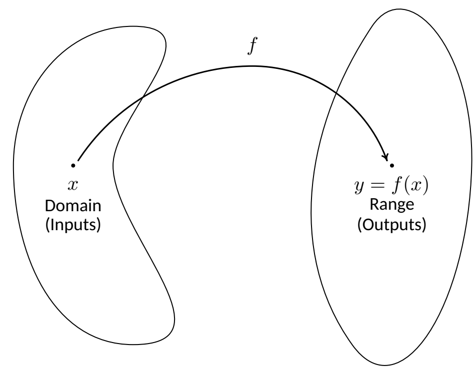


Figure 2.1.1: Graphical depiction of a function

It is common in many areas of mathematics to use the notation $f : A \to B$ to denote a function f with domain A and codomain B. However, this notation is less common in Calculus, where the domain and codomain are almost always subsets of $\mathbb{R}$. It is more common in calculus to specify a function using the formula by which each element of the domain is assigned to an element in the codomain. For example, $f(x) = x^2$ describes the function $f : \mathbb{R} \to \mathbb{R}$ that assigns each real number $x \in \mathbb{R}$ to its square.

As we shall see, the process of a function f is usually described using an algebraic formula. For example, suppose a function f takes a real number and performs the following two steps, in sequence

1. Multiply by 3
2. Add 4

If we choose 5 as our input, in Step 1 we multiply by 3 to get $(5)(3) = 15$. In Step 2, we add 4 to our result from Step 1 which yields $15 + 4 = 19$. Using function notation, we would write $f(5) = 19$ to indicate that the result of applying the process f to the input 5 gives the output 19. In general, if we use x for the input, applying Step 1 produces $3x$. Following with Step 2 produces $3x + 4$ as our final output. Hence for an input x, we get the output $f(x) = 3x + 4$. Notice that to check our formula for the case $x = 5$, we replace the occurrence of x in the formula for $f(x)$ with 5 to get $f(5) = 3(5) + 4 = 15 + 4 = 19$, as required.

Generally, we prefer to define functions of a real variable using a formula, rather than giving a verbal description, as in the following example.

Example 2.1.1 Using function notation

Let $f(x) = -x^2 + 3x + 4$

1. Find and simplify the following.

   (a) $f(-1), f(0), f(2)$

(b) $f(2x)$, $2f(x)$

(c) $f(x+2)$, $f(x)+2$, $f(x)+f(2)$

2. Solve $f(x)=4$.

Solution

1. (a) To find $f(-1)$, we replace every occurrence of x in the expression $f(x)$ with -1

$$\begin{aligned} f(-1) &= -(-1)^2+3(-1)+4 \\ &= -(1)+(-3)+4 \\ &= 0 \end{aligned}$$

Similarly, $f(0)=-(0)^2+3(0)+4=4$, and $f(2)=-(2)^2+3(2)+4=-4+6+4=6$.

(b) To find $f(2x)$, we replace every occurrence of x with the quantity $2x$

$$\begin{aligned} f(2x) &= -(2x)^2+3(2x)+4 \\ &= -(4x^2)+(6x)+4 \\ &= -4x^2+6x+4 \end{aligned}$$

The expression $2f(x)$ means we multiply the expression $f(x)$ by 2

$$\begin{aligned} 2f(x) &= 2\left(-x^2+3x+4\right) \\ &= -2x^2+6x+8 \end{aligned}$$

(c) To find $f(x+2)$, we replace every occurrence of x with the quantity $x+2$

$$\begin{aligned} f(x+2) &= -(x+2)^2+3(x+2)+4 \\ &= -\left(x^2+4x+4\right)+(3x+6)+4 \\ &= -x^2-4x-4+3x+6+4 \\ &= -x^2-x+6 \end{aligned}$$

To find $f(x)+2$, we add 2 to the expression for $f(x)$

$$\begin{aligned} f(x)+2 &= \left(-x^2+3x+4\right)+2 \\ &= -x^2+3x+6 \end{aligned}$$

From our work above, we see $f(2)=6$ so that

$$\begin{aligned} f(x)+f(2) &= \left(-x^2+3x+4\right)+6 \\ &= -x^2+3x+10 \end{aligned}$$

2. Since $f(x)=-x^2+3x+4$, the equation $f(x)=4$ is equivalent to $-x^2+3x+4=4$. Solving we get $-x^2+3x=0$, or $x(-x+3)=0$. We get $x=0$ or $x=3$, and we can verify these answers by checking that $f(0)=4$ and $f(3)=4$.

A few notes about Example 2.1.1 are in order. First note the difference between the answers for $f(2x)$ and $2f(x)$. For $f(2x)$, we are multiplying the *input* by 2; for $2f(x)$, we are multiplying the *output* by 2. As we see, we get entirely different results. Along these lines, note that $f(x+2)$, $f(x)+2$ and $f(x)+f(2)$ are three *different* expressions as well. Even though function notation uses parentheses, as does multiplication, there is *no* general 'distributive property' of function notation. Finally, note the practice of using parentheses when substituting one algebraic expression into another; we highly recommend this practice as it will reduce careless errors.

Suppose now we wish to find $r(3)$ for $r(x) = \dfrac{2x}{x^2-9}$. Substitution gives

$$r(3) = \frac{2(3)}{(3)^2-9} = \frac{6}{0},$$

which is undefined. (Why is this, again?) The number 3 is not an allowable input to the function r; in other words, 3 is not in the domain of r. Which other real numbers are forbidden in this formula? We think back to arithmetic. The reason $r(3)$ is undefined is because substitution results in a division by 0. To determine which other numbers result in such a transgression, we set the denominator equal to 0 and solve

$$\begin{aligned} x^2 - 9 &= 0 \\ x^2 &= 9 \\ \sqrt{x^2} &= \sqrt{9} && \text{extract square roots} \\ x &= \pm 3 \end{aligned}$$

As long as we substitute numbers other than 3 and -3, the expression $r(x)$ is a real number. Hence, we write our domain in interval notation (see the Exercises for Section 1.2) as $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$. When a formula for a function is given, we assume that the function is valid for all real numbers which make arithmetic sense when substituted into the formula. This set of numbers is often called the **implied domain** (or 'implicit domain') of the function. At this stage, there are only two mathematical sins we need to avoid: division by 0 and extracting even roots of negative numbers. The following example illustrates these concepts.

Example 2.1.2 Determining an implied domain

Find the domain of the following functions.

1. $g(x) = \sqrt{4-3x}$
2. $h(x) = \sqrt[5]{4-3x}$
3. $f(x) = \dfrac{2}{1 - \dfrac{4x}{x-3}}$

SOLUTION

1. The potential disaster for g is if the radicand is negative. To avoid this, we set $4 - 3x \geq 0$. From this, we get $3x \leq 4$ or $x \leq \frac{4}{3}$. What this shows is that as long as $x \leq \frac{4}{3}$, the expression $4 - 3x \geq 0$, and the formula $g(x)$ returns a real number. Our domain is $\left(-\infty, \frac{4}{3}\right]$.

The 'radicand' is the expression 'inside' the radical.

2. The formula for $h(x)$ is hauntingly close to that of $g(x)$ with one key difference $-$ whereas the expression for $g(x)$ includes an even indexed root (namely a square root), the formula for $h(x)$ involves an odd indexed root (the fifth root). Since odd roots of real numbers (even negative real numbers) are real numbers, there is no restriction on the inputs to h. Hence, the domain is $(-\infty, \infty)$.

3. In the expression for f, there are two denominators. We need to make sure neither of them is 0. To that end, we set each denominator equal to 0 and solve. For the 'small' denominator, we get $x - 3 = 0$ or $x = 3$. For the 'large' denominator

$$\begin{aligned} 1 - \frac{4x}{x-3} &= 0 \\ 1 &= \frac{4x}{x-3} \\ (1)(x-3) &= \left(\frac{4x}{\cancel{x-3}}\right)\cancel{(x-3)} && \text{clear denominators} \\ x - 3 &= 4x \\ -3 &= 3x \\ -1 &= x \end{aligned}$$

So we get two real numbers which make denominators 0, namely $x = -1$ and $x = 3$. Our domain is all real numbers except -1 and 3:

$$(-\infty, -1) \cup (-1, 3) \cup (3, \infty).$$

It is worth reiterating the importance of finding the domain of a function *before* simplifying, as evidenced by the function l in the previous example. Even though the formula $l(x)$ simplifies to $3x$, it would be inaccurate to write $l(x) = 3x$ without adding the stipulation that $x \neq 0$. It would be analogous to not reporting taxable income or some other sin of omission.

Exercises 2.1

Problems

In Exercises 1 – 8, use the given function f to find and simplify the following:

- $f(3)$
- $f(-1)$
- $f\left(\frac{3}{2}\right)$
- $f(4x)$
- $4f(x)$
- $f(-x)$
- $f(x-4)$
- $f(x)-4$
- $f\left(x^2\right)$

1. $f(x) = 2x + 1$

2. $f(x) = 3 - 4x$

3. $f(x) = 2 - x^2$

4. $f(x) = x^2 - 3x + 2$

5. $f(x) = \dfrac{x}{x-1}$

6. $f(x) = \dfrac{2}{x^3}$

7. $f(x) = 6$

8. $f(x) = 0$

In Exercises 9 – 16, use the given function f to find and simplify the following:

- $f(2)$
- $f(-2)$
- $f(2a)$
- $2f(a)$
- $f(a+2)$
- $f(a) + f(2)$
- $f\left(\frac{2}{a}\right)$
- $\frac{f(a)}{2}$
- $f(a+h)$

9. $f(x) = 2x - 5$

10. $f(x) = 5 - 2x$

11. $f(x) = 2x^2 - 1$

12. $f(x) = 3x^2 + 3x - 2$

13. $f(x) = \sqrt{2x+1}$

14. $f(x) = 117$

15. $f(x) = \dfrac{x}{2}$

16. $f(x) = \dfrac{2}{x}$

In Exercises 17 – 24, use the given function f to find $f(0)$ and solve $f(x) = 0$.

17. $f(x) = 2x - 1$

18. $f(x) = 3 - \frac{2}{5}x$

19. $f(x) = 2x^2 - 6$

20. $f(x) = x^2 - x - 12$

21. $f(x) = \sqrt{x+4}$

22. $f(x) = \sqrt{1-2x}$

23. $f(x) = \dfrac{3}{4-x}$

24. $f(x) = \dfrac{3x^2 - 12x}{4 - x^2}$

25. Let $f(x) = \begin{cases} x+5 & \text{if} \quad x \le -3 \\ \sqrt{9-x^2} & \text{if} \quad -3 < x \le 3 \\ -x+5 & \text{if} \quad x > 3 \end{cases}$ Compute the following function values.

   (a) $f(-4)$
   (b) $f(-3)$
   (c) $f(3)$
   (d) $f(3.001)$
   (e) $f(-3.001)$
   (f) $f(2)$

26. Let $f(x) = \begin{cases} x^2 & \text{if} \quad x \le -1 \\ \sqrt{1-x^2} & \text{if} \quad -1 < x \le 1 \\ x & \text{if} \quad x > 1 \end{cases}$ Compute the following function values.

   (a) $f(4)$
   (b) $f(-3)$
   (c) $f(1)$
   (d) $f(0)$
   (e) $f(-1)$
   (f) $f(-0.999)$

In Exercises 27 – 52, find the (implied) domain of the function.

27. $f(x) = x^4 - 13x^3 + 56x^2 - 19$

28. $f(x) = x^2 + 4$

29. $f(x) = \dfrac{x-2}{x+1}$

30. $f(x) = \dfrac{3x}{x^2 + x - 2}$

31. $f(x) = \dfrac{2x}{x^2 + 3}$

32. $f(x) = \dfrac{2x}{x^2 - 3}$

33. $f(x) = \dfrac{x+4}{x^2 - 36}$

34. $f(x) = \dfrac{x-2}{x-2}$

35. $f(x) = \sqrt{3-x}$

36. $f(x) = \sqrt{2x+5}$

37. $f(x) = 9x\sqrt{x+3}$

38. $f(x) = \dfrac{\sqrt{7-x}}{x^2+1}$

39. $f(x) = \sqrt{6x-2}$

40. $f(x) = \dfrac{6}{\sqrt{6x-2}}$

41. $f(x) = \sqrt[3]{6x-2}$

42. $f(x) = \dfrac{6}{4-\sqrt{6x-2}}$

43. $f(x) = \dfrac{\sqrt{6x-2}}{x^2-36}$

44. $f(x) = \dfrac{\sqrt[3]{6x-2}}{x^2+36}$

45. $s(t) = \dfrac{t}{t-8}$

46. $Q(r) = \dfrac{\sqrt{r}}{r-8}$

47. $b(\theta) = \dfrac{\theta}{\sqrt{\theta-8}}$

48. $A(x) = \sqrt{x-7} + \sqrt{9-x}$

49. $\alpha(y) = \sqrt[3]{\dfrac{y}{y-8}}$

50. $g(v) = \dfrac{1}{4-\dfrac{1}{v^2}}$

51. $T(t) = \dfrac{\sqrt{t}-8}{5-t}$

52. $u(w) = \dfrac{w-8}{5-\sqrt{w}}$

2.2 Operations on Functions

2.2.1 Arithmetic with Functions

In the previous section we used the newly defined function notation to make sense of expressions such as '$f(x)+2$' and '$2f(x)$' for a given function f. It would seem natural, then, that functions should have their own arithmetic which is consistent with the arithmetic of real numbers. The following definitions allow us to add, subtract, multiply and divide functions using the arithmetic we already know for real numbers.

Definition 2.2.1 Function Arithmetic

Suppose f and g are functions and x is in both the domain of f and the domain of g.

- The **sum** of f and g, denoted $f+g$, is the function defined by the formula
$$(f+g)(x) = f(x)+g(x)$$
- The **difference** of f and g, denoted $f-g$, is the function defined by the formula
$$(f-g)(x) = f(x)-g(x)$$
- The **product** of f and g, denoted fg, is the function defined by the formula
$$(fg)(x) = f(x)g(x)$$
- The **quotient** of f and g, denoted $\dfrac{f}{g}$, is the function defined by the formula
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$
provided $g(x) \neq 0$.

Recall that if x is in the domains of both f and g, then we can say that x is an element of the intersection of the two domains.

In other words, to add two functions, we add their outputs; to subtract two functions, we subtract their outputs, and so on. Note that while the formula $(f+g)(x) = f(x)+g(x)$ looks suspiciously like some kind of distributive property, it is nothing of the sort; the addition on the left hand side of the equation is *function* addition, and we are using this equation to *define* the output of the new function $f+g$ as the sum of the real number outputs from f and g.

Example 2.2.1 Arithmetic with functions

Let $f(x) = 6x^2 - 2x$ and $g(x) = 3 - \dfrac{1}{x}$.

1. Find $(f+g)(-1)$
2. Find $(fg)(2)$
3. Find the domain of $g-f$ then find and simplify a formula for $(g-f)(x)$.
4. Find the domain of $\left(\dfrac{g}{f}\right)$ then find and simplify a formula for $\left(\dfrac{g}{f}\right)(x)$.

SOLUTION

1. To find $(f+g)(-1)$ we first find $f(-1) = 8$ and $g(-1) = 4$. By definition, we have that $(f+g)(-1) = f(-1) + g(-1) = 8 + 4 = 12$.

2. To find $(fg)(2)$, we first need $f(2)$ and $g(2)$. Since $f(2) = 20$ and $g(2) = \frac{5}{2}$, our formula yields $(fg)(2) = f(2)g(2) = (20)\left(\frac{5}{2}\right) = 50$.

3. One method to find the domain of $g - f$ is to find the domain of g and of f separately, then find the intersection of these two sets. Owing to the denominator in the expression $g(x) = 3 - \frac{1}{x}$, we get that the domain of g is $(-\infty, 0) \cup (0, \infty)$. Since $f(x) = 6x^2 - 2x$ is valid for all real numbers, we have no further restrictions. Thus the domain of $g - f$ matches the domain of g, namely, $(-\infty, 0) \cup (0, \infty)$.

   A second method is to analyze the formula for $(g-f)(x)$ *before simplifying* and look for the usual domain issues. In this case,

$$(g - f)(x) = g(x) - f(x) = \left(3 - \frac{1}{x}\right) - \left(6x^2 - 2x\right),$$

   so we find, as before, the domain is $(-\infty, 0) \cup (0, \infty)$.

   Moving along, we need to simplify a formula for $(g - f)(x)$. One issue here is that what it means to 'simplify' this function may depend on the context. On a most basic level, we could simply clear the parentheses:

$$(g - f)(x) = \left(3 - \frac{1}{x}\right) - \left(6x^2 - 2x\right) = 3 - \frac{1}{x} - 6x^2 + 2x.$$

   In many contexts (computing a derivative comes to mind), this would be the preferred result. In other contexts, we may instead want to express our result as a single fraction. Getting a common denominator, we would write

$$(g - f)(x) = \frac{3x}{x} - \frac{1}{x} - \frac{6x^3}{x} + \frac{2x^2}{x} = \frac{-6x^3 - 2x^2 + 3x - 1}{x}.$$

4. As in the previous example, we have two ways to approach finding the domain of $\frac{g}{f}$. First, we can find the domain of g and f separately, and find the intersection of these two sets. In addition, since $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$, we are introducing a new denominator, namely $f(x)$, so we need to guard against this being 0 as well. Our previous work tells us that the domain of g is $(-\infty, 0) \cup (0, \infty)$ and the domain of f is $(-\infty, \infty)$. Setting $f(x) = 0$ gives $6x^2 - 2x = 0$ or $x = 0, \frac{1}{3}$. As a result, the domain of $\frac{g}{f}$ is all real numbers except $x = 0$ and $x = \frac{1}{3}$, or $(-\infty, 0) \cup \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$.

   Alternatively, we may proceed as above and analyze the expression $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$ *before* simplifying. In this case,

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{3 - \frac{1}{x}}{6x^2 - 2x}$$

   We see immediately from the 'little' denominator that $x \neq 0$. To keep the 'big' denominator away from 0, we solve $6x^2 - 2x = 0$ and get $x = 0$ or

$x = \frac{1}{3}$. Hence, as before, we find the domain of $\dfrac{g}{f}$ to be

$$(-\infty, 0) \cup \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right).$$

Next, we find and simplify a formula for $\left(\dfrac{g}{f}\right)(x)$.

$$\begin{aligned}
\left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)} = \frac{3 - \frac{1}{x}}{6x^2 - 2x} \\
&= \frac{3 - \frac{1}{x}}{6x^2 - 2x} \cdot \frac{x}{x} && \text{simplify compound fractions} \\
&= \frac{\left(3 - \frac{1}{x}\right) x}{(6x^2 - 2x)\, x} = \frac{3x - 1}{(6x^2 - 2x)\, x} \\
&= \frac{3x - 1}{2x^2(3x - 1)} && \text{factor} \\
&= \frac{\cancel{(3x - 1)}^{\,1}}{2x^2\cancel{(3x - 1)}} && \text{cancel} \\
&= \frac{1}{2x^2}
\end{aligned}$$

Please note the importance of finding the domain of a function *before* simplifying its expression. In number 4 in Example 2.2.1 above, had we waited to find the domain of $\dfrac{g}{f}$ until after simplifying, we'd just have the formula $\dfrac{1}{2x^2}$ to go by, and we would (incorrectly!) state the domain as $(-\infty, 0) \cup (0, \infty)$, since the other troublesome number, $x = \frac{1}{3}$, was cancelled away.

2.2.2 Function Composition

The four types of arithmetic operations with functions described so far are not the only ways to combine functions. There is one more especially important operation, known as function composition.

Definition 2.2.2 Composition of Functions

Suppose f and g are two functions. The **composite** of g with f, denoted $g \circ f$, is defined by the formula $(g \circ f)(x) = g(f(x))$, provided x is an element of the domain of f and $f(x)$ is an element of the domain of g.

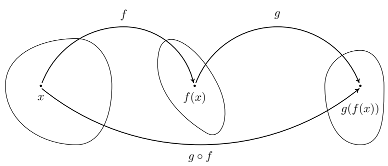


Figure 2.2.1: Composition of functions

The quantity $g \circ f$ is also read 'g composed with f' or, more simply 'g of f.' At its most basic level, Definition 2.2.2 tells us to obtain the formula for $(g \circ f)(x)$, we replace every occurrence of x in the formula for $g(x)$ with the formula we have for $f(x)$. If we take a step back and look at this from a procedural, 'inputs and outputs' perspective, Defintion 2.2.2 tells us the output from $g \circ f$ is found by taking the output from f, $f(x)$, and then making that the input to g. The result, $g(f(x))$, is the output from $g \circ f$. From this perspective, we see $g \circ f$ as a two step process taking an input x and first applying the procedure f then applying the procedure g. This is diagrammed abstractly in Figure 2.2.1.

Example 2.2.2 Evaluating composite functions

Let $f(x) = x^2 - 4x$ and $g(x) = 2 - \sqrt{x+3}$.

Find the indicated function value for each of the following:

1. $(f \circ g)(1)$
2. $(g \circ f)(1)$
3. $(g \circ f)(2)$

Solution

1. As before, we use Definition 2.2.2 to write $(f \circ g)(1) = f(g(1))$. We find $g(1) = 0$, so
$$(f \circ g)(1) = f(g(1)) = f(0) = 0$$
2. Using Definition 2.2.2, $(g \circ f)(1) = g(f(1))$. We find $f(1) = -3$, so
$$(g \circ f)(1) = g(f(1)) = g(-3) = 2$$
3. We proceed as in the previous example by first finding $f(2) = -4$. However, we now run into trouble, since $(g \circ f)(2) = g(f(2)) = g(-4)$ is undefined! We can't compute $\sqrt{(-4+3)} = \sqrt{-1}$ if we are working over the real numbers. Here we see the importance of domain for composite functions: it is not enough for x to be in the domain of f: only those x values such that $f(x)$ belongs to the domain of g are permitted. We consider this problem more generally in the next example.

Example 2.2.3 Domain of composite functions

With $f(x) = x^2 - 4x$, $g(x) = 2 - \sqrt{x+3}$ as in Example 2.2.2 find and simplify the composite functions $(g \circ f)(x)$ and $(f \circ g)(x)$. State the domain of each function.

SOLUTION By definition, $(g \circ f)(x) = g(f(x))$. We insert the expression $f(x)$ into g to get

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) = g\left(x^2 - 4x\right) = 2 - \sqrt{(x^2 - 4x) + 3} \\ &= 2 - \sqrt{x^2 - 4x + 3}\end{aligned}$$

Hence, $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$.

To find the domain of $g \circ f$, we need to find the elements in the domain of f whose outputs $f(x)$ are in the domain of g. We accomplish this by following the rule set forth in Section 2.1, that is, we find the domain *before* we simplify. To that end, we examine $(g \circ f)(x) = 2 - \sqrt{(x^2 - 4x) + 3}$. To keep the square root happy, we solve the inequality $x^2 - 4x + 3 \geq 0$ by creating a sign diagram. If we let $r(x) = x^2 - 4x + 3$, we find the zeros of r to be $x = 1$ and $x = 3$. We obtain the sign diagram in Figure 2.2.2.

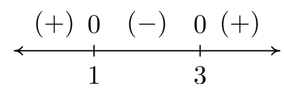


Figure 2.2.2: The sign diagram of $r(x) = x^2 - 4x + 3$

Our solution to $x^2 - 4x + 3 \geq 0$, and hence the domain of $g \circ f$, is $(-\infty, 1] \cup [3, \infty)$.

To find $(f \circ g)(x)$, we find $f(g(x))$. We insert the expression $g(x)$ into f to get

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) = f\left(2 - \sqrt{x+3}\right) \\ &= \left(2 - \sqrt{x+3}\right)^2 - 4\left(2 - \sqrt{x+3}\right) \\ &= 4 - 4\sqrt{x+3} + \left(\sqrt{x+3}\right)^2 - 8 + 4\sqrt{x+3} \\ &= 4 + x + 3 - 8 \\ &= x - 1\end{aligned}$$

Thus we get $(f \circ g)(x) = x - 1$. To find the domain of $(f \circ g)$, we look to the step before we did any simplification and find $(f \circ g)(x) = \left(2 - \sqrt{x+3}\right)^2 - 4\left(2 - \sqrt{x+3}\right)$. To keep the square root happy, we set $x + 3 \geq 0$ and find our domain to be $[-3, \infty)$.

Notice that in Example 2.2.3, we found $(g \circ f)(x) \neq (f \circ g)(x)$. In Example 2.2.4 we add evidence that this is the rule, rather than the exception.

Example 2.2.4 Comparing order of composition

Find and simplify the functions $(g \circ h)(x)$ and $(h \circ g)(x)$, where we take $g(x) = 2 - \sqrt{x+3}$ and $h(x) = \dfrac{2x}{x+1}$. State the domain of each function.

SOLUTION To find $(g \circ h)(x)$, we compute $g(h(x))$. We insert the expression $h(x)$ into g first to get

$$\begin{aligned}(g\circ h)(x) = g(h(x)) &= g\left(\frac{2x}{x+1}\right)\\ &= 2-\sqrt{\left(\frac{2x}{x+1}\right)+3}\\ &= 2-\sqrt{\frac{2x}{x+1}+\frac{3(x+1)}{x+1}} \qquad \text{get common denominators}\\ &= 2-\sqrt{\frac{5x+3}{x+1}}\end{aligned}$$

To find the domain of $(g\circ h)$, we look to the step before we began to simplify:

$$(g\circ h)(x) = 2-\sqrt{\left(\frac{2x}{x+1}\right)+3}$$

To avoid division by zero, we need $x \neq -1$. To keep the radical happy, we need to solve

$$\frac{2x}{x+1}+3 = \frac{5x+3}{x+1} \geq 0$$

Defining $r(x) = \dfrac{5x+3}{x+1}$, we see r is undefined at $x=-1$ and $r(x)=0$ at $x=-\frac{3}{5}$. Our sign diagram is given in Figure 2.2.3.

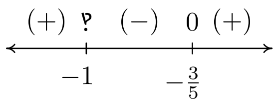


Figure 2.2.3: The sign diagram of $r(x) = \dfrac{5x+3}{x+1}$

Our domain is $(-\infty,-1)\cup\left[-\frac{3}{5},\infty\right)$.

Next, we find $(h\circ g)(x)$ by finding $h(g(x))$. We insert the expression $g(x)$ into h first to get

$$\begin{aligned}(h\circ g)(x) = h(g(x)) &= h\left(2-\sqrt{x+3}\right)\\ &= \frac{2\left(2-\sqrt{x+3}\right)}{\left(2-\sqrt{x+3}\right)+1}\\ &= \frac{4-2\sqrt{x+3}}{3-\sqrt{x+3}}\end{aligned}$$

To find the domain of $h\circ g$, we look to the step before any simplification:

$$(h\circ g)(x) = \frac{2\left(2-\sqrt{x+3}\right)}{\left(2-\sqrt{x+3}\right)+1}$$

To keep the square root happy, we require $x+3\geq 0$ or $x\geq -3$. Setting the denominator equal to zero gives $\left(2-\sqrt{x+3}\right)+1=0$ or $\sqrt{x+3}=3$. Squaring both sides gives us $x+3=9$, or $x=6$. Since $x=6$ checks in the original equation, $\left(2-\sqrt{x+3}\right)+1=0$, we know $x=6$ is the only zero of the denominator. Hence, the domain of $h\circ g$ is $[-3,6)\cup(6,\infty)$.

A useful skill in Calculus is to be able to take a complicated function and break it down into a composition of easier functions which our last example illustrates.

Example 2.2.5 Decomposing functions

Write each of the following functions as a composition of two or more (non-identity) functions. Check your answer by performing the function composition.

1. $F(x) = |3x - 1|$
2. $G(x) = \dfrac{2}{x^2 + 1}$
3. $H(x) = \dfrac{\sqrt{x} + 1}{\sqrt{x} - 1}$

Solution There are many approaches to this kind of problem, and we showcase a different methodology in each of the solutions below.

1. Our goal is to express the function F as $F = g \circ f$ for functions g and f. From Definition 2.2.2, we know $F(x) = g(f(x))$, and we can think of $f(x)$ as being the 'inside' function and g as being the 'outside' function. Looking at $F(x) = |3x - 1|$ from an 'inside versus outside' perspective, we can think of $3x - 1$ being inside the absolute value symbols. Taking this cue, we define $f(x) = 3x - 1$. At this point, we have $F(x) = |f(x)|$. What is the outside function? The function which takes the absolute value of its input, $g(x) = |x|$. Sure enough, $(g \circ f)(x) = g(f(x)) = |f(x)| = |3x - 1| = F(x)$, so we are done.

2. We attack deconstructing G from an operational approach. Given an input x, the first step is to square x, then add 1, then divide the result into 2. We will assign each of these steps a function so as to write G as a composite of three functions: f, g and h. Our first function, f, is the function that squares its input, $f(x) = x^2$. The next function is the function that adds 1 to its input, $g(x) = x + 1$. Our last function takes its input and divides it into 2, $h(x) = \frac{2}{x}$. The claim is that $G = h \circ g \circ f$. We find

$$(h \circ g \circ f)(x) = h(g(f(x))) = h(g\left(x^2\right)) = h\left(x^2 + 1\right) = \frac{2}{x^2 + 1} = G(x),$$

so we are done.

3. If we look $H(x) = \dfrac{\sqrt{x} + 1}{\sqrt{x} - 1}$ with an eye towards building a complicated function from simpler functions, we see the expression $\sqrt{x}$ is a simple piece of the larger function. If we define $f(x) = \sqrt{x}$, we have $H(x) = \frac{f(x)+1}{f(x)-1}$. If we want to decompose $H = g \circ f$, then we can glean the formula for $g(x)$ by looking at what is being done to $f(x)$. We take $g(x) = \frac{x+1}{x-1}$, so

$$(g \circ f)(x) = g(f(x)) = \frac{f(x) + 1}{f(x) - 1} = \frac{\sqrt{x} + 1}{\sqrt{x} - 1} = H(x),$$

as required.

2.2.3 Inverse Functions

Thinking of a function as a process like we did in Section 2.1, in this section we seek another function which might reverse that process. As in real life, we will find that some processes (like putting on socks and shoes) are reversible while some (like cooking a steak) are not. We start by discussing a very basic function which is reversible, $f(x) = 3x + 4$. Thinking of f as a process, we start with an input x and apply two steps, as we saw in Section 2.1

1. multiply by 3
2. add 4

To reverse this process, we seek a function g which will undo each of these steps and take the output from f, $3x + 4$, and return the input x. If we think of the real-world reversible two-step process of first putting on socks then putting on shoes, to reverse the process, we first take off the shoes, and then we take off the socks. In much the same way, the function g should undo the second step of f first. That is, the function g should

1. *subtract* 4
2. *divide* by 3

Following this procedure, we get $g(x) = \dfrac{x-4}{3}$. Let's check to see if the function g does the job. If $x = 5$, then $f(5) = 3(5) + 4 = 15 + 4 = 19$. Taking the output 19 from f, we substitute it into g to get $g(19) = \frac{19-4}{3} = \frac{15}{3} = 5$, which is our original input to f. To check that g does the job for all x in the domain of f, we take the generic output from f, $f(x) = 3x + 4$, and substitute that into g. That is, $g(f(x)) = g(3x + 4) = \dfrac{(3x+4)-4}{3} = \frac{3x}{3} = x$, which is our original input to f. If we carefully examine the arithmetic as we simplify $g(f(x))$, we actually see g first 'undoing' the addition of 4, and then 'undoing' the multiplication by 3. Not only does g undo f, but f also undoes g. That is, if we take the output from g, $g(x) = \dfrac{x-4}{3}$, and put that into f, we get $f(g(x)) = f\left(\dfrac{x-4}{3}\right) = 3\left(\dfrac{x-4}{3}\right) + 4 = (x - 4) + 4 = x$. Using the language of function composition developed in Section 2.2.2, the statements $g(f(x)) = x$ and $f(g(x)) = x$ can be written as $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$, respectively. Abstractly, we can visualize the relationship between f and g in Figure 2.2.4.

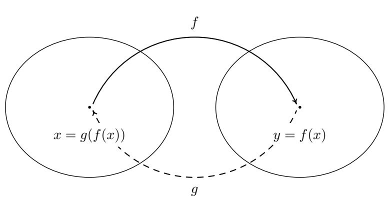


Figure 2.2.4: The relationship between a function and its inverse

The main idea to get from Figure 2.2.4 is that g takes the outputs from f and returns them to their respective inputs, and conversely, f takes outputs from g and returns them to their respective inputs. We now have enough background to state the central definition of the section.

Definition 2.2.3 Inverse of a function

Suppose f and g are two functions such that

1. $(g \circ f)(x) = x$ for all x in the domain of f **and**
2. $(f \circ g)(x) = x$ for all x in the domain of g

then f and g are **inverses** of each other and the functions f and g are said to be **invertible**.

We now formalize the concept that inverse functions exchange inputs and outputs.

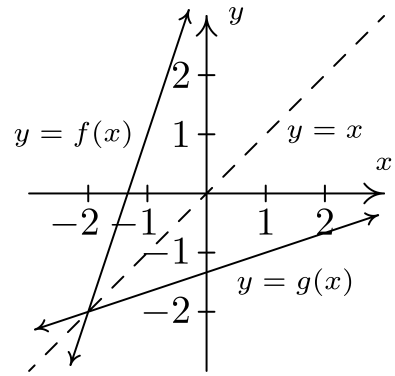


Figure 2.2.5: Reflecting $y = f(x)$ across $y = x$ to obtain $y = g(x)$

Theorem 2.2.1 Properties of Inverse Functions

Suppose f and g are inverse functions.

- The range (recall this is the set of all outputs of a function) of f is the domain of g and the domain of f is the range of g
- $f(a) = b$ if and only if $g(b) = a$
- (a, b) is on the graph of f if and only if (b, a) is on the graph of g

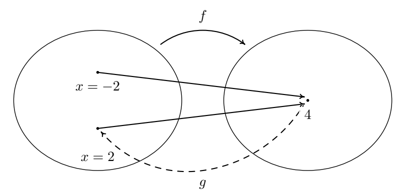


Figure 2.2.6: The function $f(x) = x^2$ is not invertible

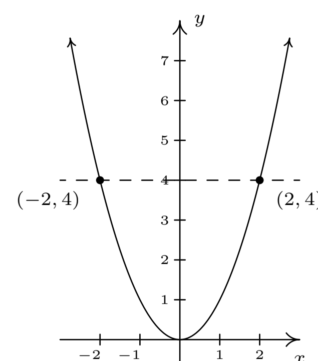
Theorem 2.2.2 Uniqueness of Inverse Functions and Their Graphs

Suppose f is an invertible function.

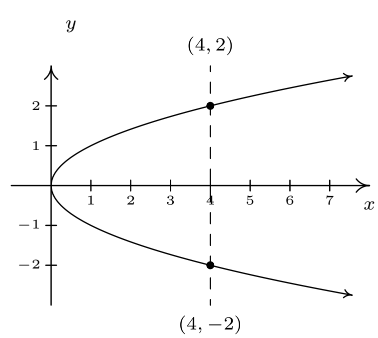
- There is exactly one inverse function for f, denoted f^{-1} (read f-inverse)
- The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across the line $y = x$.

Let's turn our attention to the function $f(x) = x^2$. Is f invertible? A likely candidate for the inverse is the function $g(x) = \sqrt{x}$. Checking the composition yields $(g \circ f)(x) = g(f(x)) = \sqrt{x^2} = |x|$, which is not equal to x for all x in the domain $(-\infty, \infty)$. For example, when $x = -2$, $f(-2) = (-2)^2 = 4$, but $g(4) = \sqrt{4} = 2$, which means g failed to return the input -2 from its output 4. What g did, however, is match the output 4 to a *different* input, namely 2, which satisfies $f(2) = 4$. This issue is presented schematically in Figure 2.2.6.

We see from the diagram that since both $f(-2)$ and $f(2)$ are 4, it is impossible to construct a *function* which takes 4 back to *both* $x = 2$ and $x = -2$. (By definition, a function matches a real number with exactly one other real number.) From a graphical standpoint, we know that if $y = f^{-1}(x)$ exists, its graph can be obtained by reflecting $y = x^2$ about the line $y = x$, in accordance with Theorem 2.2.2. Doing so takes the graph in Figure 2.2.7 (a) to the one in Figure 2.2.7 (b).



(a) $y = f(x) = x^2$



(b) $y = f^{-1}(x)$?

Figure 2.2.7: Reflecting $y = x^2$ across the line $y = x$ does not produce a function

We see that the line $x = 4$ intersects the graph of the supposed inverse twice - meaning the graph fails the Vertical Line Test, and as such, does not represent y as a function of x. The vertical line $x = 4$ on the graph on the right corresponds to the *horizontal line* $y = 4$ on the graph of $y = f(x)$. The fact that the horizontal line $y = 4$ intersects the graph of f twice means two *different* inputs, namely $x = -2$ and $x = 2$, are matched with the *same* output, 4, which is the cause of all of the trouble. In general, for a function to have an inverse, *different* inputs must go to *different* outputs, or else we will run into the same problem we did with $f(x) = x^2$. We give this property a name.

Definition 2.2.4 One-to-one function

A function f is said to be **one-to-one** if f matches different inputs to different outputs. Equivalently, f is one-to-one if and only if whenever $f(c) = f(d)$, then $c = d$.

Graphically, we detect one-to-one functions using the test below.

Theorem 2.2.3 The Horizontal Line Test

A function f is one-to-one if and only if no horizontal line intersects the graph of f more than once.

We say that the graph of a function **passes** the Horizontal Line Test if no horizontal line intersects the graph more than once; otherwise, we say the graph of the function **fails** the Horizontal Line Test. We have argued that if f is invertible, then f must be one-to-one, otherwise the graph given by reflecting the graph of $y = f(x)$ about the line $y = x$ will fail the Vertical Line Test. It turns out that being one-to-one is also enough to guarantee invertibility. To see this, we think of f as the set of ordered pairs which constitute its graph. If switching the x- and y-coordinates of the points results in a function, then f is invertible and we have found f^{-1}. This is precisely what the Horizontal Line Test does for us: it checks to see whether or not a set of points describes x as a function of y. We summarize these results below.

Theorem 2.2.4 Equivalent Conditions for Invertibility

Suppose f is a function. The following statements are equivalent.

- f is invertible
- f is one-to-one
- The graph of f passes the Horizontal Line Test

We put this result to work in the next example.

Example 2.2.6 Finding one-to-one functions

Determine if the following functions are one-to-one in two ways: (a) analytically using Definition 2.2.4 and (b) graphically using the Horizontal Line Test.

1. $f(x) = \dfrac{1-2x}{5}$
2. $g(x) = \dfrac{2x}{1-x}$
3. $h(x) = x^2 - 2x + 4$

Solution

1. (a) To determine if f is one-to-one analytically, we assume $f(c) = f(d)$ and attempt to deduce that $c = d$.

$$\begin{aligned} f(c) &= f(d) \\ \frac{1-2c}{5} &= \frac{1-2d}{5} \\ 1-2c &= 1-2d \\ -2c &= -2d \\ c &= d \checkmark \end{aligned}$$

Hence, f is one-to-one.

(b) To check if f is one-to-one graphically, we look to see if the graph of $y = f(x)$ passes the Horizontal Line Test. We have that f is a non-constant linear function, which means its graph is a non-horizontal line. Thus the graph of f passes the Horizontal Line Test: see Figure 2.2.8.

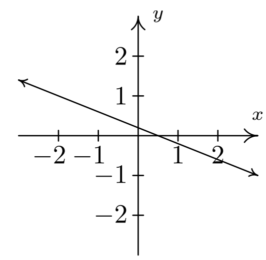


Figure 2.2.8: The function f is one-to-one

2. (a) We begin with the assumption that $g(c) = g(d)$ and try to show $c = d$.

$$\begin{aligned} g(c) &= g(d) \\ \frac{2c}{1-c} &= \frac{2d}{1-d} \\ 2c(1-d) &= 2d(1-c) \\ 2c-2cd &= 2d-2dc \\ 2c &= 2d \\ c &= d \checkmark \end{aligned}$$

We have shown that g is one-to-one.

(b) The graph of g is shown in Figure 2.2.9. We get the sole intercept at $(0, 0)$, a vertical asymptote $x = 1$ and a horizontal asymptote (which the graph never crosses) $y = -2$. We see from that the graph of g in Figure 2.2.9 that g passes the Horizontal Line Test.

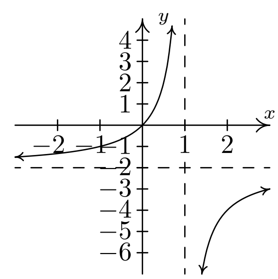


Figure 2.2.9: The function g is one-to-one

3. (a) We begin with $h(c) = h(d)$. As we work our way through the problem, we encounter a nonlinear equation. We move the non-zero terms to the left, leave a 0 on the right and factor accordingly.

$$\begin{aligned} h(c) &= h(d) \\ c^2-2c+4 &= d^2-2d+4 \\ c^2-2c &= d^2-2d \\ c^2-d^2-2c+2d &= 0 \\ (c+d)(c-d)-2(c-d) &= 0 \\ (c-d)((c+d)-2) &= 0 \qquad \text{factor by grouping} \\ c-d=0 \quad &\text{or} \quad c+d-2=0 \\ c=d \quad &\text{or} \quad c=2-d \end{aligned}$$

We get $c = d$ as one possibility, but we also get the possibility that $c = 2 - d$. This suggests that f may not be one-to-one. Taking $d = 0$, we get $c = 0$ or $c = 2$. With $h(0) = 4$ and $h(2) = 4$, we have produced two different inputs with the same output meaning h is not one-to-one.

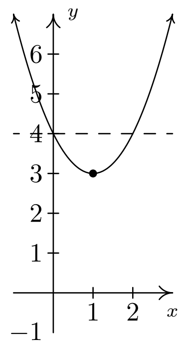


Figure 2.2.10: The function h is not one-to-one

(b) We note that h is a quadratic function and we graph $y = h(x)$ using the techniques presented in Section 3.1.3. The vertex is $(1, 3)$ and the parabola opens upwards. We see immediately from the graph in Figure 2.2.10 that h is not one-to-one, since there are several horizontal lines which cross the graph more than once.

We have shown that the functions f and g in Example 2.2.6 are one-to-one. This means they are invertible, so it is natural to wonder what $f^{-1}(x)$ and $g^{-1}(x)$ would be. For $f(x) = \frac{1-2x}{5}$, we can think our way through the inverse since there is only one occurrence of x. We can track step-by-step what is done to x and reverse those steps as we did at the beginning of the chapter. The function $g(x) = \frac{2x}{1-x}$ is a bit trickier since x occurs in two places. When one evaluates $g(x)$ for a specific value of x, which is first, the $2x$ or the $1 - x$? We can imagine functions more complicated than these so we need to develop a general methodology to attack this problem. Theorem 2.2.1 tells us equation $y = f^{-1}(x)$ is equivalent to $f(y) = x$ and this is the basis of our algorithm.

Key Idea 2.2.1 Steps for finding the Inverse of a One-to-one Function

1. Write $y = f(x)$
2. Interchange x and y
3. Solve $x = f(y)$ for y to obtain $y = f^{-1}(x)$

Note that we could have simply written 'Solve $x = f(y)$ for y' and be done with it. The act of interchanging the x and y is there to remind us that we are finding the inverse function by switching the inputs and outputs.

Example 2.2.7 Computing inverse functions

Find the inverse of the following one-to-one functions. Check your answers analytically using function composition and graphically.

1. $f(x) = \dfrac{1-2x}{5}$
2. $g(x) = \dfrac{2x}{1-x}$

Solution

1. As we mentioned earlier, it is possible to think our way through the inverse of f by recording the steps we apply to x and the order in which we apply them and then reversing those steps in the reverse order. We encourage the reader to do this. We, on the other hand, will practice the algorithm. We write $y = f(x)$ and proceed to switch x and y

$$\begin{aligned}
y &= f(x) \\
y &= \frac{1-2x}{5} \\
x &= \frac{1-2y}{5} \qquad \text{switch } x \text{ and } y \\
5x &= 1-2y \\
5x-1 &= -2y \\
\frac{5x-1}{-2} &= y \\
y &= -\frac{5}{2}x+\frac{1}{2}
\end{aligned}$$

We have $f^{-1}(x) = -\frac{5}{2}x + \frac{1}{2}$. To check this answer analytically, we first check that $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f, which is all real numbers.

$$\begin{aligned}
\left(f^{-1} \circ f\right)(x) &= f^{-1}(f(x)) \\
&= -\frac{5}{2}f(x)+\frac{1}{2} \\
&= -\frac{5}{2}\left(\frac{1-2x}{5}\right)+\frac{1}{2} \\
&= -\frac{1}{2}(1-2x)+\frac{1}{2} \\
&= -\frac{1}{2}+x+\frac{1}{2} \\
&= x \checkmark
\end{aligned}$$

We now check that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the range of f which is also all real numbers. (Recall that the domain of f^{-1}) is the range of f.)

$$\begin{aligned}
\left(f \circ f^{-1}\right)(x) = f(f^{-1}(x)) &= \frac{1-2f^{-1}(x)}{5} \\
&= \frac{1-2\left(-\frac{5}{2}x+\frac{1}{2}\right)}{5} = \frac{1+5x-1}{5} \\
&= \frac{5x}{5} = x \checkmark
\end{aligned}$$

To check our answer graphically, we graph $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes in Figure 2.2.11. They appear to be reflections across the line $y = x$.

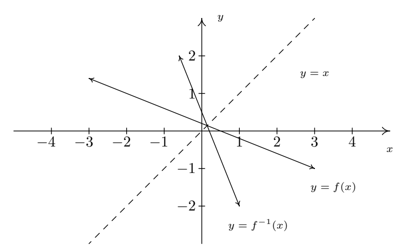


Figure 2.2.11: The graphs of f and f^{-1} from Example 2.2.7

2. To find $g^{-1}(x)$, we start with $y = g(x)$. We note that the domain of g is $(-\infty, 1) \cup (1, \infty)$.

$$\begin{aligned}
y &= g(x)\frac{2x}{1-x} \\
x &= \frac{2y}{1-y} \qquad \text{switch } x \text{ and } y \\
x(1-y) &= 2y \\
x-xy &= 2y \\
x &= xy+2y = y(x+2) \qquad \text{factor} \\
y &= \frac{x}{x+2}
\end{aligned}$$

We obtain $g^{-1}(x) = \dfrac{x}{x+2}$. To check this analytically, we first check $\left(g^{-1} \circ g\right)(x) = x$ for all x in the domain of g, that is, for all $x \neq 1$.

$$\begin{aligned}
\left(g^{-1} \circ g\right)(x) &= g^{-1}(g(x)) = g^{-1}\left(\frac{2x}{1-x}\right) \\
&= \frac{\left(\frac{2x}{1-x}\right)}{\left(\frac{2x}{1-x}\right)+2} \\
&= \frac{\left(\frac{2x}{1-x}\right)}{\left(\frac{2x}{1-x}\right)+2} \cdot \frac{(1-x)}{(1-x)} && \text{clear denominators} \\
&= \frac{2x}{2x+2(1-x)} = \frac{2x}{2x+2-2x} \\
&= \frac{2x}{2} = x \checkmark
\end{aligned}$$

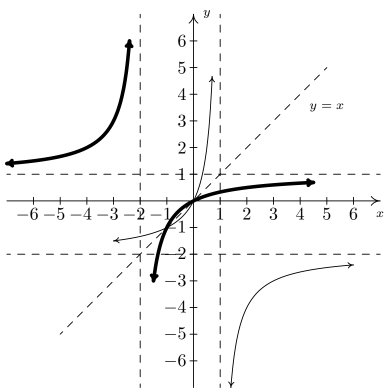


Figure 2.2.12: The graphs of g and g^{-1} from Example 2.2.7

Next, we check $g\left(g^{-1}(x)\right) = x$ for all x in the range of g. From the graph of g in Example 2.2.6, we have that the range of g is $(-\infty, -2) \cup (-2, \infty)$. This matches the domain we get from the formula $g^{-1}(x) = \frac{x}{x+2}$, as it should.

$$\begin{aligned}
\left(g \circ g^{-1}\right)(x) &= g\left(g^{-1}(x)\right) = g\left(\frac{x}{x+2}\right) \\
&= \frac{2\left(\frac{x}{x+2}\right)}{1-\left(\frac{x}{x+2}\right)} \\
&= \frac{2\left(\frac{x}{x+2}\right)}{1-\left(\frac{x}{x+2}\right)} \cdot \frac{(x+2)}{(x+2)} && \text{clear denominators} \\
&= \frac{2x}{(x+2)-x} = \frac{2x}{2} \\
&= x \checkmark
\end{aligned}$$

Graphing $y = g(x)$ and $y = g^{-1}(x)$ on the same set of axes is busy, but we can see the symmetric relationship if we thicken the curve for $y = g^{-1}(x)$. Note that the vertical asymptote $x = 1$ of the graph of g corresponds to the horizontal asymptote $y = 1$ of the graph of g^{-1}, as it should since x and y are switched. Similarly, the horizontal asymptote $y = -2$ of the graph of g corresponds to the vertical asymptote $x = -2$ of the graph of g^{-1}. See Figure 2.2.12

Exercises 2.2

Problems

In Exercises 1 – 10, use the pair of functions f and g to find the following values if they exist:

- $(f+g)(2)$
- $(f-g)(-1)$
- $(g-f)(1)$
- $(fg)\left(\frac{1}{2}\right)$
- $\left(\frac{f}{g}\right)(0)$
- $\left(\frac{g}{f}\right)(-2)$

1. $f(x) = 3x + 1$ and $g(x) = 4 - x$
2. $f(x) = x^2$ and $g(x) = -2x + 1$
3. $f(x) = x^2 - x$ and $g(x) = 12 - x^2$
4. $f(x) = 2x^3$ and $g(x) = -x^2 - 2x - 3$
5. $f(x) = \sqrt{x+3}$ and $g(x) = 2x - 1$
6. $f(x) = \sqrt{4-x}$ and $g(x) = \sqrt{x+2}$
7. $f(x) = 2x$ and $g(x) = \dfrac{1}{2x+1}$
8. $f(x) = x^2$ and $g(x) = \dfrac{3}{2x-3}$
9. $f(x) = x^2$ and $g(x) = \dfrac{1}{x^2}$
10. $f(x) = x^2 + 1$ and $g(x) = \dfrac{1}{x^2+1}$

In Exercises 11 – 20, use the pair of functions f and g to find the domain of the indicated function then find and simplify an expression for it.

- $(f+g)(x)$
- $(f-g)(x)$
- $(fg)(x)$
- $\left(\frac{f}{g}\right)(x)$

11. $f(x) = 2x + 1$ and $g(x) = x - 2$
12. $f(x) = 1 - 4x$ and $g(x) = 2x - 1$
13. $f(x) = x^2$ and $g(x) = 3x - 1$
14. $f(x) = x^2 - x$ and $g(x) = 7x$
15. $f(x) = x^2 - 4$ and $g(x) = 3x + 6$
16. $f(x) = -x^2 + x + 6$ and $g(x) = x^2 - 9$
17. $f(x) = \dfrac{x}{2}$ and $g(x) = \dfrac{2}{x}$
18. $f(x) = x - 1$ and $g(x) = \dfrac{1}{x-1}$
19. $f(x) = x$ and $g(x) = \sqrt{x+1}$
20. $f(x) = \sqrt{x-5}$ and $g(x) = f(x) = \sqrt{x-5}$

In Exercises 21 – 32, let f be the function defined by

$$f = \{(-3,4), (-2,2), (-1,0), (0,1), (1,3), (2,4), (3,-1)\}$$

and let g be the function defined

$$g = \{(-3,-2), (-2,0), (-1,-4), (0,0), (1,-3), (2,1), (3,2)\}.$$

Compute the indicated value if it exists.

21. $(f+g)(-3)$
22. $(f-g)(2)$
23. $(fg)(-1)$
24. $(g+f)(1)$
25. $(g-f)(3)$
26. $(gf)(-3)$
27. $\left(\dfrac{f}{g}\right)(-2)$
28. $\left(\dfrac{f}{g}\right)(-1)$
29. $\left(\dfrac{f}{g}\right)(2)$
30. $\left(\dfrac{g}{f}\right)(-1)$
31. $\left(\dfrac{g}{f}\right)(3)$
32. $\left(\dfrac{g}{f}\right)(-3)$

In Exercises 33 – 44, use the given pair of functions to find the following values if they exist.

- $(g \circ f)(0)$
- $(f \circ g)(-1)$
- $(f \circ f)(2)$
- $(g \circ f)(-3)$
- $(f \circ g)\left(\frac{1}{2}\right)$
- $(f \circ f)(-2)$

33. $f(x) = x^2$, $g(x) = 2x + 1$
34. $f(x) = 4 - x$, $g(x) = 1 - x^2$
35. $f(x) = 4 - 3x$, $g(x) = |x|$

36. $f(x) = |x-1|$, $g(x) = x^2 - 5$

37. $f(x) = 4x+5$, $g(x) = \sqrt{x}$

38. $f(x) = \sqrt{3-x}$, $g(x) = x^2+1$

39. $f(x) = 6 - x - x^2$, $g(x) = x\sqrt{x+10}$

40. $f(x) = \sqrt[3]{x+1}$, $g(x) = 4x^2 - x$

41. $f(x) = \dfrac{3}{1-x}$, $g(x) = \dfrac{4x}{x^2+1}$

42. $f(x) = \dfrac{x}{x+5}$, $g(x) = \dfrac{2}{7-x^2}$

43. $f(x) = \dfrac{2x}{5-x^2}$, $g(x) = \sqrt{4x+1}$

44. $f(x) = \sqrt{2x+5}$, $g(x) = \dfrac{10x}{x^2+1}$

In Exercises 45 – 56, use the given pair of functions to find and simplify expressions for the following functions and state the domain of each using interval notation.

- $(g \circ f)(x)$
- $(f \circ g)(x)$
- $(f \circ f)(x)$

45. $f(x) = 2x+3$, $g(x) = x^2 - 9$

46. $f(x) = x^2 - x + 1$, $g(x) = 3x - 5$

47. $f(x) = x^2 - 4$, $g(x) = |x|$

48. $f(x) = 3x - 5$, $g(x) = \sqrt{x}$

49. $f(x) = |x+1|$, $g(x) = \sqrt{x}$

50. $f(x) = 3 - x^2$, $g(x) = \sqrt{x+1}$

51. $f(x) = |x|$, $g(x) = \sqrt{4-x}$

52. $f(x) = x^2 - x - 1$, $g(x) = \sqrt{x-5}$

53. $f(x) = 3x - 1$, $g(x) = \dfrac{1}{x+3}$

54. $f(x) = \dfrac{3x}{x-1}$, $g(x) = \dfrac{x}{x-3}$

55. $f(x) = \dfrac{x}{2x+1}$, $g(x) = \dfrac{2x+1}{x}$

56. $f(x) = \dfrac{2x}{x^2-4}$, $g(x) = \sqrt{1-x}$

In Exercises 57 – 62, use $f(x) = -2x$, $g(x) = \sqrt{x}$ and $h(x) = |x|$ to find and simplify expressions for the following functions and state the domain of each using interval notation.

57. $(h \circ g \circ f)(x)$

58. $(h \circ f \circ g)(x)$

59. $(g \circ f \circ h)(x)$

60. $(g \circ h \circ f)(x)$

61. $(f \circ h \circ g)(x)$

62. $(f \circ g \circ h)(x)$

In Exercises 63 – 72, write the given function as a composition of two or more non-identity functions. (There are several correct answers, so check your answer using function composition.)

63. $p(x) = (2x+3)^3$

64. $P(x) = \left(x^2 - x + 1\right)^5$

65. $h(x) = \sqrt{2x-1}$

66. $H(x) = |7 - 3x|$

67. $r(x) = \dfrac{2}{5x+1}$

68. $R(x) = \dfrac{7}{x^2-1}$

69. $q(x) = \dfrac{|x|+1}{|x|-1}$

70. $Q(x) = \dfrac{2x^3+1}{x^3-1}$

71. $v(x) = \dfrac{2x+1}{3-4x}$

72. $w(x) = \dfrac{x^2}{x^4+1}$

In Exercises 73 – 92, show that the given function is one-to-one and find its inverse. Check your answers algebraically and graphically. Verify that the range of f is the domain of f^{-1} and vice-versa.

73. $f(x) = 6x - 2$

74. $f(x) = 42 - x$

75. $f(x) = \dfrac{x-2}{3} + 4$

76. $f(x) = 1 - \dfrac{4+3x}{5}$

77. $f(x) = \sqrt{3x-1} + 5$

78. $f(x) = 2 - \sqrt{x-5}$

79. $f(x) = 3\sqrt{x-1} - 4$

80. $f(x) = 1 - 2\sqrt{2x+5}$

81. $f(x) = \sqrt[5]{3x-1}$

82. $f(x) = 3 - \sqrt[3]{x-2}$

83. $f(x) = x^2 - 10x, x \geq 5$

84. $f(x) = 3(x+4)^2 - 5,\ x \leq -4$

85. $f(x) = x^2 - 6x + 5,\ x \leq 3$

86. $f(x) = 4x^2 + 4x + 1, x < -1$

87. $f(x) = \dfrac{3}{4-x}$

88. $f(x) = \dfrac{x}{1-3x}$

89. $f(x) = \dfrac{2x-1}{3x+4}$

90. $f(x) = \dfrac{4x+2}{3x-6}$

91. $f(x) = \dfrac{-3x-2}{x+3}$

92. $f(x) = \dfrac{x-2}{2x-1}$

3: Essential Functions

3.1 Linear and Quadratic Functions

3.1.1 Linear Functions

We now begin the study of families of functions. Our first family, linear functions, are old friends as we shall soon see. Recall from Geometry that two distinct points in the plane determine a unique line containing those points, as indicated in Figure 3.1.1.

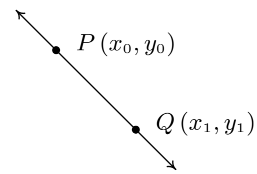


Figure 3.1.1: The line between two points *P* and *Q*

To give a sense of the 'steepness' of the line, we recall that we can compute the **slope** of the line using the formula below.

Definition 3.1.1 Slope

The **slope** *m* of the line containing the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is:

$$m = \frac{y_1 - y_0}{x_1 - x_0},$$

provided $x_1 \neq x_0$.

A couple of notes about Definition 3.1.1 are in order. First, don't ask why we use the letter '*m*' to represent slope. There are many explanations out there, but apparently no one really knows for sure. Secondly, the stipulation $x_1 \neq x_0$ ensures that we aren't trying to divide by zero. The reader is invited to pause to think about what is happening geometrically; the anxious reader can skip along to the next example.

See www.mathforum.org or www.mathworld.wolfram.com for discussions on the use of the letter *m* to indicate slope.

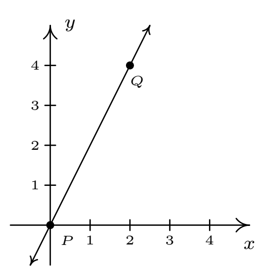
Example 3.1.1 Finding the slope of a line

Find the slope of the line containing the following pairs of points, if it exists. Plot each pair of points and the line containing them.

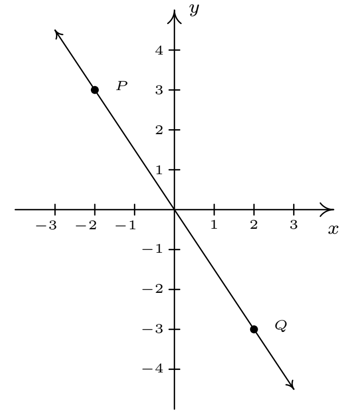
1. $P(0, 0)$, $Q(2, 4)$
2. $P(-2, 3)$, $Q(2, -3)$
3. $P(-3, 2)$, $Q(4, 2)$
4. $P(2, 3)$, $Q(2, -1)$

Solution In each of these examples, we apply the slope formula, from Definition 3.1.1.

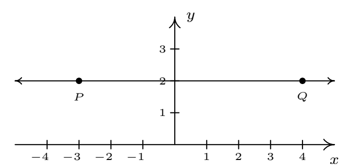
1. $m = \frac{4-0}{2-0} = \frac{4}{2} = 2$



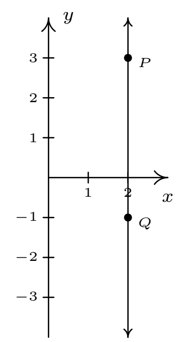
2. $m = \dfrac{-3-3}{2-(-2)} = \dfrac{-6}{4} = -\dfrac{3}{2}$



3. $m = \dfrac{2-2}{4-(-3)} = \dfrac{0}{7} = 0$



4. $m = \dfrac{-1-3}{2-2} = \dfrac{-4}{0}$, which is undefined



You may recall from high school that slope can be described as the ratio '$\frac{\text{rise}}{\text{run}}$'. For example, in the second part of Example 3.1.1, we found the slope to be $\frac{1}{2}$. We can interpret this as a rise of 1 unit upward for every 2 units to the right we travel along the line, as shown in Figure 3.1.2.

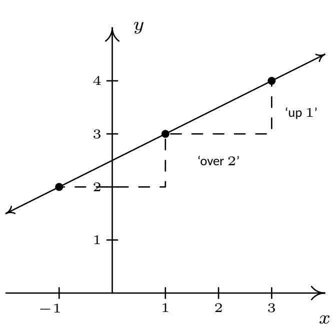


Figure 3.1.2: Slope as "rise over run"

Using more formal notation, given points (x_0, y_0) and (x_1, y_1), we use the Greek letter delta 'Δ' to write $\Delta y = y_1 - y_0$ and $\Delta x = x_1 - x_0$. In most scientific circles, the symbol Δ means 'change in'.

Hence, we may write

$$m = \frac{\Delta y}{\Delta x},$$

which describes the slope as the **rate of change** of y with respect to x. Given a slope m and a point (x_0, y_0) on a line, suppose (x, y) is another point on our line, as in Figure 3.1.3. Definition 3.1.1 yields

$$\begin{aligned} m &= \frac{y - y_0}{x - x_0} \\ m(x - x_0) &= y - y_0 \\ y - y_0 &= m(x - x_0) \end{aligned}$$

We have just derived the **point-slope form** of a line.

Key Idea 3.1.1 The point-slope form of a line

The **point-slope form** of the equation of a line with slope m containing the point (x_0, y_0) is the equation $y - y_0 = m(x - x_0)$.

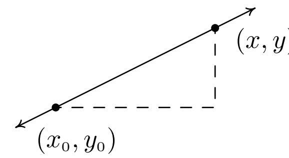


Figure 3.1.3: Deriving the point-slope formula

Example 3.1.2 Using the point-slope form

Write the equation of the line containing the points $(-1, 3)$ and $(2, 1)$.

Solution In order to use Key Idea 3.1.1 we need to find the slope of the line in question so we use Definition 3.1.1 to get $m = \frac{\Delta y}{\Delta x} = \frac{1-3}{2-(-1)} = -\frac{2}{3}$. We are spoiled for choice for a point (x_0, y_0). We'll use $(-1, 3)$ and leave it to the reader to check that using $(2, 1)$ results in the same equation. Substituting into the point-slope form of the line, we get

$$\begin{aligned}
y - y_0 &= m(x - x_0) \\
y - 3 &= -\frac{2}{3}(x - (-1)) \\
y - 3 &= -\frac{2}{3}(x + 1) \\
y - 3 &= -\frac{2}{3}x - \frac{2}{3} \\
y &= -\frac{2}{3}x + \frac{7}{3}.
\end{aligned}$$

In simplifying the equation of the line in the previous example, we produced another form of a line, the **slope-intercept form**. This is the familiar $y = mx + b$ form you have probably seen in high school. The 'intercept' in 'slope-intercept' comes from the fact that if we set $x = 0$, we get $y = b$. In other words, the y-intercept of the line $y = mx + b$ is $(0, b)$.

Key Idea 3.1.2 Slope intercept form of a line

The **slope-intercept form** of the line with slope m and y-intercept $(0, b)$ is the equation $y = mx + b$.

Note that if we have slope $m = 0$, we get the equation $y = b$. The formula given in Key Idea 3.1.2 can be used to describe all lines except vertical lines. All lines except vertical lines are functions (Why is this?) so we have finally reached a good point to introduce **linear functions**.

Definition 3.1.2 Linear function

A **linear function** is a function of the form

$$f(x) = mx + b,$$

where m and b are real numbers with $m \neq 0$. The domain of a linear function is $(-\infty, \infty)$.

For the case $m = 0$, we get $f(x) = b$. These are given their own classification.

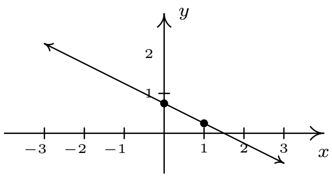


Figure 3.1.6: The graph of $f(x) = \dfrac{3-2x}{4}$

Definition 3.1.3 Constant function

A **constant function** is a function of the form

$$f(x) = b,$$

where b is real number. The domain of a constant function is $(-\infty, \infty)$.

Recall that to graph a function, f, we graph the equation $y = f(x)$. Hence, the graph of a linear function is a line with slope m and y-intercept $(0, b)$; the graph of a constant function is a horizontal line (a line with slope $m = 0$) and a y-intercept of $(0, b)$. A line with positive slope is called an increasing line because a linear function with $m > 0$ is an increasing function. Similarly, a line with a negative slope is called a decreasing line because a linear function with $m < 0$ is a decreasing function. And horizontal lines were called constant because, well, we hope you've already made the connection.

Example 3.1.3 Graphing linear functions

Graph the following functions. Identify the slope and y-intercept.

1. $f(x) = 3$
2. $f(x) = 3x - 1$
3. $f(x) = \dfrac{3-2x}{4}$
4. $f(x) = \dfrac{x^2-4}{x-2}$

Solution

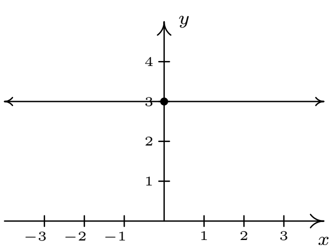


Figure 3.1.4: The graph of $f(x) = 3$

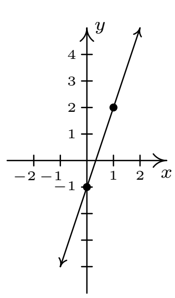


Figure 3.1.5: The graph of $f(x) = 3x - 1$

1. To graph $f(x) = 3$, we graph $y = 3$. This is a horizontal line ($m = 0$) through $(0, 3)$: see Figure 3.1.4.
2. The graph of $f(x) = 3x - 1$ is the graph of the line $y = 3x - 1$. Comparison of this equation with Equation 3.1.2 yields $m = 3$ and $b = -1$. Hence, our slope is 3 and our y-intercept is $(0, -1)$. To get another point on the line, we can plot $(1, f(1)) = (1, 2)$. Constructing the line through these points gives us Figure 3.1.5.
3. At first glance, the function $f(x) = \dfrac{3-2x}{4}$ does not fit the form in Definition 3.1.2 but after some rearranging we get $f(x) = \frac{3-2x}{4} = \frac{3}{4} - \frac{2x}{4} = -\frac{1}{2}x + \frac{3}{4}$. We identify $m = -\frac{1}{2}$ and $b = \frac{3}{4}$. Hence, our graph is a line with a slope of $-\frac{1}{2}$ and a y-intercept of $\left(0, \frac{3}{4}\right)$. Plotting an additional point, we can choose $(1, f(1))$ to get $\left(1, \frac{1}{4}\right)$: see Figure 3.1.6.
4. If we simplify the expression for f, we get

$$f(x) = \frac{x^2-4}{x-2} = \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = x + 2.$$

If we were to state $f(x) = x + 2$, we would be committing a sin of omission. Remember, to find the domain of a function, we do so **before** we simplify! In this case, f has big problems when $x = 2$, and as such, the domain of f is $(-\infty, 2) \cup (2, \infty)$. To indicate this, we write $f(x) = x + 2, x \neq 2$.

So, except at $x = 2$, we graph the line $y = x + 2$. The slope $m = 1$ and the y-intercept is $(0, 2)$. A second point on the graph is $(1, f(1)) = (1, 3)$. Since our function f is not defined at $x = 2$, we put an open circle at the point that would be on the line $y = x + 2$ when $x = 2$, namely $(2, 4)$, as shown in Figure 3.1.7.

The last two functions in the previous example showcase some of the difficulty in defining a linear function using the phrase 'of the form' as in Definition 3.1.2, since some algebraic manipulations may be needed to rewrite a given function to match 'the form'. Keep in mind that the domains of linear and constant functions are all real numbers $(-\infty, \infty)$, so while $f(x) = \frac{x^2-4}{x-2}$ simplified to a formula $f(x) = x + 2$, f is not considered a linear function since its domain excludes $x = 2$. However, we would consider

$$f(x) = \frac{2x^2 + 2}{x^2 + 1}$$

to be a constant function since its domain is all real numbers (Can you tell us why?) and

$$f(x) = \frac{2x^2 + 2}{x^2 + 1} = \frac{2\cancel{(x^2 + 1)}}{\cancel{(x^2 + 1)}} = 2.$$

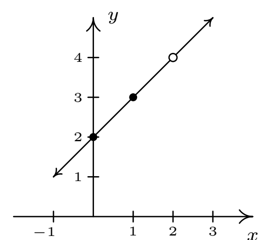
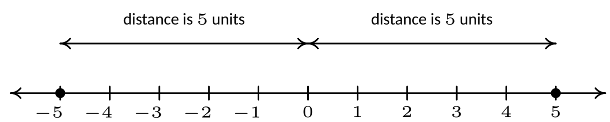


Figure 3.1.7: The graph of $f(x) = \dfrac{x^2 - 4}{x - 2}$

3.1.2 Absolute Value Functions

Before we move on to quadratic functions, we pause to consider the absolute value. The absolute value function is an example of a **piecewise** function, given by different formulas on different parts of its domain. The absolute value function is in particular a *piecewise linear* function, so we've chosen to place it between linear and quadratic functions.

There are a few ways to describe what is meant by the absolute value $|x|$ of a real number x. You may have been taught that $|x|$ is the distance from the real number x to 0 on the number line. So, for example, $|5| = 5$ and $|-5| = 5$, since each is 5 units from 0 on the number line.



Another way to define absolute value is by the equation $|x| = \sqrt{x^2}$. Using this definition, we have $|5| = \sqrt{(5)^2} = \sqrt{25} = 5$ and $|-5| = \sqrt{(-5)^2} = \sqrt{25} = 5$. The long and short of both of these procedures is that $|x|$ takes negative real numbers and assigns them to their positive counterparts while it leaves positive numbers alone. This last description is the one we shall adopt, and is summarized in the following definition.

Definition 3.1.4 Absolute value function

The **absolute value** of a real number x, denoted $|x|$, is given by

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

In Definition 3.1.4, we define $|x|$ using a piecewise-defined function. To check that this definition agrees with what we previously understood as absolute value, note that since $5 \geq 0$, to find $|5|$ we use the rule $|x| = x$, so $|5| = 5$. Similarly, since $-5 < 0$, we use the rule $|x| = -x$, so that $|-5| = -(-5) = 5$. This is one of the times when it's best to interpret the expression '$-x$' as 'the opposite of x' as opposed to 'negative x'. Before we begin studying absolute value functions, we remind ourselves of the properties of absolute value.

Theorem 3.1.1 Properties of Absolute Value

Let a, b and x be real numbers and let n be an integer. Then

- **Product Rule:** $|ab| = |a||b|$
- **Power Rule:** $|a^n| = |a|^n$ whenever a^n is defined
- **Quotient Rule:** $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, provided $b \neq 0$

Equality Properties:

- $|x| = 0$ if and only if $x = 0$.
- For $c > 0$, $|x| = c$ if and only if $x = c$ or $-x = c$.
- For $c < 0$, $|x| = c$ has no solution.

Example 3.1.4 Solving equations with absolute values

Solve each of the following equations.

1. $|3x - 1| = 6$
2. $3 - |x + 5| = 1$
3. $3|2x + 1| - 5 = 0$
4. $4 - |5x + 3| = 5$

Solution

1. The equation $|3x-1| = 6$ is of the form $|x| = c$ for $c > 0$, so by the Equality Properties, $|3x-1| = 6$ is equivalent to $3x-1 = 6$ or $3x-1 = -6$. Solving the former, we arrive at $x = \frac{7}{3}$, and solving the latter, we get $x = -\frac{5}{3}$. We may check both of these solutions by substituting them into the original equation and showing that the arithmetic works out.
2. To use the Equality Properties to solve $3 - |x+5| = 1$, we first isolate the absolute value.

$$\begin{aligned} 3 - |x+5| &= 1 & \\ -|x+5| &= -2 & \text{subtract 3} \\ |x+5| &= 2 & \text{divide by } -1 \end{aligned}$$

   From the Equality Properties, we have $x + 5 = 2$ or $x + 5 = -2$, and get our solutions to be $x = -3$ or $x = -7$. We leave it to the reader to check both answers in the original equation.
3. As in the previous example, we first isolate the absolute value in the equation $3|2x+1| - 5 = 0$ and get $|2x+1| = \frac{5}{3}$. Using the Equality Properties, we have $2x + 1 = \frac{5}{3}$ or $2x + 1 = -\frac{5}{3}$. Solving the former gives $x = \frac{1}{3}$ and solving the latter gives $x = -\frac{4}{3}$. As usual, we may substitute both answers in the original equation to check.
4. Upon isolating the absolute value in the equation $4 - |5x+3| = 5$, we get $|5x + 3| = -1$. At this point, we know there cannot be any real solution, since, by definition, the absolute value of *anything* is never negative. We are done.

Next, we turn our attention to graphing absolute value functions. Our strategy in the next example is to make liberal use of Definition 3.1.4 along with what we know about graphing linear functions (from Section 3.1.1) and piecewise-defined functions (from Section 2.1).

Example 3.1.5 Graphing the absolute value function

Graph the function $f(x) = |x|$.

Solution To find the zeros of f, we set $f(x) = 0$. We get $|x| = 0$, which, by Theorem 3.1.1 gives us $x = 0$. Since the zeros of f are the x-coordinates of the x-intercepts of the graph of $y = f(x)$, we get $(0, 0)$ as our only x-intercept, and this of course is our y-intercept as well. Using Definition 3.1.4, we get

$$f(x) = |x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}.$$

Hence, for $x < 0$, we are graphing the line $y = -x$; for $x \geq 0$, we have the line $y = x$. Plotting these gives us the first two graphs in Figure 3.1.8.

Notice that we have an 'open circle' at $(0, 0)$ in the graph when $x < 0$. As we have seen before, this is due to the fact that the points on $y = -x$ approach $(0, 0)$ as the x-values approach 0. Since x is required to be strictly less than zero on this stretch, the open circle is drawn at the origin. However, notice that when $x \geq 0$, we get to fill in the point at $(0, 0)$, which effectively 'plugs' the hole indicated by the open circle. Thus our final result is the graph at the bottom of Figure 3.1.8.

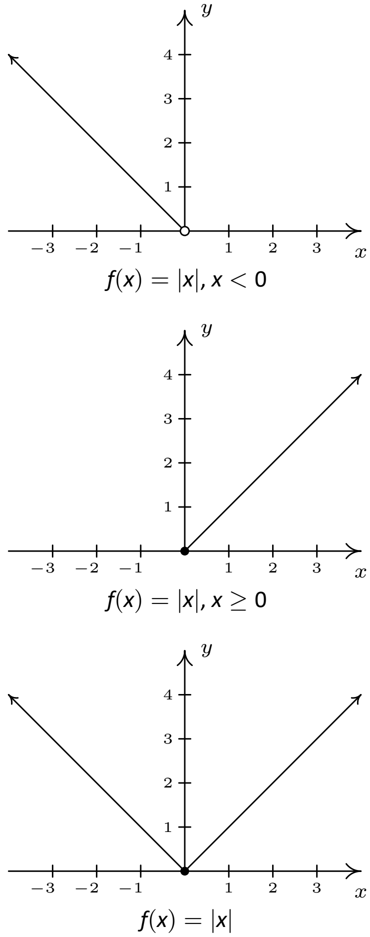


Figure 3.1.8: Constructing the graph of $f(x) = |x|$

3.1.3 Quadratic Functions

You may recall studying quadratic equations in high school. In this section, we review those equations in the context of our next family of functions: the quadratic functions.

> **Definition 3.1.5 Quadratic function**
>
> A **quadratic function** is a function of the form
>
> $$f(x) = ax^2 + bx + c,$$
>
> where a, b and c are real numbers with $a \neq 0$. The domain of a quadratic function is $(-\infty, \infty)$.

The most basic quadratic function is $f(x) = x^2$, whose graph is given in Figure 3.1.9. Its shape should look familiar from high school – it is called a **parabola**. The point $(0, 0)$ is called the **vertex** of the parabola. In this case, the vertex is a relative minimum and is also the where the absolute minimum value of f can be found.

Much like many of the absolute value functions in Section 3.1.2, knowing the graph of $f(x) = x^2$ enables us to graph an entire family of quadratic functions using transformations.

Example 3.1.6 Graphics quadratic functions

Graph the following functions starting with the graph of $f(x) = x^2$ and using transformations. Find the vertex, state the range and find the x- and y-intercepts, if any exist.

1. $g(x) = (x+2)^2 - 3$
2. $h(x) = -2(x-3)^2 + 1$

SOLUTION

1. Since $g(x) = (x+2)^2 - 3 = f(x+2) - 3$, we shift the graph of $y = f(x)$ to the *left* 2 units, and then *down* three units. We move our marked points accordingly and connect the dots in parabolic fashion to get the graph in Figure 3.1.11.

   From the graph, we see that the vertex has moved from $(0, 0)$ on the graph of $y = f(x)$ to $(-2, -3)$ on the graph of $y = g(x)$. This sets $[-3, \infty)$ as the range of g. We see that the graph of $y = g(x)$ crosses the x-axis twice, so we expect two x-intercepts. To find these, we set $y = g(x) = 0$ and solve. Doing so yields the equation $(x+2)^2 - 3 = 0$, or $(x+2)^2 = 3$. Extracting square roots gives $x + 2 = \pm\sqrt{3}$, or $x = -2 \pm \sqrt{3}$. Our x-intercepts are $(-2 - \sqrt{3}, 0) \approx (-3.73, 0)$ and $(-2 + \sqrt{3}, 0) \approx (-0.27, 0)$. The y-intercept of the graph, $(0, 1)$ was one of the points we originally plotted, so we are done.

2. To graph $h(x) = -2(x-3)^2 + 1 = -2f(x-3) + 1$, we first shift *right* 3 units. Next, we *multiply* each of our y-values first by -2 and then *add* 1 to that result. Geometrically, this is a vertical *stretch* by a factor of 2, followed by a reflection about the x-axis, followed by a vertical shift *up* 1 unit. This gives us the graph in Figure 3.1.12.

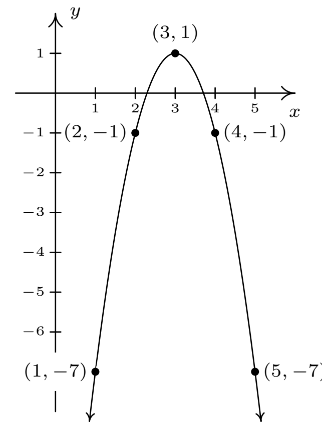


Figure 3.1.12: $h(x) = -2f(x-3) + 1 = -2(x-3)^2 + 1$

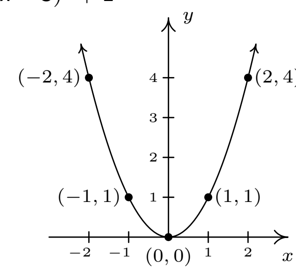


Figure 3.1.9: The graph of the basic quadratic function $f(x) = x^2$

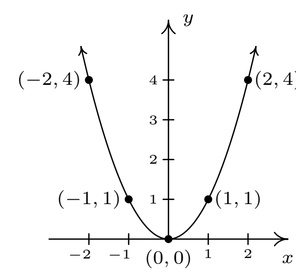


Figure 3.1.10: The graph $y = x^2$ with points labelled

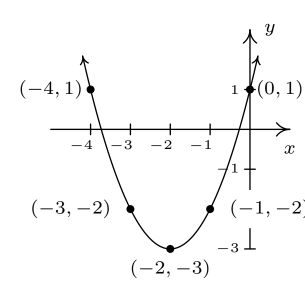


Figure 3.1.11: $g(x) = f(x+2) - 3 = (x+2)^2 - 3$

The vertex is $(3, 1)$ which makes the range of h $(-\infty, 1]$. From our graph, we know that there are two x-intercepts, so we set $y = h(x) = 0$ and solve. We get $-2(x-3)^2 + 1 = 0$ which gives $(x-3)^2 = \frac{1}{2}$. Extracting square roots gives $x - 3 = \pm\frac{1}{\sqrt{2}}$, so that when we add 3 to each side, we get $x = 3 \pm \frac{1}{\sqrt{2}}$. Although our graph doesn't show it, there is a y-intercept which can be found by setting $x = 0$. With $h(0) = -2(0-3)^2 + 1 = -17$, we have that our y-intercept is $(0, -17)$.

In the previous example, note that neither the formula given for $g(x)$ nor the one given for $h(x)$ match the form given in Definition 3.1.5. We could, of course, convert both $g(x)$ and $h(x)$ into that form by expanding and collecting like terms. Doing so, we find $g(x) = (x+2)^2 - 3 = x^2 + 4x + 1$ and $h(x) = -2(x-3)^2 + 1 = -2x^2 + 12x - 17$. While these 'simplified' formulas for $g(x)$ and $h(x)$ satisfy Definition 3.1.5, they do not lend themselves to graphing easily. For that reason, the form of g and h presented in Example 3.1.7 is given a special name, which we list below, along with the form presented in Definition 3.1.5.

Definition 3.1.6 Standard and General Form of Quadratic Functions

Suppose f is a quadratic function.

- The **general form** of the quadratic function f is $f(x) = ax^2 + bx + c$, where a, b and c are real numbers with $a \neq 0$.
- The **standard form** of the quadratic function f is $f(x) = a(x-h)^2 + k$, where a, h and k are real numbers with $a \neq 0$.

One of the advantages of the standard form is that we can immediately read off the location of the vertex:

Theorem 3.1.2 Vertex Formula for Quadratics in Standard Form

For the quadratic function $f(x) = a(x-h)^2 + k$, where a, h and k are real numbers with $a \neq 0$, the vertex of the graph of $y = f(x)$ is (h, k).

To convert a quadratic function given in general form into standard form, we employ the ancient rite of 'Completing the Square'. We remind the reader how this is done in our next example.

Example 3.1.7 Converting from general to standard form

Convert the functions below from general form to standard form.

1. $f(x) = x^2 - 4x + 3$.
2. $g(x) = 6 - x - x^2$

Solution

1. To convert from general form to standard form, we complete the square. First, we verify that the coefficient of x^2 is 1. Next, we find the coefficient

of x, in this case -4, and take half of it to get $\frac{1}{2}(-4) = -2$. This tells us that our target perfect square quantity is $(x-2)^2$. To get an expression equivalent to $(x-2)^2$, we need to add $(-2)^2 = 4$ to the $x^2 - 4x$ to create a perfect square trinomial, but to keep the balance, we must also subtract it. We collect the terms which create the perfect square and gather the remaining constant terms. Putting it all together, we get

If you forget why we do what we do to complete the square, start with $a(x-h)^2 + k$, multiply it out, step by step, and then reverse the process.

$$\begin{aligned} f(x) &= x^2 - 4x + 3 && \text{(Compute } \tfrac{1}{2}(-4) = -2.\text{)} \\ &= \left(x^2 - 4x + \underline{4} - \underline{4}\right) + 3 && \text{(Add and subtract } (-2)^2 = 4.\text{)} \\ &= \left(x^2 - 4x + 4\right) - 4 + 3 && \text{(Group the perfect square trinomial.)} \\ &= (x-2)^2 - 1 && \text{(Factor the perfect square trinomial.)} \end{aligned}$$

From the standard form we can immediately (if desired) produce a sketch of the graph of f, as shown in Figure 3.1.13.

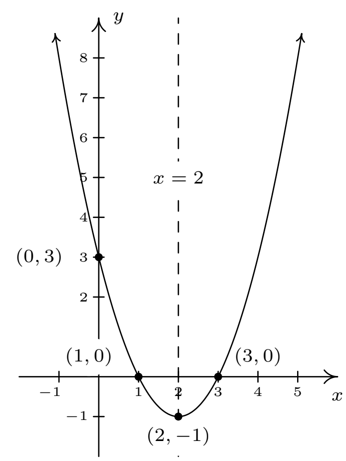


Figure 3.1.13: $f(x) = x^2 - 4x + 3$

2. To get started, we rewrite $g(x) = 6 - x - x^2 = -x^2 - x + 6$ and note that the coefficient of x^2 is -1, not 1. This means our first step is to factor out the (-1) from both the x^2 and x terms. We then follow the completing the square recipe as above.

$$\begin{aligned} g(x) &= -x^2 - x + 6 \\ &= (-1)\left(x^2 + x\right) + 6 && \text{(Factor the coefficient of } x^2 \text{ from } x^2 \text{ and } x.\text{)} \\ &= (-1)\left(x^2 + x + \underline{\frac{1}{4}} - \underline{\frac{1}{4}}\right) + 6 \\ &= (-1)\left(x^2 + x + \frac{1}{4}\right) + (-1)\left(-\frac{1}{4}\right) + 6 \\ & && \text{(Group the perfect square trinomial.)} \\ &= -\left(x + \frac{1}{2}\right)^2 + \frac{25}{4} \end{aligned}$$

Using the standard form, we can again obtain the graph of g, as shown in Figure 3.1.14.

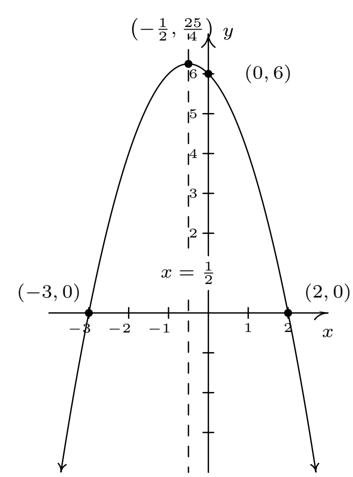


Figure 3.1.14: $g(x) = 6 - x - x^2$

In addition to making it easy for us to sketch the graph of a quadratic function by finding the standard form, completing the square is also the technique needed to obtain the famous **quadratic formula**.

> **Theorem 3.1.3 The Quadratic Formula**
>
> If a, b and c are real numbers with $a \neq 0$, then the solutions to $ax^2 + bx + c = 0$ are
> $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Assuming the conditions of Equation 3.1.3, the solutions to $ax^2 + bx + c = 0$ are precisely the zeros of $f(x) = ax^2 + bx + c$. To find these zeros (if possible), we proceed as follows:

$$
\begin{aligned}
ax^2 + bx + c &= 0 \\
a\left(x^2 + \frac{b}{a}x\right) &= -c \\
a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) &= -c + \frac{b^2}{4a} \\
a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\end{aligned}
$$

In our discussions of domain, we were warned against having negative numbers underneath the square root. Given that $\sqrt{b^2 - 4ac}$ is part of the Quadratic Formula, we will need to pay special attention to the radicand $b^2 - 4ac$. It turns out that the quantity $b^2 - 4ac$ plays a critical role in determining the nature of the solutions to a quadratic equation. It is given a special name.

Definition 3.1.7 Discriminant

If a, b and c are real numbers with $a \neq 0$, then the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$ is the quantity $b^2 - 4ac$.

The discriminant ‘discriminates’ between the kinds of solutions we get from a quadratic equation. These cases, and their relation to the discriminant, are summarized below.

Theorem 3.1.4 Discriminant Trichotomy

Let a, b and c be real numbers with $a \neq 0$.

- If $b^2 - 4ac < 0$, the equation $ax^2 + bx + c = 0$ has no real solutions.
- If $b^2 - 4ac = 0$, the equation $ax^2 + bx + c = 0$ has exactly one real solution.
- If $b^2 - 4ac > 0$, the equation $ax^2 + bx + c = 0$ has exactly two real solutions.

Exercises 3.1

Problems

In Exercises 1 – 10, find both the point-slope form and the slope-intercept form of the line with the given slope which passes through the given point.

1. $m = 3, \quad P(3, -1)$
2. $m = -2, \quad P(-5, 8)$
3. $m = -1, \quad P(-7, -1)$
4. $m = \frac{2}{3}, \quad P(-2, 1)$
5. $m = \frac{2}{3}, \quad P(-2, 1)$
6. $m = \frac{1}{7}, \quad P(-1, 4)$
7. $m = 0, \quad P(3, 117)$
8. $m = -\sqrt{2}, \quad P(0, -3)$
9. $m = -5, \quad P(\sqrt{3}, 2\sqrt{3})$
10. $m = 678, \quad P(-1, -12)$

In Exercises 11 – 20, find the slope-intercept form of the line which passes through the given points.

11. $P(0, 0),\ Q(-3, 5)$
12. $P(-1, -2),\ Q(3, -2)$
13. $P(5, 0),\ Q(0, -8)$
14. $P(3, -5),\ Q(7, 4)$
15. $P(-1, 5),\ Q(7, 5)$
16. $P(4, -8),\ Q(5, -8)$
17. $P\left(\frac{1}{2}, \frac{3}{4}\right),\ Q\left(\frac{5}{2}, -\frac{7}{4}\right)$
18. $P\left(\frac{2}{3}, \frac{7}{2}\right),\ Q\left(-\frac{1}{3}, \frac{3}{2}\right)$
19. $P\left(\sqrt{2}, -\sqrt{2}\right),\ Q\left(-\sqrt{2}, \sqrt{2}\right)$
20. $P\left(-\sqrt{3}, -1\right),\ Q\left(\sqrt{3}, 1\right)$

In Exercises 21 – 26, graph the function. Find the slope, y-intercept and x-intercept, if any exist.

21. $f(x) = 2x - 1$
22. $f(x) = 3 - x$
23. $f(x) = 3$
24. $f(x) = 0$
25. $f(x) = \frac{2}{3}x + \frac{1}{3}$
26. $f(x) = \dfrac{1 - x}{2}$

In Exercises 27 – 41, solve the equation.

27. $|x| = 6$
28. $|3x - 1| = 10$
29. $|4 - x| = 7$
30. $4 - |x| = 3$
31. $2|5x + 1| - 3 = 0$
32. $|7x - 1| + 2 = 0$
33. $\dfrac{5 - |x|}{2} = 1$
34. $\frac{2}{3}|5 - 2x| - \frac{1}{2} = 5$
35. $|x| = x + 3$
36. $|2x - 1| = x + 1$
37. $4 - |x| = 2x + 1$
38. $|x - 4| = x - 5$
39. $|x| = x^2$
40. $|x| = 12 - x^2$
41. $|x^2 - 1| = 3$

Prove that if $|f(x)| = |g(x)|$ then either $f(x) = g(x)$ or $f(x) = -g(x)$. Use that result to solve the equations in Exercises 42 – 47.

42. $|3x - 2| = |2x + 7|$
43. $|3x + 1| = |4x|$
44. $|1 - 2x| = |x + 1|$
45. $|4 - x| - |x + 2| = 0$
46. $|2 - 5x| = 5|x + 1|$
47. $3|x - 1| = 2|x + 1|$

In Exercises 48 – 59, graph the function. Find the zeros of each function and the *x*- and *y*-intercepts of each graph, if any exist. From the graph, determine the domain and range of each function, list the intervals on which the function is increasing, decreasing or constant, and find the relative and absolute extrema, if they exist.

48. $f(x) = |x + 4|$

49. $f(x) = |x| + 4$

50. $f(x) = |4x|$

51. $f(x) = -3|x|$

52. $f(x) = 3|x + 4| - 4$

53. $f(x) = \frac{1}{3}|2x - 1|$

54. $f(x) = \dfrac{|x + 4|}{x + 4}$

55. $f(x) = \dfrac{|2 - x|}{2 - x}$

56. $f(x) = x + |x| - 3$

57. $f(x) = |x + 2| - x$

58. $f(x) = |x + 2| - |x|$

59. $f(x) = |x + 4| + |x - 2|$

In Exercises 60 – 67, graph the quadratic function. Find the *x*- and *y*-intercepts of each graph, if any exist. If it is given in general form, convert it into standard form; if it is given in standard form, convert it into general form. Find the domain and range of the function and list the intervals on which the function is increasing or decreasing. Identify the vertex and the axis of symmetry and determine whether the vertex yields a relative and absolute maximum or minimum.

60. $f(x) = x^2 + 2$

61. $f(x) = -(x + 2)^2$

62. $f(x) = x^2 - 2x - 8$

63. $f(x) = -2(x + 1)^2 + 4$

64. $f(x) = 2x^2 - 4x - 1$

65. $f(x) = -3x^2 + 4x - 7$

66. $f(x) = x^2 + x + 1$

67. $f(x) = -3x^2 + 5x + 4$

In Exercises 68 – 99, solve the inequality. Write your answer using interval notation.

68. $|3x - 5| \leq 4$

69. $|7x + 2| > 10$

70. $|2x + 1| - 5 < 0$

71. $|2 - x| - 4 \geq -3$

72. $|3x + 5| + 2 < 1$

73. $2|7 - x| + 4 > 1$

74. $2 \leq |4 - x| < 7$

75. $1 < |2x - 9| \leq 3$

76. $|x + 3| \geq |6x + 9|$

77. $|x - 3| - |2x + 1| < 0$

78. $|1 - 2x| \geq x + 5$

79. $x + 5 < |x + 5|$

80. $x \geq |x + 1|$

81. $|2x + 1| \leq 6 - x$

82. $x + |2x - 3| < 2$

83. $|3 - x| \geq x - 5$

84. $x^2 + 2x - 3 \geq 0$

85. $16x^2 + 8x + 1 > 0$

86. $x^2 + 9 < 6x$

87. $9x^2 + 16 \geq 24x$

88. $x^2 + 4 \leq 4x$

89. $x^2 + 1 < 0$

90. $3x^2 \leq 11x + 4$

91. $x > x^2$

92. $2x^2 - 4x - 1 > 0$

93. $5x + 4 \leq 3x^2$

94. $2 \leq |x^2 - 9| < 9$

95. $x^2 \leq |4x - 3|$

96. $x^2 + x + 1 \geq 0$

97. $x^2 \geq |x|$

98. $x|x+5| \geq -6$

99. $x|x-3| < 2$

3.2 Polynomial Functions

3.2.1 Graphs of Polynomial Functions

Three of the families of functions studied thus far – constant, linear and quadratic – belong to a much larger group of functions called **polynomials**. We begin our formal study of general polynomials with a definition and some examples.

Definition 3.2.1 Polynomial function

A **polynomial function** is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0,$$

where a_0, a_1, ..., a_n are real numbers and $n \geq 1$ is a natural number. The domain of a polynomial function is $(-\infty, \infty)$.

There are several things about Definition 3.2.1 that may be off-putting or downright frightening. The best thing to do is look at an example. Consider $f(x) = 4x^5 - 3x^2 + 2x - 5$. Is this a polynomial function? We can re-write the formula for f as $f(x) = 4x^5 + 0x^4 + 0x^3 + (-3)x^2 + 2x + (-5)$. Comparing this with Definition 3.2.1, we identify $n = 5$, $a_5 = 4$, $a_4 = 0$, $a_3 = 0$, $a_2 = -3$, $a_1 = 2$ and $a_0 = -5$. In other words, a_5 is the coefficient of x^5, a_4 is the coefficient of x^4, and so forth; the subscript on the a's merely indicates to which power of x the coefficient belongs. The business of restricting n to be a natural number lets us focus on well-behaved algebraic animals. (Yes, there are examples of worse behaviour still to come!)

Example 3.2.1 Identifying polynomial functions

Determine if the following functions are polynomials. Explain your reasoning.

1. $g(x) = \dfrac{4 + x^3}{x}$
2. $p(x) = \dfrac{4x + x^3}{x}$
3. $q(x) = \dfrac{4x + x^3}{x^2 + 4}$
4. $f(x) = \sqrt[3]{x}$
5. $h(x) = |x|$
6. $z(x) = 0$

Solution

1. We note directly that the domain of $g(x) = \dfrac{x^3 + 4}{x}$ is $x \neq 0$. By definition, a polynomial has all real numbers as its domain. Hence, g can't be a polynomial.

2. Even though $p(x) = \dfrac{x^3 + 4x}{x}$ simplifies to $p(x) = x^2 + 4$, which certainly looks like the form given in Definition 3.2.1, the domain of p, which, as you may recall, we determine *before* we simplify, excludes 0. Alas, p is not a polynomial function for the same reason g isn't.

3. After what happened with p in the previous part, you may be a little shy about simplifying $q(x) = \dfrac{x^3 + 4x}{x^2 + 4}$ to $q(x) = x$, which certainly fits Definition 3.2.1. If we look at the domain of q before we simplified, we see

that it is, indeed, all real numbers. A function which can be written in the form of Definition 3.2.1 whose domain is all real numbers is, in fact, a polynomial.

4. We can rewrite $f(x) = \sqrt[3]{x}$ as $f(x) = x^{\frac{1}{3}}$. Since $\frac{1}{3}$ is not a natural number, f is not a polynomial.

5. The function $h(x) = |x|$ isn't a polynomial, since it can't be written as a combination of powers of x even though it can be written as a piecewise function involving polynomials. As we shall see in this section, graphs of polynomials possess a quality that the graph of h does not.

6. There's nothing in Definition 3.2.1 which prevents all the coefficients a_n, etc., from being 0. Hence, $z(x) = 0$, is an honest-to-goodness polynomial.

Once we get to calculus, we'll see that the absolute value function is the classic example of a function which is continuous everywhere, but fails to have a derivative everywhere: the graph of $h(x) = |x|$ fails to be "smooth" at the origin.

Definition 3.2.2 Polynomial terminology

Suppose f is a polynomial function.

- Given $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ with $a_n \neq 0$, we say
  - The natural number n is called the **degree** of the polynomial f.
  - The term a_nx^n is called the **leading term** of the polynomial f.
  - The real number a_n is called the **leading coefficient** of the polynomial f.
  - The real number a_0 is called the **constant term** of the polynomial f.
- If $f(x) = a_0$, and $a_0 \neq 0$, we say f has degree 0.
- If $f(x) = 0$, we say f has no degree.

The reader may well wonder why we have chosen to separate off constant functions from the other polynomials in Definition 3.2.2. Why not just lump them all together and, instead of forcing n to be a natural number, $n = 1, 2, \ldots$, allow n to be a whole number, $n = 0, 1, 2, \ldots$. We could unify all of the cases, since, after all, isn't $a_0x^0 = a_0$? The answer is 'yes, as long as $x \neq 0$.' The function $f(x) = 3$ and $g(x) = 3x^0$ are different, because their domains are different. The number $f(0) = 3$ is defined, whereas $g(0) = 3(0)^0$ is not. Indeed, much of the theory we will develop in this chapter doesn't include the constant functions, so we might as well treat them as outsiders from the start. One good thing that comes from Definition 3.2.2 is that we can now think of linear functions as degree 1 (or 'first degree') polynomial functions and quadratic functions as degree 2 (or 'second degree') polynomial functions.

In the context of limits, results such as 0^0 are known as *indeterminant forms*. These are cases where the function fails to be defined, but the methods of calculus might still be able to extract information.

Example 3.2.2 Using polynomial terminiology

Find the degree, leading term, leading coefficient and constant term of the following polynomial functions.

1. $f(x) = 4x^5 - 3x^2 + 2x - 5$
2. $g(x) = 12x + x^3$
3. $h(x) = \dfrac{4 - x}{5}$
4. $p(x) = (2x - 1)^3(x - 2)(3x + 2)$

Solution

1. There are no surprises with $f(x) = 4x^5 - 3x^2 + 2x - 5$. It is written in the form of Definition 3.2.2, and we see that the degree is 5, the leading term is $4x^5$, the leading coefficient is 4 and the constant term is -5.

2. The form given in Definition 3.2.2 has the highest power of x first. To that end, we re-write $g(x) = 12x + x^3 = x^3 + 12x$, and see that the degree of g is 3, the leading term is x^3, the leading coefficient is 1 and the constant term is 0.

3. We need to rewrite the formula for h so that it resembles the form given in Definition 3.2.2: $h(x) = \frac{4-x}{5} = \frac{4}{5} - \frac{x}{5} = -\frac{1}{5}x + \frac{4}{5}$. The degree of h is 1, the leading term is $-\frac{1}{5}x$, the leading coefficient is $-\frac{1}{5}$ and the constant term is $\frac{4}{5}$.

4. It may seem that we have some work ahead of us to get p in the form of Definition 3.2.2. However, it is possible to glean the information requested about p without multiplying out the entire expression $(2x-1)^3(x-2)(3x + 2)$. The leading term of p will be the term which has the highest power of x. The way to get this term is to multiply the terms with the highest power of x from each factor together - in other words, the leading term of $p(x)$ is the product of the leading terms of the factors of $p(x)$. Hence, the leading term of p is $(2x)^3(x)(3x) = 24x^5$. This means that the degree of p is 5 and the leading coefficient is 24. As for the constant term, we can perform a similar trick. The constant term is obtained by multiplying the constant terms from each of the factors $(-1)^3(-2)(2) = 4$.

We now consider the graphs of polynomial functions. In Figure 3.2.1 the graphs of $y = x^2$, $y = x^4$ and $y = x^6$, are shown. We have omitted the axes to allow you to see that as the exponent increases, the 'bottom' becomes 'flatter' and the 'sides' become 'steeper.' If you take the the time to graph these functions by hand, (make sure you choose some x-values between -1 and 1.) you will see why.

All of these functions are even, (Do you remember how to show this?) and it is exactly because the exponent is even. (Herein lies one of the possible origins of the term 'even' when applied to functions.) This symmetry is important, but we want to explore a different yet equally important feature of these functions which we can be seen graphically – their **end behaviour**.

The end behaviour of a function is a way to describe what is happening to the function values (the y-values) as the x-values approach the 'ends' of the x-axis. (Of course, there are no ends to the x-axis.) That is, what happens to y as x becomes small without bound (written $x \to -\infty$) and, on the flip side, as x becomes large without bound (written $x \to \infty$).

When $x \to \infty$ we think of x as moving far to the right of zero and becoming a very large *positive* number. When $x \to -\infty$ we think of x as becoming a very large (in the sense of its absolute value) *negative* number far to the left of zero.

For example, given $f(x) = x^2$, as $x \to -\infty$, we imagine substituting $x = -100$, $x = -1000$, etc., into f to get $f(-100) = 10000$, $f(-1000) = 1000000$, and so on. Thus the function values are becoming larger and larger positive numbers (without bound). To describe this behaviour, we write: as $x \to -\infty$, $f(x) \to \infty$. If we study the behaviour of f as $x \to \infty$, we see that in this case, too, $f(x) \to \infty$. (We told you that the symmetry was important!) The same can be said for any function of the form $f(x) = x^n$ where n is an even natural number. If we generalize just a bit to include vertical scalings and reflections across the x-axis, we have

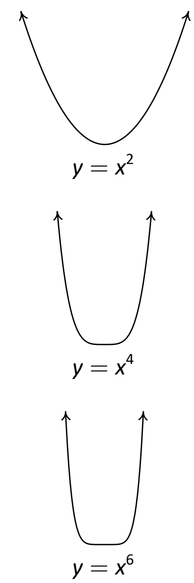


Figure 3.2.1: Graphing even powers of x

Key Idea 3.2.1 End behaviour of functions $f(x) = ax^n$, n even.

Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and n is an even natural number. The end behaviour of the graph of $y = f(x)$ matches one of the following:

- for $a > 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to \infty$
- for $a < 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$

This is illustrated graphically below:

We now turn our attention to functions of the form $f(x) = x^n$ where $n \geq 3$ is an odd natural number. (We ignore the case when $n = 1$, since the graph of $f(x) = x$ is a line and doesn't fit the general pattern of higher-degree odd polynomials.) In Figure 3.2.2 we have graphed $y = x^3$, $y = x^5$, and $y = x^7$. The 'flattening' and 'steepening' that we saw with the even powers presents itself here as well, and, it should come as no surprise that all of these functions are odd. (And are, perhaps, the inspiration for the moniker 'odd function'.) The end behaviour of these functions is all the same, with $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.

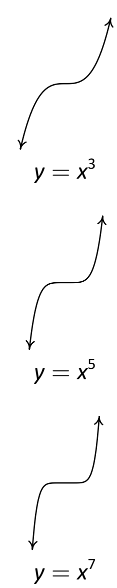


Figure 3.2.2: Graphing odd powers of x

As with the even degreed functions we studied earlier, we can generalize their end behaviour.

> **Key Idea 3.2.2 End behaviour of functions $f(x) = ax^n$, n odd.**
>
> Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and $n \geq 3$ is an odd natural number. The end behaviour of the graph of $y = f(x)$ matches one of the following:
>
> - for $a > 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$
> - for $a < 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to -\infty$
>
> This is illustrated graphically as follows:
>
> 
>
> $a > 0$ $a < 0$

In fact, when we get to Calculus, you'll find that smooth functions are automatically continuous, so that saying 'polynomials are continuous and smooth' is redundant.

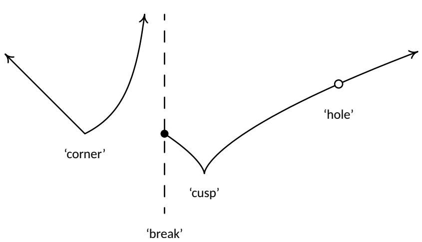


Figure 3.2.3: Pathologies not found on graphs of polynomials

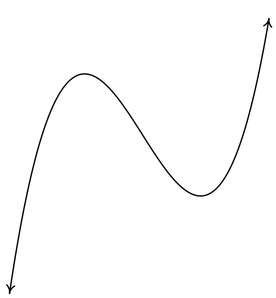


Figure 3.2.4: The graph of a polynomial

Despite having different end behaviour, all functions of the form $f(x) = ax^n$ for natural numbers n share two properties which help distinguish them from other animals in the algebra zoo: they are **continuous** and **smooth**. While these concepts are formally defined using Calculus, informally, graphs of continuous functions have no 'breaks' or 'holes' in them, and the graphs of smooth functions have no 'sharp turns'. It turns out that these traits are preserved when functions are added together, so general polynomial functions inherit these qualities. In Figure 3.2.3, we find the graph of a function which is neither smooth nor continuous, and to its right we have a graph of a polynomial, for comparison. The function whose graph appears on the left fails to be continuous where it has a 'break' or 'hole' in the graph; everywhere else, the function is continuous. The function is continuous at the 'corner' and the 'cusp', but we consider these 'sharp turns', so these are places where the function fails to be smooth. Apart from these four places, the function is smooth and continuous. Polynomial functions are smooth and continuous everywhere, as exhibited in Figure 3.2.4.

The notion of smoothness is what tells us graphically that, for example, $f(x) = |x|$, whose graph is the characteristic '$\vee$' shape, cannot be a polynomial. The notion of continuity is key to constructing sign diagrams: the zeros of a polynomial function are the only possible places where it can change sign. This last result is formalized in the following theorem.

> **Theorem 3.2.1 The Intermediate Value Theorem (Zero Version)**
>
> Suppose f is a continuous function on an interval containing $x = a$ and $x = b$ with $a < b$. If $f(a)$ and $f(b)$ have different signs, then f has at least one zero between $x = a$ and $x = b$; that is, for at least one real number c such that $a < c < b$, we have $f(c) = 0$.

The Intermediate Value Theorem is extremely profound; it gets to the heart of what it means to be a real number, and is one of the most often used and under appreciated theorems in Mathematics. With that being said, most students see the result as common sense since it says, geometrically, that the graph of a polynomial function cannot be above the x-axis at one point and below the x-

axis at another point without crossing the x-axis somewhere in between. We'll return to the Intermediate Value Theorem later in the Calculus portion of the course, when we study continuity in general. The following example uses the Intermediate Value Theorem to establish a fact that that most students take for granted. Many students, and sadly some instructors, will find it silly.

Example 3.2.3 Existence of $\sqrt{2}$

Use the Intermediate Value Theorem to establish that $\sqrt{2}$ is a real number.

SOLUTION Consider the polynomial function $f(x) = x^2 - 2$. Then $f(1) = -1$ and $f(3) = 7$. Since $f(1)$ and $f(3)$ have different signs, the Intermediate Value Theorem guarantees us a real number c between 1 and 3 with $f(c) = 0$. If $c^2 - 2 = 0$ then $c = \pm\sqrt{2}$. Since c is between 1 and 3, c is positive, so $c = \sqrt{2}$.

The validity of the result in Example 3.2.3 of course relies on having a rigorous proof of Theorem 3.2.1. Although intuitive, its proof is one of the most difficult in single variable calculus. At most universities, you don't see a proof until a first course in Analysis, like Math 3500.

Our primary use of the Intermediate Value Theorem is in the construction of sign diagrams, since it guarantees us that polynomial functions are always positive $(+)$ or always negative $(-)$ on intervals which do not contain any of its zeros. The general algorithm for polynomials is given below.

Key Idea 3.2.3 Steps for Constructing a Sign Diagram for a Polynomial Function

Suppose f is a polynomial function.

1. Find the zeros of f and place them on the number line with the number 0 above them.
2. Choose a real number, called a **test value**, in each of the intervals determined in step 1.
3. Determine the sign of $f(x)$ for each test value in step 2, and write that sign above the corresponding interval.

Example 3.2.4 Using a sign diagram to sketch a polynomial

Construct a sign diagram for $f(x) = x^3(x-3)^2(x+2)\left(x^2+1\right)$. Use it to give a rough sketch of the graph of $y = f(x)$.

SOLUTION First, we find the zeros of f by solving $x^3(x-3)^2(x+2)\left(x^2+1\right) = 0$. We get $x = 0$, $x = 3$ and $x = -2$. (The equation $x^2 + 1 = 0$ produces no real solutions.) These three points divide the real number line into four intervals: $(-\infty, -2)$, $(-2, 0)$, $(0, 3)$ and $(3, \infty)$. We select the test values $x = -3$, $x = -1$, $x = 1$ and $x = 4$. We find $f(-3)$ is $(+)$, $f(-1)$ is $(-)$ and $f(1)$ is $(+)$ as is $f(4)$. Wherever f is $(+)$, its graph is above the x-axis; wherever f is $(-)$, its graph is below the x-axis. The x-intercepts of the graph of f are $(-2, 0)$, $(0, 0)$ and $(3, 0)$. Knowing f is smooth and continuous allows us to sketch its graph in Figure 3.2.6.

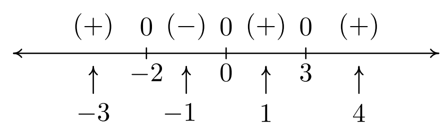


Figure 3.2.5: The sign diagram of f in Example 3.2.4

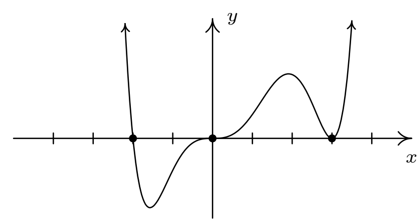


Figure 3.2.6: The graph $y = f(x)$ for Example 3.2.4

A couple of notes about the Example 3.2.4 are in order. First, note that we purposefully did not label the y-axis in the sketch of the graph of $y = f(x)$. This is because the sign diagram gives us the zeros and the relative position of the graph - it doesn't give us any information as to how high or low the graph strays from the x-axis. Furthermore, as we have mentioned earlier in the text, without Calculus, the values of the relative maximum and minimum can only be found approximately using a calculator. If we took the time to find the leading term of

f, we would find it to be x^8. Looking at the end behaviour of f, we notice that it matches the end behaviour of $y = x^8$. This is no accident, as we find out in the next theorem.

Theorem 3.2.2 End behaviour for Polynomial Functions

The end behaviour of a polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ with $a_n \neq 0$ matches the end behaviour of $y = a_nx^n$.

To see why Theorem 3.2.2 is true, let's first look at a specific example. Consider $f(x) = 4x^3 - x + 5$. If we wish to examine end behaviour, we look to see the behaviour of f as $x \to \pm\infty$. Since we're concerned with x's far down the x-axis, we are far away from $x = 0$ so can rewrite $f(x)$ for these values of x as

$$f(x) = 4x^3\left(1 - \frac{1}{4x^2} + \frac{5}{4x^3}\right)$$

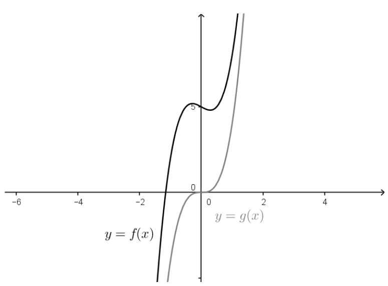
As x becomes unbounded (in either direction), the terms $\frac{1}{4x^2}$ and $\frac{5}{4x^3}$ become closer and closer to 0, as the table below indicates.

x	$\frac{1}{4x^2}$	$\frac{5}{4x^3}$
-1000	0.00000025	-0.00000000125
-100	0.000025	-0.00000125
-10	0.0025	-0.00125
10	0.0025	0.00125
100	0.000025	0.00000125
1000	0.00000025	0.00000000125

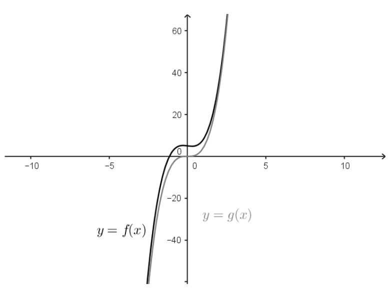
In other words, as $x \to \pm\infty$, $f(x) \approx 4x^3(1 - 0 + 0) = 4x^3$, which is the leading term of f. The formal proof of Theorem 3.2.2 works in much the same way. Factoring out the leading term leaves

$$f(x) = a_nx^n\left(1 + \frac{a_{n-1}}{a_nx} + \ldots + \frac{a_2}{a_nx^{n-2}} + \frac{a_1}{a_nx^{n-1}} + \frac{a_0}{a_nx^n}\right)$$

As $x \to \pm\infty$, any term with an x in the denominator becomes closer and closer to 0, and we have $f(x) \approx a_nx^n$. Geometrically, Theorem 3.2.2 says that if we graph $y = f(x)$ using a graphing calculator, and continue to 'zoom out', the graph of it and its leading term become indistinguishable. In Figure 3.2.7 the graphs of $y = 4x^3 - x + 5$ and $y = 4x^3$) in two different windows.



A view close to the origin



A 'zoomed out' view

Figure 3.2.7: Two views of the polynomials $f(x)$ and $g(x)$

Let's return to the function in Example 3.2.4, $f(x) = x^3(x-3)^2(x+2)\left(x^2 + 1\right)$, whose sign diagram and graph are given in Figures 3.2.5 and 3.2.6. Theorem 3.2.2 tells us that the end behaviour is the same as that of its leading term x^8. This tells us that the graph of $y = f(x)$ starts and ends above the x-axis. In other words, $f(x)$ is $(+)$ as $x \to \pm\infty$, and as a result, we no longer need to evaluate f at the test values $x = -3$ and $x = 4$. Is there a way to eliminate the need to evaluate f at the other test values? What we would really need to know is how the function behaves near its zeros - does it cross through the x-axis at these points, as it does at $x = -2$ and $x = 0$, or does it simply touch and rebound like it does at $x = 3$. From the sign diagram, the graph of f will cross the x-axis whenever the signs on either side of the zero switch (like they do at $x = -2$ and $x = 0$); it will touch when the signs are the same on either side of the zero (as is the case with $x = 3$). What we need to determine is the reason behind whether or not the sign change occurs.

Fortunately, f was given to us in factored form: $f(x) = x^3(x-3)^2(x+2)$. When we attempt to determine the sign of $f(-4)$, we are attempting to find the sign of the number $(-4)^3(-7)^2(-2)$, which works out to be $(-)(+)(-)$ which is $(+)$. If we move to the other side of $x = -2$, and find the sign of $f(-1)$, we are determining the sign of $(-1)^3(-4)^2(+1)$, which is $(-)(+)(+)$ which gives us the $(-)$. Notice that signs of the first two factors in both expressions are the same in $f(-4)$ and $f(-1)$. The only factor which switches sign is the third factor, $(x+2)$, precisely the factor which gave us the zero $x = -2$. If we move to the other side of 0 and look closely at $f(1)$, we get the sign pattern $(+1)^3(-2)^2(+3)$ or $(+)(+)(+)$ and we note that, once again, going from $f(-1)$ to $f(1)$, the only factor which changed sign was the first factor, x^3, which corresponds to the zero $x = 0$. Finally, to find $f(4)$, we substitute to get $(+4)^3(+2)^2(+5)$ which is $(+)(+)(+)$ or $(+)$. The sign didn't change for the middle factor $(x-3)^2$. Even though this is the factor which corresponds to the zero $x = 3$, the fact that the quantity is *squared* kept the sign of the middle factor the same on either side of 3. If we look back at the exponents on the factors $(x+2)$ and x^3, we see that they are both odd, so as we substitute values to the left and right of the corresponding zeros, the signs of the corresponding factors change which results in the sign of the function value changing. This is the key to the behaviour of the function near the zeros. We need a definition and then a theorem.

Definition 3.2.3 Multiplicity of a zero

Suppose f is a polynomial function and m is a natural number. If $(x-c)^m$ is a factor of $f(x)$ but $(x-c)^{m+1}$ is not, then we say $x = c$ is a zero of **multiplicity** m.

Hence, rewriting $f(x) = x^3(x-3)^2(x+2)$ as $f(x) = (x-0)^3(x-3)^2(x-(-2))^1$, we see that $x = 0$ is a zero of multiplicity 3, $x = 3$ is a zero of multiplicity 2 and $x = -2$ is a zero of multiplicity 1.

Theorem 3.2.3 The Role of Multiplicity

Suppose f is a polynomial function and $x = c$ is a zero of multiplicity m.

- If m is even, the graph of $y = f(x)$ touches and rebounds from the x-axis at $(c, 0)$.
- If m is odd, the graph of $y = f(x)$ crosses through the x-axis at $(c, 0)$.

Our last example shows how end behaviour and multiplicity allow us to sketch a decent graph without appealing to a sign diagram.

Example 3.2.5 Using end behaviour and multiplicity

Sketch the graph of $f(x) = -3(2x-1)(x+1)^2$ using end behaviour and the multiplicity of its zeros.

Solution The end behaviour of the graph of f will match that of its leading term. To find the leading term, we multiply by the leading terms of each

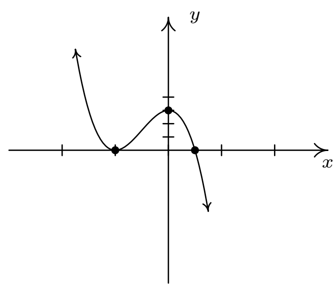


Figure 3.2.8: The graph $y = f(x)$ for Example 3.2.5

factor to get $(-3)(2x)(x)^2 = -6x^3$. This tells us that the graph will start above the x-axis, in Quadrant II, and finish below the x-axis, in Quadrant IV. Next, we find the zeros of f. Fortunately for us, f is factored. (Obtaining the factored form of a polynomial is the main focus of the next few sections.) Setting each factor equal to zero gives is $x = \frac{1}{2}$ and $x = -1$ as zeros. To find the multiplicity of $x = \frac{1}{2}$ we note that it corresponds to the factor $(2x - 1)$. This isn't strictly in the form required in Definition 3.2.3. If we factor out the 2, however, we get $(2x-1) = 2\left(x - \frac{1}{2}\right)$, and we see that the multiplicity of $x = \frac{1}{2}$ is 1. Since 1 is an odd number, we know from Theorem 3.2.3 that the graph of f will cross through the x-axis at $\left(\frac{1}{2}, 0\right)$. Since the zero $x = -1$ corresponds to the factor $(x+1)^2 = (x-(-1))^2$, we find its multiplicity to be 2 which is an even number. As such, the graph of f will touch and rebound from the x-axis at $(-1, 0)$. Though we're not asked to, we can find the y-intercept by finding $f(0) = -3(2(0)-1)(0+1)^2 = 3$. Thus $(0, 3)$ is an additional point on the graph. Putting this together gives us the graph in Figure 3.2.8.

3.2.2 Polynomial Arithmetic

The previous section introduced all the important polynomial terminology and taught us the basic techniques for graphing polynomial functions. We saw that a necessary ingredient for obtaining the graph of a polynomial function is knowledge of the zeros of the polynomial. In the next few sections, we will cover the algebraic techniques needed to obtain this information.

In this section our focus is entirely on algebraic manipulation, so we will pause briefly in our discussion of functions, and simply consider polynomial *expressions*. (That is, we simply dispense with writing "$p(x) =$" in front of every polynomial.)

We begin with (you guessed it) a bit more terminology that can come in handy when comparing polynomials.

Definition 3.2.4 Polynomial Vocabulary, Part 2

- **Like Terms:** Terms in a polynomial are called **like** terms if they have the same variables each with the same corresponding exponents.
- **Simplified:** A polynomial is said to be **simplified** if all arithmetic operations have been completed and there are no longer any like terms.
- **Classification by Number of Terms:** A simplified polynomial is called a
  - **monomial** if it has exactly one nonzero term
  - **binomial** if it has exactly two nonzero terms
  - **trinomial** if it has exactly three nonzero terms

For example, $x^2 + x\sqrt{3} + 4$ is a trinomial of degree 2. The coefficient of x^2 is 1 and the constant term is 4. The polynomial $27x^2y + \frac{7x}{2}$ is a binomial of degree 3 ($x^2y = x^2y^1$) with constant term 0.

The concept of 'like' terms really amounts to finding terms which can be combined using the Distributive Property. For example, in the polynomial $17x^2y-$

$3xy^2 + 7xy^2$, $-3xy^2$ and $7xy^2$ are like terms, since they have the same variables with the same corresponding exponents. This allows us to combine these two terms as follows:

$$17x^2y - 3xy^2 + 7xy^2 = 17x^2y + (-3)xy^2 + 7xy^2 + 17x^2y + (-3+7)xy^2 = 17x^2y + 4xy^2$$

Note that even though $17x^2y$ and $4xy^2$ have the same variables, they are not like terms since in the first term we have x^2 and $y = y^1$ but in the second we have $x = x^1$ and $y = y^2$ so the corresponding exponents aren't the same. Hence, $17x^2y + 4xy^2$ is the simplified form of the polynomial.

There are four basic operations we can perform with polynomials: addition, subtraction, multiplication and division. The first three of these operations follow directly from properties of real number arithmetic and will be discussed together first. Division, on the other hand, is a bit more complicated and will be discussed separately.

3.2.3 Polynomial Addition, Subtraction and Multiplication.

Adding and subtracting polynomials comes down to identifying like terms and then adding or subtracting the coefficients of those like terms. Multiplying polynomials comes to us courtesy of the Generalized Distributive Property.

Theorem 3.2.4 Generalized Distributive Property

To multiply a quantity of n terms by a quantity of m terms, multiply each of the n terms of the first quantity by each of the m terms in the second quantity and add the resulting $n \cdot m$ terms together.

In particular, Theorem 3.2.4 says that, before combining like terms, a product of an n-term polynomial and an m-term polynomial will generate $(n \cdot m)$-terms. For example, a binomial times a trinomial will produce six terms some of which may be like terms. Thus the simplified end result may have fewer than six terms but you will start with six terms.

A special case of Theorem 3.2.4 is the famous **F.O.I.L.**, listed here:

Key Idea 3.2.4 F.O.I.L:

The terms generated from the product of two binomials: $(a + b)(c + d)$ can be verbalized as follows "Take the sum of:

- the product of the **F**irst terms a and c, ac
- the product of the **O**uter terms a and d, ad
- the product of the **I**nner terms b and c, bc
- the product of the **L**ast terms b and d, bd."

That is, $(a + b)(c + d) = ac + ad + bc + bd$.

We caved to peer pressure on this one. Apparently all of the cool Precalculus books have FOIL in them even though it's redundant once you know how to distribute multiplication across addition. In general, we don't like mechanical shortcuts that interfere with a student's understanding of the material and FOIL is one of the worst.

Theorem 3.2.4 is best proved using the technique known as Mathematical Induction which is covered in Math 2000. The result is really nothing more than repeated applications of the Distributive Property so it seems reasonable and we'll use it without proof for now. The other major piece of polynomial multiplication is the law of exponents $a^n a^m = a^{n+m}$. The Commutative and Associative Properties of addition and multiplication are also used extensively. We put all of these properties to good use in the next example.

Example 3.2.6 Addition and subtraction of polynomials
Perform the indicated operations and simplify.

1. $\left(3x^2 - 2x + 1\right) - (7x - 3)$
2. $4xz^2 - 3z(xz - x + 4)$
3. $(2t + 1)(3t - 7)$
4. $\left(3y - \sqrt[3]{2}\right)\left(9y^2 + 3\sqrt[3]{2}y + \sqrt[3]{4}\right)$

Solution

1. We begin 'distributing the negative', then we rearrange and combine like terms:

$$\begin{aligned}\left(3x^2 - 2x + 1\right) - (7x - 3) &= 3x^2 - 2x + 1 - 7x + 3 && \text{Distribute}\\ &= 3x^2 - 2x - 7x + 1 + 3 && \text{Rearrange terms}\\ &= 3x^2 - 9x + 4 && \text{Combine like terms}\end{aligned}$$

Our answer is $3x^2 - 9x + 4$.

2. Following in our footsteps from the previous example, we first distribute the $-3z$ through, then rearrange and combine like terms.

$$\begin{aligned}4xz^2 - 3z(xz - x + 4) &= 4xz^2 - 3z(xz) + 3z(x) - 3z(4) && \text{Distribute}\\ &= 4xz^2 - 3xz^2 + 3xz - 12z && \text{Multiply}\\ &= xz^2 + 3xz - 12z && \text{Combine like terms}\end{aligned}$$

We get our final answer: $xz^2 + 3xz - 12z$

3. At last, we have a chance to use our F.O.I.L. technique:

$$\begin{aligned}(2t + 1)(3t - 7) &= (2t)(3t) + (2t)(-7) + (1)(3t) + (1)(-7) && \text{F.O.I.L.}\\ &= 6t^2 - 14t + 3t - 7 && \text{Multiply}\\ &= 6t^2 - 11t - 7 && \text{Combine like terms}\end{aligned}$$

We get $6t^2 - 11t - 7$ as our final answer.

4. We use the Generalized Distributive Property here, multiplying each term in the second quantity first by $3y$, then by $-\sqrt[3]{2}$:

$$\begin{aligned}\left(3y - \sqrt[3]{2}\right)\left(9y^2 + 3\sqrt[3]{2}y + \sqrt[3]{4}\right) &= 3y\left(9y^2\right) + 3y\left(3\sqrt[3]{2}y\right) + 3y\left(\sqrt[3]{4}\right)\\ &\quad - \sqrt[3]{2}\left(9y^2\right) - \sqrt[3]{2}\left(3\sqrt[3]{2}y\right) - \sqrt[3]{2}\left(\sqrt[3]{4}\right)\\ &= 27y^3 + 9y^2\sqrt[3]{2} - 9y^2\sqrt[3]{2} + 3y\sqrt[3]{4} - 3y\sqrt[3]{4} - 2\\ &= 27y^3 - 2\end{aligned}$$

To our surprise and delight, this product reduces to $27y^3 - 2$.

We conclude our discussion of polynomial multiplication by showcasing two special products which happen often enough they should be committed to memory.

> **Key Idea 3.2.5 Special Products**
>
> Let a and b be real numbers:
>
> - **Perfect Square:** $(a+b)^2 = a^2 + 2ab + b^2$ and $(a-b)^2 = a^2 - 2ab + b^2$
> - **Difference of Two Squares:** $(a-b)(a+b) = a^2 - b^2$

The formulas in Theorem 3.2.5 can be verified by working through the multiplication. (These are both special cases of F.O.I.L.)

3.2.4 Polynomial Long Division.

We now turn our attention to polynomial long division. Dividing two polynomials follows the same algorithm, in principle, as dividing two natural numbers so we review that process first. Suppose we wished to divide 2585 by 79. The standard division tableau is given below.

$$\begin{array}{r} 32 \\ 79\overline{)2585} \\ -\underline{2\,37}\downarrow \\ 215 \\ -\underline{158} \\ 57 \end{array}$$

In this case, 79 is called the **divisor**, 2585 is called the **dividend**, 32 is called the **quotient** and 57 is called the **remainder**. We can check our answer by showing:

$$\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$$

or in this case, $2585 = (79)(32) + 57\checkmark$. We hope that the long division tableau evokes warm, fuzzy memories of your formative years as opposed to feelings of hopelessness and frustration. If you experience the latter, keep in mind that the Division Algorithm essentially is a two-step process, iterated over and over again. First, we guess the number of times the divisor goes into the dividend and then we subtract off our guess. We repeat those steps with what's left over until what's left over (the remainder) is less than what we started with (the divisor). That's all there is to it!

The division algorithm for polynomials has the same basic two steps but when we subtract polynomials, we must take care to subtract *like terms* only. As a transition to polynomial division, let's write out our previous division tableau in expanded form.

$$\begin{array}{rl} & 3\cdot 10+2 \\ 7\cdot 10+9 \overline{)\, 2\cdot 10^3+5\cdot 10^2+8\cdot 10+5} \\ & -\underline{(2\cdot 10^3+3\cdot 10^2+7\cdot 10)} \quad \downarrow \\ & 2\cdot 10^2+1\cdot 10+5 \\ & -\underline{(1\cdot 10^2+5\cdot 10+8)} \\ & 5\cdot 10+7 \end{array}$$

Written this way, we see that when we line up the digits we are really lining up the coefficients of the corresponding powers of 10 - much like how we'll have to keep the powers of x lined up in the same columns. The big difference between polynomial division and the division of natural numbers is that the value of x is an unknown quantity. So unlike using the known value of 10, when we subtract there can be no regrouping of coefficients as in our previous example. (The subtraction $215 - 158$ requires us to 'regroup' or 'borrow' from the tens digit, then the hundreds digit.) This actually makes polynomial division easier. (In our opinion - you can judge for yourself.) Before we dive into examples, we first note that for any polynomial functions $d(x)$ and $p(x)$ such that the degree of p is greater than or equal to the degree of d, there exist unique polynomial functions $q(x)$ and $r(x)$ such that

$$p(x) = d(x)q(x) + r(x),$$

and either $r(x) = 0$, or the degree of r is less than the degree of d. This result tells us that we can divide polynomials whenever the degree of the divisor is less than or equal to the degree of the dividend. We know we're done with the division when the polynomial left over (the remainder) has a degree strictly less than the divisor. It's time to walk through a few examples to refresh your memory.

Example 3.2.7 Polynomial long division

Perform the indicated division. Check your answer by showing

$$\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$$

1. $\left(x^3 + 4x^2 - 5x - 14\right) \div (x - 2)$
2. $(2t + 7) \div (3t - 4)$
3. $\left(6y^2 - 1\right) \div (2y + 5)$
4. $\left(w^3\right) \div \left(w^2 - \sqrt{2}\right)$.

Solution

1. To begin $\left(x^3 + 4x^2 - 5x - 14\right) \div (x - 2)$, we divide the first term in the dividend, namely x^3, by the first term in the divisor, namely x, and get $\frac{x^3}{x} = x^2$. This then becomes the first term in the quotient. We proceed as in regular long division at this point: we multiply the entire divisor, $x - 2$, by this first term in the quotient to get $x^2(x - 2) = x^3 - 2x^2$. We then subtract this result from the dividend.

$$\begin{array}{rl} & x^2 \\ x-2 \overline{)\, x^3+4x^2-5x-14} \\ & -\underline{(x^3-2x^2)} \quad \downarrow \\ & 6x^2-5x \end{array}$$

Now we 'bring down' the next term of the quotient, namely $-5x$, and repeat the process. We divide $\frac{6x^2}{x} = 6x$, and add this to the quotient polynomial, multiply it by the divisor (which yields $6x(x-2) = 6x^2 - 12x$) and subtract.

$$\begin{array}{rrrrr}
 & x^2 & +\,6x & & \\ \cline{2-5}
x-2 \big) & x^3 & +\,4x^2 & -\,5x & -14 \\
-\big(x^3 & -\,2x^2\big) & & & \downarrow \\ \cline{1-2}
 & 6x^2 & -\,5x & & \downarrow \\
 & -\big(6x^2 & -\,12x\big) & & \downarrow \\ \cline{2-3}
 & & 7x & -14 &
\end{array}$$

Finally, we 'bring down' the last term of the dividend, namely -14, and repeat the process. We divide $\frac{7x}{x} = 7$, add this to the quotient, multiply it by the divisor (which yields $7(x-2) = 7x - 14$) and subtract.

$$\begin{array}{rrrrr}
 & x^2 & +\,6x & +\,7 & \\ \cline{2-5}
x-2 \big) & x^3 & +\,4x^2 & -\,5x & -\,14 \\
-\big(x^3 & -\,2x^2\big) & & & \\ \cline{1-2}
 & 6x^2 & -\,5x & & \\
 & -\big(6x^2 & -\,12x\big) & & \\ \cline{2-3}
 & & 7x & -\,14 & \\
 & & -\,(7x & -\,14) & \\ \cline{3-4}
 & & & 0 &
\end{array}$$

In this case, we get a quotient of $x^2 + 6x + 7$ with a remainder of 0. To check our answer, we compute

$$(x-2)\left(x^2 + 6x + 7\right) + 0 = x^3 + 6x^2 + 7x - 2x^2 - 12x - 14 = x^3 + 4x^2 - 5x - 14\ \checkmark$$

2. To compute $(2t+7) \div (3t-4)$, we start as before. We find $\frac{2t}{3t} = \frac{2}{3}$, so that becomes the first (and only) term in the quotient. We multiply the divisor $(3t-4)$ by $\frac{2}{3}$ and get $2t - \frac{8}{3}$. We subtract this from the divided and get $\frac{29}{3}$.

$$\begin{array}{rrr}
 & & \frac{2}{3} \\ \cline{2-3}
3t-4 \big) & 2t & +\ 7 \\
- & \left(2t - \frac{8}{3}\right) & \\ \cline{2-3}
 & & \frac{29}{3}
\end{array}$$

Our answer is $\frac{2}{3}$ with a remainder of $\frac{29}{3}$. To check our answer, we compute

$$(3t-4)\left(\frac{2}{3}\right) + \frac{29}{3} = 2t - \frac{8}{3} + \frac{29}{3} = 2t + \frac{21}{3} = 2t + 7\ \checkmark$$

3. When we set-up the tableau for $\left(6y^2 - 1\right) \div (2y+5)$, we must first issue a 'placeholder' for the 'missing' y-term in the dividend, $6y^2 - 1 = 6y^2 + 0y - 1$. We then proceed as before. Since $\frac{6y^2}{2y} = 3y$, $3y$ is the first term

in our quotient. We multiply $(2y+5)$ times $3y$ and subtract it from the dividend. We bring down the -1, and repeat.

$$\begin{array}{r|l} & 3y - \frac{15}{2} \\ \hline 2y+5 & 6y^2 + 0y - 1 \\ & -(6y^2 + 15y) \quad \downarrow \\ \hline & \quad -15y - 1 \\ & -\left(-15y - \frac{75}{2}\right) \\ \hline & \qquad \frac{73}{2} \end{array}$$

Our answer is $3y - \frac{15}{2}$ with a remainder of $\frac{73}{2}$. To check our answer, we compute:

$$(2y+5)\left(3y - \frac{15}{2}\right) + \frac{73}{2} = 6y^2 - 15y + 15y - \frac{75}{2} + \frac{73}{2} = 6y^2 - 1 \checkmark$$

4. For our last example, we need 'placeholders' for both the divisor $w^2 - \sqrt{2} = w^2 + 0w - \sqrt{2}$ and the dividend $w^3 = w^3 + 0w^2 + 0w + 0$. The first term in the quotient is $\frac{w^3}{w^2} = w$, and when we multiply and subtract this from the dividend, we're left with just $0w^2 + w\sqrt{2} + 0 = w\sqrt{2}$.

$$\begin{array}{r|l} & w \\ \hline w^2+0w-\sqrt{2} & w^3 + 0w^2 + 0w + 0 \\ & -\left(w^3 + 0w^2 - w\sqrt{2}\right) \quad \downarrow \\ \hline & \qquad 0w^2 + w\sqrt{2} + 0 \end{array}$$

Since the degree of $w\sqrt{2}$ (which is 1) is less than the degree of the divisor (which is 2), we are done. Our answer is w with a remainder of $w\sqrt{2}$. To check, we compute:

$$\left(w^2 - \sqrt{2}\right) w + w\sqrt{2} = w^3 - w\sqrt{2} + w\sqrt{2} = w^3 \checkmark$$

Exercises 3.2

Problems

In Exercises 1 – 10, solve the inequality. Write your answer using interval notation.

1. $f(x) = 4 - x - 3x^2$
2. $g(x) = 3x^5 - 2x^2 + x + 1$
3. $q(r) = 1 - 16r^4$
4. $Z(b) = 42b - b^3$
5. $f(x) = \sqrt{3}x^{17} + 22.5x^{10} - \pi x^7 + \frac{1}{3}$
6. $s(t) = -4.9t^2 + v_0t + s_0$
7. $P(x) = (x-1)(x-2)(x-3)(x-4)$
8. $p(t) = -t^2(3-5t)(t^2+t+4)$
9. $f(x) = -2x^3(x+1)(x+2)^2$
10. $G(t) = 4(t-2)^2\left(t+\frac{1}{2}\right)$

In Exercises 11 – 20, find the real zeros of the given polynomial and their corresponding multiplicities. Use this information along with a sign chart to provide a rough sketch of the graph of the polynomial. Compare your answer with the result from a graphing utility.

11. $a(x) = x(x+2)^2$
12. $g(x) = x(x+2)^3$
13. $f(x) = -2(x-2)^2(x+1)$
14. $g(x) = (2x+1)^2(x-3)$
15. $F(x) = x^3(x+2)^2$
16. $P(x) = (x-1)(x-2)(x-3)(x-4)$
17. $Q(x) = (x+5)^2(x-3)^4$
18. $h(x) = x^2(x-2)^2(x+2)^2$
19. $H(t) = (3-t)(t^2+1)$
20. $Z(b) = b(42-b^2)$
21. Here are a few other questions for you to discuss with your classmates.
    (a) How many local extrema could a polynomial of degree n have? How few local extrema can it have?
    (b) Could a polynomial have two local maxima but no local minima?
    (c) If a polynomial has two local maxima and two local minima, can it be of odd degree? Can it be of even degree?
    (d) Can a polynomial have local extrema without having any real zeros?
    (e) Why must every polynomial of odd degree have at least one real zero?
    (f) Can a polynomial have two distinct real zeros and no local extrema?
    (g) Can an x-intercept yield a local extrema? Can it yield an absolute extrema?
    (h) If the y-intercept yields an absolute minimum, what can we say about the degree of the polynomial and the sign of the leading coefficient?

In Exercises 22 – 36, perform the indicated operations and simplify.

22. $(4-3x)+(3x^2+2x+7)$
23. $t^2+4t-2(3-t)$
24. $q(200-3q)-(5q+500)$
25. $(3y-1)(2y+1)$
26. $\left(3-\dfrac{x}{2}\right)(2x+5)$
27. $-(4t+3)(t^2-2)$
28. $2w(w^3-5)(w^3+5)$
29. $(5a^2-3)(25a^4+15a^2+9)$
30. $(x^2-2x+3)(x^2+2x+3)$
31. $(\sqrt{7}-z)(\sqrt{7}+z)$
32. $(x-\sqrt[3]{5})^3$
33. $(x-\sqrt[3]{5})(x^2+x\sqrt[3]{5}+\sqrt[3]{25})$
34. $(w-3)^2-(w^2+9)$
35. $(x+h)^2-2(x+h)-(x^2-2x)$
36. $(x-[2+\sqrt{5}])(x-[2-\sqrt{5}])$

In Exercises 37 – 48, perform the indicated operations and simplify.

37. $(5x^2-3x+1)\div(x+1)$
38. $(3y^2+6y-7)\div(y-3)$

39. $(6w - 3) \div (2w + 5)$

40. $(2x + 1) \div (3x - 4)$

41. $(t^2 - 4) \div (2t + 1)$

42. $(w^3 - 8) \div (5w - 10)$

43. $(2x^2 - x + 1) \div (3x^2 + 1)$

44. $(4y^4 + 3y^2 + 1) \div (2y^2 - y + 1)$

45. $w^4 \div (w^3 - 2)$

46. $(5t^3 - t + 1) \div (t^2 + 4)$

47. $(t^3 - 4) \div (t - \sqrt[3]{4})$

48. **Perfect Cube:** $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

In Exercises 49 – 55, verify the given formula by showing the left hand side of the equation simplifies to the right hand side of the equation.

49. **Perfect Cube:** $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

50. **Difference of Cubes:** $(a - b)(a^2 + ab + b^2) = a^3 - b^3$

51. **Sum of Cubes:** $(a + b)(a^2 - ab + b^2) = a^3 + b^3$

52. **Perfect Quartic:** $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

53. **Difference of Quartics:** $(a - b)(a + b)(a^2 + b^2) = a^4 - b^4$

54. **Sum of Quartics:** $(a^2 + ab\sqrt{2} + b^2)(a^2 - ab\sqrt{2} + b^2) = a^4 + b^4$

3.3 Rational Functions

3.3.1 Introduction to Rational Functions

If we add, subtract or multiply polynomial functions according to the function arithmetic rules defined in Section 2.2.1, we will produce another polynomial function. If, on the other hand, we divide two polynomial functions, the result may not be a polynomial. In this chapter we study **rational functions** - functions which are ratios of polynomials.

Definition 3.3.1 Rational Function

A **rational function** is a function which is the ratio of polynomial functions. Said differently, r is a rational function if it is of the form

$$r(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomial functions.

According to Definition 3.3.1, all polynomial functions are also rational functions, since we can take $q(x) = 1$.

As we recall from Section 2.1, we have domain issues any time the denominator of a fraction is zero. In the example below, we review this concept as well as some of the arithmetic of rational expressions.

Example 3.3.1 Domain of rational functions

Find the domain of the following rational functions. Write them in the form $\frac{p(x)}{q(x)}$ for polynomial functions p and q and simplify.

1. $f(x) = \frac{2x-1}{x+1}$
2. $g(x) = 2 - \frac{3}{x+1}$
3. $h(x) = \frac{2x^2-1}{x^2-1} - \frac{3x-2}{x^2-1}$
4. $r(x) = \frac{2x^2-1}{x^2-1} \div \frac{3x-2}{x^2-1}$

SOLUTION

1. To find the domain of f, we proceed as we did in Section 2.1: we find the zeros of the denominator and exclude them from the domain. Setting $x+1=0$ results in $x=-1$. Hence, our domain is $(-\infty,-1)\cup(-1,\infty)$. The expression $f(x)$ is already in the form requested and when we check for common factors among the numerator and denominator we find none, so we are done.

2. Proceeding as before, we determine the domain of g by solving $x+1=0$. As before, we find the domain of g is $(-\infty,-1)\cup(-1,\infty)$. To write $g(x)$ in the form requested, we need to get a common denominator

$$\begin{aligned} g(x) &= 2 - \frac{3}{x+1} = \frac{2}{1} - \frac{3}{x+1} = \frac{(2)(x+1)}{(1)(x+1)} - \frac{3}{x+1} \\ &= \frac{(2x+2)-3}{x+1} = \frac{2x-1}{x+1} \end{aligned}$$

This formula is now completely simplified.

3. The denominators in the formula for $h(x)$ are both $x^2 - 1$ whose zeros are $x = \pm 1$. As a result, the domain of h is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. We now proceed to simplify $h(x)$. Since we have the same denominator in both terms, we subtract the numerators. We then factor the resulting numerator and denominator, and cancel out the common factor.

$$\begin{aligned} h(x) &= \frac{2x^2-1}{x^2-1} - \frac{3x-2}{x^2-1} = \frac{\left(2x^2-1\right)-(3x-2)}{x^2-1} \\ &= \frac{2x^2-1-3x+2}{x^2-1} = \frac{2x^2-3x+1}{x^2-1} \\ &= \frac{(2x-1)(x-1)}{(x+1)(x-1)} = \frac{(2x-1)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} \\ &= \frac{2x-1}{x+1} \end{aligned}$$

4. To find the domain of r, it may help to temporarily rewrite $r(x)$ as

$$r(x) = \frac{\dfrac{2x^2-1}{x^2-1}}{\dfrac{3x-2}{x^2-1}}$$

We need to set all of the denominators equal to zero which means we need to solve not only $x^2 - 1 = 0$, but also $\dfrac{3x-2}{x^2-1} = 0$. We find $x = \pm 1$ for the former and $x = \frac{2}{3}$ for the latter. Our domain is $(-\infty, -1) \cup \left(-1, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 1\right) \cup (1, \infty)$. We simplify $r(x)$ by rewriting the division as multiplication by the reciprocal and then by cancelling the common factor

$$\begin{aligned} r(x) &= \frac{2x^2-1}{x^2-1} \div \frac{3x-2}{x^2-1} = \frac{2x^2-1}{x^2-1} \cdot \frac{x^2-1}{3x-2} \\ &= \frac{\left(2x^2-1\right)\left(x^2-1\right)}{\left(x^2-1\right)(3x-2)} = \frac{\left(2x^2-1\right)\cancel{\left(x^2-1\right)}}{\cancel{\left(x^2-1\right)}(3x-2)} \\ &= \frac{2x^2-1}{3x-2} \end{aligned}$$

In Example 3.3.1, note that the expressions for $f(x)$, $g(x)$ and $h(x)$ work out to be the same. However, only two of these functions are actually equal. For two functions to be equal, they need, among other things, to have the same domain. Since $f(x) = g(x)$ and f and g have the same domain, they are equal functions. Even though the formula $h(x)$ is the same as $f(x)$, the domain of h is different than the domain of f, and thus they are different functions.

We now turn our attention to the graphs of rational functions. Consider the function $f(x) = \dfrac{2x-1}{x+1}$ from Example 3.3.1. Using GeoGebra, we obtain the graph in Figure 3.3.1

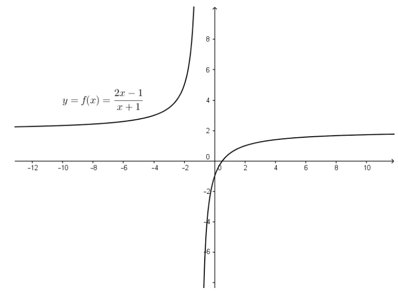


Figure 3.3.1: The graph of $f(x) = \dfrac{2x-1}{x+1}$

Two behaviours of the graph are worthy of further discussion. First, note that the graph appears to 'break' at $x = -1$. We know from our last example that $x = -1$ is not in the domain of f which means $f(-1)$ is undefined. When we make a table of values to study the behaviour of f *near* $x = -1$ we see that we can get 'near' $x = -1$ from two directions. We can choose values a little less than -1, for example $x = -1.1$, $x = -1.01$, $x = -1.001$, and so on. These values are said to 'approach -1 from the *left*.' Similarly, the values $x = -0.9$, $x = -0.99$, $x = -0.999$, etc., are said to 'approach -1 from the *right*.' If we make the two tables in Figure 3.3.2, we find that the numerical results confirm what we see graphically.

x	$f(x)$	$(x, f(x))$
-1.1	32	$(-1.1, 32)$
-1.01	302	$(-1.01, 302)$
-1.001	3002	$(-1.001, 3002)$
-1.0001	30002	$(-1.001, 30002)$

x	$f(x)$	$(x, f(x))$
-0.9	-28	$(-0.9, -28)$
-0.99	-298	$(-0.99, -298)$
-0.999	-2998	$(-0.999, -2998)$
-0.9999	-29998	$(-0.9999, -29998)$

Figure 3.3.2: Values of $f(x) = \frac{2x-1}{x+1}$ near $x = -1$

As the x values approach -1 from the left, the function values become larger and larger positive numbers. (We would need Calculus to confirm this analytically.) We express this symbolically by stating as $x \to -1^-$, $f(x) \to \infty$. Similarly, using analogous notation, we conclude from the table that as $x \to -1^+$, $f(x) \to -\infty$. For this type of unbounded behaviour, we say the graph of $y = f(x)$ has a **vertical asymptote** of $x = -1$. Roughly speaking, this means that near $x = -1$, the graph looks very much like the vertical line $x = -1$.

The other feature worthy of note about the graph of $y = f(x)$ is that it seems to 'level off' on the left and right hand sides of the screen. This is a statement about the end behaviour of the function. As we discussed in Section 3.2.1, the end behaviour of a function is its behaviour as x attains larger and larger negative values without bound (here, the word 'larger' means larger in absolute value), $x \to -\infty$, and as x becomes large without bound, $x \to \infty$.

From the tables in Figure 3.3.3, we see that as $x \to -\infty$, $f(x) \to 2^+$ and as $x \to \infty$, $f(x) \to 2^-$. Here the '$+$' means 'from above' and the '$-$' means 'from below'. In this case, we say the graph of $y = f(x)$ has a **horizontal asymptote** of $y = 2$. This means that the end behaviour of f resembles the horizontal line $y = 2$, which explains the 'levelling off' behaviour we see in Figure 3.3.1. We formalize the concepts of vertical and horizontal asymptotes in the following definitions.

x	$f(x) \approx$	$(x, f(x)) \approx$
-10	2.3333	$(-10, 2.3333)$
-100	2.0303	$(-100, 2.0303)$
-1000	2.0030	$(-1000, 2.0030)$
-10000	2.0003	$(-10000, 2.0003)$

x	$f(x) \approx$	$(x, f(x)) \approx$
10	1.7273	$(10, 1.7273)$
100	1.9703	$(100, 1.9703)$
1000	1.9970	$(1000, 1.9970)$
10000	1.9997	$(10000, 1.9997)$

Figure 3.3.3: Values of $f(x) = \dfrac{2x-1}{x+1}$ for large negative and positive values of x

Definition 3.3.2 Vertical Asymptote

The line $x = c$ is called a **vertical asymptote** of the graph of a function $y = f(x)$ if as $x \to c^-$ or as $x \to c^+$, either $f(x) \to \infty$ or $f(x) \to -\infty$.

Definition 3.3.3 Horizontal Asymptote

The line $y = c$ is called a **horizontal asymptote** of the graph of a function $y = f(x)$ if as $x \to -\infty$ or as $x \to \infty$, $f(x) \to c$.

Note that in Definition 3.3.3, we write $f(x) \to c$ (not $f(x) \to c^+$ or $f(x) \to c^-$) because we are unconcerned from which direction the values $f(x)$ approach the value c, just as long as they do so.

In our discussion following Example 3.3.1, we determined that, despite the fact that the formula for $h(x)$ reduced to the same formula as $f(x)$, the functions

x	$h(x) \approx$	$(x, h(x)) \approx$
0.9	0.4210	$(0.9, 0.4210)$
0.99	0.4925	$(0.99, 0.4925)$
0.999	0.4992	$(0.999, 0.4992)$
0.9999	0.4999	$(0.9999, 0.4999)$

x	$h(x) \approx$	$(x, h(x)) \approx$
1.1	0.5714	$(1.1, 0.5714)$
1.01	0.5075	$(1.01, 0.5075)$
1.001	0.5007	$(1.001, 0.5007)$
1.0001	0.5001	$(1.0001, 0.5001)$

Figure 3.3.4: Values of $h(x) = \frac{2x^2-1}{x^2-1} - \frac{3x-2}{x^2-1}$ near $x = 1$

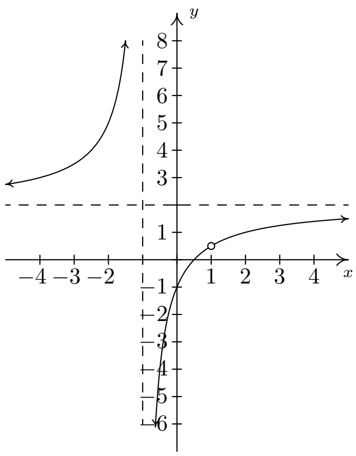


Figure 3.3.5: The graph $y = h(x)$ showing asymptotes and the 'hole'

f and h are different, since $x = 1$ is in the domain of f, but $x = 1$ is not in the domain of h. If we graph $h(x) = \frac{2x^2-1}{x^2-1} - \frac{3x-2}{x^2-1}$ using a graphing calculator, we are surprised to find that the graph looks identical to the graph of $y = f(x)$. There is a vertical asymptote at $x = -1$, but near $x = 1$, everything seem fine. Tables of values provide numerical evidence which supports the graphical observation: see Figure 3.3.4.

We see that as $x \to 1^-$, $h(x) \to 0.5^-$ and as $x \to 1^+$, $h(x) \to 0.5^+$. In other words, the points on the graph of $y = h(x)$ are approaching $(1, 0.5)$, but since $x = 1$ is not in the domain of h, it would be inaccurate to fill in a point at $(1, 0.5)$. To indicate this, we put an open circle (also called a **hole** in this case) at $(1, 0.5)$. Figure 3.3.5 is a detailed graph of $y = h(x)$, with the vertical and horizontal asymptotes as dashed lines.

Neither $x = -1$ nor $x = 1$ are in the domain of h, yet the behaviour of the graph of $y = h(x)$ is drastically different near these x-values. The reason for this lies in the second to last step when we simplified the formula for $h(x)$ in Example 3.3.1, where we had $h(x) = \frac{(2x-1)(x-1)}{(x+1)(x-1)}$. The reason $x = -1$ is not in the domain of h is because the factor $(x+1)$ appears in the denominator of $h(x)$; similarly, $x = 1$ is not in the domain of h because of the factor $(x-1)$ in the denominator of $h(x)$. The major difference between these two factors is that $(x-1)$ cancels with a factor in the numerator whereas $(x+1)$ does not. Loosely speaking, the trouble caused by $(x-1)$ in the denominator is cancelled away while the factor $(x+1)$ remains to cause mischief. This is why the graph of $y = h(x)$ has a vertical asymptote at $x = -1$ but only a hole at $x = 1$. These observations are generalized and summarized in the theorem below, whose proof is found in Calculus.

In Calculus, we will see how these 'holes' in graphs can be 'plugged' once we've made a more advanced study of continuity.

Theorem 3.3.1 Location of Vertical Asymptotes and Holes

Suppose r is a rational function which can be written as $r(x) = \frac{p(x)}{q(x)}$ where p and q have no common zeros (in other words, $r(x)$ is in lowest terms). Let c be a real number which is not in the domain of r.

- If $q(c) \neq 0$, then the graph of $y = r(x)$ has a hole at $\left(c, \frac{p(c)}{q(c)}\right)$.
- If $q(c) = 0$, then the line $x = c$ is a vertical asymptote of the graph of $y = r(x)$.

In English, Theorem 3.3.1 says that if $x = c$ is not in the domain of r but, when we simplify $r(x)$, it no longer makes the denominator 0, then we have a hole at $x = c$. Otherwise, the line $x = c$ is a vertical asymptote of the graph of $y = r(x)$. In other words, Theorem 3.3.1 tells us 'How to tell your asymptote from a hole in the graph.'

Example 3.3.2 Finding vertical asymptotes

Find the vertical asymptotes of, and/or holes in, the graphs of the following rational functions. Verify your answers using software or a graphing calculator, and describe the behaviour of the graph near them using proper notation.

1. $f(x) = \dfrac{2x}{x^2-3}$
2. $g(x) = \dfrac{x^2-x-6}{x^2-9}$
3. $h(x) = \dfrac{x^2-x-6}{x^2+9}$
4. $r(x) = \dfrac{x^2-x-6}{x^2+4x+4}$

Solution

1. To use Theorem 3.3.1, we first find all of the real numbers which aren't in the domain of f. To do so, we solve $x^2-3=0$ and get $x=\pm\sqrt{3}$. Since the expression $f(x)$ is in lowest terms, there is no cancellation possible, and we conclude that the lines $x=-\sqrt{3}$ and $x=\sqrt{3}$ are vertical asymptotes to the graph of $y=f(x)$. Plotting the function in GeoGebra verifies this claim, and from the graph in Figure 3.3.6, we see that as $x\to-\sqrt{3}^-$, $f(x)\to-\infty$, as $x\to-\sqrt{3}^+$, $f(x)\to\infty$, as $x\to\sqrt{3}^-$, $f(x)\to-\infty$, and finally as $x\to\sqrt{3}^+$, $f(x)\to\infty$.

2. Solving $x^2-9=0$ gives $x=\pm3$. In lowest terms $g(x)=\dfrac{x^2-x-6}{x^2-9}=\dfrac{(x-3)(x+2)}{(x-3)(x+3)}=\dfrac{x+2}{x+3}$. Since $x=-3$ continues to make trouble in the denominator, we know the line $x=-3$ is a vertical asymptote of the graph of $y=g(x)$. Since $x=3$ no longer produces a 0 in the denominator, we have a hole at $x=3$. To find the y-coordinate of the hole, we substitute $x=3$ into $\dfrac{x+2}{x+3}$ and find the hole is at $\left(3,\frac{5}{6}\right)$. When we graph $y=g(x)$ using GeoGebra, we clearly see the vertical asymptote at $x=-3$, but everything seems calm near $x=3$: see Figure 3.3.7. Hence, as $x\to-3^-$, $g(x)\to\infty$, as $x\to-3^+$, $g(x)\to-\infty$, as $x\to3^-$, $g(x)\to\frac{5}{6}^-$, and as $x\to3^+$, $g(x)\to\frac{5}{6}^+$.

3. The domain of h is all real numbers, since $x^2+9=0$ has no real solutions. Accordingly, the graph of $y=h(x)$ is devoid of both vertical asymptotes and holes, as see in Figure 3.3.8.

4. Setting $x^2+4x+4=0$ gives us $x=-2$ as the only real number of concern. Simplifying, we see $r(x)=\dfrac{x^2-x-6}{x^2+4x+4}=\dfrac{(x-3)(x+2)}{(x+2)^2}=\dfrac{x-3}{x+2}$. Since $x=-2$ continues to produce a 0 in the denominator of the reduced function, we know $x=-2$ is a vertical asymptote to the graph. The graph in Figure 3.3.9 bears this out, and, moreover, we see that as $x\to-2^-$, $r(x)\to\infty$ and as $x\to-2^+$, $r(x)\to-\infty$.

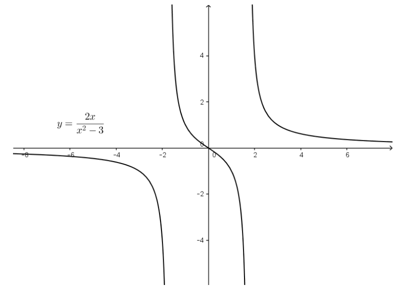


Figure 3.3.6: The graph $y=f(x)$ in Example 3.3.2

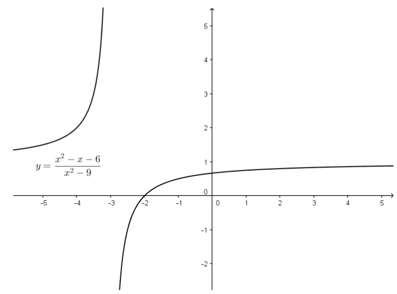


Figure 3.3.7: The graph $y=g(x)$ in Example 3.3.2

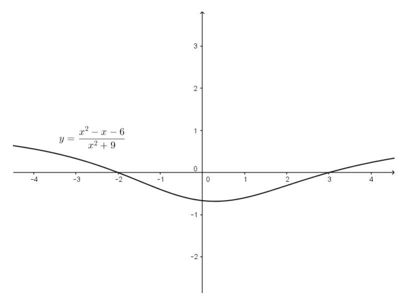


Figure 3.3.8: The graph $y=h(x)$ in Example 3.3.2

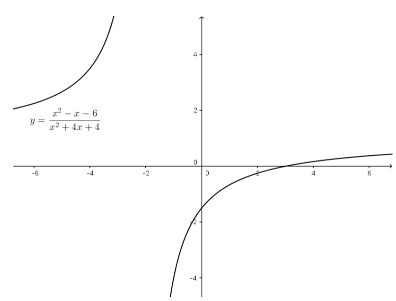


Figure 3.3.9: The graph $y=r(x)$ in Example 3.3.2

Now that we have thoroughly investigated vertical asymptotes, we can turn our attention to horizontal asymptotes. The next theorem tells us when to expect horizontal asymptotes.

Theorem 3.3.2 Location of Horizontal Asymptotes

Suppose r is a rational function and $r(x)=\dfrac{p(x)}{q(x)}$, where p and q are polynomial functions with leading coefficients a and b, respectively.

- If the degree of $p(x)$ is the same as the degree of $q(x)$, then $y=\frac{a}{b}$ is the horizontal asymptote of the graph of $y=r(x)$.
- If the degree of $p(x)$ is less than the degree of $q(x)$, then $y=0$ is the horizontal asymptote of the graph of $y=r(x)$.
- If the degree of $p(x)$ is greater than the degree of $q(x)$, then the graph of $y=r(x)$ has no horizontal asymptotes.

More specifically, as $x \to -\infty$, $f(x) \to 2^+$, and as $x \to \infty$, $f(x) \to 2^-$. Notice that the graph gets close to the same y value as $x \to -\infty$ or $x \to \infty$. This means that the graph can have only one horizontal asymptote if it is going to have one at all. Thus we were justified in using 'the' in the previous theorem.

Like Theorem 3.3.1, Theorem 3.3.2 is proved using Calculus. Nevertheless, we can understand the idea behind it using our example $f(x) = \dfrac{2x-1}{x+1}$. If we interpret $f(x)$ as a division problem, $(2x-1) \div (x+1)$, we find that the quotient is 2 with a remainder of -3. Using what we know about polynomial division, we get $2x-1 = 2(x+1)-3$. Dividing both sides by $(x+1)$ gives $\dfrac{2x-1}{x+1} = 2 - \dfrac{3}{x+1}$. As x becomes unbounded in either direction, the quantity $\dfrac{3}{x+1}$ gets closer and closer to 0 so that the values of $f(x)$ become closer and closer (as seen in the tables in Figure 3.3.3) to 2. In symbols, as $x \to \pm\infty$, $f(x) \to 2$, and we have the result.

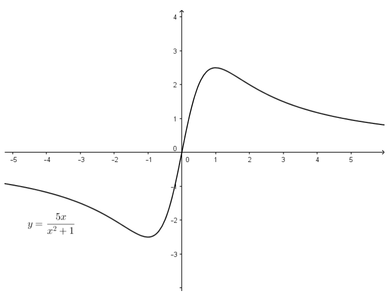
Example 3.3.3 Finding horizontal asymptotes

List the horizontal asymptotes, if any, of the graphs of the following functions. Verify your answers using a graphing calculator, and describe the behaviour of the graph near them using proper notation.

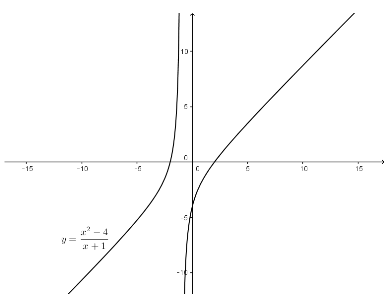
1. $f(x) = \dfrac{5x}{x^2+1}$
2. $g(x) = \dfrac{x^2-4}{x+1}$
3. $h(x) = \dfrac{6x^3-3x+1}{5-2x^3}$

SOLUTION

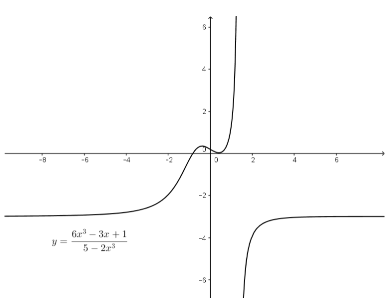
1. The numerator of $f(x)$ is $5x$, which has degree 1. The denominator of $f(x)$ is x^2+1, which has degree 2. Applying Theorem 3.3.2, $y = 0$ is the horizontal asymptote. Sure enough, we see from the graph that as $x \to -\infty$, $f(x) \to 0^-$ and as $x \to \infty$, $f(x) \to 0^+$.
2. The numerator of $g(x)$, x^2-4, has degree 2, but the degree of the denominator, $x+1$, has degree 1. By Theorem 3.3.2, there is no horizontal asymptote. From the graph, we see that the graph of $y = g(x)$ doesn't appear to level off to a constant value, so there is no horizontal asymptote. (Sit tight! We'll revisit this function and its end behaviour shortly.)
3. The degrees of the numerator and denominator of $h(x)$ are both three, so Theorem 3.3.2 tells us $y = \frac{6}{-2} = -3$ is the horizontal asymptote. We see from the calculator's graph that as $x \to -\infty$, $h(x) \to -3^+$, and as $x \to \infty$, $h(x) \to -3^-$.



$y = f(x)$



$y = g(x)$



$y = h(x)$

Figure 3.3.10: Graphs of the three functions in Example 3.3.3

We close this section with a discussion of the *third* (and final!) kind of asymptote which can be associated with the graphs of rational functions. Let us return to the function $g(x) = \frac{x^2-4}{x+1}$ in Example 3.3.3. Performing long division, (see the remarks following Theorem 3.3.2) we get $g(x) = \frac{x^2-4}{x+1} = x - 1 - \frac{3}{x+1}$. Since the term $\frac{3}{x+1} \to 0$ as $x \to \pm\infty$, it stands to reason that as x becomes unbounded, the function values $g(x) = x - 1 - \frac{3}{x+1} \approx x - 1$. Geometrically, this means that the graph of $y = g(x)$ should resemble the line $y = x - 1$ as $x \to \pm\infty$. We see this play out both numerically and graphically in Figures 3.3.11 and 3.3.12.

The way we symbolize the relationship between the end behaviour of $y = g(x)$ with that of the line $y = x - 1$ is to write 'as $x \to \pm\infty$, $g(x) \to x - 1$.' In this case, we say the line $y = x - 1$ is a **slant asymptote** (or 'oblique' asymptote) to the graph of $y = g(x)$. Informally, the graph of a rational function has a slant asymptote if, as $x \to \infty$ or as $x \to -\infty$, the graph resembles a non-horizontal, or 'slanted' line. Formally, we define a slant asymptote as follows.

x	$g(x)$	$x-1$
-10	≈ -10.6667	-11
-100	≈ -100.9697	-101
-1000	≈ -1000.9970	-1001
-10000	≈ -10000.9997	-10001

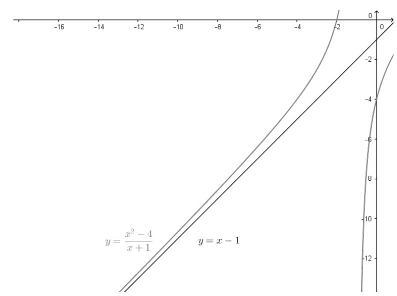


Figure 3.3.11: The graph $y = \frac{x^2-4}{x+1}$ as $x \to -\infty$

Definition 3.3.4 Slant Asymptote

The line $y = mx + b$ where $m \neq 0$ is called a **slant asymptote** of the graph of a function $y = f(x)$ if as $x \to -\infty$ or as $x \to \infty$, $f(x) \to mx + b$.

A few remarks are in order. First, note that the stipulation $m \neq 0$ in Definition 3.3.4 is what makes the 'slant' asymptote 'slanted' as opposed to the case when $m = 0$ in which case we'd have a horizontal asymptote. Secondly, while we have motivated what me mean intuitively by the notation '$f(x) \to mx + b$,' like so many ideas in this section, the formal definition requires Calculus. Another way to express this sentiment, however, is to rephrase '$f(x) \to mx + b$' as '$f(x) - (mx + b) \to 0$.' In other words, the graph of $y = f(x)$ has the *slant* asymptote $y = mx + b$ if and only if the graph of $y = f(x) - (mx + b)$ has a *horizontal* asymptote $y = 0$.

Our next task is to determine the conditions under which the graph of a rational function has a slant asymptote, and if it does, how to find it. In the case of $g(x) = \frac{x^2-4}{x+1}$, the degree of the numerator $x^2 - 4$ is 2, which is *exactly one more* than the degree if its denominator $x + 1$ which is 1. This results in a *linear* quotient polynomial, and it is this quotient polynomial which is the slant asymptote. Generalizing this situation gives us the following theorem.

x	$g(x)$	$x-1$
10	≈ 8.7273	9
100	≈ 98.9703	99
1000	≈ 998.9970	999
10000	≈ 9998.9997	9999

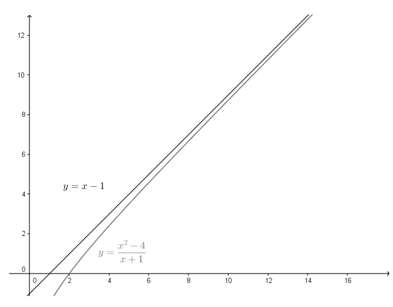


Figure 3.3.12: The graph $y = \frac{x^2-4}{x+1}$ as $x \to +\infty$

Theorem 3.3.3 Determination of Slant Asymptotes

Suppose r is a rational function and $r(x) = \frac{p(x)}{q(x)}$, where the degree of p is *exactly* one more than the degree of q. Then the graph of $y = r(x)$ has the slant asymptote $y = L(x)$ where $L(x)$ is the quotient obtained by dividing $p(x)$ by $q(x)$.

In the same way that Theorem 3.3.2 gives us an easy way to see if the graph of a rational function $r(x) = \dfrac{p(x)}{q(x)}$ has a horizontal asymptote by comparing the degrees of the numerator and denominator, Theorem 3.3.3 gives us an easy way to check for slant asymptotes. Unlike Theorem 3.3.2, which gives us a quick way to *find* the horizontal asymptotes (if any exist), Theorem 3.3.3 gives us no such 'short-cut'. If a slant asymptote exists, we have no recourse but to use long division to find it. (That's OK, though. In the next section, we'll use long division to analyze end behaviour and it's worth the effort!)

Example 3.3.4 Finding slant asymptotes

Find the slant asymptotes of the graphs of the following functions if they exist. Verify your answers using software or a graphing calculator and describe the behaviour of the graph near them using proper notation.

1. $f(x) = \dfrac{x^2 - 4x + 2}{1 - x}$
2. $g(x) = \dfrac{x^2 - 4}{x - 2}$
3. $h(x) = \dfrac{x^3 + 1}{x^2 - 4}$

SOLUTION

1. The degree of the numerator is 2 and the degree of the denominator is 1, so Theorem 3.3.3 guarantees us a slant asymptote. To find it, we divide $1 - x = -x + 1$ into $x^2 - 4x + 2$ and get a quotient of $-x + 3$, so our slant asymptote is $y = -x + 3$. We confirm this graphically in Figure 3.3.13, and we see that as $x \to -\infty$, the graph of $y = f(x)$ approaches the asymptote from below, and as $x \to \infty$, the graph of $y = f(x)$ approaches the asymptote from above.

2. As with the previous example, the degree of the numerator $g(x) = \dfrac{x^2 - 4}{x - 2}$ is 2 and the degree of the denominator is 1, so Theorem 3.3.3 applies.

$$g(x) = \frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{(x - 2)} = \frac{(x + 2)\cancel{(x - 2)}^{\,1}}{\cancel{(x - 2)}} = x + 2, \quad x \neq 2$$

so we have that the slant asymptote $y = x + 2$ is identical to the graph of $y = g(x)$ except at $x = 2$ (where the latter has a 'hole' at $(2, 4)$.) The graph (using GeoGebra) in Figure 3.3.14 supports this claim.

3. For $h(x) = \dfrac{x^3 + 1}{x^2 - 4}$, the degree of the numerator is 3 and the degree of the denominator is 2 so again, we are guaranteed the existence of a slant asymptote. The long division $\left(x^3 + 1\right) \div \left(x^2 - 4\right)$ gives a quotient of just x, so our slant asymptote is the line $y = x$. The graph confirms this, and we find that as $x \to -\infty$, the graph of $y = h(x)$ approaches the asymptote from below, and as $x \to \infty$, the graph of $y = h(x)$ approaches the asymptote from above: see Figure 3.3.15.

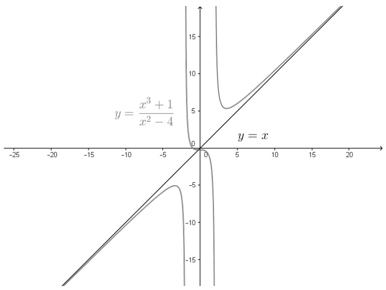


Figure 3.3.15: The graph $y = h(x)$ in Example 3.3.4

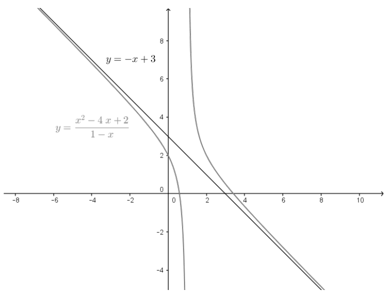


Figure 3.3.13: The graph $y = f(x)$ in Example 3.3.4

Note that we are purposefully avoiding notation like 'as $x \to \infty$, $f(x) \to (-x + 3)^+$. While it is possible to define these notions formally with Calculus, it is not standard to do so. Besides, with the introduction of the symbol '‽' in the next section, the authors feel we are in enough trouble already.

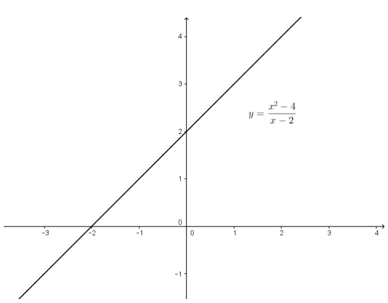


Figure 3.3.14: The graph $y = g(x)$ in Example 3.3.4

We end this section by giving a few examples of rational equations and inequalities. Particular care must be taken with rational inequalities, since the sign of both numerator and denominator can affect the solution. (Many are the students who have gone wrong by attempting to clear denominators in an inequality!)

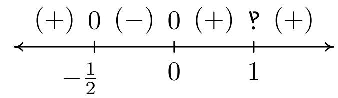


Figure 3.3.16: The sign diagram for the inequality in Example 3.3.5

Example 3.3.5 Rational equation and inequality

1. Solve $\dfrac{x^3 - 2x + 1}{x - 1} = \dfrac{1}{2}x - 1.$
2. Solve $\dfrac{x^3 - 2x + 1}{x - 1} \geq \dfrac{1}{2}x - 1.$
3. Use your computer or calculator to graphically check your answers to 1 and 2.

Solution

1. To solve the equation, we clear denominators

$$\begin{aligned}
\frac{x^3 - 2x + 1}{x - 1} &= \frac{1}{2}x - 1 \\
\left(\frac{x^3 - 2x + 1}{x - 1}\right) \cdot 2(x - 1) &= \left(\frac{1}{2}x - 1\right) \cdot 2(x - 1) \\
2x^3 - 4x + 2 &= x^2 - 3x + 2 && \text{expand} \\
2x^3 - x^2 - x &= 0 \\
x(2x + 1)(x - 1) &= 0 && \text{factor} \\
x &= -\tfrac{1}{2},\ 0,\ 1
\end{aligned}$$

Since we cleared denominators, we need to check for extraneous solutions. Sure enough, we see that $x = 1$ does not satisfy the original equation and must be discarded. Our solutions are $x = -\frac{1}{2}$ and $x = 0$.

2. To solve the inequality, it may be tempting to begin as we did with the equation — namely by multiplying both sides by the quantity $(x - 1)$. The problem is that, depending on x, $(x - 1)$ may be positive (which doesn't affect the inequality) or $(x - 1)$ could be negative (which would reverse the inequality). Instead of working by cases, we collect all of the terms on one side of the inequality with 0 on the other and make a sign diagram.

$$\begin{aligned}
\frac{x^3 - 2x + 1}{x - 1} &\geq \frac{1}{2}x - 1 \\
\frac{x^3 - 2x + 1}{x - 1} - \frac{1}{2}x + 1 &\geq 0 \\
\frac{2\left(x^3 - 2x + 1\right) - x(x - 1) + 1(2(x - 1))}{2(x - 1)} &\geq 0 && \text{get a common denominator} \\
\frac{2x^3 - x^2 - x}{2x - 2} &\geq 0 && \text{expand}
\end{aligned}$$

Viewing the left hand side as a rational function $r(x)$ we make a sign diagram. The only value excluded from the domain of r is $x = 1$ which is the solution to $2x - 2 = 0$. The zeros of r are the solutions to $2x^3 - x^2 - x = 0$, which we have already found to be $x = 0$, $x = -\frac{1}{2}$ and $x = 1$, the latter was discounted as a zero because it is not in the domain. Choosing test values in each test interval, we obtain the sign diagram in Figure 3.3.16.

We are interested in where $r(x) \geq 0$. We find $r(x) > 0$, or $(+)$, on the intervals $\left(-\infty, -\frac{1}{2}\right)$, $(0, 1)$ and $(1, \infty)$. We add to these intervals the zeros of r, $-\frac{1}{2}$ and 0, to get our final solution: $\left(-\infty, -\frac{1}{2}\right] \cup [0, 1) \cup (1, \infty)$.

3. Geometrically, if we set $f(x) = \dfrac{x^3 - 2x + 1}{x - 1}$ and $g(x) = \frac{1}{2}x - 1$, the solutions to $f(x) = g(x)$ are the x-coordinates of the points where the graphs of $y = f(x)$ and $y = g(x)$ intersect. The solution to $f(x) \geq g(x)$ represents not only where the graphs meet, but the intervals over which the graph of $y = f(x)$ is above ($>$) the graph of $g(x)$. Entering these two functions into GeoGebra gives us Figure 3.3.17.

   Zooming in and using the Intersect tool, we see in Figure 3.3.18 that the graphs cross when $x = -\frac{1}{2}$ and $x = 0$. It is clear from the calculator that the graph of $y = f(x)$ is above the graph of $y = g(x)$ on $\left(-\infty, -\frac{1}{2}\right)$ as well as on $(0, \infty)$. According to the calculator, our solution is then $\left(-\infty, -\frac{1}{2}\right] \cup [0, \infty)$ which *almost* matches the answer we found analytically. We have to remember that f is not defined at $x = 1$, and, even though it isn't shown on the calculator, there is a hole in the graph of $y = f(x)$ when $x = 1$ which is why $x = 1$ is not part of our final answer. (There is no asymptote at $x = 1$ since the graph is well behaved near $x = 1$. According to Theorem 3.3.1, there must be a hole there.)

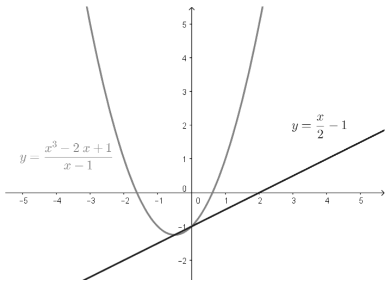


Figure 3.3.17: The initial plot of $f(x)$ and $g(x)$

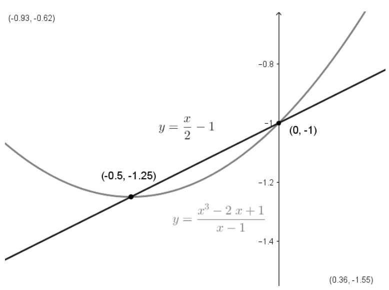


Figure 3.3.18: Zooming in to find the intersection points

Exercises 3.3

Problems

In Exercises 1 – 18, for the given rational function f:

- **Find the domain of f.**
- **Identify any vertical asymptotes of the graph of $y = f(x)$.**
- **Identify any holes in the graph.**
- **Find the horizontal asymptote, if it exists.**
- **Find the slant asymptote, if it exists.**
- **Graph the function using a graphing utility and describe the behaviour near the asymptotes.**

1. $f(x) = \dfrac{x}{3x-6}$

2. $f(x) = \dfrac{3+7x}{5-2x}$

3. $f(x) = \dfrac{x}{x^2+x-12}$

4. $f(x) = \dfrac{x}{x^2+1}$

5. $f(x) = \dfrac{x+7}{(x+3)^2}$

6. $f(x) = \dfrac{x^3+1}{x^2-1}$

7. $f(x) = \dfrac{4x}{x^2+4}$

8. $f(x) = \dfrac{4x}{x^2-4}$

9. $f(x) = \dfrac{x^2-x-12}{x^2+x-6}$

10. $f(x) = \dfrac{3x^2-5x-2}{x^2-9}$

11. $f(x) = \dfrac{x^3+2x^2+x}{x^2-x-2}$

12. $f(x) = \dfrac{x^3-3x+1}{x^2+1}$

13. $f(x) = \dfrac{2x^2+5x-3}{3x+2}$

14. $f(x) = \dfrac{-x^3+4x}{x^2-9}$

15. $f(x) = \dfrac{-5x^4-3x^3+x^2-10}{x^3-3x^2+3x-1}$

16. $f(x) = \dfrac{x^3}{1-x}$

17. $f(x) = \dfrac{18-2x^2}{x^2-9}$

18. $f(x) = \dfrac{x^3-4x^2-4x-5}{x^2+x+1}$

In Exercises 19 – 24, solve the rational equation. Be sure to check for extraneous solutions.

19. $\dfrac{x}{5x+4} = 3$

20. $\dfrac{3x-1}{x^2+1} = 1$

21. $\dfrac{1}{x+3} + \dfrac{1}{x-3} = \dfrac{x^2-3}{x^2-9}$

22. $\dfrac{2x+17}{x+1} = x+5$

23. $\dfrac{x^2-2x+1}{x^3+x^2-2x} = 1$

24. $\dfrac{-x^3+4x}{x^2-9} = 4x$

In Exercises 25 – 38, solve the rational inequality. Express your answer using interval notation.

25. $\dfrac{1}{x+2} \geq 0$

26. $\dfrac{x-3}{x+2} \leq 0$

27. $\dfrac{x}{x^2-1} > 0$

28. $\dfrac{4x}{x^2+4} \geq 0$

29. $\dfrac{x^2-x-12}{x^2+x-6} > 0$

30. $\dfrac{3x^2-5x-2}{x^2-9} < 0$

31. $\dfrac{x^3+2x^2+x}{x^2-x-2} \geq 0$

32. $\dfrac{x^2+5x+6}{x^2-1} > 0$

33. $\dfrac{3x-1}{x^2+1} \leq 1$

34. $\dfrac{2x+17}{x+1} > x+5$

35. $\dfrac{-x^3+4x}{x^2-9} \geq 4x$

36. $\dfrac{1}{x^2+1} < 0$

37. $\dfrac{x^4-4x^3+x^2-2x-15}{x^3-4x^2} \geq x$

38. $\dfrac{5x^3-12x^2+9x+10}{x^2-1} \geq 3x-1$

3.4 Exponential and Logarithmic Functions

3.4.1 Introduction to Exponential and Logarithmic Functions

Exponential and logarithmic functions frequently occur in solutions to differential equations, which are used to produce mathematical models of phenomena throughout the physical, life, and social sciences. You'll see some examples if you continue on to Calculus I and II, and even more if you take Math 3600, our first course in differential equations.

Of all of the functions we study in this text, exponential and logarithmic functions are possibly the ones which impact everyday life the most. This section introduces us to these functions while the rest of the chapter will more thoroughly explore their properties. Up to this point, we have dealt with functions which involve terms like x^2 or $x^{2/3}$, in other words, terms of the form x^p where the base of the term, x, varies but the exponent of each term, p, remains constant. In this chapter, we study functions of the form $f(x) = b^x$ where the base b is a constant and the exponent x is the variable. We start our exploration of these functions with $f(x) = 2^x$. (Apparently this is a tradition. Every textbook we have ever read starts with $f(x) = 2^x$.) We make a table of values, plot the points and connect the dots in a pleasing fashion: see Figure 3.4.1

x	$f(x)$	$(x, f(x))$
-3	$2^{-3} = \frac{1}{8}$	$\left(-3, \frac{1}{8}\right)$
-2	$2^{-2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$2^{-1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	$2^0 = 1$	$(0, 1)$
1	$2^1 = 2$	$(1, 2)$
2	$2^2 = 4$	$(2, 4)$
3	$2^3 = 8$	$(3, 8)$

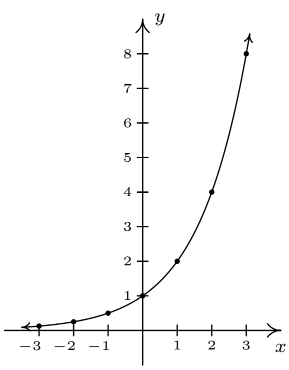


Figure 3.4.1: Plotting $f(x) = 2^x$

A few remarks about the graph of $f(x) = 2^x$ which we have constructed are in order. As $x \to -\infty$ and attains values like $x = -100$ or $x = -1000$, the function $f(x) = 2^x$ takes on values like $f(-100) = 2^{-100} = \frac{1}{2^{100}}$ or $f(-1000) = 2^{-1000} = \frac{1}{2^{1000}}$. In other words, as $x \to -\infty$,

$$2^x \approx \frac{1}{\text{very big } (+)} \approx \text{very small } (+)$$

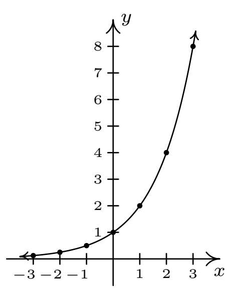
So as $x \to -\infty$, $2^x \to 0^+$. This is represented graphically using the x-axis (the line $y = 0$) as a horizontal asymptote. On the flip side, as $x \to \infty$, we find $f(100) = 2^{100}$, $f(1000) = 2^{1000}$, and so on, thus $2^x \to \infty$. As a result, our graph suggests the range of f is $(0, \infty)$. The graph of f passes the Horizontal Line Test which means f is one-to-one and hence invertible. We also note that when we 'connected the dots in a pleasing fashion', we have made the implicit assumption that $f(x) = 2^x$ is continuous (recall that this means there are no holes or other kinds of breaks in the graph) and has a domain of all real numbers. In particular, we have suggested that things like $2^{\sqrt{3}}$ exist as real numbers. We should take a moment to discuss what something like $2^{\sqrt{3}}$ might mean, and refer the interested reader to a solid course in Calculus for a more rigorous explanation. The number $\sqrt{3} = 1.73205\ldots$ is an irrational number and as such, its decimal representation neither repeats nor terminates. We can, however, approximate $\sqrt{3}$ by terminating decimals, and it stands to reason (this is where Calculus and continuity come into play) that we can use these to approximate $2^{\sqrt{3}}$. For example, if we approximate $\sqrt{3}$ by 1.73, we can approximate $2^{\sqrt{3}} \approx 2^{1.73} = 2^{\frac{173}{100}} = \sqrt[100]{2^{173}}$. It is not, by any means, a pleasant number, but it is at least a number that we understand in terms of powers and roots. It also stands to reason that better and better approximations of $\sqrt{3}$ yield better and better approximations of $2^{\sqrt{3}}$, so the value of $2^{\sqrt{3}}$ should be the result of this sequence of approximations.

To fully understand the argument we used to define 2^x when x is irrational, you'll have to proceed far enough through the Calculus sequence (Calculus III should do it) to encounter the topic of convergence of infinite sequences.

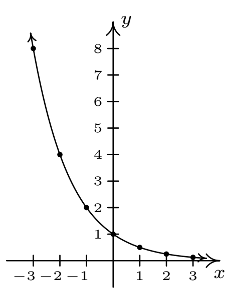
Suppose we wish to study the family of functions $f(x) = b^x$. Which bases b make sense to study? We find that we run into difficulty if $b < 0$. For example, if $b = -2$, then the function $f(x) = (-2)^x$ has trouble, for instance, at $x = \frac{1}{2}$ since $(-2)^{1/2} = \sqrt{-2}$ is not a real number. In general, if x is any rational number with an even denominator, then $(-2)^x$ is not defined, so we must restrict our attention to bases $b \geq 0$. What about $b = 0$? The function $f(x) = 0^x$ is undefined for $x \leq 0$ because we cannot divide by 0 and 0^0 is an indeterminant form. For $x > 0$, $0^x = 0$ so the function $f(x) = 0^x$ is the same as the function $f(x) = 0$, $x > 0$. We know everything we can possibly know about this function, so we exclude it from our investigations. The only other base we exclude is $b = 1$, since the function $f(x) = 1^x = 1$ is, once again, a function we have already studied. We are now ready for our definition of exponential functions.

Definition 3.4.1 Exponential function

A function of the form $f(x) = b^x$ where b is a fixed real number, $b > 0$, $b \neq 1$ is called a **base *b* exponential function**.



(a) $y = f(x) = 2^x$



(b) $y = g(x) = f(-x) = 2^{-x}$

Figure 3.4.2: Reflecting $y = 2^x$ across the y-axis to obtain the graph $y = 2^{-x}$

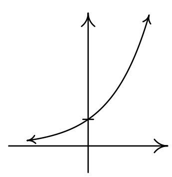
We leave it to the reader to verify (by graphing some more examples on your own) that if $b > 1$, then the exponential function $f(x) = b^x$ will share the same basic shape and characteristics as $f(x) = 2^x$. What if $0 < b < 1$? Consider $g(x) = \left(\frac{1}{2}\right)^x$. We could certainly build a table of values and connect the points, or we could take a step back and note that $g(x) = \left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x = 2^{-x} = f(-x)$, where $f(x) = 2^x$. The graph of $f(-x)$ is obtained from the graph of $f(x)$ by reflecting it across the y-axis. We get the graph in Figure 3.4.2 (b).

We see that the domain and range of g match that of f, namely $(-\infty, \infty)$ and $(0, \infty)$, respectively. Like f, g is also one-to-one. Whereas f is always increasing, g is always decreasing. As a result, as $x \to -\infty$, $g(x) \to \infty$, and on the flip side, as $x \to \infty$, $g(x) \to 0^+$. It shouldn't be too surprising that for all choices of the base $0 < b < 1$, the graph of $y = b^x$ behaves similarly to the graph of g. We summarize the basic properties of exponential functions in the following theorem. (The proof of which, like many things discussed in the text, requires Calculus.)

Theorem 3.4.1 Properties of Exponential Functions

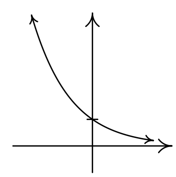
Suppose $f(x) = b^x$.

- The domain of f is $(-\infty, \infty)$ and the range of f is $(0, \infty)$.
- $(0, 1)$ is on the graph of f and $y = 0$ is a horizontal asymptote to the graph of f.
- f is one-to-one, continuous and smooth (the graph of f has no sharp turns or corners).
- If $b > 1$:
  - f is always increasing
  - As $x \to -\infty$, $f(x) \to 0^+$
  - As $x \to \infty$, $f(x) \to \infty$
  - The graph of f resembles:



$y = b^x$, $b > 1$

- If $0 < b < 1$:
  - f is always decreasing
  - As $x \to -\infty$, $f(x) \to \infty$
  - As $x \to \infty$, $f(x) \to 0^+$
  - The graph of f resembles:



$y = b^x$, $0 < b < 1$

Of all of the bases for exponential functions, two occur the most often in scientific circles. The first, base 10, is often called the **common base**. The second base is an irrational number, $e \approx 2.718$, called the **natural base**. You may encounter a more formal discussion of the number e in later Calculus courses. For now, it is enough to know that since $e > 1$, $f(x) = e^x$ is an increasing exponential function. The following examples give us an idea how these functions are used in the wild.

Example 3.4.1 Modelling vehicle depreciation

The value of a car can be modelled by $V(x) = 25\left(\frac{4}{5}\right)^x$, where $x \geq 0$ is age of the car in years and $V(x)$ is the value in thousands of dollars.

1. Find and interpret $V(0)$.
2. Sketch the graph of $y = V(x)$ using transformations.
3. Find and interpret the horizontal asymptote of the graph you found in 2.

Solution

1. To find $V(0)$, we replace x with 0 to obtain $V(0) = 25\left(\frac{4}{5}\right)^0 = 25$. Since x represents the age of the car in years, $x = 0$ corresponds to the car being brand new. Since $V(x)$ is measured in thousands of dollars, $V(0) = 25$ corresponds to a value of \$25,000. Putting it all together, we interpret $V(0) = 25$ to mean the purchase price of the car was \$25,000.

2. To graph $y = 25\left(\frac{4}{5}\right)^x$, we start with the basic exponential function $f(x) = \left(\frac{4}{5}\right)^x$. Since the base $b = \frac{4}{5}$ is between 0 and 1, the graph of $y = f(x)$ is decreasing. We plot the y-intercept $(0,1)$ and two other points, $\left(-1, \frac{5}{4}\right)$ and $\left(1, \frac{4}{5}\right)$, and label the horizontal asymptote $y = 0$. To obtain $V(x) = 25\left(\frac{4}{5}\right)^x$, $x \geq 0$, we multiply the output from f by 25, in other words, $V(x) = 25f(x)$. This results in a vertical stretch by a factor of 25. We multiply all of the y values in the graph by 25 (including the y value of the horizontal asymptote) and obtain the points $\left(-1, \frac{125}{4}\right)$, $(0,25)$ and $(1,20)$. The horizontal asymptote remains $y = 0$. Finally, we restrict the domain to $[0,\infty)$ to fit with the applied domain given to us. We have the result in Figure 3.4.3.

3. We see from the graph of V that its horizontal asymptote is $y = 0$. (We leave it to reader to verify this analytically by thinking about what happens as we take larger and larger powers of $\frac{4}{5}$.) This means as the car gets older, its value diminishes to 0.

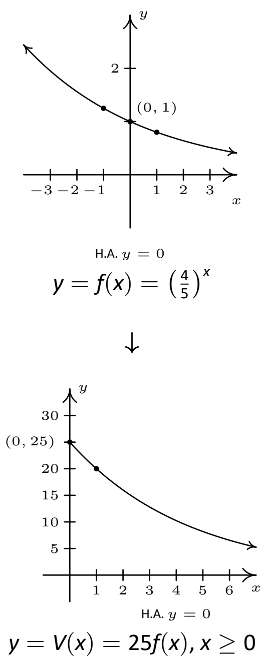


Figure 3.4.3: The graph $y = V(x)$ in Example 3.4.1

The function in the previous example is often called a 'decay curve'. Increasing exponential functions are used to model 'growth curves' many examples of which are encountered in applications of exponential functions. For now, we present another common decay curve which will serve as the basis for further study of exponential functions. Although it may look more complicated than the previous example, it is actually just a basic exponential function which has been modified by a few transformations.

Example 3.4.2 Newton's Law of Cooling

According to Newton's Law of Cooling the temperature of coffee T (in degrees Fahrenheit) t minutes after it is served can be modelled by $T(t) = 70 + 90e^{-0.1t}$.

1. Find and interpret $T(0)$.
2. Sketch the graph of $y = T(t)$ using transformations.
3. Find and interpret the horizontal asymptote of the graph.

Solution

1. To find $T(0)$, we replace every occurrence of the independent variable t with 0 to obtain $T(0) = 70 + 90e^{-0.1(0)} = 160$. This means that the coffee was served at 160°F.

2. To graph $y = T(t)$ using transformations, we start with the basic function, $f(t) = e^t$. As we have already remarked, $e \approx 2.718 > 1$ so the graph of f is an increasing exponential with y-intercept $(0,1)$ and horizontal asymptote $y = 0$. The points $\left(-1, e^{-1}\right) \approx (-1, 0.37)$ and $(1, e) \approx (1, 2.72)$ are also on the graph. We have

$$T(t) = 70 + 90e^{-0.1t} = 90e^{-0.1t} + 70 = 90f(-0.1t) + 70$$

Multiplication of the input to f, t, by -0.1 results in a horizontal expansion by a factor of 10 as well as a reflection about the y-axis. We divide each of the x values of our points by -0.1 (which amounts to multiplying them by -10) to obtain $\left(10, e^{-1}\right)$, $(0,1)$, and $(-10, e)$. Since none of these changes affected the y values, the horizontal asymptote remains $y = 0$.

Next, we see that the output from f is being multiplied by 90. This results in a vertical stretch by a factor of 90. We multiply the y-coordinates by 90 to obtain $\left(10, 90e^{-1}\right)$, $(0, 90)$, and $(-10, 90e)$. We also multiply the y value of the horizontal asymptote $y = 0$ by 90, and it remains $y = 0$. Finally, we add 70 to all of the y-coordinates, which shifts the graph upwards to obtain $\left(10, 90e^{-1} + 70\right) \approx (10, 103.11)$, $(0, 160)$, and $(-10, 90e + 70) \approx (-10, 314.64)$. Adding 70 to the horizontal asymptote shifts it upwards as well to $y = 70$. We connect these three points using the same shape in the same direction as in the graph of f and, last but not least, we restrict the domain to match the applied domain $[0, \infty)$. The result is given in Figure 3.4.4.

3. From the graph, we see that the horizontal asymptote is $y = 70$. It is worth a moment or two of our time to see how this happens analytically. As $t \to \infty$, We get $T(t) = 70 + 90e^{-0.1t} \approx 70 + 90e^{\text{very big }(-)}$. Since $e > 1$,

$$e^{\text{very big }(-)} = \frac{1}{e^{\text{very big }(+)}} \approx \frac{1}{\text{very big }(+)} \approx \text{very small }(+)$$

The larger t becomes, the smaller $e^{-0.1t}$ becomes, so the term $90e^{-0.1t} \approx$ very small $(+)$. Hence, $T(t) \approx 70+$very small $(+)$ which means the graph is approaching the horizontal line $y = 70$ from above. This means that as time goes by, the temperature of the coffee is cooling to 70°F, presumably room temperature.

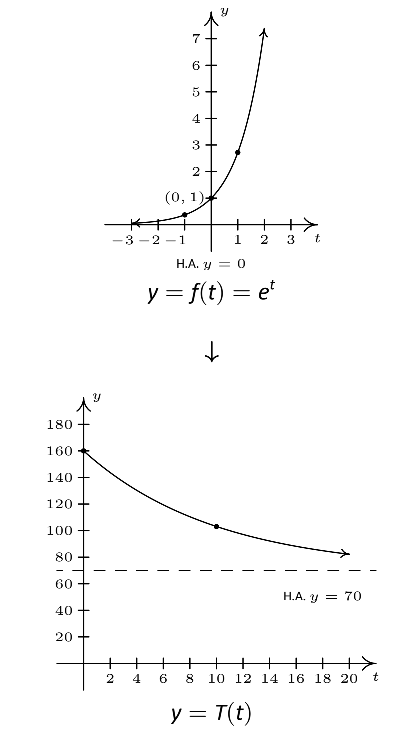


Figure 3.4.4: Graphing $T(t)$ in Example 3.4.2

As we have already remarked, the graphs of $f(x) = b^x$ all pass the Horizontal Line Test. Thus the exponential functions are invertible. We now turn our attention to these inverses, the logarithmic functions, which are called 'logs' for short.

Definition 3.4.2 Logarithm function

The inverse of the exponential function $f(x) = b^x$ is called the **base b logarithm function**, and is denoted $f^{-1}(x) = \log_b(x)$ We read '$\log_b(x)$' as 'log base b of x.'

We have special notations for the common base, $b = 10$, and the natural base, $b = e$.

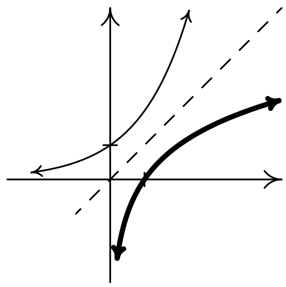
Definition 3.4.3 Common and Natural Logarithms

The **common logarithm** of a real number x is $\log_{10}(x)$ and is usually written $\log(x)$. The **natural logarithm** of a real number x is $\log_e(x)$ and is usually written $\ln(x)$.

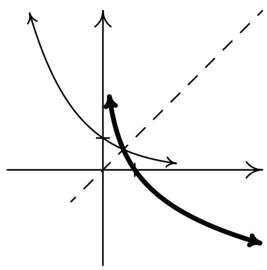
The reader is cautioned that in more advanced mathematics textbooks, the notation $\log(x)$ is often used to denote the natural logarithm (or its generalization to the complex numbers). In mathematics, the natural logarithm is preferred since it is better behaved with respect to the operations of Calculus. The base 10 logarithm tends to appear in other science fields.

Since logs are defined as the inverses of exponential functions, we can use Theorems 2.2.1 and 2.2.2 to tell us about logarithmic functions. For example, we know that the domain of a log function is the range of an exponential function, namely $(0, \infty)$, and that the range of a log function is the domain of an exponential function, namely $(-\infty, \infty)$. Since we know the basic shapes of $y = f(x) =$

b^x for the different cases of b, we can obtain the graph of $y = f^{-1}(x) = \log_b(x)$ by reflecting the graph of f across the line $y = x$ as shown below. The y-intercept $(0, 1)$ on the graph of f corresponds to an x-intercept of $(1, 0)$ on the graph of f^{-1}. The horizontal asymptotes $y = 0$ on the graphs of the exponential functions become vertical asymptotes $x = 0$ on the log graphs: see Figure 3.4.5.



$y = b^x, b > 1$
$\boldsymbol{y = \log_b(x)}, b > 1$



$y = b^x, 0 < b < 1$
$\boldsymbol{y = \log_b(x)}, 0 < b < 1$

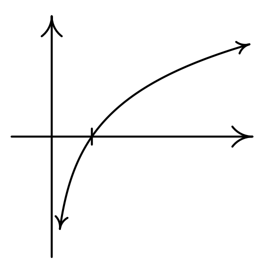
Figure 3.4.5: The logarithm is the inverse of the exponential function

On a procedural level, logs undo the exponentials. Consider the function $f(x) = 2^x$. When we evaluate $f(3) = 2^3 = 8$, the input 3 becomes the exponent on the base 2 to produce the real number 8. The function $f^{-1}(x) = \log_2(x)$ then takes the number 8 as its input and returns the exponent 3 as its output. In symbols, $\log_2(8) = 3$. More generally, $\log_2(x)$ is the exponent you put on 2 to get x. Thus, $\log_2(16) = 4$, because $2^4 = 16$. The following theorem summarizes the basic properties of logarithmic functions, all of which come from the fact that they are inverses of exponential functions.

Theorem 3.4.2 Properties of Logarithmic Functions

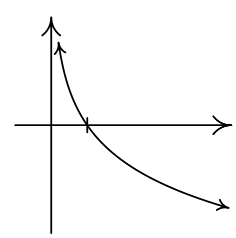
Suppose $f(x) = \log_b(x)$.

- The domain of f is $(0, \infty)$ and the range of f is $(-\infty, \infty)$.
- $(1, 0)$ is on the graph of f and $x = 0$ is a vertical asymptote of the graph of f.
- f is one-to-one, continuous and smooth
- $b^a = c$ if and only if $\log_b(c) = a$. That is, $\log_b(c)$ is the exponent you put on b to obtain c.
- $\log_b\left(b^x\right) = x$ for all x and $b^{\log_b(x)} = x$ for all $x > 0$
- If $b > 1$:
  - f is always increasing
  - As $x \to 0^+$, $f(x) \to -\infty$
  - As $x \to \infty$, $f(x) \to \infty$
  - The graph of f resembles:



$y = \log_b(x), b > 1$

- If $0 < b < 1$:
  - f is always decreasing
  - As $x \to 0^+$, $f(x) \to \infty$
  - As $x \to \infty$, $f(x) \to -\infty$
  - The graph of f resembles:



$y = \log_b(x), 0 < b < 1$

As we have mentioned, Theorem 3.4.2 is a consequence of Theorems 2.2.1 and 2.2.2. However, it is worth the reader's time to understand Theorem 3.4.2 from an exponential perspective. For instance, we know that the domain of $g(x) = \log_2(x)$ is $(0, \infty)$. Why? Because the range of $f(x) = 2^x$ is $(0, \infty)$. In a way, this says everything, but at the same time, it doesn't. For example, if we try to find $\log_2(-1)$, we are trying to find the exponent we put on 2 to give us -1. In other words, we are looking for x that satisfies $2^x = -1$. There is no such real number, since all powers of 2 are positive. While what we have said is exactly the same thing as saying 'the domain of $g(x) = \log_2(x)$ is $(0, \infty)$ because the range of $f(x) = 2^x$ is $(0, \infty)$', we feel it is in a student's best interest to understand the statements in Theorem 3.4.2 at this level instead of just merely memorizing the facts.

Example 3.4.3 Using properties of logarithms

Simplify the following.

1. $\log_3(81)$
2. $\log_2\left(\frac{1}{8}\right)$
3. $\log_{\sqrt{5}}(25)$
4. $\ln\left(\sqrt[3]{e^2}\right)$
5. $\log(0.001)$
6. $2^{\log_2(8)}$
7. $117^{-\log_{117}(6)}$

Solution

1. The number $\log_3(81)$ is the exponent we put on 3 to get 81. As such, we want to write 81 as a power of 3. We find $81 = 3^4$, so that $\log_3(81) = 4$.

2. To find $\log_2\left(\frac{1}{8}\right)$, we need rewrite $\frac{1}{8}$ as a power of 2. We find $\frac{1}{8} = \frac{1}{2^3} = 2^{-3}$, so $\log_2\left(\frac{1}{8}\right) = -3$.

3. To determine $\log_{\sqrt{5}}(25)$, we need to express 25 as a power of $\sqrt{5}$. We know $25 = 5^2$, and $5 = \left(\sqrt{5}\right)^2$, so we have $25 = \left(\left(\sqrt{5}\right)^2\right)^2 = \left(\sqrt{5}\right)^4$. We get $\log_{\sqrt{5}}(25) = 4$.

4. First, recall that the notation $\ln\left(\sqrt[3]{e^2}\right)$ means $\log_e\left(\sqrt[3]{e^2}\right)$, so we are looking for the exponent to put on e to obtain $\sqrt[3]{e^2}$. Rewriting $\sqrt[3]{e^2} = e^{2/3}$, we find $\ln\left(\sqrt[3]{e^2}\right) = \ln\left(e^{2/3}\right) = \frac{2}{3}$.

5. Rewriting $\log(0.001)$ as $\log_{10}(0.001)$, we see that we need to write 0.001 as a power of 10. We have $0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$. Hence, $\log(0.001) = \log\left(10^{-3}\right) = -3$.

6. We can use Theorem 3.4.2 directly to simplify $2^{\log_2(8)} = 8$. We can also understand this problem by first finding $\log_2(8)$. By definition, $\log_2(8)$ is the exponent we put on 2 to get 8. Since $8 = 2^3$, we have $\log_2(8) = 3$. We now substitute to find $2^{\log_2(8)} = 2^3 = 8$.

7. From Theorem 3.4.2, we know $117^{\log_{117}(6)} = 6$, but we cannot directly apply this formula to the expression $117^{-\log_{117}(6)}$. (Can you see why?) At this point, we use a property of exponents followed by Theorem 3.4.2 to get

$$117^{-\log_{117}(6)} = \frac{1}{117^{\log_{117}(6)}} = \frac{1}{6}$$

It is worth a moment of your time to think your way through why $117^{\log_{117}(6)} = 6$. By definition, $\log_{117}(6)$ is the exponent we put on 117 to get 6. What are we doing with this exponent? We are putting it on 117. By definition we get 6. In other words, the exponential function $f(x) = 117^x$ undoes the logarithmic function $g(x) = \log_{117}(x)$.

Up until this point, restrictions on the domains of functions came from avoiding division by zero and keeping negative numbers from beneath even radicals. With the introduction of logs, we now have another restriction. Since the domain of $f(x) = \log_b(x)$ is $(0, \infty)$, the argument of the log must be strictly positive.

Example 3.4.4 Domain for logarithmic functions

Find the domain of the following functions. Check your answers graphically using the computer or calculator.

1. $f(x) = 2\log(3 - x) - 1$

2. $g(x) = \ln\left(\dfrac{x}{x-1}\right)$

Solution

1. We set $3 - x > 0$ to obtain $x < 3$, or $(-\infty, 3)$. The graph in Figure 3.4.6 verifies this. Note that we could have graphed f using transformations. We rewrite $f(x) = 2\log_{10}(-x+3) - 1$ and find the main function involved is $y = h(x) = \log_{10}(x)$. We select three points to track, $\left(\frac{1}{10}, -1\right)$, $(1, 0)$ and $(10, 1)$, along with the vertical asymptote $x = 0$. Since $f(x) = 2h(-x+3) - 1$, to obtain the destinations of these points, we first subtract 3 from the x-coordinates (shifting the graph left 3 units), then divide (multiply) by the x-coordinates by -1 (causing a reflection across the y-axis). These transformations apply to the vertical asymptote $x = 0$ as well. Subtracting 3 gives us $x = -3$ as our asymptote, then multiplying by -1 gives us the vertical asymptote $x = 3$. Next, we multiply the y-coordinates by 2 which results in a vertical stretch by a factor of 2, then we finish by subtracting 1 from the y-coordinates which shifts the graph down 1 unit. We leave it to the reader to perform the indicated arithmetic on the points themselves and to verify the graph produced by the calculator below.

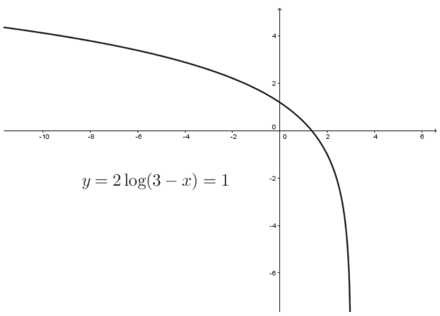


Figure 3.4.6: $y = f(x) = 2\log(3 - x) - 1$

2. To find the domain of g, we need to solve the inequality $\frac{x}{x-1} > 0$. As usual, we proceed using a sign diagram. If we define $r(x) = \dfrac{x}{x-1}$, we find r is undefined at $x = 1$ and $r(x) = 0$ when $x = 0$. Choosing some test values, we generate the sign diagram in Figure 3.4.7.

   We find $\frac{x}{x-1} > 0$ on $(-\infty, 0) \cup (1, \infty)$ to get the domain of g. The graph of $y = g(x)$ in Figure 3.4.8 confirms this. We can tell from the graph of g that it is not the result of transformations being applied to the graph $y = \ln(x)$, so barring a more detailed analysis using Calculus, the calculator graph is the best we can do. One thing worthy of note, however, is the end behaviour of g. The graph suggests that as $x \to \pm\infty$, $g(x) \to 0$. We can verify this analytically. We know that as $x \to \pm\infty$, $\frac{x}{x-1} \approx 1$. Hence, it makes sense that $g(x) = \ln\left(\frac{x}{x-1}\right) \approx \ln(1) = 0$.

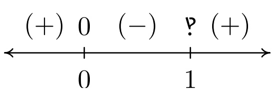


Figure 3.4.7: Sign diagram for $r(x) = \frac{x}{x-1}$

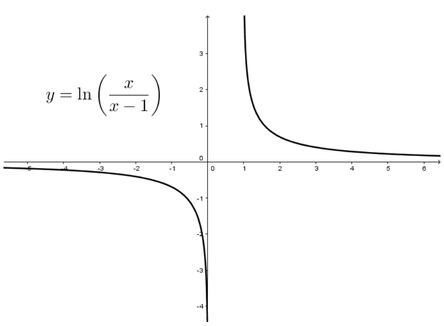


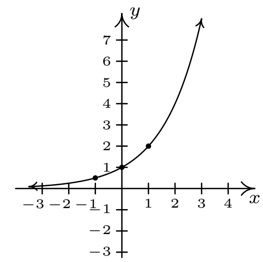
Figure 3.4.8: $y = g(x) = \ln\left(\frac{x}{x-1}\right)$

While logarithms have some interesting applications of their own which you'll explore in the exercises, their primary use to us will be to undo exponential functions. (This is, after all, how they were defined.) Our last example solidifies this and reviews all of the material in the section.

Example 3.4.5 Inverting an exponential function

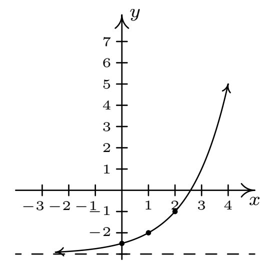
Let $f(x) = 2^{x-1} - 3$.

1. Graph f using transformations and state the domain and range of f.
2. Explain why f is invertible and find a formula for $f^{-1}(x)$.
3. Graph f^{-1} using transformations and state the domain and range of f^{-1}.
4. Verify $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.
5. Graph f and f^{-1} on the same set of axes and check the symmetry about the line $y = x$.



$y = h(x) = 2^x$

↓



$y = f(x) = 2^{x-1} - 3$

Figure 3.4.9: Graphing $f(x) = 2^{x-1} - 3$ in Example 3.4.5

SOLUTION

1. If we identify $g(x) = 2^x$, we see $f(x) = g(x-1) - 3$. We pick the points $\left(-1, \frac{1}{2}\right)$, $(0,1)$ and $(1,2)$ on the graph of g along with the horizontal asymptote $y = 0$ to track through the transformations. We first add 1 to the x-coordinates of the points on the graph of g (shifting g to the right 1 unit) to get $\left(0, \frac{1}{2}\right)$, $(1,1)$ and $(2,2)$. The horizontal asymptote remains $y = 0$. Next, we subtract 3 from the y-coordinates, shifting the graph down 3 units. We get the points $\left(0, -\frac{5}{2}\right)$, $(1,-2)$ and $(2,-1)$ with the horizontal asymptote now at $y = -3$. Connecting the dots in the order and manner as they were on the graph of g, we get the bottom graph in Figure 3.4.9. We see that the domain of f is the same as g, namely $(-\infty, \infty)$, but that the range of f is $(-3, \infty)$.

2. The graph of f passes the Horizontal Line Test so f is one-to-one, hence invertible. To find a formula for $f^{-1}(x)$, we normally set $y = f(x)$, interchange the x and y, then proceed to solve for y. Doing so in this situation leads us to the equation $x = 2^{y-1} - 3$. We have yet to discuss how to solve this kind of equation, so we will attempt to find the formula for f^{-1} from a procedural perspective. If we break $f(x) = 2^{x-1} - 3$ into a series of steps, we find f takes an input x and applies the steps

   (a) subtract 1

   (b) put as an exponent on 2

   (c) subtract 3

   Clearly, to undo subtracting 1, we will add 1, and similarly we undo subtracting 3 by adding 3. How do we undo the second step? The answer is we use the logarithm. By definition, $\log_2(x)$ undoes exponentiation by 2. Hence, f^{-1} should

   (a) add 3

   (b) take the logarithm base 2

   (c) add 1

   In symbols, $f^{-1}(x) = \log_2(x+3) + 1$.

3. To graph $f^{-1}(x) = \log_2(x+3) + 1$ using transformations, we start with $j(x) = \log_2(x)$. We track the points $\left(\frac{1}{2}, -1\right)$, $(1,0)$ and $(2,1)$ on the graph of j along with the vertical asymptote $x = 0$ through the transformations.

Since $f^{-1}(x) = j(x+3)+1$, we first subtract 3 from each of the x values (including the vertical asymptote) to obtain $\left(-\frac{5}{2}, -1\right)$, $(-2, 0)$ and $(-1, 1)$ with a vertical asymptote $x = -3$. Next, we add 1 to the y values on the graph and get $\left(-\frac{5}{2}, 0\right)$, $(-2, 1)$ and $(-1, 2)$. If you are experiencing *déjà vu*, there is a good reason for it but we leave it to the reader to determine the source of this uncanny familiarity. We obtain the graph below. The domain of f^{-1} is $(-3, \infty)$, which matches the range of f, and the range of f^{-1} is $(-\infty, \infty)$, which matches the domain of f.

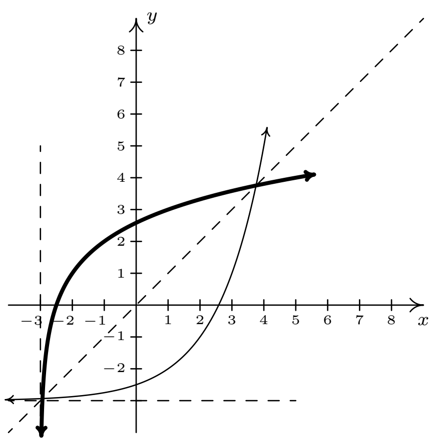
4. We now verify that $f(x) = 2^{x-1} - 3$ and $f^{-1}(x) = \log_2(x+3) + 1$ satisfy the composition requirement for inverses. For all real numbers x,

$$\begin{aligned}
\left(f^{-1} \circ f\right)(x) &= f^{-1}(f(x)) \\
&= f^{-1}\left(2^{x-1} - 3\right) \\
&= \log_2\left(\left[2^{x-1} - 3\right] + 3\right) + 1 \\
&= \log_2\left(2^{x-1}\right) + 1 \\
&= (x-1) + 1 \quad \text{Since } \log_2\left(2^u\right) = u \text{ for all real numbers } u \\
&= x \checkmark
\end{aligned}$$

For all real numbers $x > -3$, we have (pay attention - can you spot in which step below we need $x > -3$?)

$$\begin{aligned}
\left(f \circ f^{-1}\right)(x) &= f\left(f^{-1}(x)\right) \\
&= f\left(\log_2(x+3) + 1\right) \\
&= 2^{(\log_2(x+3)+1)-1} - 3 \\
&= 2^{\log_2(x+3)} - 3 \\
&= (x+3) - 3 \quad \text{Since } 2^{\log_2(u)} = u \text{ for all real numbers } u > 0 \\
&= x \checkmark
\end{aligned}$$

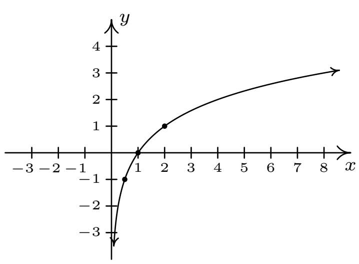
5. Last, but certainly not least, we graph $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes and see the symmetry about the line $y = x$ in Figure 3.4.11



$y = f(x) = 2^{x-1} - 3$

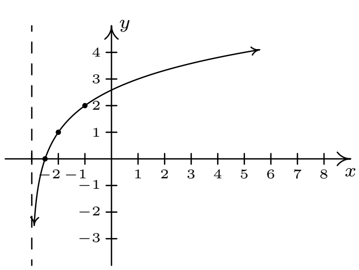
$\boldsymbol{y = f^{-1}(x) = \log_2(x+3) + 1}$

Figure 3.4.11: The graphs of f and f^{-1} in Example 3.4.5



$y = j(x) = \log_2(x)$

↓



$y = f^{-1}(x) = \log_2(x+3) + 1$

Figure 3.4.10: Graphing $f^{-1}(x) = \log_2(x+3) + 1$ in Example 3.4.5

3.4.2 Properties of Logarithms

In Section 3.4.1, we introduced the logarithmic functions as inverses of exponential functions and discussed a few of their functional properties from that perspective. In this section, we explore the algebraic properties of logarithms. Historically, these have played a huge role in the scientific development of our society since, among other things, they were used to develop analog computing devices called slide rules which enabled scientists and engineers to perform accurate calculations leading to such things as space travel and the moon landing. As we shall see shortly, logs inherit analogs of all of the properties of exponents you learned in Elementary and Intermediate Algebra. We first extract two properties from Theorem 3.4.2 to remind us of the definition of a logarithm as the inverse of an exponential function.

Theorem 3.4.3 Inverse Properties of Exponential and Logarithmic Functions

Let $b > 0$, $b \neq 1$.

- $b^a = c$ if and only if $\log_b(c) = a$
- $\log_b(b^x) = x$ for all x and $b^{\log_b(x)} = x$ for all $x > 0$

Next, we spell out what it means for exponential and logarithmic functions to be one-to-one.

Theorem 3.4.4 One-to-one Properties of Exponential and Logarithmic Functions

Let $f(x) = b^x$ and $g(x) = \log_b(x)$ where $b > 0$, $b \neq 1$. Then f and g are one-to-one and

- $b^u = b^w$ if and only if $u = w$ for all real numbers u and w.
- $\log_b(u) = \log_b(w)$ if and only if $u = w$ for all real numbers $u > 0$, $w > 0$.

We now state the algebraic properties of exponential functions which will serve as a basis for the properties of logarithms. While these properties may look identical to the ones you learned in Elementary and Intermediate Algebra, they apply to real number exponents, not just rational exponents. Note that in the theorem that follows, we are interested in the properties of exponential functions, so the base b is restricted to $b > 0$, $b \neq 1$.

Theorem 3.4.5 Algebraic Properties of Exponential Functions

Let $f(x) = b^x$ be an exponential function ($b > 0$, $b \neq 1$) and let u and w be real numbers.

- **Product Rule:** $f(u + w) = f(u)f(w)$. In other words, $b^{u+w} = b^u b^w$
- **Quotient Rule:** $f(u - w) = \dfrac{f(u)}{f(w)}$. In other words, $b^{u-w} = \dfrac{b^u}{b^w}$
- **Power Rule:** $(f(u))^w = f(uw)$. In other words, $(b^u)^w = b^{uw}$

While the properties listed in Theorem 3.4.5 are certainly believable based on similar properties of integer and rational exponents, the full proofs require Calculus. To each of these properties of exponential functions corresponds an analogous property of logarithmic functions. We list these below in our next theorem.

Theorem 3.4.6 Algebraic Properties of Logarithmic Functions

Let $g(x) = \log_b(x)$ be a logarithmic function ($b > 0$, $b \neq 1$) and let $u > 0$ and $w > 0$ be real numbers.

- **Product Rule:** $g(uw) = g(u) + g(w)$. In other words, $\log_b(uw) = \log_b(u) + \log_b(w)$
- **Quotient Rule:** $g\left(\frac{u}{w}\right) = g(u) - g(w)$. In other words, $\log_b\left(\frac{u}{w}\right) = \log_b(u) - \log_b(w)$
- **Power Rule:** $g\left(u^w\right) = wg(u)$. In other words, $\log_b\left(u^w\right) = w\log_b(u)$

Interestingly enough, expanding logarithms is the exact *opposite* process (which we will practice later) that is most useful in Algebra. The utility of expanding logarithms becomes apparent in Calculus.

There are a couple of different ways to understand why Theorem 3.4.6 is true. Consider the product rule: $\log_b(uw) = \log_b(u) + \log_b(w)$. Let $a = \log_b(uw)$, $c = \log_b(u)$, and $d = \log_b(w)$. Then, by definition, $b^a = uw$, $b^c = u$ and $b^d = w$. Hence, $b^a = uw = b^c b^d = b^{c+d}$, so that $b^a = b^{c+d}$. By the one-to-one property of b^x, we have $a = c + d$. In other words, $\log_b(uw) = \log_b(u) + \log_b(w)$. The remaining properties are proved similarly.

Example 3.4.6 Expanding logarithmic expressions

Expand the following using the properties of logarithms and simplify. Assume when necessary that all quantities represent positive real numbers.

1. $\log_2\left(\frac{8}{x}\right)$
2. $\ln\left(\frac{3}{ex}\right)^2$
3. $\log\sqrt[3]{\frac{100x^2}{yz^5}}$
4. $\log_{117}\left(x^2 - 4\right)$

Solution

1. To expand $\log_2\left(\frac{8}{x}\right)$, we use the Quotient Rule identifying $u = 8$ and $w = x$ and simplify.

$$\begin{aligned} \log_2\left(\frac{8}{x}\right) &= \log_2(8) - \log_2(x) && \text{Quotient Rule} \\ &= 3 - \log_2(x) && \text{Since } 2^3 = 8 \\ &= -\log_2(x) + 3 \end{aligned}$$

2. We have a power, quotient and product occurring in $\ln\left(\frac{3}{ex}\right)^2$. Since the exponent 2 applies to the entire quantity inside the logarithm, we begin with the Power Rule with $u = \frac{3}{ex}$ and $w = 2$. Next, we see the Quotient Rule is applicable, with $u = 3$ and $w = ex$, so we replace $\ln\left(\frac{3}{ex}\right)$ with the quantity $\ln(3) - \ln(ex)$. Since $\ln\left(\frac{3}{ex}\right)$ is being multiplied by 2, the entire quantity $\ln(3) - \ln(ex)$ is multiplied by 2. Finally, we apply the Product Rule with $u = e$ and $w = x$, and replace $\ln(ex)$ with the quantity $\ln(e) + \ln(x)$, and simplify, keeping in mind that the natural log is log base e.

$$
\begin{aligned}
\ln\left(\frac{3}{ex}\right)^2 &= 2\ln\left(\frac{3}{ex}\right) && \text{Power Rule}\\
&= 2\left[\ln(3) - \ln(ex)\right] && \text{Quotient Rule}\\
&= 2\ln(3) - 2\ln(ex)\\
&= 2\ln(3) - 2\left[\ln(e) + \ln(x)\right] && \text{Product Rule}\\
&= 2\ln(3) - 2\ln(e) - 2\ln(x)\\
&= 2\ln(3) - 2 - 2\ln(x) && \text{Since } e^1 = e\\
&= -2\ln(x) + 2\ln(3) - 2
\end{aligned}
$$

3. Recalling that a cube root is the same thing as the power $1/3$, we begin by using the Power Rule, and we keep in mind that the common log is log base 10.

$$
\begin{aligned}
\log\sqrt[3]{\frac{100x^2}{yz^5}} &= \log\left(\frac{100x^2}{yz^5}\right)^{1/3}\\
&= \frac{1}{3}\log\left(\frac{100x^2}{yz^5}\right) && \text{Power Rule}\\
&= \frac{1}{3}\left[\log\left(100x^2\right) - \log\left(yz^5\right)\right] && \text{Quotient Rule}\\
&= \frac{1}{3}\log\left(100x^2\right) - \frac{1}{3}\log\left(yz^5\right)\\
&= \frac{1}{3}\left[\log(100) + \log\left(x^2\right)\right] - \frac{1}{3}\left[\log(y) + \log\left(z^5\right)\right]\\
& && \text{Product Rule}\\
&= \frac{1}{3}\log(100) + \frac{1}{3}\log\left(x^2\right) - \frac{1}{3}\log(y) - \frac{1}{3}\log\left(z^5\right)\\
&= \frac{1}{3}\log(100) + \frac{2}{3}\log(x) - \frac{1}{3}\log(y) - \frac{5}{3}\log(z)\\
& && \text{Power Rule}\\
&= \frac{2}{3} + \frac{2}{3}\log(x) - \frac{1}{3}\log(y) - \frac{5}{3}\log(z) && \text{Since } 10^2 = 100\\
&= \frac{2}{3}\log(x) - \frac{1}{3}\log(y) - \frac{5}{3}\log(z) + \frac{2}{3}
\end{aligned}
$$

At this point in the text, the reader is encouraged to carefully read through each step and think of which quantity is playing the role of u and which is playing the role of w as we apply each property.

4. At first it seems as if we have no means of simplifying $\log_{117}\left(x^2 - 4\right)$, since none of the properties of logs addresses the issue of expanding a difference *inside* the logarithm. However, we may factor $x^2 - 4 = (x+2)(x-2)$ thereby introducing a product which gives us license to use the Product Rule.

$$
\begin{aligned}
\log_{117}\left(x^2 - 4\right) &= \log_{117}\left[(x+2)(x-2)\right] && \text{Factor}\\
&= \log_{117}(x+2) + \log_{117}(x-2) && \text{Product Rule}
\end{aligned}
$$

Example 3.4.7 Combining logarithmic expressions

Use the properties of logarithms to write the following as a single logarithm.

1. $\log_3(x-1)-\log_3(x+1)$
2. $\log(x)+2\log(y)-\log(z)$
3. $4\log_2(x)+3$
4. $-\ln(x)-\frac{1}{2}$

Solution Whereas in Example 3.4.6 we read the properties in Theorem 3.4.6 from left to right to expand logarithms, in this example we read them from right to left.

1. The difference of logarithms requires the Quotient Rule: $\log_3(x-1)-\log_3(x+1)=\log_3\left(\frac{x-1}{x+1}\right)$.

2. In the expression, $\log(x)+2\log(y)-\log(z)$, we have both a sum and difference of logarithms. However, before we use the product rule to combine $\log(x)+2\log(y)$, we note that we need to somehow deal with the coefficient 2 on $\log(y)$. This can be handled using the Power Rule. We can then apply the Product and Quotient Rules as we move from left to right. Putting it all together, we have

$$\begin{aligned}\log(x)+2\log(y)-\log(z) &= \log(x)+\log\left(y^2\right)-\log(z) && \text{Power Rule}\\ &= \log\left(xy^2\right)-\log(z) && \text{Product Rule}\\ &= \log\left(\frac{xy^2}{z}\right) && \text{Quotient Rule}\end{aligned}$$

3. We can certainly get started rewriting $4\log_2(x)+3$ by applying the Power Rule to $4\log_2(x)$ to obtain $\log_2\left(x^4\right)$, but in order to use the Product Rule to handle the addition, we need to rewrite 3 as a logarithm base 2. From Theorem 3.4.3, we know $3=\log_2\left(2^3\right)$, so we get

$$\begin{aligned}4\log_2(x)+3 &= \log_2\left(x^4\right)+3 && \text{Power Rule}\\ &= \log_2\left(x^4\right)+\log_2\left(2^3\right) && \text{Since } 3=\log_2\left(2^3\right)\\ &= \log_2\left(x^4\right)+\log_2(8) && \\ &= \log_2\left(8x^4\right) && \text{Product Rule}\end{aligned}$$

4. To get started with $-\ln(x)-\frac{1}{2}$, we rewrite $-\ln(x)$ as $(-1)\ln(x)$. We can then use the Power Rule to obtain $(-1)\ln(x)=\ln\left(x^{-1}\right)$. In order to use the Quotient Rule, we need to write $\frac{1}{2}$ as a natural logarithm. Theorem 3.4.3 gives us $\frac{1}{2}=\ln\left(e^{1/2}\right)=\ln\left(\sqrt{e}\right)$. We have

$$\begin{aligned}
-\ln(x) - \frac{1}{2} &= (-1)\ln(x) - \frac{1}{2} \\
&= \ln\left(x^{-1}\right) - \frac{1}{2} && \text{Power Rule} \\
&= \ln\left(x^{-1}\right) - \ln\left(e^{1/2}\right) && \text{Since } \tfrac{1}{2} = \ln\left(e^{1/2}\right) \\
&= \ln\left(x^{-1}\right) - \ln\left(\sqrt{e}\right) \\
&= \ln\left(\frac{x^{-1}}{\sqrt{e}}\right) && \text{Quotient Rule} \\
&= \ln\left(\frac{1}{x\sqrt{e}}\right)
\end{aligned}$$

As we would expect, the rule of thumb for re-assembling logarithms is the opposite of what it was for dismantling them. That is, if we are interested in rewriting an expression as a single logarithm, we apply log properties following the usual order of operations: deal with multiples of logs first with the Power Rule, then deal with addition and subtraction using the Product and Quotient Rules, respectively. Additionally, we find that using log properties in this fashion can increase the domain of the expression. For example, we leave it to the reader to verify the domain of $f(x) = \log_3(x-1) - \log_3(x+1)$ is $(1, \infty)$ but the domain of $g(x) = \log_3\left(\frac{x-1}{x+1}\right)$ is $(-\infty, -1) \cup (1, \infty)$.

The two logarithm buttons commonly found on calculators are the 'LOG' and 'LN' buttons which correspond to the common and natural logs, respectively. Suppose we wanted an approximation to $\log_2(7)$. The answer should be a little less than 3, (Can you explain why?) but how do we coerce the calculator into telling us a more accurate answer? We need the following theorem.

Theorem 3.4.7 Change of Base Formulas

Let $a, b > 0$, $a, b \neq 1$.

- $a^x = b^{x\log_b(a)}$ for all real numbers x.
- $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$ for all real numbers $x > 0$.

Example 3.4.8 Using change of base formulas

Use an appropriate change of base formula to convert the following expressions to ones with the indicated base. Verify your answers using a computer or calculator, as appropriate.

1. 3^2 to base 10
2. 2^x to base e
3. $\log_4(5)$ to base e
4. $\ln(x)$ to base 10

Solution

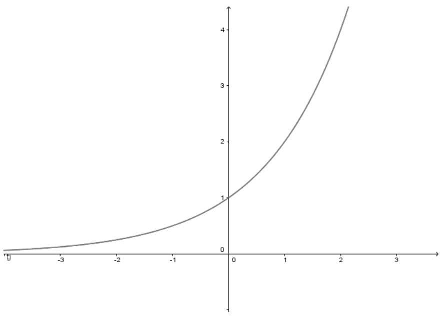


Figure 3.4.12: $y = f(x) = 2^x$ and $y = g(x) = e^{x\ln(2)}$

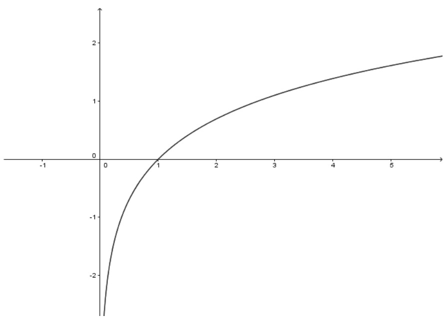


Figure 3.4.13: $y = f(x) = 2^x$ and $y = g(x) = e^{x\ln(2)}$

1. We apply the Change of Base formula with $a = 3$ and $b = 10$ to obtain $3^2 = 10^{2\log(3)}$. Typing the latter in the calculator produces an answer of 9 as required.

2. Here, $a = 2$ and $b = e$ so we have $2^x = e^{x\ln(2)}$. To verify this on our calculator, we can graph $f(x) = 2^x$ (in black) and $g(x) = e^{x\ln(2)}$ (in grey). Their graphs are indistinguishable which provides evidence that they are the same function: see Figure 3.4.12.

3. Applying the change of base with $a = 4$ and $b = e$ leads us to write $\log_4(5) = \frac{\ln(5)}{\ln(4)}$. Evaluating this in the calculator gives $\frac{\ln(5)}{\ln(4)} \approx 1.16$. How do we check this really is the value of $\log_4(5)$? By definition, $\log_4(5)$ is the exponent we put on 4 to get 5. The plot from GeoGebra in Figure 3.4.13 confirms this. (Which means if it is lying to us about the first answer it gave us, at least it is being consistent.)

4. We write $\ln(x) = \log_e(x) = \frac{\log(x)}{\log(e)}$. We graph both $f(x) = \ln(x)$ and $g(x) = \frac{\log(x)}{\log(e)}$ and find both graphs appear to be identical.

Exercises 3.4

Problems

In Exercises 1 – 15, use the property: $b^a = c$ if and only if $\log_b(c) = a$ from Theorem 3.4.2 to rewrite the given equation in the other form. That is, rewrite the exponential equations as logarithmic equations and rewrite the logarithmic equations as exponential equations.

1. $2^3 = 8$
2. $5^{-3} = \frac{1}{125}$
3. $4^{5/2} = 32$
4. $\left(\frac{1}{3}\right)^{-2} = 9$
5. $\left(\frac{4}{25}\right)^{-1/2} = \frac{5}{2}$
6. $10^{-3} = 0.001$
7. $e^0 = 1$
8. $\log_5(25) = 2$
9. $\log_{25}(5) = \frac{1}{2}$
10. $\log_3\left(\frac{1}{81}\right) = -4$
11. $\log_{\frac{4}{3}}\left(\frac{3}{4}\right) = -1$
12. $\log(100) = 2$
13. $\log(0.1) = -1$
14. $\ln(e) = 1$
15. $\ln\left(\frac{1}{\sqrt{e}}\right) = -\frac{1}{2}$

In Exercises 16 – 42, evaluate the expression.

16. $\log_3(27)$
17. $\log_6(216)$
18. $\log_2(32)$
19. $\log_6\left(\frac{1}{36}\right)$
20. $\log_8(4)$
21. $\log_{36}(216)$
22. $\log_{\frac{1}{5}}(625)$
23. $\log_{\frac{1}{6}}(216)$
24. $\log_{36}(36)$
25. $\log\left(\frac{1}{1000000}\right)$
26. $\log(0.01)$
27. $\ln\left(e^3\right)$
28. $\log_4(8)$
29. $\log_6(1)$
30. $\log_{13}\left(\sqrt{13}\right)$
31. $\log_{36}\left(\sqrt[4]{36}\right)$
32. $7^{\log_7(3)}$
33. $36^{\log_{36}(216)}$
34. $\log_{36}\left(36^{216}\right)$
35. $\ln\left(e^5\right)$
36. $\log\left(\sqrt[9]{10^{11}}\right)$
37. $\log\left(\sqrt[3]{10^5}\right)$
38. $\ln\left(\frac{1}{\sqrt{e}}\right)$
39. $\log_5\left(3^{\log_3(5)}\right)$
40. $\log\left(e^{\ln(100)}\right)$
41. $\log_2\left(3^{-\log_3(2)}\right)$
42. $\ln\left(42^{6\log(1)}\right)$

In Exercises 43 – 57, find the domain of the function.

43. $f(x) = \ln(x^2 + 1)$
44. $f(x) = \log_7(4x + 8)$
45. $f(x) = \ln(4x - 20)$
46. $f(x) = \log\left(x^2 + 9x + 18\right)$
47. $f(x) = \log\left(\dfrac{x + 2}{x^2 - 1}\right)$
48. $f(x) = \log\left(\dfrac{x^2 + 9x + 18}{4x - 20}\right)$

49. $f(x) = \ln(7 - x) + \ln(x - 4)$

50. $f(x) = \ln(4x - 20) + \ln\left(x^2 + 9x + 18\right)$

51. $f(x) = \log\left(x^2 + x + 1\right)$

52. $f(x) = \sqrt[4]{\log_4(x)}$

53. $f(x) = \log_9(|x + 3| - 4)$

54. $f(x) = \ln(\sqrt{x - 4} - 3)$

55. $f(x) = \dfrac{1}{3 - \log_5(x)}$

56. $f(x) = \dfrac{\sqrt{-1 - x}}{\log_{\frac{1}{2}}(x)}$

57. $f(x) = \ln(-2x^3 - x^2 + 13x - 6)$

In Exercises 58 – 63, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using transformations. Track at least three points of your choice and the horizontal asymptote through the transformations. State the domain and range of g.

58. $f(x) = 2^x$, $g(x) = 2^x - 1$

59. $f(x) = \left(\frac{1}{3}\right)^x$, $g(x) = \left(\frac{1}{3}\right)^{x-1}$

60. $f(x) = 3^x$, $g(x) = 3^{-x} + 2$

61. $f(x) = 10^x$, $g(x) = 10^{\frac{x+1}{2}} - 20$

62. $f(x) = e^x$, $g(x) = 8 - e^{-x}$

63. $f(x) = e^x$, $g(x) = 10e^{-0.1x}$

In Exercises 64 – 69, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using transformations. Track at least three points of your choice and the vertical asymptote through the transformations. State the domain and range of g.

64. $f(x) = \log_2(x)$, $g(x) = \log_2(x + 1)$

65. $f(x) = \log_{\frac{1}{3}}(x)$, $g(x) = \log_{\frac{1}{3}}(x) + 1$

66. $f(x) = \log_3(x)$, $g(x) = -\log_3(x - 2)$

67. $f(x) = \log(x)$, $g(x) = 2\log(x + 20) - 1$

68. $f(x) = \ln(x)$, $g(x) = -\ln(8 - x)$

69. $f(x) = \ln(x)$, $g(x) = -10\ln\left(\frac{x}{10}\right)$

In Exercises 70 – 84, expand the given logarithm and simplify. Assume when necessary that all quantities represent positive real numbers.

70. $\ln(x^3y^2)$

71. $\log_2\left(\dfrac{128}{x^2 + 4}\right)$

72. $\log_5\left(\dfrac{z}{25}\right)^3$

73. $\log(1.23 \times 10^{37})$

74. $\ln\left(\dfrac{\sqrt{z}}{xy}\right)$

75. $\log_5\left(x^2 - 25\right)$

76. $\log_{\sqrt{2}}\left(4x^3\right)$

77. $\log_{\frac{1}{3}}(9x(y^3 - 8))$

78. $\log\left(1000x^3y^5\right)$

79. $\log_3\left(\dfrac{x^2}{81y^4}\right)$

80. $\ln\left(\sqrt[4]{\dfrac{xy}{ez}}\right)$

81. $\log_6\left(\dfrac{216}{x^3y}\right)^4$

82. $\log\left(\dfrac{100x\sqrt{y}}{\sqrt[3]{10}}\right)$

83. $\log_{\frac{1}{2}}\left(\dfrac{4\sqrt[3]{x^2}}{y\sqrt{z}}\right)$

84. $\ln\left(\dfrac{\sqrt[3]{x}}{10\sqrt{yz}}\right)$

In Exercises 85 – 98, use the properties of logarithms to write the expression as a single logarithm.

85. $4\ln(x) + 2\ln(y)$

86. $\log_2(x) + \log_2(y) - \log_2(z)$

87. $\log_3(x) - 2\log_3(y)$

88. $\frac{1}{2}\log_3(x) - 2\log_3(y) - \log_3(z)$

89. $2\ln(x) - 3\ln(y) - 4\ln(z)$

90. $\log(x) - \frac{1}{3}\log(z) + \frac{1}{2}\log(y)$

91. $-\frac{1}{3}\ln(x) - \frac{1}{3}\ln(y) + \frac{1}{3}\ln(z)$

92. $\log_5(x) - 3$

93. $3 - \log(x)$

94. $\log_7(x) + \log_7(x-3) - 2$

95. $\ln(x) + \frac{1}{2}$

96. $\log_2(x) + \log_4(x)$

97. $\log_2(x) + \log_4(x-1)$

98. $\log_2(x) + \log_{\frac{1}{2}}(x-1)$

In Exercises 99 – 102, use the appropriate change of base formula to convert the given expression to an expression with the indicated base.

99. 7^{x-1} to base e

100. $\log_3(x+2)$ to base 10

101. $\left(\frac{2}{3}\right)^x$ to base e

102. $\log(x^2+1)$ to base e

In Exercises 103 – 108, use the appropriate change of base formula to approximate the logarithm.

103. $\log_3(12)$

104. $\log_5(80)$

105. $\log_6(72)$

106. $\log_4\left(\frac{1}{10}\right)$

107. $\log_{\frac{3}{5}}(1000)$

108. $\log_{\frac{2}{3}}(50)$

4: Foundations of Trigonometry

4.1 The Unit Circle: Sine and Cosine

In this section, we consider the problem of describing the position of a point on the unit circle. To that end, consider an angle θ in standard position and let P denote the point where the terminal side of θ intersects the Unit Circle, as in Figure 4.1.2. By associating the point P with the angle θ, we are assigning a *position* on the Unit Circle to the angle θ. The x-coordinate of P is called the **cosine** of θ, written $\cos(\theta)$, while the y-coordinate of P is called the **sine** of θ, written $\sin(\theta)$. The reader is encouraged to verify that these rules used to match an angle with its cosine and sine do, in fact, satisfy the definition of a function. That is, for each angle θ, there is only one associated value of $\cos(\theta)$ and only one associated value of $\sin(\theta)$.

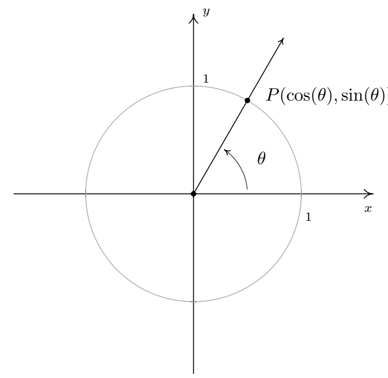


Figure 4.1.2: Defining $\cos(\theta)$ and $\sin(\theta)$

The etymology of the name 'sine' is quite colourful, and the interested reader is invited to research it; the 'co' in 'cosine' is explained in Section 4.3.

Example 4.1.1 Evaluating $\cos(\theta)$ and $\sin(\theta)$
Find the cosine and sine of the following angles.

1. $\theta = -\pi$
2. $\theta = \frac{\pi}{4}$
3. $\theta = \frac{\pi}{6}$
4. $\theta = \frac{\pi}{3}$

Solution

1. The angle $\theta = -\pi$ represents one half of a clockwise revolution so its terminal side lies on the negative x-axis. The point on the Unit Circle that lies on the negative x-axis is $(-1, 0)$ which means $\cos(-\pi) = -1$ and $\sin(-\pi) = 0$.

2. When we sketch $\theta = \frac{\pi}{4}$ in standard position, we see in Figure 4.1.1 that its terminal does not lie along any of the coordinate axes which makes our job of finding the cosine and sine values a bit more difficult. Let $P(x, y)$ denote the point on the terminal side of θ which lies on the Unit Circle. By definition, $x = \cos\left(\frac{\pi}{4}\right)$ and $y = \sin\left(\frac{\pi}{4}\right)$. If we drop a perpendicular line segment from P to the x-axis, we obtain a $45^\circ - 45^\circ - 90^\circ$ right triangle whose legs have lengths x and y units. From Geometry, we get $y = x$. (Can you show this?) Since $P(x, y)$ lies on the Unit Circle, we have $x^2 + y^2 = 1$. Substituting $y = x$ into this equation yields $2x^2 = 1$, or $x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$. Since $P(x, y)$ lies in the first quadrant, $x > 0$, so $x = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and with $y = x$ we have $y = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

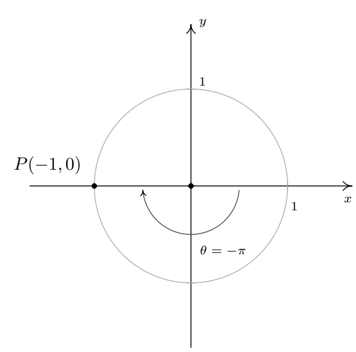
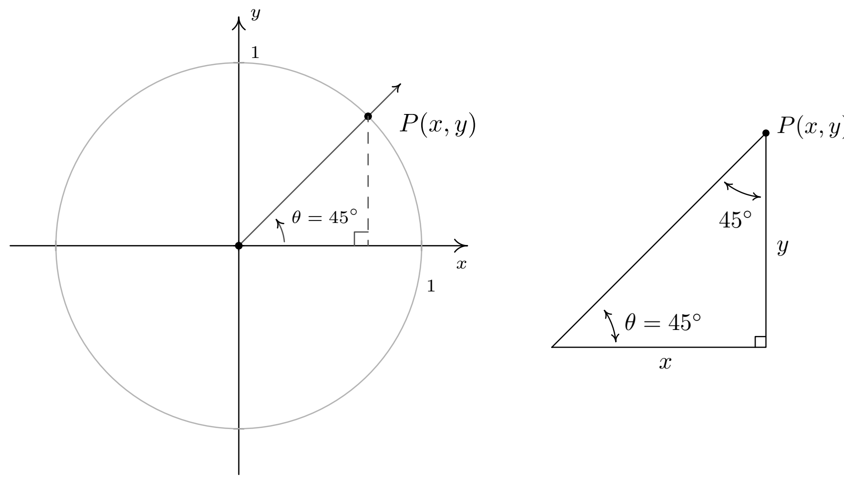


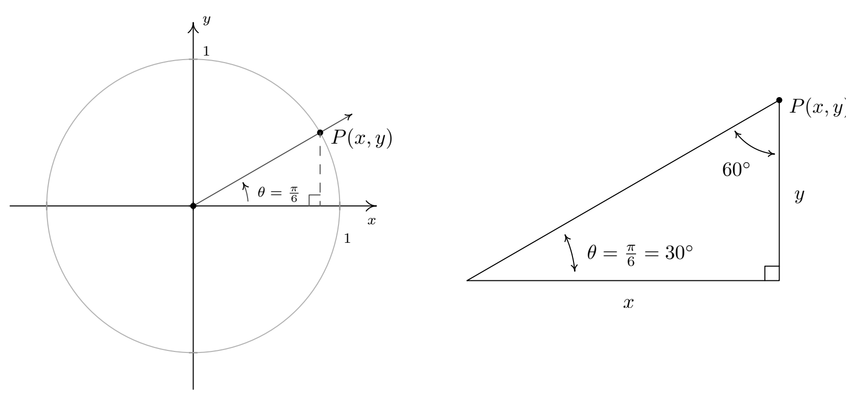
Figure 4.1.3: Finding $\cos(-\pi)$ and $\sin(-\pi)$



$\theta = \frac{\pi}{4}$ in standard position $\qquad$ $45^\circ - 45^\circ - 90^\circ$ triangle

Figure 4.1.1: Finding $\cos\left(\frac{\pi}{4}\right)$ and $\sin\left(\frac{\pi}{4}\right)$

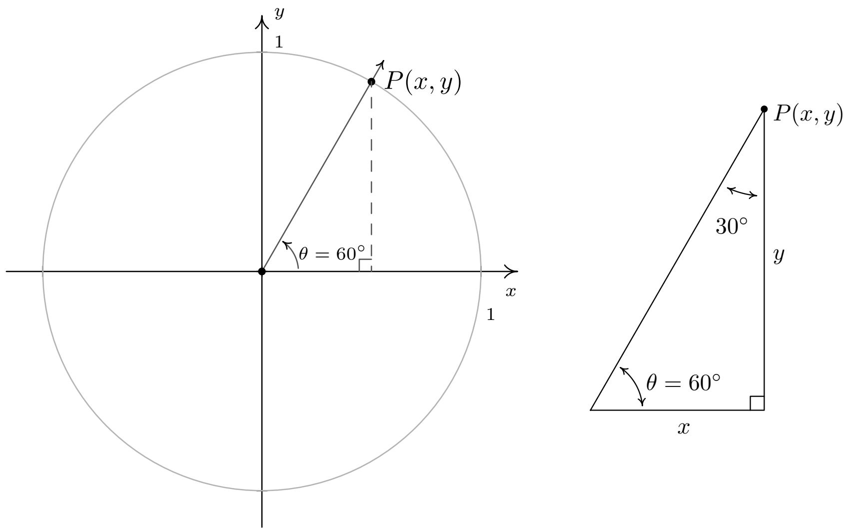
3. As before, the terminal side of $\theta = \frac{\pi}{6}$ does not lie on any of the coordinate axes, so we proceed using a triangle approach. Letting $P(x, y)$ denote the point on the terminal side of θ which lies on the Unit Circle, we drop a perpendicular line segment from P to the x-axis to form a $30^\circ - 60^\circ - 90^\circ$ right triangle: see Figure 4.1.4. After a bit of Geometry (again, can you show this?) we find $y = \frac{1}{2}$ so $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. Since $P(x, y)$ lies on the Unit Circle, we substitute $y = \frac{1}{2}$ into $x^2 + y^2 = 1$ to get $x^2 = \frac{3}{4}$, or $x = \pm\frac{\sqrt{3}}{2}$. Here, $x > 0$ so $x = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.



$\theta = \frac{\pi}{6}$ in standard position $\qquad$ $30^\circ - 60^\circ - 90^\circ$ triangle

Figure 4.1.4: Finding $\cos\left(\frac{\pi}{6}\right)$ and $\sin\left(\frac{\pi}{6}\right)$

4. Plotting $\theta = \frac{\pi}{3}$ in standard position, we find it is not a quadrantal angle and set about using a triangle approach. Once again, we get a $30^\circ - 60^\circ - 90^\circ$ right triangle and, after the usual computations, find $x = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $y = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.



$\theta = \frac{\pi}{3}$ in standard position $30° - 60° - 90°$ triangle

Figure 4.1.5: Finding $\cos\left(\frac{\pi}{3}\right)$ and $\sin\left(\frac{\pi}{3}\right)$

In Example 4.1.1, it was quite easy to find the cosine and sine of the quadrantal angles, but for non-quadrantal angles, the task was much more involved. In these latter cases, we made good use of the fact that the point $P(x, y) = (\cos(\theta), \sin(\theta))$ lies on the Unit Circle, $x^2 + y^2 = 1$. If we substitute $x = \cos(\theta)$ and $y = \sin(\theta)$ into $x^2 + y^2 = 1$, we get $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$. An unfortunate convention, which the authors are compelled to perpetuate, is to write $(\cos(\theta))^2$ as $\cos^2(\theta)$ and $(\sin(\theta))^2$ as $\sin^2(\theta)$. (This is unfortunate from a 'function notation' perspective, as you will see once you encounter the inverse trigonometric functions.) Rewriting the identity using this convention results in the following theorem, which is without a doubt one of the most important results in Trigonometry.

Theorem 4.1.1 The Pythagorean Identity

For any angle θ, $\cos^2(\theta) + \sin^2(\theta) = 1$.

The moniker 'Pythagorean' brings to mind the Pythagorean Theorem, from which both the Distance Formula and the equation for a circle are ultimately derived. The word 'Identity' reminds us that, regardless of the angle θ, the equation in Theorem 4.1.1 is always true. If one of $\cos(\theta)$ or $\sin(\theta)$ is known, Theorem 4.1.1 can be used to determine the other, up to a ($\pm$) sign. If, in addition, we know where the terminal side of θ lies when in standard position, then we can remove the ambiguity of the ($\pm$) and completely determine the missing value as the next example illustrates.

Example 4.1.2 Using the Pythagorean Identity

Using the given information about θ, find the indicated value.

1. If θ is a Quadrant II angle with $\sin(\theta) = \frac{3}{5}$, find $\cos(\theta)$.
2. If $\pi < \theta < \frac{3\pi}{2}$ with $\cos(\theta) = -\frac{\sqrt{5}}{5}$, find $\sin(\theta)$.
3. If $\sin(\theta) = 1$, find $\cos(\theta)$.

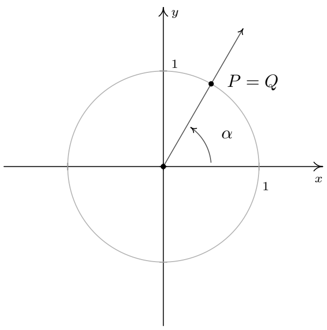


Figure 4.1.7: Reference angle α for a Quadrant I angle

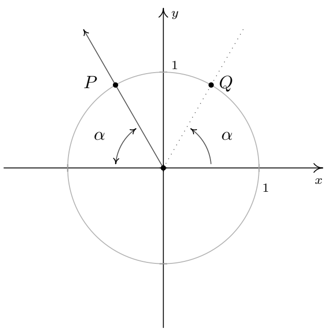


Figure 4.1.8: Reference angle α for a Quadrant II angle

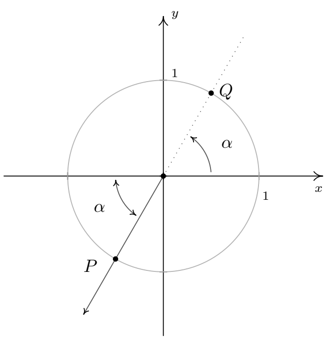


Figure 4.1.9: Reference angle α for a Quadrant III angle

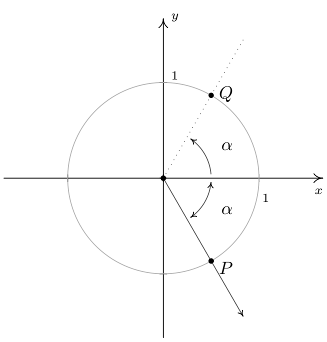


Figure 4.1.10: Reference angle α for a Quadrant IV angle

Solution

1. When we substitute $\sin(\theta) = \frac{3}{5}$ into The Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$, we obtain $\cos^2(\theta) + \frac{9}{25} = 1$. Solving, we find $\cos(\theta) = \pm\frac{4}{5}$. Since θ is a Quadrant II angle, its terminal side, when plotted in standard position, lies in Quadrant II. Since the *x*-coordinates are negative in Quadrant II, $\cos(\theta)$ is too. Hence, $\cos(\theta) = -\frac{4}{5}$.

2. Substituting $\cos(\theta) = -\frac{\sqrt{5}}{5}$ into $\cos^2(\theta) + \sin^2(\theta) = 1$ gives $\sin(\theta) = \pm\frac{2}{\sqrt{5}} = \pm\frac{2\sqrt{5}}{5}$. Since we are given that $\pi < \theta < \frac{3\pi}{2}$, we know θ is a Quadrant III angle. Hence both its sine and cosine are negative and we conclude $\sin(\theta) = -\frac{2\sqrt{5}}{5}$.

3. When we substitute $\sin(\theta) = 1$ into $\cos^2(\theta) + \sin^2(\theta) = 1$, we find $\cos(\theta) = 0$.

Another tool which helps immensely in determining cosines and sines of angles is the symmetry inherent in the Unit Circle. Suppose, for instance, we wish to know the cosine and sine of $\theta = \frac{5\pi}{6}$. We plot θ in standard position below and, as usual, let $P(x, y)$ denote the point on the terminal side of θ which lies on the Unit Circle. Note that the terminal side of θ lies $\frac{\pi}{6}$ radians short of one half revolution. In Example 4.1.1, we determined that $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. This means that the point on the terminal side of the angle $\frac{\pi}{6}$, when plotted in standard position, is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. From Figure 4.1.6, it is clear that the point $P(x, y)$ we seek can be obtained by reflecting that point about the *y*-axis. Hence, $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$.

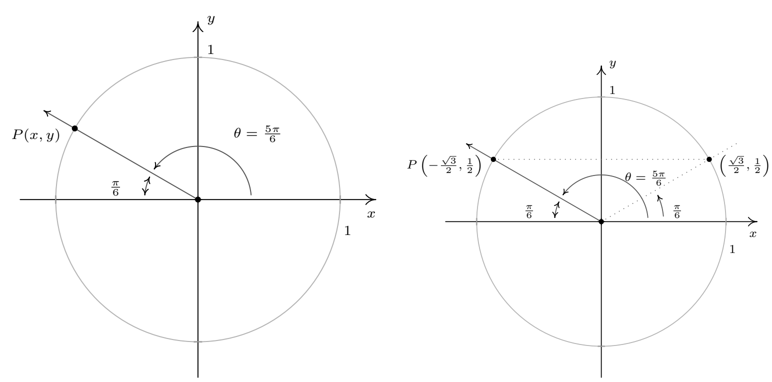


Figure 4.1.6: Refelcting $P(x, y)$ across the *y*-axis to obtain a Quadrant I angle

In the above scenario, the angle $\frac{\pi}{6}$ is called the **reference angle** for the angle $\frac{5\pi}{6}$. In general, for a non-quadrantal angle θ, the reference angle for θ (usually denoted α) is the *acute* angle made between the terminal side of θ and the *x*-axis. If θ is a Quadrant I or IV angle, α is the angle between the terminal side of θ and the *positive* *x*-axis; if θ is a Quadrant II or III angle, α is the angle between the terminal side of θ and the *negative* *x*-axis. If we let *P* denote the point $(\cos(\theta), \sin(\theta))$, then *P* lies on the Unit Circle. Since the Unit Circle possesses symmetry with respect to the *x*-axis, *y*-axis and origin, regardless of where the terminal side of θ lies, there is a point *Q* symmetric with *P* which determines θ's reference angle, α as seen below.

We have just outlined the proof of the following theorem.

Theorem 4.1.2 Reference Angle Theorem

Suppose α is the reference angle for θ. Then $\cos(\theta) = \pm\cos(\alpha)$ and $\sin(\theta) = \pm\sin(\alpha)$, where the choice of the $(\pm)$ depends on the quadrant in which the terminal side of θ lies.

In light of Theorem 4.1.2, it pays to know the cosine and sine values for certain common angles. In the table below, we summarize the values which we consider essential and must be memorized.

Cosine and Sine Values of Common Angles

θ(degrees)	θ(radians)	$\cos(\theta)$	$\sin(\theta)$
0°	0	1	0
30°	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
90°	$\frac{\pi}{2}$	0	1

Example 4.1.3 Using reference angles

Find the cosine and sine of the following angles.

1. $\theta = \frac{5\pi}{4}$
2. $\theta = \frac{11\pi}{6}$
3. $\theta = -\frac{5\pi}{4}$
4. $\theta = \frac{7\pi}{3}$

Solution

1. We begin by plotting $\theta = \frac{5\pi}{4}$ in standard position and find its terminal side overshoots the negative *x*-axis to land in Quadrant III. Hence, we obtain θ's reference angle α by subtracting: $\alpha = \theta - \pi = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$. Since θ is a Quadrant III angle, both $\cos(\theta) < 0$ and $\sin(\theta) < 0$. The Reference Angle Theorem yields: $\cos\left(\frac{5\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\sin\left(\frac{5\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

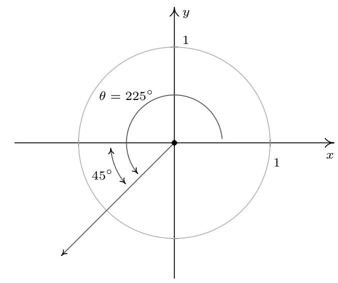


Figure 4.1.11: Finding $\cos\left(\frac{5\pi}{4}\right)$ and $\sin\left(\frac{5\pi}{4}\right)$

2. The terminal side of $\theta = \frac{11\pi}{6}$, when plotted in standard position, lies in Quadrant IV, just shy of the positive *x*-axis. To find θ's reference angle α, we subtract: $\alpha = 2\pi - \theta = 2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$. Since θ is a Quadrant IV angle, $\cos(\theta) > 0$ and $\sin(\theta) < 0$, so the Reference Angle Theorem gives: $\cos\left(\frac{11\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{11\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$.

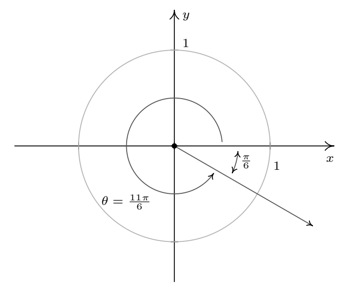


Figure 4.1.12: Finding $\cos\left(\frac{11\pi}{6}\right)$ and $\sin\left(\frac{11\pi}{6}\right)$

3. To plot $\theta = -\frac{5\pi}{4}$, we rotate *clockwise* an angle of $\frac{5\pi}{4}$ from the positive *x*-axis. The terminal side of θ, therefore, lies in Quadrant II making an angle of $\alpha = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$ radians with respect to the negative *x*-axis. Since θ is a Quadrant II angle, the Reference Angle Theorem gives: $\cos\left(-\frac{5\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\sin\left(-\frac{5\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

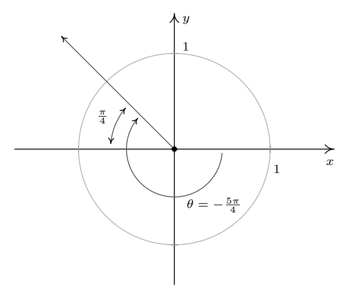


Figure 4.1.13: Finding $\cos\left(-\frac{5\pi}{4}\right)$ and $\sin\left(-\frac{5\pi}{4}\right)$

4. Since the angle $\theta = \frac{7\pi}{3}$ measures more than $2\pi = \frac{6\pi}{3}$, we find the terminal side of θ by rotating one full revolution followed by an additional $\alpha = \frac{7\pi}{3} - 2\pi = \frac{\pi}{3}$ radians. Since θ and α are coterminal, $\cos\left(\frac{7\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(\frac{7\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

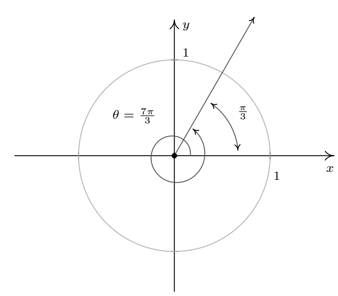


Figure 4.1.14: Finding $\cos\left(\frac{7\pi}{3}\right)$ and $\sin\left(\frac{7\pi}{3}\right)$

The reader may have noticed that when expressed in radian measure, the reference angle is easy to spot. Reduced fraction multiples of π with a denominator of 6 have $\frac{\pi}{6}$ as a reference angle, those with a denominator of 4 have $\frac{\pi}{4}$ as their reference angle, and those with a denominator of 3 have $\frac{\pi}{3}$ as their reference angle. The Reference Angle Theorem in conjunction with the table of cosine and sine values on Page 109 can be used to generate the following figure, which the authors feel should be committed to memory. (At the very least, one should memorize the first quadrant and learn to make use of Theorem 4.1.2.)

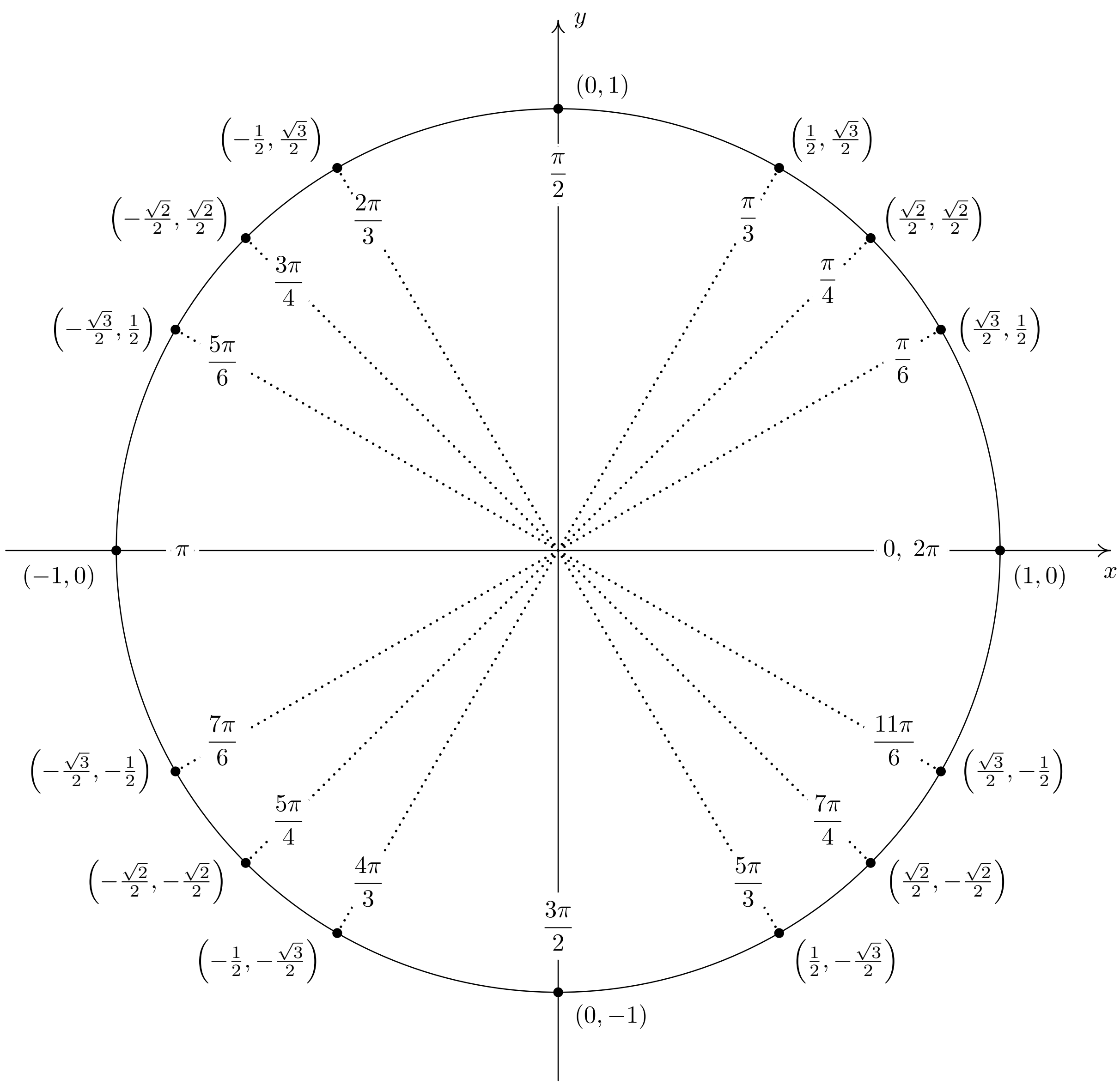


Figure 4.1.15: Important Points on the Unit Circle

Our next example asks us to solve some very basic trigonometric equations.

Example 4.1.4 Solving basic trigonometric equations

Find all of the angles which satisfy the given equation.

1. $\cos(\theta) = \dfrac{1}{2}$
2. $\sin(\theta) = -\dfrac{1}{2}$
3. $\cos(\theta) = 0.$

Solution

1. If $\cos(\theta) = \frac{1}{2}$, then the terminal side of θ, when plotted in standard position, intersects the Unit Circle at $x = \frac{1}{2}$. This means θ is a Quadrant I or IV angle with reference angle $\frac{\pi}{3}$.

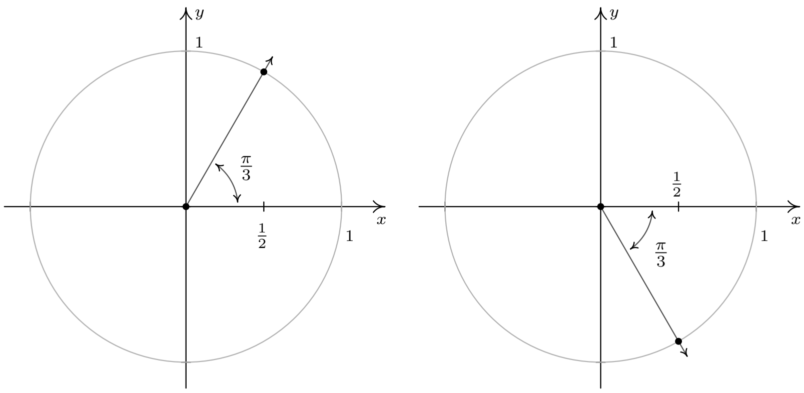


Figure 4.1.16: Angles with $\cos(\theta) = \frac{1}{2}$

One solution in Quadrant I is $\theta = \frac{\pi}{3}$, and since all other Quadrant I solutions must be coterminal with $\frac{\pi}{3}$, we find $\theta = \frac{\pi}{3} + 2\pi k$ for integers k. Proceeding similarly for the Quadrant IV case, we find the solution to $\cos(\theta) = \frac{1}{2}$ here is $\frac{5\pi}{3}$, so our answer in this Quadrant is $\theta = \frac{5\pi}{3} + 2\pi k$ for integers k.

2. If $\sin(\theta) = -\frac{1}{2}$, then when θ is plotted in standard position, its terminal side intersects the Unit Circle at $y = -\frac{1}{2}$. From this, we determine θ is a Quadrant III or Quadrant IV angle with reference angle $\frac{\pi}{6}$.

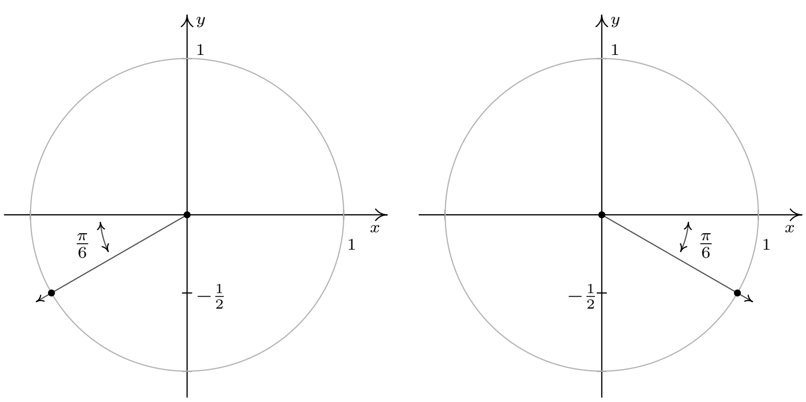


Figure 4.1.17: Angles with $\sin(\theta) = -\frac{1}{2}$

In Quadrant III, one solution is $\frac{7\pi}{6}$, so we capture all Quadrant III solutions by adding integer multiples of 2π: $\theta = \frac{7\pi}{6} + 2\pi k$. In Quadrant IV, one solution is $\frac{11\pi}{6}$ so all the solutions here are of the form $\theta = \frac{11\pi}{6} + 2\pi k$ for integers k.

3. The angles with $\cos(\theta) = 0$ are quadrantal angles whose terminal sides, when plotted in standard position, lie along the y-axis.

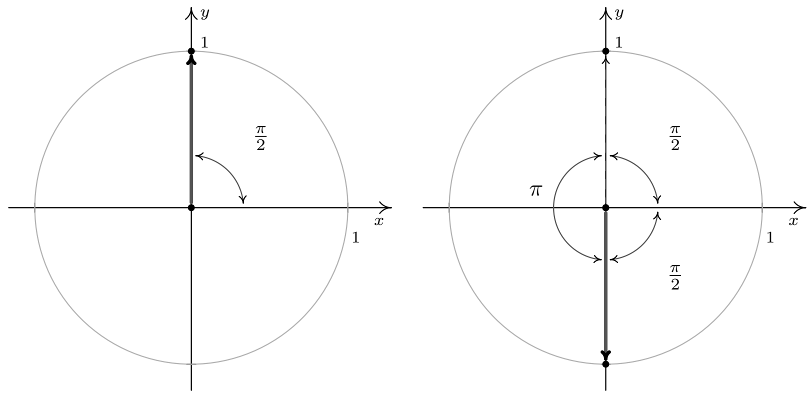


Figure 4.1.18: Angles with $\cos(\theta) = 0$

While, technically speaking, $\frac{\pi}{2}$ isn't a reference angle we can nonetheless use it to find our answers. If we follow the procedure set forth in the previous examples, we find $\theta = \frac{\pi}{2} + 2\pi k$ and $\theta = \frac{3\pi}{2} + 2\pi k$ for integers, k. While this solution is correct, it can be shortened to $\theta = \frac{\pi}{2} + \pi k$ for integers k. (Can you see why this works from the diagram?)

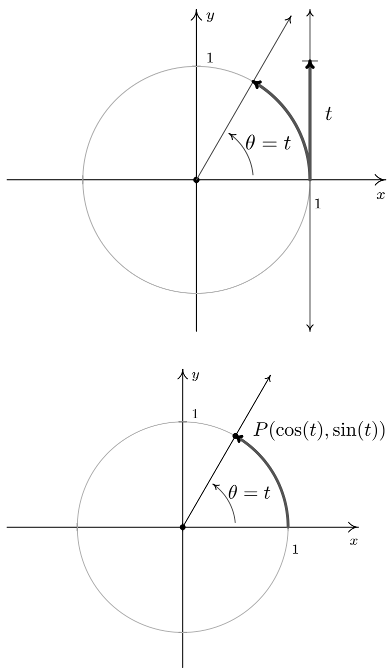


Figure 4.1.19: Defining $\cos(t)$ and $\sin(t)$ as functions of a real variable

One of the key items to take from Example 4.1.4 is that, in general, solutions to trigonometric equations consist of infinitely many answers. The reader is encouraged write out as many of these answers as necessary to get a feel for them. This is especially important when checking answers to the exercises. For example, another Quadrant IV solution to $\sin(\theta) = -\frac{1}{2}$ is $\theta = -\frac{\pi}{6}$. Hence, the family of Quadrant IV answers to number 2 above could just have easily been written $\theta = -\frac{\pi}{6} + 2\pi k$ for integers k. While on the surface, this family may look different than the stated solution of $\theta = \frac{11\pi}{6} + 2\pi k$ for integers k, we leave it to the reader to show they represent the same list of angles.

We close this section by noting that we can easily extend the functions cosine and sine to real numbers by identifying a real number t with the angle $\theta = t$ radians. Using this identification, we define $\cos(t) = \cos(\theta)$ and $\sin(t) = \sin(\theta)$. In practice this means expressions like $\cos(\pi)$ and $\sin(2)$ can be found by regarding the inputs as angles in radian measure or real numbers; the choice is the reader's.

In the same way we studied polynomial, rational, exponential, and logarithmic functions, we will study the trigonometric functions $f(t) = \cos(t)$ and $g(t) = \sin(t)$. The first order of business is to find the domains and ranges of these functions. Whether we think of identifying the real number t with the angle $\theta = t$ radians, or think of wrapping an oriented arc around the Unit Circle to find coordinates on the Unit Circle, it should be clear that both the cosine and sine functions are defined for all real numbers t. In other words, the domain of $f(t) = \cos(t)$ and of $g(t) = \sin(t)$ is $(-\infty, \infty)$. Since $\cos(t)$ and $\sin(t)$ represent x- and y-coordinates, respectively, of points on the Unit Circle, they both take on all of the values between -1 an 1, inclusive. In other words, the range of $f(t) = \cos(t)$ and of $g(t) = \sin(t)$ is the interval $[-1, 1]$. To summarize:

Theorem 4.1.3 Domain and Range of the Cosine and Sine Functions

- The function $f(t) = \cos(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
- The function $g(t) = \sin(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$

Suppose, as in the Exercises, we are asked to solve an equation such as $\sin(t) = -\frac{1}{2}$. As we have already mentioned, the distinction between t as a real number and as an angle $\theta = t$ radians is often blurred. Indeed, we solve $\sin(t) = -\frac{1}{2}$ in the exact same manner as we did in Example 4.1.4 number 2. Our solution is only cosmetically different in that the variable used is t rather than θ: $t = \frac{7\pi}{6} + 2\pi k$ or $t = \frac{11\pi}{6} + 2\pi k$ for integers, k. We will study the cosine and sine functions in greater detail in Section 4.4. Until then, keep in mind that any properties of cosine and sine developed in the following sections which regard them as functions of *angles* in *radian* measure apply equally well if the inputs are regarded as *real numbers*.

Exercises 4.1

Problems

In Exercises 1 – 20, find the exact value of the cosine and sine of the given angle.

1. $\theta = 0$
2. $\theta = \frac{\pi}{4}$
3. $\theta = \frac{\pi}{3}$
4. $\theta = \frac{\pi}{2}$
5. $\theta = \frac{2\pi}{3}$
6. $\theta = \frac{3\pi}{4}$
7. $\theta = \pi$
8. $\theta = \frac{7\pi}{6}$
9. $\theta = \frac{5\pi}{4}$
10. $\theta = \frac{4\pi}{3}$
11. $\theta = \frac{3\pi}{2}$
12. $\theta = \frac{5\pi}{3}$
13. $\theta = \frac{7\pi}{4}$
14. $\theta = \frac{23\pi}{6}$
15. $\theta = -\frac{13\pi}{2}$
16. $\theta = -\frac{43\pi}{6}$
17. $\theta = -\frac{3\pi}{4}$
18. $\theta = -\frac{\pi}{6}$
19. $\theta = \frac{10\pi}{3}$
20. $\theta = 117\pi$

In Exercises 21 – 30, use the results developed throughout the section to find the requested value.

21. If $\sin(\theta) = -\frac{7}{25}$ with θ in Quadrant IV, what is $\cos(\theta)$?
22. If $\cos(\theta) = \frac{4}{9}$ with θ in Quadrant I, what is $\sin(\theta)$?
23. If $\sin(\theta) = \frac{5}{13}$ with θ in Quadrant II, what is $\cos(\theta)$?
24. If $\cos(\theta) = -\frac{2}{11}$ with θ in Quadrant III, what is $\sin(\theta)$?
25. If $\sin(\theta) = -\frac{2}{3}$ with θ in Quadrant III, what is $\cos(\theta)$?
26. If $\cos(\theta) = \frac{28}{53}$ with θ in Quadrant IV, what is $\sin(\theta)$?
27. If $\sin(\theta) = \frac{2\sqrt{5}}{5}$ and $\frac{\pi}{2} < \theta < \pi$, what is $\cos(\theta)$?
28. If $\cos(\theta) = \frac{\sqrt{10}}{10}$ and $2\pi < \theta < \frac{5\pi}{2}$, what is $\sin(\theta)$?
29. If $\sin(\theta) = -0.42$ and $\pi < \theta < \frac{3\pi}{2}$, what is $\cos(\theta)$?
30. If $\cos(\theta) = -0.98$ and $\frac{\pi}{2} < \theta < \pi$, what is $\sin(\theta)$?

In Exercises 31 – 39, find all of the angles which satisfy the given equation.

31. $\sin(\theta) = \frac{1}{2}$
32. $\cos(\theta) = -\frac{\sqrt{3}}{2}$
33. $\sin(\theta) = 0$
34. $\cos(\theta) = \frac{\sqrt{2}}{2}$
35. $\sin(\theta) = \frac{\sqrt{3}}{2}$
36. $\cos(\theta) = -1$
37. $\sin(\theta) = -1$
38. $\cos(\theta) = \frac{\sqrt{3}}{2}$
39. $\cos(\theta) = -1.001$

4.2 The Six Circular Functions and Fundamental Identities

In section 4.1, we defined $\cos(\theta)$ and $\sin(\theta)$ for angles θ using the coordinate values of points on the Unit Circle. As such, these functions earn the moniker **circular functions**. It turns out that cosine and sine are just two of the six commonly used circular functions which we define below.

Definition 4.2.1 The Circular Functions

Suppose θ is an angle plotted in standard position and $P(x, y)$ is the point on the terminal side of θ which lies on the Unit Circle.

- The **cosine** of θ, denoted $\cos(\theta)$, is defined by $\cos(\theta) = x$.
- The **sine** of θ, denoted $\sin(\theta)$, is defined by $\sin(\theta) = y$.
- The **secant** of θ, denoted $\sec(\theta)$, is defined by $\sec(\theta) = \dfrac{1}{x}$, provided $x \neq 0$.
- The **cosecant** of θ, denoted $\csc(\theta)$, is defined by $\csc(\theta) = \dfrac{1}{y}$, provided $y \neq 0$.
- The **tangent** of θ, denoted $\tan(\theta)$, is defined by $\tan(\theta) = \dfrac{y}{x}$, provided $x \neq 0$.
- The **cotangent** of θ, denoted $\cot(\theta)$, is defined by $\cot(\theta) = \dfrac{x}{y}$, provided $y \neq 0$.

The functions in Definition 4.2.1 are also (and perhaps, more commonly) known as *trigonometric* functions, owing to the fact that the can also be defined in terms of ratios of the three sides of a right-angle triangle

While we left the history of the name 'sine' as an interesting research project in Section 4.1, the names 'tangent' and 'secant' can be explained using the diagram below. Consider the acute angle θ below in standard position. Let $P(x, y)$ denote, as usual, the point on the terminal side of θ which lies on the Unit Circle and let $Q(1, y')$ denote the point on the terminal side of θ which lies on the vertical line $x = 1$, as in Figure 4.2.1.

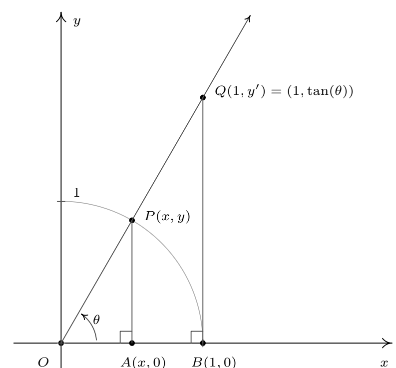


Figure 4.2.1: Explaining the tangent and secant functions

The word 'tangent' comes from the Latin meaning 'to touch,' and for this reason, the line $x = 1$ is called a *tangent* line to the Unit Circle since it intersects, or 'touches', the circle at only one point, namely $(1, 0)$. Dropping perpendiculars from P and Q creates a pair of similar triangles $\triangle OPA$ and $\triangle OQB$. Thus $\frac{y'}{y} = \frac{1}{x}$ which gives $y' = \frac{y}{x} = \tan(\theta)$, where this last equality comes from applying Definition 4.2.1. We have just shown that for acute angles θ, $\tan(\theta)$ is the y-coordinate of the point on the terminal side of θ which lies on the line $x = 1$ which is *tangent* to the Unit Circle. Now the word 'secant' means 'to cut', so a secant line is any line that 'cuts through' a circle at two points. (Compare this with the definition given in Section 3.1.1.) The line containing the terminal side of θ is a secant line since it intersects the Unit Circle in Quadrants I and III. With the point P lying on the Unit Circle, the length of the hypotenuse of $\triangle OPA$ is 1. If we let h denote the length of the hypotenuse of $\triangle OQB$, we have from similar triangles that $\frac{h}{1} = \frac{1}{x}$, or $h = \frac{1}{x} = \sec(\theta)$. Hence for an acute angle θ, $\sec(\theta)$ is the length of the line segment which lies on the secant line determined by the terminal side of θ and 'cuts off' the tangent line $x = 1$. Not only do these observations help explain the names of these functions, they serve as the basis for a fundamental inequality needed for Calculus which we'll explore in the Exercises.

Of the six circular functions, only cosine and sine are defined for all angles. Since $\cos(\theta) = x$ and $\sin(\theta) = y$ in Definition 4.2.1, it is customary to rephrase the remaining four circular functions in terms of cosine and sine. The following theorem is a result of simply replacing x with $\cos(\theta)$ and y with $\sin(\theta)$ in Definition 4.2.1.

Theorem 4.2.1 Reciprocal and Quotient Identities

- $\sec(\theta) = \dfrac{1}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$; if $\cos(\theta) = 0$, $\sec(\theta)$ is undefined.
- $\csc(\theta) = \dfrac{1}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$; if $\sin(\theta) = 0$, $\csc(\theta)$ is undefined.
- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$; if $\cos(\theta) = 0$, $\tan(\theta)$ is undefined.
- $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$; if $\sin(\theta) = 0$, $\cot(\theta)$ is undefined.

Example 4.2.1 Evaluating circular functions

Find the indicated value, if it exists.

1. $\csc\left(\frac{7\pi}{4}\right)$
2. $\cot(3)$
3. $\tan(\theta)$, where θ is any angle coterminal with $\frac{3\pi}{2}$.
4. $\cos(\theta)$, where $\csc(\theta) = -\sqrt{5}$ and θ is a Quadrant IV angle.
5. $\sin(\theta)$, where $\tan(\theta) = 3$ and $\pi < \theta < \frac{3\pi}{2}$.

Solution

1. Since $\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $\csc\left(\frac{7\pi}{4}\right) = \frac{1}{\sin\left(\frac{7\pi}{4}\right)} = \frac{1}{-\sqrt{2}/2} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$.
2. Since $\theta = 3$ radians is not one of the 'common angles' from Section 4.1, we resort to the calculator for a decimal approximation. Ensuring that the calculator is in radian mode, we find $\cot(3) = \frac{\cos(3)}{\sin(3)} \approx -7.015$.
3. If θ is coterminal with $\frac{3\pi}{2}$, then $\cos(\theta) = \cos\left(\frac{3\pi}{2}\right) = 0$ and $\sin(\theta) = \sin\left(\frac{3\pi}{2}\right) = -1$. Attempting to compute $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ results in $\frac{-1}{0}$, so $\tan(\theta)$ is undefined.
4. We are given that $\csc(\theta) = \frac{1}{\sin(\theta)} = -\sqrt{5}$ so $\sin(\theta) = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$. As we saw in Section 4.1, we can use the Pythagorean Identity, $\cos^2(\theta) +$

$\sin^2(\theta) = 1$, to find $\cos(\theta)$ by knowing $\sin(\theta)$. Substituting, we get $\cos^2(\theta) + \left(-\frac{\sqrt{5}}{5}\right)^2 = 1$, which gives $\cos^2(\theta) = \frac{4}{5}$, or $\cos(\theta) = \pm\frac{2\sqrt{5}}{5}$. Since θ is a Quadrant IV angle, $\cos(\theta) > 0$, so $\cos(\theta) = \frac{2\sqrt{5}}{5}$.

5. If $\tan(\theta) = 3$, then $\frac{\sin(\theta)}{\cos(\theta)} = 3$. Be careful - this does **NOT** mean we can take $\sin(\theta) = 3$ and $\cos(\theta) = 1$. Instead, from $\frac{\sin(\theta)}{\cos(\theta)} = 3$ we get: $\sin(\theta) = 3\cos(\theta)$. To relate $\cos(\theta)$ and $\sin(\theta)$, we once again employ the Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$. Solving $\sin(\theta) = 3\cos(\theta)$ for $\cos(\theta)$, we find $\cos(\theta) = \frac{1}{3}\sin(\theta)$. Substituting this into the Pythagorean Identity, we find $\sin^2(\theta) + \left(\frac{1}{3}\sin(\theta)\right)^2 = 1$. Solving, we get $\sin^2(\theta) = \frac{9}{10}$ so $\sin(\theta) = \pm\frac{3\sqrt{10}}{10}$. Since $\pi < \theta < \frac{3\pi}{2}$, θ is a Quadrant III angle. This means $\sin(\theta) < 0$, so our final answer is $\sin(\theta) = -\frac{3\sqrt{10}}{10}$.

Our next step is to provide versions of the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ for the remaining circular functions. Assuming $\cos(\theta) \neq 0$, we may start with $\cos^2(\theta) + \sin^2(\theta) = 1$ and divide both sides by $\cos^2(\theta)$ to obtain $1 + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$. Using properties of exponents along with the Reciprocal and Quotient Identities, this reduces to $1 + \tan^2(\theta) = \sec^2(\theta)$. If $\sin(\theta) \neq 0$, we can divide both sides of the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ by $\sin^2(\theta)$, apply Theorem 4.2.1 once again, and obtain $\cot^2(\theta) + 1 = \csc^2(\theta)$. These three Pythagorean Identities are worth memorizing and they, along with some of their other common forms, are summarized in the following theorem.

Theorem 4.2.2 The Pythagorean Identities

1. $\cos^2(\theta) + \sin^2(\theta) = 1$.

   Common Alternate Forms:

   - $1 - \sin^2(\theta) = \cos^2(\theta)$
   - $1 - \cos^2(\theta) = \sin^2(\theta)$

2. $1 + \tan^2(\theta) = \sec^2(\theta)$, provided $\cos(\theta) \neq 0$.

   Common Alternate Forms:

   - $\sec^2(\theta) - \tan^2(\theta) = 1$
   - $\sec^2(\theta) - 1 = \tan^2(\theta)$

3. $1 + \cot^2(\theta) = \csc^2(\theta)$, provided $\sin(\theta) \neq 0$.

   Common Alternate Forms:

   - $\csc^2(\theta) - \cot^2(\theta) = 1$
   - $\csc^2(\theta) - 1 = \cot^2(\theta)$

Example 4.2.2 Verifying trigonometric identities

Verify the following identities. Assume that all quantities are defined.

1. $\dfrac{1}{\csc(\theta)} = \sin(\theta)$
2. $\tan(\theta) = \sin(\theta)\sec(\theta)$
3. $(\sec(\theta) - \tan(\theta))(\sec(\theta) + \tan(\theta)) = 1$
4. $\dfrac{\sec(\theta)}{1-\tan(\theta)} = \dfrac{1}{\cos(\theta)-\sin(\theta)}$

Solution In verifying identities, we typically start with the more complicated side of the equation and use known identities to *transform* it into the other side of the equation.

1. To verify $\frac{1}{\csc(\theta)} = \sin(\theta)$, we start with the left side. Using $\csc(\theta) = \frac{1}{\sin(\theta)}$, we get:
$$\frac{1}{\csc(\theta)} = \frac{1}{\frac{1}{\sin(\theta)}} = \sin(\theta),$$
which is what we were trying to prove.

2. Starting with the right hand side of $\tan(\theta) = \sin(\theta)\sec(\theta)$, we use $\sec(\theta) = \frac{1}{\cos(\theta)}$ and find:
$$\sin(\theta)\sec(\theta) = \sin(\theta)\frac{1}{\cos(\theta)} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta),$$
where the last equality is courtesy of Theorem 4.2.1.

3. Expanding the left hand side of the equation gives: $(\sec(\theta) - \tan(\theta))(\sec(\theta) + \tan(\theta)) = \sec^2(\theta) - \tan^2(\theta)$. According to Theorem 4.2.2, $\sec^2(\theta) - \tan^2(\theta) = 1$. Putting it all together,
$$(\sec(\theta) - \tan(\theta))(\sec(\theta) + \tan(\theta)) = \sec^2(\theta) - \tan^2(\theta) = 1.$$

4. While both sides of our last identity contain fractions, the left side affords us more opportunities to use our identities. Substituting $\sec(\theta) = \frac{1}{\cos(\theta)}$ and $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, we get:
$$\begin{aligned}
\frac{\sec(\theta)}{1-\tan(\theta)} &= \frac{\frac{1}{\cos(\theta)}}{1-\frac{\sin(\theta)}{\cos(\theta)}} = \frac{\frac{1}{\cos(\theta)}}{1-\frac{\sin(\theta)}{\cos(\theta)}} \cdot \frac{\cos(\theta)}{\cos(\theta)} \\
&= \frac{\left(\frac{1}{\cos(\theta)}\right)(\cos(\theta))}{\left(1-\frac{\sin(\theta)}{\cos(\theta)}\right)(\cos(\theta))} \\
&= \frac{1}{(1)(\cos(\theta)) - \left(\frac{\sin(\theta)}{\cos(\theta)}\right)(\cos(\theta))} \\
&= \frac{1}{\cos(\theta)-\sin(\theta)},
\end{aligned}$$
which is exactly what we had set out to show.

Verifying trigonometric identities requires a healthy mix of tenacity and inspiration. You will need to spend many hours struggling with them just to become proficient in the basics. Like many things in life, there is no short-cut here – there is no complete algorithm for verifying identities. Nevertheless, a summary of some strategies which may be helpful (depending on the situation) is provided below and ample practice is provided for you in the Exercises.

Key Idea 4.2.1 Strategies for Verifying Identities

- Try working on the more complicated side of the identity.
- Use the Reciprocal and Quotient Identities in Theorem 4.2.1 to write functions on one side of the identity in terms of the functions on the other side of the identity. Simplify the resulting complex fractions.
- Add rational expressions with unlike denominators by obtaining common denominators.
- Use the Pythagorean Identities in Theorem 4.2.2 to 'exchange' sines and cosines, secants and tangents, cosecants and cotangents, and simplify sums or differences of squares to one term.
- Multiply numerator **and** denominator by Pythagorean Conjugates in order to take advantage of the Pythagorean Identities in Theorem 4.2.2.
- If you find yourself stuck working with one side of the identity, try starting with the other side of the identity and see if you can find a way to bridge the two parts of your work.

Exercises 4.2

Problems

In Exercises 1 – 20, find the exact value of the cosine and sine of the given angle.

1. $\theta = 0$
2. $\theta = \dfrac{\pi}{4}$
3. $\theta = \dfrac{\pi}{3}$
4. $\theta = \dfrac{\pi}{2}$
5. $\theta = \dfrac{2\pi}{3}$
6. $\theta = \dfrac{3\pi}{4}$
7. $\theta = \pi$
8. $\theta = \dfrac{7\pi}{6}$
9. $\theta = \dfrac{5\pi}{4}$
10. $\theta = \dfrac{4\pi}{3}$
11. $\theta = \dfrac{3\pi}{2}$
12. $\theta = \dfrac{5\pi}{3}$
13. $\theta = \dfrac{7\pi}{4}$
14. $\theta = \dfrac{23\pi}{6}$
15. $\theta = -\dfrac{13\pi}{2}$
16. $\theta = -\dfrac{43\pi}{6}$
17. $\theta = -\dfrac{3\pi}{4}$
18. $\theta = -\dfrac{\pi}{6}$
19. $\theta = \dfrac{10\pi}{3}$
20. $\theta = 117\pi$

In Exercises 21 – 34, use the given the information to find the exact values of the remaining circular functions of θ.

21. $\sin(\theta) = \dfrac{3}{5}$ with θ in Quadrant II
22. $\tan(\theta) = \dfrac{12}{5}$ with θ in Quadrant III
23. $\csc(\theta) = \dfrac{25}{24}$ with θ in Quadrant I
24. $\sec(\theta) = 7$ with θ in Quadrant IV
25. $\csc(\theta) = -\dfrac{10\sqrt{91}}{91}$ with θ in Quadrant III
26. $\cot(\theta) = -23$ with θ in Quadrant II
27. $\tan(\theta) = -2$ with θ in Quadrant IV.
28. $\sec(\theta) = -4$ with θ in Quadrant II.
29. $\cot(\theta) = \sqrt{5}$ with θ in Quadrant III.
30. $\cos(\theta) = \dfrac{1}{3}$ with θ in Quadrant I.
31. $\cot(\theta) = 2$ with $0 < \theta < \dfrac{\pi}{2}$.
32. $\csc(\theta) = 5$ with $\dfrac{\pi}{2} < \theta < \pi$.
33. $\tan(\theta) = \sqrt{10}$ with $\pi < \theta < \dfrac{3\pi}{2}$.
34. $\sec(\theta) = 2\sqrt{5}$ with $\dfrac{3\pi}{2} < \theta < 2\pi$.

In Exercises 35 – 49, find all of the angles which satisfy the equation.

35. $\tan(\theta) = \sqrt{3}$
36. $\sec(\theta) = 2$
37. $\csc(\theta) = -1$
38. $\cot(\theta) = \dfrac{\sqrt{3}}{3}$
39. $\tan(\theta) = 0$
40. $\sec(\theta) = 1$
41. $\csc(\theta) = 2$
42. $\cot(\theta) = 0$

43. $\tan(\theta) = -1$

44. $\sec(\theta) = 0$

45. $\csc(\theta) = -\dfrac{1}{2}$

46. $\sec(\theta) = -1$

47. $\tan(\theta) = -\sqrt{3}$

48. $\csc(\theta) = -2$

49. $\cot(\theta) = -1$

In Exercises 50 – 57, solve the equation for t. Give exact values.

50. $\cot(t) = 1$

51. $\tan(t) = \dfrac{\sqrt{3}}{3}$

52. $\sec(t) = -\dfrac{2\sqrt{3}}{3}$

53. $\csc(t) = 0$

54. $\cot(t) = -\sqrt{3}$

55. $\tan(t) = -\dfrac{\sqrt{3}}{3}$

56. $\sec(t) = \dfrac{2\sqrt{3}}{3}$

57. $\csc(t) = \dfrac{2\sqrt{3}}{3}$

In Exercises 58 – 104, verify the identity. Assume that all quantities are defined.

58. $\cos(\theta)\sec(\theta) = 1$

59. $\tan(\theta)\cos(\theta) = \sin(\theta)$

60. $\sin(\theta)\csc(\theta) = 1$

61. $\tan(\theta)\cot(\theta) = 1$

62. $\csc(\theta)\cos(\theta) = \cot(\theta)$

63. $\dfrac{\sin(\theta)}{\cos^2(\theta)} = \sec(\theta)\tan(\theta)$

64. $\dfrac{\cos(\theta)}{\sin^2(\theta)} = \csc(\theta)\cot(\theta)$

65. $\dfrac{1+\sin(\theta)}{\cos(\theta)} = \sec(\theta) + \tan(\theta)$

66. $\dfrac{1-\cos(\theta)}{\sin(\theta)} = \csc(\theta) - \cot(\theta)$

67. $\dfrac{\cos(\theta)}{1-\sin^2(\theta)} = \sec(\theta)$

68. $\dfrac{\sin(\theta)}{1-\cos^2(\theta)} = \csc(\theta)$

69. $\dfrac{\sec(\theta)}{1+\tan^2(\theta)} = \cos(\theta)$

70. $\dfrac{\csc(\theta)}{1+\cot^2(\theta)} = \sin(\theta)$

71. $\dfrac{\tan(\theta)}{\sec^2(\theta)-1} = \cot(\theta)$

72. $\dfrac{\cot(\theta)}{\csc^2(\theta)-1} = \tan(\theta)$

73. $4\cos^2(\theta) + 4\sin^2(\theta) = 4$

74. $9 - \cos^2(\theta) - \sin^2(\theta) = 8$

75. $\tan^3(\theta) = \tan(\theta)\sec^2(\theta) - \tan(\theta)$

76. $\sin^5(\theta) = \left(1-\cos^2(\theta)\right)^2\sin(\theta)$

77. $\sec^{10}(\theta) = \left(1+\tan^2(\theta)\right)^4\sec^2(\theta)$

78. $\cos^2(\theta)\tan^3(\theta) = \tan(\theta) - \sin(\theta)\cos(\theta)$

79. $\sec^4(\theta) - \sec^2(\theta) = \tan^2(\theta) + \tan^4(\theta)$

80. $\dfrac{\cos(\theta)+1}{\cos(\theta)-1} = \dfrac{1+\sec(\theta)}{1-\sec(\theta)}$

81. $\dfrac{\sin(\theta)+1}{\sin(\theta)-1} = \dfrac{1+\csc(\theta)}{1-\csc(\theta)}$

82. $\dfrac{1-\cot(\theta)}{1+\cot(\theta)} = \dfrac{\tan(\theta)-1}{\tan(\theta)+1}$

83. $\dfrac{1-\tan(\theta)}{1+\tan(\theta)} = \dfrac{\cos(\theta)-\sin(\theta)}{\cos(\theta)+\sin(\theta)}$

84. $\tan(\theta) + \cot(\theta) = \sec(\theta)\csc(\theta)$

85. $\csc(\theta) - \sin(\theta) = \cot(\theta)\cos(\theta)$

86. $\cos(\theta) - \sec(\theta) = -\tan(\theta)\sin(\theta)$

87. $\cos(\theta)(\tan(\theta) + \cot(\theta)) = \csc(\theta)$

88. $\sin(\theta)(\tan(\theta) + \cot(\theta)) = \sec(\theta)$

89. $\dfrac{1}{1-\cos(\theta)} + \dfrac{1}{1+\cos(\theta)} = 2\csc^2(\theta)$

90. $\dfrac{1}{\sec(\theta)+1}+\dfrac{1}{\sec(\theta)-1}=2\csc(\theta)\cot(\theta)$

91. $\dfrac{1}{\csc(\theta)+1}+\dfrac{1}{\csc(\theta)-1}=2\sec(\theta)\tan(\theta)$

92. $\dfrac{1}{\csc(\theta)-\cot(\theta)}-\dfrac{1}{\csc(\theta)+\cot(\theta)}=2\cot(\theta)$

93. $\dfrac{\cos(\theta)}{1-\tan(\theta)}+\dfrac{\sin(\theta)}{1-\cot(\theta)}=\sin(\theta)+\cos(\theta)$

94. $\dfrac{1}{\sec(\theta)+\tan(\theta)}=\sec(\theta)-\tan(\theta)$

95. $\dfrac{1}{\sec(\theta)-\tan(\theta)}=\sec(\theta)+\tan(\theta)$

96. $\dfrac{1}{\csc(\theta)-\cot(\theta)}=\csc(\theta)+\cot(\theta)$

97. $\dfrac{1}{\csc(\theta)+\cot(\theta)}=\csc(\theta)-\cot(\theta)$

98. $\dfrac{1}{1-\sin(\theta)}=\sec^2(\theta)+\sec(\theta)\tan(\theta)$

99. $\dfrac{1}{1+\sin(\theta)}=\sec^2(\theta)-\sec(\theta)\tan(\theta)$

100. $\dfrac{1}{1-\cos(\theta)}=\csc^2(\theta)+\csc(\theta)\cot(\theta)$

101. $\dfrac{1}{1+\cos(\theta)}=\csc^2(\theta)-\csc(\theta)\cot(\theta)$

102. $\dfrac{\cos(\theta)}{1+\sin(\theta)}=\dfrac{1-\sin(\theta)}{\cos(\theta)}$

103. $\csc(\theta)-\cot(\theta)=\dfrac{\sin(\theta)}{1+\cos(\theta)}$

104. $\dfrac{1-\sin(\theta)}{1+\sin(\theta)}=(\sec(\theta)-\tan(\theta))^2$

4.3 Trigonometric Identities

In Section 4.2, we saw the utility of the Pythagorean Identities in Theorem 4.2.2 along with the Quotient and Reciprocal Identities in Theorem 4.2.1. Not only did these identities help us compute the values of the circular functions for angles, they were also useful in simplifying expressions involving the circular functions. In this section, we introduce several collections of identities which have uses in this course and beyond. Our first set of identities is the 'Even / Odd' identities.

As mentioned at the end of Section 4.1, properties of the circular functions when thought of as functions of angles in radian measure hold equally well if we view these functions as functions of real numbers. Not surprisingly, the Even / Odd properties of the circular functions are so named because they identify cosine and secant as even functions, while the remaining four circular functions are odd.

Theorem 4.3.1 Even / Odd Identities

For all applicable angles θ,

- $\cos(-\theta) = \cos(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\tan(-\theta) = -\tan(\theta)$
- $\sec(-\theta) = \sec(\theta)$
- $\csc(-\theta) = -\csc(\theta)$
- $\cot(-\theta) = -\cot(\theta)$

In light of the Quotient and Reciprocal Identities, Theorem 4.2.1, it suffices to show $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$. The remaining four circular functions can be expressed in terms of $\cos(\theta)$ and $\sin(\theta)$ so the proofs of their Even / Odd Identities are left as exercises.

By adding the appropriate multiple of 2π, we may replace θ by the coterminal angle θ_0 with $0 \leq \theta_0 < 2\pi$; the reader can verify that the angles $-\theta$ and $-\theta_0$ are then also coterminal. The Evan / Odd identities then follow by observing that the points $P = (\cos(\theta_0), \sin(\theta_0))$ and $Q = (\cos(-\theta_0), \sin(-\theta_0))$ lie on opposite sides of the *x*-axis, as shown in Figure 4.3.1.

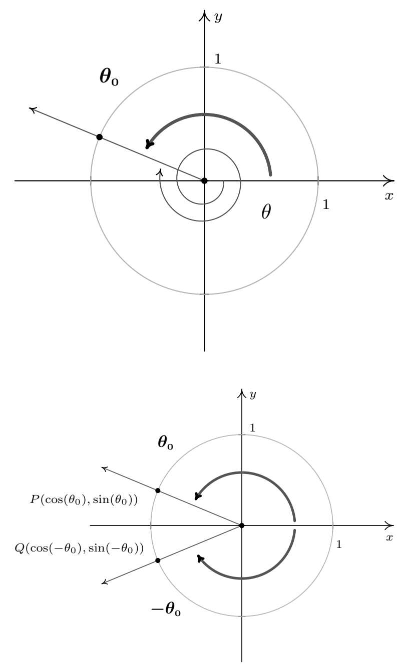


Figure 4.3.1: Establishing Theorem 4.3.1

The Even / Odd Identities are readily demonstrated using any of the 'common angles' noted in Section 4.1. Their true utility, however, lies not in computation, but in simplifying expressions involving the circular functions. In fact, our next batch of identities makes heavy use of the Even / Odd Identities.

Theorem 4.3.2 Sum and Difference Identities for Cosine

For all angles α and β,

- $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

We first prove the result for differences. As in the proof of the Even / Odd Identities, we can reduce the proof for general angles α and β to angles α_0 and β_0, coterminal with α and β, respectively, each of which measure between 0 and 2π radians. Since α and α_0 are coterminal, as are β and β_0, it follows that $\alpha - \beta$ is coterminal with $\alpha_0 - \beta_0$. Consider the case in Figure 4.3.2 where $\alpha_0 \geq \beta_0$.

Since the angles *POQ* and *AOB* are congruent, the distance between *P* and *Q* is equal to the distance between *A* and *B*. The distance formula, Equation 1.2.3, yields

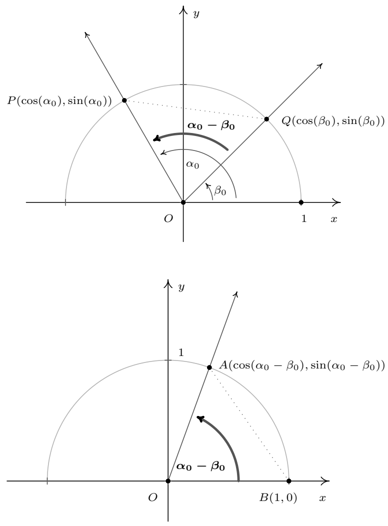


Figure 4.3.2: Establishing Theorem 4.3.2

In Figure 4.3.2, the triangles *POQ* and *AOB* are congruent, which is even better. However, $\alpha_0 - \beta_0$ could be 0 or it could be π, neither of which makes a triangle. It could also be larger than π, which makes a triangle, just not the one we've drawn. You should think about those three cases.

$$\sqrt{(\cos(\alpha_0) - \cos(\beta_0))^2 + (\sin(\alpha_0) - \sin(\beta_0))^2} = \sqrt{(\cos(\alpha_0 - \beta_0) - 1)^2 + (\sin(\alpha_0 - \beta_0) - 0)^2}$$

Squaring both sides, we expand the left hand side of this equation as

$$\begin{aligned}(\cos(\alpha_0) - \cos(\beta_0))^2 + (\sin(\alpha_0) - \sin(\beta_0))^2 &= \cos^2(\alpha_0) - 2\cos(\alpha_0)\cos(\beta_0) + \cos^2(\beta_0) \\ &\quad + \sin^2(\alpha_0) - 2\sin(\alpha_0)\sin(\beta_0) + \sin^2(\beta_0) \\ &= \cos^2(\alpha_0) + \sin^2(\alpha_0) + \cos^2(\beta_0) + \sin^2(\beta_0) \\ &\quad - 2\cos(\alpha_0)\cos(\beta_0) - 2\sin(\alpha_0)\sin(\beta_0)\end{aligned}$$

From the Pythagorean Identities we have $\cos^2(\alpha_0) + \sin^2(\alpha_0) = 1$ and $\cos^2(\beta_0) + \sin^2(\beta_0) = 1$, so

$$(\cos(\alpha_0) - \cos(\beta_0))^2 + (\sin(\alpha_0) - \sin(\beta_0))^2 = 2 - 2\cos(\alpha_0)\cos(\beta_0) - 2\sin(\alpha_0)\sin(\beta_0)$$

Turning our attention to the right hand side of our equation, we find

$$\begin{aligned}(\cos(\alpha_0 - \beta_0) - 1)^2 + (\sin(\alpha_0 - \beta_0) - 0)^2 &= \cos^2(\alpha_0 - \beta_0) - 2\cos(\alpha_0 - \beta_0) + 1 + \sin^2(\alpha_0 - \beta_0) \\ &= 1 + \cos^2(\alpha_0 - \beta_0) + \sin^2(\alpha_0 - \beta_0) - 2\cos(\alpha_0 - \beta_0)\end{aligned}$$

Once again, we simplify $\cos^2(\alpha_0 - \beta_0) + \sin^2(\alpha_0 - \beta_0) = 1$, so that

$$(\cos(\alpha_0 - \beta_0) - 1)^2 + (\sin(\alpha_0 - \beta_0) - 0)^2 = 2 - 2\cos(\alpha_0 - \beta_0)$$

Putting it all together, we get $2 - 2\cos(\alpha_0)\cos(\beta_0) - 2\sin(\alpha_0)\sin(\beta_0) = 2 - 2\cos(\alpha_0 - \beta_0)$, which simplifies to: $\cos(\alpha_0 - \beta_0) = \cos(\alpha_0)\cos(\beta_0) + \sin(\alpha_0)\sin(\beta_0)$. Since α and α_0, β and β_0 and $\alpha - \beta$ and $\alpha_0 - \beta_0$ are all coterminal pairs of angles, we have $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$. For the case where $\alpha_0 \le \beta_0$, we can apply the above argument to the angle $\beta_0 - \alpha_0$ to obtain the identity $\cos(\beta_0 - \alpha_0) = \cos(\beta_0)\cos(\alpha_0) + \sin(\beta_0)\sin(\alpha_0)$. Applying the Even Identity of cosine, we get $\cos(\beta_0 - \alpha_0) = \cos(-(\alpha_0 - \beta_0)) = \cos(\alpha_0 - \beta_0)$, and we get the identity in this case, too.

To get the sum identity for cosine, we use the difference formula along with the Even/Odd Identities

$$\begin{aligned}\cos(\alpha + \beta) = \cos(\alpha - (-\beta)) &= \cos(\alpha)\cos(-\beta) + \sin(\alpha)\sin(-\beta) \\ &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\end{aligned}$$

Example 4.3.1 Using Theorem 4.3.2

1. Find the exact value of $\cos(15^\circ)$.
2. Verify the identity: $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$.

Solution

1. In order to use Theorem 4.3.2 to find $\cos(15^\circ)$, we need to write 15° as a sum or difference of angles whose cosines and sines we know. One way to do so is to write $15^\circ = 45^\circ - 30^\circ$.

$$\begin{aligned}
\cos(15^\circ) &= \cos(45^\circ - 30^\circ) \\
&= \cos(45^\circ)\cos(30^\circ) + \sin(45^\circ)\sin(30^\circ) \\
&= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{6}+\sqrt{2}}{4}
\end{aligned}$$

2. In a straightforward application of Theorem 4.3.2, we find

$$\begin{aligned}
\cos\left(\frac{\pi}{2} - \theta\right) &= \cos\left(\frac{\pi}{2}\right)\cos(\theta) + \sin\left(\frac{\pi}{2}\right)\sin(\theta) \\
&= (0)(\cos(\theta)) + (1)(\sin(\theta)) \\
&= \sin(\theta)
\end{aligned}$$

The identity verified in Example 4.3.1, namely, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$, is the first of what are called the 'cofunction' identities. From $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$, we get:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\left(\frac{\pi}{2} - \left[\frac{\pi}{2} - \theta\right]\right) = \cos(\theta),$$

which says, in words, that the 'co'sine of an angle is the sine of its 'co'mplement. Now that these identities have been established for cosine and sine, the remaining circular functions follow suit. The remaining proofs are left as exercises.

Theorem 4.3.3 Cofunction Identities

For all applicable angles θ,

- $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$
- $\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$
- $\sec\left(\frac{\pi}{2} - \theta\right) = \csc(\theta)$
- $\csc\left(\frac{\pi}{2} - \theta\right) = \sec(\theta)$
- $\tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta)$
- $\cot\left(\frac{\pi}{2} - \theta\right) = \tan(\theta)$

With the Cofunction Identities in place, we are now in the position to derive the sum and difference formulas for sine. To derive the sum formula for sine, we convert to cosines using a cofunction identity, then expand using the difference formula for cosine

$$\begin{aligned}\sin(\alpha+\beta) &= \cos\left(\frac{\pi}{2}-(\alpha+\beta)\right)\\ &= \cos\left(\left[\frac{\pi}{2}-\alpha\right]-\beta\right)\\ &= \cos\left(\frac{\pi}{2}-\alpha\right)\cos(\beta)+\sin\left(\frac{\pi}{2}-\alpha\right)\sin(\beta)\\ &= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)\end{aligned}$$

We can derive the difference formula for sine by rewriting $\sin(\alpha-\beta)$ as $\sin(\alpha+(-\beta))$ and using the sum formula and the Even / Odd Identities. Again, we leave the details to the reader.

Theorem 4.3.4 Sum and Difference Identities for Sine

For all angles α and β,

- $\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$
- $\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$

Example 4.3.2 Using Theorem 4.3.4

1. Find the exact value of $\sin\left(\frac{19\pi}{12}\right)$
2. If α is a Quadrant II angle with $\sin(\alpha)=\frac{5}{13}$, and β is a Quadrant III angle with $\tan(\beta)=2$, find $\sin(\alpha-\beta)$.
3. Derive a formula for $\tan(\alpha+\beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$.

Solution

1. As in Example 4.3.1, we need to write the angle $\frac{19\pi}{12}$ as a sum or difference of common angles. The denominator of 12 suggests a combination of angles with denominators 3 and 4. One such combination is $\frac{19\pi}{12}=\frac{4\pi}{3}+\frac{\pi}{4}$. Applying Theorem 4.3.4, we get

$$\begin{aligned}\sin\left(\frac{19\pi}{12}\right) &= \sin\left(\frac{4\pi}{3}+\frac{\pi}{4}\right)\\ &= \sin\left(\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)+\cos\left(\frac{4\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)\\ &= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+\left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{-\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

2. In order to find $\sin(\alpha-\beta)$ using Theorem 4.3.4, we need to find $\cos(\alpha)$ and both $\cos(\beta)$ and $\sin(\beta)$. To find $\cos(\alpha)$, we use the Pythagorean Identity

$\cos^2(\alpha)+\sin^2(\alpha)=1$. Since $\sin(\alpha)=\frac{5}{13}$, we have $\cos^2(\alpha)+\left(\frac{5}{13}\right)^2=1$, or $\cos(\alpha)=\pm\frac{12}{13}$. Since α is a Quadrant II angle, $\cos(\alpha)=-\frac{12}{13}$. We now set about finding $\cos(\beta)$ and $\sin(\beta)$. We have several ways to proceed, but the Pythagorean Identity $1+\tan^2(\beta)=\sec^2(\beta)$ is a quick way to get $\sec(\beta)$, and hence, $\cos(\beta)$. With $\tan(\beta)=2$, we get $1+2^2=\sec^2(\beta)$ so that $\sec(\beta)=\pm\sqrt{5}$. Since β is a Quadrant III angle, we choose $\sec(\beta)=-\sqrt{5}$ so $\cos(\beta)=\frac{1}{\sec(\beta)}=\frac{1}{-\sqrt{5}}=-\frac{\sqrt{5}}{5}$. We now need to determine $\sin(\beta)$. We could use The Pythagorean Identity $\cos^2(\beta)+\sin^2(\beta)=1$, but we opt instead to use a quotient identity. From $\tan(\beta)=\frac{\sin(\beta)}{\cos(\beta)}$, we have $\sin(\beta)=\tan(\beta)\cos(\beta)$ so we get $\sin(\beta)=(2)\left(-\frac{\sqrt{5}}{5}\right)=-\frac{2\sqrt{5}}{5}$. We now have all the pieces needed to find $\sin(\alpha-\beta)$:

$$\begin{aligned}\sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)\\ &= \left(\frac{5}{13}\right)\left(-\frac{\sqrt{5}}{5}\right)-\left(-\frac{12}{13}\right)\left(-\frac{2\sqrt{5}}{5}\right)\\ &= -\frac{29\sqrt{5}}{65}\end{aligned}$$

3. We can start expanding $\tan(\alpha+\beta)$ using a quotient identity and our sum formulas

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}\\ &= \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)}\end{aligned}$$

Since $\tan(\alpha)=\frac{\sin(\alpha)}{\cos(\alpha)}$ and $\tan(\beta)=\frac{\sin(\beta)}{\cos(\beta)}$, it looks as though if we divide both numerator and denominator by $\cos(\alpha)\cos(\beta)$ we will have what we want

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)}\cdot\frac{\frac{1}{\cos(\alpha)\cos(\beta)}}{\frac{1}{\cos(\alpha)\cos(\beta)}}\\ &= \frac{\frac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}+\frac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}{\frac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}-\frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}\\ &= \frac{\frac{\sin(\alpha)\cancel{\cos(\beta)}}{\cos(\alpha)\cancel{\cos(\beta)}}+\frac{\cancel{\cos(\alpha)}\sin(\beta)}{\cancel{\cos(\alpha)}\cos(\beta)}}{\frac{\cancel{\cos(\alpha)}\cancel{\cos(\beta)}}{\cancel{\cos(\alpha)}\cancel{\cos(\beta)}}-\frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}\\ &= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}\end{aligned}$$

Note: As with any trigonometric identity, this formula is limited to those cases where all of the tangents are defined.

The formula developed in Exercise 4.3.2 for $\tan(\alpha+\beta)$ can be used to find a formula for $\tan(\alpha-\beta)$ by rewriting the difference as a sum, $\tan(\alpha+(-\beta))$, and the reader is encouraged to fill in the details. Below we summarize all of the sum and difference formulas for cosine, sine and tangent.

Theorem 4.3.5 Sum and Difference Identities

For all applicable angles α and β,

- $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
- $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$
- $\tan(\alpha \pm \beta) = \dfrac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$

In the statement of Theorem 4.3.5, we have combined the cases for the sum '$+$' and difference '$-$' of angles into one formula. The convention here is that if you want the formula for the sum '$+$' of two angles, you use the top sign in the formula; for the difference, '$-$', use the bottom sign. For example,

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$$

If we specialize the sum formulas in Theorem 4.3.5 to the case when $\alpha = \beta$, we obtain the following 'Double Angle' Identities.

Theorem 4.3.6 Double Angle Identities

For all applicable angles θ,

- $\cos(2\theta) = \begin{cases} \cos^2(\theta) - \sin^2(\theta) \\ 2\cos^2(\theta) - 1 \\ 1 - 2\sin^2(\theta) \end{cases}$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\tan(2\theta) = \dfrac{2\tan(\theta)}{1 - \tan^2(\theta)}$

The three different forms for $\cos(2\theta)$ can be explained by our ability to 'exchange' squares of cosine and sine via the Pythagorean Identity $\cos^2(\theta)+\sin^2(\theta) = 1$ and we leave the details to the reader. It is interesting to note that to determine the value of $\cos(2\theta)$, only *one* piece of information is required: either $\cos(\theta)$ or $\sin(\theta)$. To determine $\sin(2\theta)$, however, it appears that we must know both $\sin(\theta)$ and $\cos(\theta)$. In the next example, we show how we can find $\sin(2\theta)$ knowing just one piece of information, namely $\tan(\theta)$.

Example 4.3.3 Using Theorem 4.3.6

1. Suppose $P(-3,4)$ lies on the terminal side of θ when θ is plotted in standard position. Find $\cos(2\theta)$ and $\sin(2\theta)$ and determine the quadrant in which the terminal side of the angle 2θ lies when it is plotted in standard position.

2. If $\sin(\theta) = x$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.

3. Verify the identity: $\sin(2\theta) = \dfrac{2\tan(\theta)}{1+\tan^2(\theta)}$.

4. Express $\cos(3\theta)$ as a polynomial in terms of $\cos(\theta)$.

Solution

1. The point $(-3,4)$ lies on a circle of radius $r = \sqrt{x^2+y^2} = 5$. Hence, $\cos(\theta) = -\frac{3}{5}$ and $\sin(\theta) = \frac{4}{5}$. Applying Theorem 4.3.6, we get $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \left(-\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = -\frac{7}{25}$, and $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$. Since both cosine and sine of 2θ are negative, the terminal side of 2θ, when plotted in standard position, lies in Quadrant III.

2. If your first reaction to '$\sin(\theta) = x$' is 'No it's not, $\cos(\theta) = x$!' then you have indeed learned something, and we take comfort in that. However, context is everything. Here, 'x' is just a variable - it does not necessarily represent the x-coordinate of the point on The Unit Circle which lies on the terminal side of θ, assuming θ is drawn in standard position. Here, x represents the quantity $\sin(\theta)$, and what we wish to know is how to express $\sin(2\theta)$ in terms of x. Since $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, we need to write $\cos(\theta)$ in terms of x to finish the problem. We substitute $x = \sin(\theta)$ into the Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$, to get $\cos^2(\theta) + x^2 = 1$, or $\cos(\theta) = \pm\sqrt{1-x^2}$. Since $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\cos(\theta) \geq 0$, and thus $\cos(\theta) = \sqrt{1-x^2}$. Our final answer is $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2x\sqrt{1-x^2}$.

3. We start with the right hand side of the identity and note that $1+\tan^2(\theta) = \sec^2(\theta)$. From this point, we use the Reciprocal and Quotient Identities to rewrite $\tan(\theta)$ and $\sec(\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$:

$$\begin{aligned} \frac{2\tan(\theta)}{1+\tan^2(\theta)} &= \frac{2\tan(\theta)}{\sec^2(\theta)} = \frac{2\left(\dfrac{\sin(\theta)}{\cos(\theta)}\right)}{\dfrac{1}{\cos^2(\theta)}} = 2\left(\frac{\sin(\theta)}{\cos(\theta)}\right)\cos^2(\theta) \\ &= 2\left(\frac{\sin(\theta)}{\cancel{\cos(\theta)}}\right)\cancel{\cos(\theta)}\cos(\theta) = 2\sin(\theta)\cos(\theta) = \sin(2\theta) \end{aligned}$$

4. In Theorem 4.3.6, the formula $\cos(2\theta) = 2\cos^2(\theta) - 1$ expresses $\cos(2\theta)$ as a polynomial in terms of $\cos(\theta)$. We are now asked to find such an identity for $\cos(3\theta)$. Using the sum formula for cosine, we begin with

$$\begin{aligned} \cos(3\theta) &= \cos(2\theta+\theta) \\ &= \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta) \end{aligned}$$

Our ultimate goal is to express the right hand side in terms of $\cos(\theta)$ only. We substitute $\cos(2\theta) = 2\cos^2(\theta) - 1$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ which yields

$$\begin{aligned}
\cos(3\theta) &= \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta) \\
&= \left(2\cos^2(\theta) - 1\right)\cos(\theta) - (2\sin(\theta)\cos(\theta))\sin(\theta) \\
&= 2\cos^3(\theta) - \cos(\theta) - 2\sin^2(\theta)\cos(\theta)
\end{aligned}$$

Finally, we exchange $\sin^2(\theta)$ for $1 - \cos^2(\theta)$ courtesy of the Pythagorean Identity, and get

$$\begin{aligned}
\cos(3\theta) &= 2\cos^3(\theta) - \cos(\theta) - 2\sin^2(\theta)\cos(\theta) \\
&= 2\cos^3(\theta) - \cos(\theta) - 2\left(1 - \cos^2(\theta)\right)\cos(\theta) \\
&= 2\cos^3(\theta) - \cos(\theta) - 2\cos(\theta) + 2\cos^3(\theta) \\
&= 4\cos^3(\theta) - 3\cos(\theta)
\end{aligned}$$

and we are done.

In the last problem in Example 4.3.3, we saw how we could rewrite $\cos(3\theta)$ as sums of powers of $\cos(\theta)$. In Calculus, we have occasion to do the reverse; that is, reduce the power of cosine and sine. Solving the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$ for $\cos^2(\theta)$ and the identity $\cos(2\theta) = 1 - 2\sin^2(\theta)$ for $\sin^2(\theta)$ results in the aptly-named 'Power Reduction' formulas below.

Theorem 4.3.7 Power Reduction Formulas

For all angles θ,

- $\cos^2(\theta) = \dfrac{1 + \cos(2\theta)}{2}$
- $\sin^2(\theta) = \dfrac{1 - \cos(2\theta)}{2}$

Example 4.3.4 Using Theorem 4.3.7

Rewrite $\sin^2(\theta)\cos^2(\theta)$ as a sum and difference of cosines to the first power.

Solution We begin with a straightforward application of Theorem 4.3.7

$$\begin{aligned}
\sin^2(\theta)\cos^2(\theta) &= \left(\frac{1 - \cos(2\theta)}{2}\right)\left(\frac{1 + \cos(2\theta)}{2}\right) \\
&= \frac{1}{4}\left(1 - \cos^2(2\theta)\right) \\
&= \frac{1}{4} - \frac{1}{4}\cos^2(2\theta)
\end{aligned}$$

Next, we apply the power reduction formula to $\cos^2(2\theta)$ to finish the reduction

$$
\begin{aligned}
\sin^2(\theta)\cos^2(\theta) &= \frac{1}{4} - \frac{1}{4}\cos^2(2\theta) \\
&= \frac{1}{4} - \frac{1}{4}\left(\frac{1+\cos(2(2\theta))}{2}\right) \\
&= \frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos(4\theta) \\
&= \frac{1}{8} - \frac{1}{8}\cos(4\theta)
\end{aligned}
$$

Another application of the Power Reduction Formulas is the Half Angle Formulas. To start, we apply the Power Reduction Formula to $\cos^2\left(\frac{\theta}{2}\right)$

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\left(2\left(\frac{\theta}{2}\right)\right)}{2} = \frac{1+\cos(\theta)}{2}.$$

We can obtain a formula for $\cos\left(\frac{\theta}{2}\right)$ by extracting square roots. In a similar fashion, we may obtain a half angle formula for sine, and by using a quotient formula, obtain a half angle formula for tangent. We summarize these formulas below.

Theorem 4.3.8 Half Angle Formulas

For all applicable angles θ,

- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$
- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$
- $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$

where the choice of $\pm$ depends on the quadrant in which the terminal side of $\frac{\theta}{2}$ lies.

Example 4.3.5 Using Theorem 4.3.8

1. Use a half angle formula to find the exact value of $\cos(15^\circ)$.
2. Suppose $-\pi \leq \theta \leq 0$ with $\cos(\theta) = -\frac{3}{5}$. Find $\sin\left(\frac{\theta}{2}\right)$.
3. Use the identity given in number 3 of Example 4.3.3 to derive the identity

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1+\cos(\theta)}$$

Solution

1. To use the half angle formula, we note that $15^\circ = \frac{30^\circ}{2}$ and since 15° is a Quadrant I angle, its cosine is positive. Thus we have

$$\begin{aligned}\cos\left(15^\circ\right) &= +\sqrt{\frac{1+\cos\left(30^\circ\right)}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} \\ &= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}\cdot\frac{2}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}\end{aligned}$$

Note: Back in Example 4.3.1, we found $\cos\left(15^\circ\right)$ by using the difference formula for cosine. In that case, we determined $\cos\left(15^\circ\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$. The reader is encouraged to prove that these two expressions are equal.

2. If $-\pi \leq \theta \leq 0$, then $-\frac{\pi}{2} \leq \frac{\theta}{2} \leq 0$, which means $\sin\left(\frac{\theta}{2}\right) < 0$. Theorem 4.3.8 gives

$$\begin{aligned}\sin\left(\frac{\theta}{2}\right) &= -\sqrt{\frac{1-\cos\left(\theta\right)}{2}} = -\sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}} \\ &= -\sqrt{\frac{1+\frac{3}{5}}{2}\cdot\frac{5}{5}} = -\sqrt{\frac{8}{10}} = -\frac{2\sqrt{5}}{5}\end{aligned}$$

3. Instead of our usual approach to verifying identities, namely starting with one side of the equation and trying to transform it into the other, we will start with the identity we proved in number 3 of Example 4.3.3 and manipulate it into the identity we are asked to prove. The identity we are asked to start with is $\sin(2\theta) = \frac{2\tan(\theta)}{1+\tan^2(\theta)}$. If we are to use this to derive an identity for $\tan\left(\frac{\theta}{2}\right)$, it seems reasonable to proceed by replacing each occurrence of θ with $\frac{\theta}{2}$

$$\begin{aligned}\sin\left(2\left(\frac{\theta}{2}\right)\right) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} \\ \sin(\theta) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\end{aligned}$$

We now have the $\sin(\theta)$ we need, but we somehow need to get a factor of $1+\cos(\theta)$ involved. To get cosines involved, recall that $1+\tan^2\left(\frac{\theta}{2}\right) = \sec^2\left(\frac{\theta}{2}\right)$. We continue to manipulate our given identity by converting secants to cosines and using a power reduction formula

$$\begin{aligned}\sin(\theta) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} \\ \sin(\theta) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{\sec^2\left(\frac{\theta}{2}\right)} \\ \sin(\theta) &= 2\tan\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right) \\ \sin(\theta) &= 2\tan\left(\frac{\theta}{2}\right)\left(\frac{1+\cos\left(2\left(\frac{\theta}{2}\right)\right)}{2}\right) \\ \sin(\theta) &= \tan\left(\frac{\theta}{2}\right)\left(1+\cos(\theta)\right) \\ \tan\left(\frac{\theta}{2}\right) &= \frac{\sin(\theta)}{1+\cos(\theta)}\end{aligned}$$

Our next batch of identities, the Product to Sum Formulas, are easily verified by expanding each of the right hand sides in accordance with Theorem 4.3.5 and as you should expect by now we leave the details as exercises. They are of particular use in Calculus, and we list them here for reference.

Theorem 4.3.9 Product to Sum Formulas

For all angles α and β,

- $\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$
- $\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$
- $\sin(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$

Related to the Product to Sum Formulas are the Sum to Product Formulas, which come in handy when attempting to solve equations involving trigonometric functions. These are easily verified using the Product to Sum Formulas, and as such, their proofs are left as exercises.

Theorem 4.3.10 Sum to Product Formulas

For all angles α and β,

- $\cos(\alpha)+\cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
- $\cos(\alpha)-\cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$
- $\sin(\alpha)\pm\sin(\beta) = 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$

The identities in Theorem 4.3.9 are also known as the Prosthaphaeresis Formulas and have a rich history. The authors recommend that you conduct some research on them as your schedule allows.

Example 4.3.6 Using Theorems 4.3.9 and 4.3.10

1. Write $\cos(2\theta)\cos(6\theta)$ as a sum.
2. Write $\sin(\theta)-\sin(3\theta)$ as a product.

Solution

1. Identifying $\alpha = 2\theta$ and $\beta = 6\theta$, we find

$$\begin{aligned}\cos(2\theta)\cos(6\theta) &= \tfrac{1}{2}\left[\cos(2\theta-6\theta)+\cos(2\theta+6\theta)\right]\\ &= \tfrac{1}{2}\cos(-4\theta)+\tfrac{1}{2}\cos(8\theta)\\ &= \tfrac{1}{2}\cos(4\theta)+\tfrac{1}{2}\cos(8\theta),\end{aligned}$$

where the last equality is courtesy of the even identity for cosine, $\cos(-4\theta) = \cos(4\theta)$.

2. Identifying $\alpha = \theta$ and $\beta = 3\theta$ yields

$$\begin{aligned} \sin(\theta) - \sin(3\theta) &= 2\sin\left(\frac{\theta - 3\theta}{2}\right)\cos\left(\frac{\theta + 3\theta}{2}\right) \\ &= 2\sin\left(-\theta\right)\cos\left(2\theta\right) \\ &= -2\sin\left(\theta\right)\cos\left(2\theta\right), \end{aligned}$$

where the last equality is courtesy of the odd identity for sine, $\sin(-\theta) = -\sin(\theta)$.

This section and the one before it present a rather large volume of trigonometric identities, leading to a very common student question: "Do I have to memorize **all** of these?" The answer, of course, is no. The indispensable identities are the Pythagorean identities (Theorem 4.1.1), and the sum/difference identities (Theorems 4.3.2 and 4.3.4). They are the most common, and all other identities can be derived from them. That said, there are a number of topics in Calculus (trig integration comes to mind) where having other identities like the power reduction formulas in Theorem 4.3.7 at your fingertips will come in handy.

The reader is reminded that all of the identities presented in this section which regard the circular functions as functions of angles (in radian measure) apply equally well to the circular (trigonometric) functions regarded as functions of real numbers. In Exercises 36 - 41 in Section 4.4, we see how some of these identities manifest themselves geometrically as we study the graphs of the these functions. In the upcoming Exercises, however, you need to do all of your work analytically without graphs.

Exercises 4.3

Problems

In Exercises 1 – 6, use the Even / Odd Identities to verify the identity. Assume all quantities are defined.

1. $\sin(3\pi - 2\theta) = -\sin(2\theta - 3\pi)$
2. $\cos\left(-\frac{\pi}{4} - 5t\right) = \cos\left(5t + \frac{\pi}{4}\right)$
3. $\tan(-t^2 + 1) = -\tan(t^2 - 1)$
4. $\csc(-\theta - 5) = -\csc(\theta + 5)$
5. $\sec(-6t) = \sec(6t)$
6. $\cot(9 - 7\theta) = -\cot(7\theta - 9)$

In Exercises 7 – 21, use the Sum and Difference Identities to find the exact value. You may have need of the Quotient, Reciprocal or Even / Odd Identities as well.

7. $\cos(75^\circ)$
8. $\sec(165^\circ)$
9. $\sin(105^\circ)$
10. $\csc(195^\circ)$
11. $\cot(255^\circ)$
12. $\tan(375^\circ)$
13. $\cos\left(\frac{13\pi}{12}\right)$
14. $\sin\left(\frac{11\pi}{12}\right)$
15. $\tan\left(\frac{13\pi}{12}\right)$
16. $\cos\left(\frac{7\pi}{12}\right)$
17. $\tan\left(\frac{17\pi}{12}\right)$
18. $\sin\left(\frac{\pi}{12}\right)$
19. $\cot\left(\frac{11\pi}{12}\right)$
20. $\csc\left(\frac{5\pi}{12}\right)$
21. $\sec\left(-\frac{\pi}{12}\right)$

22. If α is a Quadrant IV angle with $\cos(\alpha) = \frac{\sqrt{5}}{5}$, and $\sin(\beta) = \frac{\sqrt{10}}{10}$, where $\frac{\pi}{2} < \beta < \pi$, find
    (a) $\cos(\alpha + \beta)$
    (b) $\sin(\alpha + \beta)$
    (c) $\tan(\alpha + \beta)$
    (d) $\cos(\alpha - \beta)$
    (e) $\sin(\alpha - \beta)$
    (f) $\tan(\alpha - \beta)$

23. If $\csc(\alpha) = 3$, where $0 < \alpha < \frac{\pi}{2}$, and β is a Quadrant II angle with $\tan(\beta) = -7$, find
    (a) $\cos(\alpha + \beta)$
    (b) $\sin(\alpha + \beta)$
    (c) $\tan(\alpha + \beta)$
    (d) $\cos(\alpha - \beta)$
    (e) $\sin(\alpha - \beta)$
    (f) $\tan(\alpha - \beta)$

24. If $\sin(\alpha) = \frac{3}{5}$, where $0 < \alpha < \frac{\pi}{2}$, and $\cos(\beta) = \frac{12}{13}$ where $\frac{3\pi}{2} < \beta < 2\pi$, find
    (a) $\sin(\alpha + \beta)$
    (b) $\cos(\alpha - \beta)$
    (c) $\tan(\alpha - \beta)$

25. If $\sec(\alpha) = -\frac{5}{3}$, where $\frac{\pi}{2} < \alpha < \pi$, and $\tan(\beta) = \frac{24}{7}$, where $\pi < \beta < \frac{3\pi}{2}$, find
    (a) $\csc(\alpha - \beta)$
    (b) $\sec(\alpha + \beta)$
    (c) $\cot(\alpha + \beta)$

In Exercises 26 – 38, verify the identity.

26. $\cos(\theta - \pi) = -\cos(\theta)$
27. $\sin(\pi - \theta) = \sin(\theta)$
28. $\tan\left(\theta + \frac{\pi}{2}\right) = -\cot(\theta)$
29. $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin(\alpha)\cos(\beta)$
30. $\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos(\alpha)\sin(\beta)$
31. $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos(\alpha)\cos(\beta)$
32. $\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin(\alpha)\sin(\beta)$
33. $\frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \frac{1 + \cot(\alpha)\tan(\beta)}{1 - \cot(\alpha)\tan(\beta)}$
34. $\frac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} = \frac{1 - \tan(\alpha)\tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$

35. $\dfrac{\tan(\alpha+\beta)}{\tan(\alpha-\beta)} = \dfrac{\sin(\alpha)\cos(\alpha)+\sin(\beta)\cos(\beta)}{\sin(\alpha)\cos(\alpha)-\sin(\beta)\cos(\beta)}$

36. $\dfrac{\sin(t+h)-\sin(t)}{h} = \cos(t)\left(\dfrac{\sin(h)}{h}\right) + \sin(t)\left(\dfrac{\cos(h)-1}{h}\right)$

37. $\dfrac{\cos(t+h)-\cos(t)}{h} = \cos(t)\left(\dfrac{\cos(h)-1}{h}\right) - \sin(t)\left(\dfrac{\sin(h)}{h}\right)$

38. $\dfrac{\tan(t+h)-\tan(t)}{h} = \left(\dfrac{\tan(h)}{h}\right)\left(\dfrac{\sec^2(t)}{1-\tan(t)\tan(h)}\right)$

In Exercises 39 – 48, use the Half Angle Formulas to find the exact value. You may have need of the Quotient, Reciprocal or Even / Odd Identities as well.

39. $\cos(75^\circ)$ (compare with Exercise 7)

40. $\sin(105^\circ)$ (compare with Exercise 9)

41. $\cos(67.5^\circ)$

42. $\sin(157.5^\circ)$

43. $\tan(112.5^\circ)$

44. $\cos\left(\dfrac{7\pi}{12}\right)$ (compare with Exercise 16)

45. $\sin\left(\dfrac{\pi}{12}\right)$ (compare with Exercise 18)

46. $\cos\left(\dfrac{\pi}{8}\right)$

47. $\sin\left(\dfrac{5\pi}{8}\right)$

48. $\tan\left(\dfrac{7\pi}{8}\right)$

In Exercises 49 – 58, use the given information about θ to find the exact values of

- $\sin(2\theta)$
- $\cos(2\theta)$
- $\tan(2\theta)$
- $\sin\left(\dfrac{\theta}{2}\right)$
- $\cos\left(\dfrac{\theta}{2}\right)$
- $\tan\left(\dfrac{\theta}{2}\right)$

49. $\sin(\theta) = -\dfrac{7}{25}$ where $\dfrac{3\pi}{2} < \theta < 2\pi$

50. $\cos(\theta) = \dfrac{28}{53}$ where $0 < \theta < \dfrac{\pi}{2}$

51. $\tan(\theta) = \dfrac{12}{5}$ where $\pi < \theta < \dfrac{3\pi}{2}$

52. $\csc(\theta) = 4$ where $\dfrac{\pi}{2} < \theta < \pi$

53. $\cos(\theta) = \dfrac{3}{5}$ where $0 < \theta < \dfrac{\pi}{2}$

54. $\sin(\theta) = -\dfrac{4}{5}$ where $\pi < \theta < \dfrac{3\pi}{2}$

55. $\cos(\theta) = \dfrac{12}{13}$ where $\dfrac{3\pi}{2} < \theta < 2\pi$

56. $\sin(\theta) = \dfrac{5}{13}$ where $\dfrac{\pi}{2} < \theta < \pi$

57. $\sec(\theta) = \sqrt{5}$ where $\dfrac{3\pi}{2} < \theta < 2\pi$

58. $\tan(\theta) = -2$ where $\dfrac{\pi}{2} < \theta < \pi$

In Exercises 59 – 73, verify the identity. Assume all quantities are defined.

59. $(\cos(\theta)+\sin(\theta))^2 = 1+\sin(2\theta)$

60. $(\cos(\theta)-\sin(\theta))^2 = 1-\sin(2\theta)$

61. $\tan(2\theta) = \dfrac{1}{1-\tan(\theta)} - \dfrac{1}{1+\tan(\theta)}$

62. $\csc(2\theta) = \dfrac{\cot(\theta)+\tan(\theta)}{2}$

63. $8\sin^4(\theta) = \cos(4\theta) - 4\cos(2\theta) + 3$

64. $8\cos^4(\theta) = \cos(4\theta) + 4\cos(2\theta) + 3$

65. $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$

66. $\sin(4\theta) = 4\sin(\theta)\cos^3(\theta) - 4\sin^3(\theta)\cos(\theta)$

67. $32\sin^2(\theta)\cos^4(\theta) = 2+\cos(2\theta) - 2\cos(4\theta) - \cos(6\theta)$

68. $32\sin^4(\theta)\cos^2(\theta) = 2-\cos(2\theta) - 2\cos(4\theta) + \cos(6\theta)$

69. $\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$

70. $\cos(8\theta) = 128\cos^8(\theta) - 256\cos^6(\theta) + 160\cos^4(\theta) - 32\cos^2(\theta) + 1$ (HINT: Use the result to 69.)

71. $\sec(2\theta) = \dfrac{\cos(\theta)}{\cos(\theta)+\sin(\theta)} + \dfrac{\sin(\theta)}{\cos(\theta)-\sin(\theta)}$

72. $\dfrac{1}{\cos(\theta)-\sin(\theta)} + \dfrac{1}{\cos(\theta)+\sin(\theta)} = \dfrac{2\cos(\theta)}{\cos(2\theta)}$

73. $\dfrac{1}{\cos(\theta)-\sin(\theta)} - \dfrac{1}{\cos(\theta)+\sin(\theta)} = \dfrac{2\sin(\theta)}{\cos(2\theta)}$

In Exercises 74 – 79, write the given product as a sum. You may need to use an Even/Odd Identity.

74. $\cos(3\theta)\cos(5\theta)$

75. $\sin(2\theta)\sin(7\theta)$

76. $\sin(9\theta)\cos(\theta)$

77. $\cos(2\theta)\cos(6\theta)$

78. $\sin(3\theta)\sin(2\theta)$

79. $\cos(\theta)\sin(3\theta)$

In Exercises 80 – 85, write the given sum as a product. You may need to use an Even/Odd or Cofunction Identity.

80. $\cos(3\theta)+\cos(5\theta)$

81. $\sin(2\theta)-\sin(7\theta)$

82. $\cos(5\theta)-\cos(6\theta)$

83. $\sin(9\theta)-\sin(-\theta)$

84. $\sin(\theta)+\cos(\theta)$

85. $\cos(\theta)-\sin(\theta)$

86. Suppose θ is a Quadrant I angle with $\sin(\theta)=x$. Verify the following formulas

    (a) $\cos(\theta)=\sqrt{1-x^2}$
    (b) $\sin(2\theta)=2x\sqrt{1-x^2}$
    (c) $\cos(2\theta)=1-2x^2$

87. Discuss with your classmates how each of the formulas, if any, in Exercise 86 change if we change assume θ is a Quadrant II, III, or IV angle.

88. Suppose θ is a Quadrant I angle with $\tan(\theta)=x$. Verify the following formulas

    (a) $\cos(\theta)=\dfrac{1}{\sqrt{x^2+1}}$
    (b) $\sin(\theta)=\dfrac{x}{\sqrt{x^2+1}}$
    (c) $\sin(2\theta)=\dfrac{2x}{x^2+1}$
    (d) $\cos(2\theta)=\dfrac{1-x^2}{x^2+1}$

89. Discuss with your classmates how each of the formulas, if any, in Exercise 88 change if we change assume θ is a Quadrant II, III, or IV angle.

90. If $\sin(\theta)=\dfrac{x}{2}$ for $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$, find an expression for $\cos(2\theta)$ in terms of x.

91. If $\tan(\theta)=\dfrac{x}{7}$ for $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.

92. If $\sec(\theta)=\dfrac{x}{4}$ for $0<\theta<\dfrac{\pi}{2}$, find an expression for $\ln|\sec(\theta)+\tan(\theta)|$ in terms of x.

93. Show that $\cos^2(\theta)-\sin^2(\theta)=2\cos^2(\theta)-1=1-2\sin^2(\theta)$ for all θ.

94. Let θ be a Quadrant III angle with $\cos(\theta)=-\dfrac{1}{5}$. Show that this is not enough information to determine the sign of $\sin\left(\dfrac{\theta}{2}\right)$ by first assuming $3\pi<\theta<\dfrac{7\pi}{2}$ and then assuming $\pi<\theta<\dfrac{3\pi}{2}$ and computing $\sin\left(\dfrac{\theta}{2}\right)$ in both cases.

95. Without using your calculator, show that $\dfrac{\sqrt{2+\sqrt{3}}}{2}=\dfrac{\sqrt{6}+\sqrt{2}}{4}$

96. In part 4 of Example 4.3.3, we wrote $\cos(3\theta)$ as a polynomial in terms of $\cos(\theta)$. In Exercise 69, we had you verify an identity which expresses $\cos(4\theta)$ as a polynomial in terms of $\cos(\theta)$. Can you find a polynomial in terms of $\cos(\theta)$ for $\cos(5\theta)$? $\cos(6\theta)$? Can you find a pattern so that $\cos(n\theta)$ could be written as a polynomial in cosine for any natural number n?

97. In Exercise 65, we has you verify an identity which expresses $\sin(3\theta)$ as a polynomial in terms of $\sin(\theta)$. Can you do the same for $\sin(5\theta)$? What about for $\sin(4\theta)$? If not, what goes wrong?

98. Verify the Even / Odd Identities for tangent, secant, cosecant and cotangent.

99. Verify the Cofunction Identities for tangent, secant, cosecant and cotangent.

100. Verify the Difference Identities for sine and tangent.

101. Verify the Product to Sum Identities.

102. Verify the Sum to Product Identities.

4.4 Graphs of the Trigonometric Functions

4.4.1 Graphs of the Cosine and Sine Functions

Since radian measure allows us to identify angles with real numbers, and the sine and cosine functions are defined for any angle, we know that the domain of $f(t) = \cos(t)$ and of $g(t) = \sin(t)$ is all real numbers, $(-\infty, \infty)$, and the range of both functions is $[-1, 1]$. The Even / Odd Identities in Theorem 4.3.1 tell us $\cos(-t) = \cos(t)$ for all real numbers t and $\sin(-t) = -\sin(t)$ for all real numbers t. This means $f(t) = \cos(t)$ is an even function, while $g(t) = \sin(t)$ is an odd function. Another important property of these functions is that $\cos(t + 2\pi k) = \cos(t)$ and $\sin(t + 2\pi k) = \sin(t)$ for all real numbers t and any integer k. This last property is given a special name.

> **Definition 4.4.1 Periodic Function**
>
> A function f is said to be **periodic** if there is a real number c so that $f(t + c) = f(t)$ for all real numbers t in the domain of f. The smallest positive number p for which $f(t + p) = f(t)$ for all real numbers t in the domain of f, if it exists, is called the **period** of f.

We have already seen a family of periodic functions in Section 3.1.1: the constant functions. However, despite being periodic, a constant function has no period. (We'll leave that odd gem as an exercise for you.) Returning to the circular functions, we see that by Definition 4.4.1, $f(t) = \cos(t)$ is periodic with period 2π, since $\cos(t + 2\pi k) = \cos(t)$ for any integer k, in particular, for $k = 1$. Similarly, we can show $g(t) = \sin(t)$ is also periodic with 2π as its period. Having period 2π essentially means that we can completely understand everything about the functions $f(t) = \cos(t)$ and $g(t) = \sin(t)$ by studying one interval of length 2π, say $[0, 2\pi]$.

To see that $p = 2\pi$ is the smallest value such that $\cos(t + p) = \cos(t)$, notice that when $t = 0$, we would need to have $\cos(p) = \cos(0) = 1$, and we know that there are no numbers p between 0 and 2π such that $\cos(p) = 1$.

One last property of the functions $f(t) = \cos(t)$ and $g(t) = \sin(t)$ is worth pointing out: both of these functions are continuous and smooth. Recall from Section 3.2.1 that geometrically this means the graphs of the cosine and sine functions have no jumps, gaps, holes in the graph, asymptotes, corners or cusps. As we shall see, the graphs of both $f(t) = \cos(t)$ and $g(t) = \sin(t)$ meander nicely and don't cause any trouble. We summarize these facts in the following theorem.

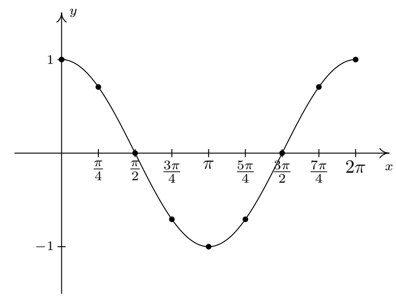
Technically, we should study the interval $[0, 2\pi)$, since whatever happens at $t = 2\pi$ is the same as what happens at $t = 0$. As we will see shortly, $t = 2\pi$ gives us an extra 'check' when we go to graph these functions. In some texts, the interval of choice is $[-\pi, \pi)$.

> **Theorem 4.4.1 Properties of the Cosine and Sine Functions**
>
> - The function $f(x) = \cos(x)$
>   - has domain $(-\infty, \infty)$
>   - has range $[-1, 1]$
>   - is continuous and smooth
>   - is even
>   - has period 2π
> - The function $f(x) = \sin(x)$
>   - has domain $(-\infty, \infty)$
>   - has range $[-1, 1]$
>   - is continuous and smooth
>   - is odd
>   - has period 2π

In this section, we follow the usual graphing convention and use x as the independent variable and y as the dependent variable. This allows us to turn our attention to graphing the cosine and sine functions in the Cartesian Plane. (**Caution**: the use of x and y in this context is not to be confused with the x- and y-coordinates of points on the Unit Circle which define cosine and sine. Using the term 'trigonometric function' as opposed to 'circular function' can help with that, but one could then ask, "Hey, where's the triangle?") To graph $y = \cos(x)$, we make a table using some of the 'common values' of x in the interval $[0, 2\pi]$. This generates a *portion* of the cosine graph, which we call the '**fundamental cycle**' of $y = \cos(x)$.

x	$\cos(x)$	$(x, \cos(x))$
0	1	$(0, 1)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{3\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
π	-1	$(\pi, -1)$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{7\pi}{4}, \frac{\sqrt{2}}{2}\right)$
2π	1	$(2\pi, 1)$

Values of $f(x) = \cos(x)$ on $[0, 2\pi]$



The 'fundamental cycle' of $y = \cos(x)$.

Figure 4.4.3: Graphing $y = \cos(x)$

A few things about the graph above are worth mentioning. First, this graph represents only part of the graph of $y = \cos(x)$. To get the entire graph, we imagine 'copying and pasting' this graph end to end infinitely in both directions (left and right) on the x-axis. Secondly, the vertical scale here has been greatly exaggerated for clarity and aesthetics. Below is an accurate-to-scale graph of $y = \cos(x)$ showing several cycles with the 'fundamental cycle' plotted thicker than the others. The graph of $y = \cos(x)$ is usually described as 'wavelike' – indeed, many of the applications involving the cosine and sine functions feature modelling wavelike phenomena.

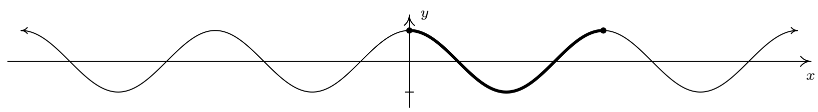
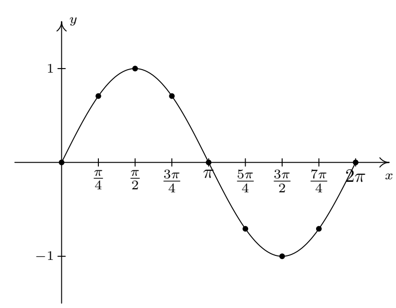


Figure 4.4.1: An accurately scaled graph of $y = \cos(x)$.

We can plot the fundamental cycle of the graph of $y = \sin(x)$ similarly, with similar results.

x	$\sin(x)$	$(x, \sin(x))$
0	0	$(0, 0)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{2}$	1	$\left(\frac{\pi}{2}, 1\right)$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{3\pi}{4}, \frac{\sqrt{2}}{2}\right)$
π	0	$(\pi, 0)$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
$\frac{3\pi}{2}$	-1	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{7\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
2π	0	$(2\pi, 0)$

Values of $f(x) = \sin(x)$ on $[0, 2\pi]$



The 'fundamental cycle' of $y = \sin(x)$

Figure 4.4.4: Graphing $y = \sin(x)$

As with the graph of $y = \cos(x)$, we provide an accurately scaled graph of $y = \sin(x)$ below with the fundamental cycle highlighted.

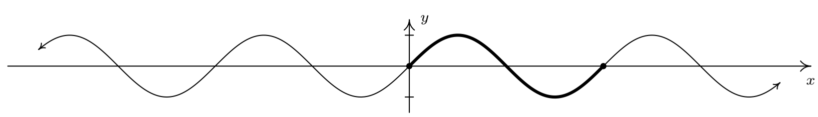


Figure 4.4.2: An accurately scaled graph of $y = \sin(x)$.

It is no accident that the graphs of $y = \cos(x)$ and $y = \sin(x)$ are so similar. Using a cofunction identity along with the even property of cosine, we have

$$\sin(x) = \cos\left(\frac{\pi}{2} - x\right) = \cos\left(-\left(x - \frac{\pi}{2}\right)\right) = \cos\left(x - \frac{\pi}{2}\right),$$

so that the graph of $y = \sin(x)$ is the result of shifting the graph of $y = \cos(x)$ to the right $\frac{\pi}{2}$ units. A visual inspection confirms this.

Now that we know the basic shapes of the graphs of $y = \cos(x)$ and $y = \sin(x)$, we can graph transformations to graph more complicated curves. To do so, we need to keep track of the movement of some key points on the original graphs. We choose to track the values $x = 0$, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and 2π. These 'quarter marks' correspond to quadrantal angles, and as such, mark the location of the zeros and the local extrema of these functions over exactly one period. Before we begin our next example, we need to review the concept of the 'argument' of a function as first introduced in Section 2.1. For the function $f(x) = 1 - 5\cos(2x - \pi)$, the argument of f is x. We shall have occasion, however, to refer to the argument of the *cosine*, which in this case is $2x - \pi$. Loosely stated, the argument of a trigonometric function is the expression 'inside' the function.

Example 4.4.1 Plotting cosine and sine functions

Graph one cycle of the following functions. State the period of each.

1. $f(x) = 3\cos\left(\frac{\pi x - \pi}{2}\right) + 1$
2. $g(x) = \frac{1}{2}\sin(\pi - 2x) + \frac{3}{2}$

Solution

1. We set the argument of the cosine, $\frac{\pi x - \pi}{2}$, equal to each of the values: 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π and solve for x. We summarize the results in Figure 4.4.7.

   Next, we substitute each of these x values into $f(x) = 3\cos\left(\frac{\pi x - \pi}{2}\right) + 1$ to determine the corresponding y-values and connect the dots in a pleasing wavelike fashion.

x	$f(x)$	$(x, f(x))$
1	4	$(1, 4)$
2	1	$(2, 1)$
3	-2	$(3, -2)$
4	1	$(4, 1)$
5	4	$(5, 4)$

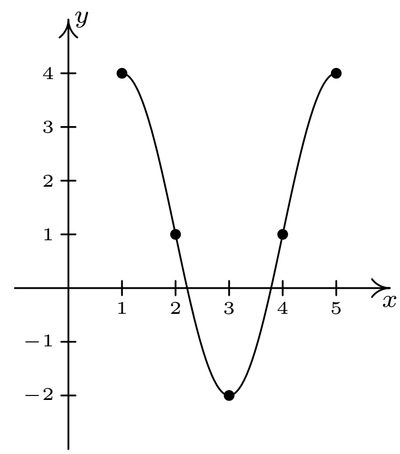


Figure 4.4.5: Plotting one cycle of $y = f(x)$ in Example 4.4.1

One cycle is graphed on $[1, 5]$ so the period is the length of that interval which is 4.

2. Proceeding as above, we set the argument of the sine, $\pi - 2x$, equal to each of our quarter marks and solve for x in Figure 4.4.8.

   We now find the corresponding y-values on the graph by substituting each of these x-values into $g(x) = \frac{1}{2}\sin(\pi - 2x) + \frac{3}{2}$. Once again, we connect the dots in a wavelike fashion.

x	$g(x)$	$(x, g(x))$
$\frac{\pi}{2}$	$\frac{3}{2}$	$\left(\frac{\pi}{2}, \frac{3}{2}\right)$
$\frac{\pi}{4}$	2	$\left(\frac{\pi}{4}, 2\right)$
0	$\frac{3}{2}$	$\left(0, \frac{3}{2}\right)$
$-\frac{\pi}{4}$	1	$\left(-\frac{\pi}{4}, 1\right)$
$-\frac{\pi}{2}$	$\frac{3}{2}$	$\left(-\frac{\pi}{2}, \frac{3}{2}\right)$

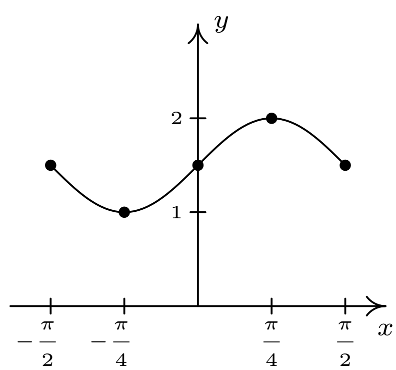


Figure 4.4.6: Plotting one cycle of $y = g(x)$ in Example 4.4.1

a	$\pi - 2x = a$	x
0	$\pi - 2x = 0$	$\frac{\pi}{2}$
$\frac{\pi}{2}$	$\pi - 2x = \frac{\pi}{2}$	$\frac{\pi}{4}$
π	$\pi - 2x = \pi$	0
$\frac{3\pi}{2}$	$\pi - 2x = \frac{3\pi}{2}$	$-\frac{\pi}{4}$
2π	$\pi - 2x = 2\pi$	$-\frac{\pi}{2}$

Figure 4.4.8: Reference points for $g(x)$ in Example 4.4.1

a	$\frac{\pi x - \pi}{2} = a$	x
0	$\frac{\pi x - \pi}{2} = 0$	1
$\frac{\pi}{2}$	$\frac{\pi x - \pi}{2} = \frac{\pi}{2}$	2
π	$\frac{\pi x - \pi}{2} = \pi$	3
$\frac{3\pi}{2}$	$\frac{\pi x - \pi}{2} = \frac{3\pi}{2}$	4
2π	$\frac{\pi x - \pi}{2} = 2\pi$	5

Figure 4.4.7: Reference points for $f(x)$ in Example 4.4.1

One cycle was graphed on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so the period is $\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$.

The functions in Example 4.4.1 are examples of **sinusoids**. Sinusoids can be characterized by four properties: period, amplitude, phase shift and vertical shift. We have already discussed period, that is, how long it takes for the sinusoid to complete one cycle. The standard period of both $f(x) = \cos(x)$ and $g(x) = \sin(x)$ is 2π, but horizontal scalings will change the period of the resulting sinusoid. The **amplitude** of the sinusoid is a measure of how 'tall' the wave is, as indicated in the figure below. The amplitude of the standard cosine and sine functions is 1, but vertical scalings can alter this: see Figure 4.4.9.

The **phase shift** of the sinusoid is the horizontal shift experienced by the fundamental cycle. We have seen that a phase (horizontal) shift of $\frac{\pi}{2}$ to the right takes $f(x) = \cos(x)$ to $g(x) = \sin(x)$ since $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$. As the reader can verify, a phase shift of $\frac{\pi}{2}$ to the left takes $g(x) = \sin(x)$ to $f(x) = \cos(x)$. In most contexts, the vertical shift of a sinusoid is assumed to be 0, but we state the more general case below. The following theorem shows how to find these four fundamental quantities from the formula of the given sinusoid.

Theorem 4.4.2 Standard form of sinusoids

For $\omega > 0$, the functions

$$C(x) = A\cos(\omega x + \phi) + B \quad \text{and} \quad S(x) = A\sin(\omega x + \phi) + B$$

- have period $\dfrac{2\pi}{\omega}$
- have phase shift $-\dfrac{\phi}{\omega}$
- have amplitude $|A|$
- have vertical shift B

We have already seen how the Even/Odd and Cofunction Identities can be used to rewrite $g(x) = \sin(x)$ as a transformed version of $f(x) = \cos(x)$, so of course, the reverse is true: $f(x) = \cos(x)$ can be written as a transformed version of $g(x) = \sin(x)$. The authors have seen some instances where sinusoids are always converted to cosine functions while in other disciplines, the sinusoids are always written in terms of sine functions.

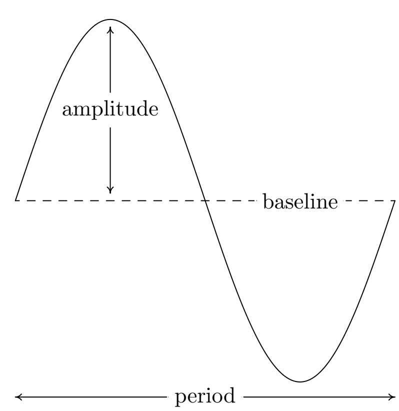


Figure 4.4.9: Properties of sinusoids

We note that in some scientific and engineering circles, the quantity ϕ mentioned in Theorem 4.4.2 is called the **phase** of the sinusoid. Since our interest in this book is primarily with *graphing* sinusoids, we focus our attention on the horizontal shift $-\frac{\phi}{\omega}$ induced by ϕ.

The parameter ω, which is stipulated to be positive, is called the (**angular**) **frequency** of the sinusoid and is the number of cycles the sinusoid completes over a 2π interval. We can always ensure $\omega > 0$ using the Even/Odd Identities. (Try using the formulas in Theorem 4.4.2 applied to $C(x) = \cos(-x + \pi)$ to see why we need $\omega > 0$.)

Example 4.4.2 Converting a sinusoid to standard form

Consider the function $f(x) = \cos(2x) - \sqrt{3}\sin(2x)$. Find a formula for $f(x)$:

1. in the form $C(x) = A\cos(\omega x + \phi) + B$ for $\omega > 0$
2. in the form $S(x) = A\sin(\omega x + \phi) + B$ for $\omega > 0$

Solution

1. The key to this problem is to use the expanded forms of the sinusoid formulas and match up corresponding coefficients. Equating $f(x) = \cos(2x) - \sqrt{3}\sin(2x)$ with the expanded form of $C(x) = A\cos(\omega x + \phi) + B$, we get

$$\cos(2x) - \sqrt{3}\sin(2x) = A\cos(\omega x)\cos(\phi) - A\sin(\omega x)\sin(\phi) + B$$

It should be clear that we can take $\omega = 2$ and $B = 0$ to get

$$\cos(2x) - \sqrt{3}\sin(2x) = A\cos(2x)\cos(\phi) - A\sin(2x)\sin(\phi)$$

To determine A and ϕ, a bit more work is involved. We get started by equating the coefficients of the trigonometric functions on either side of the equation. On the left hand side, the coefficient of $\cos(2x)$ is 1, while on the right hand side, it is $A\cos(\phi)$. Since this equation is to hold for all real numbers, we must have that $A\cos(\phi) = 1$. Similarly, we find by equating the coefficients of $\sin(2x)$ that $A\sin(\phi) = \sqrt{3}$. What we have here is a system of nonlinear equations! We can temporarily eliminate the dependence on ϕ by using the Pythagorean Identity. We know $\cos^2(\phi) + \sin^2(\phi) = 1$, so multiplying this by A^2 gives $A^2\cos^2(\phi) + A^2\sin^2(\phi) = A^2$. Since $A\cos(\phi) = 1$ and $A\sin(\phi) = \sqrt{3}$, we get $A^2 = 1^2 + (\sqrt{3})^2 = 4$ or $A = \pm 2$. Choosing $A = 2$, we have $2\cos(\phi) = 1$ and $2\sin(\phi) = \sqrt{3}$ or, after some rearrangement, $\cos(\phi) = \frac{1}{2}$ and $\sin(\phi) = \frac{\sqrt{3}}{2}$. One such angle ϕ which satisfies this criteria is $\phi = \frac{\pi}{3}$. Hence, one way to write $f(x)$ as a sinusoid is $f(x) = 2\cos\left(2x + \frac{\pi}{3}\right)$. We can easily check our answer using the sum formula for cosine

$$\begin{aligned} f(x) &= 2\cos\left(2x + \tfrac{\pi}{3}\right) \\ &= 2\left[\cos(2x)\cos\left(\tfrac{\pi}{3}\right) - \sin(2x)\sin\left(\tfrac{\pi}{3}\right)\right] \\ &= 2\left[\cos(2x)\left(\tfrac{1}{2}\right) - \sin(2x)\left(\tfrac{\sqrt{3}}{2}\right)\right] \\ &= \cos(2x) - \sqrt{3}\sin(2x) \end{aligned}$$

2. Proceeding as before, we equate $f(x) = \cos(2x) - \sqrt{3}\sin(2x)$ with the expanded form of $S(x) = A\sin(\omega x + \phi) + B$ to get

$$\cos(2x) - \sqrt{3}\sin(2x) = A\sin(\omega x)\cos(\phi) + A\cos(\omega x)\sin(\phi) + B$$

Once again, we may take $\omega = 2$ and $B = 0$ so that

$$\cos(2x) - \sqrt{3}\sin(2x) = A\sin(2x)\cos(\phi) + A\cos(2x)\sin(\phi)$$

We equate (be careful here!) the coefficients of $\cos(2x)$ on either side and get $A\sin(\phi) = 1$ and $A\cos(\phi) = -\sqrt{3}$. Using $A^2\cos^2(\phi) + A^2\sin^2(\phi) = A^2$ as before, we get $A = \pm 2$, and again we choose $A = 2$. This means $2\sin(\phi) = 1$, or $\sin(\phi) = \frac{1}{2}$, and $2\cos(\phi) = -\sqrt{3}$, which means $\cos(\phi) = -\frac{\sqrt{3}}{2}$. One such angle which meets these criteria is $\phi = \frac{5\pi}{6}$. Hence, we have $f(x) = 2\sin\left(2x + \frac{5\pi}{6}\right)$. Checking our work analytically, we have

$$\begin{aligned} f(x) &= 2\sin\left(2x + \tfrac{5\pi}{6}\right) \\ &= 2\left[\sin(2x)\cos\left(\tfrac{5\pi}{6}\right) + \cos(2x)\sin\left(\tfrac{5\pi}{6}\right)\right] \\ &= 2\left[\sin(2x)\left(-\tfrac{\sqrt{3}}{2}\right) + \cos(2x)\left(\tfrac{1}{2}\right)\right] \\ &= \cos(2x) - \sqrt{3}\sin(2x) \end{aligned}$$

It is important to note that in order for the technique presented in Example 4.4.2 to fit a function into one of the forms in Theorem 4.4.2, the arguments of the cosine and sine function much match. That is, while $f(x) = \cos(2x) - \sqrt{3}\sin(2x)$ is a sinusoid, $g(x) = \cos(2x) - \sqrt{3}\sin(3x)$ is not.(This graph does, however, exhibit sinusoid-like characteristics! Check it out!) It is also worth mentioning that, had we chosen $A = -2$ instead of $A = 2$ as we worked through Example 4.4.2, our final answers would have *looked* different. The reader is encouraged to rework Example 4.4.2 using $A = -2$ to see what these differences are, and then for a challenging exercise, use identities to show that the formulas are all equivalent. The general equations to fit a function of the form $f(x) = a\cos(\omega x) + b\sin(\omega x) + B$ into one of the forms in Theorem 4.4.2 are explored in Exercise 35.

4.4.2 Graphs of the Secant and Cosecant Functions

We now turn our attention to graphing $y = \sec(x)$. Since $\sec(x) = \frac{1}{\cos(x)}$, we can use our table of values for the graph of $y = \cos(x)$ and take reciprocals. We run into trouble at odd multiples of $\frac{\pi}{2}$ such as $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ since $\cos(x) = 0$ at these values. This results in vertical asymptotes at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. Since $\cos(x)$ is periodic with period 2π, it follows that $\sec(x)$ is also. Below we graph a fundamental cycle of $y = \sec(x)$ along with a more complete graph obtained by the usual 'copying and pasting.'

x	$\cos(x)$	$\sec(x)$	$(x, \sec(x))$
0	1	1	$(0, 1)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{\pi}{4}, \sqrt{2}\right)$
$\frac{\pi}{2}$	0	undefined	
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{3\pi}{4}, -\sqrt{2}\right)$
π	-1	-1	$(\pi, -1)$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{5\pi}{4}, -\sqrt{2}\right)$
$\frac{3\pi}{2}$	0	undefined	
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{7\pi}{4}, \sqrt{2}\right)$
2π	1	1	$(2\pi, 1)$

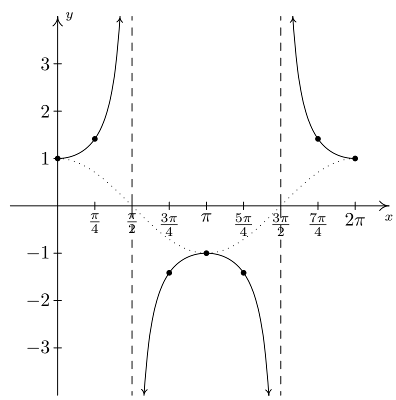


Figure 4.4.10: The 'fundamental cycle' of $y = \sec(x)$.

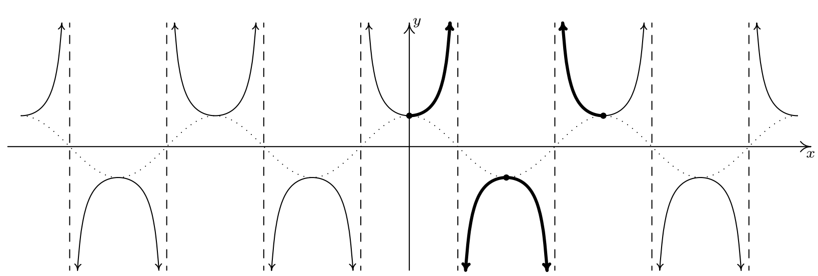


Figure 4.4.11: The graph of $y = \sec x$

Note: provided that $\sec(\alpha)$ and $\sec(\beta)$ are defined, $\sec(\alpha) = \sec(\beta)$ if and only if $\cos(\alpha) = \cos(\beta)$. Hence, $\sec(x)$ inherits its period from $\cos(x)$.

As one would expect, to graph $y = \csc(x)$ we begin with $y = \sin(x)$ and take reciprocals of the corresponding y-values. Here, we encounter issues at $x = 0$, $x = \pi$ and $x = 2\pi$. Proceeding with the usual analysis, we graph the fundamental cycle of $y = \csc(x)$ below along with the dotted graph of $y = \sin(x)$ for reference. Since $y = \sin(x)$ and $y = \cos(x)$ are merely phase shifts of each other, so too are $y = \csc(x)$ and $y = \sec(x)$.

x	$\sin(x)$	$\csc(x)$	$(x, \csc(x))$
0	0	undefined	
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{\pi}{4}, \sqrt{2}\right)$
$\frac{\pi}{2}$	1	1	$\left(\frac{\pi}{2}, 1\right)$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{3\pi}{4}, \sqrt{2}\right)$
π	0	undefined	
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{5\pi}{4}, -\sqrt{2}\right)$
$\frac{3\pi}{2}$	-1	-1	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{7\pi}{4}, -\sqrt{2}\right)$
2π	0	undefined	

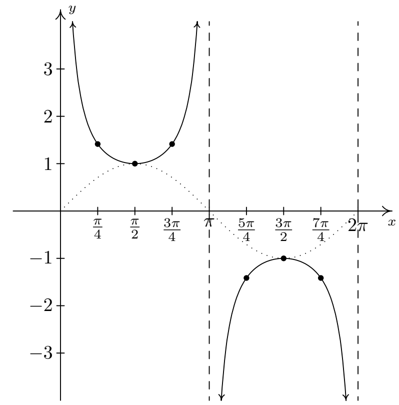


Figure 4.4.12: The ‘fundamental cycle’ of $y = \csc(x)$.

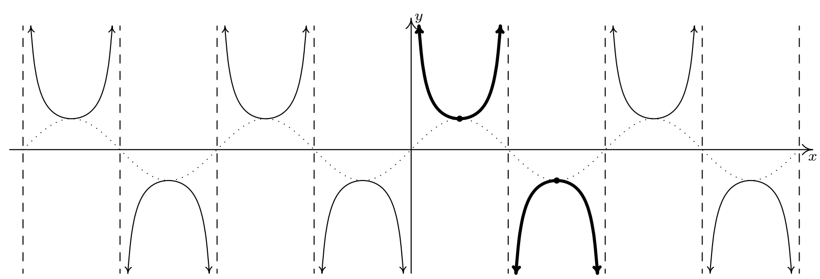


Figure 4.4.13: The graph of $y = \csc x$

Note that, on the intervals between the vertical asymptotes, both $F(x) = \sec(x)$ and $G(x) = \csc(x)$ are continuous and smooth. In other words, they are continuous and smooth *on their domains*. The following theorem summarizes the properties of the secant and cosecant functions. Note that all of these properties are direct results of them being reciprocals of the cosine and sine functions, respectively.

Theorem 4.4.3 Properties of the Secant and Cosecant Functions

- The function $F(x) = \sec(x)$
  - has domain $\left\{x : x \neq \frac{\pi}{2} + \pi k,\ k \text{ is an integer}\right\} = \bigcup_{k=-\infty}^{\infty} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2}\right)$
  - has range $\{y : |y| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - is continuous and smooth on its domain
  - is even
  - has period 2π
- The function $G(x) = \csc(x)$
  - has domain $\{x : x \neq \pi k,\ k \text{ is an integer}\} = \bigcup_{k=-\infty}^{\infty} (k\pi, (k+1)\pi)$
  - has range $\{y : |y| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - is continuous and smooth on its domain
  - is odd
  - has period 2π

In the next example, we discuss graphing more general secant and cosecant curves.

Example 4.4.3 Graphing secant and cosecant curves

Graph one cycle of the following functions. State the period of each.

1. $f(x) = 1 - 2\sec(2x)$
2. $g(x) = \dfrac{\csc(\pi - \pi x) - 5}{3}$

SOLUTION

1. To graph $y = 1 - 2\sec(2x)$, we follow the same procedure as in Example 4.4.1. First, we set the argument of secant, $2x$, equal to the 'quarter marks' 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and 2π and solve for x in Figure 4.4.15.

   Next, we substitute these x values into $f(x)$. If $f(x)$ exists, we have a point on the graph; otherwise, we have found a vertical asymptote. In addition to these points and asymptotes, we have graphed the associated cosine curve – in this case $y = 1 - 2\cos(2x)$ – dotted in the picture below. Since one cycle is graphed over the interval $[0, \pi]$, the period is $\pi - 0 = \pi$.

a	$2x = a$	x
0	$2x = 0$	0
$\frac{\pi}{2}$	$2x = \frac{\pi}{2}$	$\frac{\pi}{4}$
π	$2x = \pi$	$\frac{\pi}{2}$
$\frac{3\pi}{2}$	$2x = \frac{3\pi}{2}$	$\frac{3\pi}{4}$
2π	$2x = 2\pi$	π

Figure 4.4.15: Reference points for $f(x)$ in Example 4.4.3

x	$f(x)$	$(x, f(x))$
0	-1	$(0, -1)$
$\frac{\pi}{4}$	undefined	
$\frac{\pi}{2}$	3	$\left(\frac{\pi}{2}, 3\right)$
$\frac{3\pi}{4}$	undefined	
π	-1	$(\pi, -1)$

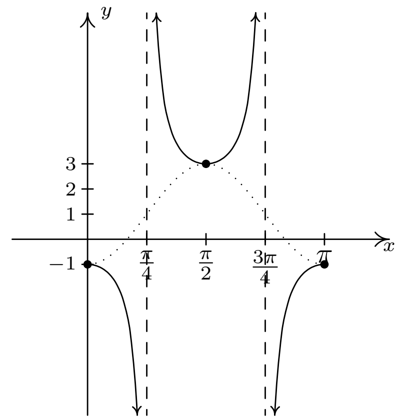


Figure 4.4.14: Plotting one cycle of $y = f(x)$ in Example 4.4.3

2. Proceeding as before, we set the argument of cosecant in $g(x) = \frac{\csc(\pi - \pi x) - 5}{3}$ equal to the quarter marks and solve for x in Figure 4.4.18.

   Substituting these x-values into $g(x)$, we generate the graph below and find the period to be $1 - (-1) = 2$. The associated sine curve, $y = \frac{\sin(\pi - \pi x) - 5}{3}$, is dotted in as a reference.

a	$\pi - \pi x = a$	x
0	$\pi - \pi x = 0$	1
$\frac{\pi}{2}$	$\pi - \pi x = \frac{\pi}{2}$	$\frac{1}{2}$
π	$\pi - \pi x = \pi$	0
$\frac{3\pi}{2}$	$\pi - \pi x = \frac{3\pi}{2}$	$-\frac{1}{2}$
2π	$\pi - \pi x = 2\pi$	-1

Figure 4.4.18: Reference points for $g(x)$ in Example 4.4.3

x	$g(x)$	$(x, g(x))$
1	undefined	
$\frac{1}{2}$	$-\frac{4}{3}$	$\left(\frac{1}{2}, -\frac{4}{3}\right)$
0	undefined	
$-\frac{1}{2}$	-2	$\left(-\frac{1}{2}, -2\right)$
-1	undefined	

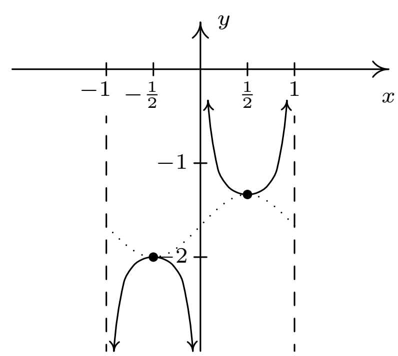


Figure 4.4.16: Plotting one cycle of $y = g(x)$ in Example 4.4.3

Before moving on, we note that it is possible to speak of the period, phase shift and vertical shift of secant and cosecant graphs and use even/odd identities to put them in a form similar to the sinusoid forms mentioned in Theorem 4.4.2. Since these quantities match those of the corresponding cosine and sine curves, we do not spell this out explicitly. Finally, since the ranges of secant and cosecant are unbounded, there is no amplitude associated with these curves.

4.4.3 Graphs of the Tangent and Cotangent Functions

Finally, we turn our attention to the graphs of the tangent and cotangent functions. When constructing a table of values for the tangent function, we see that $J(x) = \tan(x)$ is undefined at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$, and we have vertical asymptotes at these points. Plotting this information and performing the usual 'copy and paste' produces Figures 4.4.17 and 4.4.19 below.

x	$\tan(x)$	$(x, \tan(x))$
0	0	$(0, 0)$
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{2}$	undefined	
$\frac{3\pi}{4}$	-1	$\left(\frac{3\pi}{4}, -1\right)$
π	0	$(\pi, 0)$
$\frac{5\pi}{4}$	1	$\left(\frac{5\pi}{4}, 1\right)$
$\frac{3\pi}{2}$	undefined	
$\frac{7\pi}{4}$	-1	$\left(\frac{7\pi}{4}, -1\right)$
2π	0	$(2\pi, 0)$

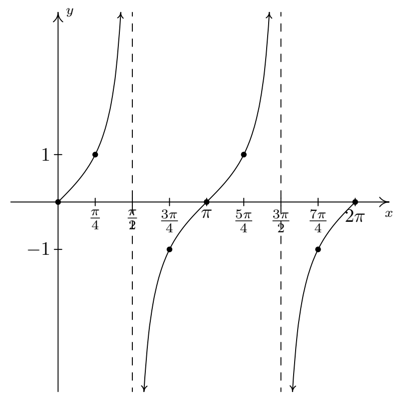


Figure 4.4.17: The graph of $y = \tan(x)$ over $[0, 2\pi]$

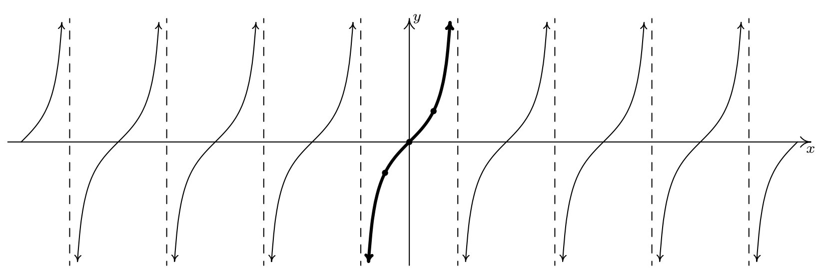


Figure 4.4.19: The graph of $y = \tan(x)$

From the graph, it appears as if the tangent function is periodic with period π. To prove that this is the case, we appeal to the sum formula for tangents. We have:

$$\tan(x + \pi) = \frac{\tan(x) + \tan(\pi)}{1 - \tan(x)\tan(\pi)} = \frac{\tan(x) + 0}{1 - (\tan(x))(0)} = \tan(x),$$

which tells us the period of $\tan(x)$ is at most π. To show that it is exactly π, suppose p is a positive real number so that $\tan(x + p) = \tan(x)$ for all real numbers x. For $x = 0$, we have $\tan(p) = \tan(0 + p) = \tan(0) = 0$, which means p is a multiple of π. The smallest positive multiple of π is π itself, so we have established the result. We take as our fundamental cycle for $y = \tan(x)$ the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and use as our 'quarter marks' $x = -\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}$ and $\frac{\pi}{2}$.

It should be no surprise that $K(x) = \cot(x)$ behaves similarly to $J(x) = \tan(x)$. Plotting $\cot(x)$ over the interval $[0, 2\pi]$ results in the graph in Figure 4.4.20 below.

x	$\cot(x)$	$(x, \cot(x))$
0	undefined	
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{3\pi}{4}$	-1	$\left(\frac{3\pi}{4}, -1\right)$
π	undefined	
$\frac{5\pi}{4}$	1	$\left(\frac{5\pi}{4}, 1\right)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{7\pi}{4}$	-1	$\left(\frac{7\pi}{4}, -1\right)$
2π	undefined	

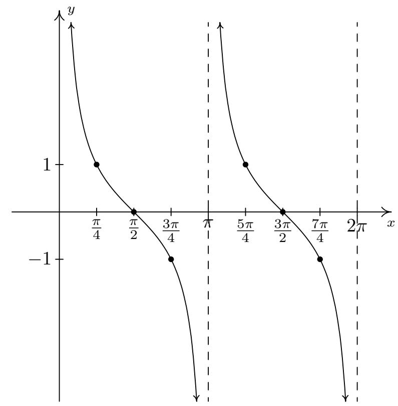


Figure 4.4.20: The graph of $y = \cot(x)$ over $[0, 2\pi]$

From these data, it clearly appears as if the period of $\cot(x)$ is π, and we leave it to the reader to prove this. (Certainly, mimicking the proof that the period of $\tan(x)$ is an option; for another approach, consider transforming $\tan(x)$ to $\cot(x)$ using identities.) We take as one fundamental cycle the interval $(0, \pi)$ with quarter marks: $x = 0$, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$ and π. A more complete graph of $y = \cot(x)$ is below, along with the fundamental cycle highlighted as usual.

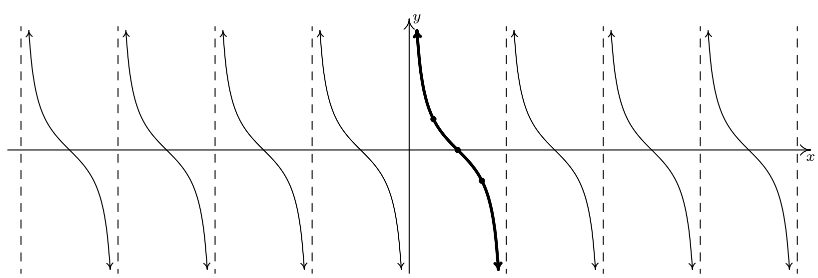


Figure 4.4.21: The graph of $y = \cot(x)$

The properties of the tangent and cotangent functions are summarized below. As with Theorem 4.4.3, each of the results below can be traced back to properties of the cosine and sine functions and the definition of the tangent and cotangent functions as quotients thereof.

Theorem 4.4.4 Properties of the Tangent and Cotangent Functions

- The function $J(x) = \tan(x)$
  - has domain $\left\{x : x \neq \frac{\pi}{2} + \pi k,\ k \text{ is an integer}\right\} = \bigcup_{k=-\infty}^{\infty} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2}\right)$
  - has range $(-\infty, \infty)$
  - is continuous and smooth on its domain
  - is odd
  - has period π
- The function $K(x) = \cot(x)$
  - has domain $\{x : x \neq \pi k,\ k \text{ is an integer}\} = \bigcup_{k=-\infty}^{\infty} (k\pi, (k+1)\pi)$
  - has range $(-\infty, \infty)$
  - is continuous and smooth on its domain
  - is odd
  - has period π

Example 4.4.4 Plotting tangent and cotangent curves

Graph one cycle of the following functions. Find the period.

1. $f(x) = 1 - \tan\left(\frac{x}{2}\right)$.
2. $g(x) = 2\cot\left(\frac{\pi}{2}x + \pi\right) + 1$.

Solution

1. We proceed as we have in all of the previous graphing examples by setting the argument of tangent in $f(x) = 1 - \tan\left(\frac{x}{2}\right)$, namely $\frac{x}{2}$, equal to each of the 'quarter marks' $-\frac{\pi}{2}$, $-\frac{\pi}{4}$, 0, $\frac{\pi}{4}$ and $\frac{\pi}{2}$, and solving for x: see Figure 4.4.23.

   Substituting these x-values into $f(x)$, we find points on the graph and the vertical asymptotes.

a	$\frac{x}{2} = a$	x
$-\frac{\pi}{2}$	$\frac{x}{2} = -\frac{\pi}{2}$	$-\pi$
$-\frac{\pi}{4}$	$\frac{x}{2} = -\frac{\pi}{4}$	$-\frac{\pi}{2}$
0	$\frac{x}{2} = 0$	0
$\frac{\pi}{4}$	$\frac{x}{2} = \frac{\pi}{4}$	$\frac{\pi}{2}$
$\frac{\pi}{2}$	$\frac{x}{2} = \frac{\pi}{2}$	π

Figure 4.4.23: Reference points for $f(x)$ in Example 4.4.4

x	$f(x)$	$(x, f(x))$
$-\pi$	undefined	
$-\frac{\pi}{2}$	2	$\left(-\frac{\pi}{2}, 2\right)$
0	1	$(0, 1)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
π	undefined	

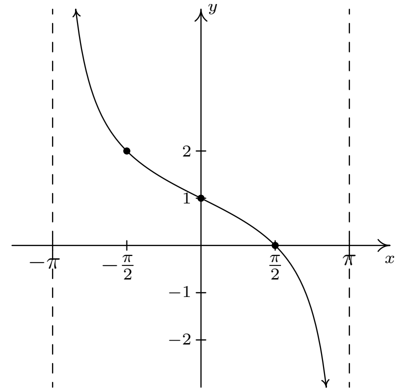


Figure 4.4.22: Plotting one cycle of $y = f(x)$ in Example 4.4.4

We see that the period is $\pi - (-\pi) = 2\pi$.

2. The 'quarter marks' for the fundamental cycle of the cotangent curve are $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ and π. To graph $g(x) = 2\cot\left(\frac{\pi}{2}x + \pi\right) + 1$, we begin by setting $\frac{\pi}{2}x + \pi$ equal to each quarter mark and solving for x in Figure 4.4.25.

a	$\frac{\pi}{2}x + \pi = a$	x
0	$\frac{\pi}{2}x + \pi = 0$	-2
$\frac{\pi}{4}$	$\frac{\pi}{2}x + \pi = \frac{\pi}{4}$	$-\frac{3}{2}$
$\frac{\pi}{2}$	$\frac{\pi}{2}x + \pi = \frac{\pi}{2}$	-1
$\frac{3\pi}{4}$	$\frac{\pi}{2}x + \pi = \frac{3\pi}{4}$	$-\frac{1}{2}$
π	$\frac{\pi}{2}x + \pi = \pi$	0

Figure 4.4.25: Reference points for $g(x)$ in Example 4.4.4

We now use these x-values to generate our graph.

x	$g(x)$	$(x, g(x))$
-2	undefined	
$-\frac{3}{2}$	3	$\left(-\frac{3}{2}, 3\right)$
-1	1	$(-1, 1)$
$-\frac{1}{2}$	-1	$\left(-\frac{1}{2}, -1\right)$
0	undefined	

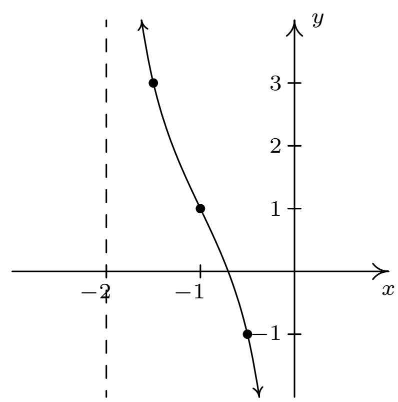


Figure 4.4.24: Plotting one cycle of $y = g(x)$ in Example 4.4.4

We find the period to be $0 - (-2) = 2$.

As with the secant and cosecant functions, it is possible to extend the notion of period, phase shift and vertical shift to the tangent and cotangent functions as we did for the cosine and sine functions in Theorem 4.4.2. Since the number of classical applications involving sinusoids far outnumber those involving tangent and cotangent functions, we omit this. The ambitious reader is invited to formulate such a theorem, however.

Exercises 4.4

Problems

In Exercises 1 – 12, graph one cycle of the given function. State the period, amplitude, phase shift and vertical shift of the function.

1. $y = 3\sin(x)$
2. $y = \sin(3x)$
3. $y = -2\cos(x)$
4. $y = \cos\left(x - \frac{\pi}{2}\right)$
5. $y = -\sin\left(x + \frac{\pi}{3}\right)$
6. $y = \sin(2x - \pi)$
7. $y = -\frac{1}{3}\cos\left(\frac{1}{2}x + \frac{\pi}{3}\right)$
8. $y = \cos(3x - 2\pi) + 4$
9. $y = \sin\left(-x - \frac{\pi}{4}\right) - 2$
10. $y = \frac{2}{3}\cos\left(\frac{\pi}{2} - 4x\right) + 1$
11. $y = -\frac{3}{2}\cos\left(2x + \frac{\pi}{3}\right) - \frac{1}{2}$
12. $y = 4\sin(-2\pi x + \pi)$

In Exercises 13 – 24, graph one cycle of the given function. State the period of the function.

13. $y = \tan\left(x - \frac{\pi}{3}\right)$
14. $y = 2\tan\left(\frac{1}{4}x\right) - 3$
15. $y = \frac{1}{3}\tan(-2x - \pi) + 1$
16. $y = \sec\left(x - \frac{\pi}{2}\right)$
17. $y = -\csc\left(x + \frac{\pi}{3}\right)$
18. $y = -\frac{1}{3}\sec\left(\frac{1}{2}x + \frac{\pi}{3}\right)$
19. $y = \csc(2x - \pi)$
20. $y = \sec(3x - 2\pi) + 4$
21. $y = \csc\left(-x - \frac{\pi}{4}\right) - 2$
22. $y = \cot\left(x + \frac{\pi}{6}\right)$
23. $y = -11\cot\left(\frac{1}{5}x\right)$
24. $y = \frac{1}{3}\cot\left(2x + \frac{3\pi}{2}\right) + 1$

In Exercises 25 – 34, use Example 4.4.2 as a guide to show that the function is a sinusoid by rewriting it in the forms $C(x) = A\cos(\omega x + \phi) + B$ and $S(x) = A\sin(\omega x + \phi) + B$ for $\omega > 0$ and $0 \le \phi < 2\pi$.

25. $f(x) = \sqrt{2}\sin(x) + \sqrt{2}\cos(x) + 1$
26. $f(x) = 3\sqrt{3}\sin(3x) - 3\cos(3x)$
27. $f(x) = -\sin(x) + \cos(x) - 2$
28. $f(x) = -\frac{1}{2}\sin(2x) - \frac{\sqrt{3}}{2}\cos(2x)$
29. $f(x) = 2\sqrt{3}\cos(x) - 2\sin(x)$
30. $f(x) = \frac{3}{2}\cos(2x) - \frac{3\sqrt{3}}{2}\sin(2x) + 6$
31. $f(x) = -\frac{1}{2}\cos(5x) - \frac{\sqrt{3}}{2}\sin(5x)$
32. $f(x) = -6\sqrt{3}\cos(3x) - 6\sin(3x) - 3$
33. $f(x) = \frac{5\sqrt{2}}{2}\sin(x) - \frac{5\sqrt{2}}{2}\cos(x)$
34. $f(x) = 3\sin\left(\frac{x}{6}\right) - 3\sqrt{3}\cos\left(\frac{x}{6}\right)$
35. you should have noticed a relationship between the phases ϕ for the $S(x)$ and $C(x)$. Show that if $f(x) = A\sin(\omega x + \alpha) + B$, then $f(x) = A\cos(\omega x + \beta) + B$ where $\beta = \alpha - \frac{\pi}{2}$.

In Exercises 36 – 41, verify the identity by graphing the right and left hand sides on a computer or calculator.

36. $\sin^2(x) + \cos^2(x) = 1$
37. $\sec^2(x) - \tan^2(x) = 1$
38. $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$
39. $\tan(x + \pi) = \tan(x)$
40. $\sin(2x) = 2\sin(x)\cos(x)$

41. $\tan\left(\frac{x}{2}\right) = \dfrac{\sin(x)}{1+\cos(x)}$

In Exercises 42 – 48, graph the function with the help of your computer or calculator and discuss the given questions with your classmates.

42. $f(x) = \cos(3x) + \sin(x)$. Is this function periodic? If so, what is the period?

43. $f(x) = \frac{\sin(x)}{x}$. What appears to be the horizontal asymptote of the graph?

44. $f(x) = x\sin(x)$. Graph $y = \pm x$ on the same set of axes and describe the behaviour of f.

45. $f(x) = \sin\left(\frac{1}{x}\right)$. What's happening as $x \to 0$?

46. $f(x) = x - \tan(x)$. Graph $y = x$ on the same set of axes and describe the behaviour of f.

47. $f(x) = e^{-0.1x}\left(\cos(2x) + \sin(2x)\right)$. Graph $y = \pm e^{-0.1x}$ on the same set of axes and describe the behaviour of f.

48. $f(x) = e^{-0.1x}\left(\cos(2x) + 2\sin(x)\right)$. Graph $y = \pm e^{-0.1x}$ on the same set of axes and describe the behaviour of f.

49. Show that a constant function f is periodic by showing that $f(x+117) = f(x)$ for all real numbers x. Then show that f has no period by showing that you cannot find a *smallest* number p such that $f(x+p) = f(x)$ for all real numbers x. Said another way, show that $f(x+p) = f(x)$ for all real numbers x for ALL values of $p > 0$, so no smallest value exists to satisfy the definition of 'period'.

4.5 Inverse Trigonometric Functions

As the title indicates, in this section we concern ourselves with finding inverses of the (circular) trigonometric functions. Our immediate problem is that, owing to their periodic nature, none of the six circular functions is one-to-one. To remedy this, we restrict the domains of the circular functions to obtain a one-to-one function. We first consider $f(x) = \cos(x)$. Choosing the interval $[0, \pi]$ allows us to keep the range as $[-1, 1]$ as well as the properties of being smooth and continuous.

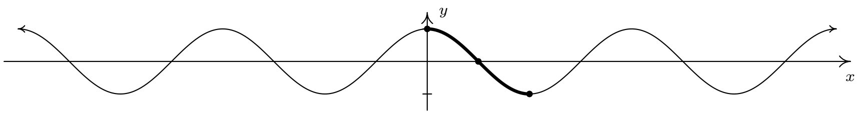


Figure 4.5.1: Restricting the domain of $f(x) = \cos(x)$ to $[0, \pi]$.

Recall from Section 2.2.3 that the inverse of a function f is typically denoted f^{-1}. For this reason, some textbooks use the notation $f^{-1}(x) = \cos^{-1}(x)$ for the inverse of $f(x) = \cos(x)$. The obvious pitfall here is our convention of writing $(\cos(x))^2$ as $\cos^2(x)$, $(\cos(x))^3$ as $\cos^3(x)$ and so on. It is far too easy to confuse $\cos^{-1}(x)$ with $\frac{1}{\cos(x)} = \sec(x)$ so we will not use this notation in our text. (But be aware that many books do! As always, be sure to check the context!) Instead, we use the notation $f^{-1}(x) = \arccos(x)$, read 'arc-cosine of x'. To understand the 'arc' in 'arccosine', recall that an inverse function, by definition, reverses the process of the original function. The function $f(t) = \cos(t)$ takes a real number input t, associates it with the angle $\theta = t$ radians, and returns the value $\cos(\theta)$. Digging deeper, we have that $\cos(\theta) = \cos(t)$ is the x-coordinate of the terminal point on the Unit Circle of an oriented arc of length $|t|$ whose initial point is $(1, 0)$. Hence, we may view the inputs to $f(t) = \cos(t)$ as oriented arcs and the outputs as x-coordinates on the Unit Circle. The function f^{-1}, then, would take x-coordinates on the Unit Circle and return oriented arcs, hence the 'arc' in arccosine. Figure 4.5.3 shows the graphs of $f(x) = \cos(x)$ and $f^{-1}(x) = \arccos(x)$, where we obtain the latter from the former by reflecting it across the line $y = x$, in accordance with Theorem 2.2.2.

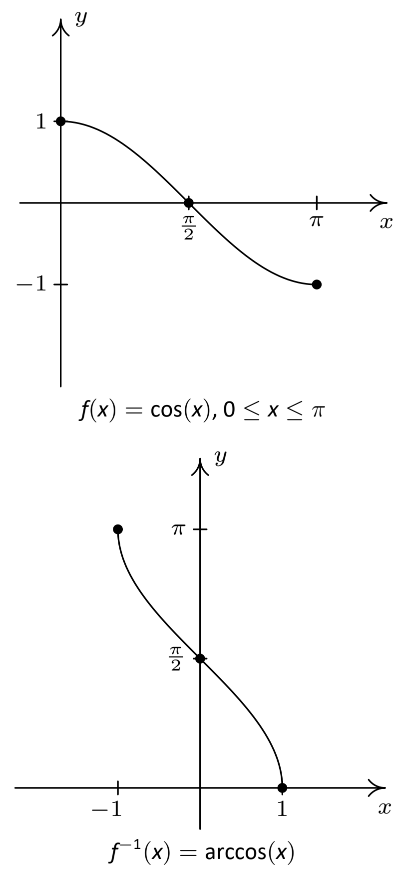


Figure 4.5.3: Reflecting $y = \cos(x)$ across $y = x$ yields $y = \arccos(x)$

We restrict $g(x) = \sin(x)$ in a similar manner, although the interval of choice is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

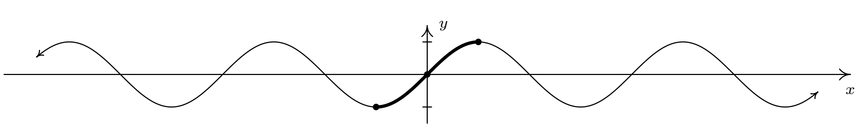
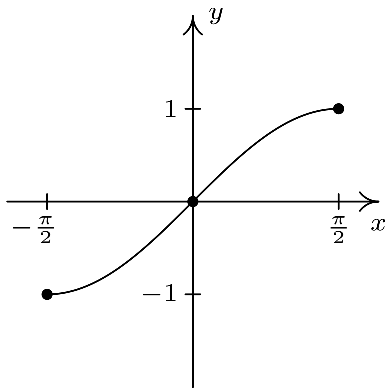


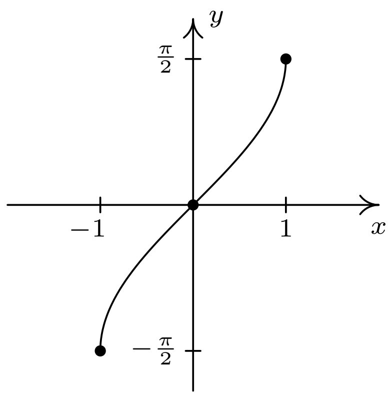
Figure 4.5.2: Restricting the domain of $f(x) = \sin(x)$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

It should be no surprise that we call $g^{-1}(x) = \arcsin(x)$, which is read 'arcsine of x'.

We list some important facts about the arccosine and arcsine functions in the following theorem.



$g(x) = \sin(x)$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$



$g^{-1}(x) = \arcsin(x)$

Figure 4.5.4: Reflecting $y = \sin(x)$ across $y = x$ yields $y = \arcsin(x)$

Theorem 4.5.1 Properties of the Arccosine and Arcsine Functions

- Properties of $F(x) = \arccos(x)$
  - Domain: $[-1, 1]$
  - Range: $[0, \pi]$
  - $\arccos(x) = t$ if and only if $0 \leq t \leq \pi$ and $\cos(t) = x$
  - $\cos(\arccos(x)) = x$ provided $-1 \leq x \leq 1$
  - $\arccos(\cos(x)) = x$ provided $0 \leq x \leq \pi$
- Properties of $G(x) = \arcsin(x)$
  - Domain: $[-1, 1]$
  - Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
  - $\arcsin(x) = t$ if and only if $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$ and $\sin(t) = x$
  - $\sin(\arcsin(x)) = x$ provided $-1 \leq x \leq 1$
  - $\arcsin(\sin(x)) = x$ provided $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
  - additionally, arcsine is odd

Everything in Theorem 4.5.1 is a direct consequence of the facts that $f(x) = \cos(x)$ for $0 \leq x \leq \pi$ and $F(x) = \arccos(x)$ are inverses of each other as are $g(x) = \sin(x)$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and $G(x) = \arcsin(x)$. It's about time for an example.

Example 4.5.1 Evaluating the arcsine and arccosine functions

1. Find the exact values of the following.

   (a) $\arccos\left(\frac{1}{2}\right)$
   (b) $\arcsin\left(\frac{\sqrt{2}}{2}\right)$
   (c) $\arccos\left(-\frac{\sqrt{2}}{2}\right)$
   (d) $\arcsin\left(-\frac{1}{2}\right)$
   (e) $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right)$
   (f) $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right)$
   (g) $\cos\left(\arccos\left(-\frac{3}{5}\right)\right)$
   (h) $\sin\left(\arccos\left(-\frac{3}{5}\right)\right)$

2. Rewrite the following as algebraic expressions of x and state the domain on which the equivalence is valid.

   (a) $\tan(\arccos(x))$
   (b) $\cos(2\arcsin(x))$

Solution

1. (a) To find $\arccos\left(\frac{1}{2}\right)$, we need to find the real number t (or, equivalently, an angle measuring t radians) which lies between 0 and π with $\cos(t) = \frac{1}{2}$. We know $t = \frac{\pi}{3}$ meets these criteria, so $\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

   (b) The value of $\arcsin\left(\frac{\sqrt{2}}{2}\right)$ is a real number t between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\sin(t) = \frac{\sqrt{2}}{2}$. The number we seek is $t = \frac{\pi}{4}$. Hence, $\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$.

(c) The number $t = \arccos\left(-\frac{\sqrt{2}}{2}\right)$ lies in the interval $[0, \pi]$ with $\cos(t) = -\frac{\sqrt{2}}{2}$. Our answer is $\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$.

(d) To find $\arcsin\left(-\frac{1}{2}\right)$, we seek the number t in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $\sin(t) = -\frac{1}{2}$. The answer is $t = -\frac{\pi}{6}$ so that $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

(e) Since $0 \le \frac{\pi}{6} \le \pi$, one option would be to simply invoke Theorem 4.5.1 to get $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$. However, in order to make sure we understand *why* this is the case, we choose to work the example through using the definition of arccosine. Working from the inside out, $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \arccos\left(\frac{\sqrt{3}}{2}\right)$. Now, $\arccos\left(\frac{\sqrt{3}}{2}\right)$ is the real number t with $0 \le t \le \pi$ and $\cos(t) = \frac{\sqrt{3}}{2}$. We find $t = \frac{\pi}{6}$, so that $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$.

(f) Since $\frac{11\pi}{6}$ does not fall between 0 and π, Theorem 4.5.1 does not apply. We are forced to work through from the inside out starting with $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right) = \arccos\left(\frac{\sqrt{3}}{2}\right)$. From the previous problem, we know $\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$. Hence, $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right) = \frac{\pi}{6}$.

(g) One way to simplify $\cos\left(\arccos\left(-\frac{3}{5}\right)\right)$ is to use Theorem 4.5.1 directly. Since $-\frac{3}{5}$ is between -1 and 1, we have that $\cos\left(\arccos\left(-\frac{3}{5}\right)\right) = -\frac{3}{5}$ and we are done. However, as before, to really understand *why* this cancellation occurs, we let $t = \arccos\left(-\frac{3}{5}\right)$. Then, by definition, $\cos(t) = -\frac{3}{5}$. Hence, $\cos\left(\arccos\left(-\frac{3}{5}\right)\right) = \cos(t) = -\frac{3}{5}$, and we are finished in (nearly) the same amount of time.

(h) As in the previous example, we let $t = \arccos\left(-\frac{3}{5}\right)$ so that $\cos(t) = -\frac{3}{5}$ for some t where $0 \le t \le \pi$. Since $\cos(t) < 0$, we can narrow this down a bit and conclude that $\frac{\pi}{2} < t < \pi$, so that t corresponds to an angle in Quadrant II. In terms of t, then, we need to find $\sin\left(\arccos\left(-\frac{3}{5}\right)\right) = \sin(t)$. Using the Pythagorean Identity $\cos^2(t) + \sin^2(t) = 1$, we get $\left(-\frac{3}{5}\right)^2 + \sin^2(t) = 1$ or $\sin(t) = \pm\frac{4}{5}$. Since t corresponds to a Quadrants II angle, we choose $\sin(t) = \frac{4}{5}$. Hence, $\sin\left(\arccos\left(-\frac{3}{5}\right)\right) = \frac{4}{5}$.

2. (a) We begin this problem in the same manner we began the previous two problems. To help us see the forest for the trees, we let $t = \arccos(x)$, so our goal is to find a way to express $\tan(\arccos(x)) = \tan(t)$ in terms of x. Since $t = \arccos(x)$, we know $\cos(t) = x$ where $0 \le t \le \pi$, but since we are after an expression for $\tan(t)$, we know we need to throw out $t = \frac{\pi}{2}$ from consideration. Hence, either $0 \le t < \frac{\pi}{2}$ or $\frac{\pi}{2} < t \le \pi$ so that, geometrically, t corresponds to an angle in Quadrant I or Quadrant II. One approach to finding $\tan(t)$ is to use the quotient identity $\tan(t) = \frac{\sin(t)}{\cos(t)}$. Substituting $\cos(t) = x$ into the Pythagorean Identity $\cos^2(t) + \sin^2(t) = 1$ gives $x^2 + \sin^2(t) = 1$, from which we get $\sin(t) = \pm\sqrt{1-x^2}$. Since t corresponds to angles in Quadrants I and II, $\sin(t) \ge 0$, so we choose $\sin(t) = \sqrt{1-x^2}$. Thus,

$$\tan(t) = \frac{\sin(t)}{\cos(t)} = \frac{\sqrt{1-x^2}}{x}$$

To determine the values of x for which this equivalence is valid, we consider our substitution $t = \arccos(x)$. Since the domain of $\arccos(x)$

An alternative approach to finding $\tan(t)$ is to use the identity $1 + \tan^2(t) = \sec^2(t)$. Since $x = \cos(t)$, $\sec(t) = \frac{1}{\cos(t)} = \frac{1}{x}$. The reader is invited to work through this approach to see what, if any, difficulties arise.

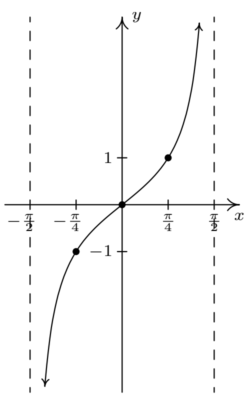
is $[-1, 1]$, we know we must restrict $-1 \leq x \leq 1$. Additionally, since we had to discard $t = \frac{\pi}{2}$, we need to discard $x = \cos\left(\frac{\pi}{2}\right) = 0$. Hence, $\tan(\arccos(x)) = \frac{\sqrt{1-x^2}}{x}$ is valid for x in $[-1, 0) \cup (0, 1]$.

(b) We proceed as in the previous problem by writing $t = \arcsin(x)$ so that t lies in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $\sin(t) = x$. We aim to express $\cos(2\arcsin(x)) = \cos(2t)$ in terms of x. Since $\cos(2t)$ is defined everywhere, we get no additional restrictions on t as we did in the previous problem. We have three choices for rewriting $\cos(2t)$: $\cos^2(t) - \sin^2(t)$, $2\cos^2(t) - 1$ and $1 - 2\sin^2(t)$. Since we know $x = \sin(t)$, it is easiest to use the last form:

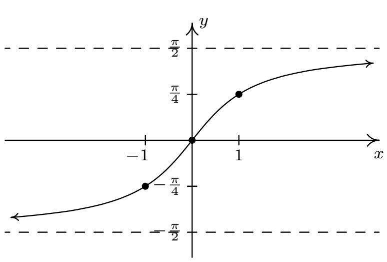
$$\cos(2\arcsin(x)) = \cos(2t) = 1 - 2\sin^2(t) = 1 - 2x^2$$

To find the restrictions on x, we once again appeal to our substitution $t = \arcsin(x)$. Since $\arcsin(x)$ is defined only for $-1 \leq x \leq 1$, the equivalence $\cos(2\arcsin(x)) = 1 - 2x^2$ is valid only on $[-1, 1]$.

A few remarks about Example 4.5.1 are in order. Most of the common errors encountered in dealing with the inverse circular functions come from the need to restrict the domains of the original functions so that they are one-to-one. One instance of this phenomenon is the fact that $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right) = \frac{\pi}{6}$ as opposed to $\frac{11\pi}{6}$. This is the exact same phenomenon discussed in Section 2.2.3 when we saw $\sqrt{(-2)^2} = 2$ as opposed to -2. Additionally, even though the expression we arrived at in part 2b above, namely $1 - 2x^2$, is defined for all real numbers, the equivalence $\cos(2\arcsin(x)) = 1 - 2x^2$ is valid for only $-1 \leq x \leq 1$. This is akin to the fact that while the expression x is defined for all real numbers, the equivalence $\left(\sqrt{x}\right)^2 = x$ is valid only for $x \geq 0$. For this reason, it pays to be careful when we determine the intervals where such equivalences are valid.



$f(x) = \tan(x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$



$f^{-1}(x) = \arctan(x)$

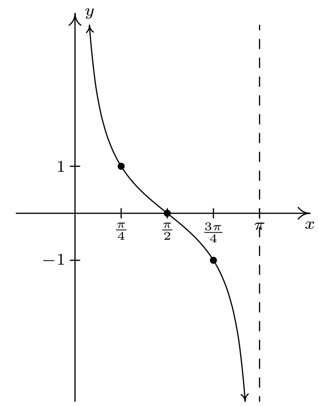
Figure 4.5.5: Reflecting $y = \tan(x)$ across $y = x$ yields $y = \arctan(x)$

The next pair of functions we wish to discuss are the inverses of tangent and cotangent, which are named arctangent and arccotangent, respectively. First, we restrict $f(x) = \tan(x)$ to its fundamental cycle on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to obtain $f^{-1}(x) = \arctan(x)$. Among other things, note that the *vertical* asymptotes $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ of the graph of $f(x) = \tan(x)$ become the *horizontal* asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ of the graph of $f^{-1}(x) = \arctan(x)$: see Figure 4.5.5.

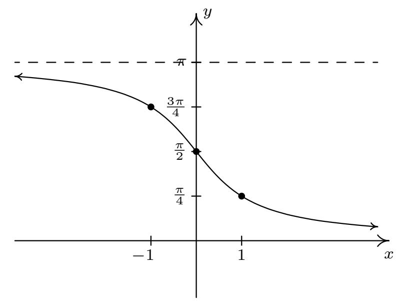
Next, we restrict $g(x) = \cot(x)$ to its fundamental cycle on $(0, \pi)$ to obtain $g^{-1}(x) = \operatorname{arccot}(x)$. Once again, the vertical asymptotes $x = 0$ and $x = \pi$ of the graph of $g(x) = \cot(x)$ become the horizontal asymptotes $y = 0$ and $y = \pi$ of the graph of $g^{-1}(x) = \operatorname{arccot}(x)$. We show these graphs in Figure 4.5.6; the basic properties of the arctangent and arccotangent functions are given in the following theorem.

Theorem 4.5.2 Properties of the Arctangent and Arccotangent Functions

- Properties of $F(x) = \arctan(x)$
  - Domain: $(-\infty, \infty)$
  - Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
  - as $x \to -\infty$, $\arctan(x) \to -\frac{\pi}{2}^+$; as $x \to \infty$, $\arctan(x) \to \frac{\pi}{2}^-$
  - $\arctan(x) = t$ if and only if $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and $\tan(t) = x$
  - $\arctan(x) = \text{arccot}\left(\frac{1}{x}\right)$ for $x > 0$
  - $\tan(\arctan(x)) = x$ for all real numbers x
  - $\arctan(\tan(x)) = x$ provided $-\frac{\pi}{2} < x < \frac{\pi}{2}$
  - additionally, arctangent is odd
- Properties of $G(x) = \text{arccot}(x)$
  - Domain: $(-\infty, \infty)$
  - Range: $(0, \pi)$
  - as $x \to -\infty$, $\text{arccot}(x) \to \pi^-$; as $x \to \infty$, $\text{arccot}(x) \to 0^+$
  - $\text{arccot}(x) = t$ if and only if $0 < t < \pi$ and $\cot(t) = x$
  - $\text{arccot}(x) = \arctan\left(\frac{1}{x}\right)$ for $x > 0$
  - $\cot(\text{arccot}(x)) = x$ for all real numbers x
  - $\text{arccot}(\cot(x)) = x$ provided $0 < x < \pi$



$g(x) = \cot(x)$, $0 < x < \pi$



$g^{-1}(x) = \text{arccot}(x)$

Figure 4.5.6: Reflecting $y = \cot(x)$ across $y = x$ yields $y = \text{arccot}(x)$

Example 4.5.2 Evaluating the arctangent and arccotangent functions

1. Find the exact values of the following.

   (a) $\arctan(\sqrt{3})$ (b) $\text{arccot}(-\sqrt{3})$

   (c) $\cot(\text{arccot}(-5))$ (d) $\sin\left(\arctan\left(-\frac{3}{4}\right)\right)$

2. Rewrite the following as algebraic expressions of x and state the domain on which the equivalence is valid.

   (a) $\tan(2\arctan(x))$ (b) $\cos(\text{arccot}(2x))$

Solution

1. (a) We know $\arctan(\sqrt{3})$ is the real number t between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\tan(t) = \sqrt{3}$. We find $t = \frac{\pi}{3}$, so $\arctan(\sqrt{3}) = \frac{\pi}{3}$.

   (b) The real number $t = \text{arccot}(-\sqrt{3})$ lies in the interval $(0, \pi)$ with $\cot(t) = -\sqrt{3}$. We get $\text{arccot}(-\sqrt{3}) = \frac{5\pi}{6}$.

(c) We can apply Theorem 4.5.2 directly and obtain $\cot(\operatorname{arccot}(-5)) = -5$. However, working it through provides us with yet another opportunity to understand why this is the case. Letting $t = \operatorname{arccot}(-5)$, we have that t belongs to the interval $(0, \pi)$ and $\cot(t) = -5$. Hence, $\cot(\operatorname{arccot}(-5)) = \cot(t) = -5$.

(d) We start simplifying $\sin\left(\arctan\left(-\frac{3}{4}\right)\right)$ by letting $t = \arctan\left(-\frac{3}{4}\right)$. Then $\tan(t) = -\frac{3}{4}$ for some $-\frac{\pi}{2} < t < \frac{\pi}{2}$. Since $\tan(t) < 0$, we know, in fact, $-\frac{\pi}{2} < t < 0$. One way to proceed is to use The Pythagorean Identity, $1 + \cot^2(t) = \csc^2(t)$, since this relates the reciprocals of $\tan(t)$ and $\sin(t)$ and is valid for all t under consideration. From $\tan(t) = -\frac{3}{4}$, we get $\cot(t) = -\frac{4}{3}$. Substituting, we get $1 + \left(-\frac{4}{3}\right)^2 = \csc^2(t)$ so that $\csc(t) = \pm\frac{5}{3}$. Since $-\frac{\pi}{2} < t < 0$, we choose $\csc(t) = -\frac{5}{3}$, so $\sin(t) = -\frac{3}{5}$. Hence, $\sin\left(\arctan\left(-\frac{3}{4}\right)\right) = -\frac{3}{5}$.

2. (a) If we let $t = \arctan(x)$, then $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and $\tan(t) = x$. We look for a way to express $\tan(2\arctan(x)) = \tan(2t)$ in terms of x. Before we get started using identities, we note that $\tan(2t)$ is undefined when $2t = \frac{\pi}{2} + \pi k$ for integers k. Dividing both sides of this equation by 2 tells us we need to exclude values of t where $t = \frac{\pi}{4} + \frac{\pi}{2}k$, where k is an integer. The only members of this family which lie in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ are $t = \pm\frac{\pi}{4}$, which means the values of t under consideration are $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right) \cup \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Returning to $\arctan(2t)$, we note the double angle identity $\tan(2t) = \frac{2\tan(t)}{1-\tan^2(t)}$, is valid for all the values of t under consideration, hence we get

$$\tan(2\arctan(x)) = \tan(2t) = \frac{2\tan(t)}{1-\tan^2(t)} = \frac{2x}{1-x^2}$$

To find where this equivalence is valid we check back with our substitution $t = \arctan(x)$. Since the domain of $\arctan(x)$ is all real numbers, the only exclusions come from the values of t we discarded earlier, $t = \pm\frac{\pi}{4}$. Since $x = \tan(t)$, this means we exclude $x = \tan\left(\pm\frac{\pi}{4}\right) = \pm 1$. Hence, the equivalence $\tan(2\arctan(x)) = \frac{2x}{1-x^2}$ holds for all x in $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

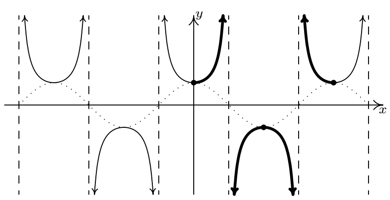
It's always a good idea to make sure the identities used in these situations are valid for all values t under consideration. Check our work back in Example 4.5.1. Were the identities we used there valid for all t under consideration? A pedantic point, to be sure, but what else do you expect from this book?

(b) To get started, we let $t = \operatorname{arccot}(2x)$ so that $\cot(t) = 2x$ where $0 < t < \pi$. In terms of t, $\cos(\operatorname{arccot}(2x)) = \cos(t)$, and our goal is to express the latter in terms of x. Since $\cos(t)$ is always defined, there are no additional restrictions on t, so we can begin using identities to relate $\cot(t)$ to $\cos(t)$. The identity $\cot(t) = \frac{\cos(t)}{\sin(t)}$ is valid for t in $(0, \pi)$, so our strategy is to obtain $\sin(t)$ in terms of x, then write $\cos(t) = \cot(t)\sin(t)$. The identity $1 + \cot^2(t) = \csc^2(t)$ holds for all t in $(0, \pi)$ and relates $\cot(t)$ and $\csc(t) = \frac{1}{\sin(t)}$. Substituting $\cot(t) = 2x$, we get $1 + (2x)^2 = \csc^2(t)$, or $\csc(t) = \pm\sqrt{4x^2+1}$. Since t is between 0 and π, $\csc(t) > 0$, so $\csc(t) = \sqrt{4x^2+1}$ which gives $\sin(t) = \frac{1}{\sqrt{4x^2+1}}$. Hence,

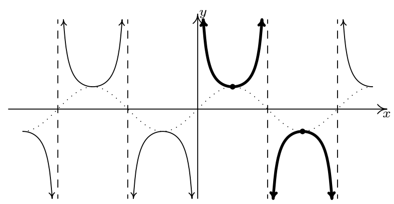
$$\cos(\operatorname{arccot}(2x)) = \cos(t) = \cot(t)\sin(t) = \frac{2x}{\sqrt{4x^2+1}}$$

Since $\operatorname{arccot}(2x)$ is defined for all real numbers x and we encountered no additional restrictions on t, we have $\cos\left(\operatorname{arccot}(2x)\right) = \frac{2x}{\sqrt{4x^2+1}}$ for all real numbers x.

The last two functions to invert are secant and cosecant. A portion of each of their graphs, which were first discussed in Subsection 4.4.2, are given in Figure 4.5.7 below with the fundamental cycles highlighted.



The graph $y = \sec(x)$



The graph $y = \csc(x)$

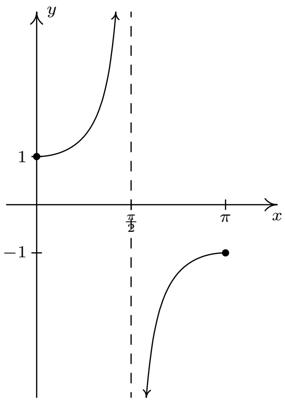
Figure 4.5.7: The fundamental cycles of $f(x) = \sec(x)$ and $g(x) = \csc(x)$

It is clear from the graph of secant that we cannot find one single continuous piece of its graph which covers its entire range of $(-\infty, -1] \cup [1, \infty)$ and restricts the domain of the function so that it is one-to-one. The same is true for cosecant. Thus in order to define the arcsecant and arccosecant functions, we must settle for a piecewise approach wherein we choose one piece to cover the top of the range, namely $[1, \infty)$, and another piece to cover the bottom, namely $(-\infty, -1]$. There are two generally accepted ways make these choices which restrict the domains of these functions so that they are one-to-one. One approach simplifies the Trigonometry associated with the inverse functions, but complicates the Calculus; the other makes the Calculus easier, but the Trigonometry less so. We present both points of view.

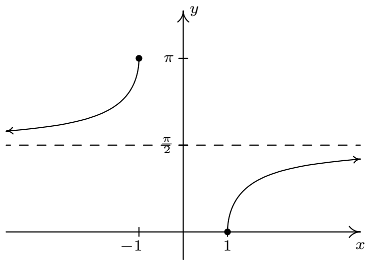
4.5.1 Inverses of Secant and Cosecant: Trigonometry Friendly Approach

In this subsection, we restrict the secant and cosecant functions to coincide with the restrictions on cosine and sine, respectively. For $f(x) = \sec(x)$, we restrict the domain to $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ (Figure 4.5.8) and we restrict $g(x) = \csc(x)$ to $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ (Figure 4.5.9.

Note that for both arcsecant and arccosecant, the domain is $(-\infty, -1] \cup [1, \infty)$. Taking a page from Section 3.1.2, we can rewrite this as $\{x : |x| \geq 1\}$. This is often done in Calculus textbooks, so we include it here for completeness. Using these definitions, we get the following properties of the arcsecant and arccosecant functions.

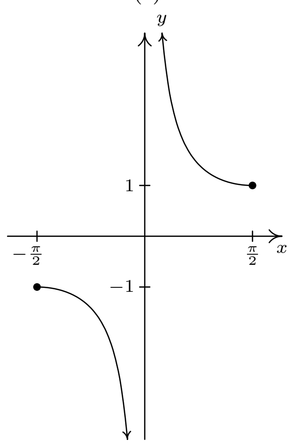


$f(x) = \sec(x)$ on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

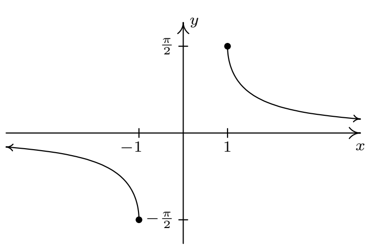


$f^{-1}(x) = \operatorname{arcsec}(x)$

Figure 4.5.8: The "Trigonometry Friendly" definition of $\operatorname{arcsec}(x)$



$g(x) = \csc(x)$ on $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$



$g^{-1}(x) = \operatorname{arccsc}(x)$

Figure 4.5.9: The "Trigonometry Friendly" definition of $\operatorname{arccsc}(x)$

Theorem 4.5.3 Properties of the Arcsecant and Arccosecant Functions ("Trigonometry Friendly" version)

- Properties of $F(x) = \operatorname{arcsec}(x)$
  - Domain: $\{x : |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
  - as $x \to -\infty$, $\operatorname{arcsec}(x) \to \frac{\pi}{2}^{+}$; as $x \to \infty$, $\operatorname{arcsec}(x) \to \frac{\pi}{2}^{-}$
  - $\operatorname{arcsec}(x) = t$ if and only if $0 \leq t < \frac{\pi}{2}$ or $\frac{\pi}{2} < t \leq \pi$ and $\sec(t) = x$
  - $\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ provided $|x| \geq 1$
  - $\sec(\operatorname{arcsec}(x)) = x$ provided $|x| \geq 1$
  - $\operatorname{arcsec}(\sec(x)) = x$ provided $0 \leq x < \frac{\pi}{2}$ or $\frac{\pi}{2} < x \leq \pi$
- Properties of $G(x) = \operatorname{arccsc}(x)$
  - Domain: $\{x : |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
  - as $x \to -\infty$, $\operatorname{arccsc}(x) \to 0^{-}$; as $x \to \infty$, $\operatorname{arccsc}(x) \to 0^{+}$
  - $\operatorname{arccsc}(x) = t$ if and only if $-\frac{\pi}{2} \leq t < 0$ or $0 < t \leq \frac{\pi}{2}$ and $\csc(t) = x$
  - $\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$ provided $|x| \geq 1$
  - $\csc(\operatorname{arccsc}(x)) = x$ provided $|x| \geq 1$
  - $\operatorname{arccsc}(\csc(x)) = x$ provided $-\frac{\pi}{2} \leq x < 0$ or $0 < x \leq \frac{\pi}{2}$
  - additionally, arccosecant is odd

Example 4.5.3 Evaluating the arcsecant and arccosecant functions

1. Find the exact values of the following.
   (a) $\operatorname{arcsec}(2)$
   (b) $\operatorname{arccsc}(-2)$
   (c) $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right)$
   (d) $\cot(\operatorname{arccsc}(-3))$
2. Rewrite the following as algebraic expressions of x and state the domain on which the equivalence is valid.
   (a) $\tan(\operatorname{arcsec}(x))$
   (b) $\cos(\operatorname{arccsc}(4x))$

Solution

1. (a) Using Theorem 4.5.3, we have $\operatorname{arcsec}(2) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.
   (b) Once again, Theorem 4.5.3 gives us $\operatorname{arccsc}(-2) = \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.
   (c) Since $\frac{5\pi}{4}$ doesn't fall between 0 and $\frac{\pi}{2}$ or $\frac{\pi}{2}$ and π, we cannot use the inverse property stated in Theorem 4.5.3. We can, nevertheless, begin by working 'inside out' which yields $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right) = \operatorname{arcsec}(-\sqrt{2}) = \arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$.

(d) One way to begin to simplify $\cot(\operatorname{arccsc}(-3))$ is to let $t = \operatorname{arccsc}(-3)$. Then, $\csc(t) = -3$ and, since this is negative, we have that t lies in the interval $\left[-\frac{\pi}{2}, 0\right)$. We are after $\cot(\operatorname{arccsc}(-3)) = \cot(t)$, so we use the Pythagorean Identity $1 + \cot^2(t) = \csc^2(t)$. Substituting, we have $1 + \cot^2(t) = (-3)^2$, or $\cot(t) = \pm\sqrt{8} = \pm 2\sqrt{2}$. Since $-\frac{\pi}{2} \leq t < 0$, $\cot(t) < 0$, so we get $\cot(\operatorname{arccsc}(-3)) = -2\sqrt{2}$.

2. (a) We begin simplifying $\tan(\operatorname{arcsec}(x))$ by letting $t = \operatorname{arcsec}(x)$. Then, $\sec(t) = x$ for t in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, and we seek a formula for $\tan(t)$. Since $\tan(t)$ is defined for all t values under consideration, we have no additional restrictions on t. To relate $\sec(t)$ to $\tan(t)$, we use the identity $1 + \tan^2(t) = \sec^2(t)$. This is valid for all values of t under consideration, and when we substitute $\sec(t) = x$, we get $1 + \tan^2(t) = x^2$. Hence, $\tan(t) = \pm\sqrt{x^2 - 1}$. If t belongs to $\left[0, \frac{\pi}{2}\right)$ then $\tan(t) \geq 0$; if, on the the other hand, t belongs to $\left(\frac{\pi}{2}, \pi\right]$ then $\tan(t) \leq 0$. As a result, we get a piecewise defined function for $\tan(t)$

$$\tan(t) = \begin{cases} \sqrt{x^2 - 1}, & \text{if } 0 \leq t < \frac{\pi}{2} \\ -\sqrt{x^2 - 1}, & \text{if } \frac{\pi}{2} < t \leq \pi \end{cases}$$

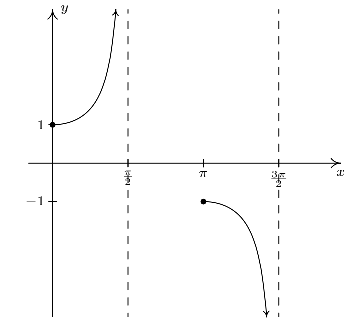
Now we need to determine what these conditions on t mean for x. Since $x = \sec(t)$, when $0 \leq t < \frac{\pi}{2}$, $x \geq 1$, and when $\frac{\pi}{2} < t \leq \pi$, $x \leq -1$. Since we encountered no further restrictions on t, the equivalence below holds for all x in $(-\infty, -1] \cup [1, \infty)$.

$$\tan(\operatorname{arcsec}(x)) = \begin{cases} \sqrt{x^2 - 1}, & \text{if } x \geq 1 \\ -\sqrt{x^2 - 1}, & \text{if } x \leq -1 \end{cases}$$

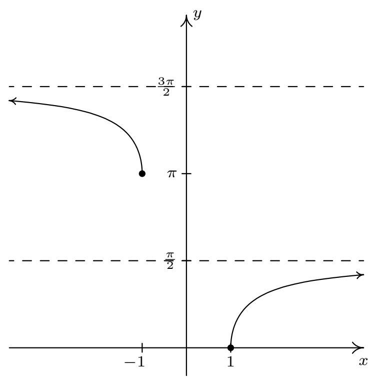
(b) To simplify $\cos(\operatorname{arccsc}(4x))$, we start by letting $t = \operatorname{arccsc}(4x)$. Then $\csc(t) = 4x$ for t in $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, and we now set about finding an expression for $\cos(\operatorname{arccsc}(4x)) = \cos(t)$. Since $\cos(t)$ is defined for all t, we do not encounter any additional restrictions on t. From $\csc(t) = 4x$, we get $\sin(t) = \frac{1}{4x}$, so to find $\cos(t)$, we can make use if the identity $\cos^2(t) + \sin^2(t) = 1$. Substituting $\sin(t) = \frac{1}{4x}$ gives $\cos^2(t) + \left(\frac{1}{4x}\right)^2 = 1$. Solving, we get

$$\cos(t) = \pm\sqrt{\frac{16x^2 - 1}{16x^2}} = \pm\frac{\sqrt{16x^2 - 1}}{4|x|}$$

Since t belongs to $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, we know $\cos(t) \geq 0$, so we choose $\cos(t) = \frac{\sqrt{16 - x^2}}{4|x|}$. (The absolute values here are necessary, since x could be negative.) To find the values for which this equivalence is valid, we look back at our original substution, $t = \operatorname{arccsc}(4x)$. Since the domain of $\operatorname{arccsc}(x)$ requires its argument x to satisfy $|x| \geq 1$, the domain of $\operatorname{arccsc}(4x)$ requires $|4x| \geq 1$. We rewrite this inequality and solve to get $x \leq -\frac{1}{4}$ or $x \geq \frac{1}{4}$. Since we had no additional restrictions on t, the equivalence $\cos(\operatorname{arccsc}(4x)) = \frac{\sqrt{16x^2 - 1}}{4|x|}$ holds for all x in $\left(-\infty, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, \infty\right)$.

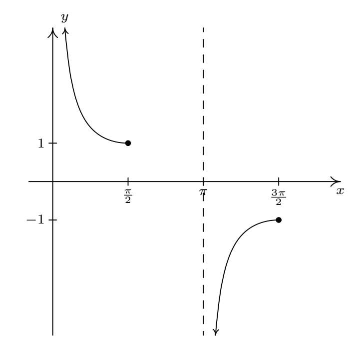


$f(x) = \sec(x)$ on $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$

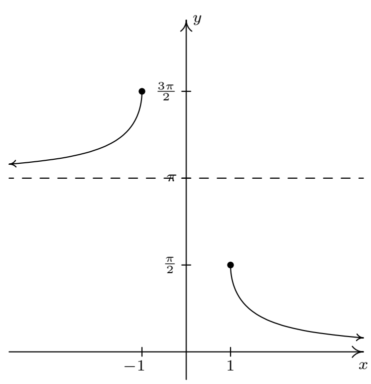


$f^{-1}(x) = \operatorname{arcsec}(x)$

Figure 4.5.10: The "Calculus Friendly" definition of $\operatorname{arcsec}(x)$



$g(x) = \csc(x)$ on $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$



$g^{-1}(x) = \operatorname{arccsc}(x)$

Figure 4.5.11: The "Calculus Friendly definition of $\operatorname{arccsc}(x)$

4.5.2 Inverses of Secant and Cosecant: Calculus Friendly Approach

In this subsection, we restrict $f(x) = \sec(x)$ to $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$, and we restrict $g(x) = \csc(x)$ to $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$.

Using these definitions, we get the following result.

Theorem 4.5.4 Properties of the Arcsecant and Arccosecant Functions ("Calculus Friendly" version)

- Properties of $F(x) = \operatorname{arcsec}(x)$
  - Domain: $\{x : |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$
  - as $x \to -\infty$, $\operatorname{arcsec}(x) \to \frac{3\pi}{2}^{-}$; as $x \to \infty$, $\operatorname{arcsec}(x) \to \frac{\pi}{2}^{-}$
  - $\operatorname{arcsec}(x) = t$ if and only if $0 \leq t < \frac{\pi}{2}$ or $\pi \leq t < \frac{3\pi}{2}$ and $\sec(t) = x$
  - $\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ for $x \geq 1$ only (Compare this with the similar result in Theorem 4.5.3.)
  - $\sec\left(\operatorname{arcsec}(x)\right) = x$ provided $|x| \geq 1$
  - $\operatorname{arcsec}(\sec(x)) = x$ provided $0 \leq x < \frac{\pi}{2}$ or $\pi \leq x < \frac{3\pi}{2}$
- Properties of $G(x) = \operatorname{arccsc}(x)$
  - Domain: $\{x : |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$
  - as $x \to -\infty$, $\operatorname{arccsc}(x) \to \pi^{+}$; as $x \to \infty$, $\operatorname{arccsc}(x) \to 0^{+}$
  - $\operatorname{arccsc}(x) = t$ if and only if $0 < t \leq \frac{\pi}{2}$ or $\pi < t \leq \frac{3\pi}{2}$ and $\csc(t) = x$
  - $\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$ for $x \geq 1$ only (Compare this with the similar result in Theorem 4.5.3.)
  - $\csc\left(\operatorname{arccsc}(x)\right) = x$ provided $|x| \geq 1$
  - $\operatorname{arccsc}(\csc(x)) = x$ provided $0 < x \leq \frac{\pi}{2}$ or $\pi < x \leq \frac{3\pi}{2}$

Our next example is a duplicate of Example 4.5.3. The interested reader is invited to compare and contrast the solution to each.

Example 4.5.4 Evaluating the arcsecant and arccosecant functions

1. Find the exact values of the following.

   (a) $\operatorname{arcsec}(2)$

   (b) $\operatorname{arccsc}(-2)$

   (c) $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right)$

   (d) $\cot\left(\operatorname{arccsc}\left(-3\right)\right)$

2. Rewrite the following as algebraic expressions of x and state the domain on which the equivalence is valid.

(a) $\tan(\text{arcsec}(x))$ (b) $\cos(\text{arccsc}(4x))$

Solution

1. (a) Since $2 \geq 1$, we can use Theorem 4.5.4 to get $\text{arcsec}(2) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

   (b) Unfortunately, -2 is not greater to or equal to 1, so we cannot apply Theorem 4.5.4 to $\text{arccsc}(-2)$ and convert this into an arcsine problem. Instead, we appeal to the definition. The real number $t = \text{arccsc}(-2)$ lies in $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ and satisfies $\csc(t) = -2$. The t we're after is $t = \frac{7\pi}{6}$, so $\text{arccsc}(-2) = \frac{7\pi}{6}$.

   (c) Since $\frac{5\pi}{4}$ lies between π and $\frac{3\pi}{2}$, we may apply Theorem 4.5.4 directly to simplify $\text{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right) = \frac{5\pi}{4}$. We encourage the reader to work this through using the definition as we have done in the previous examples to see how it goes.

   (d) To help simplify $\cot\left(\text{arccsc}\left(-3\right)\right)$ we define $t = \text{arccsc}\left(-3\right)$ so that $\cot\left(\text{arccsc}\left(-3\right)\right) = \cot(t)$. We know $\csc(t) = -3$, and since this is negative, t lies in $\left(\pi, \frac{3\pi}{2}\right]$. Using the identity $1 + \cot^2(t) = \csc^2(t)$, we find $1 + \cot^2(t) = (-3)^2$ so that $\cot(t) = \pm\sqrt{8} = \pm 2\sqrt{2}$. Since t is in the interval $\left(\pi, \frac{3\pi}{2}\right]$, we know $\cot(t) > 0$. Our answer is $\cot\left(\text{arccsc}\left(-3\right)\right) = 2\sqrt{2}$.

2. (a) We begin simplifying $\tan(\text{arcsec}(x))$ by letting $t = \text{arcsec}(x)$. Then, $\sec(t) = x$ for t in $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$, and we seek a formula for $\tan(t)$. Since $\tan(t)$ is defined for all t values under consideration, we have no additional restrictions on t. To relate $\sec(t)$ to $\tan(t)$, we use the identity $1 + \tan^2(t) = \sec^2(t)$. This is valid for all values of t under consideration, and when we substitute $\sec(t) = x$, we get $1 + \tan^2(t) = x^2$. Hence, $\tan(t) = \pm\sqrt{x^2 - 1}$. Since t lies in $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$, $\tan(t) \geq 0$, so we choose $\tan(t) = \sqrt{x^2 - 1}$. Since we found no additional restrictions on t, the equivalence $\tan(\text{arcsec}(x)) = \sqrt{x^2 - 1}$ holds for all x in the domain of $t = \text{arcsec}(x)$, namely $(-\infty, -1] \cup [1, \infty)$.

   (b) To simplify $\cos(\text{arccsc}(4x))$, we start by letting $t = \text{arccsc}(4x)$. Then $\csc(t) = 4x$ for t in $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$, and we now set about finding an expression for $\cos(\text{arccsc}(4x)) = \cos(t)$. Since $\cos(t)$ is defined for all t, we do not encounter any additional restrictions on t. From $\csc(t) = 4x$, we get $\sin(t) = \frac{1}{4x}$, so to find $\cos(t)$, we can make use if the identity $\cos^2(t) + \sin^2(t) = 1$. Substituting $\sin(t) = \frac{1}{4x}$ gives $\cos^2(t) + \left(\frac{1}{4x}\right)^2 = 1$. Solving, we get

$$\cos(t) = \pm\sqrt{\frac{16x^2 - 1}{16x^2}} = \pm\frac{\sqrt{16x^2 - 1}}{4|x|}$$

   If t lies in $\left(0, \frac{\pi}{2}\right]$, then $\cos(t) \geq 0$, and we choose $\cos(t) = \frac{\sqrt{16x^2-1}}{4|x|}$. Otherwise, t belongs to $\left(\pi, \frac{3\pi}{2}\right]$ in which case $\cos(t) \leq 0$, so, we choose $\cos(t) = -\frac{\sqrt{16x^2-1}}{4|x|}$ This leads us to a (momentarily) piecewise defined function for $\cos(t)$

$$\cos(t) = \begin{cases} \dfrac{\sqrt{16x^2-1}}{4|x|}, & \text{if } 0 \leq t \leq \frac{\pi}{2} \\ -\dfrac{\sqrt{16x^2-1}}{4|x|}, & \text{if } \pi < t \leq \frac{3\pi}{2} \end{cases}$$

We now see what these restrictions mean in terms of x. Since $4x = \csc(t)$, we get that for $0 \leq t \leq \frac{\pi}{2}$, $4x \geq 1$, or $x \geq \frac{1}{4}$. In this case, we can simplify $|x| = x$ so

$$\cos(t) = \frac{\sqrt{16x^2-1}}{4|x|} = \frac{\sqrt{16x^2-1}}{4x}$$

Similarly, for $\pi < t \leq \frac{3\pi}{2}$, we get $4x \leq -1$, or $x \leq -\frac{1}{4}$. In this case, $|x| = -x$, so we also get

$$\cos(t) = -\frac{\sqrt{16x^2-1}}{4|x|} = -\frac{\sqrt{16x^2-1}}{4(-x)} = \frac{\sqrt{16x^2-1}}{4x}$$

Hence, in all cases, $\cos(\operatorname{arccsc}(4x)) = \frac{\sqrt{16x^2-1}}{4x}$, and this equivalence is valid for all x in the domain of $t = \operatorname{arccsc}(4x)$, namely $\left(-\infty, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, \infty\right)$

Exercises 4.5

Problems

In Exercises 1 – 40, find the exact value.

1. $\arcsin(-1)$
2. $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$
3. $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$
4. $\arcsin\left(-\frac{1}{2}\right)$
5. $\arcsin(0)$
6. $\arcsin\left(\frac{1}{2}\right)$
7. $\arcsin\left(\frac{\sqrt{2}}{2}\right)$
8. $\arcsin\left(\frac{\sqrt{3}}{2}\right)$
9. $\arcsin(1)$
10. $\arccos(-1)$
11. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$
12. $\arccos\left(-\frac{\sqrt{2}}{2}\right)$
13. $\arccos\left(-\frac{1}{2}\right)$
14. $\arccos(0)$
15. $\arccos\left(\frac{1}{2}\right)$
16. $\arccos\left(\frac{\sqrt{2}}{2}\right)$
17. $\arccos\left(\frac{\sqrt{3}}{2}\right)$
18. $\arccos(1)$
19. $\arctan\left(-\sqrt{3}\right)$
20. $\arctan(-1)$
21. $\arctan\left(-\frac{\sqrt{3}}{3}\right)$
22. $\arctan(0)$
23. $\arctan\left(\frac{\sqrt{3}}{3}\right)$
24. $\arctan(1)$
25. $\arctan\left(\sqrt{3}\right)$
26. $\text{arccot}\left(-\sqrt{3}\right)$
27. $\text{arccot}(-1)$
28. $\text{arccot}\left(-\frac{\sqrt{3}}{3}\right)$
29. $\text{arccot}(0)$
30. $\text{arccot}\left(\frac{\sqrt{3}}{3}\right)$
31. $\text{arccot}(1)$
32. $\text{arccot}\left(\sqrt{3}\right)$
33. $\text{arcsec}(2)$
34. $\text{arccsc}(2)$
35. $\text{arcsec}\left(\sqrt{2}\right)$
36. $\text{arccsc}\left(\sqrt{2}\right)$
37. $\text{arcsec}\left(\frac{2\sqrt{3}}{3}\right)$
38. $\text{arccsc}\left(\frac{2\sqrt{3}}{3}\right)$
39. $\text{arcsec}(1)$
40. $\text{arccsc}(1)$

In Exercises 41 – 48, assume that the range of arcsecant is $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ and that the range of arccosecant is $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ when finding the exact value.

41. $\text{arcsec}(-2)$
42. $\text{arcsec}\left(-\sqrt{2}\right)$
43. $\text{arcsec}\left(-\frac{2\sqrt{3}}{3}\right)$
44. $\text{arcsec}(-1)$

45. $\text{arccsc}\left(-2\right)$

46. $\text{arccsc}\left(-\sqrt{2}\right)$

47. $\text{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)$

48. $\text{arccsc}\left(-1\right)$

In Exercises 49 – 56, assume that the range of arcsecant is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and that the range of arccosecant is $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ when finding the exact value.

49. $\text{arcsec}\left(-2\right)$

50. $\text{arcsec}\left(-\sqrt{2}\right)$

51. $\text{arcsec}\left(-\frac{2\sqrt{3}}{3}\right)$

52. $\text{arcsec}\left(-1\right)$

53. $\text{arccsc}\left(-2\right)$

54. $\text{arccsc}\left(-\sqrt{2}\right)$

55. $\text{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)$

56. $\text{arccsc}\left(-1\right)$

In Exercises 57 – 86, find the exact value or state that it is undefined.

57. $\sin\left(\arcsin\left(\frac{1}{2}\right)\right)$

58. $\sin\left(\arcsin\left(-\frac{\sqrt{2}}{2}\right)\right)$

59. $\sin\left(\arcsin\left(\frac{3}{5}\right)\right)$

60. $\sin\left(\arcsin\left(-0.42\right)\right)$

61. $\sin\left(\arcsin\left(\frac{5}{4}\right)\right)$

62. $\cos\left(\arccos\left(\frac{\sqrt{2}}{2}\right)\right)$

63. $\cos\left(\arccos\left(-\frac{1}{2}\right)\right)$

64. $\cos\left(\arccos\left(\frac{5}{13}\right)\right)$

65. $\cos\left(\arccos\left(-0.998\right)\right)$

66. $\cos\left(\arccos\left(\pi\right)\right)$

67. $\tan\left(\arctan\left(-1\right)\right)$

68. $\tan\left(\arctan\left(\sqrt{3}\right)\right)$

69. $\tan\left(\arctan\left(\frac{5}{12}\right)\right)$

70. $\tan\left(\arctan\left(0.965\right)\right)$

71. $\tan\left(\arctan\left(3\pi\right)\right)$

72. $\cot\left(\text{arccot}\left(1\right)\right)$

73. $\cot\left(\text{arccot}\left(-\sqrt{3}\right)\right)$

74. $\cot\left(\text{arccot}\left(-\frac{7}{24}\right)\right)$

75. $\cot\left(\text{arccot}\left(-0.001\right)\right)$

76. $\cot\left(\text{arccot}\left(\frac{17\pi}{4}\right)\right)$

77. $\sec\left(\text{arcsec}\left(2\right)\right)$

78. $\sec\left(\text{arcsec}\left(-1\right)\right)$

79. $\sec\left(\text{arcsec}\left(\frac{1}{2}\right)\right)$

80. $\sec\left(\text{arcsec}\left(0.75\right)\right)$

81. $\sec\left(\text{arcsec}\left(117\pi\right)\right)$

82. $\csc\left(\text{arccsc}\left(\sqrt{2}\right)\right)$

83. $\csc\left(\text{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)\right)$

84. $\csc\left(\text{arccsc}\left(\frac{\sqrt{2}}{2}\right)\right)$

85. $\csc\left(\text{arccsc}\left(1.0001\right)\right)$

86. $\csc\left(\text{arccsc}\left(\frac{\pi}{4}\right)\right)$

In Exercises 87 – 106, find the exact value or state that it is undefined.

87. $\arcsin\left(\sin\left(\frac{\pi}{6}\right)\right)$

88. $\arcsin\left(\sin\left(-\frac{\pi}{3}\right)\right)$

89. $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right)$

90. $\arcsin\left(\sin\left(\frac{11\pi}{6}\right)\right)$

91. $\arcsin\left(\sin\left(\frac{4\pi}{3}\right)\right)$

92. $\arccos\left(\cos\left(\frac{\pi}{4}\right)\right)$

93. $\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right)$

94. $\arccos\left(\cos\left(\frac{3\pi}{2}\right)\right)$

95. $\arccos\left(\cos\left(-\frac{\pi}{6}\right)\right)$

96. $\arccos\left(\cos\left(\frac{5\pi}{4}\right)\right)$

97. $\arctan\left(\tan\left(\frac{\pi}{3}\right)\right)$

98. $\arctan\left(\tan\left(-\frac{\pi}{4}\right)\right)$

99. $\arctan\left(\tan\left(\pi\right)\right)$

100. $\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)$

101. $\arctan\left(\tan\left(\frac{2\pi}{3}\right)\right)$

102. $\operatorname{arccot}\left(\cot\left(\frac{\pi}{3}\right)\right)$

103. $\operatorname{arccot}\left(\cot\left(-\frac{\pi}{4}\right)\right)$

104. $\operatorname{arccot}\left(\cot\left(\pi\right)\right)$

105. $\operatorname{arccot}\left(\cot\left(\frac{\pi}{2}\right)\right)$

106. $\operatorname{arccot}\left(\cot\left(\frac{2\pi}{3}\right)\right)$

In Exercises 107 – 118, assume that the range of arcsecant is $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ and that the range of arccosecant is $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ when finding the exact value.

107. $\operatorname{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right)$

108. $\operatorname{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right)$

109. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right)$

110. $\operatorname{arcsec}\left(\sec\left(-\frac{\pi}{2}\right)\right)$

111. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right)$

112. $\operatorname{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right)$

113. $\operatorname{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right)$

114. $\operatorname{arccsc}\left(\csc\left(\frac{2\pi}{3}\right)\right)$

115. $\operatorname{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right)$

116. $\operatorname{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right)$

117. $\operatorname{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right)$

118. $\operatorname{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right)$

In Exercises 119 – 130, assume that the range of arcsecant is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and that the range of arccosecant is $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ when finding the exact value.

119. $\operatorname{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right)$

120. $\operatorname{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right)$

121. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right)$

122. $\operatorname{arcsec}\left(\sec\left(-\frac{\pi}{2}\right)\right)$

123. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right)$

124. $\operatorname{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right)$

125. $\operatorname{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right)$

126. $\operatorname{arccsc}\left(\csc\left(\frac{2\pi}{3}\right)\right)$

127. $\operatorname{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right)$

128. $\operatorname{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right)$

129. $\operatorname{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right)$

130. $\operatorname{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right)$

In Exercises 131 – 154, find the exact value or state that it is undefined.

131. $\sin\left(\arccos\left(-\frac{1}{2}\right)\right)$

132. $\sin\left(\arccos\left(\frac{3}{5}\right)\right)$

133. $\sin\left(\arctan\left(-2\right)\right)$

134. $\sin\left(\operatorname{arccot}\left(\sqrt{5}\right)\right)$

135. $\sin\left(\operatorname{arccsc}\left(-3\right)\right)$

136. $\cos\left(\arcsin\left(-\frac{5}{13}\right)\right)$

137. $\cos\left(\arctan\left(\sqrt{7}\right)\right)$

138. $\cos\left(\operatorname{arccot}\left(3\right)\right)$

139. $\cos\left(\operatorname{arcsec}\left(5\right)\right)$

140. $\tan\left(\arcsin\left(-\frac{2\sqrt{5}}{5}\right)\right)$

141. $\tan\left(\arccos\left(-\frac{1}{2}\right)\right)$

142. $\tan\left(\operatorname{arcsec}\left(\frac{5}{3}\right)\right)$

143. $\tan\left(\operatorname{arccot}\left(12\right)\right)$

144. $\cot\left(\arcsin\left(\frac{12}{13}\right)\right)$

145. $\cot\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right)$

146. $\cot\left(\operatorname{arccsc}\left(\sqrt{5}\right)\right)$

147. $\cot\left(\arctan\left(0.25\right)\right)$

148. $\sec\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right)$

149. $\sec\left(\arcsin\left(-\frac{12}{13}\right)\right)$

150. $\sec\left(\arctan\left(10\right)\right)$

151. $\sec\left(\operatorname{arccot}\left(-\frac{\sqrt{10}}{10}\right)\right)$

152. $\csc\left(\operatorname{arccot}\left(9\right)\right)$

153. $\csc\left(\arcsin\left(\frac{3}{5}\right)\right)$

154. $\csc\left(\arctan\left(-\frac{2}{3}\right)\right)$

In Exercises 155 – 164, find the exact value or state that it is undefined.

155. $\sin\left(\arcsin\left(\frac{5}{13}\right)+\frac{\pi}{4}\right)$

156. $\cos\left(\operatorname{arcsec}(3)+\arctan(2)\right)$

157. $\tan\left(\arctan(3)+\arccos\left(-\frac{3}{5}\right)\right)$

158. $\sin\left(2\arcsin\left(-\frac{4}{5}\right)\right)$

159. $\sin\left(2\operatorname{arccsc}\left(\frac{13}{5}\right)\right)$

160. $\sin\left(2\arctan\left(2\right)\right)$

161. $\cos\left(2\arcsin\left(\frac{3}{5}\right)\right)$

162. $\cos\left(2\operatorname{arcsec}\left(\frac{25}{7}\right)\right)$

163. $\cos\left(2\operatorname{arccot}\left(-\sqrt{5}\right)\right)$

164. $\sin\left(\frac{\arctan(2)}{2}\right)$

In Exercises 165 – 184, rewrite the quantity as algebraic expressions of *x* and state the domain on which the equivalence is valid.

165. $\sin\left(\arccos\left(x\right)\right)$

166. $\cos\left(\arctan\left(x\right)\right)$

167. $\tan\left(\arcsin\left(x\right)\right)$

168. $\sec\left(\arctan\left(x\right)\right)$

169. $\csc\left(\arccos\left(x\right)\right)$

170. $\sin\left(2\arctan\left(x\right)\right)$

171. $\sin\left(2\arccos\left(x\right)\right)$

172. $\cos\left(2\arctan\left(x\right)\right)$

173. $\sin(\arccos(2x))$

174. $\sin\left(\arccos\left(\frac{x}{5}\right)\right)$

175. $\cos\left(\arcsin\left(\frac{x}{2}\right)\right)$

176. $\cos\left(\arctan\left(3x\right)\right)$

177. $\sin(2\arcsin(7x))$

178. $\sin\left(2\arcsin\left(\frac{x\sqrt{3}}{3}\right)\right)$

179. $\cos(2\arcsin(4x))$

180. $\sec(\arctan(2x))\tan(\arctan(2x))$

181. $\sin\left(\arcsin(x)+\arccos(x)\right)$

182. $\cos\left(\arcsin(x)+\arctan(x)\right)$

183. $\tan\left(2\arcsin(x)\right)$

184. $\sin\left(\frac{1}{2}\arctan(x)\right)$

5: LIMITS

Calculus means "a method of calculation or reasoning." When one computes the sales tax on a purchase, one employs a simple calculus. When one finds the area of a polygonal shape by breaking it up into a set of triangles, one is using another calculus. Proving a theorem in geometry employs yet another calculus.

Despite the wonderful advances in mathematics that had taken place into the first half of the 17^{th} century, mathematicians and scientists were keenly aware of what they *could not do.* (This is true even today.) In particular, two important concepts eluded mastery by the great thinkers of that time: area and rates of change.

Area seems innocuous enough; areas of circles, rectangles, parallelograms, etc., are standard topics of study for students today just as they were then. However, the areas of *arbitrary* shapes could not be computed, even if the boundary of the shape could be described exactly.

Rates of change were also important. When an object moves at a constant rate of change, then "distance = rate $\times$ time." But what if the rate is not constant – can distance still be computed? Or, if distance is known, can we discover the rate of change?

It turns out that these two concepts were related. Two mathematicians, Sir Isaac Newton and Gottfried Leibniz, are credited with independently formulating a system of computing that solved the above problems and showed how they were connected. Their system of reasoning was "a" calculus. However, as the power and importance of their discovery took hold, it became known to many as "the" calculus. Today, we generally shorten this to discuss "calculus."

The foundation of "the calculus" is the *limit.* It is a tool to describe a particular behaviour of a function. This chapter begins our study of the limit by approximating its value graphically and numerically. After a formal definition of the limit, properties are established that make "finding limits" tractable. Once the limit is understood, then the problems of area and rates of change can be approached.

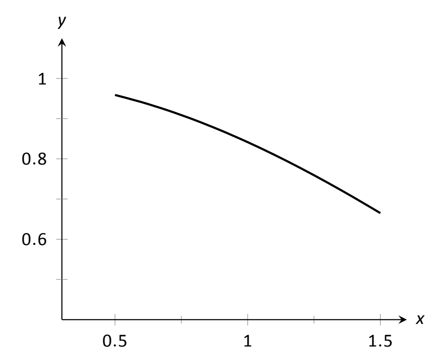


Figure 5.1.1: $\sin(x)/x$ near $x = 1$.

5.1 An Introduction To Limits

We begin our study of *limits* by considering examples that demonstrate key concepts that will be explained as we progress.

Consider the function $y = \dfrac{\sin x}{x}$. When x is near the value 1, what value (if any) is y near?

While our question is not precisely formed (what constitutes "near the value 1"?), the answer does not seem difficult to find. One might think first to look at a graph of this function to approximate the appropriate y values. Consider Figure 5.1.1, where $y = \frac{\sin x}{x}$ is graphed. For values of x near 1, it seems that y takes on values near 0.85. In fact, when $x = 1$, then $y = \frac{\sin 1}{1} \approx 0.84$, so it makes sense that when x is "near" 1, y will be "near" 0.84.

Consider this again at a different value for x. When x is near 0, what value (if any) is y near? By considering Figure 5.1.2, one can see that it seems that y takes on values near 1. But what happens when $x = 0$? We have

$$y \to \frac{\sin 0}{0} \to \text{“}\,\frac{0}{0}\,\text{”}.$$

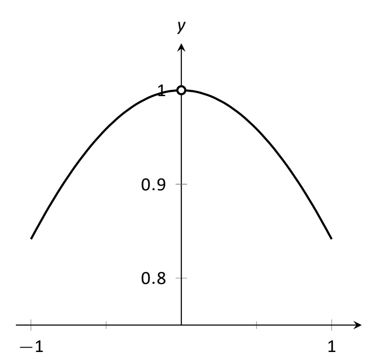


Figure 5.1.2: $\sin(x)/x$ near $x = 0$.

The expression "0/0" has no value; it is *indeterminate.* Such an expression gives no information about what is going on with the function nearby. We cannot find out how y behaves near $x = 0$ for this function simply by letting $x = 0$.

Finding a limit entails understanding how a function behaves near a particular value of x. Before continuing, it will be useful to establish some notation. Let $y = f(x)$; that is, let y be a function of x for some function f. The expression "the limit of y as x approaches 1" describes a number, often referred to as L, that y nears as x nears 1. We write all this as

$$\lim_{x\to 1} y = \lim_{x\to 1} f(x) = L.$$

This is not a complete definition; this is a pseudo-definition that will allow us to explore the idea of a limit. A more detailed, but still informal, definition of the limit is given in Definition 5.1.1 at the end of this section. A more precise definition is beyond the scope of this text.

Above, where $f(x) = \sin(x)/x$, we approximated

$$\lim_{x\to 1} \frac{\sin x}{x} \approx 0.84 \quad \text{and} \quad \lim_{x\to 0} \frac{\sin x}{x} \approx 1.$$

(We *approximated* these limits, hence used the "$\approx$" symbol, since we are working with the pseudo-definition of a limit, not the actual definition.)

In the next section, we will find limits *analytically*; that is, exactly using a variety of mathematical tools. For now, we will *approximate* limits both graphically and numerically. Graphing a function can provide a good approximation, though often not very precise. Numerical methods can provide a more accurate approximation. We have already approximated limits graphically, so we now turn our attention to numerical approximations.

x	$\sin(x)/x$
0.9	0.870363
0.99	0.844471
0.999	0.841772
1	**0.841471**
1.001	0.84117
1.01	0.838447
1.1	0.810189

Figure 5.1.3: Approximate values of $\sin(x)/x$ with x near 1.

Consider again $\lim_{x\to 1} \sin(x)/x$. To approximate this limit numerically, we can create a table of x and $f(x)$ values where x is "near" 1. This is done in Figure 5.1.3.

Notice that for values of x near 1, we have $\sin(x)/x$ near 0.841. The $x = 1$ row is in bold to highlight the fact that when considering limits, we are *not* concerned with the value of the function at that particular x value; we are only concerned with the values of the function when x is *near* 1.

x	$\sin(x)/x$
-0.1	0.9983341665
-0.01	0.9999833334
-0.001	0.9999998333
0	**not defined**
0.001	0.9999998333
0.01	0.9999833334
0.1	0.9983341665

Figure 5.1.4: Approximate values of $\sin(x)/x$ with x near 0.

Now approximate $\lim_{x\to 0} \sin(x)/x$ numerically. We already approximated the value of this limit as 1 graphically in Figure 5.1.2. The table in Figure 5.1.4 shows the value of $\sin(x)/x$ for values of x near 0. Ten places after the decimal point are shown to highlight how close to 1 the value of $\sin(x)/x$ gets as x takes on values very near 0. We include the $x = 0$ row in bold again to stress that we are not concerned with the value of our function at $x = 0$, only on the behaviour of the function *near* 0.

This numerical method gives confidence to say that 1 is a good approximation of $\lim_{x\to 0} \sin(x)/x$; that is,

$$\lim_{x\to 0} \sin(x)/x \approx 1.$$

Later we will be able to prove that the limit is *exactly* 1.

We now consider several examples that allow us explore different aspects of the limit concept.

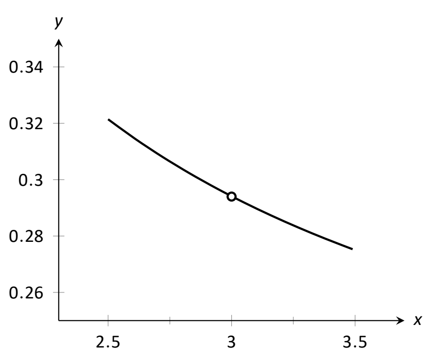


Figure 5.1.5: Graphically approximating a limit in Example 5.1.1.

Example 5.1.1 Approximating the value of a limit

Use graphical and numerical methods to approximate

$$\lim_{x\to 3} \frac{x^2 - x - 6}{6x^2 - 19x + 3}.$$

Solution To graphically approximate the limit, graph

$$y = (x^2 - x - 6)/(6x^2 - 19x + 3)$$

on a small interval that contains 3. To numerically approximate the limit, create a table of values where the x values are near 3. This is done in Figures 5.1.5 and 5.1.6, respectively.

x	$\frac{x^2-x-6}{6x^2-19x+3}$
2.9	0.29878
2.99	0.294569
2.999	0.294163
3	**not defined**
3.001	0.294073
3.01	0.293669
3.1	0.289773

Figure 5.1.6: Numerically approximating a limit in Example 5.1.1.

The graph shows that when x is near 3, the value of y is very near 0.3. By considering values of x near 3, we see that $y = 0.294$ is a better approximation. The graph and the table imply that

$$\lim_{x\to 3} \frac{x^2 - x - 6}{6x^2 - 19x + 3} \approx 0.294.$$

This example may bring up a few questions about approximating limits (and the nature of limits themselves).

1. If a graph does not produce as good an approximation as a table, why bother with it?
2. How many values of x in a table are "enough?" In the previous example, could we have just used $x = 3.001$ and found a fine approximation?

Graphs are useful since they give a visual understanding concerning the behaviour of a function. Sometimes a function may act "erratically" near certain x values which is hard to discern numerically but very plain graphically. Since graphing utilities are very accessible, it makes sense to make proper use of them.

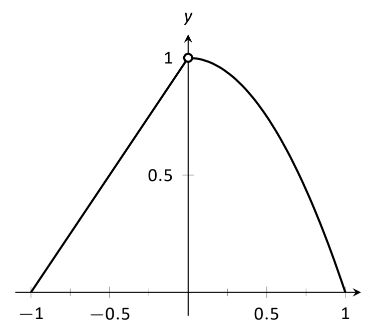


Figure 5.1.7: Graphically approximating a limit in Example 5.1.2.

Since tables and graphs are used only to *approximate* the value of a limit, there is not a firm answer to how many data points are "enough." Include enough so that a trend is clear, and use values (when possible) both less than and greater than the value in question. In Example 5.1.1, we used both values less than and greater than 3. Had we used just $x = 3.001$, we might have been tempted to conclude that the limit had a value of 0.3. While this is not far off, we could do better. Using values "on both sides of 3" helps us identify trends.

Example 5.1.2 Approximating the value of a limit

Graphically and numerically approximate the limit of $f(x)$ as x approaches 0, where

$$f(x) = \begin{cases} x + 1 & x < 0 \\ -x^2 + 1 & x > 0 \end{cases}.$$

x	$f(x)$
-0.1	0.9
-0.01	0.99
-0.001	0.999
0.001	0.999999
0.01	0.9999
0.1	0.99

Figure 5.1.8: Numerically approximating a limit in Example 5.1.2.

Solution Again we graph $f(x)$ and create a table of its values near $x = 0$ to approximate the limit. Note that this is a piecewise defined function, so it behaves differently on either side of 0. Figure 5.1.7 shows a graph of $f(x)$, and on either side of 0 it seems the y values approach 1. Note that $f(0)$ is not actually defined, as indicated in the graph with the open circle.

The table shown in Figure 5.1.8 shows values of $f(x)$ for values of x near 0. It is clear that as x takes on values very near 0, $f(x)$ takes on values very near 1. It turns out that if we let $x = 0$ for either "piece" of $f(x)$, 1 is returned; this is significant and we'll return to this idea later.

The graph and table allow us to say that $\lim_{x\to 0} f(x) \approx 1$; in fact, we are probably very sure it *equals* 1.

Identifying When Limits Do Not Exist

A function may not have a limit for all values of x. That is, we cannot say $\lim_{x\to c} f(x) = L$ for some numbers L for all values of c, for there may not be a number that $f(x)$ is approaching. There are three common ways in which a limit may fail to exist.

1. The function $f(x)$ may approach different values on either side of c.
2. The function may grow without upper or lower bound as x approaches c.
3. The function may oscillate as x approaches c without approaching a specific value.

We'll explore each of these in turn.

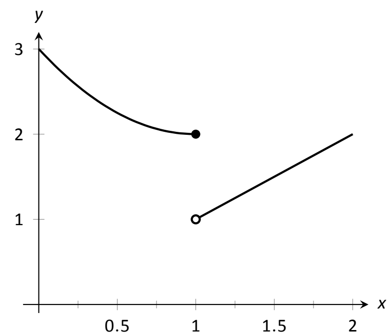


Figure 5.1.9: Observing no limit as $x \to 1$ in Example 5.1.3.

x	$f(x)$
0.9	2.01
0.99	2.0001
0.999	2.000001
1.001	1.001
1.01	1.01
1.1	1.1

Figure 5.1.10: Values of $f(x)$ near $x = 1$ in Example 5.1.3.

Example 5.1.3 Different Values Approached From Left and Right

Explore why $\lim\limits_{x\to 1} f(x)$ does not exist, where

$$f(x) = \begin{cases} x^2 - 2x + 3 & x \leq 1 \\ x & x > 1 \end{cases}$$

Solution A graph of $f(x)$ around $x = 1$ and a table are given in Figures 5.1.9 and 5.1.10, respectively. It is clear that as x approaches 1, $f(x)$ does not seem to approach a single number. Instead, it seems as though $f(x)$ approaches two different numbers. When considering values of x less than 1 (approaching 1 from the left), it seems that $f(x)$ is approaching 2; when considering values of x greater than 1 (approaching 1 from the right), it seems that $f(x)$ is approaching 1. Recognizing this behaviour is important; we'll study this in greater depth later. Right now, it suffices to say that the limit does not exist since $f(x)$ is not approaching one value as x approaches 1.

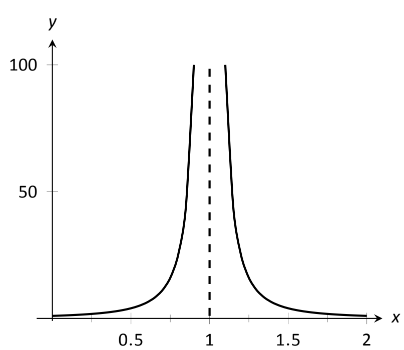


Figure 5.1.11: Observing no limit as $x \to 1$ in Example 5.1.4.

x	$f(x)$
0.9	100.
0.99	10000.
0.999	$1. \times 10^6$
1.001	$1. \times 10^6$
1.01	10000.
1.1	100.

Figure 5.1.12: Values of $f(x)$ near $x = 1$ in Example 5.1.4.

Example 5.1.4 The Function Grows Without Bound

Explore why $\lim\limits_{x\to 1} 1/(x-1)^2$ does not exist.

Solution A graph and table of $f(x) = 1/(x-1)^2$ are given in Figures 5.1.11 and 5.1.12, respectively. Both show that as x approaches 1, $f(x)$ grows larger and larger.

We can deduce this on our own, without the aid of the graph and table. If x is near 1, then $(x-1)^2$ is very small, and:

$$\frac{1}{\text{very small number}} = \text{very large number}.$$

Since $f(x)$ is not approaching a single number, we conclude that

$$\lim_{x\to 1} \frac{1}{(x-1)^2}$$

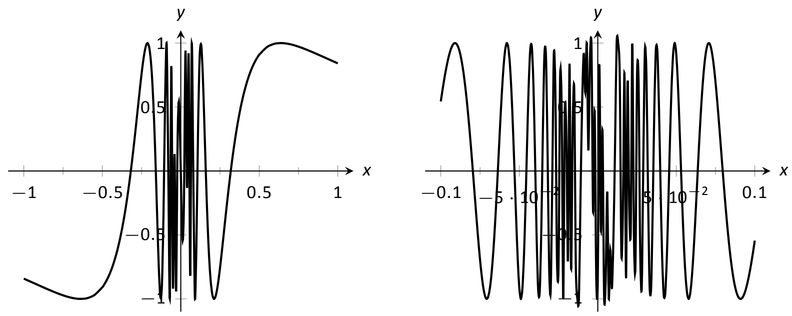
does not exist.

Example 5.1.5 The Function Oscillates

Explore why $\lim_{x\to 0} \sin(1/x)$ does not exist.

Solution Two graphs of $f(x) = \sin(1/x)$ are given in Figures 5.1.13. Figure 5.1.13(a) shows $f(x)$ on the interval $[-1, 1]$; notice how $f(x)$ seems to oscillate near $x = 0$. One might think that despite the oscillation, as x approaches 0, $f(x)$ approaches 0. However, Figure 5.1.13(b) zooms in on $\sin(1/x)$, on the interval $[-0.1, 0.1]$. Here the oscillation is even more pronounced. Finally, in the table in Figure 5.1.13(c), we see $\sin(x)/x$ evaluated for values of x near 0. As x approaches 0, $f(x)$ does not appear to approach any value.

It can be shown that in reality, as x approaches 0, $\sin(1/x)$ takes on all values between -1 and 1 infinitely many times! Because of this oscillation, $\lim_{x\to 0} \sin(1/x)$ does not exist.



x	$\sin(1/x)$
0.1	-0.544021
0.01	-0.506366
0.001	0.82688
0.0001	-0.305614
$1. \times 10^{-5}$	0.0357488
$1. \times 10^{-6}$	-0.349994
$1. \times 10^{-7}$	0.420548

(c)

Figure 5.1.13: Observing that $f(x) = \sin(1/x)$ has no limit as $x \to 0$ in Example 5.1.5.

Limits of Difference Quotients

We have approximated limits of functions as x approached a particular number. We will consider another important kind of limit after explaining a few key ideas.

Let $f(x)$ represent the position function, in feet, of some particle that is moving in a straight line, where x is measured in seconds. Let's say that when $x = 1$, the particle is at position 10 ft., and when $x = 5$, the particle is at 20 ft. Another way of expressing this is to say

$$f(1) = 10 \quad \text{and} \quad f(5) = 20.$$

Since the particle travelled 10 feet in 4 seconds, we can say the particle's *average velocity* was 2.5 ft/s. We write this calculation using a "quotient of differences," or, a *difference quotient*:

$$\frac{f(5) - f(1)}{5 - 1} = \frac{10}{4} = 2.5\text{ft/s}.$$

This difference quotient can be thought of as the familiar "rise over run" used to compute the slopes of lines. In fact, that is essentially what we are doing: given two points on the graph of f, we are finding the slope of the *secant line* through those two points. See Figure 5.1.14.

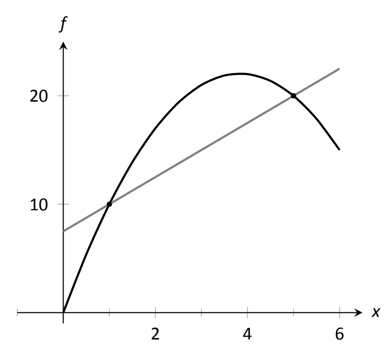


Figure 5.1.14: Interpreting a difference quotient as the slope of a secant line.

Now consider finding the average speed on another time interval. We again start at $x = 1$, but consider the position of the particle h seconds later. That is,

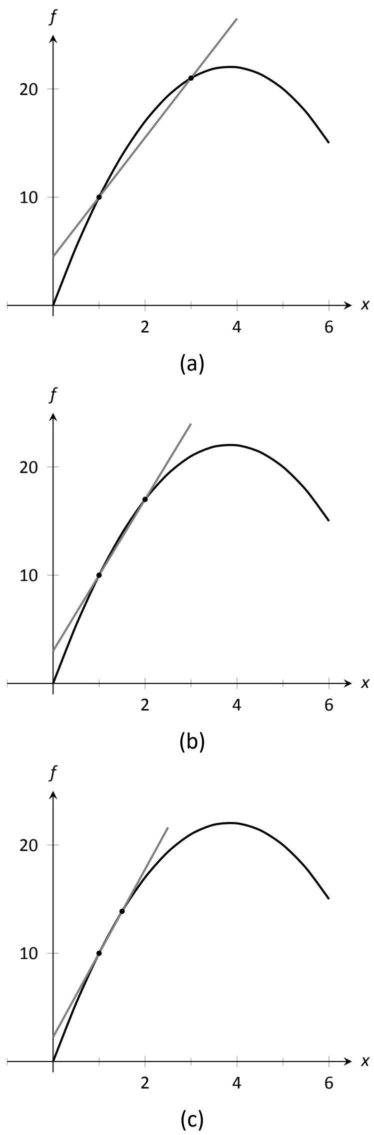


Figure 5.1.15: Secant lines of $f(x)$ at $x = 1$ and $x = 1 + h$, for shrinking values of h (i.e., $h \to 0$).

h	$\frac{f(1+h)-f(1)}{h}$
-0.5	9.25
-0.1	8.65
-0.01	8.515
0.01	8.485
0.1	8.35
0.5	7.75

Figure 5.1.16: The difference quotient evaluated at values of h near 0.

consider the positions of the particle when $x = 1$ and when $x = 1 + h$. The difference quotient is now

$$\frac{f(1+h) - f(1)}{(1+h) - 1} = \frac{f(1+h) - f(1)}{h}.$$

Let $f(x) = -1.5x^2 + 11.5x$; note that $f(1) = 10$ and $f(5) = 20$, as in our discussion. We can compute this difference quotient for all values of h (even negative values!) except $h = 0$, for then we get "0/0," the indeterminate form introduced earlier. For all values $h \neq 0$, the difference quotient computes the average velocity of the particle over an interval of time of length h starting at $x = 1$.

For small values of h, i.e., values of h close to 0, we get average velocities over very short time periods and compute secant lines over small intervals. See Figure 5.1.15. This leads us to wonder what the limit of the difference quotient is as h approaches 0. That is,

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \ ?$$

As we do not yet have a true definition of a limit nor an exact method for computing it, we settle for approximating the value. While we could graph the difference quotient (where the x-axis would represent h values and the y-axis would represent values of the difference quotient) we settle for making a table. See Figure 5.1.16. The table gives us reason to assume the value of the limit is about 8.5.

Proper understanding of limits is key to understanding calculus. With limits, we can accomplish seemingly impossible mathematical things, like adding up an infinite number of numbers (and not get infinity) and finding the slope of a line between two points, where the "two points" are actually the same point. These are not just mathematical curiosities; they allow us to link position, velocity and acceleration together, connect cross-sectional areas to volume, find the work done by a variable force, and much more.

Despite the importance of limits to calculus, we often settle for an imprecise, intuitive understanding of what the limit of a function means. The precise definition of the limit omitted from a course like Math 1560, and left for later courses, such as Math 3500. For this course, we will use the following informal definition.

Definition 5.1.1 Informal Definition of the Limit

Let I be an open interval containing c, and let f be a function defined on I, except possibly at c. We say that the **limit of $f(x)$, as x approaches c, is** L, and write

$$\lim_{x \to c} f(x) = L,$$

if we can make the value of $f(x)$ arbitrarily close to L by choosing $x \neq c$ sufficiently close to c.

The formal definition of the limit makes precise the meaning of the phrases "arbitrarily close" and "sufficiently close". The problem with the definition we have given is that, while it gives an intuitive understanding of the meaning of the limit, it's of no use for *proving* theorems about limits. In Section 5.2 we will state (but not prove) several theorems about limits which will allow use to compute their values analytically, without recourse to tables of values.

Exercises 5.1

Terms and Concepts

1. In your own words, what does it mean to "find the limit of $f(x)$ as x approaches 3"?

2. An expression of the form $\frac{0}{0}$ is called ____.

3. T/F: The limit of $f(x)$ as x approaches 5 is $f(5)$.

4. Describe three situations where $\lim\limits_{x\to c} f(x)$ does not exist.

5. In your own words, what is a difference quotient?

6. When x is near 0, $\dfrac{\sin x}{x}$ is near what value?

Problems

In Exercises 7 – 16, approximate the given limits both numerically and graphically.

7. $\lim\limits_{x\to 1} x^2 + 3x - 5$

8. $\lim\limits_{x\to 0} x^3 - 3x^2 + x - 5$

9. $\lim\limits_{x\to 0} \dfrac{x+1}{x^2+3x}$

10. $\lim\limits_{x\to 3} \dfrac{x^2-2x-3}{x^2-4x+3}$

11. $\lim\limits_{x\to -1} \dfrac{x^2+8x+7}{x^2+6x+5}$

12. $\lim\limits_{x\to 2} \dfrac{x^2+7x+10}{x^2-4x+4}$

13. $\lim\limits_{x\to 2} f(x)$, where
$$f(x) = \begin{cases} x+2 & x \le 2 \\ 3x-5 & x > 2 \end{cases}.$$

14. $\lim\limits_{x\to 3} f(x)$, where
$$f(x) = \begin{cases} x^2-x+1 & x \le 3 \\ 2x+1 & x > 3 \end{cases}.$$

15. $\lim\limits_{x\to 0} f(x)$, where
$$f(x) = \begin{cases} \cos x & x \le 0 \\ x^2+3x+1 & x > 0 \end{cases}.$$

16. $\lim\limits_{x\to \pi/2} f(x)$, where
$$f(x) = \begin{cases} \sin x & x \le \pi/2 \\ \cos x & x > \pi/2 \end{cases}.$$

In Exercises 17 – 24, a function f and a value a are given. Approximate the limit of the difference quotient, $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$, using $h = \pm 0.1, \pm 0.01$.

17. $f(x) = -7x + 2, \quad a = 3$

18. $f(x) = 9x + 0.06, \quad a = -1$

19. $f(x) = x^2 + 3x - 7, \quad a = 1$

20. $f(x) = \dfrac{1}{x+1}, \quad a = 2$

21. $f(x) = -4x^2 + 5x - 1, \quad a = -3$

22. $f(x) = \ln x, \quad a = 5$

23. $f(x) = \sin x, \quad a = \pi$

24. $f(x) = \cos x, \quad a = \pi$

5.2 Finding Limits Analytically

In Section 5.1 we explored the concept of the limit without a strict definition, meaning we could only make approximations. We ended with what we called an "informal" definition of the limit. This definition allows us to make intuitive sense of limits, but it does not allow us to prove theorems about limits.

Since we will not discuss how to formally define limits in this course, we will have to take the results in this section on faith. However, we will see that the algebraic rules given below for manipulating limits make the process of calculating limits much more straightforward.

The rigorous definition of limits is often known as the "ε, δ" definition of the limit. You might have a few brief encounters with this definition as you make your way through the calculus sequence, but a careful treatment of limits is usually not encountered until a course in Analysis.

Suppose that $\lim_{x\to 2} f(x) = 2$ and $\lim_{x\to 2} g(x) = 3$. What is $\lim_{x\to 2}(f(x) + g(x))$? Intuition tells us that the limit should be 5, as we expect limits to behave in a nice way. The following theorem states that already established limits do behave nicely.

Theorem 5.2.1 Basic Limit Properties

Let b, c, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = K.$$

The following limits hold.

1. Constants: $\lim_{x\to c} b = b$
2. Identity $\lim_{x\to c} x = c$
3. Sums/Differences: $\lim_{x\to c}(f(x) \pm g(x)) = L \pm K$
4. Scalar Multiples: $\lim_{x\to c} b \cdot f(x) = bL$
5. Products: $\lim_{x\to c} f(x) \cdot g(x) = LK$
6. Quotients: $\lim_{x\to c} f(x)/g(x) = L/K$, $(K \neq 0)$
7. Powers: $\lim_{x\to c} f(x)^n = L^n$
8. Roots: $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$ (If n is even then require $f(x) \geq 0$ on I.)
9. Compositions: Adjust our previously given limit situation to:
$$\lim_{x\to c} f(x) = L, \ \lim_{x\to L} g(x) = K \text{ and } g(L) = K.$$
Then $\lim_{x\to c} g(f(x)) = K$.

We make a note about Property #8: when n is even, L must be greater than 0. If n is odd, then the statement is true for all L.

We apply the theorem to an example.

Example 5.2.1 Using basic limit properties

Let

$$\lim_{x\to 2} f(x) = 2, \quad \lim_{x\to 2} g(x) = 3 \quad \text{and} \quad p(x) = 3x^2 - 5x + 7.$$

Find the following limits:

1. $\lim_{x\to 2} \big(f(x) + g(x)\big)$
2. $\lim_{x\to 2} \big(5f(x) + g(x)^2\big)$
3. $\lim_{x\to 2} p(x)$

Solution

1. Using the Sum/Difference rule, we know that $\lim_{x\to 2} \big(f(x)+g(x)\big) = 2+3 = 5$.
2. Using the Scalar Multiple and Sum/Difference rules, we find that $\lim_{x\to 2} \big(5f(x)+g(x)^2\big) = 5\cdot 2 + 3^2 = 19$.
3. Here we combine the Power, Scalar Multiple, Sum/Difference and Constant Rules. We show quite a few steps, but in general these can be omitted:

$$\begin{aligned}\lim_{x\to 2} p(x) &= \lim_{x\to 2}(3x^2 - 5x + 7)\\ &= \lim_{x\to 2} 3x^2 - \lim_{x\to 2} 5x + \lim_{x\to 2} 7\\ &= 3\cdot 2^2 - 5\cdot 2 + 7\\ &= 9\end{aligned}$$

Part 3 of the previous example demonstrates how the limit of a quadratic polynomial can be determined using the properties of Theorem 5.2.1. Not only that, recognize that

$$\lim_{x\to 2} p(x) = 9 = p(2);$$

i.e., the limit at 2 was found just by plugging 2 into the function. This holds true for all polynomials, and also for rational functions (which are quotients of polynomials), as stated in the following theorem.

Theorem 5.2.2 Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomials and c a real number. Then:

1. $\lim_{x\to c} p(x) = p(c)$
2. $\lim_{x\to c} \dfrac{p(x)}{q(x)} = \dfrac{p(c)}{q(c)}$, where $q(c) \neq 0$.

Example 5.2.2 Finding a limit of a rational function

Using Theorem 5.2.2, find

$$\lim_{x\to -1} \frac{3x^2 - 5x + 1}{x^4 - x^2 + 3}.$$

Solution Using Theorem 5.2.2, we can quickly state that

$$\begin{aligned}\lim_{x\to -1} \frac{3x^2 - 5x + 1}{x^4 - x^2 + 3} &= \frac{3(-1)^2 - 5(-1) + 1}{(-1)^4 - (-1)^2 + 3}\\ &= \frac{9}{3} = 3.\end{aligned}$$

Using approximations (or worse – the rigorous definition) to deal with limits such as

$$\lim_{x\to 2} x^2 = 4$$

can be frustrating, since the result seems fairly obvious. The previous theorems state that many functions behave in such an "obvious" fashion, as demonstrated by the rational function in Example 5.2.2.

Polynomial and rational functions are not the only functions to behave in such a predictable way. The following theorem gives a list of functions whose behaviour is particularly "nice" in terms of limits. In the next section, we will give a formal name to these functions that behave "nicely."

Theorem 5.2.3 Special Limits

Let c be a real number in the domain of the given function and let n be a positive integer. The following limits hold:

1. $\lim_{x\to c} \sin x = \sin c$
2. $\lim_{x\to c} \cos x = \cos c$
3. $\lim_{x\to c} \tan x = \tan c$
4. $\lim_{x\to c} \csc x = \csc c$
5. $\lim_{x\to c} \sec x = \sec c$
6. $\lim_{x\to c} \cot x = \cot c$
7. $\lim_{x\to c} a^x = a^c \ (a > 0)$
8. $\lim_{x\to c} \ln x = \ln c$
9. $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$

Example 5.2.3 Evaluating limits analytically

Evaluate the following limits.

1. $\lim_{x\to \pi} \cos x$
2. $\lim_{x\to 3} (\sec^2 x - \tan^2 x)$
3. $\lim_{x\to \pi/2} \cos x \sin x$
4. $\lim_{x\to 1} e^{\ln x}$
5. $\lim_{x\to 0} \dfrac{\sin x}{x}$

SOLUTION

1. This is a straightforward application of Theorem 5.2.3. $\lim_{x\to\pi} \cos x = \cos \pi = -1$.

2. We can approach this in at least two ways. First, by directly applying Theorem 5.2.3, we have:

$$\lim_{x\to 3} (\sec^2 x - \tan^2 x) = \sec^2 3 - \tan^2 3.$$

Using the Pythagorean Theorem, this last expression is 1; therefore

$$\lim_{x\to 3} (\sec^2 x - \tan^2 x) = 1.$$

We can also use the Pythagorean Theorem from the start.

$$\lim_{x\to 3} (\sec^2 x - \tan^2 x) = \lim_{x\to 3} 1 = 1,$$

using the Constant limit rule. Either way, we find the limit is 1.

3. Applying the Product limit rule of Theorem 5.2.1 and Theorem 5.2.3 gives

$$\lim_{x\to \pi/2} \cos x \sin x = \cos(\pi/2)\sin(\pi/2) = 0 \cdot 1 = 0.$$

4. Again, we can approach this in two ways. First, we can use the exponential/logarithmic identity that $e^{\ln x} = x$ and evaluate $\lim\limits_{x\to 1} e^{\ln x} = \lim\limits_{x\to 1} x = 1$.

   We can also use the limit Composition Rule of Theorem 5.2.1. Using Theorem 5.2.3, we have $\lim\limits_{x\to 1} \ln x = \ln 1 = 0$. Applying the Composition rule,

   $$\lim_{x\to 1} e^{\ln x} = \lim_{x\to 0} e^x = e^0 = 1.$$

   Both approaches are valid, giving the same result.

5. We encountered this limit in Section 5.1. Applying our theorems, we attempt to find the limit as

   $$\lim_{x\to 0} \frac{\sin x}{x} \rightarrow \frac{\sin 0}{0} \rightarrow \text{“}\frac{0}{0}\text{”}.$$

   This, of course, violates a condition of Theorem 5.2.1, as the limit of the denominator is not allowed to be 0. Therefore, we are still unable to evaluate this limit with tools we currently have at hand.

The section could have been titled “Using Known Limits to Find Unknown Limits.” By knowing certain limits of functions, we can find limits involving sums, products, powers, etc., of these functions. We further the development of such comparative tools with the Squeeze Theorem, a clever and intuitive way to find the value of some limits.

Before stating this theorem formally, suppose we have functions f, g and h where g always takes on values between f and h; that is, for all x in an interval,

$$f(x) \leq g(x) \leq h(x).$$

If f and h have the same limit at c, and g is always “squeezed” between them, then g must have the same limit as well. That is what the Squeeze Theorem states.

Theorem 5.2.4 Squeeze Theorem

Let f, g and h be functions on an open interval I containing c such that for all x in I,

$$f(x) \leq g(x) \leq h(x).$$

If

$$\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x),$$

then

$$\lim_{x\to c} g(x) = L.$$

It can take some work to figure out appropriate functions by which to “squeeze” the given function of which you are trying to evaluate a limit. However, that is generally the only place work is necessary; the theorem makes the “evaluating the limit part” very simple.

We use the Squeeze Theorem in the following example to finally prove that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$.

Example 5.2.4 Using the Squeeze Theorem

Use the Squeeze Theorem to show that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

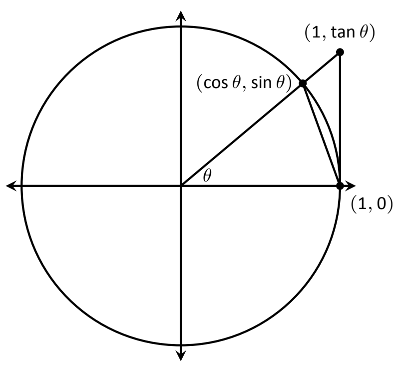
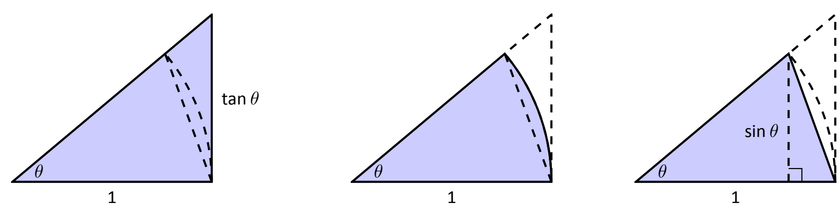


Figure 5.2.1: The unit circle and related triangles.

Solution We begin by considering the unit circle. Each point on the unit circle has coordinates $(\cos\theta, \sin\theta)$ for some angle θ as shown in Figure 5.2.1. Using similar triangles, we can extend the line from the origin through the point to the point $(1, \tan\theta)$, as shown. (Here we are assuming that $0 \leq \theta \leq \pi/2$. Later we will show that we can also consider $\theta \leq 0$.)

Figure 5.2.1 shows three regions have been constructed in the first quadrant, two triangles and a sector of a circle, which are also drawn below. The area of the large triangle is $\frac{1}{2}\tan\theta$; the area of the sector is $\theta/2$; the area of the triangle contained inside the sector is $\frac{1}{2}\sin\theta$. It is then clear from the diagram that



$$\frac{\tan\theta}{2} \geq \frac{\theta}{2} \geq \frac{\sin\theta}{2}$$

Multiply all terms by $\dfrac{2}{\sin\theta}$, giving

$$\frac{1}{\cos\theta} \geq \frac{\theta}{\sin\theta} \geq 1.$$

Taking reciprocals reverses the inequalities, giving

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1.$$

(These inequalities hold for all values of θ near 0, even negative values, since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.)

Now take limits.

$$\lim_{\theta\to 0} \cos\theta \leq \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \leq \lim_{\theta\to 0} 1$$

$$\cos 0 \leq \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \leq 1$$

$$1 \leq \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \leq 1$$

Clearly this means that $\displaystyle\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

Two notes about the previous example are worth mentioning. First, one might be discouraged by this application, thinking "I would *never* have come up with that on my own. This is too hard!" Don't be discouraged; within this text we will guide you in your use of the Squeeze Theorem. As one gains mathematical maturity, clever proofs like this are easier and easier to create.

Second, this limit tells us more than just that as x approaches 0, $\sin(x)/x$ approaches 1. Both x and $\sin x$ are approaching 0, but the *ratio* of x and $\sin x$ approaches 1, meaning that they are approaching 0 in essentially the same way. Another way of viewing this is: for small x, the functions $y = x$ and $y = \sin x$ are essentially indistinguishable.

We include this special limit, along with three others, in the following theorem.

Theorem 5.2.5 Special Limits

1. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$
2. $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$
3. $\displaystyle\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$
4. $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

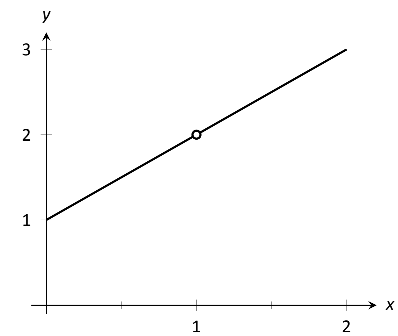


Figure 5.2.2: Graphing f in Example 5.2.5 to understand a limit.

A short word on how to interpret the latter three limits. We know that as x goes to 0, $\cos x$ goes to 1. So, in the second limit, both the numerator and denominator are approaching 0. However, since the limit is 0, we can interpret this as saying that "$\cos x$ is approaching 1 faster than x is approaching 0."

In the third limit, inside the parentheses we have an expression that is approaching 1 (though never equalling 1), and we know that 1 raised to any power is still 1. At the same time, the power is growing toward infinity. What happens to a number near 1 raised to a very large power? In this particular case, the result approaches Euler's number, e, approximately 2.718.

In the fourth limit, we see that as $x \to 0$, e^x approaches 1 "just as fast" as $x \to 0$, resulting in a limit of 1.

Our final theorem for this section will be motivated by the following example.

Example 5.2.5 Using algebra to evaluate a limit

Evaluate the following limit:

$$\lim_{x\to 1} \frac{x^2-1}{x-1}.$$

SOLUTION We begin by attempting to apply Theorem 5.2.2 and substituting 1 for x in the quotient. This gives:

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \frac{1^2-1}{1-1} = \text{“}\frac{0}{0}\text{”},$$

an indeterminate form. We cannot apply the theorem.

By graphing the function, as in Figure 5.2.2, we see that the function seems to be linear, implying that the limit should be easy to evaluate. Recognize that the numerator of our quotient can be factored:

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1}.$$

The function is not defined when $x = 1$, but for all other x,

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x+1.$$

Clearly $\lim\limits_{x\to 1} x+1 = 2$. Recall that when considering limits, we are not concerned with the value of the function at 1, only the value the function approaches as x approaches 1. Since $(x^2-1)/(x-1)$ and $x+1$ are the same at all points except $x = 1$, they both approach the same value as x approaches 1. Therefore we can conclude that

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = 2.$$

The key to the above example is that the functions $y = (x^2-1)/(x-1)$ and $y = x+1$ are identical except at $x = 1$. Since limits describe a value the function is approaching, not the value the function actually attains, the limits of the two functions are always equal.

Theorem 5.2.6 Limits of Functions Equal At All But One Point

Let $g(x) = f(x)$ for all x in an open interval, except possibly at c, and let $\lim\limits_{x\to c} g(x) = L$ for some real number L. Then

$$\lim_{x\to c} f(x) = L.$$

The Fundamental Theorem of Algebra tells us that when dealing with a rational function of the form $g(x)/f(x)$ and directly evaluating the limit $\lim\limits_{x\to c}\frac{g(x)}{f(x)}$ returns "0/0", then $(x-c)$ is a factor of both $g(x)$ and $f(x)$. One can then use algebra to factor this term out, cancel, then apply Theorem 5.2.6. We demonstrate this once more.

Example 5.2.6 Evaluating a limit using Theorem 5.2.6

Evaluate $\lim\limits_{x\to 3}\dfrac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15}$.

Solution We attempt to apply Theorem 5.2.2 by substituting 3 for x. This returns the familiar indeterminate form of "0/0". Since the numerator and denominator are each polynomials, we know that $(x-3)$ is factor of each. Using whatever method is most comfortable to you, factor out $(x-3)$ from each (using polynomial division, synthetic division, a computer algebra system, etc.). We find that

$$\frac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15} = \frac{(x-3)(x^2+x-2)}{(x-3)(2x^2+9x-5)}.$$

We can cancel the $(x-3)$ terms as long as $x \neq 3$. Using Theorem 5.2.6 we conclude:

$$\begin{aligned}\lim_{x\to 3}\frac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15} &= \lim_{x\to 3}\frac{(x-3)(x^2+x-2)}{(x-3)(2x^2+9x-5)}\\ &= \lim_{x\to 3}\frac{(x^2+x-2)}{(2x^2+9x-5)}\\ &= \frac{10}{40} = \frac{1}{4}.\end{aligned}$$

We end this section by revisiting a limit first seen in Section 5.1, a limit of a difference quotient. Let $f(x) = -1.5x^2 + 11.5x$; we approximated the limit $\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \approx 8.5$. We formally evaluate this limit in the following example.

Example 5.2.7 Evaluating the limit of a difference quotient

Let $f(x) = -1.5x^2 + 11.5x$; find $\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$.

Solution Since f is a polynomial, our first attempt should be to employ Theorem 5.2.2 and substitute 0 for h. However, we see that this gives us "$0/0$." Knowing that we have a rational function hints that some algebra will help. Consider the following steps:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0} \frac{-1.5(1+h)^2 + 11.5(1+h) - \left(-1.5(1)^2 + 11.5(1)\right)}{h} \\
&= \lim_{h\to 0} \frac{-1.5(1+2h+h^2) + 11.5 + 11.5h - 10}{h} \\
&= \lim_{h\to 0} \frac{-1.5h^2 + 8.5h}{h} \\
&= \lim_{h\to 0} \frac{h(-1.5h + 8.5)}{h} \\
&= \lim_{h\to 0} (-1.5h + 8.5) \quad \left(\text{using Theorem 5.2.6, as } h \neq 0\right) \\
&= 8.5 \quad \left(\text{using Theorem 5.2.3}\right)
\end{aligned}$$

This matches our previous approximation.

This section contains several valuable tools for evaluating limits. One of the main results of this section is Theorem 5.2.3; it states that many functions that we use regularly behave in a very nice, predictable way. In Section 5.5 we give a name to this nice behaviour; we label such functions as *continuous.* Defining that term will require us to look again at what a limit is and what causes limits to not exist.

Exercises 5.2

Terms and Concepts

1. What does the text mean when it says that certain functions' "behaviour is 'nice' in terms of limits"? What, in particular, is "nice"?

2. Sketch a graph that visually demonstrates the Squeeze Theorem.

3. You are given the following information:

   (a) $\lim_{x\to 1} f(x) = 0$

   (b) $\lim_{x\to 1} g(x) = 0$

   (c) $\lim_{x\to 1} f(x)/g(x) = 2$

   What can be said about the relative sizes of $f(x)$ and $g(x)$ as x approaches 1?

4. T/F: $\lim_{x\to 1} \ln x = 0$. Use a theorem to defend your answer.

Problems

In Exercises 5 – 12, use the following information to evaluate the given limit, when possible. If it is not possible to determine the limit, state why not.

- $\lim_{x\to 9} f(x) = 6, \quad \lim_{x\to 6} f(x) = 9, \quad f(9) = 6$
- $\lim_{x\to 9} g(x) = 3, \quad \lim_{x\to 6} g(x) = 3, \quad g(6) = 9$

5. $\lim_{x\to 9}(f(x) + g(x))$

6. $\lim_{x\to 9}(3f(x)/g(x))$

7. $\lim_{x\to 9}\left(\frac{f(x) - 2g(x)}{g(x)}\right)$

8. $\lim_{x\to 6}\left(\frac{f(x)}{3 - g(x)}\right)$

9. $\lim_{x\to 9} g\big(f(x)\big)$

10. $\lim_{x\to 6} f\big(g(x)\big)$

11. $\lim_{x\to 6} g\big(f(f(x))\big)$

12. $\lim_{x\to 6} f(x)g(x) - f^2(x) + g^2(x)$

In Exercises 13 – 16, use the following information to evaluate the given limit, when possible. If it is not possible to determine the limit, state why not.

- $\lim_{x\to 1} f(x) = 2, \quad \lim_{x\to 10} f(x) = 1, \quad f(1) = 1/5$
- $\lim_{x\to 1} g(x) = 0, \quad \lim_{x\to 10} g(x) = \pi, \quad g(10) = \pi$

13. $\lim_{x\to 1} f(x)^{g(x)}$

14. $\lim_{x\to 10} \cos\big(g(x)\big)$

15. $\lim_{x\to 1} f(x)g(x)$

16. $\lim_{x\to 1} g\big(5f(x)\big)$

In Exercises 17 – 32, evaluate the given limit.

17. $\lim_{x\to 3} x^2 - 3x + 7$

18. $\lim_{x\to \pi}\left(\frac{x-3}{x-5}\right)^7$

19. $\lim_{x\to \pi/4} \cos x \sin x$

20. $\lim_{x\to 1} \frac{2x-2}{x+4}$

21. $\lim_{x\to 0} \ln x$

22. $\lim_{x\to 3} 4^{x^3-8x}$

23. $\lim_{x\to \pi/6} \csc x$

24. $\lim_{x\to 0} \ln(1+x)$

25. $\lim_{x\to \pi} \frac{x^2+3x+5}{5x^2-2x-3}$

26. $\lim_{x\to \pi} \frac{3x+1}{1-x}$

27. $\lim_{x\to 6} \frac{x^2-4x-12}{x^2-13x+42}$

28. $\lim_{x\to 0} \frac{x^2+2x}{x^2-2x}$

29. $\lim_{x\to 2} \frac{x^2+6x-16}{x^2-3x+2}$

30. $\lim_{x\to 2} \frac{x^2-10x+16}{x^2-x-2}$

31. $\lim_{x\to -2} \frac{x^2-5x-14}{x^2+10x+16}$

32. $\lim_{x\to -1} \frac{x^2+9x+8}{x^2-6x-7}$

Use the Squeeze Theorem in Exercises 33 – 36, where appropriate, to evaluate the given limit.

33. $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$

34. $\lim_{x\to 0} \sin x \cos\left(\frac{1}{x^2}\right)$

35. $\lim_{x\to 1} f(x)$, where $3x - 2 \leq f(x) \leq x^3$.

36. $\lim_{x\to 3} f(x)$, where $6x - 9 \leq f(x) \leq x^2$.

Exercises 37 – 41 challenge your understanding of limits but can be evaluated using the knowledge gained in this section.

37. $\lim_{x\to 0} \frac{\sin 3x}{x}$

38. $\lim_{x\to 0} \frac{\sin 5x}{8x}$

39. $\lim_{x\to 0} \frac{\ln(1+x)}{x}$

40. $\lim_{x\to 0} \frac{\sin x}{x}$, where x is measured in degrees, not radians.

41. Let $f(x) = 0$ and $g(x) = \frac{x}{x}$.

    (a) Show why $\lim_{x\to 2} f(x) = 0$.

    (b) Show why $\lim_{x\to 0} g(x) = 1$.

    (c) Show why $\lim_{x\to 2} g\big(f(x)\big)$ does not exist.

    (d) Show why the answer to part (c) does not violate the Composition Rule of Theorem 5.2.1.

5.3 One Sided Limits

We introduced the concept of a limit gently, approximating their values graphically and numerically. Next came the rigorous definition of the limit, along with an admittedly tedious method for evaluating them. The previous section gave us tools (which we call theorems) that allow us to compute limits with greater ease. Chief among the results were the facts that polynomials and rational, trigonometric, exponential and logarithmic functions (and their sums, products, etc.) all behave "nicely." In this section we rigorously define what we mean by "nicely."

In Section 5.1 we saw three ways in which limits of functions failed to exist:

1. The function approached different values from the left and right,
2. The function grows without bound, and
3. The function oscillates.

In this section we explore in depth the concepts behind #1 by introducing the *one-sided limit*. We begin with definitions that are very similar to the definition of the limit given in Section 5.1, but the notation is slightly different and "$x \neq c$" is replaced with either "$x < c$" or "$x > c$."

Definition 5.3.1 One Sided Limits: Left- and Right-Hand Limits

Left-Hand Limit
Let f be a function defined on (a, c) for some $a < c$ and let L be a real number.
We say that the **limit of** $f(x)$**, as** x **approaches** c **from the left, is** L, or, **the left–hand limit of** f **at** c **is** L, and write

$$\lim_{x \to c^-} f(x) = L,$$

if we can make $f(x)$ arbitrarily close to L by choosing $x < c$ sufficiently close to c.

Right-Hand Limit
Let f be a function defined on (c, b) for some $b > c$ and let L be a real number. We say that the **limit of** $f(x)$**, as** x **approaches** c **from the right, is** L, or, **the right–hand limit of** f **at** c **is** L, and write

$$\lim_{x \to c^+} f(x) = L,$$

if we can make $f(x)$ arbitrarily close to L by choosing $x > c$ sufficiently close to c.

Practically speaking, when evaluating a left-hand limit, we consider only values of x "to the left of c," i.e., where $x < c$. The admittedly imperfect notation $x \to c^-$ is used to imply that we look at values of x to the left of c. The notation has nothing to do with positive or negative values of either x or c. A similar statement holds for evaluating right-hand limits; there we consider only values of x to the right of c, i.e., $x > c$. We can use the theorems from previous sections to help us evaluate these limits; we just restrict our view to one side of c.

We practice evaluating left- and right-hand limits through a series of examples.

Example 5.3.1 Evaluating one sided limits

Let $f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 3-x & 1 < x < 2 \end{cases}$, as shown in Figure 5.3.1. Find each of the following:

1. $\lim_{x\to 1^-} f(x)$
2. $\lim_{x\to 1^+} f(x)$
3. $\lim_{x\to 1} f(x)$
4. $f(1)$
5. $\lim_{x\to 0^+} f(x)$
6. $f(0)$
7. $\lim_{x\to 2^-} f(x)$
8. $f(2)$

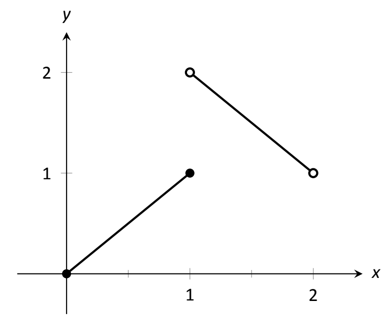


Figure 5.3.1: A graph of *f* in Example 5.3.1.

SOLUTION For these problems, the visual aid of the graph is likely more effective in evaluating the limits than using *f* itself. Therefore we will refer often to the graph.

1. As x goes to 1 *from the left*, we see that $f(x)$ is approaching the value of 1. Therefore $\lim_{x\to 1^-} f(x) = 1$.
2. As x goes to 1 *from the right*, we see that $f(x)$ is approaching the value of 2. Recall that it does not matter that there is an "open circle" there; we are evaluating a limit, not the value of the function. Therefore $\lim_{x\to 1^+} f(x) = 2$.
3. *The* limit of f as x approaches 1 does not exist, as discussed in the first section. The function does not approach one particular value, but two different values from the left and the right.
4. Using the definition and by looking at the graph we see that $f(1) = 1$.
5. As x goes to 0 from the right, we see that $f(x)$ is also approaching 0. Therefore $\lim_{x\to 0^+} f(x) = 0$. Note we cannot consider a left-hand limit at 0 as f is not defined for values of $x < 0$.
6. Using the definition and the graph, $f(0) = 0$.
7. As x goes to 2 from the left, we see that $f(x)$ is approaching the value of 1. Therefore $\lim_{x\to 2^-} f(x) = 1$.
8. The graph and the definition of the function show that $f(2)$ is not defined.

Note how the left and right-hand limits were different at $x = 1$. This, of course, causes *the* limit to not exist. The following theorem states what is fairly intuitive: *the* limit exists precisely when the left and right-hand limits are equal.

Theorem 5.3.1 Limits and One Sided Limits

Let f be a function defined on an open interval I containing c. Then

$$\lim_{x\to c} f(x) = L$$

if, and only if,

$$\lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

The phrase "if, and only if" means the two statements are *equivalent*: they are either both true or both false. If the limit equals L, then the left and right hand limits both equal L. If the limit is not equal to L, then at least one of the left and right-hand limits is not equal to L (it may not even exist).

One thing to consider in Examples 5.3.1 – 5.3.4 is that the value of the function may/may not be equal to the value(s) of its left/right-hand limits, even when these limits agree.

Example 5.3.2 Evaluating limits of a piecewise–defined function

Let $f(x) = \begin{cases} 2-x & 0<x<1 \\ (x-2)^2 & 1<x<2 \end{cases}$, as shown in Figure 5.3.2. Evaluate the following.

1. $\lim\limits_{x\to 1^-} f(x)$
2. $\lim\limits_{x\to 1^+} f(x)$
3. $\lim\limits_{x\to 1} f(x)$
4. $f(1)$
5. $\lim\limits_{x\to 0^+} f(x)$
6. $f(0)$
7. $\lim\limits_{x\to 2^-} f(x)$
8. $f(2)$

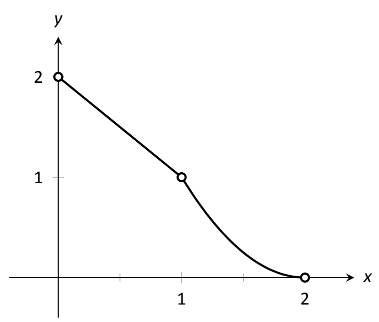


Figure 5.3.2: A graph of f from Example 5.3.2

Solution Again we will evaluate each using both the definition of f and its graph.

1. As x approaches 1 from the left, we see that $f(x)$ approaches 1. Therefore $\lim\limits_{x\to 1^-} f(x) = 1$.
2. As x approaches 1 from the right, we see that again $f(x)$ approaches 1. Therefore $\lim\limits_{x\to 1^+} f(x) = 1$.
3. *The* limit of f as x approaches 1 exists and is 1, as f approaches 1 from both the right and left. Therefore $\lim\limits_{x\to 1} f(x) = 1$.
4. $f(1)$ is not defined. Note that 1 is not in the domain of f as defined by the problem, which is indicated on the graph by an open circle when $x = 1$.
5. As x goes to 0 from the right, $f(x)$ approaches 2. So $\lim\limits_{x\to 0^+} f(x) = 2$.
6. $f(0)$ is not defined as 0 is not in the domain of f.
7. As x goes to 2 from the left, $f(x)$ approaches 0. So $\lim\limits_{x\to 2^-} f(x) = 0$.
8. $f(2)$ is not defined as 2 is not in the domain of f.

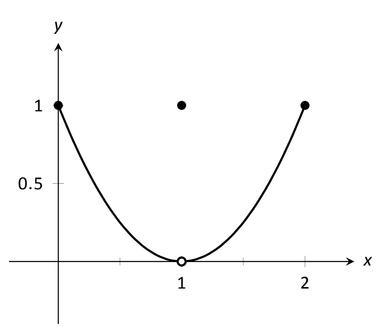


Figure 5.3.3: Graphing f in Example 5.3.3

Example 5.3.3 Evaluating limits of a piecewise–defined function

Let $f(x) = \begin{cases} (x-1)^2 & 0\leq x\leq 2, x\neq 1 \\ 1 & x=1 \end{cases}$, as shown in Figure 5.3.3. Evaluate the following.

1. $\lim\limits_{x\to 1^-} f(x)$
2. $\lim\limits_{x\to 1^+} f(x)$
3. $\lim\limits_{x\to 1} f(x)$
4. $f(1)$

Solution It is clear by looking at the graph that both the left and right-hand limits of f, as x approaches 1, are 0. Thus it is also clear that *the* limit is 0; i.e., $\lim\limits_{x\to 1} f(x) = 0$. It is also clearly stated that $f(1) = 1$.

Example 5.3.4 Evaluating limits of a piecewise–defined function

Let $f(x) = \begin{cases} x^2 & 0 \leq x \leq 1 \\ 2-x & 1 < x \leq 2 \end{cases}$, as shown in Figure 5.3.4. Evaluate the following.

1. $\lim\limits_{x\to 1^-} f(x)$
2. $\lim\limits_{x\to 1^+} f(x)$
3. $\lim\limits_{x\to 1} f(x)$
4. $f(1)$

Solution It is clear from the definition of the function and its graph that all of the following are equal:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = f(1) = 1.$$

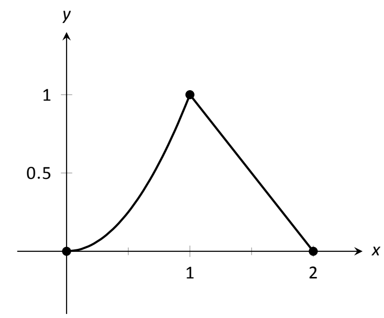


Figure 5.3.4: Graphing f in Example 5.3.4

In Examples 5.3.1 – 5.3.4 we were asked to find both $\lim\limits_{x\to 1} f(x)$ and $f(1)$. Consider the following table:

	$\lim\limits_{x\to 1} f(x)$	$f(1)$
Example 5.3.1	does not exist	1
Example 5.3.2	1	not defined
Example 5.3.3	0	1
Example 5.3.4	1	1

Only in Example 5.3.4 do both the function and the limit exist and agree. This seems "nice;" in fact, it seems "normal." This is in fact an important situation which we explore in the next section, entitled "Continuity." In short, a *continuous function* is one in which when a function approaches a value as $x \to c$ (i.e., when $\lim\limits_{x\to c} f(x) = L$), it actually *attains* that value at c. Such functions behave nicely as they are very predictable.

Exercises 5.3

Terms and Concepts

1. What are the three ways in which a limit may fail to exist?

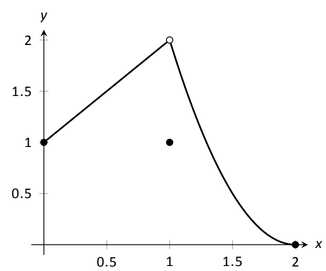
2. T/F: If $\lim\limits_{x\to 1^-} f(x) = 5$, then $\lim\limits_{x\to 1} f(x) = 5$

3. T/F: If $\lim\limits_{x\to 1^-} f(x) = 5$, then $\lim\limits_{x\to 1^+} f(x) = 5$

4. T/F: If $\lim\limits_{x\to 1} f(x) = 5$, then $\lim\limits_{x\to 1^-} f(x) = 5$

Problems

In Exercises 5 – 12, evaluate each expression using the given graph of $f(x)$.

5. 

(a) $\lim\limits_{x\to 1^-} f(x)$

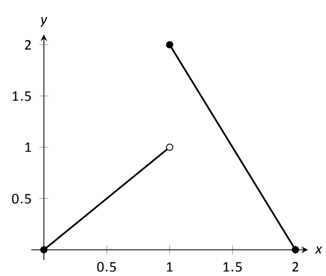
(b) $\lim\limits_{x\to 1^+} f(x)$

(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

(e) $\lim\limits_{x\to 0^-} f(x)$

(f) $\lim\limits_{x\to 0^+} f(x)$

6. 

(a) $\lim\limits_{x\to 1^-} f(x)$

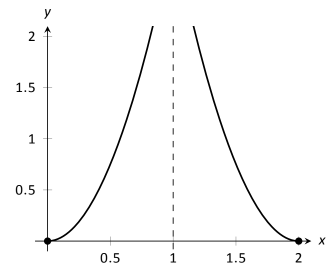
(b) $\lim\limits_{x\to 1^+} f(x)$

(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

(e) $\lim\limits_{x\to 2^-} f(x)$

(f) $\lim\limits_{x\to 2^+} f(x)$

7. 

(a) $\lim\limits_{x\to 1^-} f(x)$

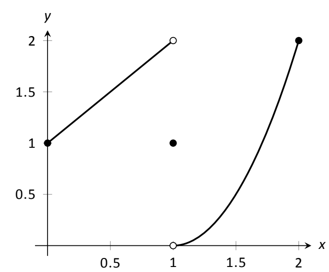
(b) $\lim\limits_{x\to 1^+} f(x)$

(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

(e) $\lim\limits_{x\to 2^-} f(x)$

(f) $\lim\limits_{x\to 0^+} f(x)$

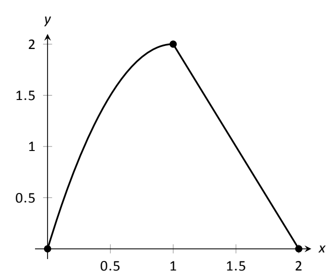
8. 

(a) $\lim\limits_{x\to 1^-} f(x)$

(b) $\lim\limits_{x\to 1^+} f(x)$

(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

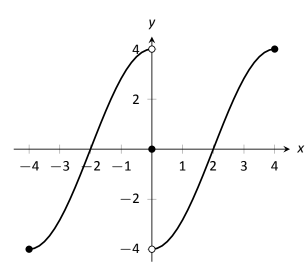
9. 

(a) $\lim\limits_{x\to 1^-} f(x)$

(b) $\lim\limits_{x\to 1^+} f(x)$

(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

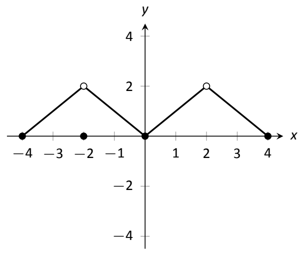
10. 

(a) $\lim\limits_{x\to 0^-} f(x)$

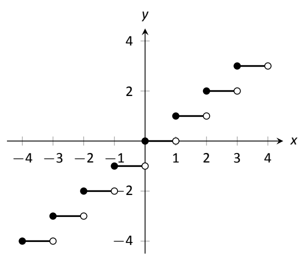
(b) $\lim\limits_{x\to 0^+} f(x)$

(c) $\lim\limits_{x\to 0} f(x)$

(d) $f(0)$

11. 

(a) $\lim_{x\to -2^-} f(x)$
(b) $\lim_{x\to -2^+} f(x)$
(c) $\lim_{x\to -2} f(x)$
(d) $f(-2)$
(e) $\lim_{x\to 2^-} f(x)$
(f) $\lim_{x\to 2^+} f(x)$
(g) $\lim_{x\to 2} f(x)$
(h) $f(2)$

12. 

Let $-3 \le a \le 3$ be an integer.

(a) $\lim_{x\to a^-} f(x)$
(b) $\lim_{x\to a^+} f(x)$
(c) $\lim_{x\to a} f(x)$
(d) $f(a)$

In Exercises 13 – 21, evaluate the given limits of the piecewise defined functions *f*.

13. $f(x) = \begin{cases} x+1 & x \le 1 \\ x^2 - 5 & x > 1 \end{cases}$

(a) $\lim_{x\to 1^-} f(x)$
(b) $\lim_{x\to 1^+} f(x)$
(c) $\lim_{x\to 1} f(x)$
(d) $f(1)$

14. $f(x) = \begin{cases} 2x^2 + 5x - 1 & x < 0 \\ \sin x & x \ge 0 \end{cases}$

(a) $\lim_{x\to 0^-} f(x)$
(b) $\lim_{x\to 0^+} f(x)$
(c) $\lim_{x\to 0} f(x)$
(d) $f(0)$

15. $f(x) = \begin{cases} x^2 - 1 & x < -1 \\ x^3 + 1 & -1 \le x \le 1 \\ x^2 + 1 & x > 1 \end{cases}$

(a) $\lim_{x\to -1^-} f(x)$
(b) $\lim_{x\to -1^+} f(x)$
(c) $\lim_{x\to -1} f(x)$
(d) $f(-1)$
(e) $\lim_{x\to 1^-} f(x)$
(f) $\lim_{x\to 1^+} f(x)$
(g) $\lim_{x\to 1} f(x)$
(h) $f(1)$

16. $f(x) = \begin{cases} \cos x & x < \pi \\ \sin x & x \ge \pi \end{cases}$

(a) $\lim_{x\to \pi^-} f(x)$
(b) $\lim_{x\to \pi^+} f(x)$
(c) $\lim_{x\to \pi} f(x)$
(d) $f(\pi)$

17. $f(x) = \begin{cases} 1 - \cos^2 x & x < a \\ \sin^2 x & x \ge a \end{cases}$,
where a is a real number.

(a) $\lim_{x\to a^-} f(x)$
(b) $\lim_{x\to a^+} f(x)$
(c) $\lim_{x\to a} f(x)$
(d) $f(a)$

18. $f(x) = \begin{cases} x+1 & x < 1 \\ 1 & x = 1 \\ x - 1 & x > 1 \end{cases}$

(a) $\lim_{x\to 1^-} f(x)$
(b) $\lim_{x\to 1^+} f(x)$
(c) $\lim_{x\to 1} f(x)$
(d) $f(1)$

19. $f(x) = \begin{cases} x^2 & x < 2 \\ x+1 & x = 2 \\ -x^2 + 2x + 4 & x > 2 \end{cases}$

(a) $\lim_{x\to 2^-} f(x)$
(b) $\lim_{x\to 2^+} f(x)$
(c) $\lim_{x\to 2} f(x)$
(d) $f(2)$

20. $f(x) = \begin{cases} a(x-b)^2 + c & x < b \\ a(x-b) + c & x \ge b \end{cases}$,
where a, b and c are real numbers.

(a) $\lim_{x\to b^-} f(x)$
(b) $\lim_{x\to b^+} f(x)$
(c) $\lim_{x\to b} f(x)$
(d) $f(b)$

21. $f(x) = \begin{cases} \frac{|x|}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$

(a) $\lim_{x\to 0^-} f(x)$
(b) $\lim_{x\to 0^+} f(x)$
(c) $\lim_{x\to 0} f(x)$
(d) $f(0)$

Review

22. Evaluate the limit: $\lim\limits_{x\to -1} \dfrac{x^2 + 5x + 4}{x^2 - 3x - 4}$.

23. Evaluate the limit: $\lim\limits_{x\to -4} \dfrac{x^2 - 16}{x^2 - 4x - 32}$.

24. Evaluate the limit: $\lim\limits_{x\to -6} \dfrac{x^2 - 15x + 54}{x^2 - 6x}$.

25. Approximate the limit numerically: $\lim\limits_{x\to 0.4} \dfrac{x^2 - 4.4x + 1.6}{x^2 - 0.4x}$.

26. Approximate the limit numerically: $\lim\limits_{x\to 0.2} \dfrac{x^2 + 5.8x - 1.2}{x^2 - 4.2x + 0.8}$.

5.4 Limits Involving Infinity

In Definition 5.1.1 we stated that in the equation $\lim_{x\to c} f(x) = L$, both c and L were numbers. In this section we relax that definition a bit by considering situations when it makes sense to let c and/or L be "infinity."

As a motivating example, consider $f(x) = 1/x^2$, as shown in Figure 5.4.1. Note how, as x approaches 0, $f(x)$ grows very, very large – in fact, it grows without bound. It seems appropriate, and descriptive, to state that

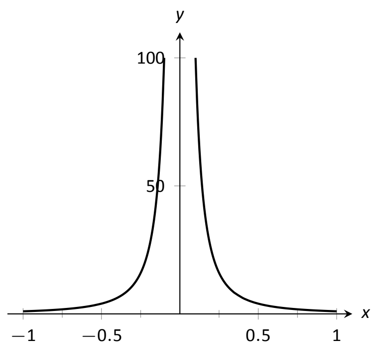


Figure 5.4.1: Graphing $f(x) = 1/x^2$ for values of x near 0.

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

Also note that as x gets very large, $f(x)$ gets very, very small. We could represent this concept with notation such as

$$\lim_{x\to \infty} \frac{1}{x^2} = 0.$$

We explore both types of use of ∞ in turn.

Definition 5.4.1 Limit of Infinity, ∞

Let I be an open interval containing c, and let f be a function defined on I, except possibly at c.

- The **limit of $f(x)$, as x approaches c, is infinity**, denoted by

$$\lim_{x\to c} f(x) = \infty,$$

  if we can obtain any arbitrarily large value for $f(x)$ by choosing $x \neq c$ sufficiently close to c.

- The **limit of $f(x)$, as x approaches c, is negative infinity**, denoted by

$$\lim_{x\to c} f(x) = -\infty,$$

  if we can obtain any arbitrarily large negative value for $f(x)$ by choosing $x \neq c$ sufficiently close to c.

This is once again an informal definition, like Definition 5.1.1: we say that if we get close enough to c, then we can make $f(x)$ as large as we want, without giving precise answers to the questions "How close?" or "How large?"

It is important to note that by saying $\lim_{x\to c} f(x) = \infty$ we are implicitly stating that *the* limit of $f(x)$, as x approaches c, *does not exist.* A limit only exists when $f(x)$ approaches an actual numeric value. We use the concept of limits that approach infinity because it is helpful and descriptive.

We define one-sided limits that approach infinity in a similar way.

Definition 5.4.2 One-Sided Limits of Infinity

- Let f be a function defined on (a, c) for some $a < c$.

  The **limit of $f(x)$, as x approaches c from the left, is infinity**, or, **the left-hand limit of f at c is infinity**, denoted by

  $$\lim_{x\to c^-} f(x) = \infty,$$

  if we can obtain any arbitrarily large value for $f(x)$ by choosing x sufficiently close to c, where $a < x < c$.

- Let f be a function defined on (c, b) for some $b > c$.

  The **limit of $f(x)$, as x approaches c from the right, is infinity**, or, **the right-hand limit of f at c is infinity**, denoted by

  $$\lim_{x\to c^+} f(x) = \infty,$$

  if we can obtain any arbitrarily large value for $f(x)$ by choosing x sufficiently close to c, where $c < x < b$.

- The term **left- (or, right-) hand limit of f at c is negative infinity** is defined in a manner similar to Definition 5.4.1.

Example 5.4.1 Evaluating limits involving infinity

Find $\lim_{x\to 1} \frac{1}{(x-1)^2}$ as shown in Figure 5.4.2.

Solution In Example 5.1.4 of Section 5.1, by inspecting values of x close to 1 we concluded that this limit does not exist. That is, it cannot equal any real number. But the limit could be infinite. And in fact, we see that the function does appear to be growing larger and larger, as $f(.99) = 10^4$, $f(.999) = 10^6$, $f(.9999) = 10^8$. A similar thing happens on the other side of 1. In general, we can see that as the difference $|x-1|$ gets smaller, the value of $f(x)$ gets larger and larger, so we may say $\lim_{x\to 1} 1/(x-1)^2 = \infty$.

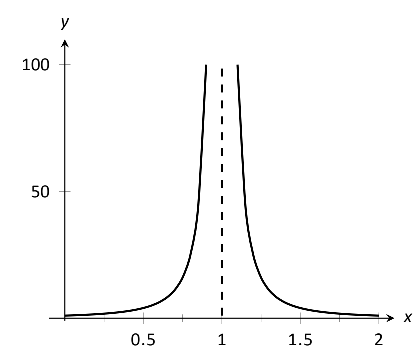


Figure 5.4.2: Observing infinite limit as $x \to 1$ in Example 5.4.1.

Example 5.4.2 Evaluating limits involving infinity

Find $\lim_{x\to 0} \frac{1}{x}$, as shown in Figure 5.4.3.

Solution It is easy to see that the function grows without bound near 0, but it does so in different ways on different sides of 0. Since its behaviour is not consistent, we cannot say that $\lim_{x\to 0} \frac{1}{x} = \infty$. However, we can make a statement about one–sided limits. We can state that $\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.

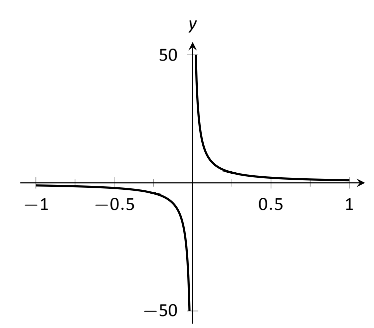


Figure 5.4.3: Evaluating $\lim_{x\to 0} \frac{1}{x}$.

Vertical asymptotes

The graphs in the two previous examples demonstrate that if a function f has a limit (or, left- or right-hand limit) of infinity at $x = c$, then the graph of f looks similar to a vertical line near $x = c$. This observation leads to a definition.

Definition 5.4.3 Vertical Asymptote

Let I be an interval that either contains c or has c as an endpoint, and let f be a function defined on I, except possibly at c.
If the limit of $f(x)$ as x approaches c from either the left or right (or both) is ∞ or $-\infty$, then the line $x = c$ is a **vertical asymptote** of f.

Example 5.4.3 Finding vertical asymptotes

Find the vertical asymptotes of $f(x) = \dfrac{3x}{x^2 - 4}$.

Solution Vertical asymptotes occur where the function grows without bound; this can occur at values of c where the denominator is 0. When x is near c, the denominator is small, which in turn can make the function take on large values. In the case of the given function, the denominator is 0 at $x = \pm 2$. Substituting in values of x close to 2 and -2 seems to indicate that the function tends toward ∞ or $-\infty$ at those points. We can graphically confirm this by looking at Figure 5.4.4. Thus the vertical asymptotes are at $x = \pm 2$.

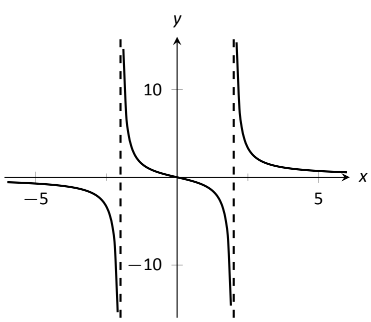


Figure 5.4.4: Graphing $f(x) = \dfrac{3x}{x^2 - 4}$.

When a rational function has a vertical asymptote at $x = c$, we can conclude that the denominator is 0 at $x = c$. However, just because the denominator is 0 at a certain point does not mean there is a vertical asymptote there. For instance, $f(x) = (x^2 - 1)/(x - 1)$ does not have a vertical asymptote at $x = 1$, as shown in Figure 5.4.5. While the denominator does get small near $x = 1$, the numerator gets small too, matching the denominator step for step. In fact, factoring the numerator, we get

$$f(x) = \frac{(x-1)(x+1)}{x-1}.$$

Cancelling the common term, we get that $f(x) = x + 1$ for $x \neq 1$. So there is clearly no asymptote; rather, a hole exists in the graph at $x = 1$.

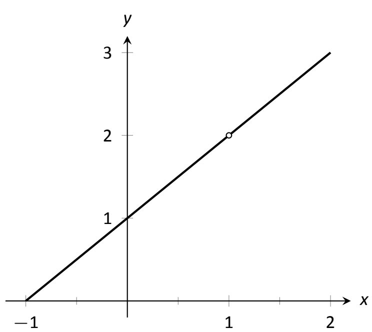


Figure 5.4.5: Graphically showing that $f(x) = \dfrac{x^2 - 1}{x - 1}$ does not have an asymptote at $x = 1$.

The above example may seem a little contrived. Another example demonstrating this important concept is $f(x) = (\sin x)/x$. We have considered this function several times in the previous sections. We found that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$; i.e., there is no vertical asymptote. No simple algebraic cancellation makes this fact obvious; we used the Squeeze Theorem in Section 5.2 to prove this.

If the denominator is 0 at a certain point but the numerator is not, then there will usually be a vertical asymptote at that point. On the other hand, if the numerator and denominator are both zero at that point, then there may or may not be a vertical asymptote at that point. This case where the numerator and denominator are both zero returns us to an important topic.

Indeterminate Forms

We have seen how the limits

$$\lim_{x\to 0}\frac{\sin x}{x} \quad \text{and} \quad \lim_{x\to 1}\frac{x^2-1}{x-1}$$

each return the indeterminate form "0/0" when we blindly plug in $x = 0$ and $x = 1$, respectively. However, $0/0$ is not a valid arithmetical expression. It gives no indication that the respective limits are 1 and 2.

With a little cleverness, one can come up with $0/0$ expressions which have a limit of ∞, 0, or any other real number. That is why this expression is called *indeterminate*.

A key concept to understand is that such limits do not really return $0/0$. Rather, keep in mind that we are taking *limits*. What is really happening is that the numerator is shrinking to 0 while the denominator is also shrinking to 0. The respective rates at which they do this are very important and determine the actual value of the limit.

An indeterminate form indicates that one needs to do more work in order to compute the limit. That work may be algebraic (such as factoring and cancelling) or it may require a tool such as the Squeeze Theorem. In a later section we will learn a technique called l'Hospital's Rule that provides another way to handle indeterminate forms.

Some other common indeterminate forms are $\infty-\infty$, $\infty\cdot 0$, ∞/∞, 0^0, ∞^0 and 1^∞. Again, keep in mind that these are the "blind" results of evaluating a limit, and each, in and of itself, has no meaning. The expression $\infty-\infty$ does not really mean "subtract infinity from infinity." Rather, it means "One quantity is subtracted from the other, but both are growing without bound." What is the result? It is possible to get every value between $-\infty$ and ∞.

Note that $1/0$ and $\infty/0$ are not indeterminate forms, though they are not exactly valid mathematical expressions, either. In each, the function is growing without bound, indicating that the limit will be ∞, $-\infty$, or simply not exist if the left- and right-hand limits do not match.

Limits at Infinity and Horizontal Asymptotes

At the beginning of this section we briefly considered what happens to $f(x) = 1/x^2$ as x grew very large. Graphically, it concerns the behaviour of the function to the "far right" of the graph. We make this notion more explicit in the following definition.

Definition 5.4.4 Limits at Infinity and Horizontal Asymptote

Let L be a real number.

1. Let f be a function defined on (a, ∞) for some number a. The **limit of f at infinity is L**, denoted $\lim\limits_{x\to\infty} f(x) = L$, if we can make the value of $f(x)$ arbitrarily close to L by choosing a sufficiently large positive value of x.

2. Let f be a function defined on $(-\infty, b)$ for some number b. The **limit of f at negative infinity is L**, denoted $\lim\limits_{x\to-\infty} f(x) = L$, if we can make the value of $f(x)$ arbitrarily close to L by choosing a sufficiently large negative value of x.

3. If $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, we say the line $y = L$ is a **horizontal asymptote** of f.

We can also define limits such as $\lim\limits_{x\to\infty} f(x) = \infty$ by combining this definition with Definition 5.4.1.

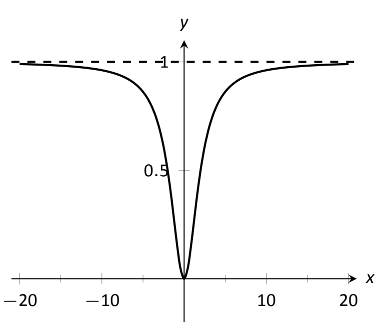
Example 5.4.4 Approximating horizontal asymptotes

Approximate the horizontal asymptote(s) of $f(x) = \dfrac{x^2}{x^2+4}$.

Solution We will approximate the horizontal asymptotes by approximating the limits

$$\lim_{x\to-\infty} \frac{x^2}{x^2+4} \quad \text{and} \quad \lim_{x\to\infty} \frac{x^2}{x^2+4}.$$

Figure 5.4.6(a) shows a sketch of f, and part (b) gives values of $f(x)$ for large magnitude values of x. It seems reasonable to conclude from both of these sources that f has a horizontal asymptote at $y = 1$. Later, we will show how to determine this analytically.



(a)

x	$f(x)$
10	0.9615
100	0.9996
10000	0.999996
−10	0.9615
−100	0.9996
−10000	0.999996

(b)

Figure 5.4.6: Using a graph and a table to approximate a horizontal asymptote in Example 5.4.4.

Horizontal asymptotes can take on a variety of forms. Figure 5.4.7(a) shows that $f(x) = x/(x^2+1)$ has a horizontal asymptote of $y = 0$, where 0 is approached from both above and below.

Figure 5.4.7(b) shows that $f(x) = x/\sqrt{x^2+1}$ has two horizontal asymptotes; one at $y = 1$ and the other at $y = -1$.

Figure 5.4.7(c) shows that $f(x) = (\sin x)/x$ has even more interesting behavior than at just $x = 0$; as x approaches $\pm\infty$, $f(x)$ approaches 0, but oscillates as it does this.

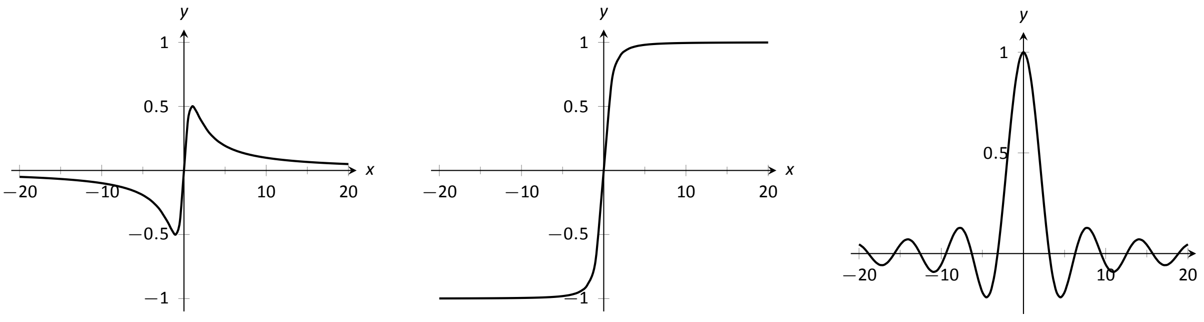


Figure 5.4.7: Considering different types of horizontal asymptotes.

We can analytically evaluate limits at infinity for rational functions once we understand $\lim\limits_{x\to\infty} 1/x$. As x gets larger and larger, $1/x$ gets smaller and smaller, approaching 0. We can, in fact, make $1/x$ as small as we want by choosing a large enough value of x.

It is now not much of a jump to conclude the following: for any positive integer n, we have

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

Now suppose we need to compute the following limit:

$$\lim_{x\to\infty} \frac{x^3+2x+1}{4x^3-2x^2+9}.$$

A good way of approaching this is to divide through the numerator and denominator by x^3 (hence multiplying by 1), which is the largest power of x to appear in the function. Doing this, we get

$$\begin{aligned}\lim_{x\to\infty} \frac{x^3+2x+1}{4x^3-2x^2+9} &= \lim_{x\to\infty} \frac{1/x^3}{1/x^3} \cdot \frac{x^3+2x+1}{4x^3-2x^2+9} \\ &= \lim_{x\to\infty} \frac{x^3/x^3+2x/x^3+1/x^3}{4x^3/x^3-2x^2/x^3+9/x^3} \\ &= \lim_{x\to\infty} \frac{1+2/x^2+1/x^3}{4-2/x+9/x^3}.\end{aligned}$$

Then using the rules for limits (which also hold for limits at infinity), as well as the fact about limits of $1/x^n$, we see that the limit becomes

$$\frac{1+0+0}{4-0+0} = \frac{1}{4}.$$

This procedure works for any rational function. In fact, it gives us the following theorem.

Theorem 5.4.1 Limits of Rational Functions at Infinity

Let $f(x)$ be a rational function of the following form:

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0},$$

where any of the coefficients may be 0 except for a_n and b_m.

1. If $n = m$, then $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = \dfrac{a_n}{b_m}$.
2. If $n < m$, then $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = 0$.
3. If $n > m$, then $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$ are both infinite.

We can see why this is true. If the highest power of x is the same in both the numerator and denominator (i.e. $n = m$), we will be in a situation like the example above, where we will divide by x^n and in the limit all the terms will approach 0 except for $a_n x^n/x^n$ and $b_m x^m/x^n$. Since $n = m$, this will leave us with the limit a_n/b_m. If $n < m$, then after dividing through by x^m, all the terms in the numerator will approach 0 in the limit, leaving us with $0/b_m$ or 0. If $n > m$, and we try dividing through by x^n, we end up with all the terms in the denominator tending toward 0, while the x^n term in the numerator does not approach 0. This is indicative of some sort of infinite limit.

Intuitively, as x gets very large, all the terms in the numerator are small in comparison to $a_n x^n$, and likewise all the terms in the denominator are small compared to $b_n x^m$. If $n = m$, looking only at these two important terms, we have $(a_n x^n)/(b_n x^m)$. This reduces to a_n/b_m. If $n < m$, the function behaves like $a_n/(b_m x^{m-n})$, which tends toward 0. If $n > m$, the function behaves like $a_n x^{n-m}/b_m$, which will tend to either ∞ or $-\infty$ depending on the values of n, m, a_n, b_m and whether you are looking for $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$.

With care, we can quickly evaluate limits at infinity for a large number of functions by considering the largest powers of x. For instance, consider again $\lim\limits_{x\to\pm\infty} \dfrac{x}{\sqrt{x^2+1}}$, graphed in Figure 5.4.7(b). When x is very large, $x^2 + 1 \approx x^2$. Thus

$$\sqrt{x^2+1} \approx \sqrt{x^2} = |x|, \quad \text{and} \quad \frac{x}{\sqrt{x^2+1}} \approx \frac{x}{|x|}.$$

This expression is 1 when x is positive and -1 when x is negative. Hence we get asymptotes of $y = 1$ and $y = -1$, respectively.

Example 5.4.5 Finding a limit of a rational function

Confirm analytically that $y = 1$ is the horizontal asymptote of $f(x) = \dfrac{x^2}{x^2+4}$, as approximated in Example 5.4.4.

Solution Before using Theorem 5.4.1, let's use the technique of evaluating limits at infinity of rational functions that led to that theorem. The largest power of x in f is 2, so divide the numerator and denominator of f by x^2, then take limits.

$$\begin{aligned}\lim_{x\to\infty}\frac{x^2}{x^2+4} &= \lim_{x\to\infty}\frac{x^2/x^2}{x^2/x^2+4/x^2}\\ &= \lim_{x\to\infty}\frac{1}{1+4/x^2}\\ &= \frac{1}{1+0}\\ &= 1.\end{aligned}$$

We can also use Theorem 5.4.1 directly; in this case $n = m$ so the limit is the ratio of the leading coefficients of the numerator and denominator, i.e., 1/1 = 1.

Example 5.4.6 Finding limits of rational functions

Use Theorem 5.4.1 to evaluate each of the following limits.

1. $\displaystyle\lim_{x\to-\infty}\frac{x^2+2x-1}{x^3+1}$
2. $\displaystyle\lim_{x\to\infty}\frac{x^2+2x-1}{1-x-3x^2}$
3. $\displaystyle\lim_{x\to\infty}\frac{x^2-1}{3-x}$

Solution

1. The highest power of x is in the denominator. Therefore, the limit is 0; see Figure 5.4.8(a).
2. The highest power of x is x^2, which occurs in both the numerator and denominator. The limit is therefore the ratio of the coefficients of x^2, which is $-1/3$. See Figure 5.4.8(b).
3. The highest power of x is in the numerator so the limit will be ∞ or $-\infty$. To see which, consider only the dominant terms from the numerator and denominator, which are x^2 and $-x$. The expression in the limit will behave like $x^2/(-x) = -x$ for large values of x. Therefore, the limit is $-\infty$. See Figure 5.4.8(c).

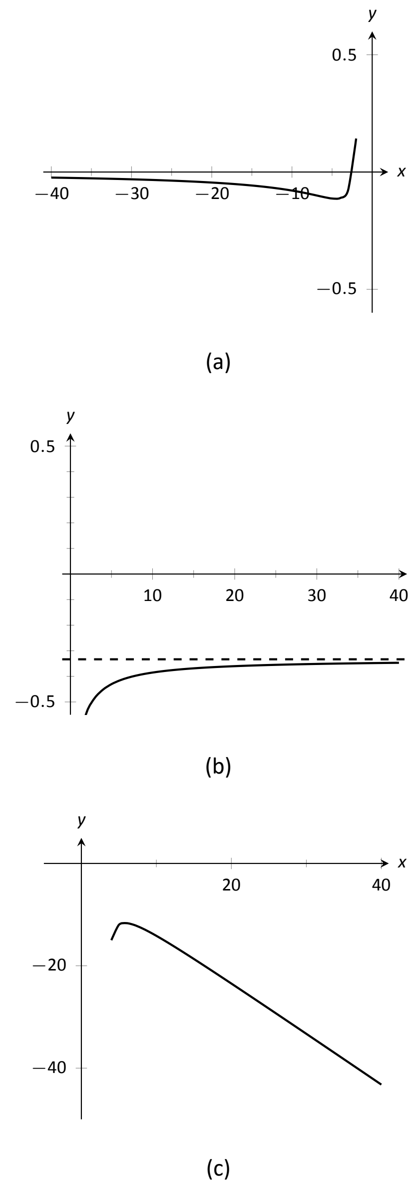


Figure 5.4.8: Visualizing the functions in Example 5.4.6.

Exercises 5.4

Terms and Concepts

1. T/F: If $\lim_{x\to 5} f(x) = \infty$, then we are implicitly stating that the limit exists.

2. T/F: If $\lim_{x\to \infty} f(x) = 5$, then we are implicitly stating that the limit exists.

3. T/F: If $\lim_{x\to 1^-} f(x) = -\infty$, then $\lim_{x\to 1^+} f(x) = \infty$

4. T/F: If $\lim_{x\to 5} f(x) = \infty$, then f has a vertical asymptote at $x = 5$.

5. T/F: $\infty/0$ is not an indeterminate form.

6. List 5 indeterminate forms.

7. Construct a function with a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = 5$.

8. Let $\lim_{x\to 7} f(x) = \infty$. Explain how we know that f is/is not continuous at $x = 7$.

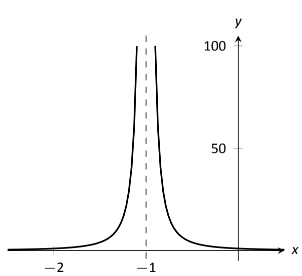
Problems

In Exercises 9 – 14, evaluate the given limits using the graph of the function.

9. $f(x) = \dfrac{1}{(x+1)^2}$

   (a) $\lim_{x\to -1^-} f(x)$

   (b) $\lim_{x\to -1^+} f(x)$



10. $f(x) = \dfrac{1}{(x-3)(x-5)^2}$.

   (a) $\lim_{x\to 3^-} f(x)$

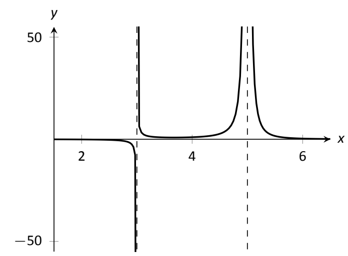
   (b) $\lim_{x\to 3^+} f(x)$

   (c) $\lim_{x\to 3} f(x)$

   (d) $\lim_{x\to 5^-} f(x)$

   (e) $\lim_{x\to 5^+} f(x)$

   (f) $\lim_{x\to 5} f(x)$



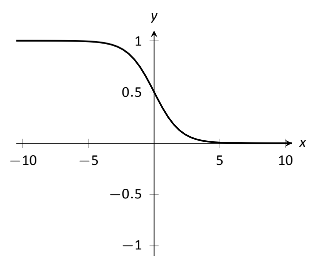
11. $f(x) = \dfrac{1}{e^x + 1}$

   (a) $\lim_{x\to -\infty} f(x)$

   (b) $\lim_{x\to \infty} f(x)$

   (c) $\lim_{x\to 0^-} f(x)$

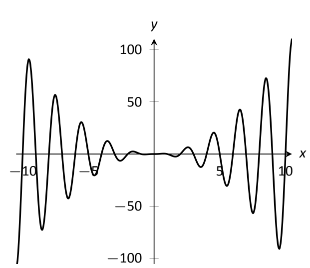
   (d) $\lim_{x\to 0^+} f(x)$



12. $f(x) = x^2 \sin(\pi x)$

   (a) $\lim_{x\to -\infty} f(x)$

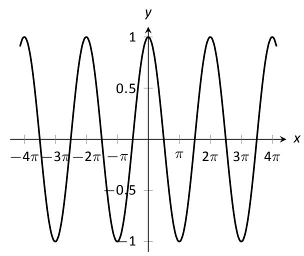
   (b) $\lim_{x\to \infty} f(x)$



13. $f(x) = \cos(x)$

(a) $\lim\limits_{x\to -\infty} f(x)$

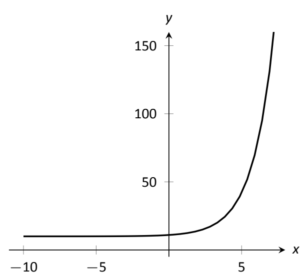
(b) $\lim\limits_{x\to \infty} f(x)$



14. $f(x) = 2^x + 10$

(a) $\lim\limits_{x\to -\infty} f(x)$

(b) $\lim\limits_{x\to \infty} f(x)$



In Exercises 15 – 18, numerically approximate the following limits:

(a) $\lim\limits_{x\to 3^-} f(x)$

(b) $\lim\limits_{x\to 3^+} f(x)$

(c) $\lim\limits_{x\to 3} f(x)$

15. $f(x) = \dfrac{x^2 - 1}{x^2 - x - 6}$

16. $f(x) = \dfrac{x^2 + 5x - 36}{x^3 - 5x^2 + 3x + 9}$

17. $f(x) = \dfrac{x^2 - 11x + 30}{x^3 - 4x^2 - 3x + 18}$

18. $f(x) = \dfrac{x^2 - 9x + 18}{x^2 - x - 6}$

In Exercises 19 – 24, identify the horizontal and vertical asymptotes, if any, of the given function.

19. $f(x) = \dfrac{2x^2 - 2x - 4}{x^2 + x - 20}$

20. $f(x) = \dfrac{-3x^2 - 9x - 6}{5x^2 - 10x - 15}$

21. $f(x) = \dfrac{x^2 + x - 12}{7x^3 - 14x^2 - 21x}$

22. $f(x) = \dfrac{x^2 - 9}{9x - 9}$

23. $f(x) = \dfrac{x^2 - 9}{9x + 27}$

24. $f(x) = \dfrac{x^2 - 1}{-x^2 - 1}$

In Exercises 25 – 28, evaluate the given limit.

25. $\lim\limits_{x\to\infty} \dfrac{x^3 + 2x^2 + 1}{x - 5}$

26. $\lim\limits_{x\to\infty} \dfrac{x^3 + 2x^2 + 1}{5 - x}$

27. $\lim\limits_{x\to -\infty} \dfrac{x^3 + 2x^2 + 1}{x^2 - 5}$

28. $\lim\limits_{x\to -\infty} \dfrac{x^3 + 2x^2 + 1}{5 - x^2}$

Review

29. Use an $\varepsilon - \delta$ proof to show that $\lim\limits_{x\to 1} 5x - 2 = 3$.

30. Let $\lim\limits_{x\to 2} f(x) = 3$ and $\lim\limits_{x\to 2} g(x) = -1$. Evaluate the following limits.

(a) $\lim\limits_{x\to 2} (f + g)(x)$

(b) $\lim\limits_{x\to 2} (fg)(x)$

(c) $\lim\limits_{x\to 2} (f/g)(x)$

(d) $\lim\limits_{x\to 2} f(x)^{g(x)}$

31. Let $f(x) = \begin{cases} x^2 - 1 & x < 3 \\ x + 5 & x \geq 3 \end{cases}$.

Is f continuous everywhere?

32. Evaluate the limit: $\lim\limits_{x\to e} \ln x$.

5.5 Continuity

As we have studied limits, we have gained the intuition that limits measure "where a function is heading." That is, if $\lim_{x\to 1} f(x) = 3$, then as x is close to 1, $f(x)$ is close to 3. We have seen, though, that this is not necessarily a good indicator of what $f(1)$ actually is. This can be problematic; functions can tend to one value but attain another. This section focuses on functions that *do not* exhibit such behaviour.

Definition 5.5.1 Continuous Function

Let f be a function defined on an open interval I containing c.

1. f is **continuous at** c if $\lim_{x\to c} f(x) = f(c)$.
2. f is **continuous on** I if f is continuous at c for all values of c in I. If f is continuous on $(-\infty, \infty)$, we say f is **continuous everywhere**.

A useful way to establish whether or not a function f is continuous at c is to verify the following three things:

1. $\lim_{x\to c} f(x)$ exists,
2. $f(c)$ is defined, and
3. $\lim_{x\to c} f(x) = f(c)$.

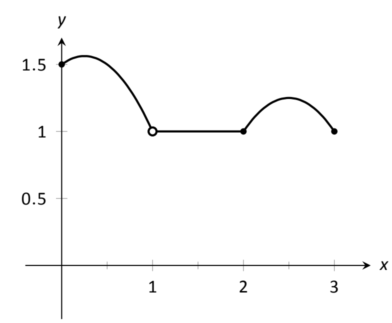


Figure 5.5.1: A graph of f in Example 5.5.1.

Example 5.5.1 Finding intervals of continuity

Let f be defined as shown in Figure 5.5.1. Give the interval(s) on which f is continuous.

Solution We proceed by examining the three criteria for continuity.

1. The limits $\lim_{x\to c} f(x)$ exists for all c between 0 and 3.
2. $f(c)$ is defined for all c between 0 and 3, *except for* $c = 1$. We know immediately that f cannot be continuous at $x = 1$.
3. The limit $\lim_{x\to c} f(x) = f(c)$ for all c between 0 and 3, except, of course, for $c = 1$.

We conclude that f is continuous at every point of $(0, 3)$ except at $x = 1$. Therefore f is continuous on $(0, 1)$ and $(1, 3)$.

Our definition of continuity (currently) only applies to open intervals. After Definition 5.5.2, we'll be able to say that f is continuous on $[0, 1)$ and $(1, 3]$.

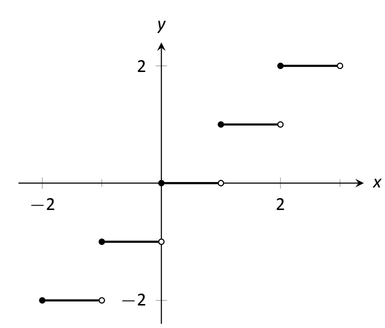


Figure 5.5.2: A graph of the step function in Example 5.5.2.

Example 5.5.2 Finding intervals of continuity

The *floor function*, $f(x) = \lfloor x \rfloor$, returns the largest integer smaller than, or equal to, the input x. (For example, $f(\pi) = \lfloor \pi \rfloor = 3$.) The graph of f in Figure 5.5.2 demonstrates why this is often called a "step function."

Give the intervals on which f is continuous.

Solution We examine the three criteria for continuity.

1. The limits $\lim_{x\to c} f(x)$ do not exist at the jumps from one "step" to the next, which occur at all integer values of c. Therefore the limits exist for all c except when c is an integer.

2. The function is defined for all values of c.

3. The limit $\lim\limits_{x\to c} f(x) = f(c)$ for all values of c where the limit exist, since each step consists of just a line.

We conclude that f is continuous everywhere except at integer values of c. So the intervals on which f is continuous are

$$\ldots,(-2,-1),(-1,0),(0,1),(1,2),\ldots.$$

Our definition of continuity on an interval specifies the interval is an open interval. We can extend the definition of continuity to closed intervals by considering the appropriate one-sided limits at the endpoints.

Definition 5.5.2 Continuity on Closed Intervals

Let f be defined on the closed interval $[a,b]$ for some real numbers $a < b$. f is **continuous on** $[a,b]$ if:

1. f is continuous on (a,b),
2. $\lim\limits_{x\to a^+} f(x) = f(a)$ and
3. $\lim\limits_{x\to b^-} f(x) = f(b)$.

We can make the appropriate adjustments to talk about continuity on half–open intervals such as $[a,b)$ or $(a,b]$ if necessary.

Using this new definition, we can adjust our answer in Example 5.5.1 by stating that f is continuous on $[0,1)$ and $(1,3]$, as mentioned in that example. We can also revisit Example 5.5.2 and state that the floor function is continuous on the following half–open intervals

$$\ldots,[-2,-1),[-1,0),[0,1),[1,2),\ldots.$$

This can tempt us to conclude that f is continuous everywhere; after all, if f is continuous on $[0,1)$ and $[1,2)$, isn't f also continuous on $[0,2)$? Of course, the answer is *no*, and the graph of the floor function immediately confirms this.

Continuous functions are important as they behave in a predictable fashion: functions attain the value they approach. Because continuity is so important, most of the functions you have likely seen in the past are continuous on their domains. This is demonstrated in the following example where we examine the intervals of continuity of a variety of common functions.

Example 5.5.3 Determining intervals on which a function is continuous

For each of the following functions, give the domain of the function and the interval(s) on which it is continuous.

1. $f(x) = 1/x$
2. $f(x) = \sin x$
3. $f(x) = \sqrt{x}$
4. $f(x) = \sqrt{1-x^2}$
5. $f(x) = |x|$

Solution We examine each in turn.

1. The domain of $f(x) = 1/x$ is $(-\infty, 0) \cup (0, \infty)$. As it is a rational function, we apply Theorem 5.2.2 to recognize that f is continuous on all of its domain.

2. The domain of $f(x) = \sin x$ is all real numbers, or $(-\infty, \infty)$. Applying Theorem 5.2.3 shows that $\sin x$ is continuous everywhere.

3. The domain of $f(x) = \sqrt{x}$ is $[0, \infty)$. Applying Theorem 5.2.3 shows that $f(x) = \sqrt{x}$ is continuous on its domain of $[0, \infty)$.

4. The domain of $f(x) = \sqrt{1-x^2}$ is $[-1, 1]$. Applying Theorems 5.2.1 and 5.2.3 shows that f is continuous on all of its domain, $[-1, 1]$.

5. The domain of $f(x) = |x|$ is $(-\infty, \infty)$. We can define the absolute value function as $f(x) = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$. Each "piece" of this piecewise defined function is continuous on all of its domain, giving that f is continuous on $(-\infty, 0)$ and $[0, \infty)$. We cannot assume this implies that f is continuous on $(-\infty, \infty)$; we need to check that $\lim\limits_{x\to 0} f(x) = f(0)$, as $x = 0$ is the point where f transitions from one "piece" of its definition to the other. It is easy to verify that this is indeed true, hence we conclude that $f(x) = |x|$ is continuous everywhere.

Continuity is inherently tied to the properties of limits. Because of this, the properties of limits found in Theorems 5.2.1 and 5.2.2 apply to continuity as well. Further, now knowing the definition of continuity we can re–read Theorem 5.2.3 as giving a list of functions that are continuous on their domains. The following theorem states how continuous functions can be combined to form other continuous functions, followed by a theorem which formally lists functions that we know are continuous on their domains.

Theorem 5.5.1 Properties of Continuous Functions

Let f and g be continuous functions on an interval I, let c be a real number and let n be a positive integer. The following functions are continuous on I.

1. Sums/Differences: $f \pm g$
2. Constant Multiples: $c \cdot f$
3. Products: $f \cdot g$
4. Quotients: f/g (as long as $g \neq 0$ on I)
5. Powers: f^n
6. Roots: $\sqrt[n]{f}$ (If n is even then require $f(x) \geq 0$ on I.)
7. Compositions: Adjust the definitions of f and g to: Let f be continuous on I, where the range of f on I is J, and let g be continuous on J. Then $g \circ f$, i.e., $g(f(x))$, is continuous on I.

Theorem 5.5.2 Continuous Functions

Let n be a positive integer. The following functions are continuous on their domains.

1. $f(x) = \sin x$
2. $f(x) = \cos x$
3. $f(x) = \tan x$
4. $f(x) = \csc x$
5. $f(x) = \sec x$
6. $f(x) = \cot x$
7. $f(x) = a^x$ $(a > 0)$
8. $f(x) = \ln x$
9. $f(x) = \sqrt[n]{x}$

We apply these theorems in the following Example.

Example 5.5.4 Determining intervals on which a function is continuous

State the interval(s) on which each of the following functions is continuous.

1. $f(x) = \sqrt{x-1} + \sqrt{5-x}$
2. $f(x) = x\sin x$
3. $f(x) = \tan x$
4. $f(x) = \sqrt{\ln x}$

SOLUTION We examine each in turn, applying Theorems 5.5.1 and 5.5.2 as appropriate.

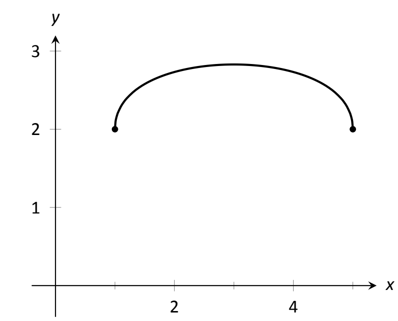


Figure 5.5.3: A graph of f in Example 5.5.4(1).

1. The square–root terms are continuous on the intervals $[1, \infty)$ and $(-\infty, 5]$, respectively. As f is continuous only where each term is continuous, f is continuous on $[1, 5]$, the intersection of these two intervals. A graph of f is given in Figure 5.5.3.

2. The functions $y = x$ and $y = \sin x$ are each continuous everywhere, hence their product is, too.

3. Theorem 5.5.2 states that $f(x) = \tan x$ is continuous "on its domain." Its domain includes all real numbers except odd multiples of $\pi/2$. Thus the intervals on which $f(x) = \tan x$ is continuous are

$$\ldots\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right),\ \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),\ \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots,$$

or, equivalently, on $D = \{x \in \mathbb{R} \mid x \neq n \cdot \frac{\pi}{2}, \text{n is an odd integer}\}$.

4. The domain of $y = \sqrt{x}$ is $[0, \infty)$. The range of $y = \ln x$ is $(-\infty, \infty)$, but if we restrict its domain to $[1, \infty)$ its range is $[0, \infty)$. So restricting $y = \ln x$ to the domain of $[1, \infty)$ restricts its output is $[0, \infty)$, on which $y = \sqrt{x}$ is defined. Thus the domain of $f(x) = \sqrt{\ln x}$ is $[1, \infty)$.

Classifying discontinuities

We now know what it means for a function to be continuous, so of course we can easily say what it means for a function to be *discontinuous*; namely, not continuous. However, to better understand continuity it is worth our time to discuss the different ways in which a function can fail to be discontinuous. By definition, a function f is continuous at a point a in its domain if $\lim\limits_{x\to a} f(x) = f(a)$. If this equality fails to hold, then f is not continuous. We note, however, that there are a number of different things that can go wrong with this equation.

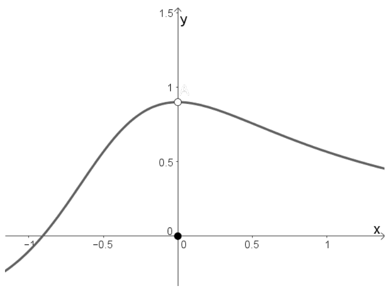


Figure 5.5.4: The graph of a function with a removable discontinuity at $x = 0$

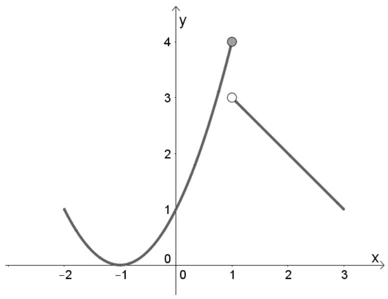


Figure 5.5.5: The graph of a function with a jump discontinuity at $x = 1$

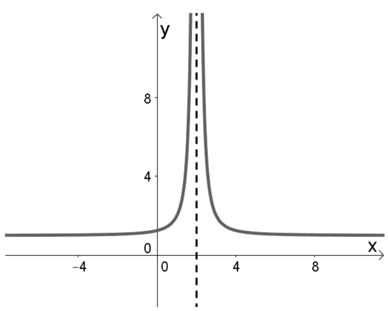


Figure 5.5.6: The graph of a function with an infinite discontinuity at $x = 2$

1. $\lim\limits_{x\to a} f(x) = L$ exists, but $L \neq f(a)$, or $f(a)$ is undefined. Such a discontinuity is called a **removable discontinuity**.

   A removable discontinuity can be pictured as a "hole" in the graph of f. The term "removable" refers to the fact that by simply redefining $f(a)$ to equal L (that is, changing the value of f at a single point), we can create a new function that is continuous at $x = a$, and agrees with f at all $x \neq a$.

2. $\lim\limits_{x\to a^+} f(x) = L$ and $\lim\limits_{x\to a^-} f(x) = M$ exist, but $L \neq M$. In this case the left and right hand limits both exist, but since they are not equal, the limit of f as $x \to a$ does not exist. Such a discontinuity is called a **jump discontinuity**.

   The phrase "jump discontinuity" is meant to represent the fact that visually, the graph of f "jumps" from one value to another as we cross the value $x = a$.

3. The function f is **unbounded** near $x = a$. This means that the value of f becomes arbitrarily large (or large and negative) as x approaches a. Such a discontinuity is called an **infinite discontinuity**.

   Infinite discontinuities are most easily understood in terms of *infinite limits*, which we will discuss in the next section.

4. $\lim_{x\to a} f(x)$ does not exist, for reasons other than the above. Such discontinuities are called **essential discontinuities** . With jump and infinite discontinuities, the limit fails to exist, but in ways that can still be described or even quantified. Essential discontinuities include examples such as $f(x) = \sin(1/x)$ as $x \to 0$, where the function oscillates infinitely often, or is otherwise so badly-behaved that the limit does not exist.

Consequences of continuity

A common way of thinking of a continuous function is that "its graph can be sketched without lifting your pencil." That is, its graph forms a "continuous" curve, without holes, breaks or jumps. While beyond the scope of this text, this pseudo–definition glosses over some of the finer points of continuity. Very strange functions are continuous that one would be hard pressed to actually sketch by hand.

This intuitive notion of continuity does help us understand another important concept as follows. Suppose f is defined on $[1, 2]$ and $f(1) = -10$ and $f(2) = 5$. If f is continuous on $[1, 2]$ (i.e., its graph can be sketched as a continuous curve from $(1, -10)$ to $(2, 5)$) then we know intuitively that somewhere on $[1, 2]$ f must be equal to -9, and -8, and $-7,\ -6,\ \ldots,\ 0,\ 1/2$, etc. In short, f takes on all *intermediate* values between -10 and 5. It may take on more values; f may actually equal 6 at some time, for instance, but we are guaranteed all values between -10 and 5.

While this notion seems intuitive, it is not trivial to prove and its importance is profound. Therefore the concept is stated in the form of a theorem.

Theorem 5.5.3 Intermediate Value Theorem

Let f be a continuous function on $[a, b]$ and, without loss of generality, let $f(a) < f(b)$. Then for every value y, where $f(a) < y < f(b)$, there is at least one value c in (a, b) such that $f(c) = y$.

One important application of the Intermediate Value Theorem is root finding. Given a function f, we are often interested in finding values of x where $f(x) = 0$. These roots may be very difficult to find exactly. Good approximations can be found through successive applications of this theorem. Suppose through direct computation we find that $f(a) < 0$ and $f(b) > 0$, where $a < b$. The Intermediate Value Theorem states that there is at least one c in (a, b) such that $f(c) = 0$. The theorem does not give us any clue as to where to find such a value in the interval (a, b), just that at least one such value exists.

There is a technique that produces a good approximation of c. Let d be the midpoint of the interval $[a, b]$ and consider $f(d)$. There are three possibilities:

1. $f(d) = 0$: We got lucky and stumbled on the actual value. We stop as we found a root.

2. $f(d) < 0$: Then we know there is a root of f on the interval $[d, b]$ – we have halved the size of our interval, hence are closer to a good approximation of the root.

3. $f(d) > 0$: Then we know there is a root of f on the interval $[a, d]$ – again,we have halved the size of our interval, hence are closer to a good approximation of the root.

Successively applying this technique is called the **Bisection Method** of root finding. We continue until the interval is sufficiently small. We demonstrate this in the following example.

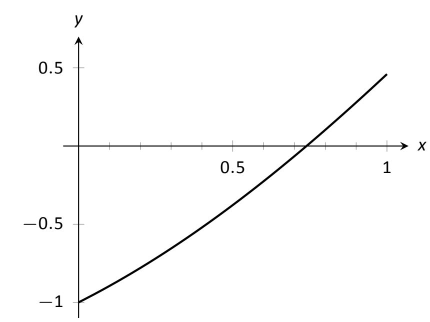


Figure 5.5.7: Graphing a root of $f(x) = x - \cos x$.

Example 5.5.5 Using the Bisection Method
Approximate the root of $f(x) = x - \cos x$, accurate to three places after the decimal.

Solution Consider the graph of $f(x) = x - \cos x$, shown in Figure 5.5.7. It is clear that the graph crosses the x-axis somewhere near $x = 0.8$. To start the Bisection Method, pick an interval that contains 0.8. We choose $[0.7, 0.9]$. Note that all we care about are signs of $f(x)$, not their actual value, so this is all we display.

Iteration 1: $f(0.7) < 0$, $f(0.9) > 0$, and $f(0.8) > 0$. So replace 0.9 with 0.8 and repeat.

Iteration 2: $f(0.7) < 0$, $f(0.8) > 0$, and at the midpoint, 0.75, we have $f(0.75) > 0$. So replace 0.8 with 0.75 and repeat. Note that we don't need to continue to check the endpoints, just the midpoint. Thus we put the rest of the iterations in Figure 5.5.8.

Iteration	Interval	Midpoint Sign
1	$[0.7, 0.9]$	$f(0.8) > 0$
2	$[0.7, 0.8]$	$f(0.75) > 0$
3	$[0.7, 0.75]$	$f(0.725) < 0$
4	$[0.725, 0.75]$	$f(0.7375) < 0$
5	$[0.7375, 0.75]$	$f(0.7438) > 0$
6	$[0.7375, 0.7438]$	$f(0.7407) > 0$
7	$[0.7375, 0.7407]$	$f(0.7391) > 0$
8	$[0.7375, 0.7391]$	$f(0.7383) < 0$
9	$[0.7383, 0.7391]$	$f(0.7387) < 0$
10	$[0.7387, 0.7391]$	$f(0.7389) < 0$
11	$[0.7389, 0.7391]$	$f(0.7390) < 0$
12	$[0.7390, 0.7391]$	

Figure 5.5.8: Iterations of the Bisection Method of Root Finding

Notice that in the 12$^{\text{th}}$ iteration we have the endpoints of the interval each starting with 0.739. Thus we have narrowed the zero down to an accuracy of the first three places after the decimal. Using a computer, we have

$$f(0.7390) = -0.00014, \quad f(0.7391) = 0.000024.$$

Either endpoint of the interval gives a good approximation of where f is 0. The Intermediate Value Theorem states that the actual zero is still within this interval. While we do not know its exact value, we know it starts with 0.739.

This type of exercise is rarely done by hand. Rather, it is simple to program a computer to run such an algorithm and stop when the endpoints differ by a preset small amount. One of the authors did write such a program and found the zero of f, accurate to 10 places after the decimal, to be 0.7390851332. While it took a few minutes to write the program, it took less than a thousandth of a second for the program to run the necessary 35 iterations. In less than 8 hundredths of a second, the zero was calculated to 100 decimal places (with less than 200 iterations).

It is a simple matter to extend the Bisection Method to solve problems similar to "Find x, where $f(x) = 0$." For instance, we can find x, where $f(x) = 1$. It actually works very well to define a new function g where $g(x) = f(x) - 1$. Then use the Bisection Method to solve $g(x) = 0$.

Similarly, given two functions f and g, we can use the Bisection Method to solve $f(x) = g(x)$. Once again, create a new function h where $h(x) = f(x) - g(x)$ and solve $h(x) = 0$.

This section formally defined what it means to be a continuous function. "Most" functions that we deal with are continuous, so often it feels odd to have to formally define this concept. Regardless, it is important, and forms the basis of the next chapter.

Chapter Summary

In this chapter we:

- defined the limit,
- found accessible ways to approximate their values numerically and graphically,
- developed a not–so–easy method of proving the value of a limit (ε-δ proofs),
- explored when limits do not exist,
- defined continuity and explored properties of continuous functions, and
- considered limits that involved infinity.

Why? Mathematics is famous for building on itself and calculus proves to be no exception. In the next chapter we will be interested in "dividing by 0." That is, we will want to divide a quantity by a smaller and smaller number and see what value the quotient approaches. In other words, we will want to find a limit. These limits will enable us to, among other things, determine *exactly* how fast something is moving when we are only given position information.

Later, we will want to add up an infinite list of numbers. We will do so by first adding up a finite list of numbers, then take a limit as the number of things we are adding approaches infinity. Surprisingly, this sum often is finite; that is, we can add up an infinite list of numbers and get, for instance, 42.

These are just two quick examples of why we are interested in limits. Many students dislike this topic when they are first introduced to it, but over time an appreciation is often formed based on the scope of its applicability.

Exercises 5.5

Terms and Concepts

1. In your own words, describe what it means for a function to be continuous.

2. In your own words, describe what the Intermediate Value Theorem states.

3. What is a "root" of a function?

4. Given functions f and g on an interval I, how can the Bisection Method be used to find a value c where $f(c) = g(c)$?

5. T/F: If f is defined on an open interval containing c, and $\lim_{x \to c} f(x)$ exists, then f is continuous at c.

6. T/F: If f is continuous at c, then $\lim_{x \to c} f(x)$ exists.

7. T/F: If f is continuous at c, then $\lim_{x \to c^+} f(x) = f(c)$.

8. T/F: If f is continuous on $[a, b]$, then $\lim_{x \to a^-} f(x) = f(a)$.

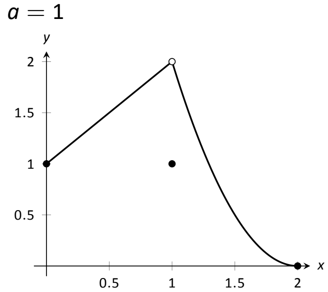
9. T/F: If f is continuous on $[0, 1)$ and $[1, 2)$, then f is continuous on $[0, 2)$.

10. T/F: The sum of continuous functions is also continuous.

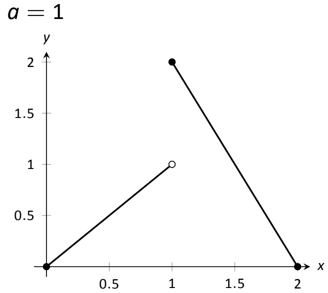
Problems

In Exercises 11 – 18, a graph of a function f is given along with a value a. Determine if f is continuous at a; if it is not, state why it is not.

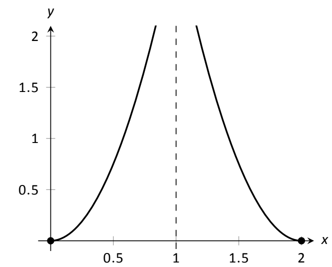
11. $a = 1$



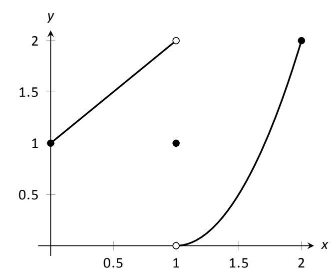
12. $a = 1$



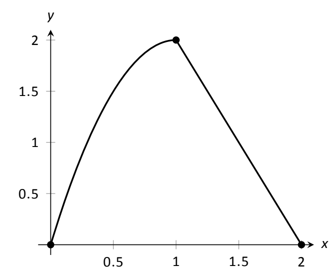
13. $a = 1$



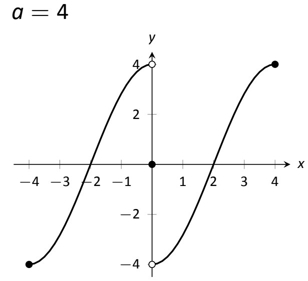
14. $a = 0$



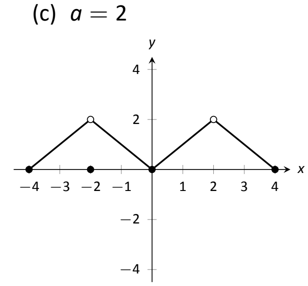
15. $a = 1$



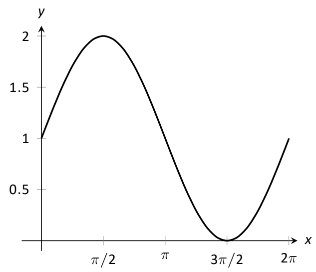
16. $a = 4$



17. (a) $a = -2$
    (b) $a = 0$
    (c) $a = 2$



18. $a = 3\pi/2$



In Exercises 19 – 22, determine if f is continuous at the indicated values. If not, explain why.

19. $f(x) = \begin{cases} 1 & x = 0 \\ \frac{\sin x}{x} & x > 0 \end{cases}$

    (a) $x = 0$

    (b) $x = \pi$

20. $f(x) = \begin{cases} x^3 - x & x < 1 \\ x - 2 & x \geq 1 \end{cases}$

    (a) $x = 0$

    (b) $x = 1$

21. $f(x) = \begin{cases} \frac{x^2+5x+4}{x^2+3x+2} & x \neq -1 \\ 3 & x = -1 \end{cases}$

    (a) $x = -1$

    (b) $x = 10$

22. $f(x) = \begin{cases} \frac{x^2-64}{x^2-11x+24} & x \neq 8 \\ 5 & x = 8 \end{cases}$

    (a) $x = 0$

    (b) $x = 8$

In Exercises 23 – 34, give the intervals on which the given function is continuous.

23. $f(x) = x^2 - 3x + 9$

24. $g(x) = \sqrt{x^2 - 4}$

25. $g(x) = \sqrt{4 - x^2}$

26. $h(k) = \sqrt{1 - k} + \sqrt{k + 1}$

27. $f(t) = \sqrt{5t^2 - 30}$

28. $g(t) = \dfrac{1}{\sqrt{1 - t^2}}$

29. $g(x) = \dfrac{1}{1 + x^2}$

30. $f(x) = e^x$

31. $g(s) = \ln s$

32. $h(t) = \cos t$

33. $f(k) = \sqrt{1 - e^k}$

34. $f(x) = \sin(e^x + x^2)$

Exercises 35 – 38 test your understanding of the Intermediate Value Theorem.

35. Let f be continuous on $[1, 5]$ where $f(1) = -2$ and $f(5) = -10$. Does a value $1 < c < 5$ exist such that $f(c) = -9$? Why/why not?

36. Let g be continuous on $[-3, 7]$ where $g(0) = 0$ and $g(2) = 25$. Does a value $-3 < c < 7$ exist such that $g(c) = 15$? Why/why not?

37. Let f be continuous on $[-1, 1]$ where $f(-1) = -10$ and $f(1) = 10$. Does a value $-1 < c < 1$ exist such that $f(c) = 11$? Why/why not?

38. Let h be a function on $[-1, 1]$ where $h(-1) = -10$ and $h(1) = 10$. Does a value $-1 < c < 1$ exist such that $h(c) = 0$? Why/why not?

In Exercises 39 – 42, use the Bisection Method to approximate, accurate to two decimal places, the value of the root of the given function in the given interval.

39. $f(x) = x^2 + 2x - 4$ on $[1, 1.5]$.

40. $f(x) = \sin x - 1/2$ on $[0.5, 0.55]$

41. $f(x) = e^x - 2$ on $[0.65, 0.7]$.

42. $f(x) = \cos x - \sin x$ on $[0.7, 0.8]$.

Review

43. Let $f(x) = \begin{cases} x^2 - 5 & x < 5 \\ 5x & x \geq 5 \end{cases}$.

    (a) $\lim_{x \to 5^-} f(x)$

    (b) $\lim_{x \to 5^+} f(x)$

    (c) $\lim_{x \to 5} f(x)$

    (d) $f(5)$

44. Numerically approximate the following limits:

    (a) $\displaystyle\lim_{x \to -4/5^+} \frac{x^2 - 8.2x - 7.2}{x^2 + 5.8x + 4}$

    (b) $\displaystyle\lim_{x \to -4/5^-} \frac{x^2 - 8.2x - 7.2}{x^2 + 5.8x + 4}$

45. Give an example of function $f(x)$ for which $\lim_{x \to 0} f(x)$ does not exist.

6: DERIVATIVES

The previous chapter introduced the most fundamental of calculus topics: the limit. This chapter introduces the second most fundamental of calculus topics: the derivative. Limits describe *where* a function is going; derivatives describe *how fast* the function is going.

6.1 Instantaneous Rates of Change: The Derivative

A common amusement park ride lifts riders to a height then allows them to free-fall a certain distance before safely stopping them. Suppose such a ride drops riders from a height of 150 feet. Students of physics may recall that the height (in feet) of the riders, t seconds after free-fall (and ignoring air resistance, etc.) can be accurately modelled by $f(t) = -16t^2 + 150$.

Using this formula, it is easy to verify that, without intervention, the riders will hit the ground at $t = 2.5\sqrt{1.5} \approx 3.06$ seconds. Suppose the designers of the ride decide to begin slowing the riders' fall after 2 seconds (corresponding to a height of 86 ft.). How fast will the riders be travelling at that time?

We have been given a *position* function, but what we want to compute is a velocity at a specific point in time, i.e., we want an *instantaneous velocity*. We do not currently know how to calculate this.

However, we do know from common experience how to calculate an *average velocity*. (If we travel 60 miles in 2 hours, we know we had an average velocity of 30 mph.) We looked at this concept in Section 5.1 when we introduced the difference quotient. We have

$$\frac{\text{change in distance}}{\text{change in time}} = \frac{\text{“ rise ”}}{\text{run}} = \text{average velocity.}$$

We can approximate the instantaneous velocity at $t = 2$ by considering the average velocity over some time period containing $t = 2$. If we make the time interval small, we will get a good approximation. (This fact is commonly used. For instance, high speed cameras are used to track fast moving objects. Distances are measured over a fixed number of frames to generate an accurate approximation of the velocity.)

Consider the interval from $t = 2$ to $t = 3$ (just before the riders hit the ground). On that interval, the average velocity is

$$\frac{f(3) - f(2)}{3 - 2} = \frac{f(3) - f(2)}{1} = -80 \text{ ft/s},$$

where the minus sign indicates that the riders are moving *down*. By narrowing the interval we consider, we will likely get a better approximation of the instantaneous velocity. On $[2, 2.5]$ we have

$$\frac{f(2.5) - f(2)}{2.5 - 2} = \frac{f(2.5) - f(2)}{0.5} = -72 \text{ ft/s}.$$

We can do this for smaller and smaller intervals of time. For instance, over a time span of $1/10^{\text{th}}$ of a second, i.e., on $[2, 2.1]$, we have

$$\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{f(2.1) - f(2)}{0.1} = -65.6 \text{ ft/s}.$$

Over a time span of 1/100th of a second, on $[2, 2.01]$, the average velocity is

$$\frac{f(2.01) - f(2)}{2.01 - 2} = \frac{f(2.01) - f(2)}{0.01} = -64.16 \text{ ft/s}.$$

What we are really computing is the average velocity on the interval $[2, 2+h]$ for small values of h. That is, we are computing

$$\frac{f(2+h) - f(2)}{h}$$

where h is small.

We really want to use $h = 0$, but this, of course, returns the familiar "0/0" indeterminate form. So we employ a limit, as we did in Section 5.1.

We can approximate the value of this limit numerically with small values of h as seen in Figure 6.1.2. It looks as though the velocity is approaching -64 ft/s. Computing the limit directly gives

h	Average Velocity ft/s
1	-80
0.5	-72
0.1	-65.6
0.01	-64.16
0.001	-64.016

Figure 6.1.2: Approximating the instantaneous velocity with average velocities over a small time period h.

$$\begin{aligned}\lim_{h\to 0}\frac{f(2+h) - f(2)}{h} &= \lim_{h\to 0}\frac{-16(2+h)^2 + 150 - (-16(2)^2 + 150)}{h}\\ &= \lim_{h\to 0}\frac{-64h - 16h^2}{h}\\ &= \lim_{h\to 0}(-64 - 16h)\\ &= -64.\end{aligned}$$

Graphically, we can view the average velocities we computed numerically as the slopes of secant lines on the graph of f going through the points $(2, f(2))$ and $(2+h, f(2+h))$. In Figure 6.1.1, the secant line corresponding to $h = 1$ is shown in three contexts. Figure 6.1.1(a) shows a "zoomed out" version of f with its secant line. In (b), we zoom in around the points of intersection between f and the secant line. Notice how well this secant line approximates f between those two points – it is a common practice to approximate functions with straight lines.

As $h \to 0$, these secant lines approach the *tangent line*, a line that goes through the point $(2, f(2))$ with the special slope of -64. In parts (c) and (d) of Figure 6.1.1, we zoom in around the point $(2, 86)$. In (c) we see the secant line, which approximates f well, but not as well the tangent line shown in (d).

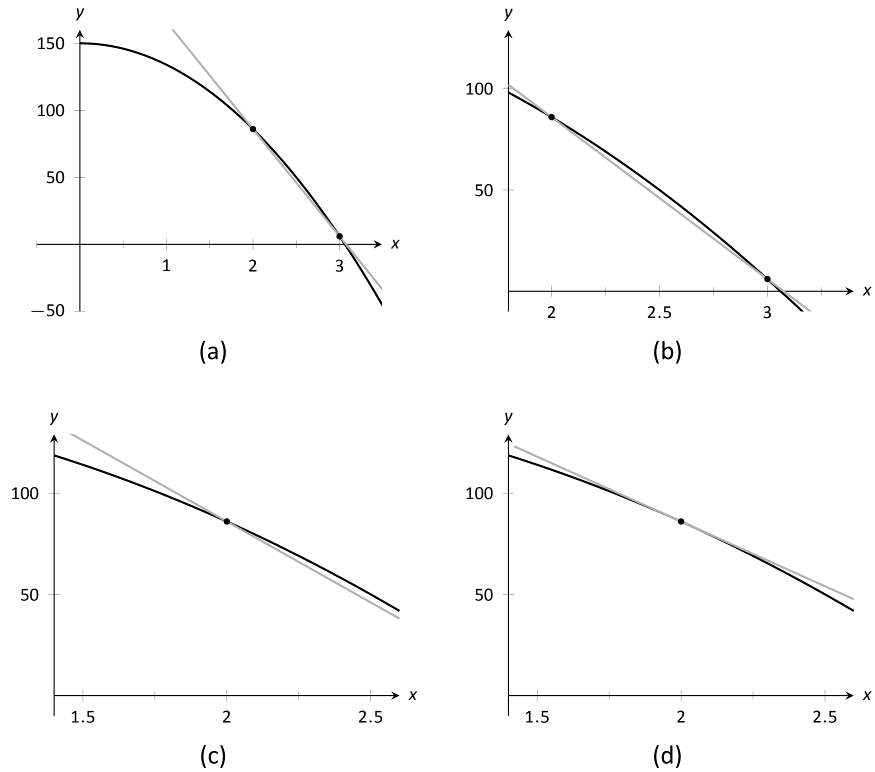


Figure 6.1.1: Parts (a), (b) and (c) show the secant line to $f(x)$ with $h = 1$, zoomed in different amounts. Part (d) shows the tangent line to f at $x = 2$.

We have just introduced a number of important concepts that we will flesh out more within this section. First, we formally define two of them.

Definition 6.1.1 Derivative at a Point

Let f be a continuous function on an open interval I and let c be in I. The **derivative of f at** c, denoted $f'(c)$, is

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h},$$

provided the limit exists. If the limit exists, we say that f **is differentiable at** c; if the limit does not exist, then f **is not differentiable at** c. If f is differentiable at every point in I, then f **is differentiable on** I.

Definition 6.1.2 Tangent Line

Let f be continuous on an open interval I and differentiable at c, for some c in I. The line with equation $\ell(x) = f'(c)(x-c)+f(c)$ is the **tangent line** to the graph of f at c; that is, it is the line through $(c, f(c))$ whose slope is the derivative of f at c.

Some examples will help us understand these definitions.

Example 6.1.1 Finding derivatives and tangent lines
Let $f(x) = 3x^2 + 5x - 7$. Find:

1. $f'(1)$
2. The equation of the tangent line to the graph of f at $x = 1$.
3. $f'(3)$
4. The equation of the tangent line to the graph f at $x = 3$.

SOLUTION

1. We compute this directly using Definition 6.1.1.

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0} \frac{3(1+h)^2+5(1+h)-7-(3(1)^2+5(1)-7)}{h} \\ &= \lim_{h\to 0} \frac{3h^2+11h}{h} \\ &= \lim_{h\to 0}(3h+11) = 11. \end{aligned}$$

2. The tangent line at $x = 1$ has slope $f'(1)$ and goes through the point $(1, f(1)) = (1, 1)$. Thus the tangent line has equation, in point-slope form, $y = 11(x-1)+1$. In slope-intercept form we have $y = 11x - 10$.

3. Again, using the definition,

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \\ &= \lim_{h\to 0} \frac{3(3+h)^2+5(3+h)-7-(3(3)^2+5(3)-7)}{h} \\ &= \lim_{h\to 0} \frac{3h^2+23h}{h} \\ &= \lim_{h\to 0}(3h+23) \\ &= 23. \end{aligned}$$

4. The tangent line at $x = 3$ has slope 23 and goes through the point $(3, f(3)) = (3, 35)$. Thus the tangent line has equation $y = 23(x-3)+35 = 23x-34$.

A graph of f is given in Figure 6.1.3 along with the tangent lines at $x = 1$ and $x = 3$.

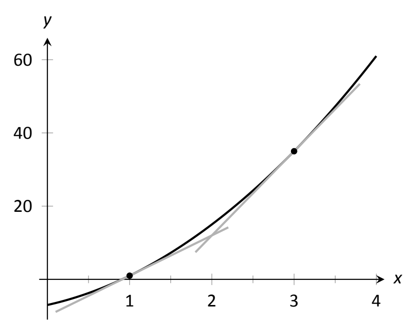


Figure 6.1.3: A graph of $f(x) = 3x^2+5x-7$ and its tangent lines at $x = 1$ and $x = 3$.

Another important line that can be created using information from the derivative is the **normal line.** It is perpendicular to the tangent line, hence its slope is the opposite–reciprocal of the tangent line's slope.

Definition 6.1.3 Normal Line

Let f be continuous on an open interval I and differentiable at c, for some c in I. The **normal line** to the graph of f at c is the line with equation

$$n(x) = \frac{-1}{f'(c)}(x - c) + f(c),$$

where $f'(c) \neq 0$. When $f'(c) = 0$, the normal line is the vertical line through $\big(c, f(c)\big)$; that is, $x = c$.

Example 6.1.2 Finding equations of normal lines

Let $f(x) = 3x^2 + 5x - 7$, as in Example 6.1.1. Find the equations of the normal lines to the graph of f at $x = 1$ and $x = 3$.

Solution In Example 6.1.1, we found that $f'(1) = 11$. Hence at $x = 1$, the normal line will have slope $-1/11$. An equation for the normal line is

$$n(x) = \frac{-1}{11}(x - 1) + 1.$$

The normal line is plotted with $y = f(x)$ in Figure 6.1.4. Note how the line looks perpendicular to f. (A key word here is "looks." Mathematically, we say that the normal line *is* perpendicular to f at $x = 1$ as the slope of the normal line is the opposite–reciprocal of the slope of the tangent line. However, normal lines may not always *look* perpendicular. The aspect ratio of the picture of the graph plays a big role in this.)

We also found that $f'(3) = 23$, so the normal line to the graph of f at $x = 3$ will have slope $-1/23$. An equation for the normal line is

$$n(x) = \frac{-1}{23}(x - 3) + 35.$$

Figure 6.1.4: A graph of $f(x) = 3x^2 + 5x - 7$, along with its normal line at $x = 1$.

Linear functions are easy to work with; many functions that arise in the course of solving real problems are not easy to work with. A common practice in mathematical problem solving is to approximate difficult functions with not–so–difficult functions. Lines are a common choice. It turns out that at any given point on the graph of a differentiable function f, the best linear approximation to f is its tangent line. That is one reason we'll spend considerable time finding tangent lines to functions.

One type of function that does not benefit from a tangent–line approximation is a line; it is rather simple to recognize that the tangent line to a line is the line itself. We look at this in the following example.

Example 6.1.3 Finding the derivative of a linear function

Consider $f(x) = 3x + 5$. Find the equation of the tangent line to f at $x = 1$ and $x = 7$.

Solution We find the slope of the tangent line by using Definition 6.1.1.

$$
\begin{aligned}
f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}\\
&= \lim_{h\to 0}\frac{3(1+h)+5-(3+5)}{h}\\
&= \lim_{h\to 0}\frac{3h}{h}\\
&= \lim_{h\to 0} 3\\
&= 3.
\end{aligned}
$$

We just found that $f'(1) = 3$. That is, we found the *instantaneous rate of change* of $f(x) = 3x + 5$ is 3. This is not surprising; lines are characterized by being the *only* functions with a *constant rate of change.* That rate of change is called the *slope* of the line. Since their rates of change are constant, their *instantaneous* rates of change are always the same; they are all the slope.

So given a line $f(x) = ax + b$, the derivative at any point x will be a; that is, $f'(x) = a$.

It is now easy to see that the tangent line to the graph of f at $x = 1$ is just f, with the same being true for $x = 7$.

We often desire to find the tangent line to the graph of a function without knowing the actual derivative of the function. In these cases, the best we may be able to do is approximate the tangent line. We demonstrate this in the next example.

Example 6.1.4 Numerical approximation of the tangent line

Approximate the equation of the tangent line to the graph of $f(x) = \sin x$ at $x = 0$.

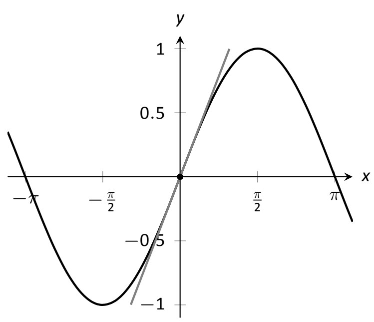


Figure 6.1.5: $f(x) = \sin x$ graphed with an approximation to its tangent line at $x = 0$.

Solution In order to find the equation of the tangent line, we need a slope and a point. The point is given to us: $(0, \sin 0) = (0, 0)$. To compute the slope, we need the derivative. This is where we will make an approximation. Recall that

$$f'(0) \approx \frac{\sin(0+h)-\sin 0}{h}$$

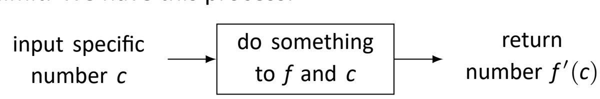
for a small value of h. We choose (somewhat arbitrarily) to let $h = 0.1$. Thus

$$f'(0) \approx \frac{\sin(0.1)-\sin 0}{0.1} \approx 0.9983.$$

Thus our approximation of the equation of the tangent line is $y = 0.9983(x - 0) + 0 = 0.9983x$; it is graphed in Figure 6.1.5. The graph seems to imply the approximation is rather good.

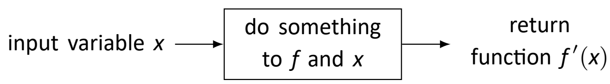
Recall from Section 5.2 that $\lim_{x\to 0}\frac{\sin x}{x} = 1$, meaning for values of x near 0, $\sin x \approx x$. Since the slope of the line $y = x$ is 1 at $x = 0$, it should seem reasonable that "the slope of $f(x) = \sin x$" is near 1 at $x = 0$. In fact, since we *approximated* the value of the slope to be 0.9983, we might guess the *actual value* is 1. We'll come back to this later.

Consider again Example 6.1.1. To find the derivative of f at $x = 1$, we needed to evaluate a limit. To find the derivative of f at $x = 3$, we needed to again evaluate a limit. We have this process:



This process describes a *function*; given one input (the value of c), we return exactly one output (the value of $f'(c)$). The "do something" box is where the tedious work (taking limits) of this function occurs.

Instead of applying this function repeatedly for different values of c, let us apply it just once to the variable x. We then take a limit just once. The process now looks like:



The output is the "derivative function," $f'(x)$. The $f'(x)$ function will take a number c as input and return the derivative of f at c. This calls for a definition.

Definition 6.1.4 Derivative Function

Let f be a differentiable function on an open interval I. The function

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

is **the derivative of** f.

Notation:
Let $y = f(x)$. The following notations all represent the derivative of f:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}(f) = \frac{d}{dx}(y).$$

Important: The notation $\frac{dy}{dx}$ is one symbol; it is **not** the fraction "dy/dx". The notation, while somewhat confusing at first, was chosen with care. A fraction–looking symbol was chosen because the derivative has many fraction–like properties. Among other places, we see these properties at work when we talk about the units of the derivative, when we discuss the Chain Rule, and when we learn about integration (topics that appear in later sections and chapters).

Examples will help us understand this definition.

Example 6.1.5 Finding the derivative of a function
Let $f(x) = 3x^2+5x-7$ as in Example 6.1.1. Find $f'(x)$.

Solution We apply Definition 6.1.4.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{3(x+h)^2+5(x+h)-7-(3x^2+5x-7)}{h} \\ &= \lim_{h\to 0}\frac{3h^2+6xh+5h}{h} \\ &= \lim_{h\to 0}(3h+6x+5) \\ &= 6x+5 \end{aligned}$$

So $f'(x) = 6x+5$. Recall earlier we found that $f'(1) = 11$ and $f'(3) = 23$. Note our new computation of $f'(x)$ affirm these facts.

Example 6.1.6 Finding the derivative of a function

Let $f(x) = \dfrac{1}{x+1}$. Find $f'(x)$.

Solution We apply Definition 6.1.4.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} \end{aligned}$$

Now find common denominator then subtract; pull $1/h$ out front to facilitate reading.

$$\begin{aligned} &= \lim_{h\to 0} \frac{1}{h}\cdot\left(\frac{x+1}{(x+1)(x+h+1)} - \frac{x+h+1}{(x+1)(x+h+1)}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\cdot\left(\frac{x+1-(x+h+1)}{(x+1)(x+h+1)}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\cdot\left(\frac{-h}{(x+1)(x+h+1)}\right) \\ &= \lim_{h\to 0} \frac{-1}{(x+1)(x+h+1)} \\ &= \frac{-1}{(x+1)(x+1)} \\ &= \frac{-1}{(x+1)^2}. \end{aligned}$$

So $f'(x) = \dfrac{-1}{(x+1)^2}$. To practice using our notation, we could also state

$$\frac{d}{dx}\left(\frac{1}{x+1}\right) = \frac{-1}{(x+1)^2}.$$

Example 6.1.7 Finding the derivative of a function

Find the derivative of $f(x) = \sin x$.

Solution Before applying Definition 6.1.4, note that once this is found, we can find the actual tangent line to $f(x) = \sin x$ at $x = 0$, whereas we settled for an approximation in Example 6.1.4.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} && \left(\begin{array}{c}\text{Use trig identity} \\ \sin(x+h) = \sin x\cos h + \cos x \sin h\end{array}\right) \\ &= \lim_{h\to 0} \frac{\sin x\cos h + \cos x\sin h - \sin x}{h} && \text{(regroup)} \\ &= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} && \text{(split into two fractions)} \\ &= \lim_{h\to 0} \left(\frac{\sin x(\cos h-1)}{h} + \frac{\cos x\sin h}{h}\right) && \left(\text{use } \lim_{h\to 0}\frac{\cos h - 1}{h} = 0 \text{ and } \lim_{h\to 0}\frac{\sin h}{h} = 1\right) \\ &= \sin x\cdot 0 + \cos x \cdot 1 \\ &= \cos x\,! \end{aligned}$$

We have found that when $f(x) = \sin x$, $f'(x) = \cos x$. This should be somewhat surprising; the result of a tedious limit process and the sine function is a nice function. Then again, perhaps this is not entirely surprising. The sine function is periodic – it repeats itself on regular intervals. Therefore its rate of change also repeats itself on the same regular intervals. We should have known the derivative would be periodic; we now know exactly which periodic function it is.

Thinking back to Example 6.1.4, we can find the slope of the tangent line to $f(x) = \sin x$ at $x = 0$ using our derivative. We approximated the slope as 0.9983; we now know the slope is *exactly* $\cos 0 = 1$.

Example 6.1.8 Finding the derivative of a piecewise defined function

Find the derivative of the absolute value function,

$$f(x) = |x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}.$$

See Figure 6.1.6.

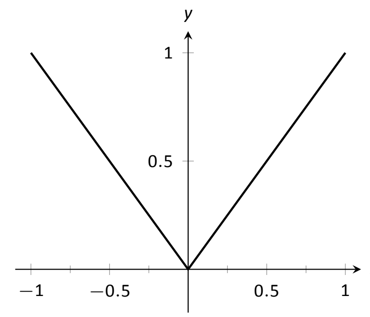


Figure 6.1.6: The absolute value function, $f(x) = |x|$. Notice how the slope of the lines (and hence the tangent lines) abruptly changes at $x = 0$.

Solution We need to evaluate $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$. As f is piecewise–defined, we need to consider separately the limits when $x < 0$ and when $x > 0$.

When $x < 0$:

$$\begin{aligned} \frac{d}{dx}\big(-x\big) &= \lim_{h\to 0} \frac{-(x+h)-(-x)}{h} \\ &= \lim_{h\to 0} \frac{-h}{h} \\ &= \lim_{h\to 0} -1 \\ &= -1. \end{aligned}$$

When $x > 0$, a similar computation shows that $\dfrac{d}{dx}\big(x\big) = 1$.

We need to also find the derivative at $x = 0$. By the definition of the derivative at a point, we have

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}.$$

Since $x = 0$ is the point where our function's definition switches from one piece to other, we need to consider left and right-hand limits. Consider the following, where we compute the left and right hand limits side by side.

$\lim_{h\to 0^-} \dfrac{f(0+h)-f(0)}{h} =$	$\lim_{h\to 0^+} \dfrac{f(0+h)-f(0)}{h} =$
$\lim_{h\to 0^-} \dfrac{-h-0}{h} =$	$\lim_{h\to 0^+} \dfrac{h-0}{h} =$
$\lim_{h\to 0^-} -1 = -1$	$\lim_{h\to 0^+} 1 = 1$

The last lines of each column tell the story: the left and right hand limits are not equal. Therefore the limit does not exist at 0, and f is not differentiable at 0. So we have

$$f'(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}.$$

At $x = 0$, $f'(x)$ does not exist; there is a jump discontinuity at 0; see Figure 6.1.7. So $f(x) = |x|$ is differentiable everywhere except at 0.

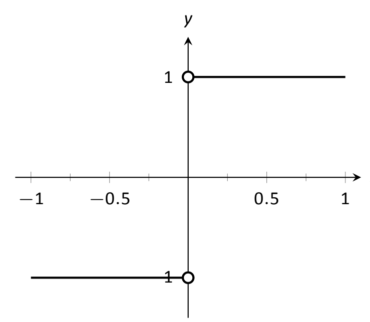


Figure 6.1.7: A graph of the derivative of $f(x) = |x|$.

The point of non-differentiability came where the piecewise defined function switched from one piece to the other. Our next example shows that this does not always cause trouble.

Example 6.1.9 Finding the derivative of a piecewise defined function

Find the derivative of $f(x)$, where $f(x) = \begin{cases} \sin x & x \leq \pi/2 \\ 1 & x > \pi/2 \end{cases}$. See Figure 6.1.8.

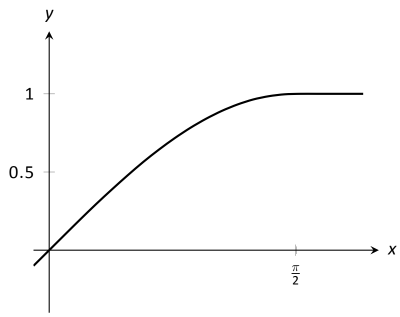


Figure 6.1.8: A graph of $f(x)$ as defined in Example 6.1.9.

Solution Using Example 6.1.7, we know that when $x < \pi/2$, $f'(x) = \cos x$. It is easy to verify that when $x > \pi/2$, $f'(x) = 0$; consider:

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1-1}{h} = \lim_{h\to 0} 0 = 0.$$

So far we have

$$f'(x) = \begin{cases} \cos x & x < \pi/2 \\ 0 & x > \pi/2 \end{cases}.$$

We still need to find $f'(\pi/2)$. Notice at $x = \pi/2$ that both pieces of f' are 0, meaning we can state that $f'(\pi/2) = 0$.

Being more rigorous, we can again evaluate the difference quotient limit at $x = \pi/2$, utilizing again left and right–hand limits:

$$\begin{aligned} \lim_{h\to 0^-} \frac{f(\pi/2+h) - f(\pi/2)}{h} &= \\ \lim_{h\to 0^-} \frac{\sin(\pi/2+h) - \sin(\pi/2)}{h} &= \\ \lim_{h\to 0^-} \frac{\sin(\frac{\pi}{2})\cos(h) + \sin(h)\cos(\frac{\pi}{2}) - \sin(\frac{\pi}{2})}{h} &= \\ \lim_{h\to 0^-} \frac{1\cdot\cos(h) + \sin(h)\cdot 0 - 1}{h} &= \\ 0. \end{aligned}$$

$$\begin{aligned} \lim_{h\to 0^+} \frac{f(\pi/2+h) - f(\pi/2)}{h} &= \\ \lim_{h\to 0^+} \frac{1-1}{h} &= \\ \lim_{h\to 0^+} \frac{0}{h} &= \\ 0. \end{aligned}$$

Since both the left and right hand limits are 0 at $x = \pi/2$, the limit exists and $f'(\pi/2)$ exists (and is 0). Therefore we can fully write f' as

$$f'(x) = \begin{cases} \cos x & x \leq \pi/2 \\ 0 & x > \pi/2 \end{cases}.$$

See Figure 6.1.9 for a graph of this function.

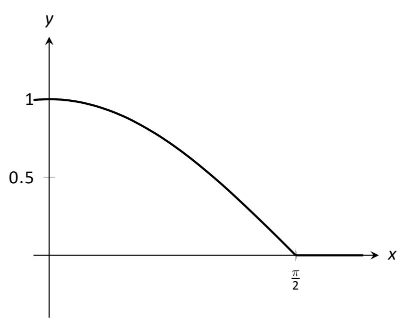


Figure 6.1.9: A graph of $f'(x)$ in Example 6.1.9.

Recall we pseudo–defined a continuous function as one in which we could sketch its graph without lifting our pencil. We can give a pseudo–definition for differentiability as well: it is a continuous function that does not have any "sharp corners." One such sharp corner is shown in Figure 6.1.6. Even though the function f in Example 6.1.9 is piecewise–defined, the transition is "smooth" hence it is differentiable. Note how in the graph of f in Figure 6.1.8 it is difficult to tell when f switches from one piece to the other; there is no "corner."

This section defined the derivative; in some sense, it answers the question of "What *is* the derivative?" The next section addresses the question "What does the derivative *mean*?"

Exercises 6.1

Terms and Concepts

1. T/F: Let f be a position function. The average rate of change on $[a, b]$ is the slope of the line through the points $(a, f(a))$ and $(b, f(b))$.

2. T/F: The definition of the derivative of a function at a point involves taking a limit.

3. In your own words, explain the difference between the average rate of change and instantaneous rate of change.

4. In your own words, explain the difference between Definitions 6.1.1 and 6.1.4.

5. Let $y = f(x)$. Give three different notations equivalent to "$f'(x)$."

6. If two lines are perpendicular, what is true of their slopes?

Problems

In Exercises 7 – 14, use the definition of the derivative to compute the derivative of the given function.

7. $f(x) = 6$

8. $f(x) = 2x$

9. $f(t) = 4 - 3t$

10. $g(x) = x^2$

11. $h(x) = x^3$

12. $f(x) = 3x^2 - x + 4$

13. $r(x) = \dfrac{1}{x}$

14. $r(s) = \dfrac{1}{s - 2}$

In Exercises 15 – 22, a function and an x–value c are given. (Note: these functions are the same as those given in Exercises 7 through 14.)

(a) Give the equation of the tangent line at $x = c$.

(b) Give the equation of the normal line at $x = c$.

15. $f(x) = 6$, at $x = -2$.

16. $f(x) = 2x$, at $x = 3$.

17. $f(x) = 4 - 3x$, at $x = 7$.

18. $g(x) = x^2$, at $x = 2$.

19. $h(x) = x^3$, at $x = 4$.

20. $f(x) = 3x^2 - x + 4$, at $x = -1$.

21. $r(x) = \dfrac{1}{x}$, at $x = -2$.

22. $r(x) = \dfrac{1}{x - 2}$, at $x = 3$.

In Exercises 23 – 26, a function f and an x–value a are given. Approximate the equation of the tangent line to the graph of f at $x = a$ by numerically approximating $f'(a)$, using $h = 0.1$.

23. $f(x) = x^2 + 2x + 1$, $x = 3$

24. $f(x) = \dfrac{10}{x + 1}$, $x = 9$

25. $f(x) = e^x$, $x = 2$

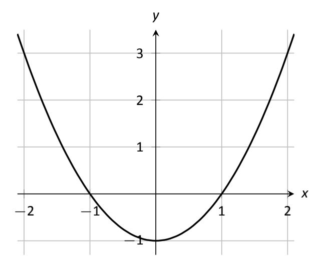
26. $f(x) = \cos x$, $x = 0$

27. The graph of $f(x) = x^2 - 1$ is shown.

    (a) Use the graph to approximate the slope of the tangent line to f at the following points: $(-1, 0)$, $(0, -1)$ and $(2, 3)$.

    (b) Using the definition, find $f'(x)$.

    (c) Find the slope of the tangent line at the points $(-1, 0)$, $(0, -1)$ and $(2, 3)$.

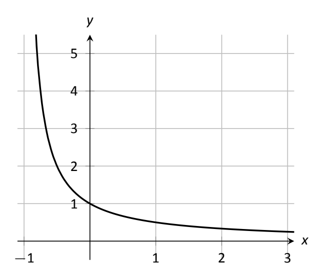


28. The graph of $f(x) = \dfrac{1}{x + 1}$ is shown.

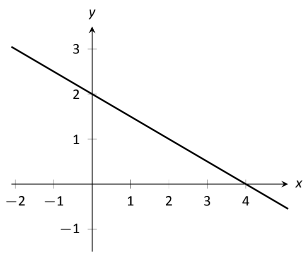
    (a) Use the graph to approximate the slope of the tangent line to f at the following points: $(0, 1)$ and $(1, 0.5)$.

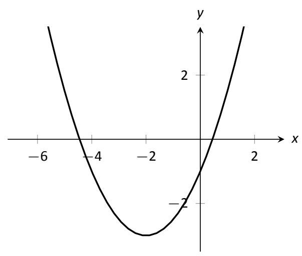
    (b) Using the definition, find $f'(x)$.

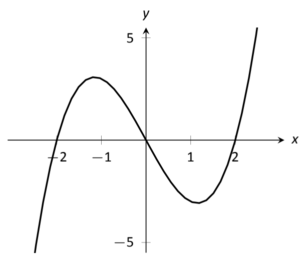
    (c) Find the slope of the tangent line at the points $(0, 1)$ and $(1, 0.5)$.

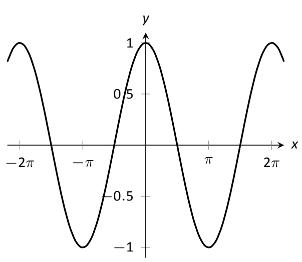


In Exercises 29 – 32, a graph of a function $f(x)$ is given. Using the graph, sketch $f'(x)$.

29. 

30. 

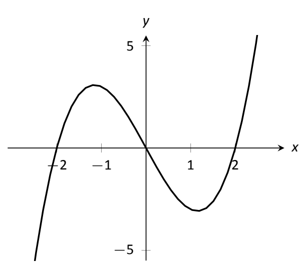
31. 

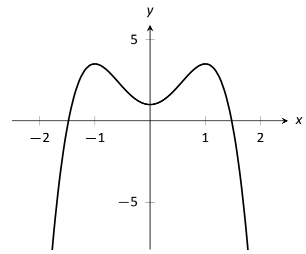
32. 

In Exercises 33 – 34, a graph of a function $g(x)$ is given. Using the graph, answer the following questions.

1. **Where is $g(x) > 0$?**
2. **Where is $g(x) < 0$?**
3. **Where is $g(x) = 0$?**

1. **Where is $g'(x) < 0$?**
2. **Where is $g'(x) > 0$?**
3. **Where is $g'(x) = 0$?**

33. 

34. 

In Exercises 35 – 36, a function $f(x)$ is given, along with its domain and derivative. Determine if $f(x)$ is differentiable on its domain.

35. $f(x) = \sqrt{x^5(1-x)}$, domain = $[0,1]$, $f'(x) = \dfrac{(5-6x)x^{3/2}}{2\sqrt{1-x}}$

36. $f(x) = \cos\left(\sqrt{x}\right)$, domain = $[0,\infty)$, $f'(x) = -\dfrac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$

Review

37. Approximate $\displaystyle\lim_{x\to 5} \frac{x^2+2x-35}{x^2-10.5x+27.5}$.

38. Use the Bisection Method to approximate, accurate to two decimal places, the root of $g(x) = x^3 + x^2 + x - 1$ on $[0.5, 0.6]$.

39. Give intervals on which each of the following functions are continuous.

(a) $\dfrac{1}{e^x+1}$

(b) $\dfrac{1}{x^2-1}$

(c) $\sqrt{5-x}$

(d) $\sqrt{5-x^2}$

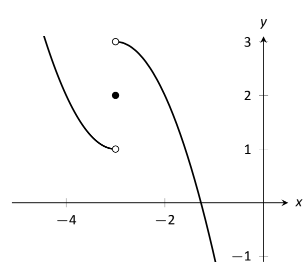
40. Use the graph of $f(x)$ provided to answer the following.

(a) $\displaystyle\lim_{x\to -3^-} f(x) = ?$

(b) $\displaystyle\lim_{x\to -3^+} f(x) = ?$

(c) $\displaystyle\lim_{x\to -3} f(x) = ?$

(d) Where is f continuous?



6.2 Interpretations of the Derivative

The previous section defined the derivative of a function and gave examples of how to compute it using its definition (i.e., using limits). The section also started with a brief motivation for this definition, that is, finding the instantaneous velocity of a falling object given its position function. The next section will give us more accessible tools for computing the derivative, tools that are easier to use than repeated use of limits.

This section falls in between the "What is the definition of the derivative?" and "How do I compute the derivative?" sections. Here we are concerned with "What does the derivative mean?", or perhaps, when read with the right emphasis, "What *is* the derivative?" We offer two interconnected interpretations of the derivative, hopefully explaining why we care about it and why it is worthy of study.

Interpretation of the Derivative #1: Instantaneous Rate of Change

The previous section started with an example of using the position of an object (in this case, a falling amusement–park rider) to find the object's velocity. This type of example is often used when introducing the derivative because we tend to readily recognize that velocity is the *instantaneous rate of change of position.* In general, if f is a function of x, then $f'(x)$ measures the instantaneous rate of change of f with respect to x. Put another way, the derivative answers "When x changes, at what rate does f change?" Thinking back to the amusement–park ride, we asked "When time changed, at what rate did the height change?" and found the answer to be "By -64 feet per second."

Now imagine driving a car and looking at the speedometer, which reads "90 km/h." Five minutes later, you wonder how far you have travelled. Certainly, lots of things could have happened in those 5 minutes; you could have intentionally sped up significantly, you might have come to a complete stop, you might have slowed to 30 km/h as you passed through construction. But suppose that you know, as the driver, none of these things happened. You know you maintained a fairly consistent speed over those 5 minutes. What is a good approximation of the distance travelled?

One could argue the *only* good approximation, given the information provided, would be based on "distance = rate $\times$ time." In this case, we assume a constant rate of 90 km/h with a time of $5/60$ hours. Hence we would approximate the distance travelled as 7.5 km.

Note: The original textbook, having been written in the USA, used primarily imperial units. We considered converting everything to metric, including the amusement park example, but this would have involved a fair amount of work, including replacing several of the diagrams in the previous section. We feel confident that the typical Canadian student is capable of working in either system of measurement.

Referring back to the falling amusement–park ride, knowing that at $t = 2$ the velocity was -64 ft/s, we could reasonably assume that 1 second later the riders' height would have dropped by about 64 feet. Knowing that the riders were *accelerating* as they fell would inform us that this is an *under–approximation*. If all we knew was that $f(2) = 86$ and $f'(2) = -64$, we'd know that we'd have to stop the riders quickly otherwise they would hit the ground!

Units of the Derivative

It is useful to recognize the *units* of the derivative function. If y is a function of x, i.e., $y = f(x)$ for some function f, and y is measured in metres and x in seconds, then the units of $y' = f'$ are "metres per second," commonly written as "m/s." In general, if y is measured in units P and x is measured in units Q, then y' will be measured in units "P per Q", or "P/Q." Here we see the fraction–like behaviour of the derivative in the notation:

$$\text{the units of } \frac{dy}{dx} \text{ are } \frac{\text{units of } y}{\text{units of } x}.$$

Example 6.2.1 The meaning of the derivative: World Population

Let $P(t)$ represent the world population t minutes after 12:00 a.m., January 1, 2012. It is fairly accurate to say that $P(0) = 7,028,734,178$ (`www.prb.org`). It is also fairly accurate to state that $P'(0) = 156$; that is, at midnight on January 1, 2012, the population of the world was growing by about 156 *people per minute* (note the units). Twenty days later (or, 28,800 minutes later) we could reasonably assume the population grew by about $28,800 \cdot 156 = 4,492,800$ people.

Example 6.2.2 The meaning of the derivative: Manufacturing

The term *widget* is an economic term for a generic unit of manufacturing output. Suppose a company produces widgets and knows that the market supports a price of \$10 per widget. Let $P(n)$ give the profit, in dollars, earned by manufacturing and selling n widgets. The company likely cannot make a (positive) profit making just one widget; the start–up costs will likely exceed \$10. Mathematically, we would write this as $P(1) < 0$.

What do $P(1000) = 500$ and $P'(1000) = 0.25$ mean? Approximate $P(1100)$.

Solution The equation $P(1000) = 500$ means that selling 1,000 widgets returns a profit of \$500. We interpret $P'(1000) = 0.25$ as meaning that the profit is increasing at rate of \$0.25 per widget (the units are "dollars per widget.") Since we have no other information to use, our best approximation for $P(1100)$ is:

$$P(1100) \approx P(1000) + P'(1000) \times 100 = \$500 + 100 \cdot 0.25 = \$525.$$

We approximate that selling 1,100 widgets returns a profit of \$525.

The previous examples made use of an important approximation tool that we first used in our previous "driving a car at 60 mph" example at the beginning of this section. Five minutes after looking at the speedometer, our best approximation for distance travelled assumed the rate of change was constant. In Examples 6.2.1 and 6.2.2 we made similar approximations. We were given rate of change information which we used to approximate total change. Notationally, we would say that

$$f(c+h) \approx f(c) + f'(c) \cdot h.$$

This approximation is best when h is "small." "Small" is a relative term; when dealing with the world population, $h = 22$ days = 28,800 minutes is small in comparison to years. When manufacturing widgets, 100 widgets is small when one plans to manufacture thousands.

The Derivative and Motion

One of the most fundamental applications of the derivative is the study of motion. Let $s(t)$ be a position function, where t is time and $s(t)$ is distance. For instance, s could measure the height of a projectile or the distance an object has travelled.

Let's let $s(t)$ measure the distance travelled, in feet, of an object after t seconds of travel. Then $s'(t)$ has units "feet per second," and $s'(t)$ measures the *instantaneous rate of distance change* – it measures **velocity**.

Now consider $v(t)$, a velocity function. That is, at time t, $v(t)$ gives the velocity of an object. The derivative of v, $v'(t)$, gives the *instantaneous rate of* *velocity change* – **acceleration**. (We often think of acceleration in terms of cars: a car may "go from 0 to 60 in 4.8 seconds." This is an *average* acceleration, a measurement of how quickly the velocity changed.) If velocity is measured in feet per second, and time is measured in seconds, then the units of acceleration (i.e., the units of $v'(t)$) are "feet per second per second," or (ft/s)/s. We often shorten this to "feet per second squared," or ft/s^2, but this tends to obscure the meaning of the units.

Perhaps the most well known acceleration is that of gravity. In this text, we use $g = 32$ ft/s^2 or $g = 9.8$ m/s^2. What do these numbers mean?

A constant acceleration of 32(ft/s)/s means that the velocity changes by 32 ft/s each second. For instance, let $v(t)$ measures the velocity of a ball thrown straight up into the air, where v has units ft/s and t is measured in seconds. The ball will have a positive velocity while travelling upwards and a negative velocity while falling down. The acceleration is thus -32 ft/s^2. If $v(1) = 20$ ft/s, then when $t = 2$, the velocity will have decreased by 32 ft/s; that is, $v(2) = -12$ ft/s. We can continue: $v(3) = -44$ ft/s, and we can also figure that $v(0) = 42$ ft/s.

These ideas are so important we write them out as a Key Idea.

Key Idea 6.2.1 The Derivative and Motion

1. Let $s(t)$ be the position function of an object. Then $s'(t)$ is the velocity function of the object.
2. Let $v(t)$ be the velocity function of an object. Then $v'(t)$ is the acceleration function of the object.

We now consider the second interpretation of the derivative given in this section. This interpretation is not independent from the first by any means; many of the same concepts will be stressed, just from a slightly different perspective.

Interpretation of the Derivative #2: The Slope of the Tangent Line

Given a function $y = f(x)$, the difference quotient $\dfrac{f(c+h) - f(c)}{h}$ gives a change in y values divided by a change in x values; i.e., it is a measure of the "rise over run," or "slope," of the line that goes through two points on the graph of f: $\big(c, f(c)\big)$ and $\big(c+h, f(c+h)\big)$. As h shrinks to 0, these two points come close together; in the limit we find $f'(c)$, the slope of a special line called the tangent line that intersects f only once near $x = c$.

Lines have a constant rate of change, their slope. Nonlinear functions do not have a constant rate of change, but we can measure their *instantaneous rate of change* at a given x value c by computing $f'(c)$. We can get an idea of how f is behaving by looking at the slopes of its tangent lines. We explore this idea in the following example.

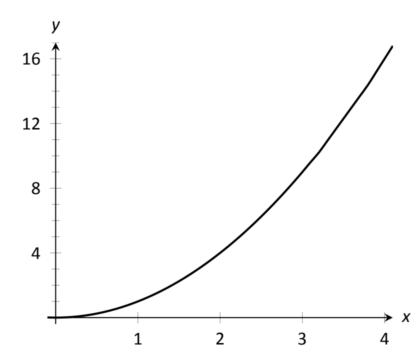


Figure 6.2.1: A graph of $f(x) = x^2$.

Example 6.2.3 Understanding the derivative: the rate of change
Consider $f(x) = x^2$ as shown in Figure 6.2.1. It is clear that at $x = 3$ the function is growing faster than at $x = 1$, as it is steeper at $x = 3$. How much faster is it growing?

Solution We can answer this directly after the following section, where

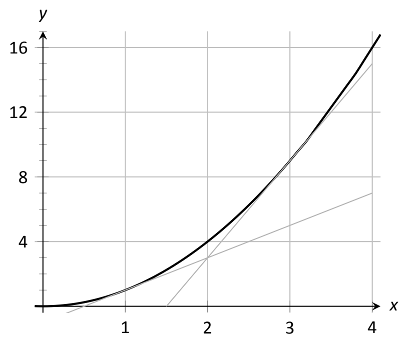
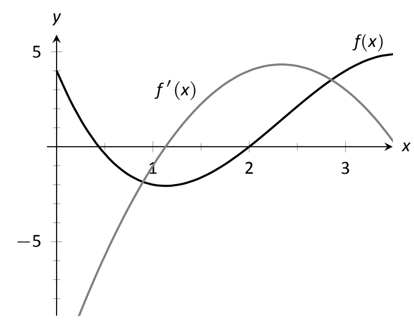
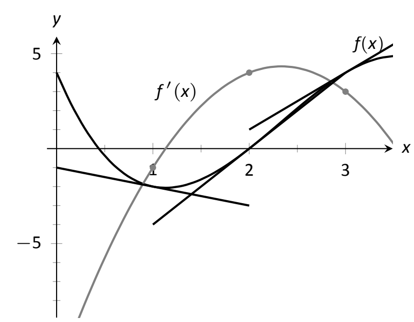


Figure 6.2.2: A graph of $f(x) = x^2$ and tangent lines.



(a)



(b)

Figure 6.2.3: Graphs of f and f' in Example 6.2.4, along with tangent lines in (b).

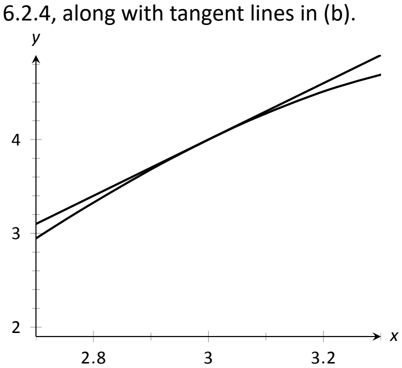


Figure 6.2.4: Zooming in on f and its tangent line at $x = 3$ for the function given in Examples 6.2.4 and 6.2.5.

we learn to quickly compute derivatives. For now, we will answer graphically, by considering the slopes of the respective tangent lines.

With practice, one can fairly effectively sketch tangent lines to a curve at a particular point. In Figure 6.2.2, we have sketched the tangent lines to f at $x = 1$ and $x = 3$, along with a grid to help us measure the slopes of these lines. At $x = 1$, the slope is 2; at $x = 3$, the slope is 6. Thus we can say not only is f growing faster at $x = 3$ than at $x = 1$, it is growing *three times as fast*.

Example 6.2.4 Understanding the graph of the derivative

Consider the graph of $f(x)$ and its derivative, $f'(x)$, in Figure 6.2.3(a). Use these graphs to find the slopes of the tangent lines to the graph of f at $x = 1$, $x = 2$, and $x = 3$.

Solution To find the appropriate slopes of tangent lines to the graph of f, we need to look at the corresponding values of f'.

The slope of the tangent line to f at $x = 1$ is $f'(1)$; this looks to be about -1.

The slope of the tangent line to f at $x = 2$ is $f'(2)$; this looks to be about 4.

The slope of the tangent line to f at $x = 3$ is $f'(3)$; this looks to be about 3.

Using these slopes, the tangent lines to f are sketched in Figure 6.2.3(b). Included on the graph of f' in this figure are filled circles where $x = 1$, $x = 2$ and $x = 3$ to help better visualize the y value of f' at those points.

Example 6.2.5 Approximation with the derivative

Consider again the graph of $f(x)$ and its derivative $f'(x)$ in Example 6.2.4. Use the tangent line to f at $x = 3$ to approximate the value of $f(3.1)$.

Solution Figure 6.2.4 shows the graph of f along with its tangent line, zoomed in at $x = 3$. Notice that near $x = 3$, the tangent line makes an excellent approximation of f. Since lines are easy to deal with, often it works well to approximate a function with its tangent line. (This is especially true when you don't actually know much about the function at hand, as we don't in this example.)

While the tangent line to f was drawn in Example 6.2.4, it was not explicitly computed. Recall that the tangent line to f at $x = c$ is $y = f'(c)(x - c) + f(c)$. While f is not explicitly given, by the graph it looks like $f(3) = 4$. Recalling that $f'(3) = 3$, we can compute the tangent line to be approximately $y = 3(x-3)+4$. It is often useful to leave the tangent line in point–slope form.

To use the tangent line to approximate $f(3.1)$, we simply evaluate y at 3.1 instead of f.

$$f(3.1) \approx y(3.1) = 3(3.1 - 3) + 4 = .1 * 3 + 4 = 4.3.$$

We approximate $f(3.1) \approx 4.3$.

To demonstrate the accuracy of the tangent line approximation, we now state that in Example 6.2.5, $f(x) = -x^3 + 7x^2 - 12x + 4$. We can evaluate $f(3.1) = 4.279$. Had we known f all along, certainly we could have just made this computation. In reality, we often only know two things:

1. what $f(c)$ is, for some value of c, and
2. what $f'(c)$ is.

For instance, we can easily observe the location of an object and its instantaneous velocity at a particular point in time. We do not have a "function f" for the location, just an observation. This is enough to create an approximating function for f.

This last example has a direct connection to our approximation method explained above after Example 6.2.2. We stated there that

$$f(c+h) \approx f(c) + f'(c) \cdot h.$$

If we know $f(c)$ and $f'(c)$ for some value $x = c$, then computing the tangent line at $(c, f(c))$ is easy: $y(x) = f'(c)(x-c) + f(c)$. In Example 6.2.5, we used the tangent line to approximate a value of f. Let's use the tangent line at $x = c$ to approximate a value of f near $x = c$; i.e., compute $y(c+h)$ to approximate $f(c+h)$, assuming again that h is "small." Note:

$$y(c+h) = f'(c)\big((c+h) - c\big) + f(c) = f'(c) \cdot h + f(c).$$

This is the exact same approximation method used above! Not only does it make intuitive sense, as explained above, it makes analytical sense, as this approximation method is simply using a tangent line to approximate a function's value.

The importance of understanding the derivative cannot be understated. When f is a function of x, $f'(x)$ measures the instantaneous rate of change of f with respect to x and gives the slope of the tangent line to f at x.

Exercises 6.2

Terms and Concepts

1. What is the instantaneous rate of change of position called?

2. Given a function $y = f(x)$, in your own words describe how to find the units of $f'(x)$.

3. What functions have a constant rate of change?

Problems

4. Given $f(5) = 10$ and $f'(5) = 2$, approximate $f(6)$.

5. Given $P(100) = -67$ and $P'(100) = 5$, approximate $P(110)$.

6. Given $z(25) = 187$ and $z'(25) = 17$, approximate $z(20)$.

7. Knowing $f(10) = 25$ and $f'(10) = 5$ and the methods described in this section, which approximation is likely to be most accurate: $f(10.1)$, $f(11)$, or $f(20)$? Explain your reasoning.

8. Given $f(7) = 26$ and $f(8) = 22$, approximate $f'(7)$.

9. Given $H(0) = 17$ and $H(2) = 29$, approximate $H'(2)$.

10. Let $V(x)$ measure the volume, in decibels, measured inside a restaurant with x customers. What are the units of $V'(x)$?

11. Let $v(t)$ measure the velocity, in ft/s, of a car moving in a straight line t seconds after starting. What are the units of $v'(t)$?

12. The height H, in feet, of a river is recorded t hours after midnight, April 1. What are the units of $H'(t)$?

13. P is the profit, in thousands of dollars, of producing and selling c cars.

    (a) What are the units of $P'(c)$?

    (b) What is likely true of $P(0)$?

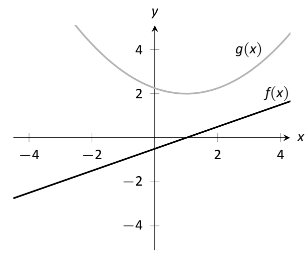
14. T is the temperature in degrees Fahrenheit, h hours after midnight on July 4 in Sidney, NE.

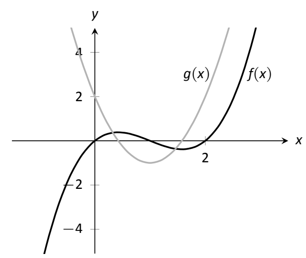
    (a) What are the units of $T'(h)$?

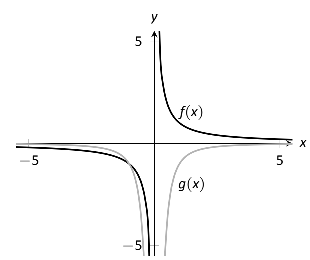
    (b) Is $T'(8)$ likely greater than or less than 0? Why?

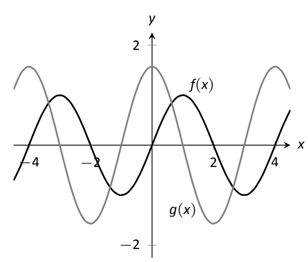
    (c) Is $T(8)$ likely greater than or less than 0? Why?

In Exercises 15 – 18, graphs of functions $f(x)$ and $g(x)$ are given. Identify which function is the derivative of the other.

15. 

16. 

17. 

18. 

Review

In Exercises 19 – 20, use the definition to compute the derivatives of the following functions.

19. $f(x) = 5x^2$

20. $f(x) = (x-2)^3$

In Exercises 21 – 22, numerically approximate the value of $f'(x)$ at the indicated x value.

21. $f(x) = \cos x$ at $x = \pi$.

22. $f(x) = \sqrt{x}$ at $x = 9$.

6.3 Basic Differentiation Rules

The derivative is a powerful tool but is admittedly awkward given its reliance on limits. Fortunately, one thing mathematicians are good at is *abstraction.* For instance, instead of continually finding derivatives at a point, we abstracted and found the derivative function.

Let's practice abstraction on linear functions, $y = mx + b$. What is y'? Without limits, recognize that linear function are characterized by being functions with a constant rate of change (the slope). The derivative, y', gives the instantaneous rate of change; with a linear function, this is constant, m. Thus $y' = m$.

Let's abstract once more. Let's find the derivative of the general quadratic function, $f(x) = ax^2 + bx + c$. Using the definition of the derivative, we have:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{a(x+h)^2 + b(x+h) + c - (ax^2 + bx + c)}{h} \\ &= \lim_{h\to 0} \frac{ah^2 + 2ahx + bh}{h} \\ &= \lim_{h\to 0} (ah + 2ax + b) \\ &= 2ax + b. \end{aligned}$$

So if $y = 6x^2 + 11x - 13$, we can immediately compute $y' = 12x + 11$.

In this section (and in some sections to follow) we will learn some of what mathematicians have already discovered about the derivatives of certain functions and how derivatives interact with arithmetic operations. We start with a theorem.

Theorem 6.3.1 Derivatives of Common Functions

1. **Constant Rule:**
   $\frac{d}{dx}(c) = 0$, where c is a constant.
2. **Power Rule:**
   $\frac{d}{dx}(x^n) = nx^{n-1}$, where n is an integer, $n > 0$.
5. $\frac{d}{dx}(\sin x) = \cos x$
6. $\frac{d}{dx}(\cos x) = -\sin x$
7. $\frac{d}{dx}(e^x) = e^x$
8. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

This theorem starts by stating an intuitive fact: constant functions have no rate of change as they are *constant*. Therefore their derivative is 0 (they change at the rate of 0). The theorem then states some fairly amazing things. The Power Rule states that the derivatives of Power Functions (of the form $y = x^n$) are very straightforward: multiply by the power, then subtract 1 from the power. We see something incredible about the function $y = e^x$: it is its own derivative. We also see a new connection between the sine and cosine functions.

One special case of the Power Rule is when $n = 1$, i.e., when $f(x) = x$. What is $f'(x)$? According to the Power Rule,

$$f'(x) = \frac{d}{dx}(x) = \frac{d}{dx}(x^1) = 1 \cdot x^0 = 1.$$

In words, we are asking "At what rate does f change with respect to x?" Since f *is* x, we are asking "At what rate does x change with respect to x?" The answer is: 1. They change at the same rate.

Let's practice using this theorem.

Example 6.3.1 Using Theorem 6.3.1 to find, and use, derivatives

Let $f(x) = x^3$.

1. Find $f'(x)$.
2. Find the equation of the line tangent to the graph of f at $x = -1$.
3. Use the tangent line to approximate $(-1.1)^3$.
4. Sketch f, f' and the found tangent line on the same axis.

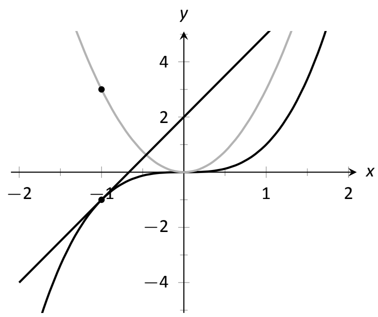


Figure 6.3.1: A graph of $f(x) = x^3$, along with its derivative $f'(x) = 3x^2$ and its tangent line at $x = -1$.

Solution

1. The Power Rule states that if $f(x) = x^3$, then $f'(x) = 3x^2$.
2. To find the equation of the line tangent to the graph of f at $x = -1$, we need a point and the slope. The point is $(-1, f(-1)) = (-1, -1)$. The slope is $f'(-1) = 3$. Thus the tangent line has equation $y = 3(x-(-1)) + (-1) = 3x + 2$.
3. We can use the tangent line to approximate $(-1.1)^3$ as -1.1 is close to -1. We have
$$(-1.1)^3 \approx 3(-1.1) + 2 = -1.3.$$
We can easily find the actual answer; $(-1.1)^3 = -1.331$.
4. See Figure 6.3.1.

It is easy to use Definition 6.1.4 to verify the Constant Rule, and with a bit of work we can confirm the Power Rule for small values of n. But how do we know that the Power Rule holds in general? One way to tackle this problem relies on a famous result from Algebra: the Binomial Theorem.

You may recall from high school that the binomial coefficients are the numbers that appear in Pascal's Triange. If we number the rows of Pascal's triangle beginning from the top at row zero, then the numbers in row n are given by $\binom{n}{k}$, for $k = 0, 1, 2, \ldots, n$.
In particular, note that:

$$\binom{n}{0} = 1, \binom{n}{1} = n, \binom{n}{2} = \frac{n(n-1)}{2}, \ldots, \binom{n}{n-1} = n, \binom{n}{n} = 1.$$

Theorem 6.3.2 Binomial Theorem

For any real numbers a and b, and any positive integer n, we have

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n,$$

where $\binom{n}{k}$ (read, "n choose k") is the binomial coefficient given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdots k}.$$

With Theorem 6.3.2 in hand, we can quickly establish the Power Rule using the definition of the derivative. Given $f(x) = x^n$, where n is a positive integer, we have:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{(x^n + nx^{n-1}h + \cdots + h^n) - x^n}{h} && \text{(Using Theorem 6.3.2)} \\ &= \lim_{h\to 0} \frac{nx^{n-1}h\binom{n}{2}x^{n-2}h^2 + \cdots + h^n}{h} && \text{(Cancelling the } x^n \text{ terms)} \\ &= \lim_{h\to 0} (nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}) && \text{(Dividing by } h\text{)} \\ &= nx^{n-1} && \text{(Setting } h = 0\text{)} \end{aligned}$$

The fact that the derivative of $\sin(x)$ is $\cos(x)$ was established in Example 6.1.7; the fact that the derivative of $\cos(x)$ is $-\sin(x)$ is established similarly, and left as an exercise. We aren't yet in a position to rigorously establish the derivative formulas for e^x and $\ln(x)$, but we can show that it's at least plausible that the exponential function is its own derivative. For $f(x) = e^x$, Definition 6.1.4 tells us:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h} && \text{(Laws of exponents)} \\ &= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} && \text{(Factoring)} \\ &= e^x \lim_{h\to 0} \frac{e^h - 1}{h}. \end{aligned}$$

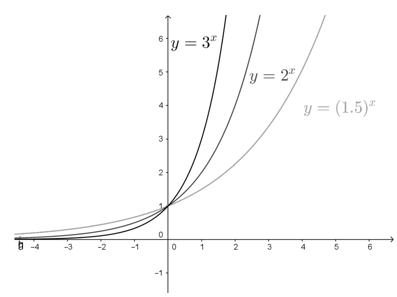


Figure 6.3.2: The graph $y = a^x$, for three values of $a > 1$

It seems we are stuck on this last limit. But notice that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = \lim_{h\to 0} \frac{e^{0+h} - e^0}{h} = f'(0),$$

so $f'(x) = f'(0)e^x$, where $f'(0)$ is simply the slope of the tangent line to the graph $y = e^x$ at $x = 0$. Looking at the graph of $y = a^x$ for several values of $a > 1$, we see that this slope depends on the value of a. One way of defining the number e used as the base of the natural exponential is that this is the value of a such that the slope of the tangent line at $x = 0$ is exactly one; that is, such that $f'(0) = 1$. With this definition, we immediately find that $f'(x) = e^x$, as expected.

The derivative of $\ln(x)$ can be obtained using the Chain Rule (Section 6.5, and the fact that $e^{\ln(x)} = x$. We will state the result here without proof.

Theorem 6.3.1 gives useful information, but we will need much more. For instance, using the theorem, we can easily find the derivative of $y = x^3$, but it does not tell how to compute the derivative of $y = 2x^3$, $y = x^3 + \sin x$ nor $y = x^3 \sin x$. The following theorem helps with the first two of these examples (the third is answered in the next section).

Theorem 6.3.3 Properties of the Derivative

Let f and g be differentiable on an open interval I and let c be a real number. Then:

1. **Sum/Difference Rule:**
$$\frac{d}{dx}\Big(f(x) \pm g(x)\Big) = \frac{d}{dx}\Big(f(x)\Big) \pm \frac{d}{dx}\Big(g(x)\Big) = f'(x) \pm g'(x)$$

2. **Constant Multiple Rule:**
$$\frac{d}{dx}\Big(c \cdot f(x)\Big) = c \cdot \frac{d}{dx}\Big(f(x)\Big) = c \cdot f'(x).$$

Theorem 6.3.3 allows us to find the derivatives of a wide variety of functions. It can be used in conjunction with the Power Rule to find the derivatives of any polynomial. Recall in Example 6.1.5 that we found, using the limit definition, the derivative of $f(x) = 3x^2 + 5x - 7$. We can now find its derivative without expressly using limits:

$$\begin{aligned}\frac{d}{dx}\Big(3x^2 + 5x + 7\Big) &= 3\frac{d}{dx}\Big(x^2\Big) + 5\frac{d}{dx}\Big(x\Big) + \frac{d}{dx}\Big(7\Big)\\ &= 3 \cdot 2x + 5 \cdot 1 + 0\\ &= 6x + 5.\end{aligned}$$

We were a bit pedantic here, showing every step. Normally we would do all the arithmetic and steps in our head and readily find $\frac{d}{dx}\Big(3x^2+5x+7\Big) = 6x+5$.

Both rules in Theorem 6.3.3 are easily established using the definition of the derivative. We will leave the Constant Multiple Rule as an exercise, and demonstrate that the Sum Rule is true. Suppose that we are given two differentiable functions f and g. Recalling how the sum $f + g$ is defined, and using Definition 6.1.4, we have:

$$\begin{aligned}(f+g)'(x) &= \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}\\ &= \lim_{h\to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h}\\ &= \lim_{h\to 0} \frac{(f(x+h) - f(x)) + (g(x+h) - g(x))}{h}\\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\\ &= f'(x) + g'(x).\end{aligned}$$

Example 6.3.2 Using the tangent line to approximate a function value

Let $f(x) = \sin x + 2x + 1$. Approximate $f(3)$ using an appropriate tangent line.

SOLUTION This problem is intentionally ambiguous; we are to *approximate* using an *appropriate* tangent line. How good of an approximation are we seeking? What does appropriate mean?

In the "real world," people solving problems deal with these issues all time. One must make a judgment using whatever seems reasonable. In this example, the actual answer is $f(3) = \sin 3 + 7$, where the real problem spot is $\sin 3$. What is $\sin 3$?

Since 3 is close to π, we can assume $\sin 3 \approx \sin \pi = 0$. Thus one guess is $f(3) \approx 7$. Can we do better? Let's use a tangent line as instructed and examine the results; it seems best to find the tangent line at $x = \pi$.

Using Theorem 6.3.1 we find $f'(x) = \cos x + 2$. The slope of the tangent line is thus $f'(\pi) = \cos \pi + 2 = 1$. Also, $f(\pi) = 2\pi + 1 \approx 7.28$. So the tangent line to the graph of f at $x = \pi$ is $y = 1(x - \pi) + 2\pi + 1 = x + \pi + 1 \approx x + 4.14$. Evaluated at $x = 3$, our tangent line gives $y = 3 + 4.14 = 7.14$. Using the tangent line, our final approximation is that $f(3) \approx 7.14$.

Using a calculator, we get an answer accurate to 4 places after the decimal: $f(3) = 7.1411$. Our initial guess was 7; our tangent line approximation was more accurate, at 7.14.

The point is *not* "Here's a cool way to do some math without a calculator." Sure, that might be handy sometime, but your phone could probably give you the answer. Rather, the point is to say that tangent lines are a good way of approximating, and many scientists, engineers and mathematicians often face problems too hard to solve directly. So they approximate.

Higher Order Derivatives

The derivative of a function f is itself a function, therefore we can take its derivative. The following definition gives a name to this concept and introduces its notation.

Definition 6.3.1 Higher Order Derivatives

Let $y = f(x)$ be a differentiable function on I. The following are defined, provided the corresponding limits exist.

1. The **second derivative of** f is:

$$f''(x) = \frac{d}{dx}\Big(f'(x)\Big) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y''.$$

2. The **third derivative of** f is:

$$f'''(x) = \frac{d}{dx}\Big(f''(x)\Big) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = y'''.$$

3. The **n^{th} derivative of** f is:

$$f^{(n)}(x) = \frac{d}{dx}\left(f^{(n-1)}(x)\right) = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = y^{(n)}.$$

Note: Definition 6.3.1 comes with the caveat "Where the corresponding limits exist." With f differentiable on I, it is possible that f' is *not* differentiable on all of I, and so on.

In general, when finding the fourth derivative and on, we resort to the $f^{(4)}(x)$ notation, not $f''''(x)$; after a while, too many ticks is confusing.

Let's practice using this new concept.

Example 6.3.3 Finding higher order derivatives

Find the first four derivatives of the following functions:

1. $f(x) = 4x^2$
2. $f(x) = \sin x$
3. $f(x) = 5e^x$

Solution

1. Using the Power and Constant Multiple Rules, we have: $f'(x) = 8x$. Continuing on, we have

$$f''(x) = \frac{d}{dx}\big(8x\big) = 8; \qquad f'''(x) = 0; \qquad f^{(4)}(x) = 0.$$

   Notice how all successive derivatives will also be 0.

2. We employ Theorem 6.3.1 repeatedly.

$$f'(x) = \cos x; \qquad f''(x) = -\sin x; \qquad f'''(x) = -\cos x; \qquad f^{(4)}(x) = \sin x.$$

   Note how we have come right back to $f(x)$ again. (Can you quickly figure what $f^{(23)}(x)$ is?)

3. Employing Theorem 6.3.1 and the Constant Multiple Rule, we can see that

$$f'(x) = f''(x) = f'''(x) = f^{(4)}(x) = 5e^x.$$

Interpreting Higher Order Derivatives

What do higher order derivatives *mean*? What is the practical interpretation?

Our first answer is a bit wordy, but is technically correct and beneficial to understand. That is,

> The second derivative of a function f is the rate of change of the rate of change of f.

One way to grasp this concept is to let f describe a position function. Then, as stated in Key Idea 6.2.1, f' describes the rate of position change: velocity. We now consider f'', which describes the rate of velocity change. Sports car enthusiasts talk of how fast a car can go from 0 to 60 mph; they are bragging about the *acceleration* of the car.

We started this chapter with amusement–park riders free–falling with position function $f(t) = -16t^2 + 150$. It is easy to compute $f'(t) = -32t$ ft/s and $f''(t) = -32$ (ft/s)/s. We may recognize this latter constant; it is the acceleration due to gravity. In keeping with the unit notation introduced in the previous section, we say the units are "feet per second per second." This is usually shortened to "feet per second squared," written as "ft/s^2."

It can be difficult to consider the meaning of the third, and higher order, derivatives. The third derivative is "the rate of change of the rate of change of the rate of change of f." That is essentially meaningless to the uninitiated. In the context of our position/velocity/acceleration example, the third derivative is the "rate of change of acceleration," commonly referred to as "jerk."

Make no mistake: higher order derivatives have great importance even if their practical interpretations are hard (or "impossible") to understand. The mathematical topic of *series* makes extensive use of higher order derivatives.

Exercises 6.3

Terms and Concepts

1. What is the name of the rule which states that $\frac{d}{dx}\big(x^n\big) = nx^{n-1}$, where $n > 0$ is an integer?

2. What is $\frac{d}{dx}\big(\ln x\big)$?

3. Give an example of a function $f(x)$ where $f'(x) = f(x)$.

4. Give an example of a function $f(x)$ where $f'(x) = 0$.

5. The derivative rules introduced in this section explain how to compute the derivative of which of the following functions?
   - $f(x) = \dfrac{3}{x^2}$
   - $g(x) = 3x^2 - x + 17$
   - $h(x) = 5\ln x$
   - $j(x) = \sin x \cos x$
   - $k(x) = e^{x^2}$
   - $m(x) = \sqrt{x}$

6. Explain in your own words how to find the third derivative of a function $f(x)$.

7. Give an example of a function where $f'(x) \neq 0$ and $f''(x) = 0$.

8. Explain in your own words what the second derivative "means."

9. If $f(x)$ describes a position function, then $f'(x)$ describes what kind of function? What kind of function is $f''(x)$?

10. Let $f(x)$ be a function measured in pounds, where x is measured in feet. What are the units of $f''(x)$?

Problems

In Exercises 11 – 26, compute the derivative of the given function.

11. $f(x) = 7x^2 - 5x + 7$

12. $g(x) = 14x^3 + 7x^2 + 11x - 29$

13. $m(t) = 9t^5 - \frac{1}{8}t^3 + 3t - 8$

14. $f(\theta) = 9\sin\theta + 10\cos\theta$

15. $f(r) = 6e^r$

16. $g(t) = 10t^4 - \cos t + 7\sin t$

17. $f(x) = 2\ln x - x$

18. $p(s) = \frac{1}{4}s^4 + \frac{1}{3}s^3 + \frac{1}{2}s^2 + s + 1$

19. $h(t) = e^t - \sin t - \cos t$

20. $f(x) = \ln(5x^2)$

21. $f(t) = \ln(17) + e^2 + \sin\pi/2$

22. $g(t) = (1 + 3t)^2$

23. $g(x) = (2x - 5)^3$

24. $f(x) = (1 - x)^3$

25. $f(x) = (2 - 3x)^2$

26. A property of logarithms is that $\log_a x = \dfrac{\log_b x}{\log_b a}$, for all bases $a, b > 0, \neq 1$.
    (a) Rewrite this identity when $b = e$, i.e., using $\log_e x = \ln x$, with $a = 10$.
    (b) Use part (a) to find the derivative of $y = \log_{10} x$.
    (c) Use part (b) to find the derivative of $y = \log_a x$, for any $a > 0, \neq 1$.

In Exercises 27 – 32, compute the first four derivatives of the given function.

27. $f(x) = x^6$

28. $g(x) = 2\cos x$

29. $h(t) = t^2 - e^t$

30. $p(\theta) = \theta^4 - \theta^3$

31. $f(\theta) = \sin\theta - \cos\theta$

32. $f(x) = 1,100$

In Exercises 33 – 38, find the equations of the tangent and normal lines to the graph of the function at the given point.

33. $f(x) = x^3 - x$ at $x = 1$

34. $f(t) = e^t + 3$ at $t = 0$

35. $g(x) = \ln x$ at $x = 1$

36. $f(x) = 4\sin x$ at $x = \pi/2$

37. $f(x) = -2\cos x$ at $x = \pi/4$

38. $f(x) = 2x + 3$ at $x = 5$

Review

39. Given that $e^0 = 1$, approximate the value of $e^{0.1}$ using the tangent line to $f(x) = e^x$ at $x = 0$.

6.4 The Product and Quotient Rules

The previous section showed that, in some ways, derivatives behave nicely. The Constant Multiple and Sum/Difference Rules established that the derivative of $f(x) = 5x^2 + \sin x$ was not complicated. We neglected computing the derivative of things like $g(x) = 5x^2 \sin x$ and $h(x) = \frac{5x^2}{\sin x}$ on purpose; their derivatives are *not* as straightforward. (If you had to guess what their respective derivatives are, you would probably guess wrong.) For these, we need the Product and Quotient Rules, respectively, which are defined in this section.

We begin with the Product Rule.

Important: $\frac{d}{dx}\Big(f(x)g(x)\Big) \neq f'(x)g'(x)$! While this answer is simpler than the Product Rule, it is wrong. If it were true, then we'd have

$$\frac{d}{dx}(x^2) = \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1!$$

In fact, we'd have $\frac{d}{dx}(x^n) = 1$ for every positive integer n, contradicting the Power Rule.

Theorem 6.4.1 Product Rule

Let f and g be differentiable functions on an open interval I. Then fg is a differentiable function on I, and

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

In the Leibniz notation, the Product Rule is written

$$\frac{d}{dx}\Big(f(x)g(x)\Big) = \left(\frac{d}{dx}f(x)\right)g(x) + f(x)\left(\frac{d}{dx}g(x)\right).$$

We practice using this new rule in an example, followed by an example that demonstrates why this theorem is true.

Example 6.4.1 Using the Product Rule

Use the Product Rule to compute the derivative of $y = 5x^2 \sin x$. Evaluate the derivative at $x = \pi/2$.

Solution To make our use of the Product Rule explicit, let's set $f(x) = 5x^2$ and $g(x) = \sin x$. We easily compute/recall that $f'(x) = 10x$ and $g'(x) = \cos x$. Employing the rule, we have

$$\frac{d}{dx}\Big(5x^2 \sin x\Big) = 10x \sin x + 5x^2 \cos x.$$

At $x = \pi/2$, we have

$$y'(\pi/2) = 10\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) + 5\left(\frac{\pi}{2}\right)^2 \cos\left(\frac{\pi}{2}\right) = 5\pi.$$

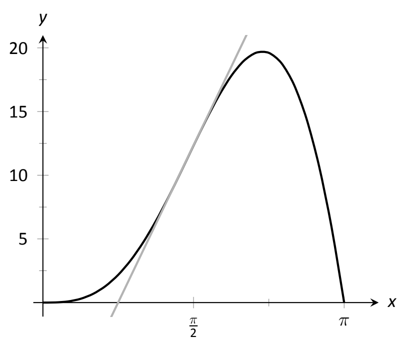


Figure 6.4.1: A graph of $y = 5x^2 \sin x$ and its tangent line at $x = \pi/2$.

We graph y and its tangent line at $x = \pi/2$, which has a slope of 5π, in Figure 6.4.1. While this does not *prove* that the Produce Rule is the correct way to handle derivatives of products, it helps validate its truth.

We now investigate why the Product Rule is true.

Example 6.4.2 A proof of the Product Rule

Use the definition of the derivative to prove Theorem 6.4.1.

Solution By the limit definition, we have

$$(fg)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

We now do something a bit unexpected; add 0 to the numerator (so that nothing is changed) in the form of $-f(x)g(x+h)+f(x)g(x+h)$, then do some regrouping as shown.

$$\begin{aligned}
(fg)'(x) &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} && \text{(now add 0 to the numerator)}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} && \text{(regroup)}\\
&= \lim_{h\to 0}\frac{\big(f(x+h)g(x+h)-f(x)g(x+h)\big)+\big(f(x)g(x+h)-f(x)g(x)\big)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)}{h}+\lim_{h\to 0}\frac{f(x)g(x+h)-f(x)g(x)}{h}\\
& && \text{(factor)}\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}g(x+h)+\lim_{h\to 0}f(x)\frac{g(x+h)-g(x)}{h} && \text{(apply limits)}\\
&= f'(x)g(x)+f(x)g'(x)
\end{aligned}$$

Notice that when we applied the limit in the last step, we relied on the fact that since g is assumed to be differentiable at x, it is continuous at x, and therefore, $\lim_{h\to 0} g(x+h) = g(x)$.

It is often true that we can recognize that a theorem is true through its proof yet somehow doubt its applicability to real problems. In the following example, we compute the derivative of a product of functions in two ways to verify that the Product Rule is indeed "right."

Example 6.4.3 Exploring alternate derivative methods

Let $y = (x^2+3x+1)(2x^2-3x+1)$. Find y' two ways: first, by expanding the given product and then taking the derivative, and second, by applying the Product Rule. Verify that both methods give the same answer.

Solution We first expand the expression for y; a little algebra shows that $y = 2x^4+3x^3-6x^2+1$. It is easy to compute y':

$$y' = 8x^3+9x^2-12x.$$

Now apply the Product Rule.

$$\begin{aligned}
y' &= (2x+3)(2x^2-3x+1)+(x^2+3x+1)(4x-3)\\
&= \big(4x^3-7x+3\big)+\big(4x^3+9x^2-5x-3\big)\\
&= 8x^3+9x^2-12x.
\end{aligned}$$

The uninformed usually assume that "the derivative of the product is the product of the derivatives." Thus we are tempted to say that $y' = (2x+3)(4x-3) = 8x^2+6x-9$. Obviously this is not correct.

Example 6.4.4 Using the Product Rule with a product of three functions

Let $y = x^3\ln x\cos x$. Find y'.

Solution We have a product of three functions while the Product Rule only specifies how to handle a product of two functions. Our method of handling this problem is to simply group the latter two functions together, and consider $y = x^3\big(\ln x\cos x\big)$. Following the Product Rule, we have

$$y' = 3x^2\big(\ln x\cos x\big)+(x^3)\frac{d}{dx}\big(\ln x\cos x\big)$$

To evaluate $\big(\ln x\cos x\big)'$, we apply the Product Rule again:

$$\begin{aligned}&= 3x^2\big(\ln x\cos x\big) + (x^3)\big(\frac{1}{x}\cos x + \ln x(-\sin x)\big)\\ &= 3x^2\ln x\cos x + x^3\frac{1}{x}\cos x + x^3\ln x(-\sin x)\end{aligned}$$

Recognize the pattern in our answer above: when applying the Product Rule to a product of three functions, there are three terms added together in the final derivative. Each term contains only one derivative of one of the original functions, and each function's derivative shows up in only one term. It is straightforward to extend this pattern to finding the derivative of a product of 4 or more functions.

We consider one more example before discussing another derivative rule.

Example 6.4.5 Using the Product Rule

Find the derivatives of the following functions.

1. $f(x) = x\ln x$
2. $g(x) = x\ln x - x$.

Solution Recalling that the derivative of $\ln x$ is $1/x$, we use the Product Rule to find our answers.

1. $\frac{d}{dx}\Big(x\ln x\Big) = 1\cdot\ln x + x\cdot 1/x = \ln x + 1$.
2. Using the result from above, we compute

$$\frac{d}{dx}\Big(x\ln x - x\Big) = \ln x + 1 - 1 = \ln x.$$

This seems significant; if the natural log function $\ln x$ is an important function (it is), it seems worthwhile to know a function whose derivative is $\ln x$. We have found one. (We leave it to the reader to find another; a correct answer will be *very* similar to this one.)

We have learned how to compute the derivatives of sums, differences, and products of functions. We now learn how to find the derivative of a quotient of functions.

The Quotient Rule is not hard to use, although it might be a bit tricky to remember. A useful mnemonic works as follows. Consider a fraction's numerator and denominator as "HI" and "LO", respectively. Then

$$\frac{d}{dx}\left(\frac{\text{HI}}{\text{LO}}\right) = \frac{\text{LO}\cdot\text{dHI} - \text{HI}\cdot\text{dLO}}{\text{LOLO}},$$

read "low dee high minus high dee low, over low low." Said fast, that phrase can roll off the tongue, making it easy to memorize. The "dee high" and "dee low" parts refer to the derivatives of the numerator and denominator, respectively. As an unexpected side benefit, you will also have an opportunity to practice your yodelling.

Theorem 6.4.2 Quotient Rule

Let f and g be differentiable functions defined on an open interval I, where $g(x)\neq 0$ on I. Then f/g is differentiable on I, and

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Let's practice using the Quotient Rule.

Example 6.4.6 Using the Quotient Rule

Let $f(x) = \dfrac{5x^2}{\sin x}$. Find $f'(x)$.

Solution Directly applying the Quotient Rule gives:

$$\frac{d}{dx}\left(\frac{5x^2}{\sin x}\right) = \frac{10x\cdot\sin x - 5x^2\cdot\cos x}{\sin^2 x}.$$

The Quotient Rule allows us to fill in holes in our understanding of derivatives of the common trigonometric functions. We start with finding the derivative of the tangent function.

Example 6.4.7 Using the Quotient Rule to find $\frac{d}{dx}\big(\tan x\big)$.

Find the derivative of $y = \tan x$.

Solution At first, one might feel unequipped to answer this question. But recall that $\tan x = \sin x/\cos x$, so we can apply the Quotient Rule.

$$\begin{aligned}\frac{d}{dx}\Big(\tan x\Big) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\ &= \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x}\\ &= \sec^2 x.\end{aligned}$$

This is a beautiful result. To confirm its truth, we can find the equation of the tangent line to $y = \tan x$ at $x = \pi/4$. The slope is $\sec^2(\pi/4) = 2$; $y = \tan x$, along with its tangent line, is graphed in Figure 6.4.2.

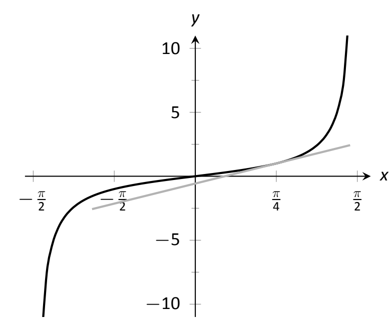


Figure 6.4.2: A graph of $y = \tan x$ along with its tangent line at $x = \pi/4$.

We include this result in the following theorem about the derivatives of the trigonometric functions. Recall we found the derivative of $y = \sin x$ in Example 6.1.7 and stated the derivative of the cosine function in Theorem 6.3.1. The derivatives of the cotangent, cosecant and secant functions can all be computed directly using Theorem 6.3.1 and the Quotient Rule.

Theorem 6.4.3 Derivatives of Trigonometric Functions

1. $\frac{d}{dx}\big(\sin x\big) = \cos x$
2. $\frac{d}{dx}\big(\cos x\big) = -\sin x$
3. $\frac{d}{dx}\big(\tan x\big) = \sec^2 x$
4. $\frac{d}{dx}\big(\cot x\big) = -\csc^2 x$
5. $\frac{d}{dx}\big(\sec x\big) = \sec x\tan x$
6. $\frac{d}{dx}\big(\csc x\big) = -\csc x\cot x$

To remember the above, it may be helpful to keep in mind that the derivatives of the trigonometric functions that start with "c" have a minus sign in them.

Example 6.4.8 Exploring alternate derivative methods

In Example 6.4.6 the derivative of $f(x) = \dfrac{5x^2}{\sin x}$ was found using the Quotient Rule. Rewriting f as $f(x) = 5x^2 \csc x$, find f' using Theorem 6.4.3 and verify the two answers are the same.

Solution We found in Example 6.4.6 that the $f'(x) = \dfrac{10x \sin x - 5x^2 \cos x}{\sin^2 x}$. We now find f' using the Product Rule, considering f as $f(x) = 5x^2 \csc x$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(5x^2 \csc x\right) \\ &= 10x \csc x + 5x^2(-\csc x \cot x) && \text{(now rewrite trig functions)} \\ &= \frac{10x}{\sin x} + 5x^2 \cdot \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x} \\ &= \frac{10x}{\sin x} + \frac{-5x^2 \cos x}{\sin^2 x} && \text{(get common denominator)} \\ &= \frac{10x \sin x - 5x^2 \cos x}{\sin^2 x} \end{aligned}$$

Finding f' using either method returned the same result. At first, the answers looked different, but some algebra verified they are the same. In general, there is not one final form that we seek; the immediate result from the Product Rule is fine. It is up to you if you wish to work to "simplify" your results into a form that is most readable and useful to you.

The only times it is really necessary – that is, worthwhile – to simplify a product or quotient rule derivative on a test is if you are trying to determine the values of x at which the derivative is zero (there will be plenty of that to come!) or in some cases, if a second derivative is required, and simplifying first makes that computation easier. (Also keep in mind that the person grading your test will be looking for the product or quotient rule pattern, so the unsimplified answer is sometimes the easiest to identify as the correct one.) However, for written assignments where you have the luxury of taking your time to perfect your presentation, a simplified answer is usually preferable.

The Quotient Rule gives other useful results, as shown in the next example.

Example 6.4.9 Using the Quotient Rule to expand the Power Rule

Find the derivatives of the following functions.

1. $f(x) = \dfrac{1}{x}$
2. $f(x) = \dfrac{1}{x^n}$, where $n > 0$ is an integer.

Solution We employ the Quotient Rule.

1. $f'(x) = \dfrac{0 \cdot x - 1 \cdot 1}{x^2} = -\dfrac{1}{x^2}$.
2. $f'(x) = \dfrac{0 \cdot x^n - 1 \cdot nx^{n-1}}{(x^n)^2} = -\dfrac{nx^{n-1}}{x^{2n}} = -\dfrac{n}{x^{n+1}}$.

The derivative of $y = \dfrac{1}{x^n}$ turned out to be rather nice. It gets better. Consider:

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{x^n}\right) &= \frac{d}{dx}\left(x^{-n}\right) && \text{(apply result from Example 6.4.9)} \\ &= -\frac{n}{x^{n+1}} && \text{(rewrite algebraically)} \\ &= -nx^{-(n+1)} \\ &= -nx^{-n-1}. \end{aligned}$$

This is reminiscent of the Power Rule: multiply by the power, then subtract 1 from the power. We now add to our previous Power Rule, which had the restriction of $n > 0$.

Theorem 6.4.4 Power Rule with Integer Exponents

Let $f(x) = x^n$, where $n \neq 0$ is an integer. Then

$$f'(x) = n \cdot x^{n-1}.$$

Taking the derivative of many functions is relatively straightforward. It is clear (with practice) what rules apply and in what order they should be applied. Other functions present multiple paths; different rules may be applied depending on how the function is treated. One of the beautiful things about calculus is that there is not "the" right way; each path, when applied correctly, leads to the same result, the derivative. We demonstrate this concept in an example.

Example 6.4.10 Exploring alternate derivative methods

Let $f(x) = \dfrac{x^2 - 3x + 1}{x}$. Find $f'(x)$ in each of the following ways:

1. By applying the Quotient Rule,
2. by viewing f as $f(x) = \left(x^2 - 3x + 1\right) \cdot x^{-1}$ and applying the Product and Power Rules, and
3. by "simplifying" first through division.

Verify that all three methods give the same result.

Solution

1. Applying the Quotient Rule gives:

$$f'(x) = \frac{\left(2x - 3\right) \cdot x - \left(x^2 - 3x + 1\right) \cdot 1}{x^2} = \frac{x^2 - 1}{x^2} = 1 - \frac{1}{x^2}.$$

2. By rewriting f, we can apply the Product and Power Rules as follows:

$$\begin{aligned} f'(x) &= \left(2x - 3\right) \cdot x^{-1} + \left(x^2 - 3x + 1\right) \cdot (-1)x^{-2} \\ &= \frac{2x - 3}{x} - \frac{x^2 - 3x + 1}{x^2} \\ &= \frac{2x^2 - 3x}{x^2} - \frac{x^2 - 3x + 1}{x^2} \\ &= \frac{x^2 - 1}{x^2} = 1 - \frac{1}{x^2}, \end{aligned}$$

the same result as above.

3. As $x \neq 0$, we can divide through by x first, giving $f(x) = x - 3 + \dfrac{1}{x}$. Now apply the Power Rule.

$$f'(x) = 1 - \frac{1}{x^2},$$

the same result as before.

Example 6.4.10 demonstrates three methods of finding f'. One is hard pressed to argue for a "best method" as all three gave the same result without too much difficulty, although it is clear that using the Product Rule required more steps. Ultimately, the important principle to take away from this is: reduce the answer to a form that seems "simple" and easy to interpret. In that example, we saw different expressions for f', including:

$$1-\frac{1}{x^2}=\frac{\big(2x-3\big)\cdot x-\big(x^2-3x+1\big)\cdot 1}{x^2}=\big(2x-3\big)\cdot x^{-1}+\big(x^2-3x+1\big)\cdot(-1)x^{-2}.$$

They are equal; they are all correct; only the first is "clear." Work to make answers clear.

In the next section we continue to learn rules that allow us to more easily compute derivatives than using the limit definition directly. We have to memorize the derivatives of a certain set of functions, such as "the derivative of $\sin x$ is $\cos x$." The Sum/Difference, Constant Multiple, Power, Product and Quotient Rules show us how to find the derivatives of certain combinations of these functions. The next section shows how to find the derivatives when we *compose* these functions together.

Exercises 6.4

Terms and Concepts

1. T/F: The Product Rule states that $\frac{d}{dx}\big(x^2\sin x\big) = 2x\cos x$.

2. T/F: The Quotient Rule states that $\frac{d}{dx}\left(\frac{x^2}{\sin x}\right) = \frac{\cos x}{2x}$.

3. T/F: The derivatives of the trigonometric functions that start with "c" have minus signs in them.

4. What derivative rule is used to extend the Power Rule to include negative integer exponents?

5. T/F: Regardless of the function, there is always exactly one right way of computing its derivative.

6. In your own words, explain what it means to make your answers "clear."

Problems

In Exercises 7 – 10:

(a) Use the Product Rule to differentiate the function.

(b) Manipulate the function algebraically and differentiate without the Product Rule.

(c) Show that the answers from (a) and (b) are equivalent.

7. $f(x) = x(x^2+3x)$

8. $g(x) = 2x^2(5x^3)$

9. $h(s) = (2s-1)(s+4)$

10. $f(x) = (x^2+5)(3-x^3)$

In Exercises 11 – 14:

(a) Use the Quotient Rule to differentiate the function.

(b) Manipulate the function algebraically and differentiate without the Quotient Rule.

(c) Show that the answers from (a) and (b) are equivalent.

11. $f(x) = \frac{x^2+3}{x}$

12. $g(x) = \frac{x^3-2x^2}{2x^2}$

13. $h(s) = \frac{3}{4s^3}$

14. $f(t) = \frac{t^2-1}{t+1}$

In Exercises 15 – 36, compute the derivative of the given function.

15. $f(x) = x\sin x$

16. $f(x) = x^2\cos x$

17. $f(x) = e^x\ln x$

18. $f(t) = \frac{1}{t^2}(\csc t - 4)$

19. $g(x) = \frac{x+7}{x-5}$

20. $g(t) = \frac{t^5}{\cos t - 2t^2}$

21. $h(x) = \cot x - e^x$

22. $f(x) = \big(\tan x\big)\ln x$

23. $h(t) = 7t^2+6t-2$

24. $f(x) = \frac{x^4+2x^3}{x+2}$

25. $f(x) = \big(3x^2+8x+7\big)e^x$

26. $g(t) = \frac{t^5-t^3}{e^t}$

27. $f(x) = (16x^3+24x^2+3x)\frac{7x-1}{16x^3+24x^2+3x}$

28. $f(t) = t^5(\sec t + e^t)$

29. $f(x) = \frac{\sin x}{\cos x + 3}$

30. $f(\theta) = \theta^3\sin\theta + \frac{\sin\theta}{\theta^3}$

31. $f(x) = \frac{\cos x}{x} + \frac{x}{\tan x}$

32. $g(x) = e^2\big(\sin(\pi/4) - 1\big)$

33. $g(t) = 4t^3e^t - \sin t\cos t$

34. $h(t) = \frac{t^2\sin t + 3}{t^2\cos t + 2}$

35. $f(x) = x^2e^x\tan x$

36. $g(x) = 2x\sin x\sec x$

In Exercises 37 – 40, find the equations of the tangent and normal lines to the graph of g at the indicated point.

37. $g(s) = e^s(s^2 + 2)$ at $(0, 2)$.

38. $g(t) = t \sin t$ at $\left(\frac{3\pi}{2}, -\frac{3\pi}{2}\right)$

39. $g(x) = \dfrac{x^2}{x-1}$ at $(2, 4)$

40. $g(\theta) = \dfrac{\cos\theta - 8\theta}{\theta + 1}$ at $(0, 1)$

In Exercises 41 – 44, find the x–values where the graph of the function has a horizontal tangent line.

41. $f(x) = 6x^2 - 18x - 24$

42. $f(x) = x \sin x$ on $[-1, 1]$

43. $f(x) = \dfrac{x}{x+1}$

44. $f(x) = \dfrac{x^2}{x+1}$

In Exercises 45 – 48, find the requested derivative.

45. $f(x) = x \sin x$; find $f''(x)$.

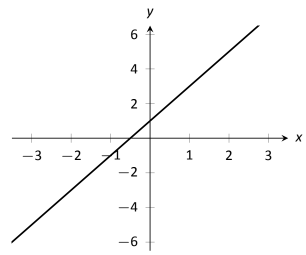
46. $f(x) = x \sin x$; find $f^{(4)}(x)$.

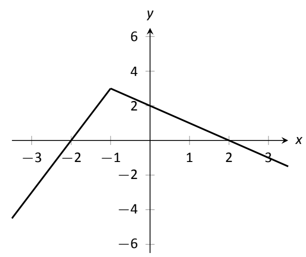
47. $f(x) = \csc x$; find $f''(x)$.

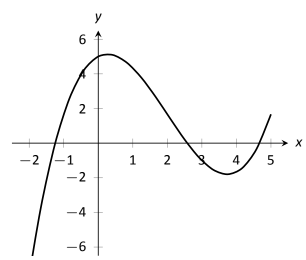
48. $f(x) = (x^3 - 5x + 2)(x^2 + x - 7)$; find $f^{(8)}(x)$.

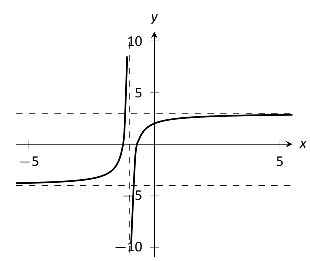
Review

In Exercises 49 – 52, use the graph of $f(x)$ to sketch $f'(x)$.

49.


50.


51.


52.


6.5 The Chain Rule

We have covered almost all of the derivative rules that deal with combinations of two (or more) functions. The operations of addition, subtraction, multiplication (including by a constant) and division led to the Sum and Difference rules, the Constant Multiple Rule, the Power Rule, the Product Rule and the Quotient Rule. To complete the list of differentiation rules, we look at the last way two (or more) functions can be combined: the process of composition (i.e. one function "inside" another).

One example of a composition of functions is $f(x) = \cos(x^2)$. We currently do not know how to compute this derivative. If forced to guess, one would likely guess $f'(x) = -\sin(2x)$, where we recognize $-\sin x$ as the derivative of $\cos x$ and $2x$ as the derivative of x^2. However, this is not the case; $f'(x) \neq -\sin(2x)$. In Example 6.5.4 we'll see the correct answer, which employs the new rule this section introduces, the **Chain Rule**.

Before we define this new rule, recall the notation for composition of functions. We write $(f \circ g)(x)$ or $f(g(x))$, read as "f of g of x," to denote composing f with g. In shorthand, we simply write $f \circ g$ or $f(g)$ and read it as "f of g." Before giving the corresponding differentiation rule, we note that the rule extends to multiple compositions like $f(g(h(x)))$ or $f(g(h(j(x))))$, etc.

To motivate the rule, let's look at three derivatives we can already compute.

Example 6.5.1 Exploring similar derivatives

Find the derivatives of $F_1(x) = (1-x)^2$, $F_2(x) = (1-x)^3$, and $F_3(x) = (1-x)^4$. (We'll see later why we are using subscripts for different functions and an uppercase F.)

SOLUTION In order to use the rules we already have, we must first expand each function as $F_1(x) = 1 - 2x + x^2$, $F_2(x) = 1 - 3x + 3x^2 - x^3$ and $F_3(x) = 1 - 4x + 6x^2 - 4x^3 + x^4$.

It is not hard to see that:

$$F_1'(x) = -2 + 2x,$$
$$F_2'(x) = -3 + 6x - 3x^2 \text{ and}$$
$$F_3'(x) = -4 + 12x - 12x^2 + 4x^3.$$

An interesting fact is that these can be rewritten as

$$F_1'(x) = -2(1-x), \quad F_2'(x) = -3(1-x)^2 \text{ and } F_3'(x) = -4(1-x)^3.$$

A pattern might jump out at you; note how the we end up multiplying by the old power and the new power is reduced by 1. We also always multiply by (-1).

Recognize that each of these functions is a composition, letting $g(x) = 1-x$:

$$\begin{aligned} F_1(x) &= f_1(g(x)), \quad \text{where } f_1(x) = x^2, \\ F_2(x) &= f_2(g(x)), \quad \text{where } f_2(x) = x^3, \\ F_3(x) &= f_3(g(x)), \quad \text{where } f_3(x) = x^4. \end{aligned}$$

We'll come back to this example after giving the formal statements of the Chain Rule; for now, we are just illustrating a pattern.

When composing functions, we need to make sure that the new function is actually defined. For instance, consider $f(x) = \sqrt{x}$ and $g(x) = -x^2 - 1$. The domain of f excludes all negative numbers, but the range of g is only negative numbers. Therefore the composition $f\big(g(x)\big) = \sqrt{-x^2-1}$ is not defined for any x, and hence is not differentiable.

The following definition takes care to ensure this problem does not arise. We'll focus more on the derivative result than on the domain/range conditions.

Theorem 6.5.1 The Chain Rule

Let g be a differentiable function on an interval I, let the range of g be a subset of the interval J, and let f be a differentiable function on J. Then $y = f(g(x))$ is a differentiable function on I, and

$$y' = f'(g(x)) \cdot g'(x).$$

To help understand the Chain Rule, we return to Example 6.5.1.

Example 6.5.2 Using the Chain Rule

Use the Chain Rule to find the derivatives of the following functions, as given in Example 6.5.1.

Solution Example 6.5.1 ended with the recognition that each of the given functions was actually a composition of functions. To avoid confusion, we ignore most of the subscripts here.

$F_1(x) = (1-x)^2$:

We found that

$$y = (1-x)^2 = f(g(x)), \text{ where } f(x) = x^2 \text{ and } g(x) = 1-x.$$

To find y', we apply the Chain Rule. We need $f'(x) = 2x$ and $g'(x) = -1$.

Part of the Chain Rule uses $f'(g(x))$. This means substitute $g(x)$ for x in the equation for $f'(x)$. That is, $f'(x) = 2(1-x)$. Finishing out the Chain Rule we have

$$y' = f'(g(x)) \cdot g'(x) = 2(1-x) \cdot (-1) = -2(1-x) = 2x-2.$$

$F_2(x) = (1-x)^3$:

Let $y = (1-x)^3 = f(g(x))$, where $f(x) = x^3$ and $g(x) = (1-x)$. We have $f'(x) = 3x^2$, so $f'(g(x)) = 3(1-x)^2$. The Chain Rule then states

$$y' = f'(g(x)) \cdot g'(x) = 3(1-x)^2 \cdot (-1) = -3(1-x)^2.$$

$F_3(x) = (1-x)^4$:

Finally, when $y = (1-x)^4$, we have $f(x) = x^4$ and $g(x) = (1-x)$. Thus $f'(x) = 4x^3$ and $f'(g(x)) = 4(1-x)^3$. Thus

$$y' = f'(g(x)) \cdot g'(x) = 4(1-x)^3 \cdot (-1) = -4(1-x)^3.$$

Example 6.5.2 demonstrated a particular pattern: when $f(x) = x^n$, then $y' = n \cdot (g(x))^{n-1} \cdot g'(x)$. This is called the Generalized Power Rule.

Theorem 6.5.2 Generalized Power Rule

Let $g(x)$ be a differentiable function and let $n \neq 0$ be an integer. Then

$$\frac{d}{dx}\Big(g(x)^n\Big) = n \cdot \big(g(x)\big)^{n-1} \cdot g'(x).$$

This allows us to quickly find the derivative of functions like $y = (3x^2 - 5x + 7 + \sin x)^{20}$. While it may look intimidating, the Generalized Power Rule states that

$$y' = 20(3x^2 - 5x + 7 + \sin x)^{19} \cdot (6x - 5 + \cos x).$$

Treat the derivative–taking process step–by–step. In the example just given, first multiply by 20, the rewrite the inside of the parentheses, raising it all to the 19th power. Then think about the derivative of the expression inside the parentheses, and multiply by that.

We now consider more examples that employ the Chain Rule.

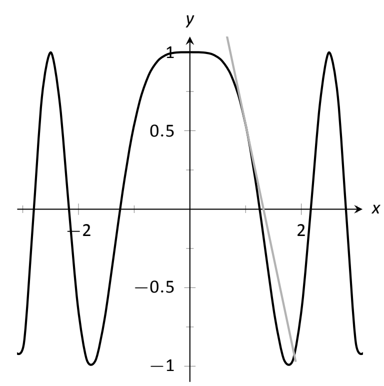


Figure 6.5.1: $f(x) = \cos x^2$ sketched along with its tangent line at $x = 1$.

Example 6.5.3 Using the Chain Rule

Find the derivatives of the following functions:

1. $y = \sin 2x$ 2. $y = \ln(4x^3 - 2x^2)$ 3. $y = e^{-x^2}$

Solution

1. Consider $y = \sin 2x$. Recognize that this is a composition of functions, where $f(x) = \sin x$ and $g(x) = 2x$. Thus

$$y' = f'(g(x)) \cdot g'(x) = \cos(2x) \cdot 2 = 2\cos 2x.$$

2. Recognize that $y = \ln(4x^3 - 2x^2)$ is the composition of $f(x) = \ln x$ and $g(x) = 4x^3 - 2x^2$. Also, recall that

$$\frac{d}{dx}\Big(\ln x\Big) = \frac{1}{x}.$$

This leads us to:

$$y' = \frac{1}{4x^3 - 2x^2} \cdot (12x^2 - 4x) = \frac{12x^2 - 4x}{4x^3 - 2x^2} = \frac{4x(3x-1)}{2x(2x^2 - x)} = \frac{2(3x-1)}{2x^2 - x}.$$

3. Recognize that $y = e^{-x^2}$ is the composition of $f(x) = e^x$ and $g(x) = -x^2$. Remembering that $f'(x) = e^x$, we have

$$y' = e^{-x^2} \cdot (-2x) = (-2x)e^{-x^2}.$$

Example 6.5.4 Using the Chain Rule to find a tangent line

Let $f(x) = \cos x^2$. Find the equation of the line tangent to the graph of f at $x = 1$.

Solution The tangent line goes through the point $(1, f(1)) \approx (1, 0.54)$ with slope $f'(1)$. To find f', we need the Chain Rule.

$f'(x) = -\sin(x^2) \cdot (2x) = -2x \sin x^2$. Evaluated at $x = 1$, we have $f'(1) = -2\sin 1 \approx -1.68$. Thus the equation of the tangent line is

$$y = -1.68(x - 1) + 0.54.$$

The tangent line is sketched along with f in Figure 6.5.1.

The Chain Rule is used often in taking derivatives. Because of this, one can become familiar with the basic process and learn patterns that facilitate finding derivatives quickly. For instance,

$$\frac{d}{dx}\Big(\ln(\text{anything})\Big) = \frac{1}{\text{anything}} \cdot (\text{anything})' = \frac{(\text{anything})'}{\text{anything}}.$$

A concrete example of this is

$$\frac{d}{dx}\Big(\ln(3x^{15} - \cos x + e^x)\Big) = \frac{45x^{14} + \sin x + e^x}{3x^{15} - \cos x + e^x}.$$

While the derivative may look intimidating at first, look for the pattern. The denominator is the same as what was inside the natural log function; the numerator is simply its derivative.

This pattern recognition process can be applied to lots of functions. In general, instead of writing "anything", we use u as a generic function of x. We then say

$$\frac{d}{dx}\Big(\ln u\Big) = \frac{u'}{u}.$$

The following is a short list of how the Chain Rule can be quickly applied to familiar functions.

1. $\frac{d}{dx}\Big(u^n\Big) = n \cdot u^{n-1} \cdot u'.$
2. $\frac{d}{dx}\Big(e^u\Big) = u' \cdot e^u.$
3. $\frac{d}{dx}\Big(\sin u\Big) = u' \cdot \cos u.$
4. $\frac{d}{dx}\Big(\cos u\Big) = -u' \cdot \sin u.$
5. $\frac{d}{dx}\Big(\tan u\Big) = u' \cdot \sec^2 u.$

Of course, the Chain Rule can be applied in conjunction with any of the other rules we have already learned. We practice this next.

Example 6.5.5 Using the Product, Quotient and Chain Rules

Find the derivatives of the following functions.

1. $f(x) = x^5 \sin 2x^3$ 2. $f(x) = \dfrac{5x^3}{e^{-x^2}}$.

Solution

1. We must use the Product and Chain Rules. Do not think that you must be able to "see" the whole answer immediately; rather, just proceed step–by–step.

$$f'(x) = x^5\big(6x^2\cos 2x^3\big) + 5x^4\big(\sin 2x^3\big) = 6x^7\cos 2x^3 + 5x^4\sin 2x^3.$$

2. We must employ the Quotient Rule along with the Chain Rule. Again, proceed step–by–step.

$$f'(x) = \frac{e^{-x^2}\big(15x^2\big) - 5x^3\big((-2x)e^{-x^2}\big)}{\big(e^{-x^2}\big)^2} = \frac{e^{-x^2}\big(10x^4 + 15x^2\big)}{e^{-2x^2}}$$
$$= e^{x^2}\big(10x^4 + 15x^2\big).$$

A key to correctly working these problems is to break the problem down into smaller, more manageable pieces. For instance, when using the Product and Chain Rules together, just consider the first part of the Product Rule at first: $f(x)g'(x)$. Just rewrite $f(x)$, then find $g'(x)$. Then move on to the $f'(x)g(x)$ part. Don't attempt to figure out both parts at once.

Likewise, using the Quotient Rule, approach the numerator in two steps and handle the denominator after completing that. Only simplify afterward.

We can also employ the Chain Rule itself several times, as shown in the next example.

Example 6.5.6 Using the Chain Rule multiple times

Find the derivative of $y = \tan^5(6x^3 - 7x)$.

Solution Recognize that we have the $g(x) = \tan(6x^3 - 7x)$ function "inside" the $f(x) = x^5$ function; that is, we have $y = \big(\tan(6x^3 - 7x)\big)^5$. We begin using the Generalized Power Rule; in this first step, we do not fully compute the derivative. Rather, we are approaching this step–by–step.

$$y' = 5\big(\tan(6x^3 - 7x)\big)^4 \cdot g'(x).$$

We now find $g'(x)$. We again need the Chain Rule;

$$g'(x) = \sec^2(6x^3 - 7x) \cdot (18x^2 - 7).$$

Combine this with what we found above to give

$$\begin{aligned} y' &= 5\big(\tan(6x^3 - 7x)\big)^4 \cdot \sec^2(6x^3 - 7x) \cdot (18x^2 - 7) \\ &= (90x^2 - 35)\sec^2(6x^3 - 7x)\tan^4(6x^3 - 7x). \end{aligned}$$

This function is frankly a ridiculous function, possessing no real practical value. It is very difficult to graph, as the tangent function has many vertical asymptotes and $6x^3 - 7x$ grows so very fast. The important thing to learn from this is that the derivative can be found. In fact, it is not "hard;" one can take several simple steps and should be careful to keep track of how to apply each of these steps.

It is a traditional mathematical exercise to find the derivatives of arbitrarily complicated functions just to demonstrate that it *can be done*. Just break everything down into smaller pieces.

Example 6.5.7 Using the Product, Quotient and Chain Rules

Find the derivative of $f(x) = \dfrac{x\cos(x^{-2}) - \sin^2(e^{4x})}{\ln(x^2 + 5x^4)}$.

Solution This function likely has no practical use outside of demonstrating derivative skills. The answer is given below without simplification. It employs the Quotient Rule, the Product Rule, and the Chain Rule three times.

$$f'(x) =$$

$$\frac{\left(\begin{array}{c} \ln(x^2+5x^4)\cdot \quad \Big[\big(x\cdot(-\sin(x^{-2}))\cdot(-2x^{-3}) + 1\cdot\cos(x^{-2})\big) \\ -2\sin(e^{4x})\cdot\cos(e^{4x})\cdot(4e^{4x})\Big] \\ -\Big(x\cos(x^{-2}) - \sin^2(e^{4x})\Big)\cdot\dfrac{2x+20x^3}{x^2+5x^4} \end{array}\right)}{\big(\ln(x^2+5x^4)\big)^2}.$$

The reader is highly encouraged to look at each term and recognize why it is there. This example demonstrates that derivatives can be computed systematically, no matter how arbitrarily complicated the function is.

The Chain Rule also has theoretic value. That is, it can be used to find the derivatives of functions that we have not yet learned as we do in the following example.

Example 6.5.8 The Chain Rule and exponential functions
Use the Chain Rule to find the derivative of $y = 2^x$.

SOLUTION We only know how to find the derivative of one exponential function, $y = e^x$. We can accomplish our goal by rewriting 2 in terms of e. Recalling that e^x and $\ln x$ are inverse functions, we can write

$$2 = e^{\ln 2} \quad \text{and so} \quad y = 2^x = \left(e^{\ln 2}\right)^x = e^{x(\ln 2)}.$$

The function is now the composition $y = f(g(x))$, with $f(x) = e^x$ and $g(x) = x(\ln 2)$. Since $f'(x) = e^x$ and $g'(x) = \ln 2$, the Chain Rule gives

$$y' = e^{x(\ln 2)} \cdot \ln 2.$$

Recall that the $e^{x(\ln 2)}$ term on the right hand side is just 2^x, our original function. Thus, the derivative contains the original function itself. We have

$$y' = y \cdot \ln 2 = 2^x \cdot \ln 2.$$

We can extend this process to use any base a, where $a > 0$ and $a \neq 1$. All we need to do is replace each "2" in our work with "a." The Chain Rule, coupled with the derivative rule of e^x, allows us to find the derivatives of all exponential functions.

The comment at the end of previous example is important and is restated formally as a theorem.

Theorem 6.5.3 Derivatives of Exponential Functions

Let $f(x) = a^x$, for $a > 0, a \neq 1$. Then f is differentiable for all real numbers and

$$f'(x) = \ln a \cdot a^x.$$

Alternate Chain Rule Notation

It is instructive to understand what the Chain Rule "looks like" using "$\frac{dy}{dx}$" notation instead of y' notation. Suppose that $y = f(u)$ is a function of u, where $u = g(x)$ is a function of x, as stated in Theorem 6.5.1. Then, through the composition $f \circ g$, we can think of y as a function of x, as $y = f(g(x))$. Thus the derivative of y with respect to x makes sense; we can talk about $\frac{dy}{dx}$. This leads to an interesting progression of notation:

$$\begin{aligned} y' &= f'(g(x)) \cdot g'(x) \\ \frac{dy}{dx} &= y'(u) \cdot u'(x) && \text{(since } y = f(u) \text{ and } u = g(x)\text{)} \\ \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{(using "fractional" notation for the derivative)} \end{aligned}$$

Here the "fractional" aspect of the derivative notation stands out. On the right hand side, it seems as though the "*du*" terms cancel out, leaving

$$\frac{dy}{dx} = \frac{dy}{dx}.$$

It is important to realize that we *are not* cancelling these terms; the derivative notation of $\frac{dy}{du}$ is one symbol. It is equally important to realize that this notation was chosen precisely because of this behaviour. It makes applying the Chain Rule easy with multiple variables. For instance,

$$\frac{dy}{dt} = \frac{dy}{d\bigcirc} \cdot \frac{d\bigcirc}{d\triangle} \cdot \frac{d\triangle}{dt}.$$

where $\bigcirc$ and $\triangle$ are any variables you'd like to use.

One of the most common ways of "visualizing" the Chain Rule is to consider a set of gears, as shown in Figure 6.5.2. The gears have 36, 18, and 6 teeth, respectively. That means for every revolution of the *x* gear, the *u* gear revolves twice. That is, the rate at which the *u* gear makes a revolution is twice as fast as the rate at which the *x* gear makes a revolution. Using the terminology of calculus, the rate of *u*-change, with respect to *x*, is $\frac{du}{dx} = 2$.

Likewise, every revolution of *u* causes 3 revolutions of *y*: $\frac{dy}{du} = 3$. How does *y* change with respect to *x*? For each revolution of *x*, *y* revolves 6 times; that is,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2 \cdot 3 = 6.$$

We can then extend the Chain Rule with more variables by adding more gears to the picture.

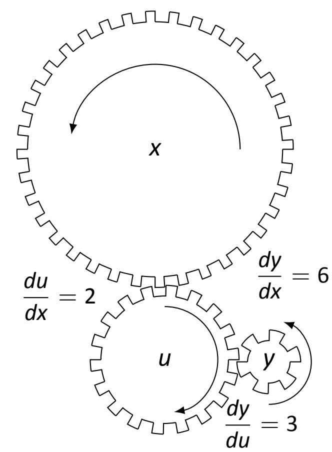


Figure 6.5.2: A series of gears to demonstrate the Chain Rule. Note how $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

It is difficult to overstate the importance of the Chain Rule. So often the functions that we deal with are compositions of two or more functions, requiring us to use this rule to compute derivatives. It is also often used in real life when actual functions are unknown. Through measurement, we can calculate (or, approximate) $\frac{dy}{du}$ and $\frac{du}{dx}$. With our knowledge of the Chain Rule, we can find $\frac{dy}{dx}$.

In the next section, we use the Chain Rule to justify another differentiation technique. There are many curves that we can draw in the plane that fail the "vertical line test." For instance, consider $x^2 + y^2 = 1$, which describes the unit circle. We may still be interested in finding slopes of tangent lines to the circle at various points. The next section shows how we can find $\frac{dy}{dx}$ without first "solving for *y*." While we can in this instance, in many other instances solving for *y* is impossible. In these situations, *implicit differentiation* is indispensable.

Exercises 6.5

Terms and Concepts

1. T/F: The Chain Rule describes how to evaluate the derivative of a composition of functions.

2. T/F: The Generalized Power Rule states that $\frac{d}{dx}\Big(g(x)^n\Big) = n\big(g(x)\big)^{n-1}$.

3. T/F: $\frac{d}{dx}\big(\ln(x^2)\big) = \frac{1}{x^2}$.

4. T/F: $\frac{d}{dx}\big(3^x\big) \approx 1.1 \cdot 3^x$.

5. T/F: $\frac{dx}{dy} = \frac{dx}{dt} \cdot \frac{dt}{dy}$

6. $f(x) = \big(\ln x + x^2\big)^3$

Problems

In Exercises 7 – 36, compute the derivative of the given function.

7. $f(x) = (4x^3 - x)^{10}$

8. $f(t) = (3t - 2)^5$

9. $g(\theta) = (\sin\theta + \cos\theta)^3$

10. $h(t) = e^{3t^2+t-1}$

11. $f(x) = \big(\ln x + x^2\big)^3$

12. $f(x) = 2^{x^3+3x}$

13. $f(x) = \big(x + \frac{1}{x}\big)^4$

14. $f(x) = \cos(3x)$

15. $g(x) = \tan(5x)$

16. $h(\theta) = \tan\big(\theta^2 + 4\theta\big)$

17. $g(t) = \sin\big(t^5 + \frac{1}{t}\big)$

18. $h(t) = \sin^4(2t)$

19. $p(t) = \cos^3(t^2 + 3t + 1)$

20. $f(x) = \ln(\cos x)$

21. $f(x) = \ln(x^2)$

22. $f(x) = 2\ln(x)$

23. $g(r) = 4^r$

24. $g(t) = 5^{\cos t}$

25. $g(t) = 15^2$

26. $m(w) = \frac{3^w}{2^w}$

27. $h(t) = \frac{2^t + 3}{3^t + 2}$

28. $m(w) = \frac{3^w + 1}{2^w}$

29. $f(x) = \frac{3^{x^2} + x}{2^{x^2}}$

30. $f(x) = x^2\sin(5x)$

31. $f(x) = (x^2 + x)^5(3x^4 + 2x)^3$

32. $g(t) = \cos(t^2 + 3t)\sin(5t - 7)$

33. $f(x) = \sin(3x + 4)\cos(5 - 2x)$

34. $g(t) = \cos(\frac{1}{t})e^{5t^2}$

35. $f(x) = \frac{\sin\big(4x + 1\big)}{(5x - 9)^3}$

36. $f(x) = \frac{(4x + 1)^2}{\tan(5x)}$

In Exercises 37 – 40, find the equations of tangent and normal lines to the graph of the function at the given point. Note: the functions here are the same as in Exercises 7 through 10.

37. $f(x) = (4x^3 - x)^{10}$ at $x = 0$

38. $f(t) = (3t - 2)^5$ at $t = 1$

39. $g(\theta) = (\sin\theta + \cos\theta)^3$ at $\theta = \pi/2$

40. $h(t) = e^{3t^2+t-1}$ at $t = -1$

41. Compute $\frac{d}{dx}\big(\ln(kx)\big)$ two ways:

    (a) Using the Chain Rule, and

    (b) by first using the logarithm rule $\ln(ab) = \ln a + \ln b$, then taking the derivative.

42. Compute $\frac{d}{dx}\big(\ln(x^k)\big)$ two ways:
    (a) Using the Chain Rule, and
    (b) by first using the logarithm rule $\ln(a^p) = p\ln a$, then taking the derivative.

Review

43. The "wind chill factor" is a measurement of how cold it "feels" during cold, windy weather. Let $W(w)$ be the wind chill factor, in degrees Fahrenheit, when it is 25°F outside with a wind of w mph.
    (a) What are the units of $W'(w)$?
    (b) What would you expect the sign of $W'(10)$ to be?

44. Find the derivatives of the following functions.
    (a) $f(x) = x^2e^x\cot x$
    (b) $g(x) = 2^x3^x4^x$

7: The Graphical Behaviour of Functions

Our study of limits led to continuous functions, a certain class of functions that behave in a particularly nice way. Limits then gave us an even nicer class of functions, functions that are differentiable.

This chapter explores many of the ways we can take advantage of the information that continuous and differentiable functions provide.

7.1 Extreme Values

Given any quantity described by a function, we are often interested in the largest and/or smallest values that quantity attains. For instance, if a function describes the speed of an object, it seems reasonable to want to know the fastest/slowest the object travelled. If a function describes the value of a stock, we might want to know the highest/lowest values the stock attained over the past year. We call such values *extreme values.*

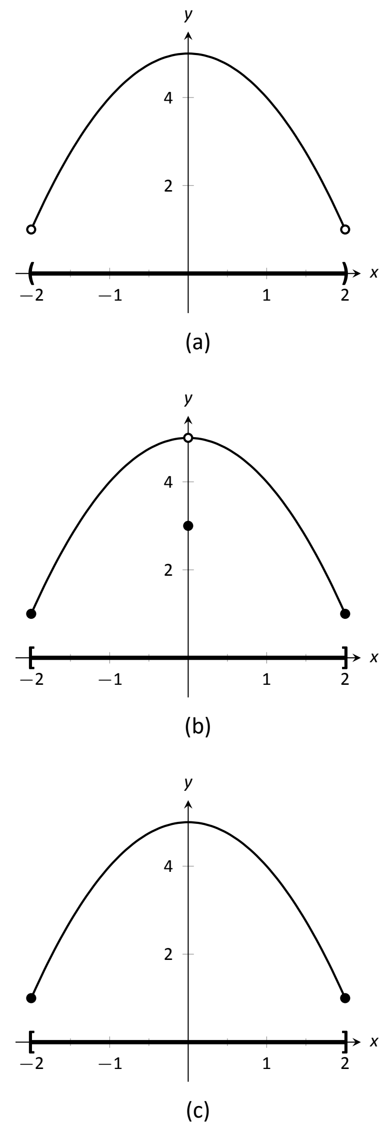


Figure 7.1.1: Graphs of functions with and without extreme values.

Definition 7.1.1 Extreme Values

Let f be defined on an interval I containing c.

1. $f(c)$ is the **minimum** (also, **absolute minimum**) of f on I if $f(c) \leq f(x)$ for all x in I.
2. $f(c)$ is the **maximum** (also, **absolute maximum**) of f on I if $f(c) \geq f(x)$ for all x in I.

The maximum and minimum values are the **extreme values**, or **extrema**, of f on I.

Consider Figure 7.1.1. The function displayed in (a) has a maximum, but no minimum, as the interval over which the function is defined is open. In (b), the function has a minimum, but no maximum; there is a discontinuity in the "natural" place for the maximum to occur. Finally, the function shown in (c) has both a maximum and a minimum; note that the function is continuous and the interval on which it is defined is closed.

It is possible for discontinuous functions defined on an open interval to have both a maximum and minimum value, but we have just seen examples where they did not. On the other hand, continuous functions on a closed interval *always* have a maximum and minimum value.

Note: The extreme values of a function are "y" values, values the function attains, not the input values.

Theorem 7.1.1 The Extreme Value Theorem

Let f be a continuous function defined on a closed interval I. Then f has both a maximum and minimum value on I.

This theorem states that f has extreme values, but it does not offer any advice about how/where to find these values. The process can seem to be fairly easy, as the next example illustrates. After the example, we will draw on lessons learned to form a more general and powerful method for finding extreme values.

Note: While Theorem 7.1.1 is intuitively plausible, a rigorous proof is actually quite technical, and beyond the scope of this text.

Example 7.1.1 Approximating extreme values

Consider $f(x) = 2x^3 - 9x^2$ on $I = [-1, 5]$, as graphed in Figure 7.1.2. Approximate the extreme values of f.

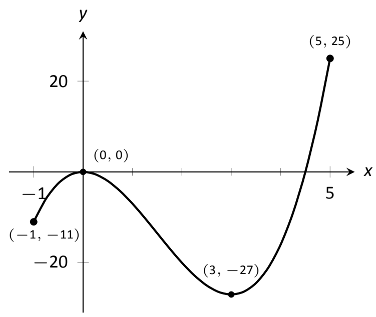


Figure 7.1.2: A graph of $f(x) = 2x^3 - 9x^2$ as in Example 7.1.1.

Solution The graph is drawn in such a way to draw attention to certain points. It certainly seems that the smallest y value is -27, found when $x = 3$. It also seems that the largest y value is 25, found at the endpoint of I, $x = 5$. We use the word *seems*, for by the graph alone we cannot be sure the smallest value is not less than -27. Since the problem asks for an approximation, we approximate the extreme values to be 25 and -27.

Notice how the minimum value came at "the bottom of a hill," and the maximum value came at an endpoint. Also note that while 0 is not an extreme value, it would be if we narrowed our interval to $[-1, 4]$. The idea that the point $(0, 0)$ is the location of an extreme value for some interval is important, leading us to a definition of a *relative maximum*. In short, a "relative max" is a y-value that's the largest y-value "nearby."

Note: The terms *local minimum* and *local maximum* are often used as synonyms for *relative minimum* and *relative maximum*.

As it makes intuitive sense that an absolute maximum is also a relative maximum, Definition 7.1.2 allows a relative maximum to occur at an interval's endpoint.

Definition 7.1.2 Relative Minimum and Relative Maximum

Let f be defined on an interval I containing c.

1. If there is an open interval containing c such that $f(c)$ is the minimum value, then $f(c)$ is a **relative minimum** of f. We also say that f has a relative minimum at $(c, f(c))$.
2. If there is an open interval containing c such that $f(c)$ is the maximum value, then $f(c)$ is a **relative maximum** of f. We also say that f has a relative maximum at $(c, f(c))$.

The relative maximum and minimum values comprise the **relative extrema** of f.

We briefly practice using these definitions.

Example 7.1.2 Approximating relative extrema

Consider $f(x) = (3x^4 - 4x^3 - 12x^2 + 5)/5$, as shown in Figure 7.1.3. Approximate the relative extrema of f. At each of these points, evaluate f'.

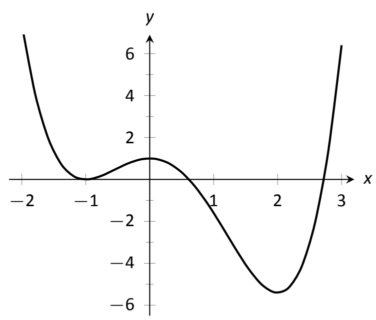


Figure 7.1.3: A graph of $f(x) = (3x^4 - 4x^3 - 12x^2 + 5)/5$ as in Example 7.1.2.

Solution We still do not have the tools to exactly find the relative extrema, but the graph does allow us to make reasonable approximations. It seems f has relative minima at $x = -1$ and $x = 2$, with values of $f(-1) = 0$ and $f(2) = -5.4$. It also seems that f has a relative maximum at the point $(0, 1)$.

We approximate the relative minima to be 0 and -5.4; we approximate the relative maximum to be 1.

It is straightforward to evaluate $f'(x) = \frac{1}{5}(12x^3 - 12x^2 - 24x)$ at $x = 0, 1$ and 2. In each case, $f'(x) = 0$.

Example 7.1.3 Approximating relative extrema

Approximate the relative extrema of $f(x) = (x-1)^{2/3} + 2$, shown in Figure 7.1.4. At each of these points, evaluate f'.

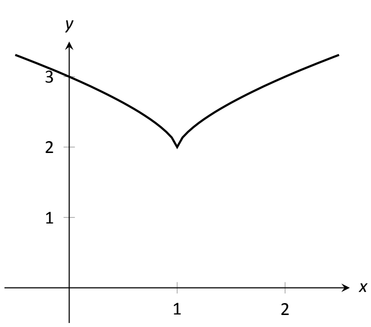


Figure 7.1.4: A graph of $f(x) = (x - 1)^{2/3} + 2$ as in Example 7.1.3.

Solution The figure implies that f does not have any relative maxima, but has a relative minimum at $(1, 2)$. In fact, the graph suggests that not only is this point a relative minimum, $y = f(1) = 2$ is *the* minimum value of the function.

We compute $f'(x) = \frac{2}{3}(x-1)^{-1/3}$. When $x = 1$, f' is undefined.

What can we learn from the previous two examples? We were able to visually approximate relative extrema, and at each such point, the derivative was either 0 or it was not defined. This observation holds for all functions, leading to a definition and a theorem.

Definition 7.1.3 Critical Numbers and Critical Points

Let f be defined at c. The value c is a **critical number** of f if $f'(c) = 0$ or $f'(c)$ is not defined. The value $f(c)$ is then referred to as a **critical value** of f.

If c is a critical number of f, then the point $(c, f(c))$ is a **critical point** of f.

Theorem 7.1.2 Relative Extrema and Critical Points

Let a function f be defined on an open interval I containing c, and let f have a relative extremum at the point $(c, f(c))$. Then c is a critical number of f.

Be careful to understand that this theorem states "Relative extrema on open intervals occur at critical points." It does not say "All critical numbers produce relative extrema." For instance, consider $f(x) = x^3$. Since $f'(x) = 3x^2$, it is straightforward to determine that $x = 0$ is a critical number of f. However, f has no relative extrema, as illustrated in Figure 7.1.5.

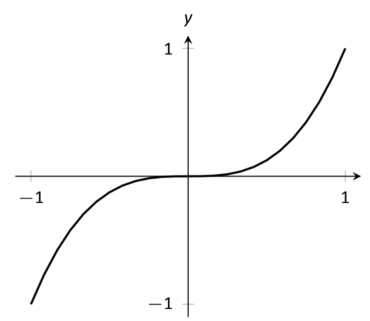


Figure 7.1.5: A graph of $f(x) = x^3$ which has a critical value of $x = 0$, but no relative extrema.

Let us pause briefly to try to understand why Theorem 7.1.2 is true. To begin, suppose that our function f has a relative maximum at the point $(c, f(c))$. (The argument for a relative minimum is similar.) If $f'(c)$ is undefined, then c is a critical number, and there is nothing to prove, so we suppose that f is differentiable at c, and try to see why it must be that $f'(c) = 0$. Consider the difference quotient

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}.$$

Since f has a relative maximum at c, we know that $f(c) \geq f(c+h)$ for sufficiently small values of h, so $f(c+h) - f(c) \leq 0$. Since $f'(c)$ exists, we know that the above limit must exist; in particular, the left-hand limit must equal the right hand limit. On the other hand, since $f(c+h) - f(c) \leq 0$, we have

$$\lim_{h\to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0,$$

since $h < 0$ in the left-hand limit, while

$$\lim_{h\to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0,$$

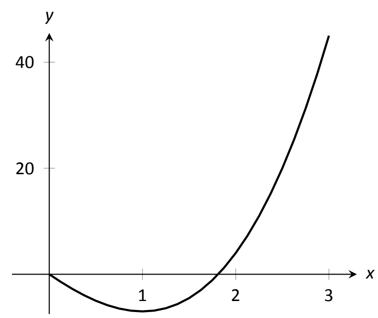
since $h > 0$ for the right-hand limit. The only way these two limits can agree is if both limits are equal to zero which proves that $f'(c) = 0$.

Theorem 7.1.1 states that a continuous function on a closed interval will have absolute extrema, that is, both an absolute maximum and an absolute minimum. These extrema occur either at the endpoints or at critical values in the interval. We combine these concepts to offer a strategy for finding extrema.

Key Idea 7.1.1 Finding Extrema on a Closed Interval

Let f be a continuous function defined on a closed interval $[a, b]$. To find the maximum and minimum values of f on $[a, b]$:

1. Evaluate f at the endpoints a and b of the interval.
2. Find the critical numbers of f in $[a, b]$.
3. Evaluate f at each critical number.
4. The absolute maximum of f is the largest of these values, and the absolute minimum of f is the least of these values.



(a)

x	$f(x)$
0	0
1	-7
3	45

(b)

Figure 7.1.6: Finding the extreme values of $f(x) = 2x^3+3x^2-12x$ in Example 7.1.4.

We practice these ideas in the next examples.

Example 7.1.4 Finding extreme values

Find the extreme values of $f(x) = 2x^3 + 3x^2 - 12x$ on $[0, 3]$, graphed in Figure 7.1.6(a).

Solution We follow the steps outlined in Key Idea 7.1.1. We first evaluate f at the endpoints:

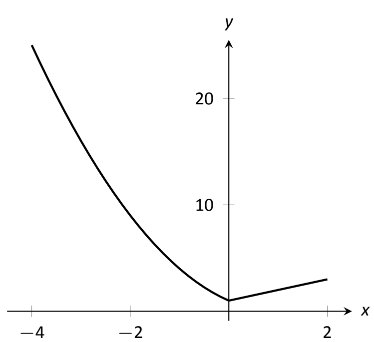
$$f(0) = 0 \quad \text{and} \quad f(3) = 45.$$

Next, we find the critical values of f on $[0, 3]$. $f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1)$; therefore the critical values of f are $x = -2$ and $x = 1$. Since $x = -2$ does not lie in the interval $[0, 3]$, we ignore it. Evaluating f at the only critical number in our interval gives: $f(1) = -7$.

The table in Figure 7.1.6(b) gives f evaluated at the "important" x values in $[0, 3]$. We can easily see the maximum and minimum values of f: the maximum value is 45 and the minimum value is -7.

Note that all this was done without the aid of a graph; this work followed an analytic algorithm and did not depend on any visualization. Figure 7.1.6 shows f and we can confirm our answer, but it is important to understand that these answers can be found without graphical assistance.

We practice again.



(a)

x	$f(x)$
-4	25
0	1
2	3

(b)

Figure 7.1.7: Finding the extreme values of a piecewise–defined function in Example 7.1.5.

Example 7.1.5 Finding extreme values

Find the maximum and minimum values of f on $[-4, 2]$, where

$$f(x) = \begin{cases} (x-1)^2 & x \leq 0 \\ x+1 & x > 0 \end{cases}.$$

Solution Here f is piecewise–defined, but we can still apply Key Idea 7.1.1. Evaluating f at the endpoints gives:

$$f(-4) = 25 \quad \text{and} \quad f(2) = 3.$$

We now find the critical numbers of f. We have to define f' in a piecewise manner; it is

$$f'(x) = \begin{cases} 2(x-1) & x < 0 \\ 1 & x > 0 \end{cases}.$$

Note that while f is defined for all of $[-4, 2]$, f' is not, as the derivative of f does not exist when $x = 0$. (From the left, the derivative approaches -2; from the right the derivative is 1.) Thus one critical number of f is $x = 0$.

We now set $f'(x) = 0$. When $x > 0$, $f'(x)$ is never 0. When $x < 0$, $f'(x)$ is also never 0, so we find no critical values from setting $f'(x) = 0$.

So we have three important x values to consider: $x = -4, 2$ and 0. Evaluating f at each gives, respectively, 25, 3 and 1, shown in Figure 7.1.7(b). Thus the absolute minimum of f is 1, the absolute maximum of f is 25, confirmed by the graph of f.

Example 7.1.6 Finding extreme values

Find the extrema of $f(x) = \cos(x^2)$ on $[-2, 2]$, graphed in Figure 7.1.8(a).

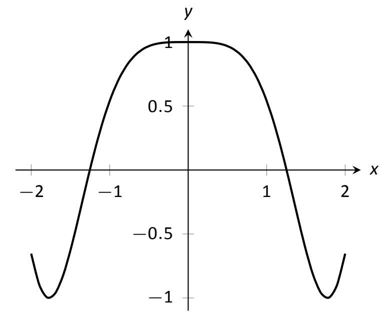
Solution We again use Key Idea 7.1.1. Evaluating f at the endpoints of the interval gives: $f(-2) = f(2) = \cos(4) \approx -0.6536$. We now find the critical values of f.

Applying the Chain Rule, we find $f'(x) = -2x\sin(x^2)$. Set $f'(x) = 0$ and solve for x to find the critical values of f.

We have $f'(x) = 0$ when $x = 0$ and when $\sin(x^2) = 0$. In general, $\sin t = 0$ when $t = \ldots - 2\pi, -\pi, 0, \pi, \ldots$ Thus $\sin(x^2) = 0$ when $x^2 = 0, \pi, 2\pi, \ldots$ (x^2 is always positive so we ignore $-\pi$, etc.) So $\sin(x^2) = 0$ when $x = 0, \pm\sqrt{\pi}, \pm\sqrt{2\pi}, \ldots$. The only values to fall in the given interval of $[-2, 2]$ are $-\sqrt{\pi}$ and $\sqrt{\pi}$, approximately ± 1.77.

We again construct a table of important values in Figure 7.1.8(b). In this example we have 5 values to consider: $x = 0, \pm 2, \pm\sqrt{\pi}$.

From the table it is clear that the maximum value of f on $[-2, 2]$ is 1; the minimum value is -1. The graph in Figure 7.1.8 confirms our results.



(a)

x	$f(x)$
-2	-0.65
$-\sqrt{\pi}$	-1
0	1
$\sqrt{\pi}$	-1
2	-0.65

(b)

Figure 7.1.8: Finding the extrema of $f(x) = \cos(x^2)$ in Example 7.1.6.

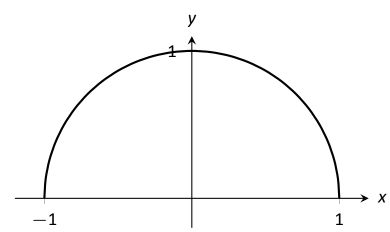
We consider one more example.

Example 7.1.7 Finding extreme values

Find the extreme values of $f(x) = \sqrt{1-x^2}$, graphed in Figure 7.1.9(a).

Solution A closed interval is not given, so we find the extreme values of f on its domain. f is defined whenever $1 - x^2 \geq 0$; thus the domain of f is $[-1, 1]$. Evaluating f at either endpoint returns 0.

Using the Chain Rule, we find $f'(x) = \dfrac{-x}{\sqrt{1-x^2}}$. The critical points of f are found when $f'(x) = 0$ or when f' is undefined. It is straightforward to find that $f'(x) = 0$ when $x = 0$, and f' is undefined when $x = \pm 1$, the endpoints of the interval. The table of important values is given in Figure 7.1.9(b). The maximum value is 1, and the minimum value is 0. (See also the marginal note.)



(a)

x	$f(x)$
-1	0
0	1
1	0

(b)

Figure 7.1.9: Finding the extrema of the half–circle in Example 7.1.7.

We have seen that continuous functions on closed intervals always have a maximum and minimum value, and we have also developed a technique to find these values. In the next section, we further our study of the information we can glean from "nice" functions with the Mean Value Theorem. On a closed interval, we can find the *average rate of change* of a function (as we did at the beginning of Chapter 2). We will see that differentiable functions always have a point at which their *instantaneous* rate of change is same as the *average* rate of change. This is surprisingly useful, as we'll see.

Exercises 7.1

Terms and Concepts

1. Describe what an "extreme value" of a function is in your own words.

2. Sketch the graph of a function f on $(-1, 1)$ that has both a maximum and minimum value.

3. Describe the difference between absolute and relative maxima in your own words.

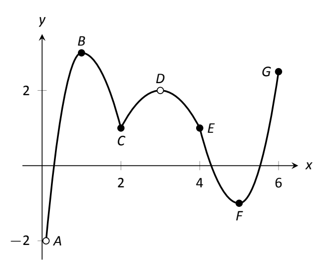
4. Sketch the graph of a function f where f has a relative maximum at $x = 1$ and $f'(1)$ is undefined.

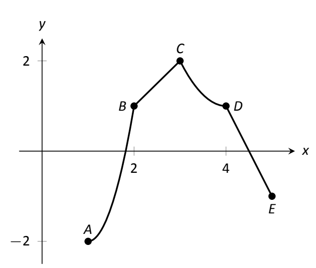
5. T/F: If c is a critical value of a function f, then f has either a relative maximum or relative minimum at $x = c$.

6. Fill in the blanks: The critical points of a function f are found where $f'(x)$ is equal to ________ or where $f'(x)$ is ________.

Problems

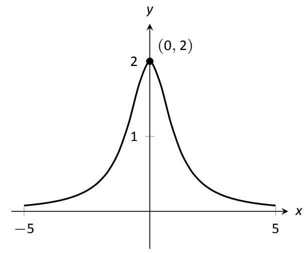
In Exercises 7 – 8, identify each of the marked points as being an absolute maximum or minimum, a relative maximum or minimum, or none of the above.

7. 

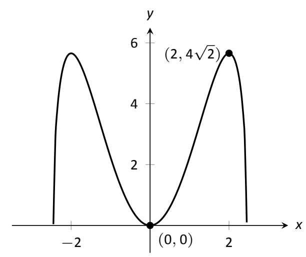
8. 

In Exercises 9 – 16, evaluate $f'(x)$ at the points indicated in the graph.

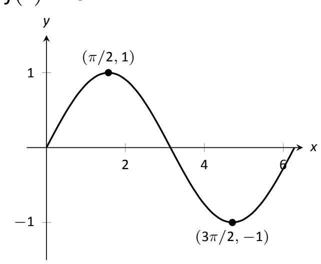
9. $f(x) = \dfrac{2}{x^2 + 1}$



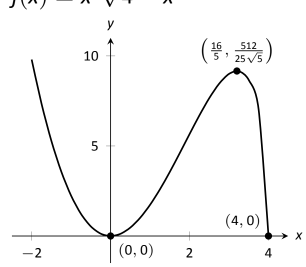
10. $f(x) = x^2\sqrt{6 - x^2}$



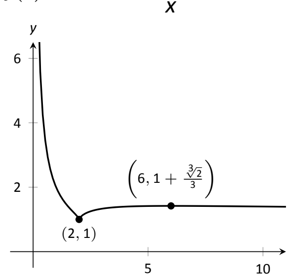
11. $f(x) = \sin x$



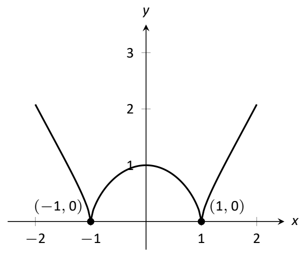
12. $f(x) = x^2\sqrt{4 - x}$



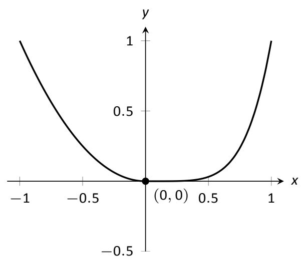
13. $f(x) = 1 + \dfrac{(x - 2)^{2/3}}{x}$



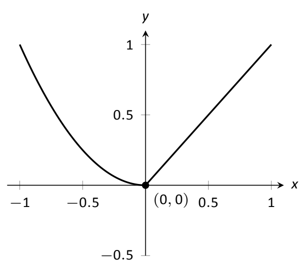
14. $f(x) = \sqrt[3]{x^4 - 2x + 1}$



15. $f(x) = \begin{cases} x^2 & x \leq 0 \\ x^5 & x > 0 \end{cases}$



16. $f(x) = \begin{cases} x^2 & x \leq 0 \\ x & x > 0 \end{cases}$



In Exercises 17 – 26, find the extreme values of the function on the given interval.

17. $f(x) = x^2 + x + 4$ on $[-1, 2]$.

18. $f(x) = x^3 - \frac{9}{2}x^2 - 30x + 3$ on $[0, 6]$.

19. $f(x) = 3\sin x$ on $[\pi/4, 2\pi/3]$.

20. $f(x) = x^2\sqrt{4 - x^2}$ on $[-2, 2]$.

21. $f(x) = x + \frac{3}{x}$ on $[1, 5]$.

22. $f(x) = \frac{x^2}{x^2 + 5}$ on $[-3, 5]$.

23. $f(x) = e^x \cos x$ on $[0, \pi]$.

24. $f(x) = e^x \sin x$ on $[0, \pi]$.

25. $f(x) = \frac{\ln x}{x}$ on $[1, 4]$.

26. $f(x) = x^{2/3} - x$ on $[0, 2]$.

Review

27. Find $\frac{dy}{dx}$, where $x^2y - y^2x = 1$.

28. Find the equation of the line tangent to the graph of $x^2 + y^2 + xy = 7$ at the point $(1, 2)$.

29. Let $f(x) = x^3 + x$.
Evaluate $\lim_{s \to 0} \frac{f(x+s) - f(x)}{s}$.

7.2 Increasing and Decreasing Functions

Our study of "nice" functions f in this chapter has so far focused on individual points: points where f is maximal/minimal, points where $f'(x) = 0$ or f' does not exist, and points c where $f'(c)$ is the average rate of change of f on some interval.

In this section we begin to study how functions behave *between* special points; we begin studying in more detail the shape of their graphs.

We start with an intuitive concept. Given the graph in Figure 7.2.1, where would you say the function is *increasing*? *Decreasing*? Even though we have not defined these terms mathematically, one likely answered that f is increasing when $x > 1$ and decreasing when $x < 1$. We formally define these terms here.

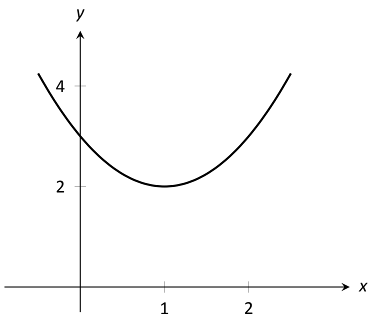


Figure 7.2.1: A graph of a function f used to illustrate the concepts of *increasing* and *decreasing*.

Definition 7.2.1 Increasing and Decreasing Functions

Let f be a function defined on an interval I.

1. f is **increasing** on I if for every $a < b$ in I, $f(a) < f(b)$.
2. f is **decreasing** on I if for every $a < b$ in I, $f(a) > f(b)$.

A function is **nonincreasing** when $a < b$ in I implies $f(a) \geq f(b)$, with a similar definition holding for **nondecreasing**.

Note: Some authors define a function to be increasing if $f(a) \leq f(b)$ whenever $a < b$ (with a similar definition for decreasing), and say that a function f satisfying our definition is *strictly increasing* (similarly, strictly decreasing). This is a perfectly reasonable definition, although it does have the odd consequence that, with this definition, a constant function would be simultaneously increasing and decreasing.

Informally, a function is increasing if as x gets larger (i.e., looking left to right) $f(x)$ gets larger.

Our interest lies in finding intervals in the domain of f on which f is either increasing or decreasing. Such information should seem useful. For instance, if f describes the speed of an object, we might want to know when the speed was increasing or decreasing (i.e., when the object was accelerating vs. decelerating). If f describes the population of a city, we should be interested in when the population is growing or declining.

To find such intervals, we again consider secant lines. Let f be an increasing, differentiable function on an open interval I, such as the one shown in Figure 7.2.2, and let $a < b$ be given in I. The secant line on the graph of f from $x = a$ to $x = b$ is drawn; it has a slope of $(f(b) - f(a))/(b - a)$. But note:

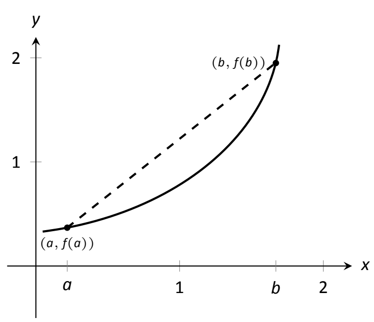


Figure 7.2.2: Examining the secant line of an increasing function.

$$\frac{f(b) - f(a)}{b - a} \Rightarrow \frac{\text{numerator } > 0}{\text{denominator } > 0} \Rightarrow \begin{array}{c}\text{slope of the}\\ \text{secant line} > 0\end{array} \Rightarrow \begin{array}{c}\text{Average rate of}\\ \text{change of } f \text{ on}\\ [a, b] \text{ is} > 0.\end{array}$$

We have shown mathematically what may have already been obvious: when f is increasing, its secant lines will have a positive slope. Now recall the Mean Value Theorem guarantees that there is a number c, where $a < c < b$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} > 0.$$

By considering all such secant lines in I, we strongly imply that $f'(x) > 0$ on I. A similar statement can be made for decreasing functions.

Our above logic can be summarized as "If f is increasing, then f' is probably positive." Theorem 7.2.1 below turns this around by stating "If f' is positive, then f is increasing." This leads us to a method for finding when functions are increasing and decreasing.

Theorem 7.2.1 Test For Increasing/Decreasing Functions

Let f be a continuous function on $[a, b]$ and differentiable on (a, b).

1. If $f'(c) > 0$ for all c in (a, b), then f is increasing on $[a, b]$.
2. If $f'(c) < 0$ for all c in (a, b), then f is decreasing on $[a, b]$.
3. If $f'(c) = 0$ for all c in (a, b), then f is constant on $[a, b]$.

Let f be differentiable on an interval I and let a and b be in I where $f'(a) > 0$ and $f'(b) < 0$. If f' is continuous on $[a, b]$, it follows from the Intermediate Value Theorem that there must be some value c between a and b where $f'(c) = 0$. If f' is not continuous on $[a, b]$, it can happen that f' changes sign at a point c where $f'(c)$ is undefined, so we should account for these points as well. This leads us to the following method for finding intervals on which a function is increasing or decreasing.

Note: Parts 1 & 2 of Theorem 7.2.1 also hold if $f'(c) = 0$ for a finite number of values of c in I.

Key Idea 7.2.1 Finding Intervals on Which f is Increasing or Decreasing

Let f be a differentiable function on an interval I. To find intervals on which f is increasing and decreasing:

1. Find the critical values of f. That is, find all c in I where $f'(c) = 0$ or f' is not defined.
2. Use the critical values to divide I into subintervals.
3. Pick any point p in each subinterval, and find the sign of $f'(p)$.
   (a) If $f'(p) > 0$, then f is increasing on that subinterval.
   (b) If $f'(p) < 0$, then f is decreasing on that subinterval.

Note: Recall that not all points c where $f'(c)$ is undefined are critical points. It could be that $f'(c)$ is undefined because c is not in the domain of f; for example, at a vertical asymptote. Even though these points are not critical points, we still include them in our sign diagram, since it's possible that f' changes sign at such a point.

To implement Key Idea 7.2.1, we use a visual aid called a **sign diagram** for f'. A sign diagram for a function g consists of the following:

- A number line representing the domain of the function g.
- A solid dot marking each point x where $g(x) = 0$.
- A hollow dot marking each point where $g(x)$ is undefined.
- Between each pair of dots, either a $+$ sign or $-$ sign to indicate whether the function is positive or negative on that interval.

We demonstrate using this process in the following example.

Example 7.2.1 Finding intervals of increasing/decreasing

Let $f(x) = x^3 + x^2 - x + 1$. Find intervals on which f is increasing or decreasing.

Solution Using Key Idea 7.2.1, we first find the critical values of f. We have $f'(x) = 3x^2 + 2x - 1 = (3x - 1)(x + 1)$, so $f'(x) = 0$ when $x = -1$ and when $x = 1/3$. f' is never undefined.

Since an interval was not specified for us to consider, we consider the entire domain of f which is $(-\infty, \infty)$. We thus break the whole real line into three subintervals based on the two critical values we just found: $(-\infty, -1)$, $(-1, 1/3)$ and $(1/3, \infty)$. This is shown in Figure 7.2.3.

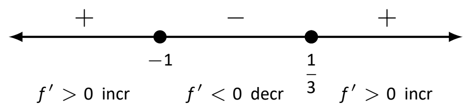


Figure 7.2.3: Sign diagram for f' in Example 7.2.1.

We now pick a value p in each subinterval and find the sign of $f'(p)$. All we care about is the sign, so we do not actually have to fully compute $f'(p)$; pick "nice" values that make this simple.

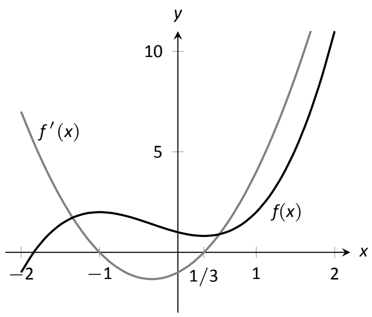


Figure 7.2.4: A graph of $f(x)$ in Example 7.2.1, showing where f is increasing and decreasing.

Subinterval 1, $(-\infty, -1)$: We (arbitrarily) pick $p = -2$. We can compute $f'(-2)$ directly: $f'(-2) = 3(-2)^2 + 2(-2) - 1 = 7 > 0$. We conclude that f is increasing on $(-\infty, -1)$.

Note we can arrive at the same conclusion without computation. For instance, we could choose $p = -100$. The first term in $f'(-100)$, i.e., $3(-100)^2$ is clearly positive and very large. The other terms are small in comparison, so we know $f'(-100) > 0$. All we need is the sign.

Subinterval 2, $(-1, 1/3)$: We pick $p = 0$ since that value seems easy to deal with. $f'(0) = -1 < 0$. We conclude f is decreasing on $(-1, 1/3)$.

Subinterval 3, $(1/3, \infty)$: Pick an arbitrarily large value for $p > 1/3$ and note that $f'(p) = 3p^2 + 2p - 1 > 0$. We conclude that f is increasing on $(1/3, \infty)$.

We can verify our calculations by considering Figure 7.2.4, where f is graphed. The graph also presents f'; note how $f' > 0$ when f is increasing and $f' < 0$ when f is decreasing.

One is justified in wondering why so much work is done when the graph seems to make the intervals very clear. We give three reasons why the above work is worthwhile.

First, the points at which f switches from increasing to decreasing are not precisely known given a graph. The graph shows us something significant happens near $x = -1$ and $x = 0.3$, but we cannot determine exactly where from the graph.

One could argue that just finding critical values is important; once we know the significant points are $x = -1$ and $x = 1/3$, the graph shows the increasing/decreasing traits just fine. That is true. However, the technique prescribed here helps reinforce the relationship between increasing/decreasing and the sign of f'. Once mastery of this concept (and several others) is obtained, one finds that either (a) just the critical points are computed and the graph shows all else that is desired, or (b) a graph is never produced, because determining increasing/decreasing using f' is straightforward and the graph is unnecessary. So our second reason why the above work is worthwhile is this: once mastery of a subject is gained, one has *options* for finding needed information.

Finally, our third reason: many problems we face "in the real world" are very complex. Solutions are tractable only through the use of computers to do many calculations for us. Computers do not solve problems "on their own," however; they need to be taught (i.e., *programmed*) to do the right things. It would be beneficial to give a function to a computer and have it return maximum and minimum values, intervals on which the function is increasing and decreasing, the locations of relative maxima, etc. The work that we are doing here is easily programmable. It is hard to teach a computer to "look at the graph and see if it is going up or down." It is easy to teach a computer to "determine if a number is greater than or less than 0."

In Section 7.1 we learned the definition of relative maxima and minima and found that they occur at critical points. We are now learning that functions can switch from increasing to decreasing (and vice–versa) at critical points. This new understanding of increasing and decreasing creates a great method of determining whether a critical point corresponds to a maximum, minimum, or neither. Imagine a function increasing until a critical point at $x = c$, after which it decreases. A quick sketch helps confirm that $f(c)$ must be a relative maximum. A similar statement can be made for relative minimums. We formalize this concept in a theorem.

Theorem 7.2.2 First Derivative Test

Let f be differentiable on an interval I and let c be a critical number in I.

1. If the sign of f' switches from positive to negative at c, then $f(c)$ is a relative maximum of f.
2. If the sign of f' switches from negative to positive at c, then $f(c)$ is a relative minimum of f.
3. If f' is positive (or, negative) before and after c, then $f(c)$ is not a relative extrema of f.

Example 7.2.2 Using the First Derivative Test

Find the intervals on which f is increasing and decreasing, and use the First Derivative Test to determine the relative extrema of f, where

$$f(x) = \frac{x^2+3}{x-1}.$$

Solution We start by noting the domain of f: $(-\infty, 1) \cup (1, \infty)$. Key Idea 7.2.1 describes how to find intervals where f is increasing and decreasing *when the domain of f is an interval.* Since the domain of f in this example is the union of two intervals, we apply the techniques of Key Idea 7.2.1 to both intervals of the domain of f.

Since f is not defined at $x = 1$, the increasing/decreasing nature of f could switch at this value. We do not formally consider $x = 1$ to be a critical value of f, but we will include it in our list of critical values that we find next.

Using the Quotient Rule, we find

$$f'(x) = \frac{x^2-2x-3}{(x-1)^2}.$$

We need to find the critical values of f; we want to know when $f'(x) = 0$ and when f' is not defined. That latter is straightforward: when the denominator of $f'(x)$ is 0, f' is undefined. That occurs when $x = 1$, which we've already recognized as an important value.

$f'(x) = 0$ when the numerator of $f'(x)$ is 0. That occurs when $x^2 - 2x - 3 = (x-3)(x+1) = 0$; i.e., when $x = -1, 3$.

We have found that f has two critical numbers, $x = -1, 3$, and at $x = 1$ something important might also happen. These three numbers divide the real number line into 4 subintervals:

$$(-\infty, -1), \quad (-1, 1), \quad (1, 3) \quad \text{and} \quad (3, \infty).$$

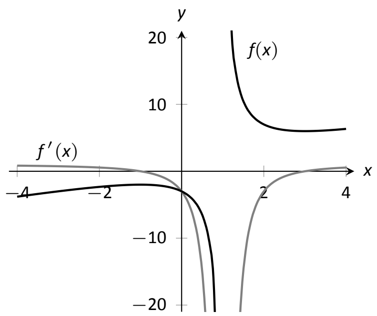


Figure 7.2.6: A graph of $f(x)$ in Example 7.2.2, showing where f is increasing and decreasing.

Pick a number p from each subinterval and test the sign of f' at p to determine whether f is increasing or decreasing on that interval. Again, we do well to avoid complicated computations; notice that the denominator of f' is *always* positive so we can ignore it during our work.

Interval 1, $(-\infty, -1)$: Choosing a very small number (i.e., a negative number with a large magnitude) p returns $p^2 - 2p - 3$ in the numerator of f'; that will be positive. Hence f is increasing on $(-\infty, -1)$.

Interval 2, $(-1, 1)$: Choosing 0 seems simple: $f'(0) = -3 < 0$. We conclude f is decreasing on $(-1, 1)$.

Interval 3, $(1, 3)$: Choosing 2 seems simple: $f'(2) = -3 < 0$. Again, f is decreasing.

Interval 4, $(3, \infty)$: Choosing an very large number p from this subinterval will give a positive numerator and (of course) a positive denominator. So f is increasing on $(3, \infty)$.

In summary, f is increasing on the set $(-\infty, -1) \cup (3, \infty)$ and is decreasing on the set $(-1, 1) \cup (1, 3)$. Since at $x = -1$, the sign of f' switched from positive to negative, Theorem 7.2.2 states that $f(-1)$ is a relative maximum of f. At $x = 3$, the sign of f' switched from negative to positive, meaning $f(3)$ is a relative minimum. At $x = 1$, f is not defined, so there is no relative extrema at $x = 1$.

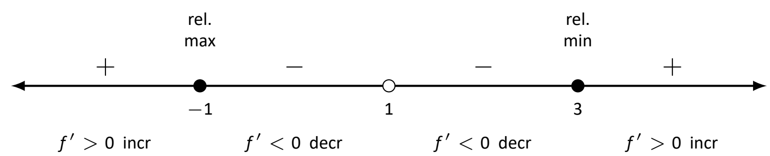


Figure 7.2.5: Sign diagram for f' in Example 7.2.2.

This is summarized in the number line shown in Figure 7.2.5. Also, Figure 7.2.6 shows a graph of f, confirming our calculations. This figure also shows f', again demonstrating that f is increasing when $f' > 0$ and decreasing when $f' < 0$.

Note: with a bit of practice, you might find that you can fill out sign diagrams quickly, without needing to use test values in each interval. One strategy is the following: start on the far left (or far right). Determine the sign in the first interval, and work left-to-right (or right-to-left). Each time you pass a point where f' is zero or undefined, check the factored expression for f'. Did this point come from an even power, or an odd power? If the power is even, leave the sign unchanged. If the power is odd, change the sign. In Example 7.2.2, the critical numbers −1 and 3 come from odd powers. (Recall $(x + 1) = (x + 1)^1$.) The vertical asymptote contributes the even power $(x - 1)^2$ in the denominator. Thus, we see sign changes at −1 and 3, but the sign is the same on either side of 1.

One is often tempted to think that functions always alternate "increasing, decreasing, increasing, decreasing,. . ." around critical values. Our previous example demonstrated that this is not always the case. While $x = 1$ was not technically a critical value, it was an important value we needed to consider. We found that f was decreasing on "both sides of $x = 1$."

We examine one more example.

Example 7.2.3 Using the First Derivative Test

Find the intervals on which $f(x) = x^{8/3} - 4x^{2/3}$ is increasing and decreasing and identify the relative extrema.

Solution We again start with taking a derivative. Since we know we want to solve $f'(x) = 0$, we will do some algebra after taking the derivative.

$$\begin{aligned} f(x) &= x^{\frac{8}{3}} - 4x^{\frac{2}{3}} \\ f'(x) &= \frac{8}{3}x^{\frac{5}{3}} - \frac{8}{3}x^{-\frac{1}{3}} \\ &= \frac{8}{3}x^{-\frac{1}{3}}\left(x^{\frac{6}{3}} - 1\right) \\ &= \frac{8}{3}x^{-\frac{1}{3}}(x^2 - 1) \\ &= \frac{8}{3}x^{-\frac{1}{3}}(x - 1)(x + 1). \end{aligned}$$

This derivation of f' shows that $f'(x) = 0$ when $x = \pm 1$ and f' is not defined when $x = 0$. Thus we have 3 critical values, breaking the number line into 4 subintervals as shown in Figure 7.2.7.

Interval 1, $(\infty, -1)$: We choose $p = -2$; we can easily verify that $f'(-2) < 0$. So f is decreasing on $(-\infty, -1)$.
Interval 2, $(-1, 0)$: Choose $p = -1/2$. Once more we practice finding the sign of $f'(p)$ without computing an actual value. We have $f'(p) = (8/3)p^{-1/3}(p-1)(p+1)$; find the sign of each of the three terms.

$$f'(p) = \frac{8}{3} \cdot \underbrace{p^{-\frac{1}{3}}}_{<0} \cdot \underbrace{(p-1)}_{<0} \underbrace{(p+1)}_{>0}.$$

We have a "negative $\times$ negative $\times$ positive" giving a positive number; f is increasing on $(-1, 0)$.
Interval 3, $(0, 1)$: We do a similar sign analysis as before, using p in $(0, 1)$.

$$f'(p) = \frac{8}{3} \cdot \underbrace{p^{-\frac{1}{3}}}_{>0} \cdot \underbrace{(p-1)}_{<0} \underbrace{(p+1)}_{>0}.$$

We have 2 positive factors and one negative factor; $f'(p) < 0$ and so f is decreasing on $(0, 1)$.
Interval 4, $(1, \infty)$: Similar work to that done for the other three intervals shows that $f'(x) > 0$ on $(1, \infty)$, so f is increasing on this interval.

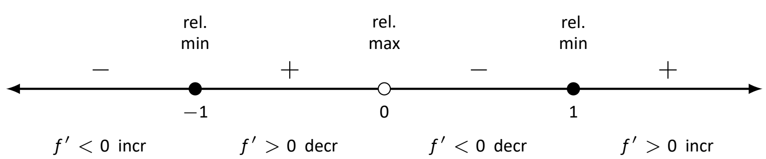


Figure 7.2.7: Sign diagram for f' in Example 7.2.3.

We conclude by stating that f is increasing on $(-1, 0) \cup (1, \infty)$ and decreasing on $(-\infty, -1) \cup (0, 1)$. The sign of f' changes from negative to positive around $x = -1$ and $x = 1$, meaning by Theorem 7.2.2 that $f(-1)$ and $f(1)$ are relative minima of f. As the sign of f' changes from positive to negative at $x = 0$, we have a relative maximum at $f(0)$. Figure 7.2.8 shows a graph of f, confirming our result. We also graph f', highlighting once more that f is increasing when $f' > 0$ and is decreasing when $f' < 0$.

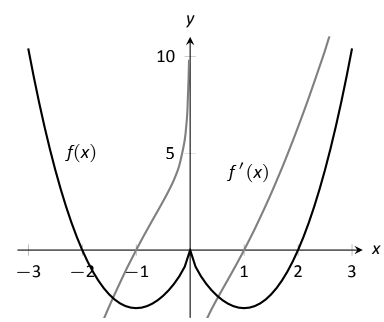


Figure 7.2.8: A graph of $f(x)$ in Example 7.2.3, showing where f is increasing and decreasing.

We have seen how the first derivative of a function helps determine when the function is going "up" or "down." In the next section, we will see how the second derivative helps determine how the graph of a function curves.

Exercises 7.2

Terms and Concepts

1. In your own words describe what it means for a function to be increasing.

2. What does a decreasing function "look like"?

3. Sketch a graph of a function on $[0, 2]$ that is increasing, where it is increasing "quickly" near $x = 0$ and increasing "slowly" near $x = 2$.

4. Give an example of a function describing a situation where it is "bad" to be increasing and "good" to be decreasing.

5. T/F: Functions always switch from increasing to decreasing, or decreasing to increasing, at critical points.

6. A function f has derivative $f'(x) = (\sin x + 2)e^{x^2+1}$, where $f'(x) > 1$ for all x. Is f increasing, decreasing, or can we not tell from the given information?

Problems

In Exercises 7 – 14, a function $f(x)$ is given.

(a) Compute $f'(x)$.

(b) Graph f and f' on the same axes (using technology is permitted) and verify Theorem 7.2.1.

7. $f(x) = 2x + 3$

8. $f(x) = x^2 - 3x + 5$

9. $f(x) = \cos x$

10. $f(x) = \tan x$

11. $f(x) = x^3 - 5x^2 + 7x - 1$

12. $f(x) = 2x^3 - x^2 + x - 1$

13. $f(x) = x^4 - 5x^2 + 4$

14. $f(x) = \dfrac{1}{x^2 + 1}$

In Exercises 15 – 24, a function $f(x)$ is given.

(a) Give the domain of f.

(b) Find the critical numbers of f.

(c) Create a number line to determine the intervals on which f is increasing and decreasing.

(d) Use the First Derivative Test to determine whether each critical point is a relative maximum, minimum, or neither.

15. $f(x) = x^2 + 2x - 3$

16. $f(x) = x^3 + 3x^2 + 3$

17. $f(x) = 2x^3 + x^2 - x + 3$

18. $f(x) = x^3 - 3x^2 + 3x - 1$

19. $f(x) = \dfrac{1}{x^2 - 2x + 2}$

20. $f(x) = \dfrac{x^2 - 4}{x^2 - 1}$

21. $f(x) = \dfrac{x}{x^2 - 2x - 8}$

22. $f(x) = \dfrac{(x-2)^{2/3}}{x}$

23. $f(x) = \sin x \cos x$ on $(-\pi, \pi)$.

24. $f(x) = x^5 - 5x$

Review

25. Consider $f(x) = x^2 - 3x + 5$ on $[-1, 2]$; find c guaranteed by the Mean Value Theorem.

26. Consider $f(x) = \sin x$ on $[-\pi/2, \pi/2]$; find c guaranteed by the Mean Value Theorem.

7.3 Concavity and the Second Derivative

Our study of "nice" functions continues. The previous section showed how the first derivative of a function, f', can relay important information about f. We now apply the same technique to f' itself, and learn what this tells us about f.

The key to studying f' is to consider its derivative, namely f'', which is the second derivative of f. When $f'' > 0$, f' is increasing. When $f'' < 0$, f' is decreasing. f' has relative maxima and minima where $f'' = 0$ or is undefined.

This section explores how knowing information about f'' gives information about f.

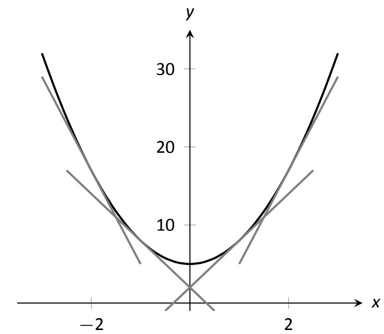


Figure 7.3.1: A function f with a concave up graph. Notice how the slopes of the tangent lines, when looking from left to right, are increasing.

Concavity

We begin with a definition, then explore its meaning.

> **Definition 7.3.1 Concave Up and Concave Down**
>
> Let f be differentiable on an interval I. The graph of f is **concave up** on I if f' is increasing. The graph of f is **concave down** on I if f' is decreasing. If f' is constant then the graph of f is said to have **no concavity**.

Note: We often state that "f is concave up" instead of "the graph of f is concave up" for simplicity.

Note: A mnemonic for remembering what concave up/down means is: "Concave up is like a cup; concave down is like a frown." It is admittedly terrible, but it works.

The graph of a function f is *concave up* when f' is *increasing.* That means as one looks at a concave up graph from left to right, the slopes of the tangent lines will be increasing. Consider Figure 7.3.1, where a concave up graph is shown along with some tangent lines. Notice how the tangent line on the left is steep, downward, corresponding to a small value of f'. On the right, the tangent line is steep, upward, corresponding to a large value of f'.

If a function is decreasing and concave up, then its rate of decrease is slowing; it is "levelling off." If the function is increasing and concave up, then the *rate* of increase is increasing. The function is increasing at a faster and faster rate.

Now consider a function which is concave down. We essentially repeat the above paragraphs with slight variation.

The graph of a function f is *concave down* when f' is *decreasing.* That means as one looks at a concave down graph from left to right, the slopes of the tangent lines will be decreasing. Consider Figure 7.3.2, where a concave down graph is shown along with some tangent lines. Notice how the tangent line on the left is steep, upward, corresponding to a large value of f'. On the right, the tangent line is steep, downward, corresponding to a small value of f'.

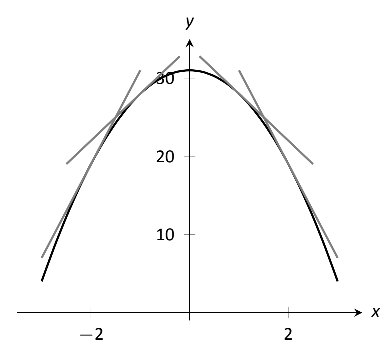


Figure 7.3.2: A function f with a concave down graph. Notice how the slopes of the tangent lines, when looking from left to right, are decreasing.

If a function is increasing and concave down, then its rate of increase is slowing; it is "levelling off." If the function is decreasing and concave down, then the *rate* of decrease is decreasing. The function is decreasing at a faster and faster rate.

Our definition of concave up and concave down is given in terms of when the first derivative is increasing or decreasing. We can apply the results of the previous section and to find intervals on which a graph is concave up or down. That is, we recognize that f' is increasing when $f'' > 0$, etc.

> **Theorem 7.3.1 Test for Concavity**
>
> Let f be twice differentiable on an interval I. The graph of f is concave up if $f'' > 0$ on I, and is concave down if $f'' < 0$ on I.

Note: Geometrically speaking, a function is concave up if its graph lies above its tangent lines. A function is concave down if its graph lies below its tangent lines.

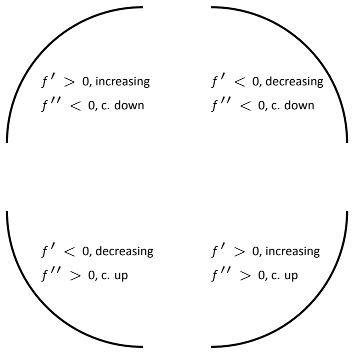


Figure 7.3.3: Demonstrating the 4 ways that concavity interacts with increasing/decreasing, along with the relationships with the first and second derivatives.

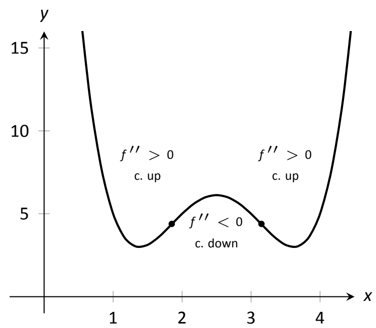


Figure 7.3.4: A graph of a function with its inflection points marked. The intervals where concave up/down are also indicated.

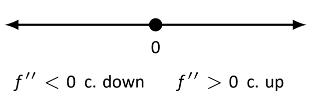


Figure 7.3.5: A sign diagram for f'' determining the concavity of f in Example 7.3.1.

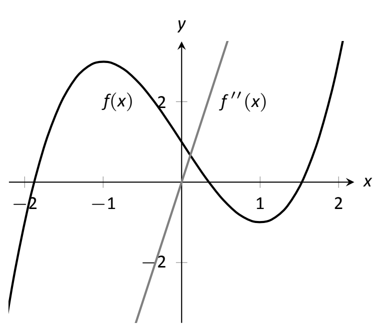


Figure 7.3.6: A graph of $f(x)$ used in Example 7.3.1.

If knowing where a graph is concave up/down is important, it makes sense that the places where the graph changes from one to the other is also important. This leads us to a definition.

> **Definition 7.3.2 Point of Inflection**
>
> A **point of inflection** is a point on the graph of f at which the concavity of f changes.

Figure 7.3.4 shows a graph of a function with inflection points labelled.

If the concavity of f changes at a point $(c, f(c))$, then f' is changing from increasing to decreasing (or, decreasing to increasing) at $x = c$. That means that the sign of f'' is changing from positive to negative (or, negative to positive) at $x = c$. This leads to the following theorem.

> **Theorem 7.3.2 Points of Inflection**
>
> If $(c, f(c))$ is a point of inflection on the graph of f, then either $f''(c) = 0$ or f'' is not defined at c.

We have identified the concepts of concavity and points of inflection. It is now time to practice using these concepts; given a function, we should be able to find its points of inflection and identify intervals on which it is concave up or down. We do so in the following examples.

Example 7.3.1 Finding intervals of concave up/down, inflection points

Let $f(x) = x^3 - 3x + 1$. Find the inflection points of f and the intervals on which it is concave up/down.

Solution We start by finding $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$. To find the inflection points, we use Theorem 7.3.2 and find where $f''(x) = 0$ or where f'' is undefined. We find f'' is always defined, and is 0 only when $x = 0$. So the point $(0, 1)$ is the only possible point of inflection.

This possible inflection point divides the real line into two intervals, $(-\infty, 0)$ and $(0, \infty)$. We use a process similar to the one used in the previous section to determine increasing/decreasing. Pick any $c < 0$; $f''(c) < 0$ so f is concave down on $(-\infty, 0)$. Pick any $c > 0$; $f''(c) > 0$ so f is concave up on $(0, \infty)$. Since the concavity changes at $x = 0$, the point $(0, 1)$ is an inflection point.

The number line in Figure 7.3.5 illustrates the process of determining concavity; Figure 7.3.6 shows a graph of f and f'', confirming our results. Notice how f is concave down precisely when $f''(x) < 0$ and concave up when $f''(x) > 0$.

Example 7.3.2 Finding intervals of concave up/down, inflection points

Let $f(x) = x/(x^2 - 1)$. Find the inflection points of f and the intervals on which it is concave up/down.

Solution We need to find f' and f''. Using the Quotient Rule and simplifying, we find

$$f'(x) = \frac{-(1 + x^2)}{(x^2 - 1)^2} \quad \text{and} \quad f''(x) = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}.$$

To find the possible points of inflection, we seek to find where $f''(x) = 0$ and where f'' is not defined. Solving $f''(x) = 0$ reduces to solving $2x(x^2 + 3) = 0$; we find $x = 0$. We find that f'' is not defined when $x = \pm 1$, for then the denominator of f'' is 0. We also note that f itself is not defined at $x = \pm 1$, having a domain of $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. Since the domain of f is the union of three intervals, it makes sense that the concavity of f could switch across intervals. We technically cannot say that f has a point of inflection at $x = \pm 1$ as they are not part of the domain, but we must still consider these x-values to be important and will include them in our number line.

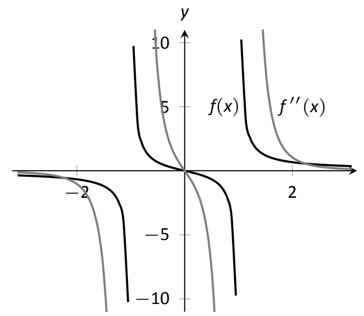


Figure 7.3.8: A graph of $f(x)$ and $f''(x)$ in Example 7.3.2.

The important x-values at which concavity might switch are $x = -1$, $x = 0$ and $x = 1$, which split the number line into four intervals as shown in Figure 7.3.7. We determine the concavity on each. Keep in mind that all we are concerned with is the *sign* of f'' on the interval.

Interval 1, $(-\infty, -1)$: Select a number c in this interval with a large magnitude (for instance, $c = -100$). The denominator of $f''(x)$ will be positive. In the numerator, the $(c^2 + 3)$ will be positive and the $2c$ term will be negative. Thus the numerator is negative and $f''(c)$ is negative. We conclude f is concave down on $(-\infty, -1)$.
Interval 2, $(-1, 0)$: For any number c in this interval, the term $2c$ in the numerator will be negative, the term $(c^2 + 3)$ in the numerator will be positive, and the term $(c^2 - 1)^3$ in the denominator will be negative. Thus $f''(c) > 0$ and f is concave up on this interval.
Interval 3, $(0, 1)$: Any number c in this interval will be positive and "small." Thus the numerator is positive while the denominator is negative. Thus $f''(c) < 0$ and f is concave down on this interval.
Interval 4, $(1, \infty)$: Choose a large value for c. It is evident that $f''(c) > 0$, so we conclude that f is concave up on $(1, \infty)$.

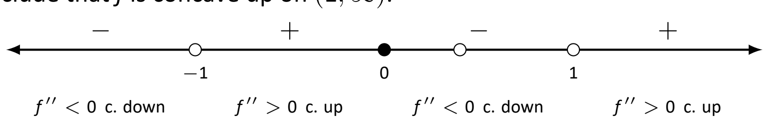


Figure 7.3.7: Sign diagram for f'' in Example 7.3.2.

We conclude that f is concave up on $(-1, 0) \cup (1, \infty)$ and concave down on $(-\infty, -1) \cup (0, 1)$. There is only one point of inflection, $(0, 0)$, as f is not defined at $x = \pm 1$. Our work is confirmed by the graph of f in Figure 7.3.8. Notice how f is concave up whenever f'' is positive, and concave down when f'' is negative.

Recall that relative maxima and minima of f are found at critical points of f; that is, they are found when $f'(x) = 0$ or when f' is undefined. Likewise, the relative maxima and minima of f' are found when $f''(x) = 0$ or when f'' is undefined; note that these are the inflection points of f.

What does a "relative maximum of f'" *mean*? The derivative measures the rate of change of f; maximizing f' means finding where f is increasing the most – where f has the steepest tangent line. A similar statement can be made for minimizing f'; it corresponds to where f has the steepest negatively–sloped tangent line.

We utilize this concept in the next example.

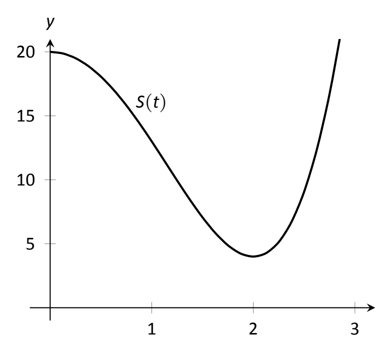


Figure 7.3.9: A graph of $S(t)$ in Example 7.3.3, modelling the sale of a product over time.

Example 7.3.3 Understanding inflection points
The sales of a certain product over a three-year span are modelled by $S(t) = t^4 - 8t^2 + 20$, where t is the time in years, shown in Figure 7.3.9. Over the first two years, sales are decreasing. Find the point at which sales are decreasing at their greatest rate.

Solution We want to maximize the rate of decrease, which is to say, we want to find where S' has a minimum. To do this, we find where S'' is 0. We find $S'(t) = 4t^3 - 16t$ and $S''(t) = 12t^2 - 16$. Setting $S''(t) = 0$ and solving, we get $t = \sqrt{4/3} \approx 1.16$ (we ignore the negative value of t since it does not lie in the domain of our function S).

This is both the inflection point and the point of maximum decrease. This is the point at which things first start looking up for the company. After the inflection point, it will still take some time before sales start to increase, but at least sales are not decreasing quite as quickly as they had been.

A graph of $S(t)$ and $S'(t)$ is given in Figure 7.3.10. When $S'(t) < 0$, sales are decreasing; note how at $t \approx 1.16$, $S'(t)$ is minimized. That is, sales are decreasing at the fastest rate at $t \approx 1.16$. On the interval of $(1.16, 2)$, S is decreasing but concave up, so the decline in sales is "levelling off."

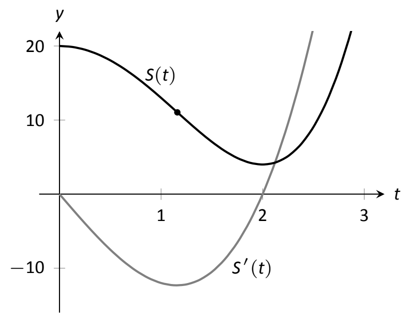


Figure 7.3.10: A graph of $S(t)$ in Example 7.3.3 along with $S'(t)$.

Not every critical point corresponds to a relative extrema; $f(x) = x^3$ has a critical point at $(0, 0)$ but no relative maximum or minimum. Likewise, just because $f''(x) = 0$ we cannot conclude concavity changes at that point. We were careful before to use terminology "*possible* point of inflection" since we needed to check to see if the concavity changed. The canonical example of $f''(x) = 0$ *without* concavity changing is $f(x) = x^4$. At $x = 0$, $f''(x) = 0$ but f is always concave up, as shown in Figure 7.3.11.

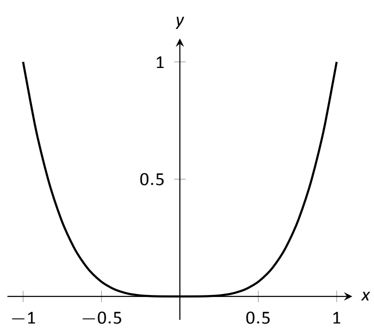


Figure 7.3.11: A graph of $f(x) = x^4$. Clearly f is always concave up, despite the fact that $f''(x) = 0$ when $x = 0$. It this example, the *possible* point of inflection $(0, 0)$ is not a point of inflection.

The Second Derivative Test

The first derivative of a function gave us a test to find if a critical value corresponded to a relative maximum, minimum, or neither. The second derivative gives us another way to test if a critical point is a local maximum or minimum. The following theorem officially states something that is intuitive: if a critical value occurs in a region where a function f is concave up, then that critical value must correspond to a relative minimum of f, etc. See Figure 7.3.12 for a visualization of this.

Theorem 7.3.3 The Second Derivative Test

Let c be a critical value of f where $f''(c)$ is defined.

1. If $f''(c) > 0$, then f has a local minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a local maximum at $(c, f(c))$.

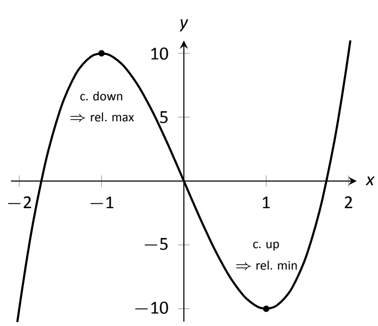


Figure 7.3.12: Demonstrating the fact that relative maxima occur when the graph is concave down and relative minima occur when the graph is concave up.

The Second Derivative Test relates to the First Derivative Test in the following way. If $f''(c) > 0$, then the graph is concave up at a critical point c and f' itself is growing. Since $f'(c) = 0$ and f' is growing at c, then it must go from negative to positive at c. This means the function goes from decreasing to increasing, indicating a local minimum at c.

Example 7.3.4 Using the Second Derivative Test

Let $f(x) = 100/x + x$. Find the critical points of f and use the Second Derivative Test to label them as relative maxima or minima.

Solution We find $f'(x) = -100/x^2 + 1$ and $f''(x) = 200/x^3$. We set $f'(x) = 0$ and solve for x to find the critical values (note that f' is not defined at $x = 0$, but neither is f so this is not a critical value.) We find the critical values are $x = \pm 10$. Evaluating f'' at $x = 10$ gives $0.1 > 0$, so there is a local minimum at $x = 10$. Evaluating $f''(-10) = -0.1 < 0$, determining a relative maximum at $x = -10$. These results are confirmed in Figure 7.3.13.

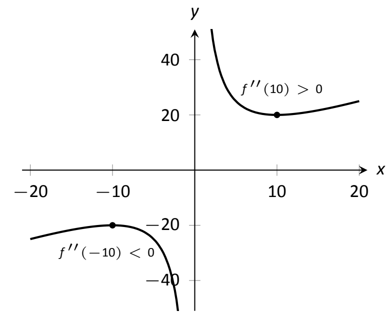


Figure 7.3.13: A graph of $f(x)$ in Example 7.3.4. The second derivative is evaluated at each critical point. When the graph is concave up, the critical point represents a local minimum; when the graph is concave down, the critical point represents a local maximum.

We have been learning how the first and second derivatives of a function relate information about the graph of that function. We have found intervals of increasing and decreasing, intervals where the graph is concave up and down, along with the locations of relative extrema and inflection points. In Chapter 5 we saw how limits explained asymptotic behaviour. In the next section we combine all of this information to produce accurate sketches of functions.

Exercises 7.3

Terms and Concepts

1. Sketch a graph of a function $f(x)$ that is concave up on $(0,1)$ and is concave down on $(1,2)$.

2. Sketch a graph of a function $f(x)$ that is:

   (a) Increasing, concave up on $(0,1)$,

   (b) increasing, concave down on $(1,2)$,

   (c) decreasing, concave down on $(2,3)$ and

   (d) increasing, concave down on $(3,4)$.

3. Is is possible for a function to be increasing and concave down on $(0,\infty)$ with a horizontal asymptote of $y=1$? If so, give a sketch of such a function.

4. Is is possible for a function to be increasing and concave up on $(0,\infty)$ with a horizontal asymptote of $y=1$? If so, give a sketch of such a function.

Problems

In Exercises 5 – 14, a function $f(x)$ is given.

(a) Compute $f''(x)$.

(b) Graph f and f'' on the same axes (using technology is permitted) and verify Theorem 7.3.1.

5. $f(x)=-7x+3$

6. $f(x)=-4x^2+3x-8$

7. $f(x)=4x^2+3x-8$

8. $f(x)=x^3-3x^2+x-1$

9. $f(x)=-x^3+x^2-2x+5$

10. $f(x)=\sin x$

11. $f(x)=\tan x$

12. $f(x)=\dfrac{1}{x^2+1}$

13. $f(x)=\dfrac{1}{x}$

14. $f(x)=\dfrac{1}{x^2}$

In Exercises 15 – 28, a function $f(x)$ is given.

(a) Find the possible points of inflection of f.

(b) Create a number line to determine the intervals on which f is concave up or concave down.

15. $f(x)=x^2-2x+1$

16. $f(x)=-x^2-5x+7$

17. $f(x)=x^3-x+1$

18. $f(x)=2x^3-3x^2+9x+5$

19. $f(x)=\dfrac{x^4}{4}+\dfrac{x^3}{3}-2x+3$

20. $f(x)=-3x^4+8x^3+6x^2-24x+2$

21. $f(x)=x^4-4x^3+6x^2-4x+1$

22. $f(x)=\sec x$ on $(-3\pi/2, 3\pi/2)$

23. $f(x)=\dfrac{1}{x^2+1}$

24. $f(x)=\dfrac{x}{x^2-1}$

25. $f(x)=\sin x+\cos x$ on $(-\pi,\pi)$

26. $f(x)=x^2e^x$

27. $f(x)=x^2\ln x$

28. $f(x)=e^{-x^2}$

In Exercises 29 – 42, a function $f(x)$ is given. Find the critical points of f and use the Second Derivative Test, when possible, to determine the relative extrema. (Note: these are the same functions as in Exercises 15 – 28.)

29. $f(x)=x^2-2x+1$

30. $f(x)=-x^2-5x+7$

31. $f(x)=x^3-x+1$

32. $f(x)=2x^3-3x^2+9x+5$

33. $f(x)=\dfrac{x^4}{4}+\dfrac{x^3}{3}-2x+3$

34. $f(x)=-3x^4+8x^3+6x^2-24x+2$

35. $f(x)=x^4-4x^3+6x^2-4x+1$

36. $f(x)=\sec x$ on $(-3\pi/2, 3\pi/2)$

37. $f(x) = \dfrac{1}{x^2+1}$

38. $f(x) = \dfrac{x}{x^2-1}$

39. $f(x) = \sin x + \cos x$ on $(-\pi, \pi)$

40. $f(x) = x^2e^x$

41. $f(x) = x^2 \ln x$

42. $f(x) = e^{-x^2}$

In Exercises 43 – 56, a function $f(x)$ is given. Find the x values where $f'(x)$ has a relative maximum or minimum. (Note: these are the same functions as in Exercises 15 – 28.)

43. $f(x) = x^2 - 2x + 1$

44. $f(x) = -x^2 - 5x + 7$

45. $f(x) = x^3 - x + 1$

46. $f(x) = 2x^3 - 3x^2 + 9x + 5$

47. $f(x) = \dfrac{x^4}{4} + \dfrac{x^3}{3} - 2x + 3$

48. $f(x) = -3x^4 + 8x^3 + 6x^2 - 24x + 2$

49. $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$

50. $f(x) = \sec x$ on $(-3\pi/2, 3\pi/2)$

51. $f(x) = \dfrac{1}{x^2+1}$

52. $f(x) = \dfrac{x}{x^2-1}$

53. $f(x) = \sin x + \cos x$ on $(-\pi, \pi)$

54. $f(x) = x^2e^x$

55. $f(x) = x^2 \ln x$

56. $f(x) = e^{-x^2}$

7.4 Curve Sketching

We have been learning how we can understand the behaviour of a function based on its first and second derivatives. While we have been treating the properties of a function separately (increasing and decreasing, concave up and concave down, etc.), we combine them here to produce an accurate graph of the function without plotting lots of extraneous points.

Why bother? Graphing utilities are very accessible, whether on a computer, a hand–held calculator, or a smartphone. These resources are usually very fast and accurate. We will see that our method is not particularly fast – it will require time (but it is not *hard*). So again: why bother?

We are attempting to understand the behaviour of a function f based on the information given by its derivatives. While all of a function's derivatives relay information about it, it turns out that "most" of the behaviour we care about is explained by f' and f''. Understanding the interactions between the graph of f and f' and f'' is important. To gain this understanding, one might argue that all that is needed is to look at lots of graphs. This is true to a point, but is somewhat similar to stating that one understands how an engine works after looking only at pictures. It is true that the basic ideas will be conveyed, but "hands–on" access increases understanding.

The following Key Idea summarizes what we have learned so far that is applicable to sketching graphs of functions and gives a framework for putting that information together. It is followed by several examples.

Key Idea 7.4.1 Curve Sketching

To produce an accurate sketch a given function f, consider the following steps.

1. Find the domain of f. Generally, we assume that the domain is the entire real line then find restrictions, such as where a denominator is 0 or where negatives appear under the radical.
2. Find the x- and y-intercepts of f, if possible; construct a sign diagram for f.
3. Find the location of any vertical asymptotes of f (usually done in conjunction with item 2 above). Use your sign diagram to determine whether $f(x)$ is approaching ∞ or *infty* on either side of each vertical asymptote.
4. Consider the limits $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to\infty} f(x)$ to determine the end behaviour of the function.
5. Compute f', and find the critical points of f.

(continued)

Key Idea 7.4.1 Curve Sketching – Continued

6. Construct a sign diagram for f'; classify the critical points using the first derivative test. Determine the intervals on which f is increasing or decreasing.
7. Compute f'' and find the possible points of inflection of f.
8. Construct a sign diagram for f'', and determine the intervals on which the graph of f is concave up or concave down.
9. Plot the intercepts and asymptotes of f on a set of coordinate axes. Roughly sketch the behaviour of f near the asymptotes. Then plot the critical points and inflection points.
10. Sketch the graph of f by connecting the points plotted so far with curves exhibiting the proper concavity. Sketch asymptotes and x and y intercepts where applicable.

Example 7.4.1 Curve sketching

Use Key Idea 7.4.1 to sketch $f(x) = 3x^3 - 10x^2 + 7x + 5$.

Solution We follow the steps outlined in the Key Idea.

1. The domain of f is the entire real line; there are no values x for which $f(x)$ is not defined.
2. The y-intercept is given by $f(0) = 5$. Determining the x-intercepts would involve finding the (quite likely irrational) zeros of a cubic polynomial, so we skip this step for now. (We may have to settle for approximate zeros later.) Since we don't know the zeros of f, we can't construct a sign diagram for f.
3. There are no vertical asymptotes, since the domain of f is $\mathbb{R}$.
4. We determine the end behaviour using limits as x approaches $\pm$infinity.
$$\lim_{x\to-\infty} f(x) = -\infty \qquad \lim_{x\to\infty} f(x) = \infty.$$
We do not have any horizontal asymptotes. (But it is still useful to know the direction in which the graph is headed at either end.)
5. Find the critical points of f. We compute $f'(x) = 9x^2 - 20x + 7$. Use the Quadratic Formula to find the roots of f':
$$x = \frac{20 \pm \sqrt{(-20)^2 - 4(9)(7)}}{2(9)} = \frac{1}{9}\left(10 \pm \sqrt{37}\right) \Rightarrow x \approx 0.435, 1.787.$$
6. Construct a sign diagram for f'. We found that the critical points of f are
$$c_1 = \frac{10 - \sqrt{37}}{9} < \frac{10 + \sqrt{37}}{9} = c_2.$$
With $f'(x) = 9(x - c_1)(x - c_2)$ we quickly see that $f'(x) > 0$ for $x < c_1$ or $x > c_2$, and $f'(x) < 0$ for $c_1 < x < c_2$.

The sign diagram for f' is given by:

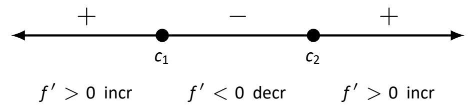


Figure 7.4.1: Sign diagram for f' in Example 7.4.1.

From the sign diagram, we see that f is increasing on $(-\infty, c_1) \cup (c_2, \infty)$ (where $f'(x) > 0$, and f is decreasing on (c_1, c_2) (where $f'(x) < 0$).

Since f' changes from positive to negative at c_1, we know that $(c_1, f(c_1))$ is a local maximum, and since f' changes from negative to positive at c_2, we know that $(c_2, f(c_2))$ is a local minimum.

7. Find the possible points of inflection of f. We compute $f''(x) = 18x - 20$. We have

$$f''(x) = 0 \Rightarrow x = 10/9 \approx 1.111.$$

8. Construct a sign diagram for f''. We have only one zero for f'', and we easily see that $f''(x) > 0$ for $x > 10/9$, and $f''(x) < 0$ for $x < 10/9$. The sign diagram for f'' is given below, with the critical points also indicated for reference:

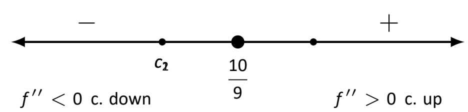


Figure 7.4.2: Sign diagram for f'' in Example 7.4.1.

9. We plot the appropriate points on axes as shown in Figure 7.4.4(a) and connect the points with straight lines. In Figure 7.4.4(b) we adjust these lines to demonstrate the proper concavity. Our curve crosses the y axis at $y = 5$ and crosses the x axis near $x = -0.424$. In Figure 7.4.4(c) we show a graph of f drawn with a computer program, verifying the accuracy of our sketch.

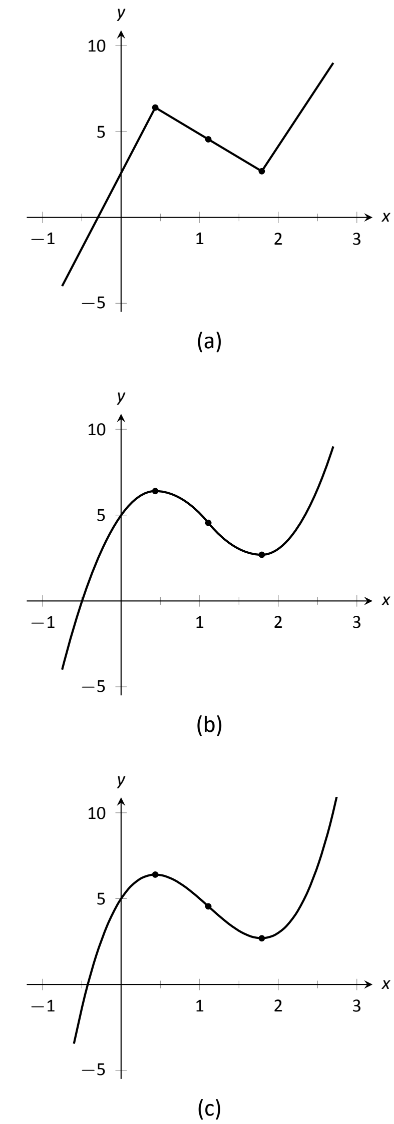


Figure 7.4.4: Sketching f in Example 7.4.1.

Example 7.4.2 Curve sketching

Sketch $f(x) = \dfrac{x^2 - x - 2}{x^2 - x - 6}$.

Solution We again follow the steps outlined in Key Idea 7.4.1.

1. In determining the domain, we assume it is all real numbers and look for restrictions. We find that at $x = -2$ and $x = 3$, $f(x)$ is not defined. So the domain of f is $D = \{\text{real numbers } x \mid x \neq -2, 3\}$.

2. The numerator of f factors as $(x - 2)(x + 1)$, so $f(x) = 0$ for $x = -1$ and $x = 2$; these are the x-intercepts of f. The y-intercept is given by $f(0) = 1/3$.

   Our function has two zeros and two points at which it is undefined. Note that $f(x)$ changes sign at each of these points, so we need to indicate each of them in our sign diagram. We use hollow dots to indicate the points at which f is undefined, giving us the following sign diagram:

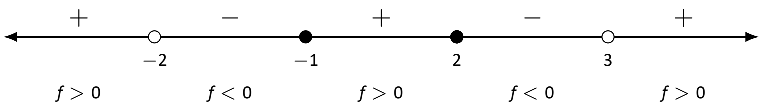


Figure 7.4.3: Sign diagram for f in Example 7.4.2.

3. We see from the sign diagram for f in Figure 7.4.3 that f has vertical asymptotes at $x = -2$ and $x = 3$; moreover, we can deduce the following asymptotic behaviour: at $x = -2$
$$\lim_{x\to -2^-} f(x) = +\infty \quad \text{and} \quad \lim_{x\to -2^+} f(x) = -\infty,$$
and at $x = 3$
$$\lim_{x\to 3^-} f(x) = -\infty \quad \text{and} \quad \lim_{x\to 3^+} f(x) = +\infty.$$

4. There is a horizontal asymptote of $y = 1$, as $\lim_{x\to -\infty} f(x) = 1$ and $\lim_{x\to\infty} f(x) = 1$.

5. To find the critical values of f, we first find $f'(x)$. Using the Quotient Rule, we find
$$f'(x) = \frac{-8x+4}{(x^2+x-6)^2} = \frac{-8x+4}{(x-3)^2(x+2)^2},$$
so $f'(x) = 0$ when $x = 1/2$, and f' is undefined when $x = -2, 3$. Since f' is undefined only when f is, these are not critical values. The only critical value is $x = 1/2$. The sign diagram for f' is given as follows:

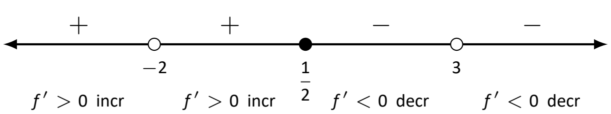


Figure 7.4.5: Sign diagram for f' in Example 7.4.2.

From the sign diagram for f', we see that $f'(x)$ changes from positive to negative at $x = 1/2$, so we have a local maximum at $(1/2, f(1/2))$. We also see that f is increasing on $(-\infty, -2) \cup (-2, 1/2)$ and decreasing on $(1/2, 3) \cup (3, \infty)$.

6. To find the possible points of inflection, we find $f''(x)$, again employing the Quotient Rule:
$$f''(x) = \frac{24x^2 - 24x + 56}{(x-3)^3(x+2)^3}.$$

7. We find that $f''(x)$ is never 0 (setting the numerator equal to 0 and solving for x, we find the only roots to this quadratic are imaginary) and f'' is undefined when $x = -2, 3$. Thus concavity will possibly only change at $x = -2$ and $x = 3$. The sign diagram is given by:

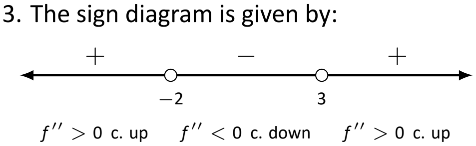
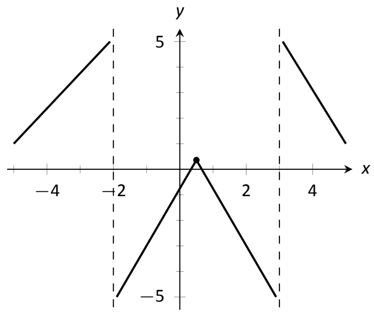
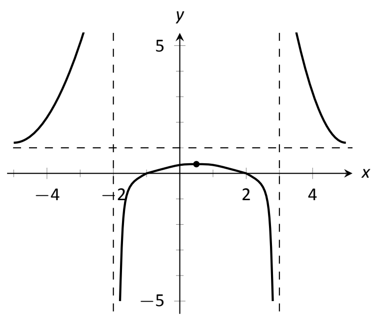


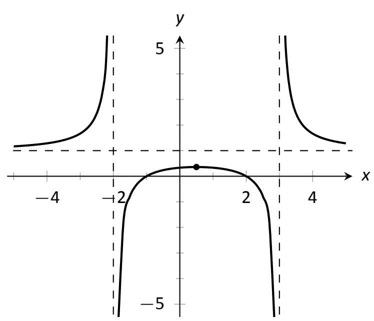
Figure 7.4.6: Sign diagram for f'' in Example 7.4.2.



(a)



(b)



(c)

Figure 7.4.9: Sketching f in Example 7.4.2.

From the sign diagram we see that the graph of f is concave up on $(-\infty,-2)\cup(3,\infty)$ and concave down on $(-2,3)$

8. In Figure 7.4.9(a), we plot the points from the number line on a set of axes and connect the points with straight lines to get a general idea of what the function looks like (these lines effectively only convey increasing/decreasing information). In Figure 7.4.9(b), we adjust the graph with the appropriate concavity. We also show f crossing the x axis at $x=-1$ and $x=2$.

Figure 7.4.9(c) shows a computer generated graph of f, which verifies the accuracy of our sketch.

Example 7.4.3 Curve sketching

Sketch $f(x)=\dfrac{5(x-2)(x+1)}{x^2+2x+4}$.

Solution We again follow Key Idea 7.4.1.

1. We assume that the domain of f is all real numbers and consider restrictions. The only restrictions come when the denominator is 0, but this never occurs. Therefore the domain of f is all real numbers, $\mathbb{R}$.

2. The x-intercepts of f are $(-1,0)$, and $(2,0)$, and the y-intercept is $(0,-5/2)$. The sign diagram of f is given below:

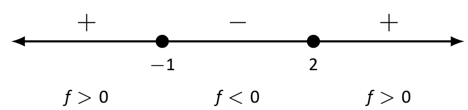


Figure 7.4.7: Sign diagram for f in Example 7.4.3.

3. Since the domain of f is $\mathbb{R}$, there are no vertical asymptotes.

4. We have a horizontal asymptote of $y=5$, as $\lim\limits_{x\to-\infty}f(x)=\lim\limits_{x\to\infty}f(x)=5$.

5. We find the critical values of f by setting $f'(x)=0$ and solving for x. We find

$$f'(x)=\frac{15x(x+4)}{(x^2+2x+4)^2}\quad\Rightarrow\quad f'(x)=0\text{ when }x=-4,0.$$

6. The sign diagram for f' is given by:

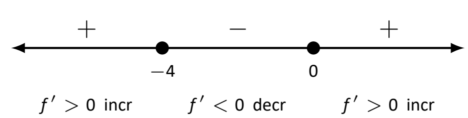


Figure 7.4.8: Sign diagram for f' in Example 7.4.3.

From the sign diagram, we see that $f'(x)$ changes from positive to negative at $x=-4$, so $(-4,f(-4))$ is a relative maximum, and $f'(x)$ changes from negative to positive at $x=0$, so $(0,f(0))$ is a relative minimum. We also see that f is increasing on $(-\infty,-4)\cup(0,\infty)$, and decreasing on $(-4,0)$.

7. We find the possible points of inflection by solving $f''(x) = 0$ for x. We find

$$f''(x) = -\frac{30x^3 + 180x^2 - 240}{(x^2 + 2x + 4)^3}.$$

The cubic in the numerator does not factor very "nicely." We instead approximate the roots (with the help of a computer) at $c_1 = -5.759$, $c_2 = -1.305$ and $c_3 = 1.064$. The sign diagram for f'' is given by:

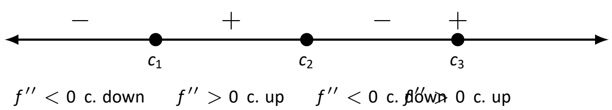


Figure 7.4.10: Sign diagram for f'' in Example 7.4.3.

8. In Figure 7.4.12(a) we plot the significant points from the number line as well as the two roots of f, $x = -1$ and $x = 2$, and connect the points with straight lines to get a general impression about the graph. In Figure 7.4.12(b), we add concavity. Figure 7.4.12(c) shows a computer generated graph of f, affirming our results.

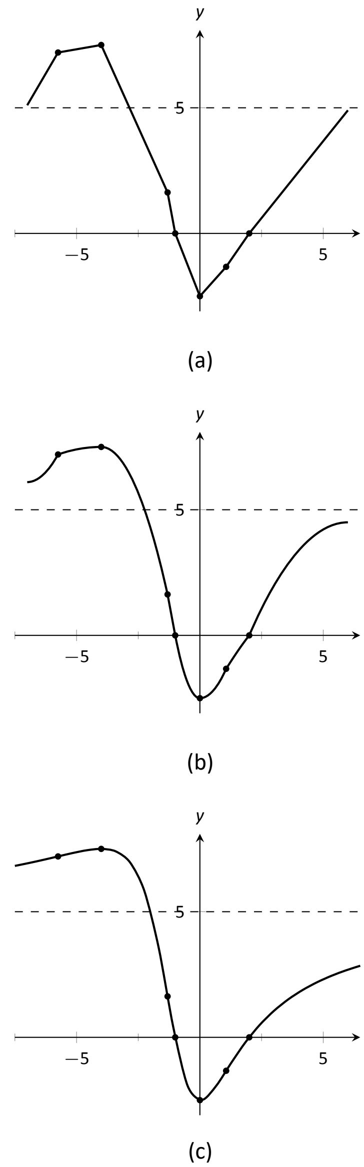


Figure 7.4.12: Sketching f in Example 7.4.3.

In each of our examples, we found a few, significant points on the graph of f that corresponded to changes in increasing/decreasing or concavity. We connected these points with straight lines, then adjusted for concavity, and finished by showing a very accurate, computer generated graph.

Why are computer graphics so good? It is not because computers are "smarter" than we are. Rather, it is largely because computers are much faster at computing than we are. In general, computers graph functions much like most students do when first learning to draw graphs: they plot equally spaced points, then connect the dots using lines. By using lots of points, the connecting lines are short and the graph looks smooth.

This does a fine job of graphing in most cases (in fact, this is the method used for many graphs in this text). However, in regions where the graph is very "curvy," this can generate noticeable sharp edges on the graph unless a large number of points are used. High quality computer algebra systems, such as *Mathematica*, use special algorithms to plot lots of points only where the graph is "curvy."

In Figure 7.4.11, a graph of $y = \sin x$ is given, generated by *Mathematica*. The small points represent each of the places *Mathematica* sampled the function. Notice how at the "bends" of $\sin x$, lots of points are used; where $\sin x$ is relatively straight, fewer points are used. (Many points are also used at the endpoints to ensure the "end behaviour" is accurate.) In fact, in the interval of length 0.2 centered around $\pi/2$, *Mathematica* plots 72 of the 431 points plotted; that is, it plots about 17% of its points in a subinterval that accounts for about 3% of the total interval length.

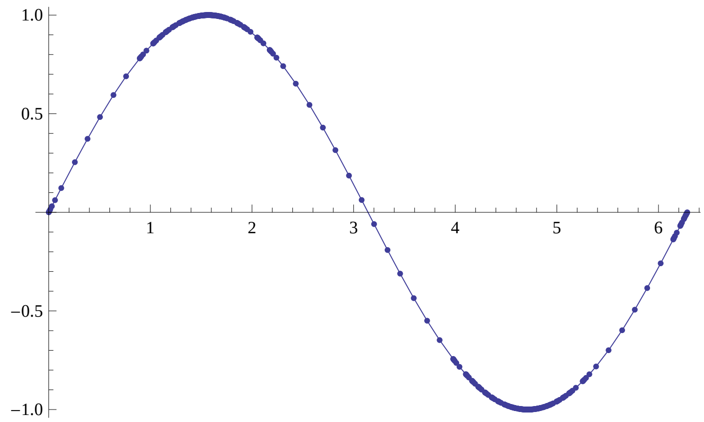


Figure 7.4.11: A graph of $y = \sin x$ generated by *Mathematica*.

How does *Mathematica* know where the graph is "curvy"? Calculus. When we study *curvature* in a later chapter, we will see how the first and second derivatives of a function work together to provide a measurement of "curviness." *Mathematica* employs algorithms to determine regions of "high curvature" and plots extra points there.

Again, the goal of this section is not "How to graph a function when there is no computer to help." Rather, the goal is "Understand that the shape of the graph of a function is largely determined by understanding the behaviour of the function at a few key places." In Example 7.4.3, we were able to accurately sketch a complicated graph using only 5 points and knowledge of asymptotes!

There are many applications of our understanding of derivatives beyond curve sketching. The next chapter explores some of these applications, demonstrating just a few kinds of problems that can be solved with a basic knowledge of differentiation.

Exercises 7.4

Terms and Concepts

1. Why is sketching curves by hand beneficial even though technology is ubiquitous?

2. What does "ubiquitous" mean?

3. T/F: When sketching graphs of functions, it is useful to find the critical points.

4. T/F: When sketching graphs of functions, it is useful to find the possible points of inflection.

5. T/F: When sketching graphs of functions, it is useful to find the horizontal and vertical asymptotes.

6. T/F: When sketching graphs of functions, one need not plot any points at all.

Problems

In Exercises 7 – 12, practice using Key Idea 7.4.1 by applying the principles to the given functions with familiar graphs.

7. $f(x) = 2x + 4$

8. $f(x) = -x^2 + 1$

9. $f(x) = \sin x$

10. $f(x) = e^x$

11. $f(x) = \dfrac{1}{x}$

12. $f(x) = \dfrac{1}{x^2}$

In Exercises 13 – 26, sketch a graph of the given function using Key Idea 7.4.1. Show all work; check your answer with technology.

13. $f(x) = x^3 - 2x^2 + 4x + 1$

14. $f(x) = -x^3 + 5x^2 - 3x + 2$

15. $f(x) = x^3 + 3x^2 + 3x + 1$

16. $f(x) = x^3 - x^2 - x + 1$

17. $f(x) = (x - 2)\ln(x - 2)$

18. $f(x) = (x - 2)^2 \ln(x - 2)$

19. $f(x) = \dfrac{x^2 - 4}{x^2}$

20. $f(x) = \dfrac{x^2 - 4x + 3}{x^2 - 6x + 8}$

21. $f(x) = \dfrac{x^2 - 2x + 1}{x^2 - 6x + 8}$

22. $f(x) = x\sqrt{x + 1}$

23. $f(x) = x^2 e^x$

24. $f(x) = \sin x \cos x$ on $[-\pi, \pi]$

25. $f(x) = (x - 3)^{2/3} + 2$

26. $f(x) = \dfrac{(x - 1)^{2/3}}{x}$

In Exercises 27 – 30, a function with the parameters a and b are given. Describe the critical points and possible points of inflection of f in terms of a and b.

27. $f(x) = \dfrac{a}{x^2 + b^2}$

28. $f(x) = ax^2 + bx + 1$

29. $f(x) = \sin(ax + b)$

30. $f(x) = (x - a)(x - b)$

31. Given $x^2 + y^2 = 1$, use implicit differentiation to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Use this information to justify the sketch of the unit circle.

We have spent considerable time considering the derivatives of a function and their applications. In this section, we are going to starting thinking in "the other direction." That is, given a function $f(x)$, we are going to consider functions $F(x)$ such that $F'(x) = f(x)$. Here, we will only consider very basic examples, and leave most of the heavy lifting to later courses. The importance of antiderivatives becomes apparent in Math 1560, once integration and the Fundamental Theorem of Calculus have been introduced. More advanced techniques for finding antiderivatives are taught in Math 2560.

7.5 Antiderivatives and Indefinite Integration

Given a function $y = f(x)$, a *differential equation* is one that incorporates y, x, and the derivatives of y. For instance, a simple differential equation is:

$$y' = 2x.$$

Solving a differential equation amounts to finding a function y that satisfies the given equation. Take a moment and consider that equation; can you find a function y such that $y' = 2x$?

Can you find another?

And yet another?

Hopefully one was able to come up with at least one solution: $y = x^2$. "Finding another" may have seemed impossible until one realizes that a function like $y = x^2 + 1$ also has a derivative of $2x$. Once that discovery is made, finding "yet another" is not difficult; the function $y = x^2 + 123,456,789$ also has a derivative of $2x$. The differential equation $y' = 2x$ has many solutions. This leads us to some definitions.

Definition 7.5.1 Antiderivatives and Indefinite Integrals

Let a function $f(x)$ be given. An **antiderivative** of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

The set of all antiderivatives of $f(x)$ is the **indefinite integral of** f, denoted by

$$\int f(x)\, dx.$$

Make a note about our definition: we refer to *an* antiderivative of f, as opposed to *the* antiderivative of f, since there is *always* an infinite number of them. We often use upper-case letters to denote antiderivatives.

Knowing one antiderivative of f allows us to find infinitely more, simply by adding a constant. Not only does this give us *more* antiderivatives, it gives us *all* of them.

Theorem 7.5.1 Antiderivative Forms

Let $F(x)$ and $G(x)$ be antiderivatives of $f(x)$ on an interval I. Then there exists a constant C such that, on I,

$$G(x) = F(x) + C.$$

Given a function f defined on an interval I and one of its antiderivatives F, we know *all* antiderivatives of f on I have the form $F(x) + C$ for some constant C. Using Definition 7.5.1, we can say that

$$\int f(x)\,dx = F(x) + C.$$

Let's analyze this indefinite integral notation.

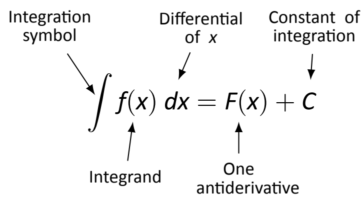


Figure 7.5.1: Understanding the indefinite integral notation.

Figure 7.5.1 shows the typical notation of the indefinite integral. The integration symbol, $\int$, is in reality an "elongated S," representing "take the sum." We will later see how *sums* and *antiderivatives* are related.

The function we want to find an antiderivative of is called the *integrand*. It contains the differential of the variable we are integrating with respect to. The $\int$ symbol and the differential dx are not "bookends" with a function sandwiched in between; rather, the symbol $\int$ means "find all antiderivatives of what follows," and the function $f(x)$ and dx are multiplied together; the dx does not "just sit there."

Let's practice using this notation.

Example 7.5.1 Evaluating indefinite integrals

Evaluate $\int \sin x\,dx$.

SOLUTION We are asked to find all functions $F(x)$ such that $F'(x) = \sin x$. Some thought will lead us to one solution: $F(x) = -\cos x$, because $\frac{d}{dx}(-\cos x) = \sin x$.

The indefinite integral of $\sin x$ is thus $-\cos x$, plus a constant of integration. So:

$$\int \sin x\,dx = -\cos x + C.$$

A commonly asked question is "What happened to the dx?" The unenlightened response is "Don't worry about it. It just goes away." A full understanding includes the following.

This process of *antidifferentiation* is really solving a *differential* question. The integral

$$\int \sin x\,dx$$

presents us with a differential, $dy = \sin x\,dx$. It is asking: "What is y?" We found lots of solutions, all of the form $y = -\cos x + C$.

Letting $dy = \sin x\,dx$, rewrite

$$\int \sin x\,dx \quad \text{as} \quad \int dy.$$

This is asking: "What functions have a differential of the form dy?" The answer is "Functions of the form $y + C$, where C is a constant." What is y? We have lots of choices, all differing by a constant; the simplest choice is $y = -\cos x$.

Understanding all of this is more important later as we try to find antiderivatives of more complicated functions. In this section, we will simply explore the rules of indefinite integration, and one can succeed for now with answering "What happened to the *dx*?" with "It went away."

Let's practice once more before stating integration rules.

Example 7.5.2 Evaluating indefinite integrals

Evaluate $\int (3x^2 + 4x + 5)\ dx$.

Solution We seek a function $F(x)$ whose derivative is $3x^2 + 4x + 5$. When taking derivatives, we can consider functions term–by–term, so we can likely do that here.

What functions have a derivative of $3x^2$? Some thought will lead us to a cubic, specifically $x^3 + C_1$, where C_1 is a constant.

What functions have a derivative of $4x$? Here the x term is raised to the first power, so we likely seek a quadratic. Some thought should lead us to $2x^2 + C_2$, where C_2 is a constant.

Finally, what functions have a derivative of 5? Functions of the form $5x + C_3$, where C_3 is a constant.

Our answer appears to be

$$\int (3x^2 + 4x + 5)\ dx = x^3 + C_1 + 2x^2 + C_2 + 5x + C_3.$$

We do not need three separate constants of integration; combine them as one constant, giving the final answer of

$$\int (3x^2 + 4x + 5)\ dx = x^3 + 2x^2 + 5x + C.$$

It is easy to verify our answer; take the derivative of $x^3 + 2x^3 + 5x + C$ and see we indeed get $3x^2 + 4x + 5$.

This final step of "verifying our answer" is important both practically and theoretically. In general, taking derivatives is easier than finding antiderivatives so checking our work is easy and vital as we learn.

We also see that taking the derivative of our answer returns the function in the integrand. Thus we can say that:

$$\frac{d}{dx}\left(\int f(x)\ dx\right) = f(x).$$

Differentiation "undoes" the work done by antidifferentiation.

For ease of reference, and to stress the relationship between derivatives and antiderivatives, we include below a list of many of the common differentiation rules we have learned, along with the corresponding antidifferentiation rules.

Theorem 7.5.2 Derivatives and Antiderivatives

Common Differentiation Rules	Common Indefinite Integral Rules
1. $\frac{d}{dx}\big(cf(x)\big) = c \cdot f'(x)$	1. $\int c \cdot f(x)\ dx = c \cdot \int f(x)\ dx$
2. $\frac{d}{dx}\big(f(x) \pm g(x)\big) = f'(x) \pm g'(x)$	2. $\int \big(f(x) \pm g(x)\big)\ dx = \int f(x)\ dx \pm \int g(x)\ dx$
3. $\frac{d}{dx}\big(C\big) = 0$	3. $\int 0\ dx = C$
4. $\frac{d}{dx}\big(x\big) = 1$	4. $\int 1\ dx = \int dx = x + C$
5. $\frac{d}{dx}\big(x^n\big) = n \cdot x^{n-1}$	5. $\int x^n\ dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$
6. $\frac{d}{dx}\big(\sin x\big) = \cos x$	6. $\int \cos x\ dx = \sin x + C$
7. $\frac{d}{dx}\big(\cos x\big) = -\sin x$	7. $\int \sin x\ dx = -\cos x + C$
8. $\frac{d}{dx}\big(\tan x\big) = \sec^2 x$	8. $\int \sec^2 x\ dx = \tan x + C$
9. $\frac{d}{dx}\big(\csc x\big) = -\csc x \cot x$	9. $\int \csc x \cot x\ dx = -\csc x + C$
10. $\frac{d}{dx}\big(\sec x\big) = \sec x \tan x$	10. $\int \sec x \tan x\ dx = \sec x + C$
11. $\frac{d}{dx}\big(\cot x\big) = -\csc^2 x$	11. $\int \csc^2 x\ dx = -\cot x + C$
12. $\frac{d}{dx}\big(e^x\big) = e^x$	12. $\int e^x\ dx = e^x + C$
13. $\frac{d}{dx}\big(a^x\big) = \ln a \cdot a^x$	13. $\int a^x\ dx = \frac{1}{\ln a} \cdot a^x + C$
14. $\frac{d}{dx}\big(\ln x\big) = \frac{1}{x}$	14. $\int \frac{1}{x}\ dx = \ln\lvert x\rvert + C$

We highlight a few important points from Theorem 7.5.2:

- Rule #1 states $\int c \cdot f(x)\ dx = c \cdot \int f(x)\ dx$. This is the Constant Multiple Rule: we can temporarily ignore constants when finding antiderivatives, just as we did when computing derivatives (i.e., $\frac{d}{dx}\big(3x^2\big)$ is just as easy to compute as $\frac{d}{dx}\big(x^2\big)$). An example:

  $$\int 5\cos x\ dx = 5 \cdot \int \cos x\ dx = 5 \cdot (\sin x + C) = 5\sin x + C.$$

  In the last step we can consider the constant as also being multiplied by 5, but "5 times a constant" is still a constant, so we just write "C".

- Rule #2 is the Sum/Difference Rule: we can split integrals apart when the integrand contains terms that are added/subtracted, as we did in Example 7.5.2. So:

  $$\begin{aligned}\int (3x^2 + 4x + 5)\ dx &= \int 3x^2\ dx + \int 4x\ dx + \int 5\ dx\\ &= 3\int x^2\ dx + 4\int x\ dx + \int 5\ dx\\ &= 3 \cdot \frac{1}{3}x^3 + 4 \cdot \frac{1}{2}x^2 + 5x + C\\ &= x^3 + 2x^2 + 5x + C\end{aligned}$$

In practice we generally do not write out all these steps, but we demonstrate them here for completeness.

- Rule #5 is the Power Rule of indefinite integration. There are two important things to keep in mind:

  1. Notice the restriction that $n \neq -1$. This is important: $\int \frac{1}{x}\,dx \neq$ "$\frac{1}{0}x^0 + C$"; rather, see Rule #14.
  2. We are presenting antidifferentiation as the "inverse operation" of differentiation. Here is a useful quote to remember:

     > "Inverse operations do the opposite things in the opposite order."

     When taking a derivative using the Power Rule, we **first** *multiply* by the power, then **second** *subtract* 1 from the power. To find the antiderivative, do the opposite things in the opposite order: **first** *add* one to the power, then **second** *divide* by the power.

- Note that Rule #14 incorporates the absolute value of x. The exercises will work the reader through why this is the case; for now, know the absolute value is important and cannot be ignored.

Initial Value Problems

In Section 6.3 we saw that the derivative of a position function gave a velocity function, and the derivative of a velocity function describes acceleration. We can now go "the other way:" the antiderivative of an acceleration function gives a velocity function, etc. While there is just one derivative of a given function, there are infinitely many antiderivatives. Therefore we cannot ask "What is *the* velocity of an object whose acceleration is -32ft/s^2?", since there is more than one answer.

We can find *the* answer if we provide more information with the question, as done in the following example. Often the additional information comes in the form of an *initial value*, a value of the function that one knows beforehand.

Example 7.5.3 Solving initial value problems

The acceleration due to gravity of a falling object is -32 ft/s^2. At time $t = 3$, a falling object had a velocity of -10 ft/s. Find the equation of the object's velocity.

Solution We want to know a velocity function, $v(t)$. We know two things:

- The acceleration, i.e., $v'(t) = -32$, and
- the velocity at a specific time, i.e., $v(3) = -10$.

Using the first piece of information, we know that $v(t)$ is an antiderivative of $v'(t) = -32$. So we begin by finding the indefinite integral of -32:

$$\int (-32)\,dt = -32t + C = v(t).$$

Now we use the fact that $v(3) = -10$ to find C:

$$\begin{aligned} v(t) &= -32t + C \\ v(3) &= -10 \\ -32(3) + C &= -10 \\ C &= 86 \end{aligned}$$

Thus $v(t) = -32t + 86$. We can use this equation to understand the motion of the object: when $t = 0$, the object had a velocity of $v(0) = 86$ ft/s. Since the velocity is positive, the object was moving upward.

When did the object begin moving down? Immediately after $v(t) = 0$:

$$-32t + 86 = 0 \quad \Rightarrow \quad t = \frac{43}{16} \approx 2.69\text{s}.$$

Recognize that we are able to determine quite a bit about the path of the object knowing just its acceleration and its velocity at a single point in time.

Example 7.5.4 Solving initial value problems

Find $f(t)$, given that $f''(t) = \cos t$, $f'(0) = 3$ and $f(0) = 5$.

Solution We start by finding $f'(t)$, which is an antiderivative of $f''(t)$:

$$\int f''(t)\,dt = \int \cos t\,dt = \sin t + C = f'(t).$$

So $f'(t) = \sin t + C$ for the correct value of C. We are given that $f'(0) = 3$, so:

$$f'(0) = 3 \quad \Rightarrow \quad \sin 0 + C = 3 \quad \Rightarrow \quad C = 3.$$

Using the initial value, we have found $f'(t) = \sin t + 3$.

We now find $f(t)$ by integrating again.

$$f(t) = \int f'(t)\,dt = \int (\sin t + 3)\,dt = -\cos t + 3t + C.$$

We are given that $f(0) = 5$, so

$$\begin{aligned} -\cos 0 + 3(0) + C &= 5 \\ -1 + C &= 5 \\ C &= 6 \end{aligned}$$

Thus $f(t) = -\cos t + 3t + 6$.

This section introduced antiderivatives and the indefinite integral. We found they are needed when finding a function given information about its derivative(s). For instance, we found a velocity function given an acceleration function.

If you continue on to Math 1560, you will see how position and velocity are unexpectedly related by the areas of certain regions on a graph of the velocity function, and how the Fundamental Theorem of Calculus ties together areas and antiderivatives.

Exercises 7.5

Terms and Concepts

1. Define the term "antiderivative" in your own words.

2. Is it more accurate to refer to "the" antiderivative of $f(x)$ or "an" antiderivative of $f(x)$?

3. Use your own words to define the indefinite integral of $f(x)$.

4. Fill in the blanks: "Inverse operations do the ________ things in the ________ order."

5. What is an "initial value problem"?

6. The derivative of a position function is a ________ function.

7. The antiderivative of an acceleration function is a ________ function.

8. If $F(x)$ is an antiderivative of $f(x)$, and $G(x)$ is an antiderivative of $g(x)$, give an antiderivative of $f(x)+g(x)$.

Problems

In Exercises 9 – 27, evaluate the given indefinite integral.

9. $\displaystyle\int 3x^3\,dx$

10. $\displaystyle\int x^8\,dx$

11. $\displaystyle\int (10x^2-2)\,dx$

12. $\displaystyle\int dt$

13. $\displaystyle\int 1\,ds$

14. $\displaystyle\int \frac{1}{3t^2}\,dt$

15. $\displaystyle\int \frac{3}{t^2}\,dt$

16. $\displaystyle\int \frac{1}{\sqrt{x}}\,dx$

17. $\displaystyle\int \sec^2\theta\,d\theta$

18. $\displaystyle\int \sin\theta\,d\theta$

19. $\displaystyle\int (\sec x\tan x+\csc x\cot x)\,dx$

20. $\displaystyle\int 5e^\theta\,d\theta$

21. $\displaystyle\int 3^t\,dt$

22. $\displaystyle\int \frac{5^t}{2}\,dt$

23. $\displaystyle\int (2t+3)^2\,dt$

24. $\displaystyle\int (t^2+3)(t^3-2t)\,dt$

25. $\displaystyle\int x^2x^3\,dx$

26. $\displaystyle\int e^\pi\,dx$

27. $\displaystyle\int a\,dx$

28. This problem investigates why Theorem 7.5.2 states that $\displaystyle\int \frac{1}{x}\,dx=\ln|x|+C.$
    (a) What is the domain of $y=\ln x$?
    (b) Find $\frac{d}{dx}\big(\ln x\big)$.
    (c) What is the domain of $y=\ln(-x)$?
    (d) Find $\frac{d}{dx}\big(\ln(-x)\big)$.
    (e) You should find that $1/x$ has two types of antiderivatives, depending on whether $x>0$ or $x<0$. In one expression, give a formula for $\displaystyle\int\frac{1}{x}\,dx$ that takes these different domains into account, and explain your answer.

In Exercises 29 – 39, find $f(x)$ described by the given initial value problem.

29. $f'(x)=\sin x$ and $f(0)=2$

30. $f'(x)=5e^x$ and $f(0)=10$

31. $f'(x)=4x^3-3x^2$ and $f(-1)=9$

32. $f'(x)=\sec^2 x$ and $f(\pi/4)=5$

33. $f'(x)=7^x$ and $f(2)=1$

34. $f''(x)=5$ and $f'(0)=7$, $f(0)=3$

35. $f''(x)=7x$ and $f'(1)=-1$, $f(1)=10$

36. $f''(x) = 5e^x$ and $f'(0) = 3, f(0) = 5$

37. $f''(\theta) = \sin\theta$ and $f'(\pi) = 2, f(\pi) = 4$

38. $f''(x) = 24x^2 + 2^x - \cos x$ and $f'(0) = 5, f(0) = 0$

39. $f''(x) = 0$ and $f'(1) = 3, f(1) = 1$

Review

40. Use information gained from the first and second derivatives to sketch $f(x) = \dfrac{1}{e^x + 1}$.

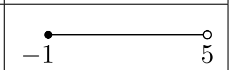
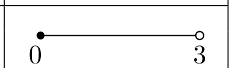
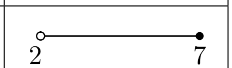
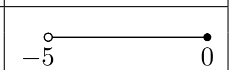
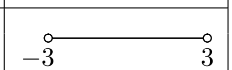
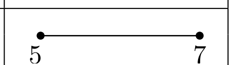
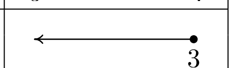
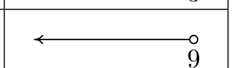
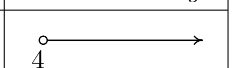
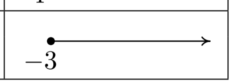
41. Given $y = x^2 e^x \cos x$, find dy.

A: Answers To Selected Problems

Chapter 1

Section 1.1

1.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \le x < 5\}$	$[-1, 5)$	
$\{x \mid 0 \le x < 3\}$	$[0, 3)$	
$\{x \mid 2 < x \le 7\}$	$(2, 7]$	
$\{x \mid -5 < x \le 0\}$	$(-5, 0]$	
$\{x \mid -3 < x < 3\}$	$(-3, 3)$	
$\{x \mid 5 \le x \le 7\}$	$[5, 7]$	
$\{x \mid x \le 3\}$	$(-\infty, 3]$	
$\{x \mid x < 9\}$	$(-\infty, 9)$	
$\{x \mid x > 4\}$	$(4, \infty)$	
$\{x \mid x \ge -3\}$	$[-3, \infty)$	

3. $(-1, 1) \cup [0, 6] = (-1, 6]$

5. $(-\infty, 0) \cap [1, 5] = \emptyset$

7. $(-\infty, 5] \cap [5, 8) = \{5\}$

9. $(-\infty, -1) \cup (-1, \infty)$

11. $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$

13. $(-\infty, -4) \cup (-4, 0) \cup (0, 4) \cup (4, \infty)$

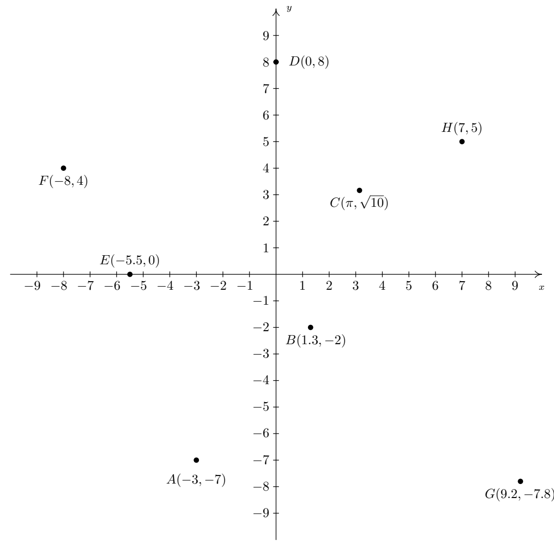
15. $(-\infty, \infty)$

17. $(-\infty, 5] \cup \{6\}$

19. $(-3, 3) \cup \{4\}$

Section 1.2

1. The required points $A(-3, -7)$, $B(1.3, -2)$, $C(\pi, \sqrt{10})$, $D(0, 8)$, $E(-5.5, 0)$, $F(-8, 4)$, $G(9.2, -7.8)$, and $H(7, 5)$ are plotted in the Cartesian Coordinate Plane below.



3. $d = 5$, $M = \left(-1, \frac{7}{2}\right)$

5. $d = \sqrt{26}$, $M = \left(1, \frac{3}{2}\right)$

7. $d = \sqrt{74}$, $M = \left(\frac{13}{10}, -\frac{13}{10}\right)$

9. $d = \sqrt{83}$, $M = \left(4\sqrt{5}, \frac{5\sqrt{3}}{2}\right)$

11. $(3 + \sqrt{7}, -1)$, $(3 - \sqrt{7}, -1)$

13. $(-1 + \sqrt{3}, 0)$, $(-1 - \sqrt{3}, 0)$

15. $(-3, -4)$, 5 miles, $(4, -4)$

17.

19.

21.

Chapter 2

Section 2.1

1. For $f(x) = 2x + 1$

- $f(3) = 7$
- $f(-1) = -1$
- $f\left(\frac{3}{2}\right) = 4$
- $f(4x) = 8x + 1$
- $4f(x) = 8x + 4$
- $f(-x) = -2x + 1$
- $f(x - 4) = 2x - 7$
- $f(x) - 4 = 2x - 3$
- $f\left(x^2\right) = 2x^2 + 1$

3. For $f(x) = 2 - x^2$

- $f(3) = -7$
- $f(-1) = 1$
- $f\left(\frac{3}{2}\right) = -\frac{1}{4}$
- $f(4x) = 2 - 16x^2$
- $4f(x) = 8 - 4x^2$
- $f(-x) = 2 - x^2$
- $f(x - 4) = -x^2 + 8x - 14$
- $f(x) - 4 = -x^2 - 2$
- $f\left(x^2\right) = 2 - x^4$

5. For $f(x) = \dfrac{x}{x - 1}$

- $f(3) = \frac{3}{2}$
- $f(-1) = \frac{1}{2}$
- $f\left(\frac{3}{2}\right) = 3$
- $f(4x) = \dfrac{4x}{4x-1}$
- $4f(x) = \dfrac{4x}{x-1}$
- $f(-x) = \dfrac{x}{x+1}$
- $f(x-4) = \dfrac{x-4}{x-5}$
- $f(x) - 4 = \dfrac{x}{x-1} - 4 = \frac{4-3x}{x-1}$
- $f\left(x^2\right) = \dfrac{x^2}{x^2-1}$

7. For $f(x) = 6$

- $f(3) = 6$
- $f(-1) = 6$
- $f\left(\frac{3}{2}\right) = 6$
- $f(4x) = 6$
- $4f(x) = 24$
- $f(-x) = 6$
- $f(x-4) = 6$
- $f(x) - 4 = 2$
- $f\left(x^2\right) = 6$

9. For $f(x) = 2x - 5$

- $f(2) = -1$
- $f(-2) = -9$
- $f(2a) = 4a - 5$
- $2f(a) = 4a - 10$
- $f(a+2) = 2a - 1$
- $f(a) + f(2) = 2a - 6$
- $f\left(\frac{2}{a}\right) = \frac{4}{a} - 5 = \frac{4-5a}{a}$
- $\frac{f(a)}{2} = \frac{2a-5}{2}$
- $f(a+h) = 2a + 2h - 5$

11. For $f(x) = 2x^2 - 1$

- $f(2) = 7$
- $f(-2) = 7$
- $f(2a) = 8a^2 - 1$
- $2f(a) = 4a^2 - 2$
- $f(a+2) = 2a^2 + 8a + 7$
- $f(a) + f(2) = 2a^2 + 6$
- $f\left(\frac{2}{a}\right) = \frac{8}{a^2} - 1 = \frac{8-a^2}{a^2}$
- $\frac{f(a)}{2} = \frac{2a^2-1}{2}$
- $f(a+h) = 2a^2 + 4ah + 2h^2 - 1$

13. For $f(x) = \sqrt{2x+1}$

- $f(2) = \sqrt{5}$
- $f(-2)$ is not real
- $f(2a) = \sqrt{4a+1}$
- $2f(a) = 2\sqrt{2a+1}$
- $f(a+2) = \sqrt{2a+5}$
- $f(a) + f(2) = \sqrt{2a+1} + \sqrt{5}$
- $f\left(\frac{2}{a}\right) = \sqrt{\frac{4}{a} + 1} = \sqrt{\frac{a+4}{a}}$
- $\frac{f(a)}{2} = \frac{\sqrt{2a+1}}{2}$
- $f(a+h) = \sqrt{2a + 2h + 1}$

15. For $f(x) = \frac{x}{2}$

- $f(2) = 1$
- $f(-2) = -1$
- $f(2a) = a$
- $2f(a) = a$
- $f(a+2) = \frac{a+2}{2}$
- $f(a) + f(2) = \frac{a}{2} + 1 = \frac{a+2}{2}$
- $f\left(\frac{2}{a}\right) = \frac{1}{a}$
- $\frac{f(a)}{2} = \frac{a}{4}$
- $f(a+h) = \frac{a+h}{2}$

17. For $f(x) = 2x - 1$, $f(0) = -1$ and $f(x) = 0$ when $x = \frac{1}{2}$
19. For $f(x) = 2x^2 - 6$, $f(0) = -6$ and $f(x) = 0$ when $x = \pm\sqrt{3}$
21. For $f(x) = \sqrt{x+4}$, $f(0) = 2$ and $f(x) = 0$ when $x = -4$
23. For $f(x) = \frac{3}{4-x}$, $f(0) = \frac{3}{4}$ and $f(x)$ is never equal to 0
25. (a) $f(-4) = 1$
    (b) $f(-3) = 2$
    (c) $f(3) = 0$
    (d) $f(3.001) = 1.999$
    (e) $f(-3.001) = 1.999$
    (f) $f(2) = \sqrt{5}$
27. $(-\infty, \infty)$
29. $(-\infty, -1) \cup (-1, \infty)$
31. $(-\infty, \infty)$
33. $(-\infty, -6) \cup (-6, 6) \cup (6, \infty)$
35. $(-\infty, 3]$
37. $[-3, \infty)$
39. $\left[\frac{1}{3}, \infty\right)$
41. $(-\infty, \infty)$
43. $\left[\frac{1}{3}, 6\right) \cup (6, \infty)$
45. $(-\infty, 8) \cup (8, \infty)$
47. $(8, \infty)$
49. $(-\infty, 8) \cup (8, \infty)$
51. $[0, 5) \cup (5, \infty)$

Section 2.2

1. For $f(x) = 3x + 1$ and $g(x) = 4 - x$

- $(f+g)(2) = 9$
- $(f-g)(-1) = -7$
- $(g-f)(1) = -1$
- $(fg)\left(\frac{1}{2}\right) = \frac{35}{4}$
- $\left(\frac{f}{g}\right)(0) = \frac{1}{4}$
- $\left(\frac{g}{f}\right)(-2) = -\frac{6}{5}$

3. For $f(x) = x^2 - x$ and $g(x) = 12 - x^2$

- $(f+g)(2) = 10$
- $(f-g)(-1) = -9$
- $(g-f)(1) = 11$
- $(fg)\left(\frac{1}{2}\right) = -\frac{47}{16}$
- $\left(\frac{f}{g}\right)(0) = 0$
- $\left(\frac{g}{f}\right)(-2) = \frac{4}{3}$

5. For $f(x) = \sqrt{x+3}$ and $g(x) = 2x - 1$

- $(f+g)(2) = 3 + \sqrt{5}$
- $(f-g)(-1) = 3 + \sqrt{2}$
- $(g-f)(1) = -1$
- $(fg)\left(\frac{1}{2}\right) = 0$
- $\left(\frac{f}{g}\right)(0) = -\sqrt{3}$
- $\left(\frac{g}{f}\right)(-2) = -5$

7. For $f(x) = 2x$ and $g(x) = \dfrac{1}{2x+1}$

- $(f+g)(2) = \frac{21}{5}$
- $(f-g)(-1) = -1$
- $(g-f)(1) = -\frac{5}{3}$
- $(fg)\left(\frac{1}{2}\right) = \frac{1}{2}$
- $\left(\frac{f}{g}\right)(0) = 0$
- $\left(\frac{g}{f}\right)(-2) = \frac{1}{12}$

9. For $f(x) = x^2$ and $g(x) = \dfrac{1}{x^2}$

- $(f+g)(2) = \frac{17}{4}$
- $(f-g)(-1) = 0$
- $(g-f)(1) = 0$
- $(fg)\left(\frac{1}{2}\right) = 1$
- $\left(\frac{f}{g}\right)(0)$ is undefined.
- $\left(\frac{g}{f}\right)(-2) = \frac{1}{16}$

11. For $f(x) = 2x+1$ and $g(x) = x-2$

- $(f+g)(x) = 3x-1$ Domain: $(-\infty, \infty)$
- $(f-g)(x) = x+3$ Domain: $(-\infty, \infty)$
- $(fg)(x) = 2x^2-3x-2$ Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{2x+1}{x-2}$ Domain: $(-\infty, 2) \cup (2, \infty)$

13. For $f(x) = x^2$ and $g(x) = 3x-1$

- $(f+g)(x) = x^2+3x-1$ Domain: $(-\infty, \infty)$
- $(f-g)(x) = x^2-3x+1$ Domain: $(-\infty, \infty)$
- $(fg)(x) = 3x^3-x^2$ Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x^2}{3x-1}$ Domain: $\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$

15. For $f(x) = x^2-4$ and $g(x) = 3x+6$

- $(f+g)(x) = x^2+3x+2$ Domain: $(-\infty, \infty)$
- $(f-g)(x) = x^2-3x-10$ Domain: $(-\infty, \infty)$
- $(fg)(x) = 3x^3+6x^2-12x-24$ Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x-2}{3}$ Domain: $(-\infty, -2) \cup (-2, \infty)$

17. For $f(x) = \frac{x}{2}$ and $g(x) = \frac{2}{x}$

- $(f+g)(x) = \frac{x^2+4}{2x}$ Domain: $(-\infty, 0) \cup (0, \infty)$
- $(f-g)(x) = \frac{x^2-4}{2x}$ Domain: $(-\infty, 0) \cup (0, \infty)$
- $(fg)(x) = 1$ Domain: $(-\infty, 0) \cup (0, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x^2}{4}$ Domain: $(-\infty, 0) \cup (0, \infty)$

19. For $f(x) = x$ and $g(x) = \sqrt{x+1}$

- $(f+g)(x) = x+\sqrt{x+1}$ Domain: $[-1, \infty)$
- $(f-g)(x) = x-\sqrt{x+1}$ Domain: $[-1, \infty)$
- $(fg)(x) = x\sqrt{x+1}$ Domain: $[-1, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x}{\sqrt{x+1}}$ Domain: $(-1, \infty)$

21. $(f+g)(-3) = 2$

23. $(fg)(-1) = 0$

25. $(g-f)(3) = 3$

27. $\left(\dfrac{f}{g}\right)(-2)$ does not exist

29. $\left(\dfrac{f}{g}\right)(2) = 4$

31. $\left(\dfrac{g}{f}\right)(3) = -2$

33. For $f(x) = x^2$ and $g(x) = 2x+1$,

- $(g \circ f)(0) = 1$
- $(f \circ g)(-1) = 1$
- $(f \circ f)(2) = 16$
- $(g \circ f)(-3) = 19$
- $(f \circ g)\left(\frac{1}{2}\right) = 4$
- $(f \circ f)(-2) = 16$

35. For $f(x) = 4-3x$ and $g(x) = |x|$,

- $(g \circ f)(0) = 4$
- $(f \circ g)(-1) = 1$
- $(f \circ f)(2) = 10$
- $(g \circ f)(-3) = 13$
- $(f \circ g)\left(\frac{1}{2}\right) = \frac{5}{2}$
- $(f \circ f)(-2) = -26$

37. For $f(x) = 4x+5$ and $g(x) = \sqrt{x}$,

- $(g \circ f)(0) = \sqrt{5}$
- $(f \circ g)(-1)$ is not real
- $(f \circ f)(2) = 57$
- $(g \circ f)(-3)$ is not real
- $(f \circ g)\left(\frac{1}{2}\right) = 5+2\sqrt{2}$
- $(f \circ f)(-2) = -7$

39. For $f(x) = 6-x-x^2$ and $g(x) = x\sqrt{x+10}$,

- $(g \circ f)(0) = 24$
- $(f \circ g)(-1) = 0$
- $(f \circ f)(2) = 6$
- $(g \circ f)(-3) = 0$
- $(f \circ g)\left(\frac{1}{2}\right) = \frac{27-2\sqrt{42}}{8}$
- $(f \circ f)(-2) = -14$

41. For $f(x) = \frac{3}{1-x}$ and $g(x) = \frac{4x}{x^2+1}$,

- $(g \circ f)(0) = \frac{6}{5}$
- $(f \circ g)(-1) = 1$
- $(f \circ f)(2) = \frac{3}{4}$
- $(g \circ f)(-3) = \frac{48}{25}$
- $(f \circ g)\left(\frac{1}{2}\right) = -5$
- $(f \circ f)(-2)$ is undefined

43. For $f(x) = \frac{2x}{5-x^2}$ and $g(x) = \sqrt{4x+1}$,

- $(g \circ f)(0) = 1$
- $(f \circ g)(-1)$ is not real
- $(f \circ f)(2) = -\frac{8}{11}$
- $(g \circ f)(-3) = \sqrt{7}$
- $(f \circ g)\left(\frac{1}{2}\right) = \sqrt{3}$
- $(f \circ f)(-2) = \frac{8}{11}$

45. For $f(x) = 2x+3$ and $g(x) = x^2-9$

- $(g \circ f)(x) = 4x^2+12x$, domain: $(-\infty, \infty)$
- $(f \circ g)(x) = 2x^2-15$, domain: $(-\infty, \infty)$
- $(f \circ f)(x) = 4x+9$, domain: $(-\infty, \infty)$

47. For $f(x) = x^2-4$ and $g(x) = |x|$

- $(g \circ f)(x) = |x^2-4|$, domain: $(-\infty, \infty)$
- $(f \circ g)(x) = |x|^2-4 = x^2-4$, domain: $(-\infty, \infty)$
- $(f \circ f)(x) = x^4-8x^2+12$, domain: $(-\infty, \infty)$

49. For $f(x) = |x+1|$ and $g(x) = \sqrt{x}$

- $(g \circ f)(x) = \sqrt{|x+1|}$, domain: $(-\infty, \infty)$
- $(f \circ g)(x) = |\sqrt{x}+1| = \sqrt{x}+1$, domain: $[0, \infty)$
- $(f \circ f)(x) = ||x+1|+1| = |x+1|+1$, domain: $(-\infty, \infty)$

51. For $f(x) = |x|$ and $g(x) = \sqrt{4-x}$

- $(g \circ f)(x) = \sqrt{4-|x|}$, domain: $[-4, 4]$
- $(f \circ g)(x) = |\sqrt{4-x}| = \sqrt{4-x}$, domain: $(-\infty, 4]$

- $(f \circ f)(x) = ||x|| = |x|$, domain: $(-\infty, \infty)$

53. For $f(x) = 3x - 1$ and $g(x) = \frac{1}{x+3}$
    - $(g \circ f)(x) = \frac{1}{3x+2}$, domain: $\left(-\infty, -\frac{2}{3}\right) \cup \left(-\frac{2}{3}, \infty\right)$
    - $(f \circ g)(x) = -\frac{x}{x+3}$, domain: $(-\infty, -3) \cup (-3, \infty)$
    - $(f \circ f)(x) = 9x - 4$, domain: $(-\infty, \infty)$

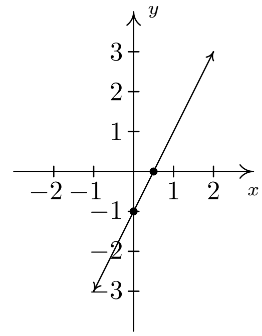
55. For $f(x) = \frac{x}{2x+1}$ and $g(x) = \frac{2x+1}{x}$
    - $(g \circ f)(x) = \frac{4x+1}{x}$, domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right), \cup (0, \infty)$
    - $(f \circ g)(x) = \frac{2x+1}{5x+2}$, domain: $\left(-\infty, -\frac{2}{5}\right) \cup \left(-\frac{2}{5}, 0\right) \cup (0, \infty)$
    - $(f \circ f)(x) = \frac{x}{4x+1}$, domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, -\frac{1}{4}\right) \cup \left(-\frac{1}{4}, \infty\right)$

57. $(h \circ g \circ f)(x) = |\sqrt{-2x}| = \sqrt{-2x}$, domain: $(-\infty, 0]$
59. $(g \circ f \circ h)(x) = \sqrt{-2|x|}$, domain: $\{0\}$
61. $(f \circ h \circ g)(x) = -2|\sqrt{x}| = -2\sqrt{x}$, domain: $[0, \infty)$
63. Let $f(x) = 2x + 3$ and $g(x) = x^3$, then $p(x) = (g \circ f)(x)$.
65. Let $f(x) = 2x - 1$ and $g(x) = \sqrt{x}$, then $h(x) = (g \circ f)(x)$.
67. Let $f(x) = 5x + 1$ and $g(x) = \frac{2}{x}$, then $r(x) = (g \circ f)(x)$.
69. Let $f(x) = |x|$ and $g(x) = \frac{x+1}{x-1}$, then $q(x) = (g \circ f)(x)$.
71. Let $f(x) = 2x$ and $g(x) = \frac{x+1}{3-2x}$, then $v(x) = (g \circ f)(x)$.
73. $f^{-1}(x) = \dfrac{x+2}{6}$
75. $f^{-1}(x) = 3x - 10$
77. $f^{-1}(x) = \frac{1}{3}(x-5)^2 + \frac{1}{3}, x \geq 5$
79. $f^{-1}(x) = \frac{1}{9}(x+4)^2 + 1, x \geq -4$
81. $f^{-1}(x) = \frac{1}{3}x^5 + \frac{1}{3}$
83. $f^{-1}(x) = 5 + \sqrt{x+25}$
85. $f^{-1}(x) = 3 - \sqrt{x+4}$
87. $f^{-1}(x) = \dfrac{4x-3}{x}$
89. $f^{-1}(x) = \dfrac{4x+1}{2-3x}$
91. $f^{-1}(x) = \dfrac{-3x-2}{x+3}$

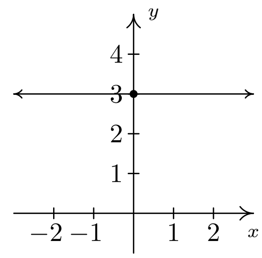
Chapter 3

Section 3.1

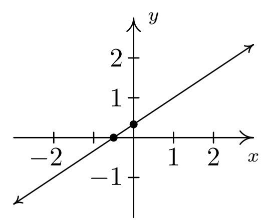
1. $y + 1 = 3(x - 3)$
   $y = 3x - 10$
3. $y + 1 = -(x + 7)$
   $y = -x - 8$
5. $y - 4 = -\frac{1}{5}(x - 10)$
   $y = -\frac{1}{5}x + 6$
7. $y - 117 = 0$
   $y = 117$
9. $y - 2\sqrt{3} = -5(x - \sqrt{3})$
   $y = -5x + 7\sqrt{3}$
11. $y = -\frac{5}{3}x$
13. $y = \frac{8}{5}x - 8$
15. $y = 5$
17. $y = -\frac{5}{4}x + \frac{11}{8}$
19. $y = -x$
21. $f(x) = 2x - 1$
    slope: $m = 2$
    y-intercept: $(0, -1)$
    x-intercept: $\left(\frac{1}{2}, 0\right)$



23. $f(x) = 3$
    slope: $m = 0$
    y-intercept: $(0, 3)$
    x-intercept: none

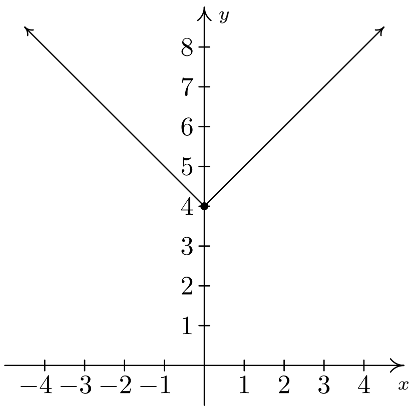


25. $f(x) = \frac{2}{3}x + \frac{1}{3}$
    slope: $m = \frac{2}{3}$
    y-intercept: $\left(0, \frac{1}{3}\right)$
    x-intercept: $\left(-\frac{1}{2}, 0\right)$

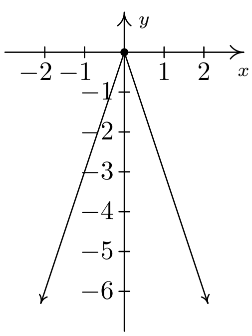


27. $x = -6$ or $x = 6$
29. $x = -3$ or $x = 11$
31. $x = -\frac{1}{2}$ or $x = \frac{1}{10}$
33. $x = -3$ or $x = 3$
35. $x = -\frac{3}{2}$
37. $x = 1$
39. $x = -1$, $x = 0$ or $x = 1$
41. $x = -2$ or $x = 2$
43. $x = -\frac{1}{7}$ or $x = 1$
45. $x = 1$
47. $x = \frac{1}{5}$ or $x = 5$

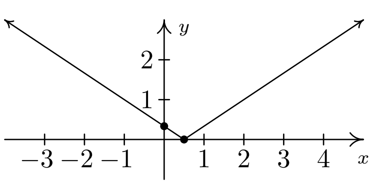
49. $f(x) = |x| + 4$
No zeros
No x-intercepts
y-intercept $(0, 4)$
Domain $(-\infty, \infty)$
Range $[4, \infty)$
Decreasing on $(-\infty, 0]$
Increasing on $[0, \infty)$
Relative and absolute minimum at $(0, 4)$
No relative or absolute maximum



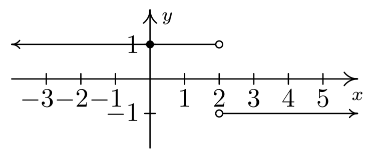
51. $f(x) = -3|x|$
$f(0) = 0$
x-intercept $(0, 0)$
y-intercept $(0, 0)$
Domain $(-\infty, \infty)$
Range $(-\infty, 0]$
Increasing on $(-\infty, 0]$
Decreasing on $[0, \infty)$
Relative and absolute maximum at $(0, 0)$
No relative or absolute minimum



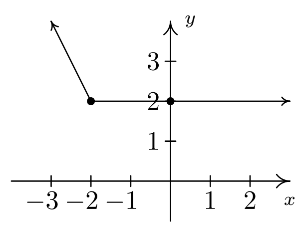
53. $f(x) = \frac{1}{3}|2x - 1|$
$f\left(\frac{1}{2}\right) = 0$
x-intercepts $\left(\frac{1}{2}, 0\right)$
y-intercept $\left(0, \frac{1}{3}\right)$
Domain $(-\infty, \infty)$
Range $[0, \infty)$
Decreasing on $\left(-\infty, \frac{1}{2}\right]$
Increasing on $\left[\frac{1}{2}, \infty\right)$
Relative and absolute min. at $\left(\frac{1}{2}, 0\right)$
No relative or absolute maximum



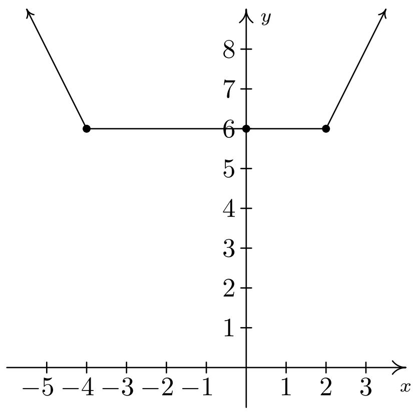
55. $f(x) = \dfrac{|2 - x|}{2 - x}$
No zeros
No x-intercept
y-intercept $(0, 1)$
Domain $(-\infty, 2) \cup (2, \infty)$
Range $\{-1, 1\}$
Constant on $(-\infty, 2)$
Constant on $(2, \infty)$
Absolute minimum at every point $(x, -1)$ where $x > 2$
Absolute maximum at every point $(x, 1)$ where $x < 2$
Relative maximum AND minimum at every point on the graph



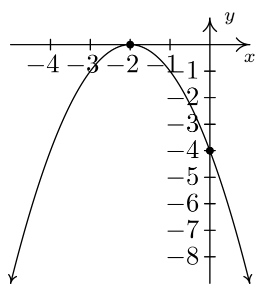
57. Re-write $f(x) = |x + 2| - x$ as
$$f(x) = \begin{cases} -2x - 2 & \text{if} \quad x < -2 \\ 2 & \text{if} \quad x \geq -2 \end{cases}$$
No zeros
No x-intercepts
y-intercept $(0, 2)$
Domain $(-\infty, \infty)$
Range $[2, \infty)$
Decreasing on $(-\infty, -2]$
Constant on $[-2, \infty)$
Absolute minimum at every point $(x, 2)$ where $x \geq -2$
No absolute maximum
Relative minimum at every point $(x, 2)$ where $x \geq -2$
Relative maximum at every point $(x, 2)$ where $x > -2$



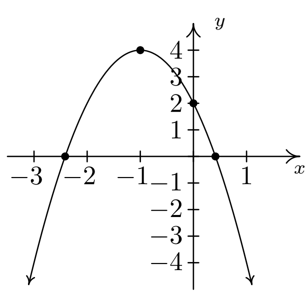
59. Re-write $f(x) = |x + 4| + |x - 2|$ as
$$f(x) = \begin{cases} -2x - 2 & \text{if} \quad x < -4 \\ 6 & \text{if} \quad -4 \leq x < 2 \\ 2x + 2 & \text{if} \quad x \geq 2 \end{cases}$$
No zeros
No x-intercept
y-intercept $(0, 6)$
Domain $(-\infty, \infty)$
Range $[6, \infty)$
Decreasing on $(-\infty, -4]$
Constant on $[-4, 2]$
Increasing on $[2, \infty)$
Absolute minimum at every point $(x, 6)$ where $-4 \leq x \leq 2$
No absolute maximum
Relative minimum at every point $(x, 6)$ where $-4 \leq x \leq 2$
Relative maximum at every point $(x, 6)$ where $-4 < x < 2$



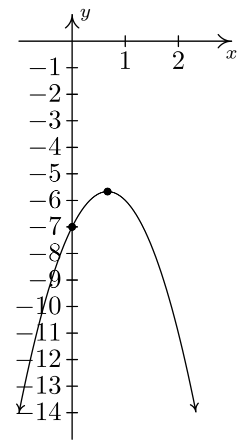
61. $f(x) = -(x+2)^2 = -x^2 - 4x - 4$
x-intercept $(-2, 0)$
y-intercept $(0, -4)$
Domain: $(-\infty, \infty)$
Range: $(-\infty, 0]$
Increasing on $(-\infty, -2]$
Decreasing on $[-2, \infty)$
Vertex $(-2, 0)$ is a maximum
Axis of symmetry $x = -2$



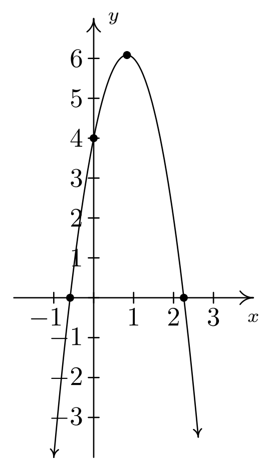
63. $f(x) = -2(x+1)^2 + 4 = -2x^2 - 4x + 2$
x-intercepts $(-1-\sqrt{2}, 0)$ and $(-1+\sqrt{2}, 0)$
y-intercept $(0, 2)$
Domain: $(-\infty, \infty)$
Range: $(-\infty, 4]$
Increasing on $(-\infty, -1]$
Decreasing on $[-1, \infty)$
Vertex $(-1, 4)$ is a maximum
Axis of symmetry $x = -1$



65. $f(x) = -3x^2 + 4x - 7 = -3\left(x - \frac{2}{3}\right)^2 - \frac{17}{3}$
No x-intercepts
y-intercept $(0, -7)$
Domain: $(-\infty, \infty)$
Range: $\left(-\infty, -\frac{17}{3}\right]$
Increasing on $\left(-\infty, \frac{2}{3}\right]$
Decreasing on $\left[\frac{2}{3}, \infty\right)$
Vertex $\left(\frac{2}{3}, -\frac{17}{3}\right)$ is a maximum
Axis of symmetry $x = \frac{2}{3}$



67. $f(x) = -3x^2 + 5x + 4 = -3\left(x - \frac{5}{6}\right)^2 + \frac{73}{12}$
x-intercepts $\left(\frac{5-\sqrt{73}}{6}, 0\right)$ and $\left(\frac{5+\sqrt{73}}{6}, 0\right)$
y-intercept $(0, 4)$
Domain: $(-\infty, \infty)$
Range: $\left(-\infty, \frac{73}{12}\right]$
Increasing on $\left(-\infty, \frac{5}{6}\right]$
Decreasing on $\left[\frac{5}{6}, \infty\right)$
Vertex $\left(\frac{5}{6}, \frac{73}{12}\right)$ is a maximum
Axis of symmetry $x = \frac{5}{6}$



69. $\left(-\infty, -\frac{12}{7}\right) \cup \left(\frac{8}{7}, \infty\right)$
71. $(-\infty, 1] \cup [3, \infty)$
73. $(-\infty, \infty)$
75. $[3, 4) \cup (5, 6]$
77. $(-\infty, -4) \cup \left(\frac{2}{3}, \infty\right)$
79. $(-\infty, -5)$
81. $\left[-7, \frac{5}{3}\right]$
83. $(-\infty, \infty)$
85. $\left(-\infty, -\frac{1}{4}\right) \cup \left(-\frac{1}{4}, \infty\right)$
87. $(-\infty, \infty)$
89. No solution
91. $(0, 1)$
93. $\left(-\infty, \frac{5-\sqrt{73}}{6}\right] \cup \left[\frac{5+\sqrt{73}}{6}, \infty\right)$
95. $\left[-2-\sqrt{7}, -2+\sqrt{7}\right] \cup [1, 3]$
97. $(-\infty, -1] \cup \{0\} \cup [1, \infty)$
99. $(-\infty, 1) \cup \left(2, \frac{3+\sqrt{17}}{2}\right)$

Section 3.2

1. $f(x) = 4 - x - 3x^2$
   Degree 2
   Leading term $-3x^2$
   Leading coefficient -3
   Constant term 4
   As $x \to -\infty,\ f(x) \to -\infty$
   As $x \to \infty,\ f(x) \to -\infty$

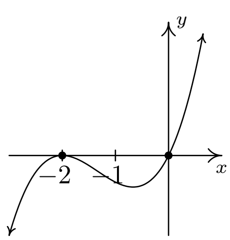
3. $q(r) = 1 - 16r^4$
   Degree 4
   Leading term $-16r^4$
   Leading coefficient -16
   Constant term 1
   As $r \to -\infty,\ q(r) \to -\infty$
   As $r \to \infty,\ q(r) \to -\infty$

5. $f(x) = \sqrt{3}x^{17} + 22.5x^{10} - \pi x^7 + \frac{1}{3}$
   Degree 17
   Leading term $\sqrt{3}x^{17}$
   Leading coefficient $\sqrt{3}$
   Constant term $\frac{1}{3}$
   As $x \to -\infty,\ f(x) \to -\infty$
   As $x \to \infty,\ f(x) \to \infty$

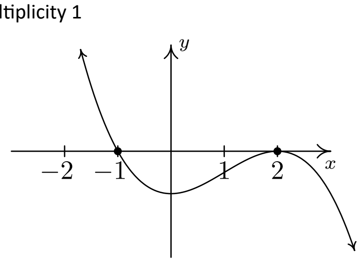
7. $P(x) = (x-1)(x-2)(x-3)(x-4)$
   Degree 4
   Leading term x^4
   Leading coefficient 1
   Constant term 24
   As $x \to -\infty,\ P(x) \to \infty$
   As $x \to \infty,\ P(x) \to \infty$

9. $f(x) = -2x^3(x+1)(x+2)^2$
   Degree 6
   Leading term $-2x^6$
   Leading coefficient -2
   Constant term 0
   As $x \to -\infty,\ f(x) \to -\infty$
   As $x \to \infty,\ f(x) \to -\infty$

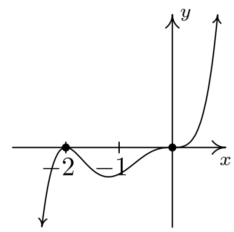
11. $a(x) = x(x+2)^2$
   $x = 0$ multiplicity 1
   $x = -2$ multiplicity 2



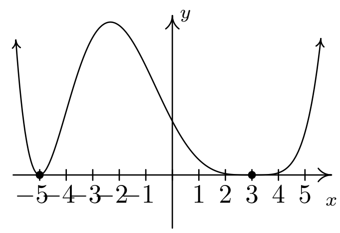
13. $f(x) = -2(x-2)^2(x+1)$
   $x = 2$ multiplicity 2
   $x = -1$ multiplicity 1



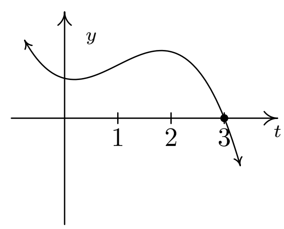
15. $F(x) = x^3(x+2)^2$
   $x = 0$ multiplicity 3
   $x = -2$ multiplicity 2



17. $Q(x) = (x+5)^2(x-3)^4$
   $x = -5$ multiplicity 2
   $x = 3$ multiplicity 4



19. $H(t) = (3-t)\left(t^2+1\right)$
   $x = 3$ multiplicity 1



21.

23. $t^2 + 6t - 6$

25. $6y^2 + y - 1$

27. $-4t^3 - 3t^2 + 8t + 6$

29. $125a^6 - 27$

31. $7 - z^2$

33. $x^3 - 5$

35. $h^2 + 2xh - 2h$

37. quotient: $5x - 8$, remainder: 9

39. quotient: 3, remainder: 18

41. quotient: $\frac{t}{2} - \frac{1}{4}$, remainder: $-\frac{15}{4}$

43. quotient: $\frac{2}{3}$, remainder: $-x + \frac{1}{3}$

45. quotient: w, remainder: $2w$

47. quotient:[1] $t^2 + t\sqrt[3]{4} + 2\sqrt[3]{2}$, remainder: 0

49.

51.

53.

Section 3.3

[1]Note: $\sqrt[3]{16} = 2\sqrt[3]{2}$.

1. $f(x) = \dfrac{x}{3x-6}$
Domain: $(-\infty, 2) \cup (2, \infty)$
Vertical asymptote: $x = 2$
As $x \to 2^-, f(x) \to -\infty$
As $x \to 2^+, f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = \frac{1}{3}$
As $x \to -\infty, f(x) \to \frac{1}{3}^-$
As $x \to \infty, f(x) \to \frac{1}{3}^+$

3. $f(x) = \dfrac{x}{x^2+x-12} = \dfrac{x}{(x+4)(x-3)}$
Domain: $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$
Vertical asymptotes: $x = -4, x = 3$
As $x \to -4^-, f(x) \to -\infty$
As $x \to -4^+, f(x) \to \infty$
As $x \to 3^-, f(x) \to -\infty$
As $x \to 3^+, f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty, f(x) \to 0^-$
As $x \to \infty, f(x) \to 0^+$

5. $f(x) = \dfrac{x+7}{(x+3)^2}$
Domain: $(-\infty, -3) \cup (-3, \infty)$
Vertical asymptote: $x = -3$
As $x \to -3^-, f(x) \to \infty$
As $x \to -3^+, f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty, f(x) \to 0^-$
As $x \to \infty, f(x) \to 0^+$

7. $f(x) = \dfrac{4x}{x^2+4}$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty, f(x) \to 0^-$
As $x \to \infty, f(x) \to 0^+$

9. $f(x) = \dfrac{x^2-x-12}{x^2+x-6} = \dfrac{x-4}{x-2}$
Domain: $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$
Vertical asymptote: $x = 2$
As $x \to 2^-, f(x) \to \infty$
As $x \to 2^+, f(x) \to -\infty$
Hole at $\left(-3, \frac{7}{5}\right)$
Horizontal asymptote: $y = 1$
As $x \to -\infty, f(x) \to 1^+$
As $x \to \infty, f(x) \to 1^-$

11. $f(x) = \dfrac{x^3+2x^2+x}{x^2-x-2} = \dfrac{x(x+1)}{x-2}$
Domain: $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$
Vertical asymptote: $x = 2$
As $x \to 2^-, f(x) \to -\infty$
As $x \to 2^+, f(x) \to \infty$
Hole at $(-1, 0)$
Slant asymptote: $y = x + 3$
As $x \to -\infty$, the graph is below $y = x + 3$
As $x \to \infty$, the graph is above $y = x + 3$

13. $f(x) = \dfrac{2x^2+5x-3}{3x+2}$
Domain: $\left(-\infty, -\frac{2}{3}\right) \cup \left(-\frac{2}{3}, \infty\right)$
Vertical asymptote: $x = -\frac{2}{3}$
As $x \to -\frac{2}{3}^-, f(x) \to \infty$
As $x \to -\frac{2}{3}^+, f(x) \to -\infty$
No holes in the graph
Slant asymptote: $y = \frac{2}{3}x + \frac{11}{9}$
As $x \to -\infty$, the graph is above $y = \frac{2}{3}x + \frac{11}{9}$
As $x \to \infty$, the graph is below $y = \frac{2}{3}x + \frac{11}{9}$

15. $f(x) = \dfrac{-5x^4-3x^3+x^2-10}{x^3-3x^2+3x-1} = \dfrac{-5x^4-3x^3+x^2-10}{(x-1)^3}$
Domain: $(-\infty, 1) \cup (1, \infty)$
Vertical asymptotes: $x = 1$
As $x \to 1^-, f(x) \to \infty$
As $x \to 1^+, f(x) \to -\infty$
No holes in the graph
Slant asymptote: $y = -5x - 18$
As $x \to -\infty$, the graph is above $y = -5x - 18$
As $x \to \infty$, the graph is below $y = -5x - 18$

17. $f(x) = \dfrac{18-2x^2}{x^2-9} = -2$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
No vertical asymptotes
Holes in the graph at $(-3, -2)$ and $(3, -2)$
Horizontal asymptote $y = -2$
As $x \to \pm\infty, f(x) = -2$

19. $x = -\frac{6}{7}$

21. $x = -1$

23. No solution

25. $(-2, \infty)$

27. $(-1, 0) \cup (1, \infty)$

29. $(-\infty, -3) \cup (-3, 2) \cup (4, \infty)$

31. $(-1, 0] \cup (2, \infty)$

33. $(-\infty, 1] \cup [2, \infty)$

35. $(-\infty, -3) \cup \left[-2\sqrt{2}, 0\right] \cup \left[2\sqrt{2}, 3\right)$

37. $[-3, 0) \cup (0, 4) \cup [5, \infty)$

Section 3.4

1. $\log_2(8) = 3$
3. $\log_4(32) = \frac{5}{2}$
5. $\log_{\frac{4}{25}}\left(\frac{5}{2}\right) = -\frac{1}{2}$
7. $\ln(1) = 0$
9. $(25)^{\frac{1}{2}} = 5$
11. $\left(\frac{4}{3}\right)^{-1} = \frac{3}{4}$
13. $10^{-1} = 0.1$
15. $e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$
17. $\log_6(216) = 3$
19. $\log_6\left(\frac{1}{36}\right) = -2$
21. $\log_{36}(216) = \frac{3}{2}$
23. $\log_{\frac{1}{6}}(216) = -3$
25. $\log \frac{1}{1000000} = -6$
27. $\ln\left(e^3\right) = 3$
29. $\log_6(1) = 0$
31. $\log_{36}\left(\sqrt[4]{36}\right) = \frac{1}{4}$
33. $36^{\log_{36}(216)} = 216$
35. $\ln(e^5) = 5$

37. $\log\left(\sqrt[3]{10^5}\right) = \frac{5}{3}$

39. $\log_5\left(3^{\log_3 5}\right) = 1$

41. $\log_2\left(3^{-\log_3(2)}\right) = -1$

43. $(-\infty, \infty)$

45. $(5, \infty)$

47. $(-2, -1) \cup (1, \infty)$

49. $(4, 7)$

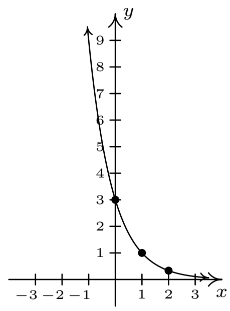
51. $(-\infty, \infty)$

53. $(-\infty, -7) \cup (1, \infty)$

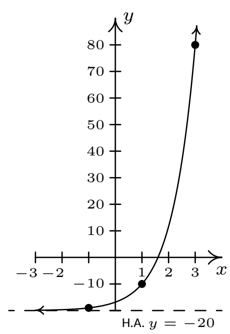
55. $(0, 125) \cup (125, \infty)$

57. $(-\infty, -3) \cup \left(\frac{1}{2}, 2\right)$

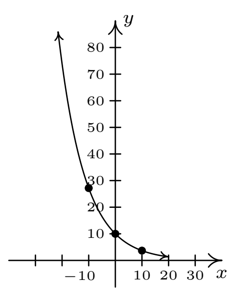
59. Domain of g: $(-\infty, \infty)$
Range of g: $(0, \infty)$



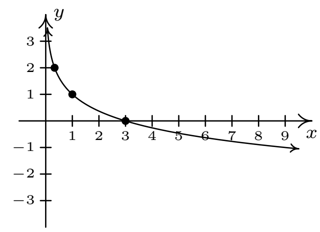
61. Domain of g: $(-\infty, \infty)$
Range of g: $(-20, \infty)$



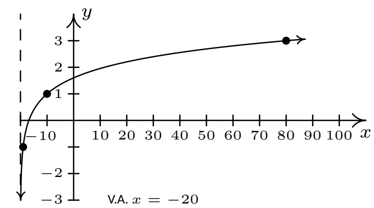
63. Domain of g: $(-\infty, \infty)$
Range of g: $(0, \infty)$



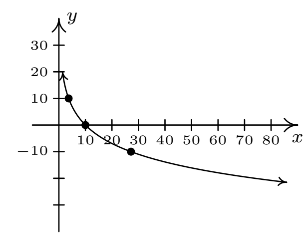
65. Domain of g: $(0, \infty)$
Range of g: $(-\infty, \infty)$



67. Domain of g: $(-20, \infty)$
Range of g: $(-\infty, \infty)$



69. Domain of g: $(0, \infty)$
Range of g: $(-\infty, \infty)$



71. $7 - \log_2(x^2 + 4)$

73. $\log(1.23) + 37$

75. $\log_5(x - 5) + \log_5(x + 5)$

77. l$-2 + \log_{\frac{1}{3}}(x) + \log_{\frac{1}{3}}(y - 2) + \log_{\frac{1}{3}}(y^2 + 2y + 4)$

79. $2\log_3(x) - 4 - 4\log_3(y)$

81. $12 - 12\log_6(x) - 4\log_6(y)$

83. $-2 + \frac{2}{3}\log_{\frac{1}{2}}(x) - \log_{\frac{1}{2}}(y) - \frac{1}{2}\log_{\frac{1}{2}}(z)$

85. $\ln(x^4y^2)$

87. $\log_3\left(\frac{x}{y^2}\right)$

89. $\ln\left(\frac{x^2}{y^3z^4}\right)$

91. $\ln\left(\sqrt[3]{\frac{z}{xy}}\right)$

93. $\log\left(\frac{1000}{x}\right)$

95. $\ln\left(x\sqrt{e}\right)$

97. $\log_2\left(x\sqrt{x-1}\right)$

99. $7^{x-1} = e^{(x-1)\ln(7)}$

101. $\left(\frac{2}{3}\right)^x = e^{x\ln\left(\frac{2}{3}\right)}$

103. $\log_3(12) \approx 2.26186$

105. $\log_6(72) \approx 2.38685$

107. $\log_{\frac{3}{5}}(1000) \approx -13.52273$

Chapter 4

Section 4.1

1. $\cos(0) = 1,\ \sin(0) = 0$

3. $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2},\ \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

5. $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2},\ \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

7. $\cos(\pi) = -1,\ \sin(\pi) = 0$

9. $\cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2},\ \sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

11. $\cos\left(\frac{3\pi}{2}\right) = 0,\ \sin\left(\frac{3\pi}{2}\right) = -1$

13. $\cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2},\ \sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

15. $\cos\left(-\frac{13\pi}{2}\right) = 0,\ \sin\left(-\frac{13\pi}{2}\right) = -1$

17. $\cos\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2},\ \sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

19. $\cos\left(\frac{10\pi}{3}\right) = -\frac{1}{2},\ \sin\left(\frac{10\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

21. If $\sin(\theta) = -\frac{7}{25}$ with θ in Quadrant IV, then $\cos(\theta) = \frac{24}{25}$.

23. If $\sin(\theta) = \frac{5}{13}$ with θ in Quadrant II, then $\cos(\theta) = -\frac{12}{13}$.

25. If $\sin(\theta) = -\frac{2}{3}$ with θ in Quadrant III, then $\cos(\theta) = -\frac{\sqrt{5}}{3}$.

27. If $\sin(\theta) = \frac{2\sqrt{5}}{5}$ and $\frac{\pi}{2} < \theta < \pi$, then $\cos(\theta) = -\frac{\sqrt{5}}{5}$.

29. If $\sin(\theta) = -0.42$ and $\pi < \theta < \frac{3\pi}{2}$, then $\cos(\theta) = -\sqrt{0.8236} \approx -0.9075$.

31. $\sin(\theta) = \frac{1}{2}$ when $\theta = \frac{\pi}{6} + 2\pi k$ or $\theta = \frac{5\pi}{6} + 2\pi k$ for any integer k.

33. $\sin(\theta) = 0$ when $\theta = \pi k$ for any integer k.

35. $\sin(\theta) = \frac{\sqrt{3}}{2}$ when $\theta = \frac{\pi}{3} + 2\pi k$ or $\theta = \frac{2\pi}{3} + 2\pi k$ for any integer k.

37. $\sin(\theta) = -1$ when $\theta = \frac{3\pi}{2} + 2\pi k$ for any integer k.

39. $\cos(\theta) = -1.001$ never happens

Section 4.2

1. $\cos(0) = 1,\ \sin(0) = 0$

3. $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2},\ \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

5. $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2},\ \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

7. $\cos(\pi) = -1,\ \sin(\pi) = 0$

9. $\cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2},\ \sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

11. $\cos\left(\frac{3\pi}{2}\right) = 0,\ \sin\left(\frac{3\pi}{2}\right) = -1$

13. $\cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2},\ \sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

15. $\cos\left(-\frac{13\pi}{2}\right) = 0,\ \sin\left(-\frac{13\pi}{2}\right) = -1$

17. $\cos\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2},\ \sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

19. $\cos\left(\frac{10\pi}{3}\right) = -\frac{1}{2},\ \sin\left(\frac{10\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

21. $\sin(\theta) = \frac{3}{5}, \cos(\theta) = -\frac{4}{5}, \tan(\theta) = -\frac{3}{4}, \csc(\theta) = \frac{5}{3}, \sec(\theta) = -\frac{5}{4}, \cot(\theta) = -\frac{4}{3}$

23. $\sin(\theta) = \frac{24}{25}, \cos(\theta) = \frac{7}{25}, \tan(\theta) = \frac{24}{7}, \csc(\theta) = \frac{25}{24}, \sec(\theta) = \frac{25}{7}, \cot(\theta) = \frac{7}{24}$

25. $\sin(\theta) = -\frac{\sqrt{91}}{10}, \cos(\theta) = -\frac{3}{10}, \tan(\theta) = \frac{\sqrt{91}}{3}, \csc(\theta) = -\frac{10\sqrt{91}}{91}, \sec(\theta) = -\frac{10}{3}, \cot(\theta) = \frac{3\sqrt{91}}{91}$

27. $\sin(\theta) = -\frac{2\sqrt{5}}{5}, \cos(\theta) = \frac{\sqrt{5}}{5}, \tan(\theta) = -2, \csc(\theta) = -\frac{\sqrt{5}}{2}, \sec(\theta) = \sqrt{5}, \cot(\theta) = -\frac{1}{2}$

29. $\sin(\theta) = -\frac{\sqrt{6}}{6}, \cos(\theta) = -\frac{\sqrt{30}}{6}, \tan(\theta) = \frac{\sqrt{5}}{5}, \csc(\theta) = -\sqrt{6}, \sec(\theta) = -\frac{\sqrt{30}}{5}, \cot(\theta) = \sqrt{5}$

31. $\sin(\theta) = \frac{\sqrt{5}}{5}, \cos(\theta) = \frac{2\sqrt{5}}{5}, \tan(\theta) = \frac{1}{2}, \csc(\theta) = \sqrt{5}, \sec(\theta) = \frac{\sqrt{5}}{2}, \cot(\theta) = 2$

33. $\sin(\theta) = -\frac{\sqrt{110}}{11}, \cos(\theta) = -\frac{\sqrt{11}}{11}, \tan(\theta) = \sqrt{10}, \csc(\theta) = -\frac{\sqrt{110}}{10}, \sec(\theta) = -\sqrt{11}, \cot(\theta) = \frac{\sqrt{10}}{10}$

35. $\tan(\theta) = \sqrt{3}$ when $\theta = \dfrac{\pi}{3} + \pi k$ for any integer k

37. $\csc(\theta) = -1$ when $\theta = \dfrac{3\pi}{2} + 2\pi k$ for any integer k.

39. $\tan(\theta) = 0$ when $\theta = \pi k$ for any integer k

41. $\csc(\theta) = 2$ when $\theta = \dfrac{\pi}{6} + 2\pi k$ or $\theta = \dfrac{5\pi}{6} + 2\pi k$ for any integer k.

43. $\tan(\theta) = -1$ when $\theta = \dfrac{3\pi}{4} + \pi k$ for any integer k

45. $\csc(\theta) = -\dfrac{1}{2}$ never happens

47. $\tan(\theta) = -\sqrt{3}$ when $\theta = \dfrac{2\pi}{3} + \pi k$ for any integer k

49. $\cot(\theta) = -1$ when $\theta = \dfrac{3\pi}{4} + \pi k$ for any integer k

51. $\tan(t) = \dfrac{\sqrt{3}}{3}$ when $t = \dfrac{\pi}{6} + \pi k$ for any integer k

53. $\csc(t) = 0$ never happens

55. $\tan(t) = -\dfrac{\sqrt{3}}{3}$ when $t = \dfrac{5\pi}{6} + \pi k$ for any integer k

57. $\csc(t) = \dfrac{2\sqrt{3}}{3}$ when $t = \dfrac{\pi}{3} + 2\pi k$ or $t = \dfrac{2\pi}{3} + 2\pi k$ for any integer k

59.

61.

63.

65.

67.

69.

71.

73.

75.

77.

79.

81.

83.

85.

87.

89.

91.

93.

95.

97.

99.

101.

103.

Section 4.3

1.

3.

5.

7. $\cos(75^\circ) = \dfrac{\sqrt{6} - \sqrt{2}}{4}$

9. $\sin(105^\circ) = \dfrac{\sqrt{6} + \sqrt{2}}{4}$

11. $\cot(255^\circ) = \dfrac{\sqrt{3} - 1}{\sqrt{3} + 1} = 2 - \sqrt{3}$

13. $\cos\left(\dfrac{13\pi}{12}\right) = -\dfrac{\sqrt{6} + \sqrt{2}}{4}$

15. $\tan\left(\dfrac{13\pi}{12}\right) = \dfrac{3 - \sqrt{3}}{3 + \sqrt{3}} = 2 - \sqrt{3}$

17. $\tan\left(\dfrac{17\pi}{12}\right) = 2 + \sqrt{3}$

19. $\cot\left(\dfrac{11\pi}{12}\right) = -(2 + \sqrt{3})$

21. $\sec\left(-\dfrac{\pi}{12}\right) = \sqrt{6} - \sqrt{2}$

23. (a) $\cos(\alpha + \beta) = -\dfrac{4 + 7\sqrt{2}}{30}$

 (b) $\sin(\alpha + \beta) = \dfrac{28 - \sqrt{2}}{30}$

 (c) $\tan(\alpha + \beta) = \dfrac{-28 + \sqrt{2}}{4 + 7\sqrt{2}} = \dfrac{63 - 100\sqrt{2}}{41}$

 (d) $\cos(\alpha - \beta) = \dfrac{-4 + 7\sqrt{2}}{30}$

 (e) $\sin(\alpha - \beta) = -\dfrac{28 + \sqrt{2}}{30}$

 (f) $\tan(\alpha - \beta) = \dfrac{28 + \sqrt{2}}{4 - 7\sqrt{2}} = -\dfrac{63 + 100\sqrt{2}}{41}$

25. (a) $\csc(\alpha - \beta) = -\dfrac{5}{4}$

 (b) $\sec(\alpha + \beta) = \dfrac{125}{117}$

 (c) $\cot(\alpha + \beta) = \dfrac{117}{44}$

27.

29.

31.

33.

35.

37.

39. $\cos(75°) = \dfrac{\sqrt{2-\sqrt{3}}}{2}$

41. $\cos(67.5°) = \dfrac{\sqrt{2-\sqrt{2}}}{2}$

43. $\tan(112.5°) = -\sqrt{\dfrac{2+\sqrt{2}}{2-\sqrt{2}}} = -1-\sqrt{2}$

45. $\sin\left(\frac{\pi}{12}\right) = \dfrac{\sqrt{2-\sqrt{3}}}{2}$

47. $\sin\left(\frac{5\pi}{8}\right) = \dfrac{\sqrt{2+\sqrt{2}}}{2}$

49.
- $\sin(2\theta) = -\frac{336}{625}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{2}}{10}$
- $\cos(2\theta) = \frac{527}{625}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{7\sqrt{2}}{10}$
- $\tan(2\theta) = -\frac{336}{527}$
- $\tan\left(\frac{\theta}{2}\right) = -\frac{1}{7}$

51.
- $\sin(2\theta) = \frac{120}{169}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{3\sqrt{13}}{13}$
- $\cos(2\theta) = -\frac{119}{169}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{2\sqrt{13}}{13}$
- $\tan(2\theta) = -\frac{120}{119}$
- $\tan\left(\frac{\theta}{2}\right) = -\frac{3}{2}$

53.
- $\sin(2\theta) = \frac{24}{25}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{5}}{5}$
- $\cos(2\theta) = -\frac{7}{25}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{2\sqrt{5}}{5}$
- $\tan(2\theta) = -\frac{24}{7}$
- $\tan\left(\frac{\theta}{2}\right) = \frac{1}{2}$

55.
- $\sin(2\theta) = -\frac{120}{169}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{26}}{26}$
- $\cos(2\theta) = \frac{119}{169}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{5\sqrt{26}}{26}$
- $\tan(2\theta) = -\frac{120}{119}$
- $\tan\left(\frac{\theta}{2}\right) = -\frac{1}{5}$

57.
- $\sin(2\theta) = -\frac{4}{5}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{50-10\sqrt{5}}}{10}$
- $\cos(2\theta) = -\frac{3}{5}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{\sqrt{50+10\sqrt{5}}}{10}$
- $\tan(2\theta) = \frac{4}{3}$
- $\tan\left(\frac{\theta}{2}\right) = -\sqrt{\frac{5-\sqrt{5}}{5+\sqrt{5}}} = \frac{5-5\sqrt{5}}{10}$

59.

61.

63.

65.

67.

69.

71.

73.

75. $\dfrac{\cos(5\theta)-\cos(9\theta)}{2}$

77. $\dfrac{\cos(4\theta)+\cos(8\theta)}{2}$

79. $\dfrac{\sin(2\theta)+\sin(4\theta)}{2}$

81. $-2\cos\left(\frac{9}{2}\theta\right)\sin\left(\frac{5}{2}\theta\right)$

83. $2\cos(4\theta)\sin(5\theta)$

85. $-\sqrt{2}\sin\left(\theta-\frac{\pi}{4}\right)$

87.

89.

91. $\dfrac{14x}{x^2+49}$

93.

95.

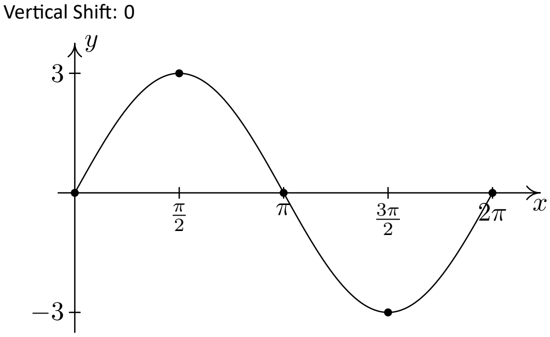
97.

99.

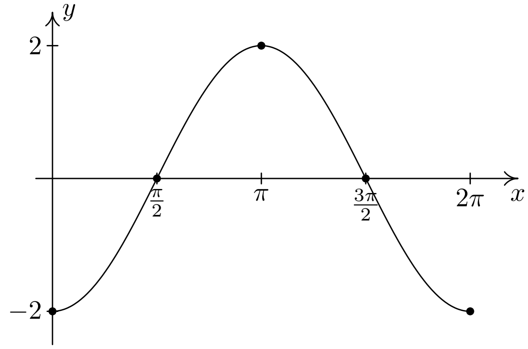
101.

Section 4.4

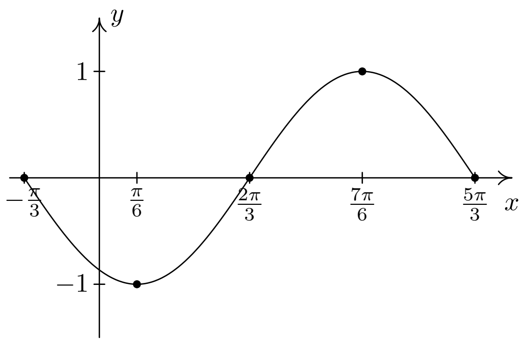
1. $y = 3\sin(x)$
Period: 2π
Amplitude: 3
Phase Shift: 0
Vertical Shift: 0



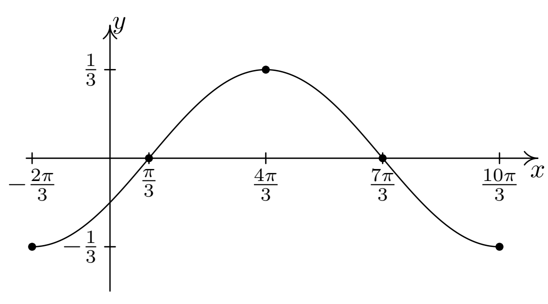
3. $y = -2\cos(x)$
Period: 2π
Amplitude: 2
Phase Shift: 0
Vertical Shift: 0



5. $y = -\sin\left(x + \frac{\pi}{3}\right)$
Period: 2π
Amplitude: 1
Phase Shift: $-\frac{\pi}{3}$
Vertical Shift: 0

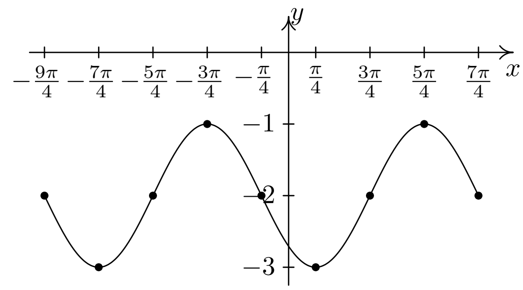


7. $y = -\frac{1}{3}\cos\left(\frac{1}{2}x + \frac{\pi}{3}\right)$
Period: 4π
Amplitude: $\frac{1}{3}$
Phase Shift: $-\frac{2\pi}{3}$
Vertical Shift: 0

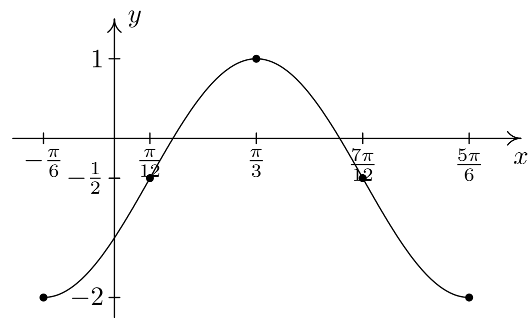


9. $y = \sin\left(-x - \frac{\pi}{4}\right) - 2$
Period: 2π
Amplitude: 1
Phase Shift: $-\frac{\pi}{4}$ (You need to use

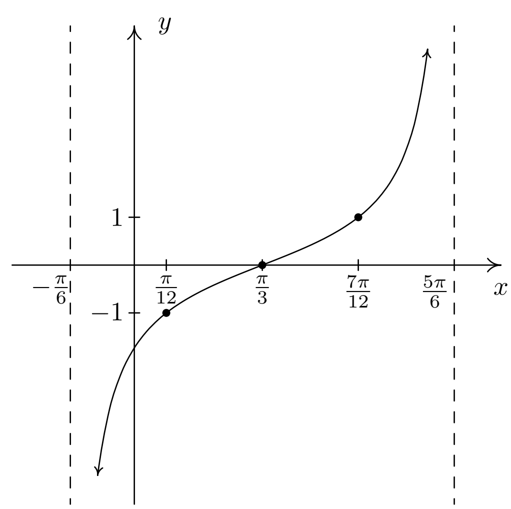
$y = -\sin\left(x + \frac{\pi}{4}\right) - 2$ to find this.)
Vertical Shift: -2



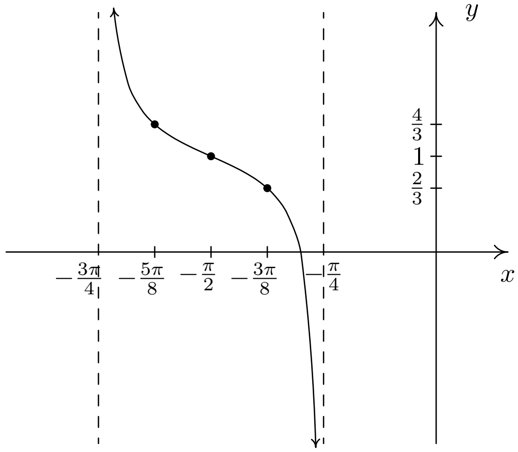
11. $y = -\frac{3}{2}\cos\left(2x + \frac{\pi}{3}\right) - \frac{1}{2}$
Period: π
Amplitude: $\frac{3}{2}$
Phase Shift: $-\frac{\pi}{6}$
Vertical Shift: $-\frac{1}{2}$



13. $y = \tan\left(x - \frac{\pi}{3}\right)$
Period: π



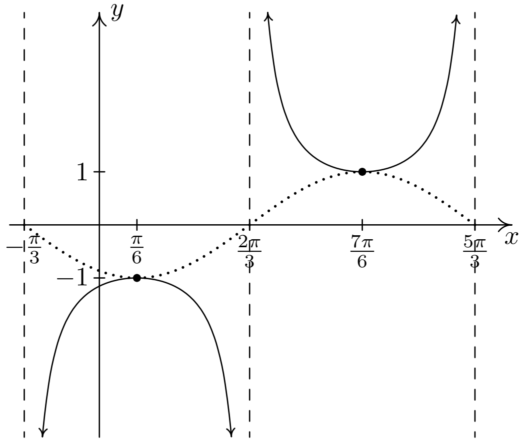
15. $y = \frac{1}{3}\tan(-2x - \pi) + 1$
is equivalent to
$y = -\frac{1}{3}\tan(2x + \pi) + 1$
via the Even / Odd identity for tangent.
Period: $\frac{\pi}{2}$



17. $y = -\csc\left(x + \frac{\pi}{3}\right)$

Start with $y = -\sin\left(x + \frac{\pi}{3}\right)$

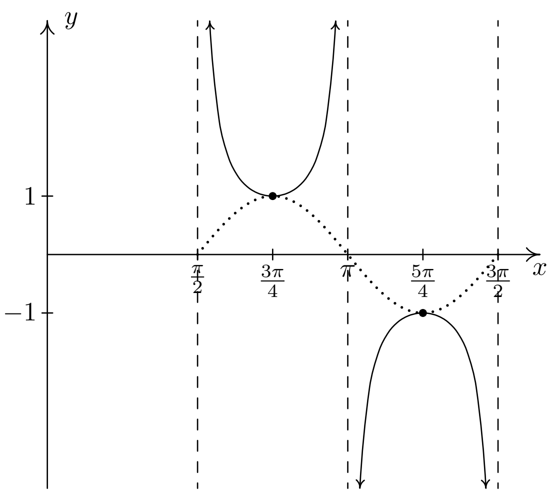
Period: 2π



19. $y = \csc(2x - \pi)$

Start with $y = \sin(2x - \pi)$

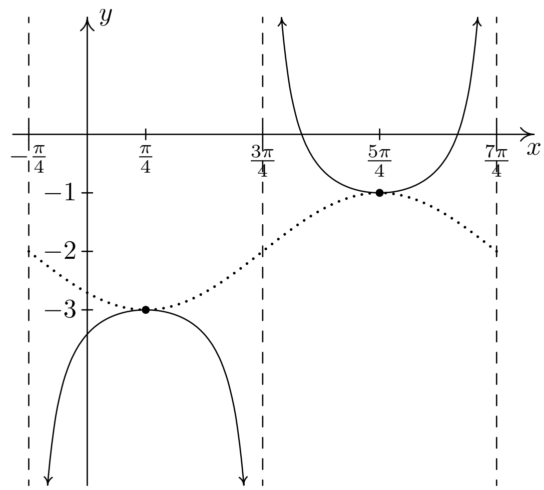
Period: π



21. $y = \csc\left(-x - \frac{\pi}{4}\right) - 2$

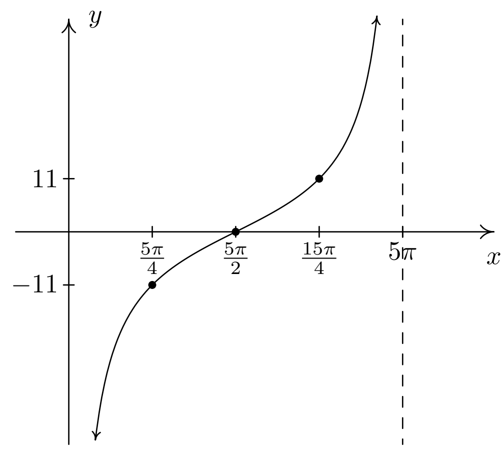
Start with $y = \sin\left(-x - \frac{\pi}{4}\right) - 2$

Period: 2π



23. $y = -11\cot\left(\frac{1}{5}x\right)$

Period: 5π



25. $f(x) = \sqrt{2}\sin(x) + \sqrt{2}\cos(x) + 1 = 2\sin\left(x + \frac{\pi}{4}\right) + 1 = 2\cos\left(x + \frac{7\pi}{4}\right) + 1$

27. $f(x) = -\sin(x) + \cos(x) - 2 = \sqrt{2}\sin\left(x + \frac{3\pi}{4}\right) - 2 = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right) - 2$

29. $f(x) = 2\sqrt{3}\cos(x) - 2\sin(x) = 4\sin\left(x + \frac{2\pi}{3}\right) = 4\cos\left(x + \frac{\pi}{6}\right)$

31. $f(x) = -\frac{1}{2}\cos(5x) - \frac{\sqrt{3}}{2}\sin(5x) = \sin\left(5x + \frac{7\pi}{6}\right) = \cos\left(5x + \frac{2\pi}{3}\right)$

33. $f(x) = \frac{5\sqrt{2}}{2}\sin(x) - \frac{5\sqrt{2}}{2}\cos(x) = 5\sin\left(x + \frac{7\pi}{4}\right) = 5\cos\left(x + \frac{5\pi}{4}\right)$

35.

37.

39.

41.

43.

45.

47.

49.

Section 4.5

1. $\arcsin(-1) = -\frac{\pi}{2}$

3. $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

5. $\arcsin(0) = 0$

7. $\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

9. $\arcsin(1) = \frac{\pi}{2}$

11. $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

13. $\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

15. $\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$

17. $\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

19. $\arctan\left(-\sqrt{3}\right) = -\frac{\pi}{3}$

21. $\arctan\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6}$

23. $\arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$

25. $\arctan\left(\sqrt{3}\right) = \frac{\pi}{3}$

27. $\operatorname{arccot}(-1) = \frac{3\pi}{4}$

29. $\operatorname{arccot}(0) = \frac{\pi}{2}$

31. $\operatorname{arccot}(1) = \frac{\pi}{4}$

33. $\operatorname{arcsec}(2) = \frac{\pi}{3}$

35. $\operatorname{arcsec}\left(\sqrt{2}\right) = \frac{\pi}{4}$

37. $\operatorname{arcsec}\left(\frac{2\sqrt{3}}{3}\right) = \frac{\pi}{6}$

39. $\operatorname{arcsec}(1) = 0$

41. $\operatorname{arcsec}(-2) = \frac{4\pi}{3}$

43. $\operatorname{arcsec}\left(-\frac{2\sqrt{3}}{3}\right) = \frac{7\pi}{6}$

45. $\operatorname{arccsc}(-2) = \frac{7\pi}{6}$

47. $\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right) = \frac{4\pi}{3}$

49. $\operatorname{arcsec}(-2) = \frac{2\pi}{3}$

51. $\operatorname{arcsec}\left(-\frac{2\sqrt{3}}{3}\right) = \frac{5\pi}{6}$

53. $\operatorname{arccsc}(-2) = -\frac{\pi}{6}$

55. $\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right) = -\frac{\pi}{3}$

57. $\sin\left(\arcsin\left(\frac{1}{2}\right)\right) = \frac{1}{2}$

59. $\sin\left(\arcsin\left(\frac{3}{5}\right)\right) = \frac{3}{5}$

61. $\sin\left(\arcsin\left(\frac{5}{4}\right)\right)$ is undefined.

63. $\cos\left(\arccos\left(-\frac{1}{2}\right)\right) = -\frac{1}{2}$

65. $\cos(\arccos(-0.998)) = -0.998$

67. $\tan(\arctan(-1)) = -1$

69. $\tan\left(\arctan\left(\frac{5}{12}\right)\right) = \frac{5}{12}$

71. $\tan(\arctan(3\pi)) = 3\pi$

73. $\cot\left(\operatorname{arccot}\left(-\sqrt{3}\right)\right) = -\sqrt{3}$

75. $\cot(\operatorname{arccot}(-0.001)) = -0.001$

77. $\sec(\operatorname{arcsec}(2)) = 2$

79. $\sec\left(\operatorname{arcsec}\left(\frac{1}{2}\right)\right)$ is undefined.

81. $\sec(\operatorname{arcsec}(117\pi)) = 117\pi$

83. $\csc\left(\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)\right) = -\frac{2\sqrt{3}}{3}$

85. $\csc(\operatorname{arccsc}(1.0001)) = 1.0001$

87. $\arcsin\left(\sin\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$

89. $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}$

91. $\arcsin\left(\sin\left(\frac{4\pi}{3}\right)\right) = -\frac{\pi}{3}$

93. $\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$

95. $\arccos\left(\cos\left(-\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$

97. $\arctan\left(\tan\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$

99. $\arctan(\tan(\pi)) = 0$

101. $\arctan\left(\tan\left(\frac{2\pi}{3}\right)\right) = -\frac{\pi}{3}$

103. $\operatorname{arccot}\left(\cot\left(-\frac{\pi}{4}\right)\right) = \frac{3\pi}{4}$

105. $\operatorname{arccot}\left(\cot\left(\frac{3\pi}{2}\right)\right) = \frac{\pi}{2}$

107. $\operatorname{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$

109. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right) = \frac{7\pi}{6}$

111. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right) = \frac{\pi}{3}$

113. $\operatorname{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right) = \frac{5\pi}{4}$

115. $\operatorname{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right) = \frac{3\pi}{2}$

117. $\operatorname{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right) = \frac{13\pi}{12}$

119. $\operatorname{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$

121. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right) = \frac{5\pi}{6}$

123. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right) = \frac{\pi}{3}$

125. $\operatorname{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right) = -\frac{\pi}{4}$

127. $\operatorname{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right) = -\frac{\pi}{2}$

129. $\operatorname{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right) = \frac{11\pi}{12}$

131. $\sin\left(\arccos\left(-\frac{1}{2}\right)\right) = \frac{\sqrt{3}}{2}$

133. $\sin\left(\arctan\left(-2\right)\right) = -\frac{2\sqrt{5}}{5}$

135. $\sin\left(\operatorname{arccsc}\left(-3\right)\right) = -\frac{1}{3}$

137. $\cos\left(\arctan\left(\sqrt{7}\right)\right) = \frac{\sqrt{2}}{4}$

139. $\cos\left(\operatorname{arcsec}\left(5\right)\right) = \frac{1}{5}$

141. $\tan\left(\arccos\left(-\frac{1}{2}\right)\right) = -\sqrt{3}$

143. $\tan\left(\operatorname{arccot}\left(12\right)\right) = \frac{1}{12}$

145. $\cot\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right) = \sqrt{3}$

147. $\cot\left(\arctan\left(0.25\right)\right) = 4$

149. $\sec\left(\arcsin\left(-\frac{12}{13}\right)\right) = \frac{13}{5}$

151. $\sec\left(\operatorname{arccot}\left(-\frac{\sqrt{10}}{10}\right)\right) = -\sqrt{11}$

153. $\csc\left(\arcsin\left(\frac{3}{5}\right)\right) = \frac{5}{3}$

155. $\sin\left(\arcsin\left(\frac{5}{13}\right) + \frac{\pi}{4}\right) = \frac{17\sqrt{2}}{26}$

157. $\tan\left(\arctan(3) + \arccos\left(-\frac{3}{5}\right)\right) = \frac{1}{3}$

159. $\sin\left(2\operatorname{arccsc}\left(\frac{13}{5}\right)\right) = \frac{120}{169}$

161. $\cos\left(2\arcsin\left(\frac{3}{5}\right)\right) = \frac{7}{25}$

163. $\cos\left(2\operatorname{arccot}\left(-\sqrt{5}\right)\right) = \frac{2}{3}$

165. $\sin\left(\arccos\left(x\right)\right) = \sqrt{1-x^2}$ for $-1 \leq x \leq 1$

167. $\tan\left(\arcsin\left(x\right)\right) = \frac{x}{\sqrt{1-x^2}}$ for $-1 < x < 1$

169. $\csc\left(\arccos\left(x\right)\right) = \frac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$

171. $\sin\left(2\arccos\left(x\right)\right) = 2x\sqrt{1-x^2}$ for $-1 \leq x \leq 1$

173. $\sin(\arccos(2x)) = \sqrt{1-4x^2}$ for $-\frac{1}{2} \leq x \leq \frac{1}{2}$

175. $\cos\left(\arcsin\left(\frac{x}{2}\right)\right) = \frac{\sqrt{4-x^2}}{2}$ for $-2 \leq x \leq 2$

177. $\sin(2\arcsin(7x)) = 14x\sqrt{1-49x^2}$ for $-\frac{1}{7} \leq x \leq \frac{1}{7}$

179. $\cos(2\arcsin(4x)) = 1 - 32x^2$ for $-\frac{1}{4} \leq x \leq \frac{1}{4}$

181. $\sin\left(\arcsin(x) + \arccos(x)\right) = 1$ for $-1 \leq x \leq 1$

183. $\tan\left(2\arcsin(x)\right) = \frac{2x\sqrt{1-x^2}}{1-2x^2}$ for x in $\left(-1, -\frac{\sqrt{2}}{2}\right) \cup \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cup \left(\frac{\sqrt{2}}{2}, 1\right)$[2]

Chapter 5

Section 5.1

1. Answers will vary.
3. F
5. Answers will vary.
7. -1
9. Limit does not exist
11. 1.5
13. Limit does not exist.
15. 1

17.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-7
-0.01	-7
0.01	-7
0.1	-7

The limit seems to be exactly 7.

19.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	4.9
-0.01	4.99
0.01	5.01
0.1	5.1

The limit is approx. 5.

21.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	29.4
-0.01	29.04
0.01	28.96
0.1	28.6

The limit is approx. 29.

23.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-0.998334
-0.01	-0.999983
0.01	-0.999983
0.1	-0.998334

The limit is approx. -1.

Section 5.2

1. Answers will vary.

[2] The equivalence for $x = \pm 1$ can be verified independently of the derivation of the formula, but Calculus is required to fully understand what is happening at those x values. You'll see what we mean when you work through the details of the identity for $\tan(2t)$. For now, we exclude $x = \pm 1$ from our answer.

3. As x is near 1, both f and g are near 0, but f is approximately twice the size of g. (I.e., $f(x) \approx 2g(x)$.)
5. 9
7. 0
9. 3
11. 3
13. 1
15. 0
17. 7
19. $1/2$
21. Limit does not exist
23. 2
25. $\frac{\pi^2+3\pi+5}{5\pi^2-2\pi-3} \approx 0.6064$
27. -8
29. 10
31. $-3/2$
33. 0
35. 1
37. 3
39. 1
41. (a) Apply Part 1 of Theorem 5.2.1.
    (b) Apply Theorem 5.2.6; $g(x) = \frac{x}{x}$ is the same as $g(x) = 1$ everywhere except at $x = 0$. Thus $\lim_{x\to 0} g(x) = \lim_{x\to 0} 1 = 1$.
    (c) The function $f(x)$ is always 0, so $g\big(f(x)\big)$ is never defined as $g(x)$ is not defined at $x = 0$. Therefore the limit does not exist.
    (d) The Composition Rule requires that $\lim_{x\to 0} g(x)$ be equal to $g(0)$. They are not equal, so the conditions of the Composition Rule are not satisfied, and hence the rule is not violated.

Section 5.3

1. The function approaches different values from the left and right; the function grows without bound; the function oscillates.
3. F
5. (a) 2
   (b) 2
   (c) 2
   (d) 1
   (e) As f is not defined for $x < 0$, this limit is not defined.
   (f) 1
7. (a) Does not exist.
   (b) Does not exist.
   (c) Does not exist.
   (d) Not defined.
   (e) 0
   (f) 0
9. (a) 2
   (b) 2
   (c) 2
   (d) 2
11. (a) 2
    (b) 2
    (c) 2
    (d) 0
    (e) 2
    (f) 2
    (g) 2
    (h) Not defined
13. (a) 2
    (b) -4
    (c) Does not exist.
    (d) 2
15. (a) 0
    (b) 0
    (c) 0
    (d) 0
    (e) 2
    (f) 2
    (g) 2
    (h) 2
17. (a) $1 - \cos^2 a = \sin^2 a$
    (b) $\sin^2 a$
    (c) $\sin^2 a$
    (d) $\sin^2 a$
19. (a) 4
    (b) 4
    (c) 4
    (d) 3
21. (a) -1
    (b) 1
    (c) Does not exist
    (d) 0
23. $2/3$
25. -9

Section 5.4

1. F
3. F
5. T
7. Answers will vary.
9. (a) ∞
   (b) ∞
11. (a) 1
    (b) 0
    (c) $1/2$
    (d) $1/2$
13. (a) Limit does not exist
    (b) Limit does not exist
15. Tables will vary.

(a)

x	$f(x)$
2.9	-15.1224
2.99	-159.12
2.999	-1599.12

It seems $\lim_{x\to 3^-} f(x) = -\infty$.

(b)

x	$f(x)$
3.1	16.8824
3.01	160.88
3.001	1600.88

It seems $\lim_{x\to 3^+} f(x) = \infty$.

(c) It seems $\lim_{x\to 3} f(x)$ does not exist.

17. Tables will vary.

(a)

x	$f(x)$
2.9	132.857
2.99	12124.4

It seems $\lim_{x\to 3^-} f(x) = \infty$.

(b)

x	$f(x)$
3.1	108.039
3.01	11876.4

It seems $\lim_{x\to 3^+} f(x) = \infty$.

(c) It seems $\lim_{x\to 3} f(x) = \infty$.

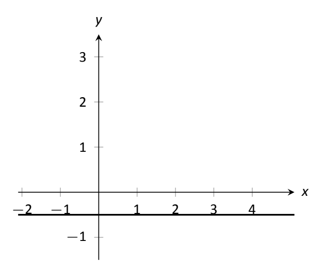
19. Horizontal asymptote at $y = 2$; vertical asymptotes at $x = -5, 4$.
21. Horizontal asymptote at $y = 0$; vertical asymptotes at $x = -1, 0$.
23. No horizontal or vertical asymptotes.
25. ∞
27. $-\infty$
29. Solution omitted.
31. Yes. The only "questionable" place is at $x = 3$, but the left and right limits agree.

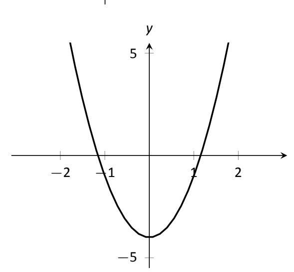
Section 5.5

1. Answers will vary.
3. A root of a function f is a value c such that $f(c) = 0$.
5. F
7. T
9. F
11. No; $\lim_{x\to 1} f(x) = 2$, while $f(1) = 1$.
13. No; $f(1)$ does not exist.
15. Yes
17. (a) No; $\lim_{x\to -2} f(x) \neq f(-2)$
    (b) Yes
    (c) No; $f(2)$ is not defined.
19. (a) Yes
    (b) Yes
21. (a) Yes
    (b) Yes
23. $(-\infty, \infty)$
25. $[-2, 2]$
27. $(-\infty, -\sqrt{6}]$ and $[\sqrt{6}, \infty)$
29. $(-\infty, \infty)$
31. $(0, \infty)$
33. $(-\infty, 0]$
35. Yes, by the Intermediate Value Theorem.
37. We cannot say; the Intermediate Value Theorem only applies to function values between -10 and 10; as 11 is outside this range, we do not know.
39. Approximate root is $x = 1.23$. The intervals used are:
    $[1, 1.5]$ $[1, 1.25]$ $[1.125, 1.25]$
    $[1.1875, 1.25]$ $[1.21875, 1.25]$ $[1.234375, 1.25]$
    $[1.234375, 1.2421875]$ $[1.234375, 1.2382813]$
41. Approximate root is $x = 0.69$. The intervals used are:
    $[0.65, 0.7]$ $[0.675, 0.7]$ $[0.6875, 0.7]$
    $[0.6875, 0.69375]$ $[0.690625, 0.69375]$
43. (a) 20
    (b) 25
    (c) Limit does not exist
    (d) 25
45. Answers will vary.

Chapter 6

Section 6.1

1. T
3. Answers will vary.
5. Answers will vary.
7. $f'(x) = 0$
9. $f'(t) = -3$
11. $h'(x) = 3x^2$
13. $r'(x) = \frac{-1}{x^2}$
15. (a) $y = 6$
    (b) $x = -2$
17. (a) $y = -3x + 4$
    (b) $y = 1/3(x - 7) - 17$
19. (a) $y = 48(x - 4) + 64$
    (b) $y = -\frac{1}{48}(x - 4) + 64$
21. (a) $y = -1/4(x + 2) - 1/2$
    (b) $y = 4(x + 2) - 1/2$
23. $y = 8.1(x - 3) + 16$
25. $y = 7.77(x - 2) + e^2$, or $y = 7.77(x - 2) + 7.39$.
27. (a) Approximations will vary; they should match (c) closely.
    (b) $f'(x) = 2x$
    (c) At $(-1, 0)$, slope is -2. At $(0, -1)$, slope is 0. At $(2, 3)$, slope is 4.
29. 

31. 

33. (a) Approximately on $(-2,0)$ and $(2,\infty)$.
    (b) Approximately on $(-\infty,-2)$ and $(0,2)$.
    (c) Approximately at $x=0,\ \pm 2$.
    (d) Approximately on $(-\infty,-1)$ and $(1,\infty)$.
    (e) Approximately on $(-1,1)$.
    (f) Approximately at $x=\pm 1$.

35. $\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=0$; note also that $\lim_{x\to 0^+}f'(x)=0$. So f is differentiable at $x=0$.
$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=-\infty$; note also that $\lim_{x\to 1^-}f'(x)=-\infty$. So f is not differentiable at $x=1$.
f is differentiable on $[0,1)$, not its entire domain.

37. Approximately 24.

39. (a) $(-\infty,\infty)$
    (b) $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$
    (c) $(-\infty,5]$
    (d) $[-\sqrt{5},\sqrt{5}]$

Section 6.2

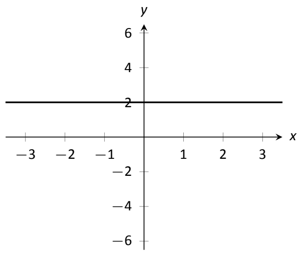
1. Velocity
3. Linear functions.
5. -17
7. $f(10.1)$ is likely most accurate, as accuracy is lost the farther from $x=10$ we go.
9. 6
11. ft/s^2
13. (a) thousands of dollars per car
    (b) It is likely that $P(0)<0$. That is, negative profit for not producing any cars.
15. $f(x)=g'(x)$
17. Either $g(x)=f'(x)$ or $f(x)=g'(x)$ is acceptable. The actual answer is $g(x)=f'(x)$, but is very hard to show that $f(x)\neq g'(x)$ given the level of detail given in the graph.
19. $f'(x)=10x$
21. $f'(\pi)\approx 0$.

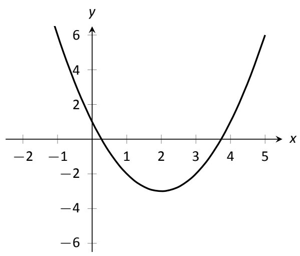
Section 6.3

1. Power Rule.
3. One answer is $f(x)=10e^x$.
5. $g(x)$ and $h(x)$
7. One possible answer is $f(x)=17x-205$.
9. $f'(x)$ is a velocity function, and $f''(x)$ is acceleration.
11. $f'(x)=14x-5$
13. $m'(t)=45t^4-\frac{3}{8}t^2+3$
15. $f'(r)=6e^r$
17. $f'(x)=\frac{2}{x}-1$
19. $h'(t)=e^t-\cos t+\sin t$
21. $f'(t)=0$
23. $g'(x)=24x^2-120x+150$
25. $f'(x)=18x-12$
27. $f'(x)=6x^5\ f''(x)=30x^4\ f'''(x)=120x^3\ f^{(4)}(x)=360x^2$
29. $h'(t)=2t-e^t\ h''(t)=2-e^t\ h'''(t)=-e^t\ h^{(4)}(t)=-e^t$
31. $f'(\theta)=\cos\theta+\sin\theta\ f''(\theta)=-\sin\theta+\cos\theta$
$f'''(\theta)=-\cos\theta-\sin\theta\ f^{(4)}(\theta)=\sin\theta-\cos\theta$
33. Tangent line: $y=2(x-1)$
Normal line: $y=-1/2(x-1)$
35. Tangent line: $y=x-1$
Normal line: $y=-x+1$
37. Tangent line: $y=\sqrt{2}(x-\frac{\pi}{4})-\sqrt{2}$
Normal line: $y=\frac{-1}{\sqrt{2}}(x-\frac{\pi}{4})-\sqrt{2}$
39. The tangent line to $f(x)=e^x$ at $x=0$ is $y=x+1$; thus $e^{0.1}\approx y(0.1)=1.1$.

Section 6.4

1. F
3. T
5. F
7. (a) $f'(x)=(x^2+3x)+x(2x+3)$
    (b) $f'(x)=3x^2+6x$
    (c) They are equal.
9. (a) $h'(s)=2(s+4)+(2s-1)(1)$
    (b) $h'(s)=4s+7$
    (c) They are equal.
11. (a) $f'(x)=\frac{x(2x)-(x^2+3)1}{x^2}$
    (b) $f'(x)=1-\frac{3}{x^2}$
    (c) They are equal.
13. (a) $h'(s)=\frac{4s^3(0)-3(12s^2)}{16s^6}$
    (b) $h'(s)=-9/4s^{-4}$
    (c) They are equal.
15. $f'(x)=\sin x+x\cos x$
17. $f'(x)=e^x\ln x+e^x\frac{1}{x}$
19. $g'(x)=\frac{-12}{(x-5)^2}$
21. $h'(x)=-\csc^2 x-e^x$
23. $h'(t)=14t+6$
25. $f'(x)=\big(6x+8\big)e^x+\big(3x^2+8x+7\big)e^x$
27. $f'(x)=7$
29. $f'(x)=\frac{\sin^2(x)+\cos^2(x)+3\cos(x)}{(\cos(x)+3)^2}$
31. $f'(x)=\frac{-x\sin x-\cos x}{x^2}+\frac{\tan x-x\sec^2 x}{\tan^2 x}$
33. $g'(t)=12t^2e^t+4t^3e^t-\cos^2 t+\sin^2 t$
35. $f'(x)=2xe^x\tan x=x^2e^x\tan x+x^2e^x\sec^2 x$
37. Tangent line: $y=2x+2$
Normal line: $y=-1/2x+2$
39. Tangent line: $y=4$
Normal line: $x=2$
41. $x=3/2$
43. $f'(x)$ is never 0.
45. $f''(x)=2\cos x-x\sin x$
47. $f''(x)=\cot^2 x\csc x+\csc^3 x$

49. 

51. 

Section 6.5

1. T
3. F
5. T
7. $f'(x) = 10(4x^3 - x)^9 \cdot (12x^2 - 1) = (120x^2 - 10)(4x^3 - x)^9$
9. $g'(\theta) = 3(\sin\theta + \cos\theta)^2(\cos\theta - \sin\theta)$
11. $f'(x) = 3\big(\ln x + x^2\big)2\left(\frac{1}{x} + 2x\right)$
13. $f'(x) = 4\left(x + \frac{1}{x}\right)^3\left(1 - \frac{1}{x^2}\right)$
15. $g'(x) = 5\sec^2(5x)$
17. $g'(t) = \cos\left(t^5 + \frac{1}{t}\right)\left(5t^4 - \frac{1}{t^3}\right)$
19. $p'(t) = -3\cos^2(t^2 + 3t + 1)\sin(t^2 + 3t + 1)(2t + 3)$
21. $f'(x) = 2/x$
23. $g'(r) = \ln 4 \cdot 4^r$
25. $g'(t) = 0$
27. $f'(x) = \frac{(3^t+2)\left((\ln 2)2^t\right) - (2^t+3)\left((\ln 3)3^t\right)}{(3^t+2)^2}$
29. $f'(x) = \frac{2^{x^2}(\ln 3 \cdot 3^{x^2}2x + 1) - (3^{x^2}+x)(\ln 2 \cdot 2^{x^2}2x)}{2^{2x^2}}$
31. $f'(x) =$ $5(x^2+x)^4(2x+1)(3x^4+2x)^3 + 3(x^2+x)^5(3x^4+2x)^2(12x^3+2)$
33. $f'(x) = 3\cos(3x+4)\cos(5-2x) + 2\sin(3x+4)\sin(5-2x)$
35. $f'(x) = \frac{4(5x-9)^3\cos(4x+1) - 15\sin(4x+1)(5x-9)^2}{(5x-9)^6}$
37. Tangent line: $y = 0$
    Normal line: $x = 0$
39. Tangent line: $y = -3(\theta - \pi/2) + 1$
    Normal line: $y = 1/3(\theta - \pi/2) + 1$
41. In both cases the derivative is the same: $1/x$.
43. (a) $^\circ$ F/mph
    (b) The sign would be negative; when the wind is blowing at 10 mph, any increase in wind speed will make it feel colder, i.e., a lower number on the Fahrenheit scale.

Chapter 7

Section 7.1

1. Answers will vary.
3. Answers will vary.
5. F
7. A: none; the function isn't defined here. B: abs. max & rel. max C: rel. min D: none; the function isn't defined here. E: none F: rel. min G: rel. max
9. $f'(0) = 0$
11. $f'(\pi/2) = 0$ $f'(3\pi/2) = 0$
13. $f'(2)$ is not defined $f'(6) = 0$
15. $f'(0) = 0$
17. min: $(-0.5, 3.75)$
    max: $(2, 10)$
19. min: $(\pi/4, 3\sqrt{2}/2)$
    max: $(\pi/2, 3)$
21. min: $(\sqrt{3}, 2\sqrt{3})$
    max: $(5, 28/5)$
23. min: $(\pi, -e^{\pi})$
    max: $(\pi/4, \frac{\sqrt{2}e^{\pi/4}}{2})$
25. min: $(1, 0)$
    max: $(e, 1/e)$
27. $\frac{dy}{dx} = \frac{y(y-2x)}{x(x-2y)}$
29. $3x^2 + 1$

Section 7.2

1. Answers will vary.
3. Answers will vary; graphs should be steeper near $x = 0$ than near $x = 2$.
5. False; for instance, $y = x^3$ is always increasing though it has a critical point at $x = 0$.
7. Graph and verify.
9. Graph and verify.
11. Graph and verify.
13. Graph and verify.
15. domain: $(-\infty, \infty)$
    c.p. at $c = -1$;
    decreasing on $(-\infty, -1)$;
    increasing on $(-1, \infty)$;
    rel. min at $x = -1$.
17. domain=$(-\infty, \infty)$
    c.p. at $c = \frac{1}{6}(-1 \pm \sqrt{7})$;
    decreasing on $(\frac{1}{6}(-1 - \sqrt{7}), \frac{1}{6}(-1 + \sqrt{7})))$;
    increasing on $(-\infty, \frac{1}{6}(-1 - \sqrt{7})) \cup (\frac{1}{6}(-1 + \sqrt{7}), \infty)$;
    rel. min at $x = \frac{1}{6}(-1 + \sqrt{7})$;
    rel. max at $x = \frac{1}{6}(-1 - \sqrt{7})$.
19. domain=$(-\infty, \infty)$
    c.p. at $c = 1$;
    decreasing on $(1, \infty)$
    increasing on $(-\infty, 1)$;
    rel. max at $x = 1$.
21. domain=$(-\infty, -2) \cup (-2, 4) \cup (4, \infty)$
    no c.p.;
    decreasing on entire domain, $(-\infty, -2) \cup (-2, 4) \cup (4, \infty)$

23. domain=$(-\infty, \infty)$
c.p. at $c = -3\pi/4, -\pi/4, \pi/4, 3\pi/4$;
decreasing on $(-3\pi/4, -\pi/4) \cup (\pi/4, 3\pi/4)$;
increasing on $(-\pi, -3\pi/4) \cup (-\pi/4, \pi/4) \cup (3\pi/4, \pi)$;
rel. min at $x = -\pi/4, 3\pi/4$;
rel. max at $x = -3\pi/4, \pi/4$.

25. $c = 1/2$

Section 7.3

1. Answers will vary.
3. Yes; Answers will vary.
5. Graph and verify.
7. Graph and verify.
9. Graph and verify.
11. Graph and verify.
13. Graph and verify.
15. Possible points of inflection: none; concave up on $(-\infty, \infty)$
17. Possible points of inflection: $x = 0$; concave down on $(-\infty, 0)$; concave up on $(0, \infty)$
19. Possible points of inflection: $x = -2/3, 0$; concave down on $(-2/3, 0)$; concave up on $(-\infty, -2/3) \cup (0, \infty)$
21. Possible points of inflection: $x = 1$; concave up on $(-\infty, \infty)$
23. Possible points of inflection: $x = \pm 1/\sqrt{3}$; concave down on $(-1/\sqrt{3}, 1/\sqrt{3})$; concave up on $(-\infty, -1/\sqrt{3}) \cup (1/\sqrt{3}, \infty)$
25. Possible points of inflection: $x = -\pi/4, 3\pi/4$; concave down on $(-\pi/4, 3\pi/4)$ concave up on $(-\pi, -\pi/4) \cup (3\pi/4, \pi)$
27. Possible points of inflection: $x = 1/e^{3/2}$; concave down on $(0, 1/e^{3/2})$ concave up on $(1/e^{3/2}, \infty)$
29. min: $x = 1$
31. max: $x = -1/\sqrt{3}$ min: $x = 1/\sqrt{3}$
33. min: $x = 1$
35. min: $x = 1$
37. max: $x = 0$
39. max: $x = \pi/4$; min: $x = -3\pi/4$
41. min: $x = 1/\sqrt{e}$
43. f' has no maximal or minimal value.
45. f' has a minimal value at $x = 0$
47. Possible points of inflection: $x = -2/3, 0$; f' has a relative min at: $x = 0$; relative max at: $x = -2/3$
49. f' has no relative extrema
51. f' has a relative max at $x = -1/\sqrt{3}$; relative min at $x = 1/\sqrt{3}$
53. f' has a relative min at $x = 3\pi/4$; relative max at $x = -\pi/4$
55. f' has a relative min at $x = 1/\sqrt{e^3} = e^{-3/2}$

Section 7.4

1. Answers will vary.
3. T
5. T
7. A good sketch will include the x and y intercepts and draw the appropriate line.
9. Use technology to verify sketch.
11. Use technology to verify sketch.
13. Use technology to verify sketch.
15. Use technology to verify sketch.
17. Use technology to verify sketch.
19. Use technology to verify sketch.
21. Use technology to verify sketch.
23. Use technology to verify sketch.
25. Use technology to verify sketch.
27. Critical point: $x = 0$ Points of inflection: $\pm b/\sqrt{3}$
29. Critical points: $x = \frac{n\pi/2 - b}{a}$, where n is an odd integer Points of inflection: $(n\pi - b)/a$, where n is an integer.
31. $\frac{dy}{dx} = -x/y$, so the function is increasing in second and fourth quadrants, decreasing in the first and third quadrants.
$\frac{d^2y}{dx^2} = -1/y - x^2/y^3$, which is positive when $y < 0$ and is negative when $y > 0$. Hence the function is concave down in the first and second quadrants and concave up in the third and fourth quadrants.

Section 7.5

1. Answers will vary.
3. Answers will vary.
5. Answers will vary.
7. velocity
9. $3/4x^4 + C$
11. $10/3x^3 - 2x + C$
13. $s + C$
15. $-3/(t) + C$
17. $\tan\theta + C$
19. $\sec x - \csc x + C$
21. $3^t/\ln 3 + C$
23. $4/3t^3 + 6t^2 + 9t + C$
25. $x^6/6 + C$
27. $ax + C$
29. $-\cos x + 3$
31. $x^4 - x^3 + 7$
33. $7^x/\ln 7 + 1 - 49/\ln 7$
35. $\frac{7x^3}{6} - \frac{9x}{2} + \frac{40}{3}$
37. $\theta - \sin(\theta) - \pi + 4$
39. $3x - 2$
41. $dy = (2xe^x\cos x + x^2e^x\cos x - x^2e^x\sin x)dx$

Index

$\in$, 2
$\notin$, 2
x-axis, 10
x-coordinate, 10
y-axis, 10
y-coordinate, 10

abscissa, 10
absolute maximum, 257
absolute minimum, 257
absolute value
    definition of, 46
    properties of, 47
acceleration, 227
amplitude, 141
angle
    reference, 108
antiderivative, 286
arccosecant
    calculus friendly
        definition of, 162
        graph of, 162
        properties of, 162
    trigonometry friendly
        definition of, 160
        graph of, 159
        properties of, 160
arccosine
    definition of, 154
    graph of, 153
    properties of, 154
arccotangent
    definition of, 157
    graph of, 156
    properties of, 157
arcsecant
    calculus friendly
        definition of, 162
        graph of, 162
        properties of, 162
    trigonometry friendly
        definition of, 160
        graph of, 159
        properties of, 160
arcsine
    definition of, 154
    graph of, 153
    properties of, 154
arctangent
    definition of, 157
    graph of, 156
    properties of, 157
argument
    of a function, 17
    of a logarithm, 92
    of a trigonometric function, 139
asymptote
    horizontal, 207
    vertical, 205
asymptote
    horizontal
        formal definition of, 75
        intuitive definition of, 75
        location of, 77
    slant
        determination of, 79
        formal definition of, 79
    slant (oblique), 79
    vertical
        formal definition of, 75
        intuitive definition of, 75
        location of, 76

Bisection Method, 199

Cartesian coordinate plane, 10
Cartesian coordinates, 10
Chain Rule, 248
    notation, 252
change of base formulas, 99
circular function, 115
codomain, 17
Cofunction Identities, 125
common base, 87
common logarithm, 89
complex number
    definition of, 4
    set of, 4
composite function
    definition of, 26
concave down, 271
concave up, 271
concavity, 271
    inflection point, 271
    test for, 271
constant function
    as a horizontal line, 44
Constant Multiple Rule
    of derivatives, 234
    of integration, 289
constant term of a polynomial, 57
continuous, 60

continuous function, 194
properties, 197
coordinates
Cartesian, 10
cosecant
graph of, 143
of an angle, 115
properties of, 145
cosine
graph of, 139
of an angle, 105, 115
properties of, 138
cotangent
graph of, 147
of an angle, 115
properties of, 149
critical number, 259
critical point, 259
curve sketching, 278

decreasing function, 264
finding intervals, 265
degree of a polynomial, 57
dependent variable, 17
depreciation, 87
derivative
acceleration, 227
as a function, 219
at a point, 215
basic rules, 231
Chain Rule, 248, 252
Constant Multiple Rule, 234
Constant Rule, 231
exponential functions, 252
First Deriv. Test, 267
Generalized Power Rule, 249
higher order, 235
interpretation, 236
interpretation, 225
motion, 227
normal line, 217
notation, 219, 235
Power Rule, 231, 243
Product Rule, 238
Quotient Rule, 240
second, 235
Second Deriv. Test, 274
Sum/Difference Rule, 234
tangent line, 215
third, 235
trigonometric functions, 241
velocity, 227
diagram
Venn Diagram, 3
Difference Identity
for cosine, 123, 128
for sine, 126, 128
for tangent, 128
differentiable, 215
discontinuity, 197
discontinuity
essential, 198
infinite, 198
jump, 198
removable, 198
discriminant
of a quadratic equation, 52
trichotomy, 52
distance
definition, 13
distance formula, 14
domain, 17
domain
implied, 19
Double Angle Identities, 128

empty set, 3, 4
end behaviour
of $f(x) = ax^n$, n even, 59
of $f(x) = ax^n$, n odd, 60
of a function graph, 58
polynomial, 62
Even/Odd Identities, 123
exponential function
algebraic properties of, 95
change of base formula, 99
common base, 87
definition of, 86
graphical properties of, 87
inverse properties of, 95
natural base, 87
one-to-one properties of, 95
extrema
absolute, 257
and First Deriv. Test, 267
and Second Deriv. Test, 274
finding, 260
relative, 258
Extreme Value Theorem, 257
extreme values, 257

First Derivative Test, 267
floor function, 194
frequency
of a sinusoid, 141
function
absolute value, 46
argument, 17
arithmetic, 23
as a process, 17, 30
circular, 115
composite
definition of, 26
constant, 44
continuous, 60
definition, 17
dependent variable of, 17
difference, 23
exponential, 86
independent variable of, 17

inverse
definition of, 30
properties of, 31
solving for, 34
uniqueness of, 31
linear, 43
logarithmic, 89
notation, 17
one-to-one, 32
periodic, 138
piecewise, 46
polynomial, 56
product, 23
quadratic, 49
quotient, 23
rational, 73
smooth, 60
sum, 23
fundamental cycle
of $y = \cos(x)$, 139

Generalized Power Rule, 249
graph
hole in, 76

Half-Angle Formulas, 131
hole
in a graph, 76
location of, 76
horizontal asymptote
formal definition of, 75
intuitive definition of, 75
location of, 77
Horizontal Line Test (HLT), 32

implied domain of a function, 19
increasing function, 264
finding intervals, 265
indefinite integral, 286
independent variable, 17
indeterminate form, 171, 206
inflection point, 272
initial value problem, 290
integer
definition of, 4
set of, 4
integration
indefinite, 286
notation, 287
Power Rule, 290
Sum/Difference Rule, 289
Intermediate Value Theorem, 198
Intermediate Value Theorem
polynomial zero version, 60
intersection of two sets, 2
interval
definition of, 6
notation for, 6
inverse
of a function
definition of, 30
properties of, 31
solving for, 34
uniqueness of, 31
invertibility
function, 32
invertible
function, 30
irrational number
definition of, 4
set of, 4

leading coefficient of a polynomial, 57
leading term of a polynomial, 57
limit
at infinity, 207
definition, 176
difference quotient, 175
does not exist, 174, 189
indeterminate form, 171, 206
informal definition, 176
left handed, 188
of infinity, 203
one sided, 188
properties, 178
pseudo-definition, 172
right handed, 188
Squeeze Theorem, 181
line
linear function, 43
point-slope form, 43
slope of, 41
slope-intercept form, 43
linear function, 43
logarithm
algebraic properties of, 96
change of base formula, 99
common, 89
general, "base *b*", 89
graphical properties of, 90
inverse properties of, 95
natural, 89
one-to-one properties of, 95

maximum
absolute, 257
and First Deriv. Test, 267
and Second Deriv. Test, 274
relative/local, 258
midpoint
definition of, 15
midpoint formula, 15
minimum
absolute, 257
and First Deriv. Test, 267, 274
relative/local, 258
multiplicity
effect on the graph of a polynomial, 63
of a zero, 63

natural base, 87
natural logarithm, 89

natural number
definition of, 4
set of, 4
Newton's Law of Cooling, 88
normal line, 217

oblique asymptote, 79
one-to-one function, 32
ordered pair, 10
ordinate, 10
origin, 10

parabola
graph of a quadratic function, 49
vertex, 49
period
of a function, 138
periodic function, 138
phase, 141
phase shift, 141
point of inflection, 272
point-slope form of a line, 43
polynomial function
constant term, 57
definition of, 56
degree, 57
end behaviour, 58
leading coefficient, 57
leading term, 57
zero
multiplicity, 63
Power Reduction Formulas, 130
Power Rule
differentiation, 231, 238, 243
integration, 290
power rule
for absolute value, 47
for exponential functions, 95
for logarithms, 96
product rule
for absolute value, 47
for exponential functions, 95
for logarithms, 96
Product to Sum Formulas, 133
Pythagorean Identities, 117

quadrants, 12
quadratic formula, 51
quadratic function
definition of, 49
general form, 50
standard form, 50
Quotient Identities, 116
Quotient Rule, 240
quotient rule
for absolute value, 47
for exponential functions, 95
for logarithms, 96

rate of change
slope of a line, 42
rational functions, 73
rational number
definition of, 4
set of, 4
real number
definition of, 3, 4
set of, 3, 4
Reciprocal Identities, 116
reference angle, 108
Reference Angle Theorem
for cosine and sine, 109
reflection
of a point, 13

secant
graph of, 143
of an angle, 115
properties of, 145
Second Derivative Test, 274
set
definition of, 1
empty, 3, 4
exclusion, 2
inclusion, 2
intersection, 2
roster method, 1
set-builder notation, 1
sets of numbers, 4
union, 2
verbal description, 1
set-builder notation, 1
sign diagram, 265
sign diagram
polynomial function, 61
sine
graph of, 139
of an angle, 105, 115
properties of, 138
sinusoid
amplitude, 141
graph of, 141
phase shift, 141
slant asymptote, 79
slant asymptote
determination of, 79
formal definition of, 79
slope
definition, 41
of a line, 41
rate of change, 42
slope-intercept form of a line, 43
smooth, 60
Squeeze Theorem, 181
subset
definition of, 2
Sum Identity
for cosine, 123, 128
for sine, 126, 128
for tangent, 128
Sum to Product Formulas, 133

Sum/Difference Rule
of derivatives, 234
of integration, 289
symmetry
about the *x*-axis, 12
about the *y*-axis, 12
about the origin, 12

tangent
graph of, 146
of an angle, 115
properties of, 149
tangent line, 215
trichotomy, 5

union of two sets, 2
Unit Circle
important points, 110

variable
dependent, 17
independent, 17
velocity, 226
Venn Diagram, 3
vertex
of a parabola, 49
vertical asymptote
formal definition of, 75
intuitive definition of, 75
location of, 76

zero
multiplicity of, 63

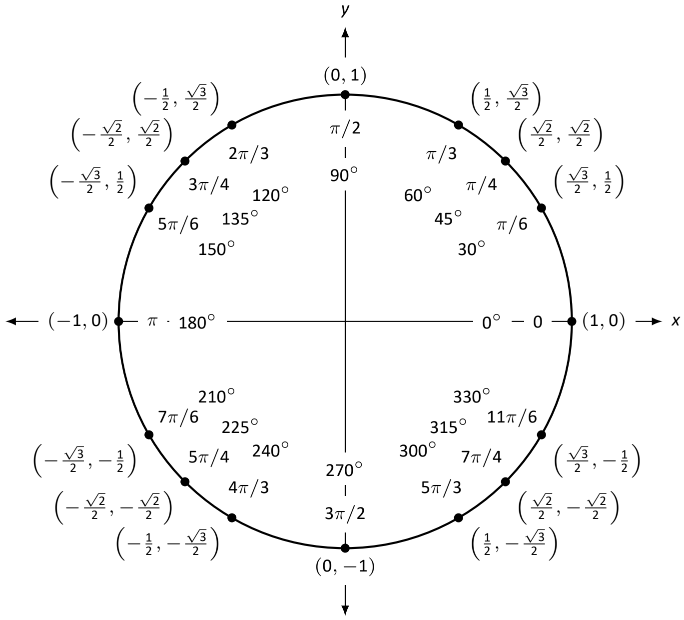
Differentiation Rules

1. $\frac{d}{dx}(cx) = c$
2. $\frac{d}{dx}(u \pm v) = u' \pm v'$
3. $\frac{d}{dx}(u \cdot v) = uv' + u'v$
4. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}(u(v)) = u'(v)v'$
6. $\frac{d}{dx}(c) = 0$
7. $\frac{d}{dx}(x) = 1$
8. $\frac{d}{dx}(x^n) = nx^{n-1}$
9. $\frac{d}{dx}(e^x) = e^x$
10. $\frac{d}{dx}(a^x) = \ln a \cdot a^x$
11. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
12. $\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$
13. $\frac{d}{dx}(\sin x) = \cos x$
14. $\frac{d}{dx}(\cos x) = -\sin x$
15. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
16. $\frac{d}{dx}(\sec x) = \sec x \tan x$
17. $\frac{d}{dx}(\tan x) = \sec^2 x$
18. $\frac{d}{dx}(\cot x) = -\csc^2 x$
19. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
20. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
21. $\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$
22. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$
23. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
24. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
25. $\frac{d}{dx}(\cosh x) = \sinh x$
26. $\frac{d}{dx}(\sinh x) = \cosh x$
27. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
28. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
29. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
30. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
31. $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$
32. $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$
33. $\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$
34. $\frac{d}{dx}(\operatorname{csch}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}$
35. $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$
36. $\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$

Integration Rules

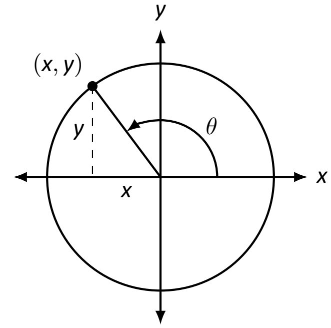
1. $\int c \cdot f(x)\, dx = c \int f(x)\, dx$
2. $\int f(x) \pm g(x)\, dx =$
   $\int f(x)\, dx \pm \int g(x)\, dx$
3. $\int 0\, dx = C$
4. $\int 1\, dx = x + C$
5. $\int x^n\, dx = \frac{1}{n+1} x^{n+1} + C,\ n \neq -1$
   $n \neq -1$
6. $\int e^x\, dx = e^x + C$
7. $\int a^x\, dx = \frac{1}{\ln a} \cdot a^x + C$
8. $\int \frac{1}{x}\, dx = \ln|x| + C$
9. $\int \cos x\, dx = \sin x + C$
10. $\int \sin x\, dx = -\cos x + C$
11. $\int \tan x\, dx = -\ln|\cos x| + C$
12. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$
13. $\int \csc x\, dx = -\ln|\csc x + \cot x| + C$
14. $\int \cot x\, dx = \ln|\sin x| + C$
15. $\int \sec^2 x\, dx = \tan x + C$
16. $\int \csc^2 x\, dx = -\cot x + C$
17. $\int \sec x \tan x\, dx = \sec x + C$
18. $\int \csc x \cot x\, dx = -\csc x + C$
19. $\int \cos^2 x\, dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$
20. $\int \sin^2 x\, dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$
21. $\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
22. $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
23. $\int \frac{1}{x\sqrt{x^2-a^2}}\, dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$
24. $\int \cosh x\, dx = \sinh x + C$
25. $\int \sinh x\, dx = \cosh x + C$
26. $\int \tanh x\, dx = \ln(\cosh x) + C$
27. $\int \coth x\, dx = \ln|\sinh x| + C$
28. $\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \ln\left|x + \sqrt{x^2-a^2}\right| + C$
29. $\int \frac{1}{\sqrt{x^2+a^2}}\, dx = \ln\left|x + \sqrt{x^2+a^2}\right| + C$
30. $\int \frac{1}{a^2-x^2}\, dx = \frac{1}{2}\ln\left|\frac{a+x}{a-x}\right| + C$
31. $\int \frac{1}{x\sqrt{a^2-x^2}}\, dx = \frac{1}{a}\ln\left(\frac{x}{a+\sqrt{a^2-x^2}}\right) + C$
32. $\int \frac{1}{x\sqrt{x^2+a^2}}\, dx = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{x^2+a^2}}\right| + C$

The Unit Circle



Definitions of the Trigonometric Functions

Unit Circle Definition

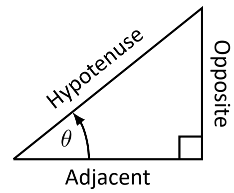


$\sin\theta = y \qquad \cos\theta = x$

$\csc\theta = \frac{1}{y} \qquad \sec\theta = \frac{1}{x}$

$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$

Right Triangle Definition



$\sin\theta = \frac{\text{O}}{\text{H}} \qquad \csc\theta = \frac{\text{H}}{\text{O}}$

$\cos\theta = \frac{\text{A}}{\text{H}} \qquad \sec\theta = \frac{\text{H}}{\text{A}}$

$\tan\theta = \frac{\text{O}}{\text{A}} \qquad \cot\theta = \frac{\text{A}}{\text{O}}$

Common Trigonometric Identities

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

Cofunction Identities

$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \csc\left(\frac{\pi}{2} - x\right) = \sec x$

$\cos\left(\frac{\pi}{2} - x\right) = \sin x \qquad \sec\left(\frac{\pi}{2} - x\right) = \csc x$

$\tan\left(\frac{\pi}{2} - x\right) = \cot x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x$

Double Angle Formulas

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$= 2\cos^2 x - 1$

$= 1 - 2\sin^2 x$

$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$

Sum to Product Formulas

$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$

$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

Power–Reducing Formulas

$\sin^2 x = \frac{1 - \cos 2x}{2}$

$\cos^2 x = \frac{1 + \cos 2x}{2}$

$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$

Even/Odd Identities

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

$\csc(-x) = -\csc x$

$\sec(-x) = \sec x$

$\cot(-x) = -\cot x$

Product to Sum Formulas

$\sin x \sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$

$\cos x \cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$

$\sin x \cos y = \frac{1}{2}\big(\sin(x+y) + \sin(x-y)\big)$

Angle Sum/Difference Formulas

$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$

$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$

$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$

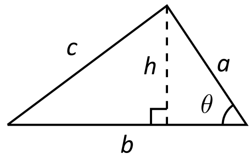
Areas and Volumes

Triangles

$h = a \sin\theta$

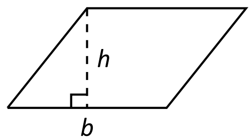
Area = $\frac{1}{2}bh$

Law of Cosines:
$c^2 = a^2 + b^2 - 2ab\cos\theta$



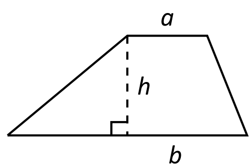
Parallelograms

Area = bh



Trapezoids

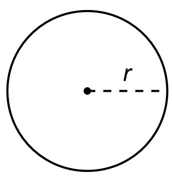
Area = $\frac{1}{2}(a+b)h$



Circles

Area = πr^2

Circumference = $2\pi r$

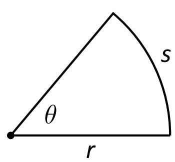


Sectors of Circles

θ in radians

Area = $\frac{1}{2}\theta r^2$

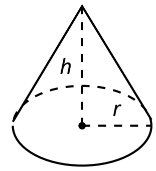
$s = r\theta$



Right Circular Cone

Volume = $\frac{1}{3}\pi r^2 h$

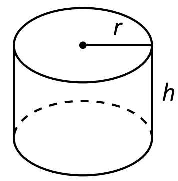
Surface Area =
$\pi r\sqrt{r^2 + h^2} + \pi r^2$



Right Circular Cylinder

Volume = $\pi r^2 h$

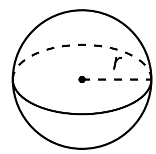
Surface Area =
$2\pi rh + 2\pi r^2$



Sphere

Volume = $\frac{4}{3}\pi r^3$

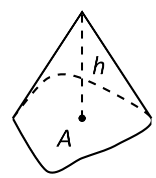
Surface Area =$4\pi r^2$



General Cone

Area of Base = A

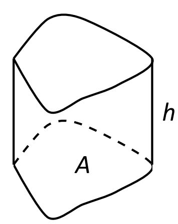
Volume = $\frac{1}{3}Ah$



General Right Cylinder

Area of Base = A

Volume = Ah



Algebra

Factors and Zeros of Polynomials

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. If $p(a) = 0$, then a is a *zero* of the polynomial and a solution of the equation $p(x) = 0$. Furthermore, $(x - a)$ is a *factor* of the polynomial.

Fundamental Theorem of Algebra

An nth degree polynomial has n (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

Quadratic Formula

If $p(x) = ax^2 + bx + c$, and $0 \le b^2 - 4ac$, then the real zeros of p are $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

Special Factors

$x^2 - a^2 = (x - a)(x + a)$ $\qquad$ $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$ $\qquad$ $x^4 - a^4 = (x^2 - a^2)(x^2 + a^2)$

$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n$

$(x - y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \cdots \pm nxy^{n-1} \mp y^n$

Binomial Theorem

$(x + y)^2 = x^2 + 2xy + y^2$ $\qquad$ $(x - y)^2 = x^2 - 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ $\qquad$ $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ $\qquad$ $(x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

Rational Zero Theorem

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every *rational zero* of p is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Factoring by Grouping

$acx^3 + adx^2 + bcx + bd = ax^2(cx + d) + b(cx + d) = (ax^2 + b)(cx + d)$

Arithmetic Operations

$$ab + ac = a(b + c) \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc} \qquad \frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc} \qquad \frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c} \qquad \frac{a - b}{c - d} = \frac{b - a}{d - c} \qquad \frac{ab + ac}{a} = b + c$$

Exponents and Radicals

$$a^0 = 1,\ \ a \neq 0 \qquad (ab)^x = a^x b^x \qquad a^x a^y = a^{x+y} \qquad \sqrt{a} = a^{1/2} \qquad \frac{a^x}{a^y} = a^{x-y} \qquad \sqrt[n]{a} = a^{1/n}$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \sqrt[n]{a^m} = a^{m/n} \qquad a^{-x} = \frac{1}{a^x} \qquad \sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b} \qquad (a^x)^y = a^{xy} \qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Additional Formulas

Summation Formulas:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^3}{12n^2}\left[\max\left|f''(x)\right|\right]$

Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \ldots + 2f(x_{n-1}) + 4f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^5}{180n^4}\left[\max\left|f^{(4)}(x)\right|\right]$

Arc Length:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\;\;dx$$

Surface of Revolution:

$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\;\;dx$$

(where $f(x) \geq 0$)

$$S = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\;\;dx$$

(where $a, b \geq 0$)

Work Done by a Variable Force:

$$W = \int_a^b F(x)\;\;dx$$

Force Exerted by a Fluid:

$$F = \int_a^b w\, d(y)\, \ell(y)\;\;dy$$

Taylor Series Expansion for $f(x)$:

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Maclaurin Series Expansion for $f(x)$, where $c = 0$:

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

Summary of Tests for Series:

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
*n*th-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} r^n$	$\|r\| < 1$	$\|r\| \geq 1$	Sum $= \dfrac{1}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+a})$	$\lim_{n\to\infty} b_n = L$		Sum $= \left(\sum_{n=1}^{a} b_n\right) - L$
p-Series	$\sum_{n=1}^{\infty} \dfrac{1}{(an+b)^p}$	$p > 1$	$p \leq 1$	
Integral Test	$\sum_{n=0}^{\infty} a_n$	$\int_1^{\infty} a(n)\, dn$ is convergent	$\int_1^{\infty} a(n)\, dn$ is divergent	$a_n = a(n)$ must be continuous
Direct Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$	$\sum_{n=0}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$	
Limit Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n\to\infty} a_n/b_n \geq 0$	$\sum_{n=0}^{\infty} b_n$ diverges and $\lim_{n\to\infty} a_n/b_n > 0$	Also diverges if $\lim_{n\to\infty} a_n/b_n = \infty$
Ratio Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} < 1$	$\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} a_{n+1}/a_n = \infty$
Root Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} (a_n)^{1/n} < 1$	$\lim_{n\to\infty} (a_n)^{1/n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} (a_n)^{1/n} = \infty$