

# Elementary Linear Algebra

For University of Lethbridge Math 1410



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# Chapter 1

## The Real and Complex Numbers

We begin with some basic set theory terminology that may pop up from time to time, followed by a reminder on the rules for arithmetic with real numbers, and a tour of the Cartesian coordinate plane. Students who are already comfortable with these topics can feel free to jump ahead to Section 1.4, where we introduce the complex numbers.

### 1.1 Some Basic Set Theory Notions

#### Definition 1.1.1 Set.

A **set** is a well-defined collection of objects which are called the “elements” of the set. Here, “well-defined” means that it is possible to determine if something belongs to the collection or not, without prejudice.

For example, the collection of letters that make up the word “pronghorns” is well-defined and is a set, but the collection of the worst math teachers in the world is *not* well-defined, and so is *not* a set. In general, there are three ways to describe sets.

One thing that student evaluations teach us is that any given Mathematics instructor can be simultaneously the best and worst teacher ever, depending on who is completing the evaluation.

#### Key Idea 1.1.2 Ways to Describe Sets.

1. *The Verbal Method:* Use a sentence to define a set.
2. *The Roster Method:* Begin with a left brace “{”, list each element of the set *only once* and then end with a right brace “}”.
3. *The Set-Builder Method:* A combination of the verbal and roster methods using a “dummy variable” such as  $x$ .

For example, let  $S$  be the set described *verbally* as the set of letters that make up the word “pronghorns”. A **roster** description of  $S$  would be  $\{p, r, o, n, g, h, s\}$ . Note that we listed “r”, “o”, and “n” only once, even though they appear twice in “pronghorns.” Also, the *order* of the elements doesn’t matter, so  $\{o, n, p, r, g, s, h\}$  is also a roster description of  $S$ . A **set-builder** description of  $S$  is:

$$\{x \mid x \text{ is a letter in the word “pronghorns”}\}.$$

The way to read this is: “The set of elements  $x$  such that  $x$  is a letter in the word ‘pronghorns.’” We define two sets to be equal if they have exactly the same elements, and denote this using the familiar equals sign “=”. Thus, we may write  $S = \{p, r, o, n, g, h, s\}$  or  $S = \{x \mid x \text{ is a letter in the word “pronghorns”}\}$ . Clearly  $r$  is an element of  $S$  and  $q$  is not an element of  $S$ . We express these sentiments mathematically by writing  $r \in S$  and  $q \notin S$ .

More precisely, we have the following.

**Definition 1.1.3 Notation for set inclusion.**

Let  $A$  be a set.

- If  $x$  is an element of  $A$  then we write  $x \in A$ , which is read “ $x$  is in  $A$ ”.
- If  $x$  is *not* an element of  $A$  then we write  $x \notin A$ , which is read “ $x$  is not in  $A$ ”.

The notation  $x \in A$  can be read as “ $x$  is in  $A$ ”, or “ $x$  is an element of  $A$ ”, or “ $x$  belongs to  $A$ ”, with similar readings for  $x \notin A$ .

Now let’s consider the set

$$C = \{x \mid x \text{ is a consonant in the word “pronghorns”}\}.$$

A roster description of  $C$  is  $C = \{p, r, n, g, h, s\}$ . Note that by construction, every element of  $C$  is also in  $S$ . We express this relationship by stating that the set  $C$  is a **subset** of the set  $S$ , which is written in symbols as  $C \subseteq S$ . The more formal definition is given below.

**Definition 1.1.4 Subset.**

Given sets  $A$  and  $B$ , we say that the set  $A$  is a **subset** of the set  $B$  and write “ $A \subseteq B$ ” if every element in  $A$  is also an element of  $B$ .

Note that in our example above  $C \subseteq S$ , but not vice-versa, since  $o \in S$  but  $o \notin C$ . Additionally, the set of vowels  $V = \{a, e, i, o, u\}$ , while it does have an element in common with  $S$ , is not a subset of  $S$ . (As an added note,  $S$  is not a subset of  $V$ , either.) We could, however, *build* a set which contains both  $S$  and  $V$  as subsets by gathering all of the elements in both  $S$  and  $V$  together into a single set, say  $U = \{p, r, o, n, g, h, s, a, e, i, u\}$ . Then  $S \subseteq U$  and  $V \subseteq U$ . The set  $U$  we have built is called the **union** of the sets  $S$  and  $V$  and is denoted  $S \cup V$ . Furthermore,  $S$  and  $V$  aren’t completely *different* sets since they both contain the letter “o.” (Since the word “different” could be ambiguous, mathematicians use the word *disjoint* to refer to two sets that have no elements in common.) The **intersection** of two sets is the set of elements (if any) the two sets have in common. In this case, the intersection of  $S$  and  $V$  is  $\{o\}$ , written  $S \cap V = \{o\}$ . We formalize these ideas below.

**Definition 1.1.5 Intersection and Union.**

Suppose  $A$  and  $B$  are sets.

- The **intersection** of  $A$  and  $B$  is  $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$ .
- The **union** of  $A$  and  $B$  is  $A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ (or both)}\}$ .

The key words in Definition 1.1.5 to focus on are the conjunctions: “intersection” corresponds to “and” meaning the elements have to be in *both* sets to be



in the intersection, whereas “union” corresponds to “or” meaning the elements have to be in one set, or the other set (or both). In other words, to belong to the union of two sets an element must belong to *at least one* of them.

Returning to the sets  $C$  and  $V$  above,  $C \cup V = \{p, r, n, g, h, s, a, e, i, o, u\}$ . When it comes to their intersection, however, we run into a bit of notational awkwardness since  $C$  and  $V$  have no elements in common. While we could write  $C \cap V = \{\}$ , this sort of thing happens often enough that we give the set with no elements a name.

**Definition 1.1.6 Empty set.**

The **Empty Set**  $\emptyset$  is the set which contains no elements. That is,

$$\emptyset = \{\} = \{x \mid x \neq x\}.$$

As promised, the empty set is the set containing no elements since no matter what “ $x$ ” is, “ $x = x$ .” Like the number “0,” the empty set plays a vital role in mathematics. We introduce it here more as a symbol of convenience as opposed to a contrivance.

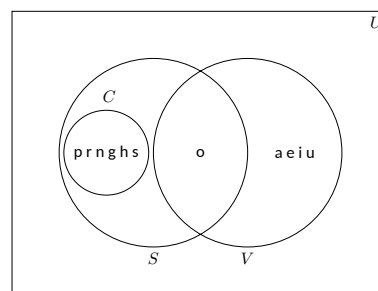
Using this new bit of notation, we have for the sets  $C$  and  $V$  above that  $C \cap V = \emptyset$ . A nice way to visualize relationships between sets and set operations is to draw a **Venn Diagram**<sup>2</sup>. A Venn Diagram for the sets  $S$ ,  $C$  and  $V$  is drawn in Figure 1.1.7.

In Figure 1.1.7 we have three circles – one for each of the sets  $C$ ,  $S$  and  $V$ . We visualize the area enclosed by each of these circles as the elements of each set. Here, we’ve spelled out the elements for definitiveness. Notice that the circle representing the set  $C$  is completely inside the circle representing  $S$ . This is a geometric way of showing that  $C \subseteq S$ . Also, notice that the circles representing  $S$  and  $V$  overlap on the letter “o”. This common region is how we visualize  $S \cap V$ . Notice that since  $C \cap V = \emptyset$ , the circles which represent  $C$  and  $V$  have no overlap whatsoever.

All of these circles lie in a rectangle labelled  $U$  (for “universal” set). A universal set contains all of the elements under discussion, so it could always be taken as the union of all of the sets in question, or an even larger set. In this case, we could take  $U = S \cup V$  or  $U$  as the set of letters in the entire alphabet. The usual triptych of Venn Diagrams indicating generic sets  $A$  and  $B$  along with  $A \cap B$  and  $A \cup B$  is given in Figure 1.1.8.

(The reader may well wonder if there is an ultimate universal set which contains *everything*. The short answer is “no”. Our definition of a set turns out to be overly simplistic, but correcting this takes us well beyond the confines of this course. If you want the longer answer, you can begin by reading about Russell’s Paradox<sup>3</sup> on Wikipedia.)

The full extent of the empty set’s role will not be explored in this text, but it is of fundamental importance in Set Theory. In fact, the empty set can be used to generate numbers – mathematicians can create something from nothing! If you’re interested, read about the von Neumann construction of the natural numbers<sup>1</sup>.



**Figure 1.1.7** A Venn diagram for  $C$ ,  $S$ , and  $V$

<sup>2</sup>[en.wikipedia.org/wiki/Venn\\_diagram](http://en.wikipedia.org/wiki/Venn_diagram)

<sup>3</sup>[en.wikipedia.org/wiki/Russell's\\_paradox](http://en.wikipedia.org/wiki/Russell's_paradox)

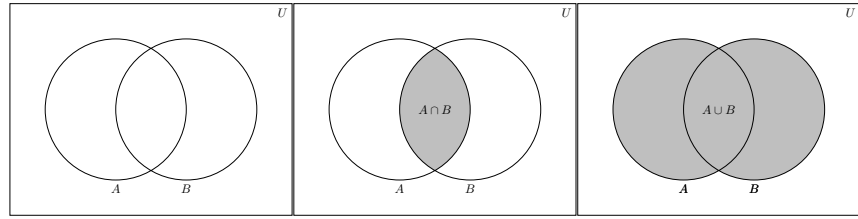


Figure 1.1.8 Venn diagrams for intersection and union

In the next section, we will review the algebraic properties of the real number system. Other properties of the real numbers (such as the order property that allows us to picture the set of real numbers as a “number line”) are essential to Calculus, but not that important for Linear Algebra, so we will leave it to other courses to handle the discussion of these topics.

## 1.2 Real Number Arithmetic

In this section we list the properties of real number arithmetic. We will focus on those aspects that are most needed for linear algebra (things like the algebraic axioms, and working with fractions), and gloss over those that are more calculus related (exponents, roots, etc.). In particular, since this is an *algebra* textbook, we will assume that the reader has encountered the real number system before, and omit a definition, focusing instead on the algebraic properties of real numbers. We begin with the axioms for addition of real numbers.

The set of real numbers is denoted usually denoted  $\mathbb{R}$ . A careful definition of  $\mathbb{R}$  is actually quite complicated, and even most calculus classes choose not to include it. If you want to *really* understand what the real numbers are, you’ll need to take a course in Real Analysis.

### Definition 1.2.1 Properties of Real Number Addition.

<b>Closure</b>	For all real numbers $a$ and $b$ , $a + b$ is also a real number.
<b>Commutativity</b>	For all real numbers $a$ and $b$ , $a + b = b + a$ .
<b>Associativity</b>	For all real numbers $a$ , $b$ and $c$ , $a + (b + c) = (a + b) + c$ .
<b>Identity</b>	There is a real number “0” so that for all real numbers $a$ , $a + 0 = a$ .
<b>Inverse</b>	For all real numbers $a$ , there is a real number $-a$ such that $a + (-a) = 0$ .
<b>Definition of Subtraction</b>	For all real numbers $a$ and $b$ , $a - b = a + (-b)$ .

While it is true that  $ab = ba$  for any pair of real numbers  $a$  and  $b$ , there are plenty of algebraic systems where this property does not hold. In particular, in Chapter 4 we’ll learn how to multiply *matrices*  $A$  and  $B$ , and see that in most cases, the matrix product  $AB$  is **not** equal to the product  $BA$ . (We’ll also see that such products are not even guaranteed to be *defined*!)

Next, we give real number multiplication a similar treatment. Recall that we may denote the product of two real numbers  $a$  and  $b$  a variety of ways:  $ab$ ,  $a \cdot b$ ,  $a(b)$ ,  $(a)(b)$  and so on. We’ll refrain from using  $a \times b$  for real number multiplication in this text.

### Definition 1.2.2 Properties of Real Number Multiplication.

<b>Closure</b>	For all real numbers $a$ and $b$ , $ab$ is also a real number.
<b>Commutativity</b>	For all real numbers $a$ and $b$ , $ab = ba$ .

<b>Associativity</b>	For all real numbers $a, b$ and $c$ , $a(bc) = (ab)c$ .
<b>Identity</b>	There is a real number “1” so that for all real numbers $a$ , $a \cdot 1 = a$ .
<b>Inverse</b>	For all real numbers $a \neq 0$ , there is a real number $\frac{1}{a}$ such that $a \left(\frac{1}{a}\right) = 1$ .
<b>Definition of Division</b>	For all real numbers $a$ and $b \neq 0$ , $a \div b = \frac{a}{b} = a \left(\frac{1}{b}\right)$ .

While most students (and some faculty) tend to skip over these properties or give them a cursory glance at best, it is important to realize that the properties stated above are what drive the symbolic manipulation for all of algebra. When listing a tally of more than two numbers,

$$1 + 2 + 3 \quad (1.2.1)$$

for example, we don’t need to specify the order in which those numbers are added. Notice though, try as we might, we can add only two numbers at a time and it is the associative property of addition which assures us that we could organize this sum as  $(1 + 2) + 3$  or  $1 + (2 + 3)$ . This brings up a note about “grouping symbols”. Recall that parentheses and brackets are used in order to specify which operations are to be performed first. In the absence of such grouping symbols, multiplication (and hence division) is given priority over addition (and hence subtraction). For example,  $1 + 2 \cdot 3 = 1 + 6 = 7$ , but  $(1 + 2) \cdot 3 = 3 \cdot 3 = 9$ . As you may recall, we can “distribute” the 3 across the addition if we really wanted to do the multiplication first:  $(1 + 2) \cdot 3 = 1 \cdot 3 + 2 \cdot 3 = 3 + 6 = 9$ . More generally, we have the following.

### Definition 1.2.3 The Distributive Property and Factoring.

For all real numbers  $a, b$  and  $c$ :

<b>Distributive Property</b>	$a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ .
<b>Factoring</b>	$ab + ac = a(b + c)$ and $ac + bc = (a + b)c$ .

**Warning:** A common source of errors for beginning students is the misuse (that is, lack of use) of parentheses. When in doubt, more is better than less: redundant parentheses add clutter, but do not change meaning, whereas writing  $2x + 1$  when you meant to write  $2(x + 1)$  is almost guaranteed to cause you to make a mistake. (Even if you’re able to proceed correctly in spite of your lack of proper notation, this is the sort of thing that will get you on your grader’s bad side, so it’s probably best to avoid the problem in the first place.)

It is worth pointing out that we didn’t really need to list the Distributive Property both for  $a(b + c)$  (distributing from the left) and  $(a + b)c$  (distributing from the right), since the commutative property of multiplication gives us one from the other. Also, “factoring” really is the same equation as the distributive property, just read from right to left. These are the first of many redundancies in

Taken together, the axioms for addition and multiplication of real numbers, along with the distributive property, tell us that algebraically, the set of real numbers is a field<sup>1</sup>.

this section, and they exist in this review section for one reason only – in our experience, many students see these things differently so we will list them as such.

It is hard to overstate the importance of the Distributive Property. For example, in the expression  $5(2 + x)$ , without knowing the value of  $x$ , we cannot perform the addition inside the parentheses first; we must rely on the distributive property here to get  $5(2 + x) = 5 \cdot 2 + 5 \cdot x = 10 + 5x$ . The Distributive Property is also responsible for combining “like terms”. Why is  $3x + 2x = 5x$ ? Because  $3x + 2x = (3 + 2)x = 5x$ .

We continue our review with summaries of other properties of arithmetic, each of which can be derived from the properties listed above. First up are properties of the additive identity 0.

The Zero Product Property drives most of the equation solving algorithms in algebra because it allows us to take complicated equations and reduce them to simpler ones. For example, you may recall that one way to solve  $x^2 + x - 6 = 0$  is by factoring the left hand side of this equation to get  $(x - 2)(x + 3) = 0$ . From here, we apply the Zero Product Property and set each factor equal to zero. This yields  $x - 2 = 0$  or  $x + 3 = 0$  so  $x = 2$  or  $x = -3$ . This type of calculation is key to finding the eigenvalues of a matrix, as we’ll see in Section 7.1.

#### Theorem 1.2.4 Properties of Zero.

Suppose  $a$  and  $b$  are real numbers.

##### Zero Product Property

$ab = 0$  if and only if  $a = 0$  or  $b = 0$  (or both)

**Note:** This not only says that  $0 \cdot a = 0$  for any real number  $a$ , it also says that the only way to get an answer of “0” when multiplying two real numbers is to have one (or both) of the numbers be “0” in the first place.

##### Zeros in Fractions

If  $a \neq 0$ ,  $\frac{0}{a} = 0 \cdot \left(\frac{1}{a}\right) = 0$ .

**Note:** The quantity  $\frac{a}{0}$  is undefined.

We now continue with a review of arithmetic with fractions.

#### Key Idea 1.2.5 Properties of Fractions.

Suppose  $a, b, c$  and  $d$  are real numbers. Assume them to be nonzero whenever necessary; for example, when they appear in a denominator.

##### Identity Properties

$a = \frac{a}{1}$  and  $\frac{a}{a} = 1$ .

##### Fraction Equality

$\frac{a}{b} = \frac{c}{d}$  if and only if  $ad = bc$ .

##### Multiplication of Fractions

$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ . In particular:  $\frac{a}{b} \cdot c = \frac{a}{b} \cdot \frac{c}{1} = \frac{ac}{b}$ .

##### Division of Fractions

$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$ .

In particular:  $1 \div \frac{a}{b} = \frac{b}{a}$  and  $\frac{a}{b} \div c =$

$\frac{a}{b} \div \frac{1}{c} = \frac{a}{b} \cdot \frac{c}{1} = \frac{ac}{b}$ .

**Note:** A common denominator is not required to multiply or divide fractions!

##### Addition and Subtraction of Fractions

$\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$ .

**Note:** A common denominator is required to add or subtract fractions!

**Equivalent Fractions**

$$\frac{a}{b} = \frac{ad}{bd}, \text{ since } \frac{a}{b} = \frac{a}{b} \cdot 1 = \frac{a}{b} \cdot \frac{d}{d} = \frac{ad}{bd}.$$

**Note:** The only way to change the denominator is to multiply both it and the numerator by the same nonzero value because we are, in essence, multiplying the fraction by 1.

**“Reducing” Fractions**

$$\frac{ad}{bd} = \frac{a}{b}, \text{ since } \frac{ad}{bd} = \frac{a}{b} \cdot \frac{d}{d} = \frac{a}{b} \cdot 1 = \frac{a}{b}.$$

$$\text{In particular, } \frac{ab}{b} = a \text{ since } \frac{ab}{b} = \frac{ab}{1 \cdot b} = \frac{a\cancel{b}}{1 \cdot \cancel{b}} = \frac{a}{1} = a \text{ and } \frac{b-a}{a-b} = \frac{(-1)(\cancel{a-b})}{(\cancel{a-b})} = -1.$$

Next up is a review of the arithmetic of “negatives”. In Definition 1.2.1 we first introduced the dash which we all recognize as the “negative” symbol in terms of the additive inverse. For example, the number  $-3$  (read “negative 3”) is defined so that  $3 + (-3) = 0$ . We then defined subtraction using the concept of the additive inverse again so that, for example,  $5 - 3 = 5 + (-3)$ .

**Key Idea 1.2.6 Properties of Negatives.**

Given real numbers  $a$  and  $b$  we have the following.

**Additive Inverse Properties**

$$-a = (-1)a \text{ and } -(-a) = a.$$

**Products of Negatives**

$$(-a)(-b) = ab.$$

**Negatives and Products**

$$-ab = -(ab) = (-a)b = a(-b).$$

**Negatives and Fractions**

$$\text{If } b \text{ is nonzero, } -\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b} \text{ and } \frac{-a}{-b} = \frac{a}{b}.$$

**“Distributing” Negatives**

$$-(a+b) = -a-b \text{ and } -(a-b) = -a+b = b-a.$$

**“Factoring” Negatives**

$$-a-b = -(a+b) \text{ and } b-a = -(a-b).$$

An important point here is that when we “distribute” negatives, we do so across addition or subtraction only. This is because we are really distributing a factor of  $-1$  across each of these terms:  $-(a+b) = (-1)(a+b) = (-1)(a) + (-1)(b) = (-a) + (-b) = -a-b$ . Negatives do not “distribute” across multiplication:  $-(2 \cdot 3) \neq (-2) \cdot (-3)$ . Instead,  $-(2 \cdot 3) = (-2) \cdot (3) = (2) \cdot (-3) = -6$ . The same sort of thing goes for fractions:  $-\frac{3}{5}$  can be written as  $\frac{-3}{5}$  or  $\frac{3}{-5}$ , but not  $\frac{-3}{-5}$ . It’s about time we did a few examples to see how these properties work in practice.

It’s always worth remembering that *division is the same as multiplication by the reciprocal*. You’d be surprised how often this comes in handy.

We reduce fractions by “cancelling” common factors – this is really just reading the previous property “from right to left”.

**Caution:** We may only cancel common *factors* from both numerator and denominator: we can cancel the twos in  $\frac{3(2)}{5(2)}$ , but not in  $\frac{3+2}{5+2}$ .

In this text we do not distinguish typographically between the dashes in the expressions “ $5 - 3$ ” and “ $-3$ ” even though they are mathematically quite different. In the expression “ $5 - 3$ ,” the dash is a *binary* operation (that is, an operation requiring two numbers) whereas in “ $-3$ ,” the dash is a *unary* operation (that is, an operation requiring only one number). You might ask, “Who cares?” Your calculator does – that’s who! In the text we can write  $-3 - 3 = -6$  but that will not work in your calculator. Instead you’d need to type  $\mp 3 - 3$  to get  $-6$  where the first dash comes from the “ $+/-$ ” key.

**Example 1.2.7 Arithmetic with fractions.**

Perform the indicated operations and simplify. By “simplify” here, we mean to have the final answer written in the form  $\frac{a}{b}$  where  $a$  and  $b$  are integers which have no common factors. Said another way, we want  $\frac{a}{b}$  in “lowest terms”.

(a)  $\frac{1}{4} + \frac{6}{7}$

**Solution.** It may seem silly to start with an example this basic but experience has taught us not to take much for granted. We start by finding the lowest common denominator and then we rewrite the fractions using that new denominator. Since 4 and 7 are **relatively prime**, meaning they have no factors in common, the lowest common denominator is  $4 \cdot 7 = 28$ .

$$\begin{aligned}\frac{1}{4} + \frac{6}{7} &= \frac{1}{4} \cdot \frac{7}{7} + \frac{6}{7} \cdot \frac{4}{4} && \text{(Equivalent Fractions)} \\ &= \frac{7}{28} + \frac{24}{28} && \text{(Multiplication of Fractions)} \\ &= \frac{31}{28} && \text{(Addition of Fractions).}\end{aligned}$$

The result is in lowest terms because 31 and 28 are relatively prime so we're done.

(b)  $\frac{5}{12} - \left( \frac{47}{30} - \frac{7}{3} \right)$

**Solution.** We could begin with the subtraction in parentheses, namely  $\frac{47}{30} - \frac{7}{3}$ , and then subtract that result from  $\frac{5}{12}$ . It's easier, however, to first distribute the negative across the quantity in parentheses and then use the Associative Property to perform all of the addition and subtraction in one step. The lowest common denominator for all three fractions is 60.

$$\begin{aligned}\frac{5}{12} - \left( \frac{47}{30} - \frac{7}{3} \right) &= \frac{5}{12} - \frac{47}{30} + \frac{7}{3} && \text{(Distribute the Negative)} \\ &= \frac{5}{12} \cdot \frac{5}{5} - \frac{47}{30} \cdot \frac{2}{2} + \frac{7}{3} \cdot \frac{20}{20} && \text{(Equivalent Fractions)} \\ &= \frac{25}{60} - \frac{94}{60} + \frac{140}{60} && \text{(Multiplication of Fractions)} \\ &= \frac{71}{60} && \text{(Addition and Subtraction of Fractions)}\end{aligned}$$

The numerator and denominator are relatively prime so the fraction is in lowest terms and we have our final answer.

(c)  $\frac{\frac{12}{5} - \frac{7}{24}}{1 + \left( \frac{12}{5} \right) \left( \frac{7}{24} \right)}$

**Solution.** What we are asked to simplify in this problem is known as a “complex” or “compound” fraction. Simply put, we have frac-

tions within a fraction. The longest division line (also called a “vinculum”) performs the same sort of grouping function as parentheses:

$$\frac{\frac{12}{5} - \frac{7}{24}}{1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)} = \frac{\left(\frac{12}{5} - \frac{7}{24}\right)}{\left(1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)}.$$

The first step to simplifying a compound fraction like this one is to see if you can simplify the little fractions inside it. There are two ways to proceed. One is to simplify the numerator and denominator separately, and then use the fact that division is the same thing as multiplication by the reciprocal, as follows:

$$\begin{aligned} \frac{\left(\frac{12}{5} - \frac{7}{24}\right)}{\left(1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} &= \frac{\left(\frac{12}{5} \cdot \frac{24}{24} - \frac{7}{24} \cdot \frac{5}{5}\right)}{\left(1 \cdot \frac{120}{120} + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} && \text{(Equivalent Fractions)} \\ &= \frac{288/120 - 35/120}{120/120 + 84/120} && \text{(Multiplication of fractions)} \\ &= \frac{253/120}{204/120} && \text{(Addition and subtraction of fractions)} \\ &= \frac{253}{120} \cdot \frac{120}{204} && \text{(Division of fractions and cancellation)} \\ &= \frac{253}{204}. \end{aligned}$$

Since  $253 = 11 \cdot 23$  and  $204 = 2 \cdot 2 \cdot 3 \cdot 17$  have no common factors our result is in lowest terms which means we are done.

While there is nothing wrong with the above approach, we can also use our Equivalent Fractions property to rid ourselves of the “compound” nature of this fraction straight away. The idea is to multiply both the numerator and denominator by the lowest common denominator of each of the “smaller” fractions – in this case,  $24 \cdot 5 = 120$ .

$$\begin{aligned} \frac{\left(\frac{12}{5} - \frac{7}{24}\right)}{\left(1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} &= \frac{\left(\frac{12}{5} - \frac{7}{24}\right) \cdot 120}{\left(1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right) \cdot 120} && \text{(Equivalent Fractions)} \\ &= \frac{\left(\frac{12}{5}\right)(120) - \left(\frac{7}{24}\right)(120)}{(1)(120) + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)(120)} && \text{(Distributive Property)} \\ &= \frac{\frac{12 \cdot 120}{5} - \frac{7 \cdot 120}{24}}{120 + \frac{12 \cdot 7 \cdot 120}{5 \cdot 24}} && \text{(Multiply fractions)} \\ &= \frac{\frac{12 \cdot 24 \cdot \cancel{5}}{\cancel{5}} - \frac{7 \cdot 5 \cdot \cancel{24}}{\cancel{24}}}{120 + \frac{12 \cdot 7 \cdot \cancel{5} \cdot \cancel{24}}{\cancel{5} \cdot \cancel{24}}} && \text{(Factor and cancel)} \end{aligned}$$

$$\begin{aligned}
 &= \frac{(12 \cdot 24) - (7 \cdot 5)}{120 + (12 \cdot 7)} \\
 &= \frac{288 - 35}{120 + 84} = \frac{253}{204},
 \end{aligned}$$

which is the same as we obtained above.

$$(d) \frac{(2(2) + 1)(-3 - (-3)) - 5(4 - 7)}{4 - 2(3)}$$

**Solution.** This fraction may look simpler than the one before it, but the negative signs and parentheses mean that we shouldn't get complacent. Again we note that the division line here acts as a grouping symbol. That is,

$$\frac{(2(2) + 1)(-3 - (-3)) - 5(4 - 7)}{4 - 2(3)} = \frac{((2(2) + 1)(-3 - (-3)) - 5(4 - 7))}{(4 - 2(3))}.$$

This means that we should simplify the numerator and denominator first, then perform the division last. We tend to what's in parentheses first, giving multiplication priority over addition and subtraction.

$$\begin{aligned}
 \frac{(2(2) + 1)(-3 - (-3)) - 5(4 - 7)}{4 - 2(3)} &= \frac{(4 + 1)(-3 + 3) - 5(-3)}{4 - 6} \\
 &= \frac{(5)(0) + 15}{-2} = \frac{15}{-2} \\
 &= -\frac{15}{2} \quad (\text{Properties of Negatives}).
 \end{aligned}$$

Since  $15 = 3 \cdot 5$  and 2 have no common factors, we are done.

$$(e) \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right)$$

**Solution.** In this problem, we have multiplication and subtraction. Multiplication takes precedence so we perform it first. Recall that to multiply fractions, we do *not* need to obtain common denominators; rather, we multiply the corresponding numerators together along with the corresponding denominators. However, when we perform the subtraction, we *do* need a common denominator, so we will resist the temptation to cancel the fives in the first term straight away.

$$\begin{aligned}
 \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) &= \frac{3 \cdot 5}{5 \cdot 13} - \frac{4 \cdot (-12)}{5 \cdot 13} \quad (\text{Multiply fractions}) \\
 &= \frac{15}{65} - \frac{-48}{65} \\
 &= \frac{15}{65} + \frac{48}{65} \quad (\text{Properties of Negatives}) \\
 &= \frac{15 + 48}{65} \quad (\text{Add numerators}) \\
 &= \frac{63}{65}.
 \end{aligned}$$

Since  $64 = 3 \cdot 3 \cdot 7$  and  $65 = 5 \cdot 13$  have no common factors, our answer  $\frac{63}{65}$  is in lowest terms and we are done.



Of the issues discussed in the previous set of examples none causes students more trouble than simplifying compound fractions. We presented two different methods for simplifying them: one in which we simplified the overall numerator and denominator and then performed the division and one in which we removed the compound nature of the fraction at the very beginning. We encourage the reader to go back and use both methods on each of the compound fractions presented. Keep in mind that when a compound fraction is encountered in the rest of the text it will usually be simplified using only one method and we may not choose your favourite method. Feel free to use the other one in your notes.

## 1.2.1 Exercises

**Exercise Group.** Perform the indicated operations and simplify.

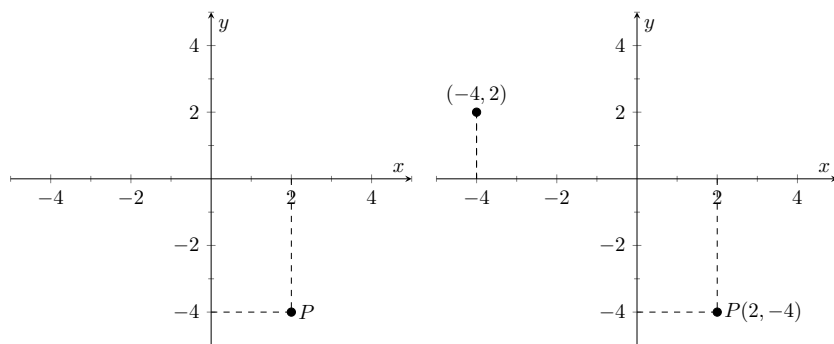
1.  $\frac{3 - \frac{4}{9}}{-2 - (-3)}$
2.  $\frac{2}{3} - \frac{4}{7}$
3.  $\left(-\frac{32}{9}\right)^{-3/5}$
4.  $\frac{1 - 2(-3)}{5(-3) + 7}$
5.  $\frac{1 + 2^{-3}}{3 - 4^{-1}}$
6.  $\frac{3}{8} + \frac{5}{12}$
7.  $\frac{-12 + \sqrt{18}}{21}$
8.  $3^{-1} - 4^{-2}$
9.  $\sqrt{(2 - (-1))^2 + \left(\frac{1}{2} - 3\right)^2}$
10.  $\frac{-(-4) + \sqrt{(-4)^2 - 4(1)(-1)}}{2(1)}$
11.  $\frac{2\left(\frac{4}{3}\right)}{1 - \left(\frac{4}{3}\right)^2}$
12.  $3\sqrt{2(4) + 1} + 3(4)\left(\frac{1}{2}\right)(2(4) + 1)^{-1/2}(2)$
13.  $5 - 2 + 3$
14.  $2(-7)\sqrt[3]{1 - (-7)} + (-7)^2\left(\frac{1}{3}\right)(1 - (-7))^{-2/3}(-1)$
15.  $\frac{4 - 5.8}{2 - 2.1}$
16.  $\frac{5(3) - 7}{2(3)^2 - 3(3) - 9}$
17.  $\frac{(-2)^2 - (-2) - 6}{(-2)^2 - 4}$
18.  $\sqrt{12} - \sqrt{75}$
19.  $2(-5)(-5 + 1)^{-1} + (-5)^2(-1)(-5 + 1)^{-2}$
20.  $(-8)^{2/3} - 9^{-3/2}$
21.  $\frac{2(3) - (4 - 1)}{2^2 + 1}$
22.  $\sqrt{(\sqrt{5} - 2\sqrt{5})^2 + (\sqrt{18} - \sqrt{8})^2}$
23.  $\frac{2(-3)}{3 - (-3)}$
24.  $\left(\frac{2}{3}\right)^{-5}$
25.  $\sqrt{3^2 + 4^2}$
26.  $\frac{2((-1)^2 - 1)}{((-1)^2 + 1)^2}$
27.  $\sqrt{(3 - 4)^2 + (5 - 2)^2}$
28.  $\frac{\frac{2}{3} - \frac{4}{5}}{4 - \frac{7}{10}}$
29.  $\frac{-2 - \sqrt{(2)^2 - 4(3)(-1)}}{2(3)}$
30.  $\frac{1 - \left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}{1 + \left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}$
31.  $\frac{5 - 3}{-2 - 4}$
32.  $5 - (2 + 3)$
33.  $\frac{3 \cdot 5^{100}}{12 \cdot 5^{98}}$

## 1.3 The Cartesian Coordinate Plane

As a warm-up for the discussions of vectors and three-dimensional geometry yet to come, we will make a quick review of the **Cartesian Coordinate Plane**. Imagine two real number lines crossing at a right angle at 0 as shown in Figure 1.3.1.

The horizontal number line is usually called the **x axis** while the vertical number line is usually called the **y axis**. As with the usual number line, we imagine these axes extending off indefinitely in both directions. Having two number lines allows us to locate the positions of points off of the number lines as well as points on the lines themselves.

For example, consider the point  $P$  in Figure 1.3.2. To use the numbers on the axes to label this point, we imagine dropping a vertical line from the  $x$ -axis to  $P$  and extending a horizontal line from the  $y$ -axis to  $P$ . This process is sometimes called “projecting” the point  $P$  to the  $x$  (respectively  $y$ ) axis. We then describe the point  $P$  using the **ordered pair**  $(2, -4)$ . The first number in the ordered pair is called the **abscissa** or **x coordinate** and the second is called the **ordinate** or **y coordinate**. Taken together, the ordered pair  $(2, -4)$  comprise the **Cartesian coordinates** of the point  $P$ . In practice, the distinction between a point and its coordinates is blurred; for example, we often speak of “the point  $(2, -4)$ .” We can think of  $(2, -4)$  as instructions on how to reach  $P$  from the **origin**  $(0, 0)$  by moving 2 units to the right and 4 units downwards. Notice that the order in the ordered pair is important — if we wish to plot the point  $(-4, 2)$ , we would move to the left 4 units from the origin and then move upwards 2 units, as below on the right.



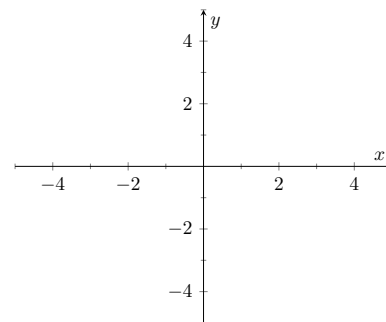
**Figure 1.3.2** Plotting points in Cartesian coordinates

When we speak of the Cartesian Coordinate Plane, we mean the set of all possible ordered pairs  $(x, y)$  as  $x$  and  $y$  take values from the real numbers. Below is a summary of important facts about Cartesian coordinates.

### Key Idea 1.3.3 Important Facts about the Cartesian Coordinate Plane.

- $(a, b)$  and  $(c, d)$  represent the same point in the plane if and only if  $a = c$  and  $b = d$ .
- $(x, y)$  lies on the  $x$ -axis if and only if  $y = 0$ .
- $(x, y)$  lies on the  $y$ -axis if and only if  $x = 0$ .
- The origin is the point  $(0, 0)$ . It is the only point common to both axes.

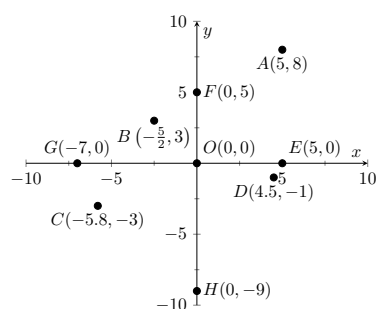
The Cartesian Plane is named in honour of René Descartes<sup>1</sup>.



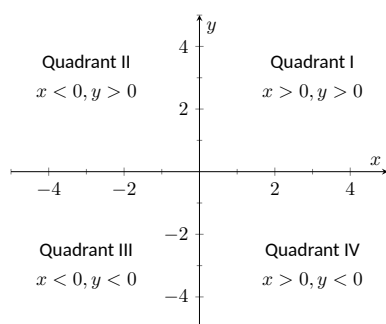
**Figure 1.3.1** The Cartesian coordinate plane

Usually extending off towards infinity is indicated by arrows, but here, the arrows are used to indicate the *direction* of increasing values of  $x$  and  $y$ .

The names of the coordinates can vary depending on the context of the application. If, for example, the horizontal axis represented time we might choose to call it the  $t$ -axis. The first number in the ordered pair would then be the  $t$ -coordinate.



**Figure 1.3.5** Plotting points in Example 1.3.4



**Figure 1.3.6** The four quadrants of the Cartesian plane

### Example 1.3.4 Plotting points in the Cartesian Plane.

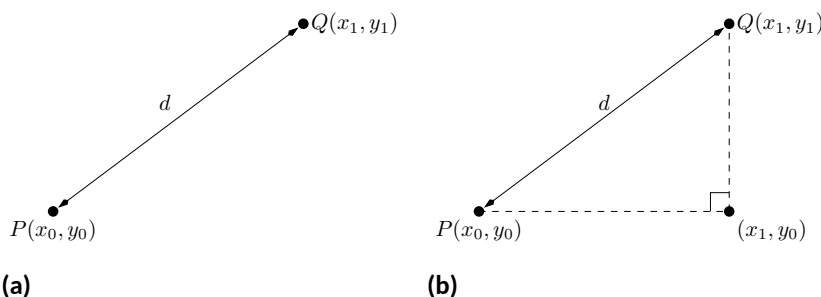
Plot the following points:  $A(5, 8)$ ,  $B(-\frac{5}{2}, 3)$ ,  $C(-5.8, -3)$ ,  $D(4.5, -1)$ ,  $E(5, 0)$ ,  $F(0, 5)$ ,  $G(-7, 0)$ ,  $H(0, -9)$ ,  $O(0, 0)$ .

**Solution.** To plot these points, we start at the origin and move to the right if the  $x$  coordinate is positive; to the left if it is negative. Next, we move up if the  $y$  coordinate is positive or down if it is negative. If the  $x$ -coordinate is 0, we start at the origin and move along the  $y$ -axis only. If the  $y$ -coordinate is 0 we move along the  $x$ -axis only.

The axes divide the plane into four regions called **quadrants**. They are labelled with Roman numerals and proceed counterclockwise around the plane: see Figure 1.3.6.

For example,  $(1, 2)$  lies in Quadrant I,  $(-1, 2)$  in Quadrant II,  $(-1, -2)$  in Quadrant III and  $(1, -2)$  in Quadrant IV. If a point other than the origin happens to lie on the axes, we typically refer to that point as lying on the positive or negative  $x$  axis (if  $y = 0$ ) or on the positive or negative  $y$  axis (if  $x = 0$ ). For example,  $(0, 4)$  lies on the positive  $y$  axis whereas  $(-117, 0)$  lies on the negative  $x$  axis. Such points do not belong to any of the four quadrants.

**Distance in the Plane.** Another important concept in Geometry is the notion of length. If we are going to unite Algebra and Geometry using the Cartesian Plane, then we need to develop an algebraic understanding of what distance in the plane means. Suppose we have two points,  $P(x_0, y_0)$  and  $Q(x_1, y_1)$ , in the plane. By the **distance**  $d$  between  $P$  and  $Q$ , we mean the length of the line segment joining  $P$  with  $Q$ . (Remember, given any two distinct points in the plane, there is a unique line containing both points.) Our goal now is to create an algebraic formula to compute the distance between these two points. Consider the generic situation in Figure 1.3.7(a).



**Figure 1.3.7** Distance between  $P$  and  $Q$

With a little more imagination, we can envision a right triangle whose hypotenuse has length  $d$  as drawn in Figure 1.3.7(b). From the latter figure, we see that the lengths of the legs of the triangle are  $|x_1 - x_0|$  and  $|y_1 - y_0|$  so the Pythagorean Theorem<sup>2</sup> gives us

$$\begin{aligned} |x_1 - x_0|^2 + |y_1 - y_0|^2 &= d^2 \\ (x_1 - x_0)^2 + (y_1 - y_0)^2 &= d^2. \end{aligned}$$

(Since the square of a number is always positive, we can drop the absolute value signs.) By extracting the square root of both sides of the second equation and using the fact that distance is never negative, we get the following result.

<sup>2</sup>en.wikipedia.org/wiki/Pythagorean\_Theorem

Recall that the **absolute value** of a real number  $a$ , denoted by  $|a|$ , measures the distance of  $a$  from the origin. Since we never want a negative distance, this means  $|a| = a$  if  $a \geq 0$ , while if  $a < 0$ ,  $|a| = -a$ . For example,  $|4| = 4$ , and  $|-7| = -(-7) = 7$ . Given two real numbers  $a$  and  $b$ ,  $|a - b|$  measures the distance between them.

**Key Idea 1.3.8 The Distance Formula.**

The distance  $d$  between the points  $P(x_0, y_0)$  and  $Q(x_1, y_1)$  is:

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}. \quad (1.3.1)$$

It is not always the case that the points  $P$  and  $Q$  lend themselves to constructing such a triangle. If the points  $P$  and  $Q$  are arranged vertically or horizontally, or describe the exact same point, we cannot use the above geometric argument to derive the distance formula. It is left to the reader in Exercise Exercise 1.3.15 to verify Equation (1.3.1) for these cases.

**Example 1.3.9 Distance between two points.**

Find and simplify the distance between  $P(-2, 3)$  and  $Q(1, -3)$ .

**Solution.** Using The Distance Formula, we have

$$\begin{aligned} d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\ &= \sqrt{(1 - (-2))^2 + (-3 - 3)^2} \\ &= \sqrt{9 + 36} \\ &= 3\sqrt{5}, \end{aligned}$$

so the distance is  $3\sqrt{5}$ .

**Example 1.3.10 Finding points at a given distance.**

Find all of the points with  $x$ -coordinate 1 which are 4 units from the point  $(3, 2)$ .

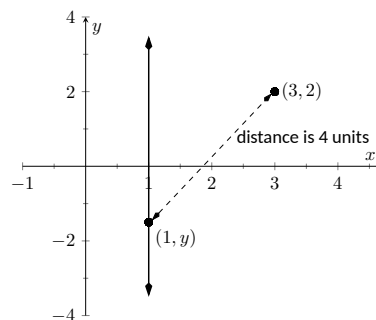
**Solution.** We shall soon see that the points we wish to find are on the line  $x = 1$ , but for now we'll just view them as points of the form  $(1, y)$ .

We require that the distance from  $(3, 2)$  to  $(1, y)$  be 4. The Distance Formula, Equation (1.3.1), yields

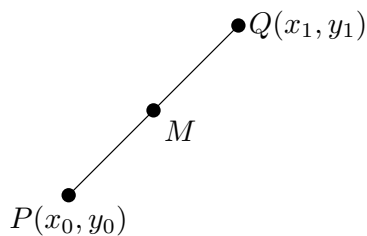
$$\begin{aligned} d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\ 4 &= \sqrt{(1 - 3)^2 + (y - 2)^2} \\ 4 &= \sqrt{4 + (y - 2)^2} \\ 4^2 &= \left( \sqrt{4 + (y - 2)^2} \right)^2 \quad (\text{squaring both sides}) \\ 16 &= 4 + (y - 2)^2 \\ 12 &= (y - 2)^2 \\ (y - 2)^2 &= 12 \\ y - 2 &= \pm\sqrt{12} \quad (\text{extracting the square root}) \\ y &= 2 \pm 2\sqrt{3}. \end{aligned}$$

We obtain two answers:  $(1, 2 + 2\sqrt{3})$  and  $(1, 2 - 2\sqrt{3})$ . The reader is encouraged to think about why there are two answers.

Related to finding the distance between two points is the problem of find-



**Figure 1.3.11** Diagram for Example 1.3.10



**Figure 1.3.12** The midpoint of a line segment

ing the **midpoint** of the line segment connecting two points. Given two points,  $P(x_0, y_0)$  and  $Q(x_1, y_1)$ , the **midpoint**  $M$  of  $P$  and  $Q$  is defined to be the point on the line segment connecting  $P$  and  $Q$  whose distance from  $P$  is equal to its distance from  $Q$ .

If we think of reaching  $M$  by going “halfway over” and “halfway up” we get the following formula.

**Key Idea 1.3.13 The Midpoint Formula.**

The midpoint  $M$  of the line segment connecting  $P(x_0, y_0)$  and  $Q(x_1, y_1)$  is

$$M = \left( \frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2} \right).$$

If we let  $d$  denote the distance between  $P$  and  $Q$ , we leave it as Exercise 1.3.16 to show that the distance between  $P$  and  $M$  is  $d/2$ , which is the same as the distance between  $M$  and  $Q$ . This suffices to show that Key Idea 1.3.13 gives the coordinates of the midpoint.

**Example 1.3.14 Finding the midpoint of a line segment.**

Find the midpoint of the line segment connecting  $P(-2, 3)$  and  $Q(1, -3)$ .

**Solution.** Using The Midpoint Formula, we have

$$\begin{aligned} M &= \left( \frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2} \right) \\ &= \left( \frac{(-2) + 1}{2}, \frac{3 + (-3)}{2} \right) = \left( -\frac{1}{2}, 0 \right) \\ &= \left( -\frac{1}{2}, 0 \right). \end{aligned}$$

The midpoint is  $(-\frac{1}{2}, 0)$ .

We close with a more abstract application of the Midpoint Formula.

**Example 1.3.15 An abstract midpoint problem.**

If  $a \neq b$ , prove that the line  $y = x$  equally divides the line segment with endpoints  $(a, b)$  and  $(b, a)$ .

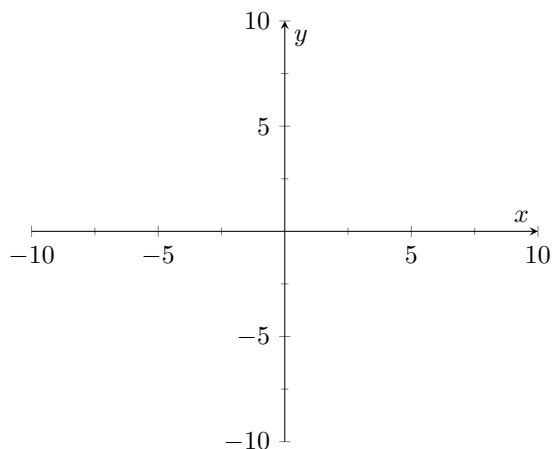
**Solution.** To prove the claim, we use The Midpoint Formula to find the midpoint.

$$\begin{aligned} M &= \left( \frac{a + b}{2}, \frac{b + a}{2} \right) \\ &= \left( \frac{a + b}{2}, \frac{a + b}{2} \right). \end{aligned}$$

Since the  $x$  and  $y$  coordinates of this point are the same, we find that the midpoint lies on the line  $y = x$ , as required.

## 1.3.1 Exercises

1. Plot and label the points  $A(-3, -7)$ ,  $B(1.3, -2)$ ,  $C(\pi, \sqrt{10})$ ,  $D(0, 8)$ ,  $E(-5.5, 0)$ ,  $F(-8, 4)$ ,  $G(9.2, -7.8)$  and  $H(7, 5)$  in the Cartesian Coordinate Plane given below.



**Exercise Group.** Find the distance  $d$  between the points and the midpoint  $M$  of the line segment which connects them.

2.  $(1, 2), (-3, 5)$
3.  $(3, -10), (-1, 2)$
4.  $\left(\frac{1}{2}, 4\right), \left(\frac{3}{2}, -1\right)$
5.  $\left(-\frac{2}{3}, \frac{3}{2}\right), \left(\frac{7}{3}, 2\right)$
6.  $\left(\frac{24}{5}, \frac{6}{5}\right), \left(-\frac{11}{5}, -\frac{19}{5}\right)$
7.  $(\sqrt{2}, \sqrt{3}), (-\sqrt{8}, -\sqrt{12})$
8.  $(2\sqrt{45}, \sqrt{12}), (\sqrt{20}, \sqrt{27})$
9.  $(0, 0), (x, y)$
10. Find all of the points of the form  $(x, -1)$  which are 4 units from the point  $(3, 2)$ .
11. Find all of the points on the  $y$ -axis which are 5 units from the point  $(-5, 3)$ .
12. Find all of the points on the  $x$ -axis which are 2 units from the point  $(-1, 1)$ .
13. Find all of the points of the form  $(x, -x)$  which are 1 unit from the origin.
14. Let's assume for a moment that we are standing at the origin and the positive  $y$ -axis points due North while the positive  $x$ -axis points due East. Our Sasquatch-o-meter tells us that Sasquatch is 3 miles West and 4 miles South of our current position. What are the coordinates of his position? How far away is he from us? If he runs 7 miles due East what would his new position be?
15. Verify The Distance Formula for the cases when:
  - (a) The points are arranged vertically. (Hint: Use  $P(a, y_0)$  and  $Q(a, y_1)$ .)
  - (b) The points are arranged horizontally. (Hint: Use  $P(x_0, b)$  and  $Q(x_1, b)$ .)
  - (c) The points are actually the same point. (You shouldn't need a hint for this one.)
16. Verify the Midpoint Formula by showing the distance between  $P(x_1, y_1)$  and  $M$  and the distance between  $M$  and  $Q(x_2, y_2)$  are both half of the distance between  $P$  and  $Q$ .
17. Show that the points  $A$ ,  $B$  and  $C$  below are the vertices of a right triangle.
  - (a)  $A(-3, 2)$ ,  $B(-6, 4)$ , and  $C(1, 8)$
  - (b)  $A(-3, 1)$ ,  $B(4, 0)$  and  $C(0, -3)$

<sup>3</sup>[en.wikipedia.org/wiki/Pythagorean\\_theorem\#Converse](http://en.wikipedia.org/wiki/Pythagorean_theorem\#Converse)

18. Find a point  $D(x, y)$  such that the points  $A(-3, 1)$ ,  $B(4, 0)$ ,  $C(0, -3)$  and  $D$  are the corners of a square. Justify your answer.
19. The world is not flat. (There are those who disagree with this statement. Look them up on the Internet some time when you're bored.) Thus the Cartesian Plane cannot possibly be the end of the story. Discuss with your classmates how you would extend Cartesian Coordinates to represent the three dimensional world. What would the Distance and Midpoint formulas look like, assuming those concepts make sense at all?



## 1.4 Complex Numbers

We now move on to the study of the set of **complex numbers**. As you may recall, the complex numbers fill an algebraic gap left by the real numbers. There is no real number  $x$  with  $x^2 = -1$ , since for any real number  $x^2 \geq 0$ . However, we could formally extract square roots and write  $x = \pm\sqrt{-1}$ . We build the complex numbers by relabelling the quantity  $\sqrt{-1}$  as  $i$ , the unfortunately misnamed **imaginary unit**,  $i$ . The number  $i$ , while not a real number, is defined so that it plays along well with real numbers and acts very much like any other radical expression. For instance,  $3(2i) = 6i$ ,  $7i - 3i = 4i$ ,  $(2 - 7i) + (3 + 4i) = 5 - 3i$ , and so forth. The key properties which distinguish  $i$  from the real numbers are listed below.

### Definition 1.4.1 The imaginary unit.

The imaginary unit  $i$  satisfies the two following properties:

1.  $i^2 = -1$
2. If  $c$  is a real number with  $c \geq 0$  then  $\sqrt{-c} = i\sqrt{c}$

Property 1 in Definition 1.4.1 establishes that  $i$  does act as a square root of  $-1$ , and property 2 establishes what we mean by the “principal square root” of a negative real number. In property 2, it is important to remember the restriction on  $c$ . For example, it is perfectly acceptable to say  $\sqrt{-4} = i\sqrt{4} = i(2) = 2i$ . However,  $\sqrt{-(-4)} \neq i\sqrt{-4}$ , otherwise, we’d get

$$2 = \sqrt{4} = \sqrt{-(-4)} = i\sqrt{-4} = i(2i) = 2i^2 = 2(-1) = -2,$$

which is unacceptable. The moral of this story is that the general properties of radicals do not apply for even roots of negative quantities. With Definition 1.4.1 in place, we are now in position to define the **complex numbers**.

### Definition 1.4.2 Complex numbers.

A **complex number** is a number of the form  $a + bi$ , where  $a$  and  $b$  are real numbers and  $i$  is the imaginary unit. The set of complex numbers is denoted  $\mathbb{C}$ .

Complex numbers include things you’d normally expect, like  $3 + 2i$  and  $\frac{2}{5} - i\sqrt{3}$ . However, don’t forget that  $a$  or  $b$  could be zero, which means numbers like  $3i$  and  $6$  are also complex numbers. In other words, don’t forget that the complex numbers *include* the real numbers, so  $0$  and  $\pi - \sqrt{21}$  are both considered complex numbers. We want to study the arithmetic of complex numbers, but before we can do so, we first need to make sure we understand what it means for two complex numbers to be equal.

### Definition 1.4.3 Equality of complex numbers.

Let  $z = a + ib$  and  $w = c + id$  be two complex numbers. We say that  $z$  and  $w$  are **equal**, and write  $z = w$ , if and only if  $a = c$  and  $b = d$ .

The arithmetic of complex numbers is as you would expect. The definitions of addition and multiplication are as follows.

Historically, the lack of solutions to the equation  $x^2 = -1$  had nothing to do with the development of the complex numbers. Until the 19th century, equations such as  $x^2 = -1$  would have been considered in the context of the analytic geometry of Descartes. The lack of solutions simply indicated that the graph  $y = x^2$  did not intersect the line  $y = -1$ . The more remarkable case was that of *cubic* equations, of the form  $x^3 = ax + b$ . In this case a real solution is *guaranteed*, but there are cases where one needs complex numbers to find it! For details, see the excellent book *Visual Complex Analysis*, by Tristan Needham.

Note the use of the indefinite article “a”. Whatever beast is chosen to be  $i$ ,  $-i$  is the other square root of  $-1$ .

Some technical mathematics textbooks label the imaginary unit “ $j$ ”, usually to avoid confusion with the use of the letter  $i$  to denote electric current. While it carries the adjective “imaginary”, these numbers have essential real-world implications. For example, every electronic device owes its existence to the study of “imaginary” numbers.

**Definition 1.4.4 Addition of complex numbers.**

Given two complex numbers  $z = a + ib$  and  $w = c + id$ , we define their **sum** to be the complex number given by

$$z + w = (a + c) + i(b + d).$$

**Definition 1.4.5 Multiplication of complex numbers.**

Given two complex numbers  $z = a + ib$  and  $w = c + id$ , we define their **product** to be the complex number

$$zw = (ac - bd) + i(ad + bc).$$

Addition of complex numbers is defined by simply adding the corresponding parts. The definition of multiplication looks complicated, but it's simply an application of the "F.O.I.L." rule for multiplying binomials, where we have to account for the fact that  $i^2 = -1$ .

As long as we remember the two properties in Definition 1.4.1, we can treat expressions involving  $i = \sqrt{-1}$  as we would with any other radical. Let's work through an example to see how this works.

**Example 1.4.6 Arithmetic with complex numbers.**

Perform the indicated operations.

- |                            |                                   |
|----------------------------|-----------------------------------|
| 1. $(1 - 2i) - (3 + 4i)$   | 4. $\sqrt{-3}\sqrt{-12}$          |
| 2. $(1 - 2i)(3 + 4i)$      | 5. $\sqrt{(-3)(-12)}$             |
| 3. $\frac{1 - 2i}{3 - 4i}$ | 6. $(x - [1 + 2i])(x - [1 - 2i])$ |

**Solution.**

- Subtraction is simply a variation on addition: We distribute the minus sign across the second complex number and combine like terms:

$$\begin{aligned}(1 - 2i) - (3 + 4i) &= 1 - 2i - 3 - 4i && \text{(Distribute)} \\ &= -2 - 6i && \text{(Gather like terms)}\end{aligned}$$

Technically, we'd have to rewrite our answer  $-2 - 6i$  as  $(-2) + (-6)i$  to be (in the strictest sense) "in the form  $a + bi$ ". That being said, even pedants have their limits, and we'll consider  $-2 - 6i$  good enough.

- The key to multiplying complex numbers is to forget about Definition 1.4.5 above, and simply treat this as a product of two binomials. Using the Distributive Property (a.k.a. F.O.I.L.), we get

$$\begin{aligned}(1 - 2i)(3 + 4i) &= (1)(3) + (1)(4i) - (2i)(3) - (2i)(4i) && \text{(F.O.I.L.)} \\ &= 3 + 4i - 6i - 8i^2 \\ &= 3 - 2i - 8(-1) && i^2 = -1\end{aligned}$$

$$= 3 - 2i + 8$$

$$= 11 - 2i.$$

3. How in the world are we supposed to simplify  $\frac{1-2i}{3-4i}$ ? Well, we deal with the denominator  $3 - 4i$  as we would any other denominator containing two terms, one of which is a square root: we then multiply both numerator and denominator by  $3 + 4i$ , the (complex) conjugate of  $3 - 4i$ . Doing so produces

$$\begin{aligned}\frac{1-2i}{3-4i} &= \frac{(1-2i)(3+4i)}{(3-4i)(3+4i)} && \text{(Equivalent Fractions)} \\ &= \frac{3+4i-6i-8i^2}{9-16i^2} && \text{(F.O.I.L.)} \\ &= \frac{3-2i-8(-1)}{9-16(-1)} && (i^2 = -1) \\ &= \frac{11-2i}{25} = \frac{11}{25} - \frac{2}{25}i.\end{aligned}$$

4. We use property 2 of Definition 1.4.1 first, then apply the rules of radicals applicable to real numbers to get  $\sqrt{-3}\sqrt{-12} = (i\sqrt{3})(i\sqrt{12}) = i^2\sqrt{3 \cdot 12} = -\sqrt{36} = -6$ .
5. We adhere to the order of operations here and perform the multiplication before the radical to get  $\sqrt{(-3)(-12)} = \sqrt{36} = 6$ .
6. We can brute force multiply using the distributive property and see that

$$\begin{aligned}(x - [1 + 2i])(x - [1 - 2i]) &= x^2 - x[1 - 2i] - x[1 + 2i] + [1 - 2i][1 + 2i] \\ &= x^2 - x + 2ix - x - 2ix + 1 - 2i + 2i - 4i^2 \\ &= x^2 - 2x + 1 - 4(-1) \\ &= x^2 - 2x + 5.\end{aligned}$$

In the previous example, we used the idea of a “conjugate” to divide two complex numbers. More generally, the **complex conjugate** of a complex number  $a+bi$  is the number  $a-bi$ . The notation commonly used for complex conjugation is a “bar”:  $\overline{a+bi} = a-bi$ . For example,

$$\overline{3+2i} = 3-2i \quad \text{and} \quad \overline{3-2i} = 3+2i.$$

To find  $\overline{6}$ , we note that  $\overline{6} = \overline{6+0i} = 6-0i = 6$ , so  $\overline{6} = 6$ . Similarly,  $\overline{4i} = -4i$ , since  $\overline{4i} = \overline{0+4i} = 0-4i = -4i$ . The properties of the conjugate are summarized in the following theorem.

**Theorem 1.4.7 Properties of the Complex Conjugate.**

Let  $z$  and  $w$  be complex numbers.

- $\overline{\overline{z}} = z$
- $\overline{z+w} = \overline{z} + \overline{w}$
- $\overline{zw} = \overline{z}\overline{w}$

You may recall using conjugates to rationalize expressions involving square roots. For example, we have

$$\frac{3}{\sqrt{2}+1} = \frac{3(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)} = 3\sqrt{2}-3.$$

The key observation is that multiplying by the conjugate sets up a difference of squares: the terms involving the radical cancel out. In some ways, multiplication by a complex conjugate is even more convenient than with real radicals: since

$$(a+ib)(a-ib) = a^2 - iab + iab - i^2b^2 = a^2 + b^2,$$

the product  $z\bar{z}$  is never negative, and only vanishes if  $z = 0$ .

- $\overline{z^n} = (\bar{z})^n$ , for any natural number  $n$
- $z$  is a real number if and only if  $\bar{z} = z$ .

Essentially, Theorem 1.4.7 says that complex conjugation works well with addition, multiplication and powers. The proofs of these properties can best be achieved by writing out  $z = a+bi$  and  $w = c+di$  for real numbers  $a, b, c$  and  $d$ . Next, we compute the left and right sides of each equation and verify that they are the same.

Verifying the first property is a very quick exercise. To prove the second property, we compare  $\overline{z+w}$  with  $\bar{z} + \bar{w}$ . We have  $\bar{z} + \bar{w} = \overline{a+bi} + \overline{c+di} = a-bi + c-di$ . To find  $\overline{z+w}$ , we first compute

$$\begin{aligned} z+w &= (a+bi) + (c+di) \\ &= (a+c) + (b+d)i, \end{aligned}$$

so

$$\begin{aligned} \overline{z+w} &= \overline{(a+c) + (b+d)i} \\ &= (a+c) - (b+d)i \\ &= a-bi + c-di \\ &= \bar{z} + \bar{w}. \end{aligned}$$

As such, we have established  $\overline{z+w} = \bar{z} + \bar{w}$ .

The proof for multiplication works similarly: we have

$$\begin{aligned} \overline{zw} &= \overline{(ac-bd) + i(ad+bc)} \\ &= (ac-bd) - i(ad+bc) \\ &= (ac - (-b)(-d)) + i(a(-d) + b(-c)) \\ &= (a-ib)(c-id) \\ &= \bar{z}\bar{w}. \end{aligned}$$

The proof that the conjugate works well with powers can be viewed as a repeated application of the product rule.

The last property is a characterization of real numbers. If  $z$  is real, then  $z = a+0i$ , so  $\bar{z} = a-0i = a = z$ . On the other hand, if  $z = \bar{z}$ , then  $a+bi = a-bi$  which means  $b = -b$  so  $b = 0$ . Hence,  $z = a+0i = a$  and is real.

It is worth noting that although the arithmetic of complex numbers seems, at first impression, to be very different and strange compared to the arithmetic of real numbers, it actually satisfies all the same properties, as outlined in the following theorem.

#### Theorem 1.4.8 Properties of Complex Arithmetic.

The addition and multiplication of complex numbers satisfy the following properties:

<b>Closure under addition</b>	For any complex numbers $z$ and $w$ , $z+w$ is a complex number.
<b>Commutativity of addition</b>	For any complex numbers $z$ and $w$ , $z+w = w+z$ .

<b>Associativity of addition</b>	For any complex numbers $z_1, z_2, z_3$ , $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ .
<b>Additive identity</b>	There exists a complex number $0$ such that $z + 0 = 0 + z = z$ for every complex number $z$ .
<b>Additive inverses</b>	For every complex number $z$ there exists a complex number $-z$ such that $z + (-z) = -z + z = 0$ .
<b>Closure under multiplication</b>	For any complex numbers $z$ and $w$ , $zw$ is a complex number.
<b>Commutativity of multiplication</b>	For any complex numbers $z$ and $w$ , $zw = wz$ .
<b>Associativity of multiplication</b>	For any complex numbers $z_1, z_2, z_3$ , $z_1(z_2 z_3) = (z_1 z_2) z_3$ .
<b>Multiplicative identity</b>	There exists a complex number $1$ such that $1 \cdot z = z \cdot 1 = z$ for every complex number $z$ .
<b>Multiplicative inverses</b>	For every complex number $z \neq 0$ , there exists a complex number $z^{-1}$ such that $zz^{-1} = z^{-1}z = 1$ .
<b>Distributive property</b>	For all complex numbers $z_1, z_2, z_3$ , we have $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ .

We leave the proof of Theorem 1.4.8 as a long (but straightforward) exercise. Working through the proof is a good way to confirm for yourself that you understand the corresponding rules for real number arithmetic from Section 1.2, and how the properties for complex arithmetic are inherited from their real counterparts.

We now consider the problem of solving quadratic equations. Consider  $x^2 - 2x + 5 = 0$ . The discriminant  $b^2 - 4ac = -16$  is negative, so we know from the quadratic formula that there are no *real* solutions, since the Quadratic Formula would involve the term  $\sqrt{-16}$ . Complex numbers, however, are built just for such situations, so we can go ahead and apply the Quadratic Formula to get:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

#### Example 1.4.9 Finding complex solutions.

Find the complex solutions to the following equations.

$$1. \frac{2x}{x+1} = x+3 \quad 2. 2t^4 = 9t^2 + 5 \quad 3. z^3 + 1 = 0$$

#### Solution.

1. Clearing fractions yields a quadratic equation so we collect all terms on one side and apply the quadratic formula.

$$\begin{aligned} \frac{2x}{x+1} &= x+3 \\ 2x &= (x+3)(x+1) \quad (\text{Clear denominators}) \end{aligned}$$

$$2x = x^2 + x + 3x + 3 \quad (\text{F.O.I.L.})$$

$$2x = x^2 + 4x + 3 \quad (\text{Gather like terms})$$

$$0 = x^2 + 2x + 3 \quad (\text{Subtract } 2x).$$

From here, we apply the Quadratic Formula

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{2^2 - 4(1)(3)}}{2(1)} && (\text{Quadratic Formula}) \\ &= \frac{-2 \pm \sqrt{-8}}{2} && (\text{Simplify}) \\ &= \frac{-2 \pm i\sqrt{8}}{2} && (\text{Definition of } i) \\ &= \frac{-2 \pm i2\sqrt{2}}{2} && (\text{Product Rule for Radicals}) \\ &= \frac{\cancel{2}(-1 \pm i\sqrt{2})}{\cancel{2}} && (\text{Factor and reduce}) \\ &= -1 \pm i\sqrt{2}. \end{aligned}$$

We get two answers:  $x = -1 + i\sqrt{2}$  and its conjugate  $x = -1 - i\sqrt{2}$ . Checking both of these answers reviews all of the salient points about complex number arithmetic and is therefore strongly encouraged.

2. Since we have three terms, and the exponent on one term ("4" on  $t^4$ ) is exactly twice the exponent on the other ("2" on  $t^2$ ), we have a "Quadratic in Disguise". We proceed accordingly.

$$2t^4 = 9t^2 + 5$$

$$2t^4 - 9t^2 - 5 = 0 \quad (\text{Subtract } 9t^2 \text{ and } 5)$$

$$(2t^2 + 1)(t^2 - 5) = 0 \quad (\text{Factor})$$

$$2t^2 + 1 = 0 \quad \text{or} \quad t^2 = 5 \quad (\text{Zero Product Property}).$$

From  $2t^2 + 1 = 0$  we get  $2t^2 = -1$ , or  $t^2 = -\frac{1}{2}$ . We extract square roots as follows:

$$t = \pm \sqrt{-\frac{1}{2}} = \pm i \sqrt{\frac{1}{2}} = \pm i \frac{\sqrt{1}}{\sqrt{2}} = \pm i \frac{1}{\sqrt{2}} = \pm \frac{i\sqrt{2}}{2},$$

where we have rationalized the denominator per convention. From  $t^2 = 5$ , we get  $t = \pm\sqrt{5}$ . In total, we have four complex solutions - two real:  $t = \pm\sqrt{5}$  and two non-real:  $t = \pm \frac{i\sqrt{2}}{2}$ .

3. To find the *real* solutions to  $z^3 + 1 = 0$ , we can subtract the 1 from both sides and extract cube roots:  $z^3 = -1$ , so  $z = \sqrt[3]{-1} = -1$ . It turns out there are two more non-real complex number solutions to this equation. To get at these, we factor:

$$z^3 + 1 = 0$$

$$(z + 1)(z^2 - z + 1) = 0 \quad (\text{Factor (Sum of Two Cubes)})$$

$$z + 1 = 0 \quad \text{or} \quad z^2 - z + 1 = 0.$$

From  $z+1 = 0$ , we get our real solution  $z = -1$ . From  $z^2 - z + 1 = 0$ , we apply the Quadratic Formula to get:

$$z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2}.$$

Thus we get *three* solutions to  $z^3 + 1 = 0$  - one real:  $z = -1$  and two non-real:  $z = \frac{1 \pm i\sqrt{3}}{2}$ . As always, the reader is encouraged to test their algebraic mettle and check these solutions.

It is no coincidence that the non-real solutions to the equations in Example 1.4.9 appear in complex conjugate pairs. Any time we use the Quadratic Formula to solve an equation with *real* coefficients, the answers will form a complex conjugate pair owing to the  $\pm$  in the Quadratic Formula. This is stated formally in the following theorem.

**Theorem 1.4.10 Discriminant Theorem.**

Given a Quadratic Equation  $AX^2 + BX + C = 0$ , where  $A$ ,  $B$  and  $C$  are real numbers, let  $D = B^2 - 4AC$  be the discriminant.

- If  $D > 0$ , there are two distinct real number solutions to the equation.
- If  $D = 0$ , there is one (repeated) real number solution.

Note: "Repeated" here comes from the fact that "both" solutions  $\frac{-B \pm 0}{2A}$  reduce to  $-\frac{B}{2A}$ .

- If  $D < 0$ , there are two non-real solutions which form a complex conjugate pair.

Theorem 1.4.10 tells us that if ever we obtain non-real zeros to a quadratic function with real coefficients, the zeros will be a complex conjugate pair. (Do you see why?) Next, we note that in Example 1.4.6, part 6, we found  $(x - [1 + 2i])(x - [1 - 2i]) = x^2 - 2x + 5$ . This demonstrates that the factor theorem holds even for non-real zeros, i.e.,  $x = 1 + 2i$  is a zero of  $f(x) = x^2 - 2x + 5$ , and, sure enough,  $(x - [1 + 2i])$  is a factor of  $f(x)$ . It turns out that polynomial division works the same way for all complex numbers, real and non-real alike, so the Factor and Remainder Theorems hold as well. But how do we know if a general polynomial has any complex zeros at all? We have many examples of polynomials with no real zeros. Can there be polynomials with no zeros whatsoever? The answer to that last question is "No." and the theorem which provides that answer is The Fundamental Theorem of Algebra.

**Theorem 1.4.11 The Fundamental Theorem of Algebra.**

If  $f$  is a polynomial function with complex number coefficients of degree  $n \geq 1$ , then  $f$  has at least one complex zero.

The Fundamental Theorem of Algebra is an example of an "existence" theorem in mathematics. It guarantees the existence of at least one zero, but gives us no algorithm to use in finding it. It took mathematicians literally hundreds of years to prove the theorem in its full generality, and some of that history can

be found by looking up the Fundamental Theorem on Wikipedia<sup>1</sup>. Note that the Fundamental Theorem of Algebra applies to not only polynomial functions with real coefficients, but to those with complex number coefficients as well.

The Fundamental Theorem of Algebra has since been proved many times, using many different methods, by many mathematicians. There are probably very few, if any, results in mathematics with the variety of proofs this result has. Unfortunately, none of the proofs can be understood within the realm of this text, but if the reader is sufficiently interested, a collection of proofs can be found at [www.cut-the-knot.org/fta/analytic.shtml](http://www.cut-the-knot.org/fta/analytic.shtml)

Suppose  $f$  is a polynomial of degree  $n \geq 1$ . The Fundamental Theorem of Algebra guarantees us at least one complex zero,  $z_1$ , and as such, the Factor Theorem guarantees that  $f(x)$  factors as  $f(x) = (x - z_1) q_1(x)$  for a polynomial function  $q_1$ , of degree exactly  $n - 1$ . If  $n - 1 \geq 1$ , then the Fundamental Theorem of Algebra guarantees a complex zero of  $q_1$  as well, say  $z_2$ , so then the Factor Theorem gives us  $q_1(x) = (x - z_2) q_2(x)$ , and hence  $f(x) = (x - z_1)(x - z_2) q_2(x)$ . We can continue this process exactly  $n$  times, at which point our quotient polynomial  $q_n$  has degree 0 so it's a constant. This argument gives us the following factorization theorem.

**Theorem 1.4.12 Complex Factorization Theorem.**

*Suppose  $f$  is a polynomial function with complex number coefficients. If the degree of  $f$  is  $n$  and  $n \geq 1$ , then  $f$  has exactly  $n$  complex zeros, counting multiplicity. If  $z_1, z_2, \dots, z_k$  are the distinct zeros of  $f$ , with multiplicities  $m_1, m_2, \dots, m_k$ , respectively, then  $f(x) = a(x - z_1)^{m_1}(x - z_2)^{m_2} \cdots (x - z_k)^{m_k}$ .*

<sup>1</sup>[en.wikipedia.org/wiki/Fundamental\\_theorem\\_of\\_algebra](http://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra)



## 1.4.1 Exercises

**Exercise Group.** Use the given complex numbers  $z$  and  $w$  to find and simplify the following:

- |  |  |
|--|--|
| • $z + w$  | • $\frac{w}{z}$  |
| • $zw$   | • $\bar{z}$  |
| • $z^2$  | • $z\bar{z}$   |
| • $\frac{1}{z}$  | • $(\bar{z})^2$  |
| • $\frac{z}{w}$  |  |
| 1. $z = 2 + 3i, w = 4i$  | 2. $z = 1 + i, w = -i$   |
| 3. $z = i, w = -1 + 2i$  | 4. $z = 4i, w = 2 - 2i$  |
| 5. $z = 3 - 5i, w = 2 + 7i$  | 6. $z = -5 + i, w = 4 + 2i$  |
| 7. $z = \sqrt{2} - i\sqrt{2}, w = \sqrt{2} + i\sqrt{2}$                            | 8. $z = 1 - i\sqrt{3}, w = -1 - i\sqrt{3}$   |
| 9. $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i, w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ | 10. $z = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, w = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ |

**Exercise Group.** Simplify the given quantity.

- |                           |                        |
|---------------------------|------------------------|
| 11. $\sqrt{-49}$          | 12. $\sqrt{-9}$        |
| 13. $\sqrt{-25}\sqrt{-4}$ | 14. $\sqrt{(-25)(-4)}$ |
| 15. $\sqrt{-9}\sqrt{-16}$ | 16. $\sqrt{(-9)(-16)}$ |
| 17. $\sqrt{-(-9)}$        | 18. $-\sqrt{(-9)}$     |

**Exercise Group.** We know that  $i^2 = -1$  which means  $i^3 = i^2 \cdot i = (-1) \cdot i = -i$  and  $i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$ . Use this information to simplify the given power of  $i$ .

- |               |               |
|---------------|---------------|
| 19. $i^5$     | 20. $i^6$     |
| 21. $i^7$     | 22. $i^8$     |
| 23. $i^{15}$  | 24. $i^{26}$  |
| 25. $i^{117}$ | 26. $i^{304}$ |

**Exercise Group.** Find all complex solutions.

- |                                     |                                       |
|-------------------------------------|---------------------------------------|
| 27. $3x^2 + 6 = 4x$                 | 28. $15t^2 + 2t + 5 = 3t(t^2 + 1)$    |
| 29. $3y^2 + 4 = y^4$                | 30. $\frac{2}{1-w} = w$               |
| 31. $\frac{y}{3} - \frac{3}{y} = y$ | 32. $\frac{x^3}{2x-1} = \frac{x}{3}$  |
| 33. $x = \frac{2}{\sqrt{5-x}}$      | 34. $\frac{5y^4 + 1}{y^2 - 1} = 3y^2$ |
| 35. $z^4 = 16$                      |                                       |
36. Multiply and simplify:  $(x - [3 - i\sqrt{23}])(x - [3 + i\sqrt{23}])$



## Chapter 2

# Vectors

This chapter introduces a new mathematical object, the *vector*. Defined in Section 2.2, we will see that vectors provide a powerful language for describing quantities that have magnitude and direction aspects. A simple example of such a quantity is force: when applying a force, one is generally interested in how much force is applied (i.e., the magnitude of the force) and the direction in which the force was applied. Vectors will play an important role in many of the subsequent chapters in this text.

This chapter begins with moving our mathematics out of the plane and into “space.” That is, we begin to think mathematically not only in two dimensions, but in three. With this foundation, we can explore vectors both in the plane and in space.

### 2.1 Introduction to Cartesian Coordinates in Space

We reviewed the two-dimensional Cartesian Plane in Section 1.3. This is the familiar background on which much of your high school mathematics played out, and it provides the setting for the Calculus of one variable encountered in an introductory calculus course.

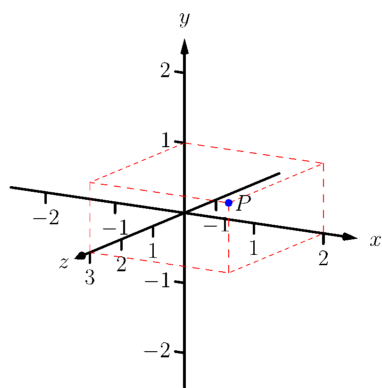
While there is wonderful mathematics to explore in “2D,” we live in a “3D” world and eventually we will want to apply mathematics involving this third dimension. In this section we introduce Cartesian coordinates in space and explore basic surfaces. This will lay a foundation for much of what we do in the remainder of the text.

Each point  $P$  in space can be represented with an ordered triple,  $P = (a, b, c)$ , where  $a$ ,  $b$  and  $c$  represent the relative position of  $P$  along the  $x$ -,  $y$ - and  $z$ -axes, respectively. Each axis is perpendicular to the other two.

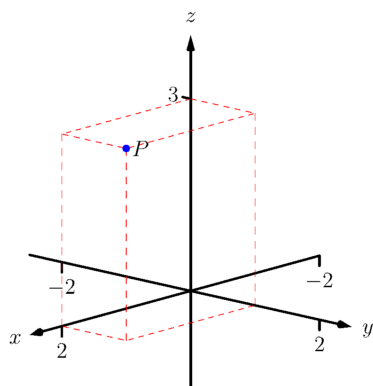
Visualizing points in space on paper can be problematic, as we are trying to represent a 3-dimensional concept on a 2-dimensional medium. We cannot draw three lines representing the three axes in which each line is perpendicular to the other two. Despite this issue, standard conventions exist for plotting shapes in space that we will discuss that are more than adequate.

One convention is that the axes must conform to the *right hand rule*. This rule states that when the index finger of the right hand is extended in the direction of the positive  $x$  axis, and the middle finger (bent “inward” so it is perpendicular to the palm) points along the positive  $y$  axis, then the extended thumb will point in the direction of the positive  $z$  axis. (It may take some thought to verify this, but this system is inherently different from the one created by using the “left hand rule.”)

Until the last section of this chapter we’ll restrict ourselves to vectors in two and three dimensions so that we’re able to understand things visually. However, we’ll also see that the *algebraic* behaviour of vectors is the same in *any* dimension, including dimension four or greater. The only thing that changes is the number of coordinates involved. This is one of the great powers of mathematics: we are aided by our visual imagination, but not limited by it.



**Figure 2.1.1** Plotting the point  $P = (2, 1, 3)$  in space



**Figure 2.1.2** Plotting the point  $P = (2, 1, 3)$  in space with a perspective used in this text

As long as the coordinate axes are positioned so that they follow this rule, it does not matter how the axes are drawn on paper. There are two popular methods that we briefly discuss.

In Figure 2.1.1 we see the point  $P = (2, 1, 3)$  plotted on a set of axes. The basic convention here is that the  $xy$ -plane is drawn in its standard way, with the  $z$ -axis down to the left. The perspective is that the paper represents the  $xy$ -plane and the positive  $z$  axis is coming up, off the page. This method is preferred by many engineers. Because it can be hard to tell where a single point lies in relation to all the axes, dashed lines have been added to let one see how far along each axis the point lies.

One can also consider the  $xy$ -plane as being a horizontal plane in, say, a room, where the positive  $z$ -axis is pointing up. When one steps back and looks at this room, one might draw the axes as shown in Figure 2.1.2. The same point  $P$  is drawn, again with dashed lines. This point of view is preferred by most mathematicians, and is the convention adopted by this text.

Just as the  $x$ - and  $y$ -axes divide the plane into four *quadrants*, the  $x$ -,  $y$ -, and  $z$ -coordinate planes divide space into eight *octants*. The octant in which  $x$ ,  $y$ , and  $z$  are positive is called the *first octant*. We do not name the other seven octants in this text.

### 2.1.1 Measuring Distances

It is of critical importance to know how to measure distances between points in space. The formula for doing so is based on measuring distance in the plane, and is known (in both contexts) as the Euclidean measure of distance.

#### Definition 2.1.3 Distance In Space.

Let  $P = (x_1, y_1, z_1)$  and  $Q = (x_2, y_2, z_2)$  be points in space. The distance  $D$  between  $P$  and  $Q$  is

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

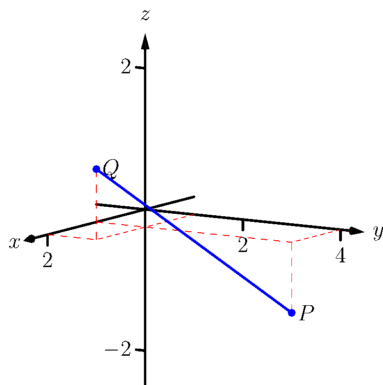
We refer to the line segment that connects points  $P$  and  $Q$  in space as  $\overline{PQ}$ , and refer to the length of this segment as  $\|\overline{PQ}\|$ . The above distance formula allows us to compute the length of this segment.

#### Example 2.1.4 Length of a line segment.

Let  $P = (1, 4, -1)$  and let  $Q = (2, 1, 1)$ . Draw the line segment  $\overline{PQ}$  and find its length.

**Solution.** The points  $P$  and  $Q$  are plotted in Figure 2.1.5; no special consideration need be made to draw the line segment connecting these two points; simply connect them with a straight line. One *cannot* actually measure this line on the page and deduce anything meaningful; its true length must be measured analytically. Applying Definition 2.1.3, we have

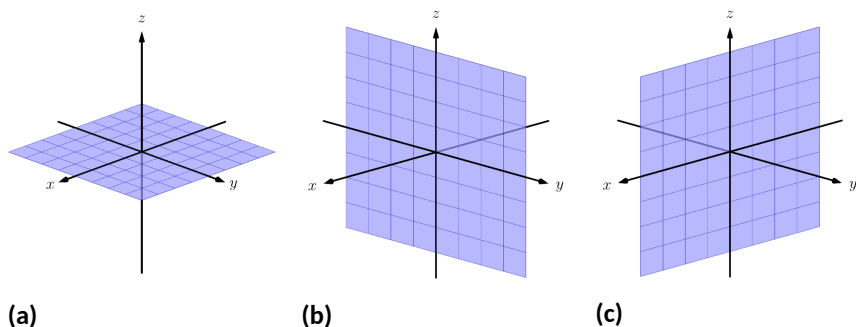
$$\|\overline{PQ}\| = \sqrt{(2 - 1)^2 + (1 - 4)^2 + (1 - (-1))^2} = \sqrt{14} \approx 3.74.$$



**Figure 2.1.5** Plotting points  $P$  and  $Q$  in Example 2.1.4

### 2.1.2 Introduction to Planes in Space

The coordinate axes naturally define three planes (shown in Figure 2.1.6), the *coordinate planes*: the  $xy$ -plane, the  $yz$ -plane and the  $xz$ -plane. The  $xy$ -plane is characterized as the set of all points in space where the  $z$ -value is 0. This, in fact, gives us an equation that describes this plane:  $z = 0$ . Likewise, the  $xz$ -plane is all points where the  $y$ -value is 0, characterized by  $y = 0$ .



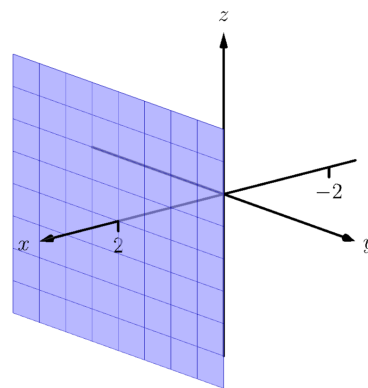
**Figure 2.1.6** The  $xy$ -plane in (a), the  $yz$ -plane in (b) and the  $xz$ -plane in (c)

The equation  $x = 2$  describes all points in space where the  $x$ -value is 2. This is a plane, parallel to the  $yz$ -coordinate plane, shown in Figure 2.1.7.

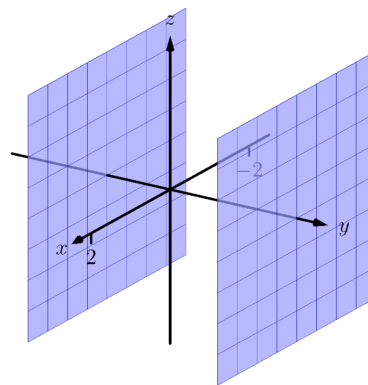
#### Example 2.1.8 Regions defined by planes.

Sketch the region defined by the inequalities  $-1 \leq y \leq 2$ .

**Solution.** The region is all points between the planes  $y = -1$  and  $y = 2$ . These planes are sketched in Figure 2.1.9, which are parallel to the  $xz$ -plane. Thus the region extends infinitely in the  $x$  and  $z$  directions, and is bounded by planes in the  $y$  direction.



**Figure 2.1.7** The plane  $x = 2$



**Figure 2.1.9** Sketching the boundaries of a region in Example 2.1.8

**2.1.3 Exercises**

1. The points  $A = (1, 4, 2)$ ,  $B = (2, 6, 3)$  and  $C = (4, 3, 1)$  form a triangle in space. Find the distances between each pair of points and determine if the triangle is a right triangle.

$$\|\overline{AB}\| = \underline{\hspace{2cm}}$$

$$\|\overline{BC}\| = \underline{\hspace{2cm}}$$

$$\|\overline{CA}\| = \underline{\hspace{2cm}}$$

The three points ☐ ? ☐ do ☐ do not) form a right triangle.

2. The points  $A = (1, 1, 3)$ ,  $B = (3, 2, 7)$ ,  $C = (2, 0, 8)$  and  $D = (0, -1, 4)$  form a quadrilateral  $ABCD$  in space. Is this a parallelogram?

**Exercise Group.** In the following exercises, describe the region in space defined by the inequalities.

3.  $0 \leq x \leq 3$

4.  $x \geq 0, y \geq 0, z \geq 0$

5.  $y \geq 3$

## 2.2 An Introduction to Vectors

Many quantities we think about daily can be described by a single number: temperature, speed, cost, weight and height. There are also many other concepts we encounter daily that cannot be described with just one number. For instance, a weather forecaster often describes wind with its speed and its direction (“... with winds from the southeast gusting up to 30 mph ...”). When applying a force, we are concerned with both the magnitude and direction of that force. In both of these examples, *direction* is important. Because of this, we study *vectors*, mathematical objects that convey both magnitude and direction information.

One “bare-bones” definition of a vector is based on what we wrote above: “a vector is a mathematical object with magnitude and direction parameters.” This definition leaves much to be desired, as it gives no indication as to how such an object is to be used. Several other definitions exist; we choose here a definition rooted in a geometric visualization of vectors. It is very simplistic but readily permits further investigation.

### Definition 2.2.1 Vector.

A **vector** is a directed line segment.

Given points  $P$  and  $Q$  (either in the plane or in space), we denote with  $\overrightarrow{PQ}$  the vector **from**  $P$  **to**  $Q$ . The point  $P$  is said to be the **initial point** of the vector, and the point  $Q$  is the **terminal point**.

The **magnitude**, **length** or **norm** of  $\overrightarrow{PQ}$  is the length of the line segment  $\overline{PQ}$ :  $\|\overrightarrow{PQ}\| = \|\overline{PQ}\|$ .

Two vectors are **equal** if they have the same magnitude and direction.

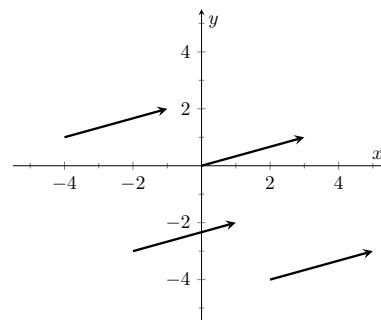


Figure 2.2.2 Drawing the same vector with different initial points

Figure 2.2.2 shows multiple instances of the same vector. Each directed line segment has the same direction and length (magnitude), hence each is the same vector.

We use  $\mathbb{R}^2$  (pronounced “r two”) to represent all the vectors in the plane, and use  $\mathbb{R}^3$  (pronounced “r three”) to represent all the vectors in space.

Consider the vectors  $\overrightarrow{PQ}$  and  $\overrightarrow{RS}$  as shown in Figure 2.2.3. The vectors look to be equal; that is, they seem to have the same length and direction. Indeed, they are. Both vectors move 2 units to the right and 1 unit up from the initial point to reach the terminal point. One can analyze this movement to measure the magnitude of the vector, and the movement itself gives direction information (one could also measure the slope of the line passing through  $P$  and  $Q$  or  $R$  and  $S$ ). Since they have the same length and direction, these two vectors are equal.

This demonstrates that inherently all we care about is *displacement*; that is, how far in the  $x$ ,  $y$  and possibly  $z$  directions the terminal point is from the initial point. Both the vectors  $\overrightarrow{PQ}$  and  $\overrightarrow{RS}$  in Figure 2.2.3 have an  $x$ -displacement of 2 and a  $y$ -displacement of 1. This suggests a standard way of describing vectors in the plane. A vector whose  $x$ -displacement is  $a$  and whose  $y$ -displacement is  $b$  will have terminal point  $(a, b)$  when the initial point is the origin,  $(0, 0)$ . This leads us to a definition of a standard and concise way of referring to vectors.

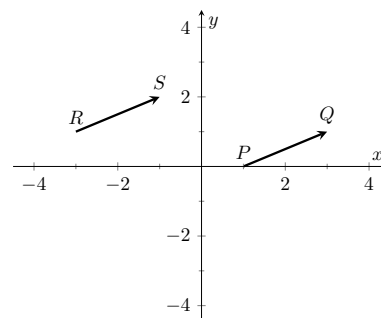


Figure 2.2.3 Illustrating how equal vectors have the same displacement

### Definition 2.2.4 Component Form of a Vector.

1. The *component form* of a vector  $\vec{v}$  in  $\mathbb{R}^2$ , whose terminal point is

$(a, b)$  when its initial point is  $(0, 0)$ , is  $\begin{bmatrix} a \\ b \end{bmatrix}$ .

2. The *component form* of a vector  $\vec{v}$  in  $\mathbb{R}^3$ , whose terminal point is

$(a, b, c)$  when its initial point is  $(0, 0, 0)$ , is  $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ .

The numbers  $a$ ,  $b$  (and  $c$ , respectively) are the **components** of  $\vec{v}$ .

It follows from the definition that the component form of the vector  $\overrightarrow{PQ}$ , where  $P = (x_1, y_1)$  and  $Q = (x_2, y_2)$  is

$$\overrightarrow{PQ} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix};$$

in space, where  $P = (x_1, y_1, z_1)$  and  $Q = (x_2, y_2, z_2)$ , the component form of  $\overrightarrow{PQ}$  is

$$\overrightarrow{PQ} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}.$$

We practice using this notation in the following example.

**Example 2.2.5 Using component form notation for vectors.**

1. Sketch the vector  $\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$  starting at  $P = (3, 2)$  and find its magnitude.
2. Find the component form of the vector  $\vec{w}$  whose initial point is  $R = (-3, -2)$  and whose terminal point is  $S = (-1, 2)$ .
3. Sketch the vector  $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$  starting at the point  $Q = (1, 1, 1)$  and find its magnitude.

**Solution.**

1. Using  $P$  as the initial point, we move 2 units in the positive  $x$ -direction and  $-1$  units in the positive  $y$ -direction to arrive at the terminal point  $P' = (5, 1)$ , as drawn in Figure 2.2.6(a). The magnitude of  $\vec{v}$  is determined directly from the component form:

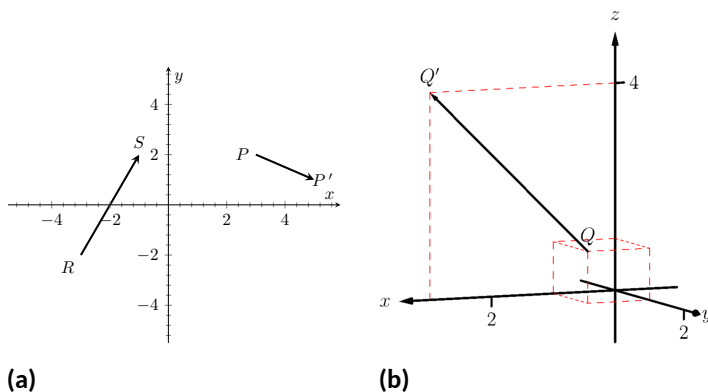
$$\|\vec{v}\| = \sqrt{2^2 + (-1)^2} = \sqrt{5}.$$

The component form of a vector allows us to identify a point  $(a, b)$  (or  $(a, b, c)$ ) with the corre-

sponding vector  $\begin{bmatrix} a \\ b \end{bmatrix}$  (or  $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ ),

so that vectors and points contain essentially the same information, presented in different contexts. This is why mathematicians don't mind using the notation  $\mathbb{R}^n$  to refer to both a set of vectors and the set of points containing those vectors.





(a) (b)

**Figure 2.2.6** Graphing vectors in Example 2.2.5

2. Using the note following Definition 2.2.4, we have

$$\overrightarrow{RS} = \begin{bmatrix} -1 - (-3) \\ 2 - (-2) \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

One can readily see from Figure 2.2.6(a) that the  $x$ - and  $y$ -displacement of  $\overrightarrow{RS}$  is 2 and 4, respectively, as the component form suggests.

3. Using  $Q$  as the initial point, we move 2 units in the positive  $x$ -direction,  $-1$  unit in the positive  $y$ -direction, and 3 units in the positive  $z$ -direction to arrive at the terminal point  $Q' = (3, 0, 4)$ , illustrated in Figure 2.2.6(b). The magnitude of  $\vec{u}$  is:

$$\|\vec{u}\| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}.$$

Now that we have defined vectors, and have created a nice notation by which to describe them, we start considering how vectors interact with each other. That is, we define an *algebra* on vectors.

**Definition 2.2.7 Vector Algebra.**

1. Let  $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  be vectors in  $\mathbb{R}^2$ , and let  $c$  be a scalar.

- (a) The addition, or sum, of the vectors  $\vec{u}$  and  $\vec{v}$  is the vector

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}.$$

- (b) The multiplication of a scalar  $c$  and a vector  $\vec{v}$  is the vector

$$c\vec{v} = c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}.$$

2. Let  $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$  be vectors in  $\mathbb{R}^3$ , and let  $c$  be a scalar.

In this section we restrict ourselves to vectors in two and three dimensions, since our ability to visualize is limited to these dimensions. However, our ability to work with vectors algebraically extends to any number of dimensions. For a general treatment of vectors in  $\mathbb{R}^n$ , see Section 2.7.

(a) The addition, or sum, of the vectors  $\vec{u}$  and  $\vec{v}$  is the vector

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}.$$

(b) The multiplication of a scalar  $c$  and a vector  $\vec{v}$  is the vector

$$c\vec{v} = c \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \\ cv_3 \end{bmatrix}.$$

In short, we say addition and scalar multiplication are computed “component-wise.”

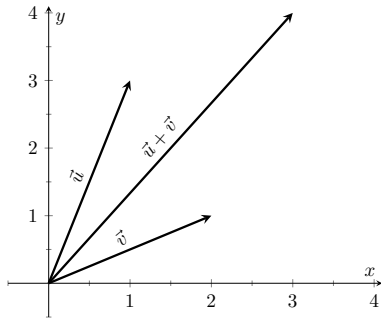
### Example 2.2.8 Adding vectors.

Sketch the vectors  $\vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and  $\vec{u} + \vec{v}$  all with initial point at the origin.

**Solution.** We first compute  $\vec{u} + \vec{v}$ .

$$\begin{aligned} \vec{u} + \vec{v} &= \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 4 \end{bmatrix}. \end{aligned}$$

These are all sketched in Figure 2.2.9.



**Figure 2.2.9** Graphing the sum of vectors in Example 2.2.8

As vectors convey magnitude and direction information, the sum of vectors also convey length and magnitude information. Adding  $\vec{u} + \vec{v}$  suggests the following idea-

“Starting at an initial point, go out  $\vec{u}$ , then go out  $\vec{v}$ .”

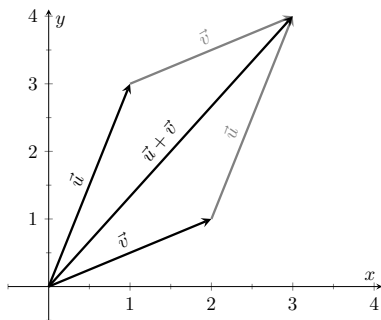
This idea is sketched in Figure 2.2.10, where the initial point of  $\vec{v}$  is the terminal point of  $\vec{u}$ . This is known as the “Head to Tail Rule” of adding vectors. Vector addition is very important. For instance, if the vectors  $\vec{u}$  and  $\vec{v}$  represent forces acting on a body, the sum  $\vec{u} + \vec{v}$  gives the resulting force. Because of various physical applications of vector addition, the sum  $\vec{u} + \vec{v}$  is often referred to as the *resultant vector*, or just the “resultant.”

Analytically, it is easy to see that  $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ . Figure 2.2.10 also gives a graphical representation of this, using gray vectors. Note that the vectors  $\vec{u}$  and  $\vec{v}$ , when arranged as in the figure, form a parallelogram. Because of this, the Head to Tail Rule is also known as the *Parallelogram Law*: the vector  $\vec{u} + \vec{v}$  is defined by forming the parallelogram defined by the vectors  $\vec{u}$  and  $\vec{v}$ ; the initial point of  $\vec{u} + \vec{v}$  is the common initial point of parallelogram, and the terminal point of the sum is the common terminal point of the parallelogram.

While not illustrated here, the Head to Tail Rule and Parallelogram Law hold for vectors in  $\mathbb{R}^3$  as well.

It follows from the properties of the real numbers and Definition 2.2.7 that

$$\vec{u} - \vec{v} = \vec{u} + (-1)\vec{v}.$$



**Figure 2.2.10** Illustrating how to add vectors using the Head to Tail Rule and Parallelogram Law

The Parallelogram Law gives us a good way to visualize this subtraction. We demonstrate this in the following example.

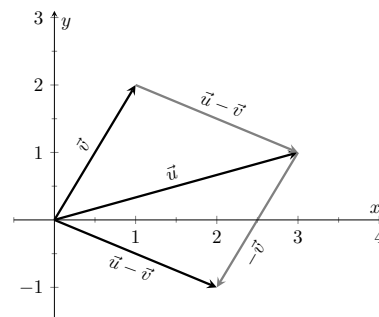
**Example 2.2.11 Vector Subtraction.**

Let  $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ . Compute and sketch  $\vec{u} - \vec{v}$ .

**Solution.** The computation of  $\vec{u} - \vec{v}$  is straightforward, and we show all steps below. Usually the formal step of multiplying by  $(-1)$  is omitted and we “just subtract.”

$$\begin{aligned}\vec{u} - \vec{v} &= \vec{u} + (-1)\vec{v} \\ &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -1 \end{bmatrix}.\end{aligned}$$

Figure 2.2.12 illustrates, using the Head to Tail Rule, how the subtraction can be viewed as the sum  $\vec{u} + (-\vec{v})$ . The figure also illustrates how  $\vec{u} - \vec{v}$  can be obtained by looking only at the terminal points of  $\vec{u}$  and  $\vec{v}$  (when their initial points are the same).



**Figure 2.2.12** Illustrating how to subtract vectors graphically

**Example 2.2.13 Scaling vectors.**

1. Sketch the vectors  $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and  $2\vec{v}$  with initial point at the origin.
2. Compute the magnitudes of  $\vec{v}$  and  $2\vec{v}$ .

**Solution.**

1. We compute  $2\vec{v}$ :

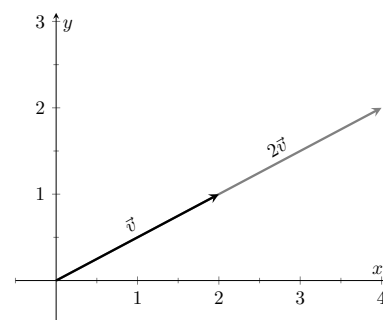
$$\begin{aligned}2\vec{v} &= 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 2 \end{bmatrix}.\end{aligned}$$

Both  $\vec{v}$  and  $2\vec{v}$  are sketched in Figure 2.2.14. Make note that  $2\vec{v}$  does not start at the terminal point of  $\vec{v}$ ; rather, its initial point is also the origin.

2. The figure suggests that  $2\vec{v}$  is twice as long as  $\vec{v}$ . We compute their magnitudes to confirm this.

$$\begin{aligned}\|\vec{v}\| &= \sqrt{2^2 + 1^2} \\ &= \sqrt{5}. \\ \|2\vec{v}\| &= \sqrt{4^2 + 2^2} \\ &= \sqrt{20} \\ &= \sqrt{4 \cdot 5} = 2\sqrt{5}.\end{aligned}$$

As we suspected,  $2\vec{v}$  is twice as long as  $\vec{v}$ .



**Figure 2.2.14** Graphing vectors  $\vec{v}$  and  $2\vec{v}$  in Example 2.2.13

Scalar multiplication of a vector  $\vec{v}$  produces a new vector that is in the same (or opposite) direction, but of a possibly different length. This leads us to the following definition.

**Definition 2.2.15 Parallel Vectors.**

We say that a vector  $\vec{w}$  is **parallel** to a vector  $\vec{v}$  if there exists a scalar  $c$  such that  $\vec{w} = c\vec{v}$ .

The *zero vector* is the vector whose initial point is also its terminal point. It is denoted by  $\vec{0}$ . Its component form, in  $\mathbb{R}^2$ , is  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ; in  $\mathbb{R}^3$ , it is  $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ . Usually the

context makes it clear whether  $\vec{0}$  is referring to a vector in the plane or in space.

Our examples have illustrated key principles in vector algebra: how to add and subtract vectors and how to multiply vectors by a scalar. The following theorem states formally the properties of these operations.

**Theorem 2.2.16 Properties of Vector Operations.**

The following are true for all scalars  $c$  and  $d$ , and for all vectors  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$ , where  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  are all in  $\mathbb{R}^2$  or where  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  are all in  $\mathbb{R}^3$ :

1.  $\vec{u} + \vec{v} = \vec{v} + \vec{u}$  *Commutative Property*
2.  $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$  *Associative Property*
3.  $\vec{v} + \vec{0} = \vec{v}$  *Additive Identity*
4.  $(cd)\vec{v} = c(d\vec{v})$
5.  $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$  *Distributive Property*
6.  $(c + d)\vec{v} = c\vec{v} + d\vec{v}$  *Distributive Property*
7.  $0\vec{v} = \vec{0}$
8.  $\|c\vec{v}\| = |c| \cdot \|\vec{v}\|$
9.  $\|\vec{u}\| = 0$  if, and only if,  $\vec{u} = \vec{0}$ .

As stated before, each nonzero vector  $\vec{v}$  conveys magnitude and direction information. We have a method of extracting the magnitude, which we write as  $\|\vec{v}\|$ . *Unit vectors* are a way of extracting just the direction information from a vector.

**Definition 2.2.17 Unit Vector.**

A **unit vector** is a vector  $\vec{v}$  with a magnitude of 1; that is,

$$\|\vec{v}\| = 1.$$

Consider this scenario: you are given a vector  $\vec{v}$  and are told to create a vector of length 10 in the direction of  $\vec{v}$ . How does one do that? If we knew that  $\vec{u}$  was the unit vector in the direction of  $\vec{v}$ , the answer would be easy:  $10\vec{u}$ . So how do we find  $\vec{u}$ ?

Property 8 of Theorem 2.2.16 holds the key. If we divide  $\vec{v}$  by its magnitude, it becomes a vector of length 1. Consider:

$$\left\| \frac{1}{\|\vec{v}\|} \vec{v} \right\| = \frac{1}{\|\vec{v}\|} \|\vec{v}\| \quad (\text{we can pull out } \frac{1}{\|\vec{v}\|} \text{ as it is a positive scalar})$$

$$= 1.$$

So the vector of length 10 in the direction of  $\vec{v}$  is  $10 \frac{1}{\|\vec{v}\|} \vec{v}$ . An example will make this more clear.

**Example 2.2.18 Using Unit Vectors.**

Let  $\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$  and let  $\vec{w} = \begin{bmatrix} 1 \\ 2, 2 \end{bmatrix}$ .

1. Find the unit vector in the direction of  $\vec{v}$ .
2. Find the unit vector in the direction of  $\vec{w}$ .
3. Find the vector in the direction of  $\vec{v}$  with magnitude 5.

**Solution.**

1. We find  $\|\vec{v}\| = \sqrt{10}$ . So the unit vector  $\vec{u}$  in the direction of  $\vec{v}$  is

$$\vec{u} = \frac{1}{\sqrt{10}} \vec{v} = \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}.$$

2. We find  $\|\vec{w}\| = 3$ , so the unit vector  $\vec{z}$  in the direction of  $\vec{w}$  is

$$\vec{z} = \frac{1}{3} \vec{w} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}.$$

3. To create a vector with magnitude 5 in the direction of  $\vec{v}$ , we multiply the unit vector  $\vec{u}$  by 5. Thus  $5\vec{u} = \begin{bmatrix} 15/\sqrt{10} \\ 5/\sqrt{10} \end{bmatrix}$  is the vector we seek. This is sketched in Figure 2.2.19.

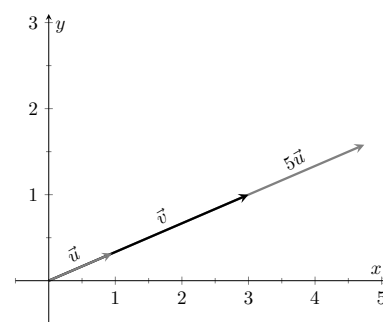
The basic formation of the unit vector  $\vec{u}$  in the direction of a vector  $\vec{v}$  leads to a interesting equation. It is:

$$\vec{v} = \|\vec{v}\| \frac{1}{\|\vec{v}\|} \vec{v}.$$

We rewrite the equation with parentheses to make a point:

$$\vec{v} = \underbrace{\|\vec{v}\|}_{\text{magnitude}} \cdot \underbrace{\left( \frac{1}{\|\vec{v}\|} \vec{v} \right)}_{\text{direction}}.$$

This equation illustrates the fact that a nonzero vector has both magnitude and direction, where we view a unit vector as supplying *only* direction information.



**Figure 2.2.19** Graphing vectors in Example 2.2.18. All vectors shown have their initial point at the origin

**Direction and the zero vector.**  $\vec{0}$  is directionless; because  $\|\vec{0}\| = 0$ , there is no unit vector in the “direction” of  $\vec{0}$ .

Some texts define two vectors as being parallel if one is a scalar multiple of the other. By this definition,  $\vec{0}$  is parallel to all vectors as  $\vec{0} = 0\vec{v}$  for all  $\vec{v}$ .

We define what it means for two vectors to be perpendicular in Definition 2.3.11, which is written to exclude  $\vec{0}$ . It could be written to include  $\vec{0}$ ; by such a definition,  $\vec{0}$  is perpendicular to all vectors. While counter-intuitive, it is mathematically sound to allow  $\vec{0}$  to be both parallel and perpendicular to all vectors.

We prefer the given definition of parallel as it is grounded in the fact that unit vectors provide direction information. One may adopt the convention that  $\vec{0}$  is parallel to all vectors if they desire. (See also the aside in Section 2.4.)

If one graphed all unit vectors in  $\mathbb{R}^2$  with the initial point at the origin, then the terminal points would all lie on the unit circle. Based on what we know from trigonometry, we can then say that the component form of all unit vectors in  $\mathbb{R}^2$  is  $\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$  for some angle  $\theta$ .

A similar construction in  $\mathbb{R}^3$  shows that the terminal points all lie on the unit sphere. These vectors also have a particular component form, but its derivation is not as straightforward as the one for unit vectors in  $\mathbb{R}^2$ . Important concepts about unit vectors are given in the following Key Idea.

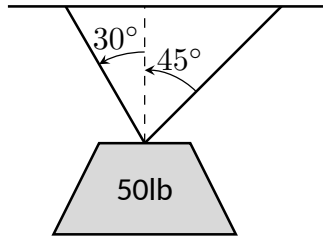
**Key Idea 2.2.20 Unit Vectors.**

1. The unit vector in the direction of a nonzero vector  $\vec{v}$  is

$$\vec{u} = \frac{1}{\|\vec{v}\|} \vec{v}.$$

2. A vector  $\vec{u}$  in  $\mathbb{R}^2$  is a unit vector if, and only if, its component form is  $\begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$  for some angle  $\theta$ .
3. A vector  $\vec{u}$  in  $\mathbb{R}^3$  is a unit vector if, and only if, its component form is  $\begin{bmatrix} \sin(\theta) \cos(\varphi) \\ \sin(\theta) \sin(\varphi) \\ \cos(\theta) \end{bmatrix}$  for some angles  $\theta$  and  $\varphi$ .

These formulas can come in handy in a variety of situations, especially the formula for unit vectors in the plane.



**Figure 2.2.22** A diagram of a weight hanging from 2 chains in Example 2.2.21

**Example 2.2.21 Finding Component Forces.**

Consider a weight of 50lb hanging from two chains, as shown in Figure 2.2.22. One chain makes an angle of  $30^\circ$  with the vertical, and the other an angle of  $45^\circ$ . Find the force applied to each chain.

**Solution.** Knowing that gravity is pulling the 50lb weight straight down, we can create a vector  $\vec{F}$  to represent this force.

$$\vec{F} = 50 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -50 \end{bmatrix}.$$

We can view each chain as “pulling” the weight up, preventing it from falling. We can represent the force from each chain with a vector. Let  $\vec{F}_1$  represent the force from the chain making an angle of  $30^\circ$  with the vertical, and let  $\vec{F}_2$  represent the force from the other chain. Convert all angles to be measured from the horizontal (as shown in Figure 2.2.23), and apply Key Idea 2.2.20. As we do not yet know the magnitudes of these vectors, (that is the problem at hand), we use  $m_1$  and  $m_2$  to represent them.

$$\vec{F}_1 = m_1 \begin{bmatrix} \cos(120^\circ) \\ \sin(120^\circ) \end{bmatrix}$$

$$\vec{F}_2 = m_2 \begin{bmatrix} \cos(45^\circ) \\ \sin(45^\circ) \end{bmatrix}$$

As the weight is not moving, we know the sum of the forces is  $\vec{0}$ . This gives:

$$\vec{F} + \vec{F}_1 + \vec{F}_2 = \vec{0}$$

$$\begin{bmatrix} 0 \\ -50 \end{bmatrix} + m_1 \begin{bmatrix} \cos(120^\circ) \\ \sin(120^\circ) \end{bmatrix} + m_2 \begin{bmatrix} \cos(45^\circ) \\ \sin(45^\circ) \end{bmatrix} = \vec{0}$$

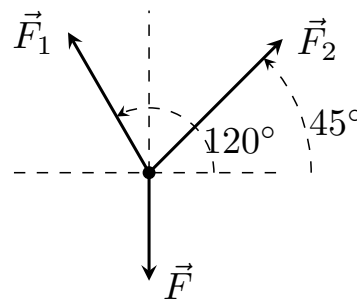
The sum of the entries in the first component is 0, and the sum of the entries in the second component is also 0. This leads us to the following two equations:

$$\begin{aligned} m_1 \cos(120^\circ) + m_2 \cos(45^\circ) &= 0 \\ m_1 \sin(120^\circ) + m_2 \sin(45^\circ) &= 50 \end{aligned}$$

This is a simple 2-equation, 2-unknown system of linear equations. We leave it to the reader to verify that the solution is

$$m_1 = 50(\sqrt{3} - 1) \approx 36.6; \quad m_2 = \frac{50\sqrt{2}}{1 + \sqrt{3}} \approx 25.88.$$

It might seem odd that the sum of the forces applied to the chains is more than 50lb. We leave it to a physics class to discuss the full details, but offer this short explanation. Our equations were established so that the *vertical* components of each force sums to 50lb, thus supporting the weight. Since the chains are at an angle, they also pull against each other, creating an “additional” horizontal force while holding the weight in place.



**Figure 2.2.23** A diagram of the force vectors from Example 2.2.21

Unit vectors were very important in the previous calculation; they allowed us to define a vector in the proper direction but with an unknown magnitude. Our computations were then computed component-wise. Because such calculations are often necessary, the *standard unit vectors* can be useful.

**Definition 2.2.24 Standard Unit Vectors.**

1. In  $\mathbb{R}^2$ , the standard unit vectors are

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

2. In  $\mathbb{R}^3$ , the standard unit vectors are

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

**Example 2.2.25 Using standard unit vectors.**

1. Rewrite  $\vec{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$  using the standard unit vectors.
2. Rewrite  $\vec{w} = 4\vec{i} - 5\vec{j} + 2\vec{k}$  in component form.

**Solution.**

$$\begin{aligned}
 \vec{v} &= \begin{bmatrix} 2 \\ -3 \end{bmatrix} \\
 1. \quad &= \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} \\
 &= 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
 &= 2\vec{i} - 3\vec{j} \\
 \vec{w} &= 4\vec{i} - 5\vec{j} + 2\vec{k} \\
 2. \quad &= \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \\
 &= \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}
 \end{aligned}$$

These two examples demonstrate that converting between component form and the standard unit vectors is rather straightforward. Many mathematicians prefer component form, and it is the preferred notation in this text. Many engineers prefer using the standard unit vectors, and many engineering text use that notation.

The algebra we have applied to vectors is already demonstrating itself to be very useful. There are two more fundamental operations we can perform with vectors, the *dot product* and the *cross product*. The next two sections explore each in turn.



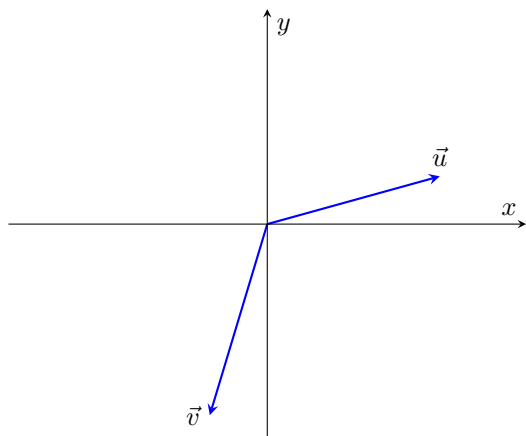
## 2.2.1 Exercises

**Exercise Group.** In the following exercises, points  $P$  and  $Q$  are given. Write the vector  $\overrightarrow{PQ}$  in component form and using the standard unit vectors.

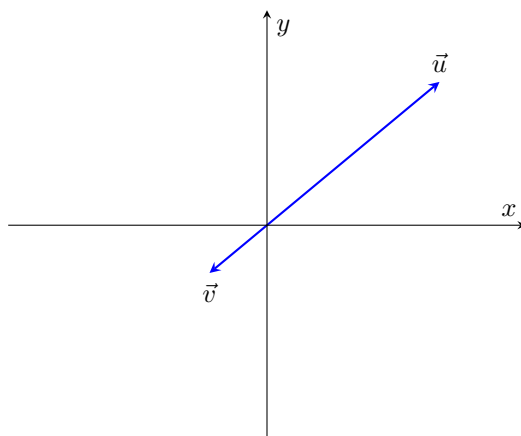
1. If  $P = (2, -1)$  and  $Q = (3, 5)$ , write the vector  $\overrightarrow{PQ}$ :
    - (a) in component form.
    - (b) using the standard unit vectors.
  2. If  $P = (3, 2)$  and  $Q = (7, -2)$ , write the vector  $\overrightarrow{PQ}$ :
    - (a) in component form.
    - (b) using the standard unit vectors.
  3. If  $P = (0, 3, -1)$  and  $Q = (6, 2, 5)$ , write the vector  $\overrightarrow{PQ}$ :
    - (a) in component form.
    - (b) using the standard unit vectors.
  4. If  $P = (2, 1, 2)$  and  $Q = (4, 3, 2)$ , write the vector  $\overrightarrow{PQ}$ :
    - (a) in component form.
    - (b) using the standard unit vectors.
5. Let  $\vec{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .
- (a) Find  $\vec{u} + \vec{v}$ ,  $\vec{u} - \vec{v}$ ,  $2\vec{u} - 3\vec{v}$ .
  - (b) Sketch the above vectors on the same axes, along with  $\vec{u}$  and  $\vec{v}$ .
  - (c) Find  $\vec{x}$  where  $\vec{u} + \vec{x} = 2\vec{v} - \vec{x}$ .
6. Let  $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ .
- (a) Find  $\vec{u} + \vec{v}$ ,  $\vec{u} - \vec{v}$ ,  $\pi\vec{u} - \sqrt{2}\vec{v}$ .
  - (b) Sketch the above vectors on the same axes, along with  $\vec{u}$  and  $\vec{v}$ .
  - (c) Find  $\vec{x}$  where  $\vec{u} + \vec{x} = \vec{v} + 2\vec{x}$ .

**Exercise Group.** In the following exercises, sketch  $\vec{u}$ ,  $\vec{v}$ ,  $\vec{u} + \vec{v}$  and  $\vec{u} - \vec{v}$  on the same axes.

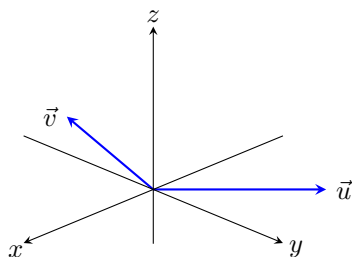
7.



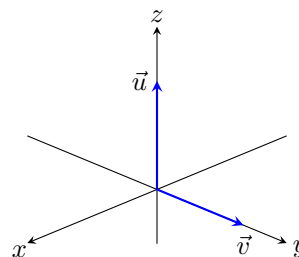
8.



9.



10.



**Exercise Group.** In the following exercises, find  $\|\vec{u}\|$ ,  $\|\vec{v}\|$ ,  $\|\vec{u} + \vec{v}\|$  and  $\|\vec{u} - \vec{v}\|$ .

11.  $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$

12.  $\vec{u} = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$

13.  $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} -3 \\ -6 \end{bmatrix}.$

14.  $\vec{u} = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}, \vec{v} = \begin{bmatrix} 10 \\ -15 \\ 30 \end{bmatrix}.$

15. Under what conditions is  $\|\vec{u}\| + \|\vec{v}\| = \|\vec{u} + \vec{v}\|$ ?

**Exercise Group.** In the following exercises, find the unit vector  $\vec{u}$  in the direction of  $\vec{v}$ .

16. Find the unit vector  $\vec{u}$  in the direction of  $\vec{v} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$

17. Find the unit vector  $\vec{u}$  in the direction of  $\vec{v} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}.$

18. Find the unit vector  $\vec{u}$  in the direction of  $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$

19. Find the unit vector  $\vec{u}$  in the direction of  $\vec{v} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}.$

20. Find the unit vector in the first quadrant of  $\mathbb{R}^2$  that makes a  $50^\circ$  angle with the  $x$ -axis.

21. Find the unit vector in the second quadrant of  $\mathbb{R}^2$  that makes a  $30^\circ$  angle with the  $y$ -axis.

22. Verify, from Key Idea 2.2.20, that

$$\vec{u} = \begin{bmatrix} \sin(\theta) \cos(\varphi) \\ \sin(\theta) \sin(\varphi) \\ \cos(\theta) \end{bmatrix}$$

is a unit vector for all angles  $\theta$  and  $\varphi$ .

## 2.3 The Dot Product

The previous section introduced vectors and described how to add them together and how to multiply them by scalars. This section introduces a multiplication on vectors called the *dot product*.

### Definition 2.3.1 Dot Product.

1. Let  $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  in  $\mathbb{R}^2$ . The *dot product* of  $\vec{u}$  and  $\vec{v}$ , denoted  $\vec{u} \cdot \vec{v}$ , is

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2.$$

2. Let  $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$  in  $\mathbb{R}^3$ . The *dot product* of  $\vec{u}$  and  $\vec{v}$ , denoted  $\vec{u} \cdot \vec{v}$ , is

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Note how this product of vectors returns a *scalar*, not another vector. We practice evaluating a dot product in the following example, then we will discuss why this product is useful.

### Example 2.3.2 Evaluating dot products.

1. Let  $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$  in  $\mathbb{R}^2$ . Find  $\vec{u} \cdot \vec{v}$ .
2. Let  $\vec{x} = \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix}$  and  $\vec{y} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$  in  $\mathbb{R}^3$ . Find  $\vec{x} \cdot \vec{y}$ .

### Solution.

1. Using Definition 2.3.1, we have

$$\vec{u} \cdot \vec{v} = 1(3) + 2(-1) = 1.$$

2. Using the definition, we have

$$\vec{x} \cdot \vec{y} = 2(-1) - 2(0) + 5(3) = 13.$$

The dot product, as shown by the preceding example, is very simple to evaluate. It is only the sum of products. While the definition gives no hint as to why we would care about this operation, there is an amazing connection between the dot product and angles formed by the vectors. Before stating this connection, we give a theorem stating some of the properties of the dot product.

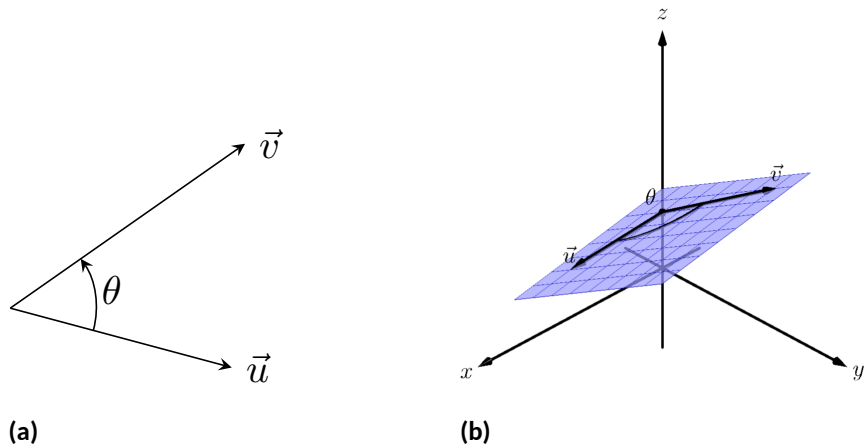
**Theorem 2.3.3 Properties of the Dot Product.**

Let  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  be vectors in  $\mathbb{R}^2$  or  $\mathbb{R}^3$  and let  $c$  be a scalar.

1.  $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$  {Commutative Property}
2.  $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$  {Distributive Property}
3.  $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
4.  $\vec{0} \cdot \vec{v} = 0$
5.  $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

The last statement of the theorem makes a handy connection between the magnitude of a vector and the dot product with itself. Our definition and theorem give properties of the dot product, but we are still likely wondering “What does the dot product *mean*?” It is helpful to understand that the dot product of a vector with itself is connected to its magnitude.

The next theorem extends this understanding by connecting the dot product to magnitudes and angles. Given vectors  $\vec{u}$  and  $\vec{v}$  in the plane, an angle  $\theta$  is clearly formed when  $\vec{u}$  and  $\vec{v}$  are drawn with the same initial point as illustrated in Figure 2.3.4(a). (We always take  $\theta$  to be the angle in  $[0, \pi]$  as two angles are actually created.)



**Figure 2.3.4** Illustrating the angle formed by two vectors with the same initial point

The same is also true of 2 vectors in space: given  $\vec{u}$  and  $\vec{v}$  in  $\mathbb{R}^3$  with the same initial point, there is a plane that contains both  $\vec{u}$  and  $\vec{v}$ . (When  $\vec{u}$  and  $\vec{v}$  are co-linear, there are infinitely many planes that contain both vectors.) In that plane, we can again find an angle  $\theta$  between them (and again,  $0 \leq \theta \leq \pi$ ). This is illustrated in Figure 2.3.4(b).

The following theorem connects this angle  $\theta$  to the dot product of  $\vec{u}$  and  $\vec{v}$ .

**Theorem 2.3.5 The Dot Product and Angles.**

Let  $\vec{u}$  and  $\vec{v}$  be nonzero vectors in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ . Then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos(\theta),$$

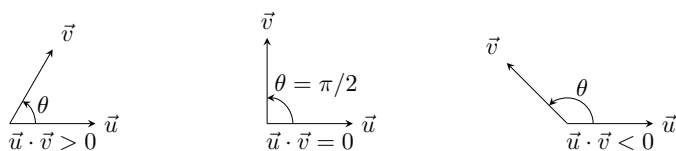
where  $\theta$ ,  $0 \leq \theta \leq \pi$ , is the angle between  $\vec{u}$  and  $\vec{v}$ .

Using Theorem 2.3.3, we can rewrite this theorem as

$$\frac{\vec{u}}{\|\vec{u}\|} \cdot \frac{\vec{v}}{\|\vec{v}\|} = \cos(\theta).$$

Note how on the left hand side of the equation, we are computing the dot product of two unit vectors. Recalling that unit vectors essentially only provide direction information, we can informally restate Theorem 2.3.5 as saying “The dot product of two directions gives the cosine of the angle between them.”

When  $\theta$  is an acute angle (i.e.,  $0 \leq \theta < \pi/2$ ),  $\cos(\theta)$  is positive; when  $\theta = \pi/2$ ,  $\cos(\theta) = 0$ ; when  $\theta$  is an obtuse angle ( $\pi/2 < \theta \leq \pi$ ),  $\cos(\theta)$  is negative. Thus the sign of the dot product gives a general indication of the angle between the vectors, illustrated in Figure 2.3.6.



**Figure 2.3.6** Illustrating the relationship between the angle between vectors and the sign of their dot product

We can use Theorem 2.3.5 to compute the dot product, but generally this theorem is used to find the angle between known vectors (since the dot product is generally easy to compute). To this end, we rewrite the theorem's equation as

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \Leftrightarrow \theta = \cos^{-1} \left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right).$$

We practice using this theorem in the following example.

**Example 2.3.7** Using the dot product to find angles.

Let  $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ , as shown in Figure 2.3.8.

Find the angles  $\alpha$ ,  $\beta$  and  $\theta$ .

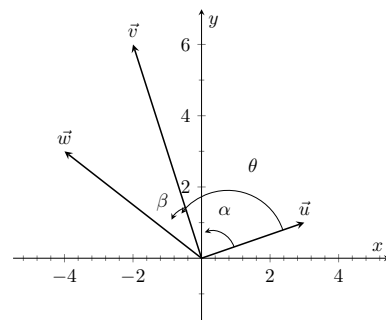
**Solution.** We start by computing the magnitude of each vector.

$$\|\vec{u}\| = \sqrt{10}; \|\vec{v}\| = 2\sqrt{10}; \|\vec{w}\| = 5.$$

We now apply Theorem 2.3.5 to find the angles.

$$\begin{aligned} \alpha &= \cos^{-1} \left( \frac{\vec{u} \cdot \vec{v}}{(\sqrt{10})(2\sqrt{10})} \right) \\ &= \cos^{-1}(0) = \frac{\pi}{2} = 90^\circ. \end{aligned}$$

$$\begin{aligned} \beta &= \cos^{-1} \left( \frac{\vec{v} \cdot \vec{w}}{(2\sqrt{10})(5)} \right) \\ &= \cos^{-1} \left( \frac{26}{10\sqrt{10}} \right) \\ &\approx 0.6055 \approx 34.7^\circ. \end{aligned}$$

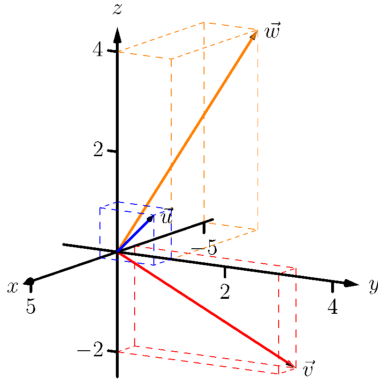


**Figure 2.3.8** Vectors used in Example 2.3.7

$$\begin{aligned}
 \theta &= \cos^{-1} \left( \frac{\vec{u} \cdot \vec{w}}{(\sqrt{10})(5)} \right) \\
 &= \cos^{-1} \left( \frac{-9}{5\sqrt{10}} \right) \\
 &\approx 2.1763 \approx 124.7^\circ
 \end{aligned}$$

We see from our computation that  $\alpha + \beta = \theta$ , as indicated by Figure 2.3.8. While we knew this should be the case, it is nice to see that this non-intuitive formula indeed returns the results we expected.

We do a similar example next in the context of vectors in space.



**Figure 2.3.10** Vectors used in Example 2.3.9

**Example 2.3.9 Using the dot product to find angles.**

Let  $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} -5 \\ 1 \\ 4 \end{bmatrix}$ , as illustrated in Figure 2.3.10. Find the angle between each pair of vectors.

**Solution.**

1. Between  $\vec{u}$  and  $\vec{v}$ :

$$\begin{aligned}
 \theta &= \cos^{-1} \left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right) \\
 &= \cos^{-1} \left( \frac{0}{\sqrt{3}\sqrt{14}} \right) \\
 &= \frac{\pi}{2}.
 \end{aligned}$$

2. Between  $\vec{u}$  and  $\vec{w}$ :

$$\begin{aligned}
 \theta &= \cos^{-1} \left( \frac{\vec{u} \cdot \vec{w}}{\|\vec{u}\| \|\vec{w}\|} \right) \\
 &= \cos^{-1} \left( \frac{0}{\sqrt{3}\sqrt{42}} \right) \\
 &= \frac{\pi}{2}.
 \end{aligned}$$

3. Between  $\vec{v}$  and  $\vec{w}$ :

$$\begin{aligned}
 \theta &= \cos^{-1} \left( \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|} \right) \\
 &= \cos^{-1} \left( \frac{0}{\sqrt{14}\sqrt{42}} \right) \\
 &= \frac{\pi}{2}.
 \end{aligned}$$

While our work shows that each angle is  $\pi/2$ , i.e.,  $90^\circ$ , none of these angles looks to be a right angle in Figure 2.3.10. Such is the case when drawing three-dimensional objects on the page.

All three angles between these vectors was  $\pi/2$ , or  $90^\circ$ . We know from geometry and everyday life that  $90^\circ$  angles are “nice” for a variety of reasons, so it should seem significant that these angles are all  $\pi/2$ . Notice the common

feature in each calculation (and also the calculation of  $\alpha$  in Example 2.3.7): the dot products of each pair of angles was 0. We use this as a basis for a definition of the term *orthogonal*, which is essentially synonymous to *perpendicular*.

**Definition 2.3.11 Orthogonal.**

Nonzero vectors  $\vec{u}$  and  $\vec{v}$  are **orthogonal** if their dot product is 0.

**Example 2.3.12 Finding orthogonal vectors.**

Let  $\vec{u} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 1 \\ 2, 3 \end{bmatrix}$ .

1. Find two vectors in  $\mathbb{R}^2$  that are orthogonal to  $\vec{u}$ .
2. Find two non-parallel vectors in  $\mathbb{R}^3$  that are orthogonal to  $\vec{v}$ .

**Solution.**

1. Recall that a line perpendicular to a line with slope  $m$  has slope  $-1/m$ , the “opposite reciprocal slope.” We can think of the slope of  $\vec{u}$  as  $5/3$ , its “rise over run.” A vector orthogonal to  $\vec{u}$  will have slope  $-3/5$ . There are many such choices, though all parallel:

$$\begin{bmatrix} -5 \\ 3 \end{bmatrix} \text{ or } \begin{bmatrix} 5 \\ -3 \end{bmatrix} \text{ or } \begin{bmatrix} -10 \\ 6 \end{bmatrix} \text{ or } \begin{bmatrix} 15 \\ -9 \end{bmatrix}, \text{ etc.}$$

2. There are infinitely many directions in space orthogonal to any given direction, so there are an infinite number of non-parallel vectors orthogonal to  $\vec{v}$ . Since there are so many, we have great leeway in finding some. One way is to arbitrarily pick values for the first two components, leaving the third unknown. For instance, let

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 7 \\ z \end{bmatrix}. \text{ If } \vec{v}_1 \text{ is to be orthogonal to } \vec{v}, \text{ then } \vec{v}_1 \cdot \vec{v} = 0, \text{ so}$$

$$2 + 14 + 3z = 0 \Rightarrow z = \frac{-16}{3}.$$

$$\text{So } \vec{v}_1 = \begin{bmatrix} 2 \\ 7 \\ -16/3 \end{bmatrix} \text{ is orthogonal to } \vec{v}. \text{ We can apply a similar tech-}$$

nique by leaving the first or second component unknown. Another method of finding a vector orthogonal to  $\vec{v}$  mirrors what we did in

$$\text{part 1. Let } \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}. \text{ Here we switched the first two compo-}$$

nents of  $\vec{v}$ , changing the sign of one of them (similar to the “opposite reciprocal” concept before). Letting the third component be 0 effectively ignores the third component of  $\vec{v}$ , and it is easy to see that

$$\vec{v}_2 \cdot \vec{v} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0.$$

Clearly  $\vec{v}_1$  and  $\vec{v}_2$  are not parallel.

**Note:** the term *perpendicular* originally referred to lines. As mathematics progressed, the concept of “being at right angles” was applied to other objects, such as vectors and planes, and the term *orthogonal* was introduced. It is especially used when discussing objects that are hard, or impossible, to visualize. Two vectors in 5-dimensional space are orthogonal if their dot product is 0. It is not wrong to say that they are perpendicular, but common convention gives preference to the word orthogonal.

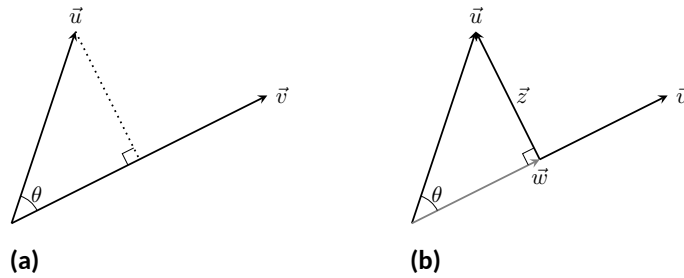
Also note that Definition 2.3.11 makes sense if  $\vec{u}$  or  $\vec{v}$  is the zero vector, but this is not the case for the usual use of the word perpendicular.

An important construction is illustrated in Figure 2.3.13, where vectors  $\vec{u}$  and  $\vec{v}$  are sketched. In Figure 2.3.13(a), a dotted line is drawn from the tip of  $\vec{u}$  to the line containing  $\vec{v}$ , where the dotted line is orthogonal to  $\vec{v}$ . In Figure 2.3.13(b), the dotted line is replaced with the vector  $\vec{z}$  and  $\vec{w}$  is formed, parallel to  $\vec{v}$ . It is clear by the diagram that  $\vec{u} = \vec{w} + \vec{z}$ . What is important about this construction is this:  $\vec{u}$  is *decomposed* as the sum of two vectors, one of which is parallel to  $\vec{v}$  and one that is perpendicular to  $\vec{v}$ . It is hard to overstate the importance of this construction (as we'll see in upcoming examples).

The vectors  $\vec{w}$ ,  $\vec{z}$  and  $\vec{u}$  as shown in Figure 2.3.13(b) form a right triangle, where the angle between  $\vec{v}$  and  $\vec{u}$  is labeled  $\theta$ . We can find  $\vec{w}$  in terms of  $\vec{v}$  and  $\vec{u}$ .

Using trigonometry, we can state that

$$\|\vec{w}\| = \|\vec{u}\| \cos(\theta). \quad (2.3.1)$$



**Figure 2.3.13** Developing the construction of the *orthogonal projection*

We also know that  $\vec{w}$  is parallel to  $\vec{v}$ ; that is, the direction of  $\vec{w}$  is the direction of  $\vec{v}$ , described by the unit vector  $\vec{v}/\|\vec{v}\|$ . The vector  $\vec{w}$  is the vector in the direction  $\vec{v}/\|\vec{v}\|$  with magnitude  $\|\vec{u}\| \cos(\theta)$ :

$$\begin{aligned} \vec{w} &= \left( \|\vec{u}\| \cos(\theta) \right) \frac{1}{\|\vec{v}\|} \vec{v} \\ &= \left( \|\vec{u}\| \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right) \frac{1}{\|\vec{v}\|} \vec{v} \quad (\text{Replacing } \cos(\theta) \text{ using Theorem 2.3.5}) \\ &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} \\ &= \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} \quad (\text{Applying Theorem 2.3.3}). \end{aligned}$$

Since this construction is so important, it is given a special name.

**Definition 2.3.14 Orthogonal Projection.**

Let nonzero vectors  $\vec{u}$  and  $\vec{v}$  be given. The **orthogonal projection** of  $\vec{u}$  onto  $\vec{v}$ , denoted  $\text{proj}_{\vec{v}} \vec{u}$ , is

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}.$$

**Example 2.3.15 Computing the orthogonal projection.**

1. Let  $\vec{u} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ . Find  $\text{proj}_{\vec{v}} \vec{u}$ , and sketch all three vectors with initial points at the origin.



2. Let  $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$  and  $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ . Find  $\text{proj}_{\vec{x}} \vec{w}$ , and sketch all three vectors with initial points at the origin.

**Solution.**

1. Applying Definition 2.3.14, we have

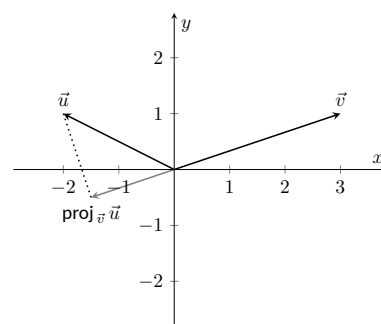
$$\begin{aligned} \text{proj}_{\vec{v}} \vec{u} &= \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} \\ &= \frac{-5}{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}. \end{aligned}$$

Vectors  $\vec{u}$ ,  $\vec{v}$  and  $\text{proj}_{\vec{v}} \vec{u}$  are sketched in Figure 2.3.16. Note how the projection is parallel to  $\vec{v}$ ; that is, it lies on the same line through the origin as  $\vec{v}$ , although it points in the opposite direction. That is because the angle between  $\vec{u}$  and  $\vec{v}$  is obtuse (i.e., greater than  $90^\circ$ ).

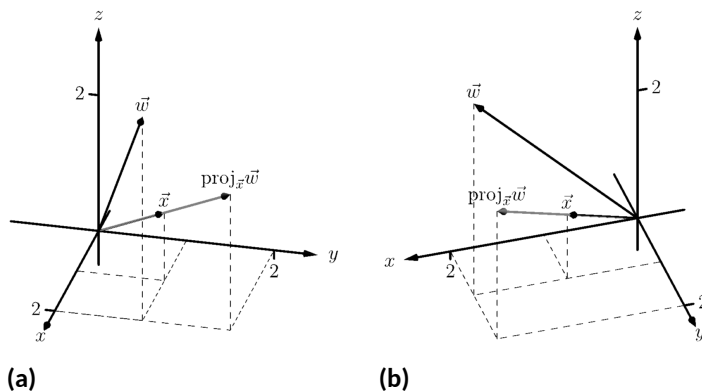
2. Apply the definition:

$$\begin{aligned} \text{proj}_{\vec{x}} \vec{w} &= \frac{\vec{w} \cdot \vec{x}}{\vec{x} \cdot \vec{x}} \vec{x} \\ &= \frac{6}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}. \end{aligned}$$

These vectors are sketched in Figure 2.3.17(a), and again in Figure 2.3.17(b) from a different perspective. Because of the nature of graphing these vectors, the sketch in Figure 2.3.17(a) makes it difficult to recognize that the drawn projection has the geometric properties it should. The graph shown in Figure 2.3.17(b) illustrates these properties better.



**Figure 2.3.16** Sketching the three vectors in Part 1 of Example 2.3.15



**Figure 2.3.17** Sketching the three vectors in Part 2 of Example 2.3.15

We can use the properties of the dot product found in Theorem 2.3.3 to rearrange the formula found in Definition 2.3.14:

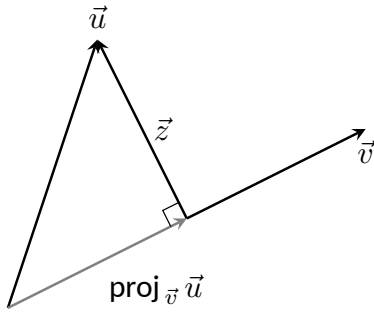
$$\begin{aligned}\text{proj}_{\vec{v}} \vec{u} &= \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} \\ &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} \\ &= \left( \vec{u} \cdot \frac{\vec{v}}{\|\vec{v}\|} \right) \frac{\vec{v}}{\|\vec{v}\|}.\end{aligned}$$

The above formula shows that the orthogonal projection of  $\vec{u}$  onto  $\vec{v}$  is only concerned with the *direction* of  $\vec{v}$ , as both instances of  $\vec{v}$  in the formula come in the form  $\vec{v}/\|\vec{v}\|$ , the unit vector in the direction of  $\vec{v}$ .

A special case of orthogonal projection occurs when  $\vec{v}$  is a unit vector. In this situation, the formula for the orthogonal projection of a vector  $\vec{u}$  onto  $\vec{v}$  reduces to just  $\text{proj}_{\vec{v}} \vec{u} = (\vec{u} \cdot \vec{v})\vec{v}$ , as  $\vec{v} \cdot \vec{v} = 1$ .

This gives us a new understanding of the dot product. When  $\vec{v}$  is a unit vector, essentially providing only direction information, the dot product of  $\vec{u}$  and  $\vec{v}$  gives “how much of  $\vec{u}$  is in the direction of  $\vec{v}$ .” This use of the dot product will be very useful in future sections.

Now consider Figure 2.3.18 where the concept of the orthogonal projection is again illustrated. It is clear that



**Figure 2.3.18** Illustrating the orthogonal projection

$$\vec{u} = \text{proj}_{\vec{v}} \vec{u} + \vec{z}. \quad (2.3.2)$$

As we know what  $\vec{u}$  and  $\text{proj}_{\vec{v}} \vec{u}$  are, we can solve for  $\vec{z}$  and state that

$$\vec{z} = \vec{u} - \text{proj}_{\vec{v}} \vec{u}.$$

This leads us to rewrite Equation (2.3.2) in a seemingly silly way:

$$\vec{u} = \text{proj}_{\vec{v}} \vec{u} + (\vec{u} - \text{proj}_{\vec{v}} \vec{u}).$$

This is not nonsense, as pointed out in the following Key Idea. (Notation note: the expression “ $\parallel \vec{y}$ ” means “is parallel to  $\vec{y}$ .” We can use this notation to state “ $\vec{x} \parallel \vec{y}$ ” which means “ $\vec{x}$  is parallel to  $\vec{y}$ .” The expression “ $\perp \vec{y}$ ” means “is orthogonal to  $\vec{y}$ ,” and is used similarly.)

**Key Idea 2.3.19 Orthogonal Decomposition of Vectors.**

Let nonzero vectors  $\vec{u}$  and  $\vec{v}$  be given. Then  $\vec{u}$  can be written as the sum of two vectors, one of which is parallel to  $\vec{v}$ , and one of which is orthogonal to  $\vec{v}$ :

$$\vec{u} = \underbrace{\text{proj}_{\vec{v}} \vec{u}}_{\parallel \vec{v}} + \underbrace{(\vec{u} - \text{proj}_{\vec{v}} \vec{u})}_{\perp \vec{v}}.$$

We illustrate the use of this equality in the following example.

**Example 2.3.20 Orthogonal decomposition of vectors.**

1. Let  $\vec{u} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$  as in Example 2.3.15. Decompose  $\vec{u}$  as the sum of a vector parallel to  $\vec{v}$  and a vector orthogonal to  $\vec{v}$ .

2. Let  $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$  and  $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  as in Example 2.3.15. Decompose  $\vec{w}$  as the sum of a vector parallel to  $\vec{x}$  and a vector orthogonal to  $\vec{x}$ .

**Solution.**

1. In Example 2.3.15, we found that  $\text{proj}_{\vec{v}} \vec{u} = \begin{bmatrix} -1.5 \\ -0.5 \end{bmatrix}$ . Let

$$\vec{z} = \vec{u} - \text{proj}_{\vec{v}} \vec{u} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} - \begin{bmatrix} -1.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 1.5 \end{bmatrix}.$$

Is  $\vec{z}$  orthogonal to  $\vec{v}$ ? (i.e., is  $\vec{z} \perp \vec{v}$ ?) We check for orthogonality with the dot product:

$$\vec{z} \cdot \vec{v} = \begin{bmatrix} -0.5 \\ 1.5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 0.$$

Since the dot product is 0, we know  $\vec{z} \perp \vec{v}$ . Thus:

$$\begin{aligned} \vec{u} &= \text{proj}_{\vec{v}} \vec{u} + (\vec{u} - \text{proj}_{\vec{v}} \vec{u}) \\ \begin{bmatrix} -2 \\ 1 \end{bmatrix} &= \underbrace{\begin{bmatrix} -1.5 \\ -0.5 \end{bmatrix}}_{\parallel \vec{v}} + \underbrace{\begin{bmatrix} -0.5 \\ 1.5 \end{bmatrix}}_{\perp \vec{v}}. \end{aligned}$$

2. We found in Example 2.3.15 that  $\text{proj}_{\vec{x}} \vec{w} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ . Applying the

Key Idea, we have:

$$\vec{z} = \vec{w} - \text{proj}_{\vec{x}} \vec{w} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

We check to see if  $\vec{z} \perp \vec{x}$ :

$$\vec{z} \cdot \vec{x} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0.$$

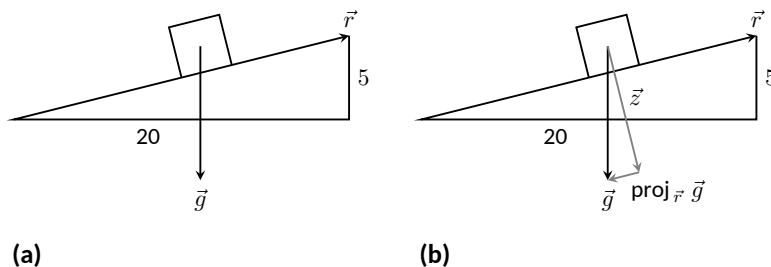
Since the dot product is 0, we know the two vectors are orthogonal. We now write  $\vec{w}$  as the sum of two vectors, one parallel and one orthogonal to  $\vec{x}$ :

$$\begin{aligned} \vec{w} &= \text{proj}_{\vec{x}} \vec{w} + (\vec{w} - \text{proj}_{\vec{x}} \vec{w}) \\ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} &= \underbrace{\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}}_{\parallel \vec{x}} + \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}_{\perp \vec{x}} \end{aligned}$$

We give an example of where this decomposition is useful.

**Example 2.3.21 Orthogonally decomposing a force vector.**

Consider Figure 2.3.22(a), showing a box weighing 50lb on a ramp that rises 5ft over a span of 20ft. Find the components of force, and their magnitudes, acting on the box (as sketched in Figure 2.3.22(b)):



**Figure 2.3.22** Sketching the ramp and box in Example 2.3.21. Note: The vectors are not drawn to scale.

1. in the direction of the ramp, and
2. orthogonal to the ramp.

**Solution.** As the ramp rises 5ft over a horizontal distance of 20ft, we can represent the direction of the ramp with the vector  $\vec{r} = \begin{bmatrix} 20 \\ 5 \end{bmatrix}$ . Gravity pulls down with a force of 50lb, which we represent with  $\vec{g} = \begin{bmatrix} 0 \\ -50 \end{bmatrix}$ .

1. To find the force of gravity in the direction of the ramp, we compute  $\text{proj}_{\vec{r}} \vec{g}$ :

$$\begin{aligned} \text{proj}_{\vec{r}} \vec{g} &= \frac{\vec{g} \cdot \vec{r}}{\vec{r} \cdot \vec{r}} \vec{r} \\ &= \frac{-250}{425} \begin{bmatrix} 20 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{200}{17} \\ -\frac{50}{17} \end{bmatrix} \approx \begin{bmatrix} -11.76 \\ -2.94 \end{bmatrix}. \end{aligned}$$

The magnitude of  $\text{proj}_{\vec{r}} \vec{g}$  is  $\|\text{proj}_{\vec{r}} \vec{g}\| = 50/\sqrt{17} \approx 12.13$  lb. Though the box weighs 50lb, a force of about 12lb is enough to keep the box from sliding down the ramp.

2. To find the component  $\vec{z}$  of gravity orthogonal to the ramp, we use Key Idea 2.3.19.

$$\begin{aligned} \vec{z} &= \vec{g} - \text{proj}_{\vec{r}} \vec{g} \\ &= \begin{bmatrix} \frac{200}{17} \\ -\frac{800}{17} \end{bmatrix} \approx \begin{bmatrix} 11.76 \\ -47.06 \end{bmatrix}. \end{aligned}$$

The magnitude of this force is  $\|\vec{z}\| \approx 48.51$ lb. In physics and engineering, knowing this force is important when computing things like static frictional force. (For instance, we could easily compute if the static frictional force alone was enough to keep the box from sliding down the ramp.)

### 2.3.1 Application to Work

In physics, the application of a force  $F$  to move an object in a straight line a distance  $d$  produces *work*; the amount of work  $W$  is  $W = Fd$ , (where  $F$  is in the direction of travel). The orthogonal projection allows us to compute work when the force is not in the direction of travel.

Consider Figure 2.3.23, where a force  $\vec{F}$  is being applied to an object moving in the direction of  $\vec{d}$ . (The distance the object travels is the magnitude of  $\vec{d}$ .) The work done is the amount of force in the direction of  $\vec{d}$ ,  $\|\text{proj}_{\vec{d}} \vec{F}\|$ , times  $\|\vec{d}\|$ :

$$\begin{aligned} \|\text{proj}_{\vec{d}} \vec{F}\| \cdot \|\vec{d}\| &= \left\| \frac{\vec{F} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \vec{d} \right\| \cdot \|\vec{d}\| \\ &= \left| \frac{\vec{F} \cdot \vec{d}}{\|\vec{d}\|^2} \right| \cdot \|\vec{d}\| \cdot \|\vec{d}\| \\ &= \frac{|\vec{F} \cdot \vec{d}|}{\|\vec{d}\|^2} \|\vec{d}\|^2 \\ &= |\vec{F} \cdot \vec{d}|. \end{aligned}$$

The expression  $\vec{F} \cdot \vec{d}$  will be positive if the angle between  $\vec{F}$  and  $\vec{d}$  is acute; when the angle is obtuse (hence  $\vec{F} \cdot \vec{d}$  is negative), the force is causing motion in the opposite direction of  $\vec{d}$ , resulting in “negative work.” We want to capture this sign, so we drop the absolute value and find that  $W = \vec{F} \cdot \vec{d}$ .

#### Definition 2.3.24 Work.

Let  $\vec{F}$  be a constant force that moves an object in a straight line from point  $P$  to point  $Q$ . Let  $\vec{d} = \overrightarrow{PQ}$ . The **work**  $W$  done by  $\vec{F}$  along  $\vec{d}$  is  $W = \vec{F} \cdot \vec{d}$ .

#### Example 2.3.25 Computing work.

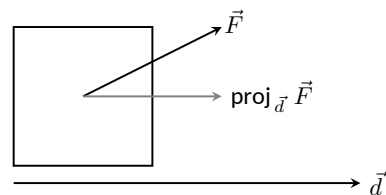
A man slides a box along a ramp that rises 3ft over a distance of 15ft by applying 50lb of force as shown in Figure 2.3.26. Compute the work done.

**Solution.** The figure indicates that the force applied makes a  $30^\circ$  angle with the horizontal, so  $\vec{F} = 50 \begin{bmatrix} \cos(30^\circ) \\ \sin(30^\circ) \end{bmatrix} \approx \begin{bmatrix} 43.3 \\ 25 \end{bmatrix}$ . The ramp is represented by  $\vec{d} = \begin{bmatrix} 15 \\ 3 \end{bmatrix}$ . The work done is simply

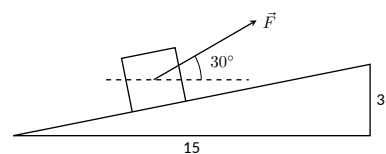
$$\vec{F} \cdot \vec{d} = 50 \begin{bmatrix} \cos(30^\circ) \\ \sin(30^\circ) \end{bmatrix} \cdot \begin{bmatrix} 15 \\ 3 \end{bmatrix} \approx 724.5 \text{ ft-lb}.$$

Note how we did not actually compute the distance the object traveled, nor the magnitude of the force in the direction of travel; this is all inherently computed by the dot product!

The dot product is a powerful way of evaluating computations that depend on angles without actually using angles. The next section explores another “prod-



**Figure 2.3.23** Finding work when the force and direction of travel are given as vectors



**Figure 2.3.26** Computing work when sliding a box up a ramp in Example 2.3.25

uct" on vectors, the *cross product*. Once again, angles play an important role, though in a much different way.

## 2.3.2 Exercises

**Exercise Group.** In the following exercises, find the dot product of the given vectors.

1.  $\vec{u} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

2.  $\vec{u} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \vec{v} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

3.  $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$

4.  $\vec{u} = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}$

5.  $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

6.  $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$   
 $\vec{u} \cdot \vec{v} = \underline{\hspace{2cm}}$

7. Create your own vectors  $\vec{u}, \vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^2$  and show that  $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ .

8. Create your own vectors  $\vec{u}$  and  $\vec{v}$  in  $\mathbb{R}^3$  and scalar  $c$  and show that  $c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$ .

**Exercise Group.** In the following exercises, find the measure of the angle between the two vectors in radians.

9.  $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .

10.  $\vec{u} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ .

11.  $\vec{u} = \begin{bmatrix} 8 \\ 1 \\ -4 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ .

12.  $\vec{u} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}$ .

**Exercise Group.** In the following exercises, a vector  $\vec{v}$  is given. Give two vectors that are orthogonal to  $\vec{v}$ .

13. Find two nonzero vectors orthogonal to  $\vec{v} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ .

14. Find two nonzero vectors orthogonal to  $\vec{v} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ .

15. Find two nonzero vectors orthogonal to  $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ .

16. Find two nonzero vectors orthogonal to  $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ .

**Exercise Group.** In the following exercises, vectors  $\vec{u}$  and  $\vec{v}$  are given. Find  $\text{proj}_{\vec{v}} \vec{u}$ , the orthogonal projection of  $\vec{u}$  onto  $\vec{v}$ , and sketch all three vectors with the same initial point.

17.  $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ .

18.  $\vec{u} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ .

19.  $\vec{u} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

20.  $\vec{u} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ .

21.  $\vec{u} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ .

22.  $\vec{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ .

**Exercise Group.** In the following exercises, vectors  $\vec{u}$  and  $\vec{v}$  are given. Write  $\vec{u}$  as the sum of two vectors, one of which is parallel to  $\vec{v}$  (or is zero) and one of which is orthogonal to  $\vec{v}$ . Note: these are the same pairs of vectors as found in Exercises 17–22.

23. Write  $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  as the sum of two vectors, one parallel to  $\vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$  (or zero) and the other perpendicular.  
 $\vec{u} = \underline{\hspace{2cm}} + \underline{\hspace{2cm}}$
24. Write  $\vec{u} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$  as the sum of two vectors, one parallel to  $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$  (or zero) and the other perpendicular.  
 $\vec{u} = \underline{\hspace{2cm}} + \underline{\hspace{2cm}}$
25. Write  $\vec{u} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$  as the sum of two vectors, one parallel to  $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  (or zero) and the other perpendicular.  
 $\vec{u} = \underline{\hspace{2cm}} + \underline{\hspace{2cm}}$
26. Write  $\vec{u} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$  as the sum of two vectors, one parallel to  $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  (or zero) and the other perpendicular.  
 $\vec{u} = \underline{\hspace{2cm}} + \underline{\hspace{2cm}}$
27. Write  $\vec{u} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$  as the sum of two vectors, one parallel to  $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  (or zero) and the other perpendicular.  
 $\vec{u} = \underline{\hspace{2cm}} + \underline{\hspace{2cm}}$
28. Write  $\vec{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$  as the sum of two vectors, one parallel to  $\vec{v} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$  (or zero) and the other perpendicular.  
 $\vec{u} = \underline{\hspace{2cm}} + \underline{\hspace{2cm}}$
29. A 10lb box sits on a ramp that rises 4ft over a distance of 20ft. How much force is required to keep the box from sliding down the ramp?
30. A 10lb box sits on a 15ft ramp that makes a  $30^\circ$  angle with the horizontal. How much force is required to keep the box from sliding down the ramp?
31. How much work is performed in moving a box horizontally 10ft with a force of 20lb applied at an angle of  $45^\circ$  to the horizontal?
32. How much work is performed in moving a box horizontally 10ft with a force of 20lb applied at an angle of  $10^\circ$  to the horizontal?
33. How much work is performed in moving a box up the length of a ramp that rises 2ft over a distance of 10ft, with a force of 50lb applied horizontally?
34. How much work is performed in moving a box up the length of a ramp that rises 2ft over a distance of 10ft, with a force of 50lb applied at an angle of  $45^\circ$  to the horizontal?
35. How much work is performed in moving a box up the length of a 10ft ramp that makes a  $5^\circ$  angle with the horizontal, with 50lb of force applied in the direction of the ramp?



## 2.4 The Cross Product

“Orthogonality” is immensely important. A quick scan of your current environment will undoubtedly reveal numerous surfaces and edges that are perpendicular to each other (including the edges of this page). The dot product provides a quick test for orthogonality: vectors  $\vec{u}$  and  $\vec{v}$  are perpendicular if, and only if,  $\vec{u} \cdot \vec{v} = 0$ .

Given two non-parallel, nonzero vectors  $\vec{u}$  and  $\vec{v}$  in space, it is very useful to find a vector  $\vec{w}$  that is perpendicular to both  $\vec{u}$  and  $\vec{v}$ . There is an operation, called the *cross product*, that creates such a vector. This section defines the cross product, then explores its properties and applications.

### Definition 2.4.1 Cross Product.

Let  $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$  be vectors in  $\mathbb{R}^3$ . The **cross product** of  $\vec{u}$  and  $\vec{v}$ , denoted  $\vec{u} \times \vec{v}$ , is the vector

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ -(u_1v_3 - u_3v_1) \\ u_1v_2 - u_2v_1 \end{bmatrix}.$$

This definition can be a bit cumbersome to remember. After an example we will give a convenient method for computing the cross product. For now, careful examination of the products and differences given in the definition should reveal a pattern that is not too difficult to remember. (For instance, in the first component only 2 and 3 appear as subscripts; in the second component, only 1 and 3 appear as subscripts. Further study reveals the order in which they appear.)

Let's practice using this definition by computing a cross product.

### Example 2.4.2 Computing a cross product.

Let  $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ . Find  $\vec{u} \times \vec{v}$ , and verify that it is orthogonal to both  $\vec{u}$  and  $\vec{v}$ .

**Solution.** Using Definition 2.4.1, we have

$$\vec{u} \times \vec{v} = \begin{bmatrix} (-1)5 - (4)2 \\ -((2)5 - (4)3) \\ (2)2 - (-1)3 \end{bmatrix} = \begin{bmatrix} -13 \\ 2 \\ 7 \end{bmatrix}.$$

(We encourage the reader to compute this product on their own, then verify their result.)

We test whether or not  $\vec{u} \times \vec{v}$  is orthogonal to  $\vec{u}$  and  $\vec{v}$  using the dot product:

$$(\vec{u} \times \vec{v}) \cdot \vec{u} = \begin{bmatrix} -13 \\ 2 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = 0,$$

$$(\vec{u} \times \vec{v}) \cdot \vec{v} = \begin{bmatrix} -13 \\ 2 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = 0.$$

The definition of the cross product may look strange (and complicated) at first, but it's more or less forced by the requirement that it be orthogonal to both  $\vec{u}$  and  $\vec{v}$ . To begin to see why, suppose  $\vec{w} = a\vec{i} + b\vec{j} + c\vec{k}$  is an arbitrary vector such that  $\vec{u} \cdot \vec{v} = 0$  and  $\vec{u} \cdot \vec{w} = 0$ . This gives us the pair of equations

$$\begin{aligned} u_1a + u_2b + u_3c &= 0 \\ v_1a + v_2b + v_3c &= 0. \end{aligned}$$

This is a *system of linear equations* in the variables  $a$ ,  $b$ , and  $c$ . We'll learn the techniques for solving such a system in Chapter 3, at which point we'll be able to see that (up to a scalar multiple), the common solution to this pair of equations is given by Definition 2.4.1.

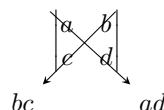
Since both dot products are zero,  $\vec{u} \times \vec{v}$  is indeed orthogonal to both  $\vec{u}$  and  $\vec{v}$ .

We now introduce a method for computing the cross-product that is easier to remember, and has the added benefit of allowing us to preview **determinants**, which we will return to in earnest in Section 6.3.

Consider a rectangular array  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  of four real numbers  $a, b, c$ , and  $d$ . A  $2 \times 2$  determinant takes any such array and assigns the number  $ad - bc$ . This is commonly denoted as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Most people find it easiest to remember this in terms of the two *diagonals* of the array: we take the product of the two numbers on the *main diagonal* (top-left to bottom-right), and subtract the product of the two numbers on the other diagonal.



For example, we have  $\begin{vmatrix} 4 & -2 \\ 6 & 3 \end{vmatrix} = 4(3) - (-2)(6) = 24$ . Once we get comfortable with  $2 \times 2$  determinants, we can write the cross product in terms of them, as follows:

$$\vec{u} \times \vec{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k} \quad (2.4.1)$$

$$= (u_2v_3 - u_3v_2)\vec{i} - (u_3v_1 - u_1v_3)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}, \quad (2.4.2)$$

as before. Now, this might not seem like much of an improvement over the previous formula, so we take things one step further. First, we form a  $3 \times 3$  array as shown below.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

The first row comprises the standard unit vectors  $\vec{i}$ ,  $\vec{j}$ , and  $\vec{k}$ . The second and third rows are the vectors  $\vec{u}$  and  $\vec{v}$ , respectively. Next, we *expand* our  $3 \times 3$  array as a vector, where the coefficient of each standard unit vector is given by the  $2 \times 2$  determinant that's left over when we delete the row and column containing that unit vector.

For example, if we use  $\vec{u}$  and  $\vec{v}$  from Example 2.4.2, we obtain the array

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 3 & 2 & 5 \end{vmatrix}.$$

The expansion process used to obtain the coefficients of  $\vec{i}, \vec{j}, \vec{k}$  looks like the following:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 3 & 2 & 5 \end{vmatrix} \rightarrow \begin{vmatrix} -1 & 4 \\ 2 & 5 \end{vmatrix} \vec{i} = -13\vec{i}$$

$$\begin{vmatrix} \vec{i} & \boxed{\vec{j}} & \vec{k} \\ 2 & -1 & 4 \\ 3 & 2 & 5 \end{vmatrix} \rightarrow \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} \vec{j} = -2\vec{j}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \boxed{\vec{k}} \\ 2 & -1 & 4 \\ 3 & 2 & 5 \end{vmatrix} \rightarrow \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} \vec{k} = 7\vec{k}$$

There is one more important detail to note: notice in Equation (2.4.1) that there is a **minus sign** in front of the coefficient of the unit vector  $\vec{j}$ . We need to make sure that the signs in front of each  $2 \times 2$  determinant follow this  $+, -, +$  pattern when we expand our array as a vector. For the vectors  $\vec{u}$  and  $\vec{v}$  in Example 2.4.2, we end up with the following:

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 3 & 2 & 5 \end{vmatrix} = \begin{vmatrix} -1 & 4 \\ 2 & 5 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} \vec{k} \\ &= -13\vec{i} - (-2)\vec{j} + 7\vec{k} = \begin{bmatrix} -13 \\ 2 \\ 7 \end{bmatrix}, \end{aligned}$$

as before. The method will become more clear with a bit of practice.

#### Example 2.4.3 Computing a cross product.

Let  $\vec{u} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ . Compute both  $\vec{u} \times \vec{v}$  and  $\vec{v} \times \vec{u}$ .

**Solution.** To compute  $\vec{u} \times \vec{v}$ , we form the  $3 \times 3$  array as prescribed above, and expand it into a vector.

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 6 \\ -1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 6 \\ 2 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 6 \\ -1 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} \vec{k} \\ &= (3(1) - 6(2))\vec{i} - (1(1) - 6(-1))\vec{j} + (1(2) - 3(-1))\vec{k} \\ &= -9\vec{i} - 7\vec{j} + 5\vec{k} = \begin{bmatrix} -9 \\ -7 \\ 5 \end{bmatrix}. \end{aligned}$$

To compute  $\vec{v} \times \vec{u}$ , we switch the second and third rows of the above matrix, then expand as before:

$$\begin{aligned} \vec{v} \times \vec{u} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 1 \\ 1 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 3 & 6 \end{vmatrix} \vec{i} - \begin{vmatrix} -1 & 1 \\ 1 & 6 \end{vmatrix} \vec{j} + \begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} \vec{k} \\ &= (2(6) - 1(3))\vec{i} - ((-1)(6) - 1(1))\vec{j} + ((-1)(3) - 2(1))\vec{k} \\ &= 9\vec{i} + 7\vec{j} - 5\vec{k} = \begin{bmatrix} 9 \\ 7 \\ -5 \end{bmatrix} = -\vec{u} \times \vec{v}. \end{aligned}$$

Note how with the rows being switched, the products that once appeared on the right now appear on the left, and vice-versa, so that the result is the opposite of  $\vec{u} \times \vec{v}$

**Note:** If the minus sign in front of the  $\vec{j}$  coefficient seems out of place to you, it might help to imagine wrapping our  $3 \times 3$  array around a cylinder (like the label on a tin can). If we read from left to right, beginning in the  $\vec{j}$  column, then we should place the  $\vec{k}$  column first, followed by the  $\vec{i}$  column. For the vectors  $\vec{u}$  and  $\vec{v}$  in Example 2.4.2, this would result in the coefficient  $\begin{vmatrix} 4 & 2 \\ 5 & 2 \end{vmatrix} = 2$  for the  $\vec{j}$  component, which has the correct sign. However, since our habit is to read starting from the far left, we tend to write the  $\vec{i}$  column first, and then introduce the minus sign to compensate.

### 2.4.1 Properties of the Cross Product

It is not coincidence that  $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$  in the preceding example; one can show using Definition 2.4.1 that this will always be the case. The following theorem states several useful properties of the cross product, each of which can be verified by referring to the definition.

#### Theorem 2.4.4 Properties of the Cross Product.

Let  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  be vectors in  $\mathbb{R}^3$  and let  $c$  be a scalar. The following identities hold:

1.  $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$  *Anticommutative Property*
2. (a)  $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$  *Distributive Properties*  
 (b)  $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
3.  $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$
4. (a)  $(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$  *Orthogonality Properties*  
 (b)  $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$
5.  $\vec{u} \times \vec{u} = \vec{0}$
6.  $\vec{u} \times \vec{0} = \vec{0}$
7.  $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$  *Triple Scalar Product*

We introduced the cross product as a way to find a vector orthogonal to two given vectors, but we did not give a proof that the construction given in Definition 2.4.1 satisfies this property. Theorem 2.4.4 asserts this property holds; we leave it as a problem in the Exercise section to verify this.

Property 5 from the theorem is also left to the reader to prove in the Exercise section, but it reveals something more interesting than “the cross product of a vector with itself is  $\vec{0}$ .” Let  $\vec{u}$  and  $\vec{v}$  be parallel vectors; that is, let there be a scalar  $c$  such that  $\vec{v} = c\vec{u}$ . Consider their cross product:

$$\begin{aligned}\vec{u} \times \vec{v} &= \vec{u} \times (c\vec{u}) \\ &= c(\vec{u} \times \vec{u}) \text{ (by Property 3 of Theorem 2.4.4)} \\ &= \vec{0} \text{ (by Property 5 of Theorem 2.4.4).}\end{aligned}$$

We have just shown that the cross product of parallel vectors is  $\vec{0}$ . This hints at something deeper. Theorem 2.3.5 related the angle between two vectors and their dot product; there is a similar relationship relating the cross product of two vectors and the angle between them, given by the following theorem.

**Parallel vectors and the cross product.** We could rewrite Definition 2.3.11 and Theorem 2.4.5 to include  $\vec{0}$ , then define that  $\vec{u}$  and  $\vec{v}$  are parallel if  $\vec{u} \times \vec{v} = \vec{0}$ . Since  $\vec{0} \cdot \vec{v} = 0$  and  $\vec{0} \times \vec{v} = \vec{0}$ , this would mean that  $\vec{0}$  is both parallel and orthogonal to all vectors. Apparent paradoxes such as this are not uncommon in mathematics and can be very useful. (See also the aside in Section 2.2.)

#### Theorem 2.4.5 The Cross Product and Angles.

Let  $\vec{u}$  and  $\vec{v}$  be nonzero vectors in  $\mathbb{R}^3$ . Then

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin(\theta),$$

where  $\theta$ ,  $0 \leq \theta \leq \pi$ , is the angle between  $\vec{u}$  and  $\vec{v}$ .

Note that this theorem makes a statement about the *magnitude* of the cross product. When the angle between  $\vec{u}$  and  $\vec{v}$  is 0 or  $\pi$  (i.e., the vectors are parallel),

the magnitude of the cross product is 0. The only vector with a magnitude of 0 is  $\vec{0}$  (see Property 9 of Theorem 2.2.16), hence the cross product of parallel vectors is  $\vec{0}$ .

We demonstrate the truth of this theorem in the following example.

**Example 2.4.6 The cross product and angles.**

Let  $\vec{u} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$  as in Example 2.4.3. Verify Theorem 2.4.5 by finding  $\theta$ , the angle between  $\vec{u}$  and  $\vec{v}$ , and the magnitude of  $\vec{u} \times \vec{v}$ .

**Solution.** We use Theorem 2.3.5 to find the angle between  $\vec{u}$  and  $\vec{v}$ .

$$\begin{aligned} \theta &= \cos^{-1} \left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right) \\ &= \cos^{-1} \left( \frac{11}{\sqrt{46}\sqrt{6}} \right) \\ &\approx 0.8471 = 48.54^\circ. \end{aligned}$$

Our work in Example 2.4.3 showed that  $\vec{u} \times \vec{v} = \begin{bmatrix} -9 \\ -7 \\ 5 \end{bmatrix}$ , hence

$\|\vec{u} \times \vec{v}\| = \sqrt{155}$ . Is  $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin(\theta)$ ? Using numerical approximations, we find:

$$\begin{aligned} \|\vec{u} \times \vec{v}\| &= \sqrt{155} & \|\vec{u}\| \|\vec{v}\| \sin(\theta) &= \sqrt{46}\sqrt{6} \sin(0.8471) \\ &\approx 12.45. & &\approx 12.45. \end{aligned}$$

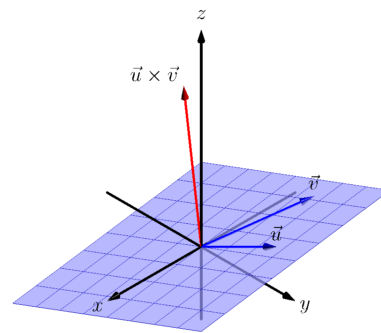
Numerically, they seem equal. Using a right triangle, one can show that

$$\sin \left( \cos^{-1} \left( \frac{11}{\sqrt{46}\sqrt{6}} \right) \right) = \frac{\sqrt{155}}{\sqrt{46}\sqrt{6}},$$

which allows us to verify the theorem exactly.

**Right Hand Rule.** The anticommutative property of the cross product demonstrates that  $\vec{u} \times \vec{v}$  and  $\vec{v} \times \vec{u}$  differ only by a sign — these vectors have the same magnitude but point in the opposite direction. When seeking a vector perpendicular to  $\vec{u}$  and  $\vec{v}$ , we essentially have two directions to choose from, one in the direction of  $\vec{u} \times \vec{v}$  and one in the direction of  $\vec{v} \times \vec{u}$ . Does it matter which we choose? How can we tell which one we will get without graphing, etc.?

Another wonderful property of the cross product, as defined, is that it follows the *right hand rule*. Given  $\vec{u}$  and  $\vec{v}$  in  $\mathbb{R}^3$  with the same initial point, point the index finger of your right hand in the direction of  $\vec{u}$  and let your middle finger point in the direction of  $\vec{v}$  (much as we did when establishing the right hand rule for the 3-dimensional coordinate system). Your thumb will naturally extend in the direction of  $\vec{u} \times \vec{v}$ . One can “practice” this using Figure 2.4.7. If you switch, and point the index finder in the direction of  $\vec{v}$  and the middle finger in the direction of  $\vec{u}$ , your thumb will now point in the opposite direction, allowing you to “visualize” the anticommutative property of the cross product.



**Figure 2.4.7** Illustrating the Right Hand Rule of the cross product

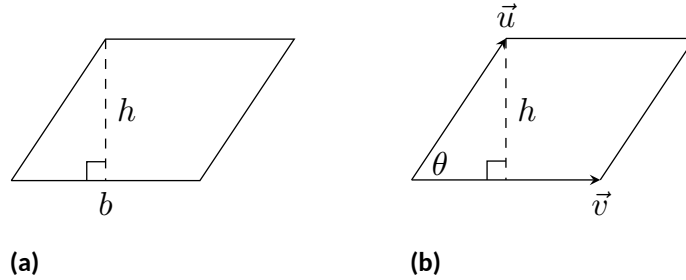
### 2.4.2 Applications of the Cross Product

There are a number of ways in which the cross product is useful in mathematics, physics and other areas of science beyond “just” finding a vector perpendicular to two others. We highlight a few here.

**Area of a Parallelogram.** It is a standard geometry fact that the area of a parallelogram is  $A = bh$ , where  $b$  is the length of the base and  $h$  is the height of the parallelogram, as illustrated in Figure 2.4.8(a). As shown when defining the Parallelogram Law of vector addition, two vectors  $\vec{u}$  and  $\vec{v}$  define a parallelogram when drawn from the same initial point, as illustrated in Figure 2.4.8(b). Trigonometry tells us that  $h = \|\vec{u}\| \sin(\theta)$ , hence the area of the parallelogram is

$$A = \|\vec{u}\| \|\vec{v}\| \sin(\theta) = \|\vec{u} \times \vec{v}\|, \quad (2.4.3)$$

where the second equality comes from Theorem 2.4.5.



**Figure 2.4.8** Using the cross product to find the area of a parallelogram

We illustrate using Equation (2.4.3) in the following example.

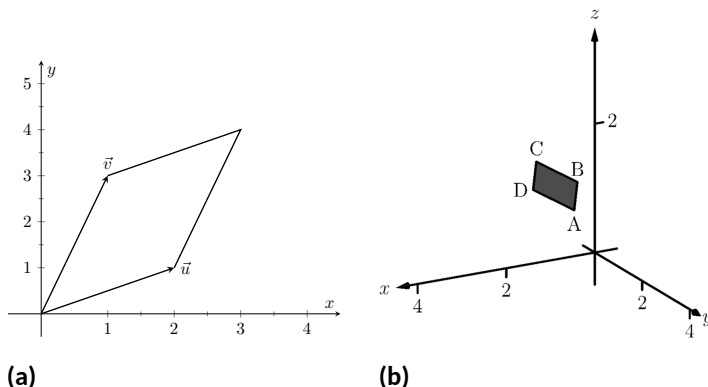
**Example 2.4.9 Finding the area of a parallelogram.**

1. Find the area of the parallelogram defined by the vectors  $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$ .
2. Verify that the points  $A = (1, 1, 1)$ ,  $B = (2, 3, 2)$ ,  $C = (4, 5, 3)$  and  $D = (3, 3, 2)$  are the vertices of a parallelogram. Find the area of the parallelogram.

**Solution.**

1. Figure 2.4.10(a) sketches the parallelogram defined by the vectors  $\vec{u}$  and  $\vec{v}$ . We have a slight problem in that our vectors exist in  $\mathbb{R}^3$ , not  $\mathbb{R}^2$ , and the cross product is only defined on vectors in  $\mathbb{R}^3$ . We skirt this issue by viewing  $\vec{u}$  and  $\vec{v}$  as vectors in the  $x - y$  plane of  $\mathbb{R}^3$ , and rewrite them as  $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ . We can now

compute the cross product. It is easy to show that  $\vec{u} \times \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$ ;  
therefore the area of the parallelogram is  $A = \|\vec{u} \times \vec{v}\| = 5$ .



**Figure 2.4.10** Sketching the parallelograms in Example 2.4.9

2. To show that the quadrilateral  $ABCD$  is a parallelogram (shown in Figure 2.4.10(b)), we need to show that the opposite sides are

parallel. We can quickly show that  $\overrightarrow{AB} = \overrightarrow{DC} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$  and  $\overrightarrow{BC} =$

$\overrightarrow{AD} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ . We find the area by computing the magnitude of the cross product of  $\overrightarrow{AB}$  and  $\overrightarrow{BC}$ :

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \Rightarrow \|\overrightarrow{AB} \times \overrightarrow{BC}\| = \sqrt{5} \approx 2.236.$$

This application is perhaps more useful in finding the area of a triangle (in short, triangles are used more often than parallelograms). We illustrate this in the following example.

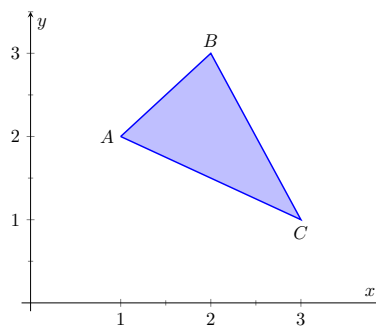
**Example 2.4.11 Area of a triangle.**

Find the area of the triangle with vertices  $A = (1, 2)$ ,  $B = (2, 3)$  and  $C = (3, 1)$ , as pictured in Figure 2.4.12.

**Solution.** We can choose any two sides of the triangle to use to form vectors; we choose  $\overrightarrow{AB} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$  and  $\overrightarrow{AC} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ . As in the previous example, we will rewrite these vectors with a third component of 0 so that we can apply the cross product. The area of the triangle is

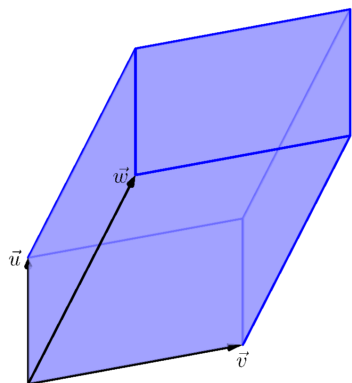
$$\frac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2} \left\| \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\| = \frac{1}{2} \left\| \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} \right\| = \frac{3}{2}.$$

We arrive at the same answer as before with less work.

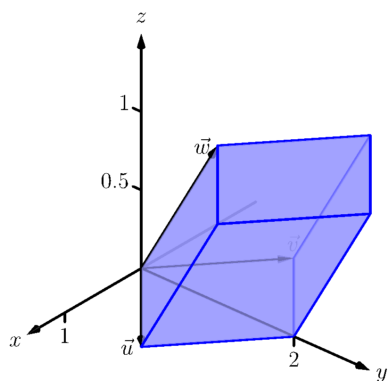


**Figure 2.4.12** Finding the area of a triangle in Example 2.4.11

The word “parallelepiped” is pronounced “parallel-eh-pipe-ed.”



**Figure 2.4.13** A parallelepiped is the three dimensional analogue to the parallelogram



**Figure 2.4.15** A parallelepiped in Example 2.4.14

**Volume of a Parallelepiped.** The three dimensional analogue to the parallelogram is the *parallelepiped*. Each face is parallel to the opposite face, as illustrated in Figure 2.4.13. By crossing  $\vec{v}$  and  $\vec{w}$ , one gets a vector whose magnitude is the area of the base. Dotting this vector with  $\vec{u}$  computes the volume of the parallelepiped! (Up to a sign; take the absolute value.)

Thus the volume of a parallelepiped defined by vectors  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  is

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})|. \quad (2.4.4)$$

Note how this is the Triple Scalar Product, first seen in Theorem 2.4.4. Applying the identities given in the theorem shows that we can apply the Triple Scalar Product in any “order” we choose to find the volume. That is,

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})| = |\vec{u} \cdot (\vec{w} \times \vec{v})| = |(\vec{u} \times \vec{v}) \cdot \vec{w}|, \text{ etc.}$$

**Example 2.4.14** Finding the volume of parallelepiped.

Find the volume of the parallelepiped defined by the vectors  $\vec{u} =$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

**Solution.** We apply Equation (2.4.4). We first find  $\vec{v} \times \vec{w} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ .

Then

$$|\vec{u} \cdot (\vec{v} \times \vec{w})| = \left| \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right| = 2.$$

So the volume of the parallelepiped is 2 cubic units.

Let’s take another look at how Equation (2.4.4) is computed in terms of our

formulas for the dot and cross products. With  $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ , and

$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ , we have

$$\begin{aligned} \vec{u} \cdot (\vec{v} \times \vec{w}) &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \\ -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \\ \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \end{bmatrix} \\ &= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}. \end{aligned}$$

Compare this with our determinant formula for computing the cross product,

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \vec{k}.$$



If we replace the unit vectors  $\vec{i}, \vec{j}, \vec{k}$  in the above equation with the components of  $\vec{u}$ , we arrive at our first instance of a  $3 \times 3$  **determinant**, along with a method for computing such an object:

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = \vec{u} \cdot (\vec{v} \times \vec{w}).$$

We will return to our study of determinants in Section 6.3, where we will learn techniques for efficiently computing determinants of any size.

**Torque.** *Torque* is a measure of the turning force applied to an object. A classic scenario involving torque is the application of a wrench to a bolt. When a force is applied to the wrench, the bolt turns. When we represent the force and wrench with vectors  $\vec{F}$  and  $\vec{\ell}$ , we see that the bolt moves (because of the threads) in a direction orthogonal to  $\vec{F}$  and  $\vec{\ell}$ . Torque is usually represented by the Greek letter  $\tau$ , or tau, and has units of N·m, a Newton-meter, or ft·lb, a foot-pound.

While a full understanding of torque is beyond the purposes of this book, when a force  $\vec{F}$  is applied to a lever arm  $\vec{\ell}$ , the resulting torque is

$$\vec{\tau} = \vec{\ell} \times \vec{F}. \quad (2.4.5)$$

#### Example 2.4.16 Computing torque.

A lever of length 2ft makes an angle with the horizontal of  $45^\circ$ . Find the resulting torque when a force of 10lb is applied to the end of the lever where:

1. the force is perpendicular to the lever, and
2. the force makes an angle of  $60^\circ$  with the lever, as shown in Figure 2.4.17.

#### Solution.

1. We start by determining vectors for the force and lever arm. Since the lever arm makes a  $45^\circ$  angle with the horizontal and is 2ft long, we can state that  $\vec{\ell} = 2 \begin{bmatrix} \cos(45^\circ) \\ \sin(45^\circ) \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$ . Since the force vector is perpendicular to the lever arm (as seen in the left hand side of Figure 2.4.17), we can conclude it is making an angle of  $-45^\circ$  with the horizontal. As it has a magnitude of 10lb, we can state  $\vec{F} = 10 \begin{bmatrix} \cos(-45^\circ) \\ \sin(-45^\circ) \end{bmatrix} = \begin{bmatrix} 5\sqrt{2} \\ -5\sqrt{2} \end{bmatrix}$ . Using Equation (2.4.5) to find the torque requires a cross product. We again let the third component of each vector be 0 and compute the cross product:

$$\begin{aligned} \vec{\tau} &= \vec{\ell} \times \vec{F} \\ &= \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix} \times \begin{bmatrix} 5\sqrt{2} \\ -5\sqrt{2} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ -20 \end{bmatrix} \end{aligned}$$

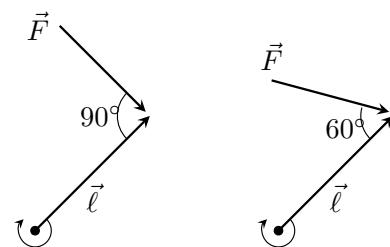


Figure 2.4.17 Showing a force being applied to a lever in Example 2.4.16

This clearly has a magnitude of 20 ft-lb. We can view the force and lever arm vectors as lying “on the page”; our computation of  $\vec{\tau}$  shows that the torque goes “into the page.” This follows the Right Hand Rule of the cross product, and it also matches well with the example of the wrench turning the bolt. Turning a bolt clockwise moves it in.

2. Our lever arm can still be represented by  $\vec{\ell} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$ . As our force vector makes a  $60^\circ$  angle with  $\vec{\ell}$ , we can see (referencing the right hand side of the figure) that  $\vec{F}$  makes a  $-15^\circ$  angle with the horizontal. Thus

$$\begin{aligned} \vec{F} &= 10 \begin{bmatrix} \cos -15^\circ \\ \sin -15^\circ \end{bmatrix} = \begin{bmatrix} \frac{5(1+\sqrt{3})}{\sqrt{2}} \\ \frac{5(-1+\sqrt{3})}{\sqrt{2}} \end{bmatrix} \\ &\approx \begin{bmatrix} 9.659 \\ -2.588 \end{bmatrix}. \end{aligned}$$

We again make the third component 0 and take the cross product to find the torque:

$$\begin{aligned} \vec{\tau} &= \vec{\ell} \times \vec{F} \\ &= \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix} \times \begin{bmatrix} \frac{5(1+\sqrt{3})}{\sqrt{2}} \\ \frac{5(-1+\sqrt{3})}{\sqrt{2}} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ -10\sqrt{3} \end{bmatrix} \\ &\approx \begin{bmatrix} 0 \\ 0 \\ -17.321 \end{bmatrix}. \end{aligned}$$

As one might expect, when the force and lever arm vectors *are* orthogonal, the magnitude of force is greater than when the vectors *are not* orthogonal.

While the cross product has a variety of applications (as noted in this chapter), its fundamental use is finding a vector perpendicular to two others. Knowing a vector is orthogonal to two others is of incredible importance, as it allows us to find the equations of lines and planes in a variety of contexts. The importance of the cross product, in some sense, relies on the importance of lines and planes, which see widespread use throughout engineering, physics and mathematics. We study lines and planes in the next two sections.

## 2.4.3 Exercises

**Exercise Group.** In the following exercises, vectors  $\vec{u}$  and  $\vec{v}$  are given. Compute  $\vec{u} \times \vec{v}$  and check that this vector is orthogonal to both  $\vec{u}$  and  $\vec{v}$ .

1. Let  $\vec{u} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$ .

2. Let  $\vec{u} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$ .

3. Let  $\vec{u} = \begin{bmatrix} 4 \\ -5 \\ -5 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$ .

4. Let  $\vec{u} = \begin{bmatrix} -4 \\ 7 \\ -10 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$ .

5. Let  $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix}$ .

6. Let  $\vec{u} = \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} -2 \\ -10 \\ 8 \end{bmatrix}$ .

7.  $\vec{u} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} c \\ d \\ 0 \end{bmatrix}$

8.  $\vec{u} = \hat{i}$ ,  $\vec{v} = \hat{j}$ .  
Check this is orthogonal to both  $\vec{u}$  and  $\vec{v}$ .

9.  $\vec{u} = \hat{i}$ ,  $\vec{v} = \hat{k}$ .

10.  $\vec{u} = \hat{j}$ ,  $\vec{v} = \hat{k}$ .  
 $\vec{u} \times \vec{v} = \underline{\hspace{2cm}}$

11. Pick any vectors  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^3$  and show that  $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$ .

12. Pick any vectors  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^3$  and show that  $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$ .

**Exercise Group.** In the following exercises, the magnitudes of vectors  $\vec{u}$  and  $\vec{v}$  in  $\mathbb{R}^3$  are given, along with the angle  $\theta$  between them. Use this information to find the magnitude of  $\vec{u} \times \vec{v}$ .

13. If  $\|\vec{u}\| = 2$ ,  $\|\vec{v}\| = 5$ , and  $\theta = 30^\circ$  is the angle between  $\vec{u}$  and  $\vec{v}$ , then  $\|\vec{u} \times \vec{v}\| = \underline{\hspace{2cm}}$

14. If  $\|\vec{u}\| = 3$ ,  $\|\vec{v}\| = 7$ , and  $\theta = \pi/2$  is the angle between  $\vec{u}$  and  $\vec{v}$ , then  $\|\vec{u} \times \vec{v}\| = \underline{\hspace{2cm}}$

15. If  $\|\vec{u}\| = 3$ ,  $\|\vec{v}\| = 4$ , and  $\theta = \pi$  is the angle between  $\vec{u}$  and  $\vec{v}$ , then  $\|\vec{u} \times \vec{v}\| = \underline{\hspace{2cm}}$

16. If  $\|\vec{u}\| = 2$ ,  $\|\vec{v}\| = 5$ , and  $\theta = 5\pi/6$  is the angle between  $\vec{u}$  and  $\vec{v}$ , then  $\|\vec{u} \times \vec{v}\| = \underline{\hspace{2cm}}$

**Exercise Group.** In the following exercises, find the area of the parallelogram defined by the given vectors.

17. Find the area of the parallelogram defined by

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \text{ and } \vec{v} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}.$$

18. Find the area of the parallelogram defined by

$$\vec{u} = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}, \text{ and } \vec{v} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}.$$

19. Find the area of the parallelogram defined by

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ and } \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

20. Find the area of the parallelogram defined by

$$\vec{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \text{ and } \vec{v} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

**Exercise Group.** In the following exercises, find the area of the triangle with the given vertices.

21. Find the area of the triangle with vertices  $(0, 0, 0)$ ,  $(1, 3, -1)$  and  $(2, 1, 1)$ .

22. Find the area of the triangle with vertices  $(5, 2, -1)$ ,  $(3, 6, 2)$  and  $(1, 0, 4)$ .

23. Find the area of the triangle with vertices  $(1, 1)$ ,  $(1, 3)$  and  $(2, 2)$ .

24. Find the area of the triangle with vertices  $(3, 1)$ ,  $(1, 2)$  and  $(4, 3)$ .

**Exercise Group.** In the following exercises, find the area of the quadrilateral with the given vertices. (Hint: break the quadrilateral into two triangles.)

25. Find the area of the quadrilateral with vertices  $(0, 0)$ ,  $(1, 2)$ ,  $(3, 0)$ , and  $(4, 3)$ .

26. Find the area of the quadrilateral with vertices  $(0, 0, 0)$ ,  $(2, 1, 1)$ ,  $(-1, 2, -8)$ , and  $(1, -1, 5)$ .

**Exercise Group.** In the following exercises, find the volume of the parallelepiped defined by the given vectors.

27. Find the volume of the parallelepiped defined

$$\text{by } \vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \text{ and } \vec{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

28. Find the volume of the parallelepiped defined

$$\text{by } \vec{u} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \text{ and } \vec{w} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}.$$

**Exercise Group.** In the following exercises, find a unit vector orthogonal to both  $\vec{u}$  and  $\vec{v}$ .

29. Find a unit vector orthogonal to both  $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ,

$$\text{and } \vec{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

30. Find a unit vector orthogonal to both

$$\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \text{ and } \vec{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

31. Find a unit vector orthogonal to both  $\vec{u} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$ ,

$$\text{and } \vec{v} = \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}.$$

32. Find a unit vector orthogonal to both

$$\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \text{ and } \vec{v} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}.$$

33. A bicycle rider applies 150lb of force, straight down, onto a pedal that extends 7in horizontally from the crankshaft. Find the magnitude of the torque applied to the crankshaft.
34. A bicycle rider applies 150lb of force, straight down, onto a pedal that extends 7in from the crankshaft, making a  $30^\circ$  angle with the horizontal. Find the magnitude of the torque applied to the crankshaft.
35. To turn a stubborn bolt, 80lb of force is applied to a 10in wrench. What is the maximum amount of torque that can be applied to the bolt?
36. To turn a stubborn bolt, 80lb of force is applied to a 10in wrench in a confined space, where the direction of applied force makes a  $10^\circ$  angle with the wrench. How much torque is subsequently applied to the wrench?
37. Show, using the definition of the Cross Product, that  $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$ ; that is, that  $\vec{u}$  is orthogonal to the cross product of  $\vec{u}$  and  $\vec{v}$ .
38. Show, using the definition of the Cross Product, that  $\vec{u} \times \vec{u} = \vec{0}$ .

## 2.5 Lines

To find the equation of a line in the  $xy$ -plane, we need two pieces of information: a point and the slope. The slope conveys *direction* information. As vertical lines have an undefined slope, the following statement is more accurate:

To define a line, one needs a point on the line and the direction of the line.

This holds true for lines in space.

### 2.5.1 Lines in space

Let  $P$  be a point in space, let  $\vec{p}$  be the vector with initial point at the origin and terminal point at  $P$  (i.e.,  $\vec{p}$  “points” to  $P$ ), and let  $\vec{d}$  be a vector. Consider the points on the line through  $P$  in the direction of  $\vec{d}$ .

Clearly one point on the line is  $P$ ; we can say that the vector  $\vec{p}$  lies at this point on the line. To find another point on the line, we can start at  $\vec{p}$  and move in a direction parallel to  $\vec{d}$ . For instance, starting at  $\vec{p}$  and traveling one length of  $\vec{d}$  places one at another point on the line. Consider Figure 2.5.1 where certain points along the line are indicated.

The figure illustrates how every point on the line can be obtained by starting with  $\vec{p}$  and moving a certain distance in the direction of  $\vec{d}$ . That is, we can define the line as a function of  $t$ :

$$\vec{\ell}(t) = \vec{p} + t\vec{d}. \quad (2.5.1)$$

In many ways, this is *not* a new concept. Compare Equation (2.5.1) to the familiar “ $y = mx + b$ ” equation of a line:

The equations exhibit the same structure: they give a starting point, define a direction, and state how far in that direction to travel.

There are other ways to represent a line. Let  $P = (x_0, y_0, z_0)$ ,  $\vec{p} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ , and let  $\vec{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ . Then the equation of the line through  $P$  in the direction of  $\vec{d}$  is:

$$\begin{aligned} \vec{\ell}(t) &= \vec{p} + t\vec{d} \\ &= \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= \begin{bmatrix} x_0 + at \\ y_0 + bt \\ z_0 + ct \end{bmatrix}. \end{aligned}$$

The last line states that the  $x$  values of the line are given by  $x = x_0 + at$ , the  $y$  values are given by  $y = y_0 + bt$ , and the  $z$  values are given by  $z = z_0 + ct$ . These three equations, taken together, are the *parametric equations of the line* through  $\vec{p}$  in the direction of  $\vec{d}$ .

Finally, each of the equations for  $x$ ,  $y$  and  $z$  above contain the variable  $t$ . We can solve for  $t$  in each equation:

$$\begin{aligned} x &= x_0 + at \Rightarrow t = \frac{x - x_0}{a}, \\ y &= y_0 + bt \Rightarrow t = \frac{y - y_0}{b}, \end{aligned}$$

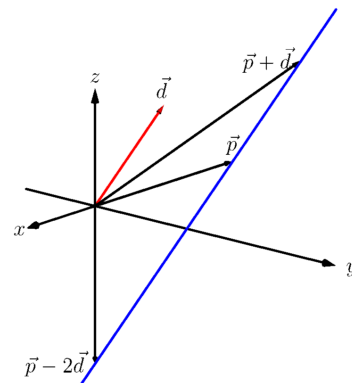


Figure 2.5.1 Defining a line in space

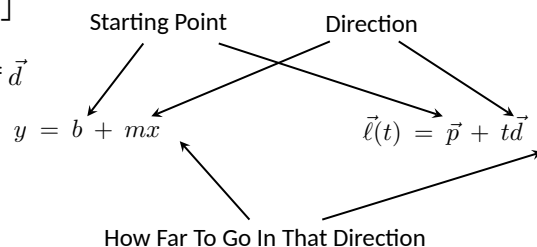


Figure 2.5.2 Understanding the vector equation of a line

$$z = z_0 + ct \Rightarrow t = \frac{z - z_0}{c},$$

assuming  $a, b, c \neq 0$ . Since  $t$  is equal to each expression on the right, we can set these equal to each other, forming the *symmetric equations of the line* through  $\vec{p}$  in the direction of  $\vec{d}$ :

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Each representation has its own advantages, depending on the context. We summarize these three forms in the following definition, then give examples of their use.

**Definition 2.5.3 Equations of Lines in Space.**

Let  $P = (x_0, y_0, z_0)$  and let  $\vec{p} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ . Consider the line in space

that passes through  $P$  in the direction of  $\vec{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ .

1. The *vector equation* of the line is

$$\vec{\ell}(t) = \vec{p} + t\vec{d}.$$

2. The *parametric equations* of the line are

$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct.$$

3. The *symmetric equations* of the line are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

**Example 2.5.4 Finding the equation of a line.**

Give all three equations, as given in Definition 2.5.3, of the line through  $P = (2, 3, 1)$  in the direction of  $\vec{d} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ . Does the point  $Q = (-1, 6, 6)$  lie on this line?

**Solution.** We identify the point  $P = (2, 3, 1)$  with the vector  $\vec{p} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ .

Following the definition, we have

- the vector equation of the line is  $\vec{\ell}(t) = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ ;

- the parametric equations of the line are

$$x = 2 - t, y = 3 + t, z = 1 + 2t; \text{ and}$$

- the symmetric equations of the line are

$$\frac{x-2}{-1} = \frac{y-3}{1} = \frac{z-1}{2}.$$

The first two equations of the line are useful when a  $t$  value is given: one can immediately find the corresponding point on the line. These forms are good when calculating with a computer; most software programs easily handle equations in these formats. (For instance, the graphics program that made Figure 2.5.5 can be given the input “(2-t, 3+t, 1+2\*t)” for  $-1 \leq t \leq 3$ .)

Does the point  $Q = (-1, 6, 6)$  lie on the line? The graph in Figure 2.5.5 makes it clear that it does not. We can answer this question without the graph using any of the three equation forms. Of the three, the symmetric equations are probably best suited for this task. Simply plug in the values of  $x$ ,  $y$  and  $z$  and see if equality is maintained:

$$\frac{-1-2}{-1} \stackrel{?}{=} \frac{6-3}{1} \stackrel{?}{=} \frac{6-1}{2} \Rightarrow 3 = 3 \neq 2.5.$$

We see that  $Q$  does not lie on the line as it did not satisfy the symmetric equations.

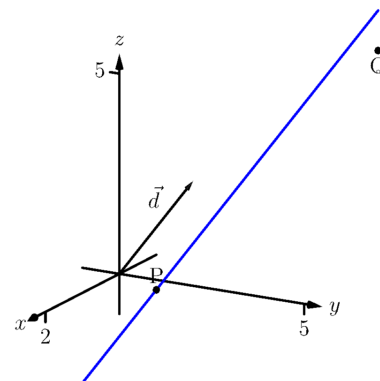


Figure 2.5.5 Graphing a line in Example 2.5.4

#### Example 2.5.6 Finding the equation of a line through two points.

Find the parametric equations of the line through the points  $P = (2, -1, 2)$  and  $Q = (1, 3, -1)$ .

**Solution.** Recall the statement made at the beginning of this section: to find the equation of a line, we need a point and a direction. We have two points; either one will suffice. The direction of the line can be found

by the vector with initial point  $P$  and terminal point  $Q$ :  $\overrightarrow{PQ} = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$ .

The parametric equations of the line  $\ell$  through  $P$  in the direction of  $\overrightarrow{PQ}$  are:

$$\ell : x = 2 - t, y = -1 + 4t, z = 2 - 3t.$$

A graph of the points and line are given in Figure 2.5.7. Note how in the given parametrization of the line,  $t = 0$  corresponds to the point  $P$ , and  $t = 1$  corresponds to the point  $Q$ . This relates to the understanding of the vector equation of a line described in Figure 2.5.2. The parametric equations “start” at the point  $P$ , and  $t$  determines how far in the direction of  $\overrightarrow{PQ}$  to travel. When  $t = 0$ , we travel 0 lengths of  $\overrightarrow{PQ}$ ; when  $t = 1$ , we travel one length of  $\overrightarrow{PQ}$ , resulting in the point  $Q$ .

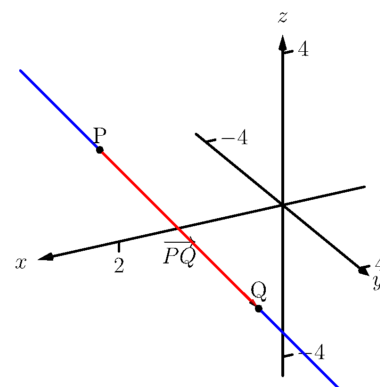


Figure 2.5.7 A graph of the line in Example 2.5.6

### 2.5.2 Parallel, Intersecting and Skew Lines

In the plane, two *distinct* lines can either be parallel or they will intersect at exactly one point. In space, given equations of two lines, it can sometimes be difficult to tell whether the lines are distinct or not (i.e., the same line can be represented in different ways). Given lines  $\vec{\ell}_1(t) = \vec{p}_1 + t\vec{d}_1$  and  $\vec{\ell}_2(t) = \vec{p}_2 + t\vec{d}_2$ , we have four possibilities:  $\vec{\ell}_1$  and  $\vec{\ell}_2$  are

the same line	they share all points
intersecting lines	they share only 1 point;
parallel lines	$\vec{d}_1 \parallel \vec{d}_2$ , no points in common;
skew lines	$\vec{d}_1 \nparallel \vec{d}_2$ , no points in common.

The next two examples investigate these possibilities.

### Example 2.5.8 Comparing lines.

Consider lines  $\ell_1$  and  $\ell_2$ , given in parametric equation form:

$$\begin{array}{rcl} x & = & 1 + 3t \\ \ell_1 : y & = & 2 - t \\ z & = & t \end{array} \qquad \begin{array}{rcl} x & = & -2 + 4s \\ \ell_2 : y & = & 3 + s \\ z & = & 5 + 2s \end{array}.$$

Determine whether  $\ell_1$  and  $\ell_2$  are the same line, intersect, are parallel, or skew.

**Solution.** We start by looking at the directions of each line. Line  $\ell_1$  has the direction given by  $\vec{d}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$  and line  $\ell_2$  has the direction given by

$$\vec{d}_2 = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

It should be clear that  $\vec{d}_1$  and  $\vec{d}_2$  are not parallel, hence  $\ell_1$  and  $\ell_2$  are not the same line, nor are they parallel. Figure 2.5.9 verifies this fact (where the points and directions indicated by the equations of each line are identified).

We next check to see if they intersect (if they do not, they are skew lines). To find if they intersect, we look for  $t$  and  $s$  values such that the respective  $x$ ,  $y$  and  $z$  values are the same. That is, we want  $s$  and  $t$  such that:

$$\begin{array}{rcl} 1 + 3t & = & -2 + 4s \\ 2 - t & = & 3 + s \\ t & = & 5 + 2s \end{array}.$$

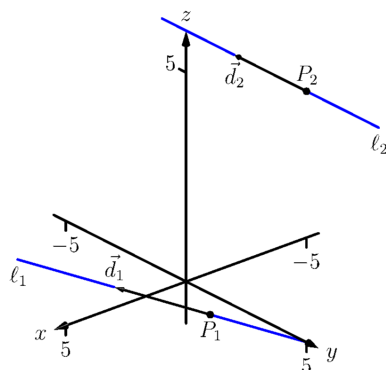
This is a relatively simple system of linear equations. Since the last equation is already solved for  $t$ , substitute that value of  $t$  into the equation above it:

$$2 - (5 + 2s) = 3 + s \Rightarrow s = -2, t = 1.$$

A key to remember is that we have *three* equations; we need to check if  $s = -2$ ,  $t = 1$  satisfies the first equation as well:

$$1 + 3(1) \neq -2 + 4(-2).$$

It does not. Therefore, we conclude that the lines  $\ell_1$  and  $\ell_2$  are skew.



**Figure 2.5.9** Sketching the lines from Example 2.5.8

We say that a system of equations with no solution, such as the one in Example 2.5.8, is **inconsistent**. Although it is possible to find values that work for any two of the three equations, there is no set of values of  $s$  and  $t$  that works for all three equations simultaneously. We'll develop general techniques for investigating such systems in Chapter 3.



**Example 2.5.10 Comparing lines.**

Consider lines  $\ell_1$  and  $\ell_2$ , given in parametric equation form:

$$\begin{array}{rcl} x & = & -0.7 + 1.6t \\ \ell_1 : y & = & 4.2 + 2.72t \\ z & = & 2.3 - 3.36t \end{array} \qquad \begin{array}{rcl} x & = & 2.8 - 2.9s \\ \ell_2 : y & = & 10.15 - 4.93s \\ z & = & -5.05 + 6.09s. \end{array}$$

Determine whether  $\ell_1$  and  $\ell_2$  are the same line, intersect, are parallel, or skew.

**Solution.** It is obviously very difficult to simply look at these equations and discern anything. This is done intentionally. In the “real world,” most equations that are used do not have nice, integer coefficients. Rather, there are lots of digits after the decimal and the equations can look “messy.”

We again start by deciding whether or not each line has the same direction. The direction of  $\ell_1$  is given by  $\vec{d}_1 = \begin{bmatrix} 1.6 \\ 2.72 \\ -3.36 \end{bmatrix}$  and the di-

rection of  $\ell_2$  is given by  $\vec{d}_2 = \begin{bmatrix} -2.9 \\ -4.93 \\ 6.09 \end{bmatrix}$ . When it is not clear through

observation whether two vectors are parallel or not, the standard way of determining this is by comparing their respective unit vectors. Using a calculator, we find:

$$\begin{aligned} \vec{u}_1 &= \frac{\vec{d}_1}{\|\vec{d}_1\|} = \begin{bmatrix} 0.3471 \\ 0.5901 \\ -0.7289 \end{bmatrix} \\ \vec{u}_2 &= \frac{\vec{d}_2}{\|\vec{d}_2\|} = \begin{bmatrix} -0.3471 \\ -0.5901 \\ 0.7289 \end{bmatrix}. \end{aligned}$$

The two vectors seem to be parallel (at least, their components are equal to 4 decimal places). In most situations, it would suffice to conclude that the lines are at least parallel, if not the same. One way to be sure is to rewrite  $\vec{d}_1$  and  $\vec{d}_2$  in terms of fractions, not decimals. We have

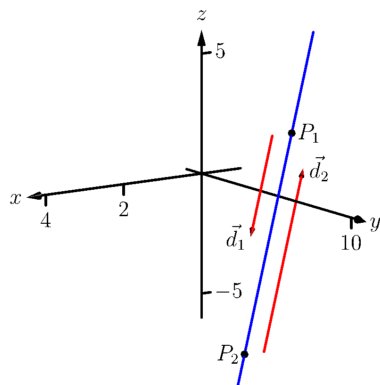
$$\vec{d}_1 = \begin{bmatrix} \frac{16}{10} \\ \frac{272}{100} \\ -\frac{336}{100} \end{bmatrix} \qquad \vec{d}_2 = \begin{bmatrix} -\frac{29}{10} \\ -\frac{493}{100} \\ \frac{609}{100} \end{bmatrix}.$$

One can then find the magnitudes of each vector in terms of fractions, then compute the unit vectors likewise. After a lot of manual arithmetic (or after briefly using a computer algebra system), one finds that

$$\vec{u}_1 = \begin{bmatrix} \frac{\sqrt{\frac{10}{83}}}{\sqrt{830}} \\ \frac{17}{\sqrt{830}} \\ -\frac{21}{\sqrt{830}} \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} -\frac{\sqrt{\frac{10}{83}}}{\sqrt{830}} \\ -\frac{17}{\sqrt{830}} \\ \frac{21}{\sqrt{830}} \end{bmatrix}.$$

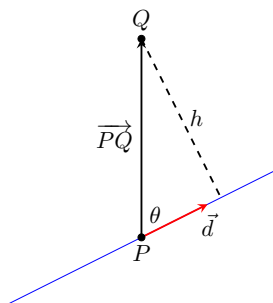
We can now say without equivocation that these lines are parallel.

Are they the same line? The parametric equations for a line describe one point that lies on the line, so we know that the point  $P_1 =$



**Figure 2.5.11** Graphing the lines in Example 2.5.10

It is possible to obtain a formula for the distance from  $Q$  to the line using a clever trick: consider the triangle that has  $\vec{d}$  and  $\vec{PQ}$  as two of its sides. The area of this triangle can be found using the usual formula  $A = \frac{1}{2}bh$ , but it can also be computed in terms of a cross product. See if you can use this observation to come up with a formula, but be cautioned that you will probably be expected to use the method outlined below.



**Figure 2.5.12** Establishing the distance from a point to a line

$(-0.7, 4.2, 2.3)$  lies on  $\ell_1$ . To determine if this point also lies on  $\ell_2$ , plug in the  $x$ ,  $y$  and  $z$  values of  $P_1$  into the symmetric equations for  $\ell_2$ :

$$\frac{(-0.7) - 2.8}{-2.9} \stackrel{?}{=} \frac{(4.2) - 10.15}{-4.93} \stackrel{?}{=} \frac{(2.3) - (-5.05)}{6.09}$$

$$1.2069 = 1.2069 = 1.2069.$$

The point  $P_1$  lies on both lines, so we conclude they are the same line, just parametrized differently. Figure 2.5.11 graphs this line along with the points and vectors described by the parametric equations. Note how  $\vec{d}_1$  and  $\vec{d}_2$  are parallel, though point in opposite directions (as indicated by their unit vectors above).

### 2.5.3 Distances

Given a point  $Q$  and a line  $\vec{\ell}(t) = \vec{p} + t\vec{d}$  in space, it is often useful to know the distance from the point to the line. (Here we use the standard definition of “distance,” i.e., the length of the shortest line segment from the point to the line.) Identifying  $\vec{p}$  with the point  $P$ , Figure 2.5.12 will help establish a general method of computing this distance  $h$ .

Our method will be based on the Orthogonal Projection from Section 2.3. Although it is possible to derive a formula that gives the distance immediately, our method will help develop geometric intuition that we can adapt to later situations, and it also has the advantage of letting us identify the point  $R$  on the line that is closest to  $Q$ .

The basic idea is that the vector  $\vec{PR}$  must be the orthogonal projection of the vector  $\vec{PQ}$  onto the direction vector  $\vec{d}$ .

#### Example 2.5.13 Finding the distance from a point to a line.

Find the distance from the point  $Q = (1, 1, 3)$  to the line  $\vec{\ell}(t) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ .

**Solution.** From the equation of the line, we know that the point  $P = (1, -1, 1)$  is on the line, and that the vector  $\vec{d} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$  is in the direction of the line. Let

$$\vec{v} = \vec{PQ} = \begin{bmatrix} 1 - 1 \\ 1 - (-1) \\ 3 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

We know that if  $R$  is the point on the line closest to  $Q$ , then the projection of  $\vec{v}$  onto  $\vec{d}$  will equal  $\vec{PR}$ .

We compute

$$\text{proj}_{\vec{d}} \vec{v} = \frac{\vec{v} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \vec{d} = \frac{0 + 6 + 2}{4 + 9 + 1} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \frac{4}{7} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

Now, if  $R = (x_1, y_1, z_1)$ , we know that  $\overrightarrow{PR} = \begin{bmatrix} x_1 - 1 \\ x_2 - (-1) \\ x_3 - 1 \end{bmatrix}$ , since  $P = (1, -1, 1)$ . This gives us  $\begin{bmatrix} x_1 - 1 \\ x_2 - (-1) \\ x_3 - 1 \end{bmatrix} = \frac{4}{7} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ , so  $x_1 = 1 + \frac{8}{7} = \frac{15}{7}$ ,  $x_2 = -1 + \frac{12}{7} = \frac{5}{7}$ , and  $x_3 = 1 + \frac{4}{7} = \frac{11}{7}$ . Therefore,

$$R = \left( \frac{15}{7}, \frac{5}{7}, \frac{11}{7} \right)$$

is the point on the line that is closest to  $Q$ .

The desired distance is therefore the distance from  $Q$  to  $R$ , which can be found using the distance formula:

$$\begin{aligned} h &= \sqrt{\left(1 - \frac{15}{7}\right)^2 + \left(1 - \frac{5}{7}\right)^2 + \left(3 - \frac{11}{7}\right)^2} \\ &= \sqrt{\left(-\frac{8}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(\frac{10}{7}\right)^2} \\ &= \frac{1}{7} \sqrt{8^2 + 2^2 + 10^2} \\ &= \frac{\sqrt{168}}{7} = 2\sqrt{\frac{6}{7}}. \end{aligned}$$

We summarize the steps involved to give the following general advice:

**Key Idea 2.5.14 Steps for solving shortest distance problems.**

Suppose you are asked to find the distance between two objects, or to determine an object (such as a point) that is closest to a given object (a line or plane). Your solution to the problem should always include the following steps:

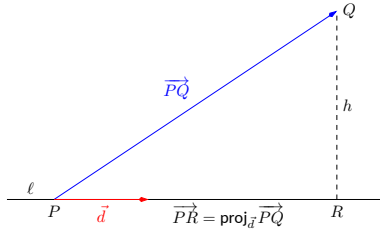
1. Make a list of all the information provided in the problem.
2. Make a note of what quantities you're asked to determine.
3. **Draw a diagram.** Label all relevant points and vectors, including those you know, and those you want to find.
4. Using your diagram as a reference, compute any unknown points or vectors.

**Note:** We can't overemphasize the fact that the diagram referred to in Key Idea 2.5.14 *does not have to be accurate* with respect to the coordinates and directions involved. It simply has to be capable of representing the information in the problem. Note that in Figure 2.5.16 in Example 2.5.15 we've drawn a line, some points, and some vectors that represent the problem, without reference to a coordinate system. The goal is to provide enough detail to allow us to set up the problem.

**Example 2.5.15 Finding the closest point on a line.**

Find the distance from the point  $Q = (1, 3, -2)$  to the line  $\ell$  that passes through the point  $P = (2, 0, -1)$  in the direction of  $\vec{d} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ , and find the point  $R$  on  $\ell$  that is closest to  $Q$ .

**Solution.** We're given a point  $P$  on the line, along with a direction vec-



**Figure 2.5.16** Setting up the solution in Example 2.5.15

tor  $\vec{d}$ , and a point  $Q$  not on the line. We seek the point  $R$  on the line that is closest to  $Q$ , as well as the distance from  $Q$  to  $R$ . We begin by diagramming the information in Figure 2.5.16.

From the given points  $P$  and  $Q$  we can immediately construct the vector

$$\vec{PQ} = \begin{bmatrix} 1-2 \\ 3-0 \\ -2-(-1) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}.$$

We begin by finding the point  $R$  on the line that is closest to  $Q$ . From our diagram, we can see that the vector  $\vec{PR}$  from  $P$  to  $R$  is equal to the projection of  $\vec{PQ}$  onto the distance vector  $\vec{d}$ :

$$\vec{PR} = \text{proj}_{\vec{d}} \vec{PQ} = \left( \frac{\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} \right) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}.$$

Now, we need to pause and take care that we don't make a very common mistake: the vector  $\vec{PR}$  does *not* give the coordinates of the point  $R$ . Instead,  $\vec{PR}$  tells us how to get *from* the point  $P$  to the point  $R$ . Letting  $O$  denote the origin, we can write  $\vec{OP}$  and  $\vec{OR}$  for the position vectors of  $P$  and  $R$ , respectively. Since  $\vec{PR} = \vec{OR} - \vec{OP}$  using the "tip minus tail" rule for computing the vector between two points, we have

$$\vec{OR} = \vec{OP} + \vec{PR} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}.$$

Thus, we have  $R = (0, 2, -1)$  as the point on the line closest to the point  $Q$ . We can now find the distance from  $Q$  to the line using the distance formula:

$$D = \sqrt{(1-0)^2 + (3-2)^2 + (-2-(-1))^2} = \sqrt{3}.$$

An alternative way of computing the distance is to make use of the orthogonal decomposition in Key Idea 2.3.19. By definition of the distance from a point to a line, we know that the vector  $\vec{RQ}$  must be orthogonal to the line, and thus to the direction vector  $\vec{d}$ . Using Key Idea 2.3.19, we have that

$$\vec{RQ} = \vec{PQ} - \vec{PR} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},$$

and the shortest distance is given by  $\|\vec{RQ}\| = \sqrt{3}$ , as before.

It is also useful to determine the distance between lines, which we define as the length of the shortest line segment that connects the two lines (an argument from geometry shows that this line segments is perpendicular to both lines). Let

lines  $\vec{\ell}_1(t) = \vec{p}_1 + t\vec{d}_1$  and  $\vec{\ell}_2(t) = \vec{p}_2 + t\vec{d}_2$  be given, as shown in Figure 2.5.17. To find the direction orthogonal to both  $\vec{d}_1$  and  $\vec{d}_2$ , we take the cross product:  $\vec{c} = \vec{d}_1 \times \vec{d}_2$ . The magnitude of the orthogonal projection of  $\overrightarrow{P_1P_2}$  onto  $\vec{c}$  is the distance  $h$  we seek:

$$h = \left\| \text{proj}_{\vec{c}} \overrightarrow{P_1P_2} \right\| = \left\| \frac{\overrightarrow{P_1P_2} \cdot \vec{c}}{\vec{c} \cdot \vec{c}} \vec{c} \right\| \quad (2.5.2)$$

$$= \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{\|\vec{c}\|^2} \|\vec{c}\| = \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{\|\vec{c}\|}. \quad (2.5.3)$$

A problem in the exercises of this section is to show that this distance is 0 when the lines intersect. Note the use of the Triple Scalar Product:  $\overrightarrow{P_1P_2} \cdot \vec{c} = \overrightarrow{P_1P_2} \cdot (\vec{d}_1 \times \vec{d}_2)$ .

### Example 2.5.18 Finding the distance between lines.

Find the distance between the lines

$$\begin{array}{ll} x = 1 + 3t & x = -2 + 4s \\ \ell_1 : y = 2 - t & \ell_2 : y = 3 + s \\ z = t & z = 5 + 2s. \end{array}$$

**Solution.** These are the same lines as given in Example 2.5.8, where we showed them to be skew. The equations allow us to identify the following points and vectors:

$$P_1 = (1, 2, 0), P_2 = (-2, 3, 5) \Rightarrow \overrightarrow{P_1P_2} = \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}.$$

$$\vec{d}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \vec{d}_2 = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \vec{c} = \vec{d}_1 \times \vec{d}_2 = \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix}.$$

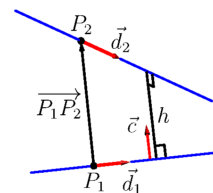
Using Equation (2.5.3) we have the distance  $h$  between the two lines is

$$\begin{aligned} h &= \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{\|\vec{c}\|} \\ &= \frac{42}{\sqrt{62}} \approx 5.334. \end{aligned}$$

The lines are approximately 5.334 units apart.

Once again, we do not recommend attempting to memorize Equation (2.5.3). Unless you somehow find yourself at a point in your life where you need to find the distances between a whole lot of pairs of skew lines, you will be better served by learning the skills required to set up and think through a problem than you will be by memorizing a formula to plug numbers into. In the case of skew lines, the key observation is that if we take the vector between *any* pair of points, one on each line, and project it onto the vector  $\vec{c} = \vec{d}_1 \times \vec{d}_2$ , the length of the resulting vector is the distance we seek.

Somewhat more challenging is the problem of finding the points on each line that actually *realize* this shortest distance.



**Figure 2.5.17** Establishing the distance between lines

It turns out that skew lines always lie in parallel planes. In the next section, we'll see that a plane can be determined by two non-parallel direction vectors and a point on the plane. The distance between two skew lines is then equal to the distance between the parallel planes.

**Example 2.5.19 Finding the closest points on skew lines.**

Find the points  $R_1$  on  $\vec{\ell}_1$  and  $R_2$  on  $\vec{\ell}_2$ , where  $\vec{\ell}_1$  and  $\vec{\ell}_2$  are the lines from Example 2.5.18, such that the distance from  $R_1$  to  $R_2$  is a minimum.

**Solution.** Since  $R_1$  is a point on  $\vec{\ell}_1$ , we know that

$$R_1 = (1 + 3t, 2 - t, t), \quad \text{for some real number } t, \quad (2.5.4)$$

and similarly,

$$R_2 = (-2 + 4s, 3 + s, 5 + 2s), \quad \text{for some real number } s. \quad (2.5.5)$$

The vector  $\overrightarrow{R_1R_2}$  is therefore given by

$$\overrightarrow{R_1R_2} = \begin{bmatrix} -3 + 4s - 3t \\ 1 + s + t \\ 5 + 2s - t \end{bmatrix},$$

for some pair of real numbers  $s$  and  $t$ .

We know that the line segment  $\overline{R_1R_2}$  must be perpendicular to both  $\vec{\ell}_1$  and  $\vec{\ell}_2$  in order to minimize the distance, so the vector  $\overrightarrow{R_1R_2}$  must be orthogonal to both  $\vec{d}_1$  and  $\vec{d}_2$ . Thus,

$$\begin{aligned} 0 = \vec{d}_1 \cdot \overrightarrow{R_1R_2} &= 3(-3 + 4s - 3t) - 1(1 + s + t) + 1(5 + 2s - t) \\ &= 13s - 11t - 5, \text{ and} \end{aligned}$$

$$\begin{aligned} 0 = \vec{d}_2 \cdot \overrightarrow{R_1R_2} &= 4(-3 + 4s - 3t) + 1(1 + s + t) + 2(5 + 2s - t) \\ &= 21s - 13t - 1. \end{aligned}$$

We end up having to solve a system of two linear equations in the two variables,  $s$  and  $t$ , given by

$$\begin{array}{rcl} 13s & - & 11t = 5 \\ 21s & - & 13t = 1 \end{array}.$$

You probably had to solve such systems in high school. One option is to solve graphically, by plotting the lines given by each equation, and seeing where they intersect. However, this method has little hope of providing an accurate answer. Instead, we try a little algebra.

Multiplying the first equation by 21 and the second by 13 gives us the equations  $273s - 231t = 105$  and  $273s - 169t = 13$ , respectively. Subtracting the second equation from the first, we have  $-62t = 92$ , so  $t = -\frac{92}{62} = -\frac{46}{31}$ . Plugging this value back into any of the previous equations gives us  $s = -\frac{351}{403} = -\frac{27}{31}$ . (We didn't promise that the numbers would work out nicely!) Plugging these values back into equations (2.5.4) and (2.5.5), we find

$$R_1 = \left(-\frac{107}{31}, \frac{108}{31}, -\frac{46}{31}\right) \quad \text{and} \quad R_2 = \left(-\frac{170}{31}, \frac{66}{31}, \frac{101}{31}\right).$$

Our vector  $\overrightarrow{R_1R_2}$  is then given by

$$\overrightarrow{R_1R_2} = \begin{bmatrix} -\frac{63}{31} \\ -\frac{42}{31} \\ \frac{147}{31} \end{bmatrix} = \frac{1}{31} \begin{bmatrix} -63 \\ -42 \\ 147 \end{bmatrix},$$

and the distance between the two lines is given by

$$\left\| \overrightarrow{R_1 R_2} \right\| = \frac{1}{31} \sqrt{63^2 + 42^2 + 147^2} = \frac{42}{\sqrt{62}},$$

as before.

Example 2.5.19 required us to solve a system of two linear equations in two unknowns  $s$  and  $t$ . Although this involved some messy fractions, the algebra involved was fairly straightforward. In many real life problems it is necessary to be able to solve systems involving hundreds or even thousands of equations and variables. We will begin our study of how to systematically solve such systems in the next chapter.

One of the key points to understand from this section is this: to describe a line, we need a point and a direction. Whenever a problem is posed concerning a line, one needs to take whatever information is offered and glean point and direction information. Many questions can be asked (and *are* asked in the Exercise section) whose answer immediately follows from this understanding.

Lines are one of two fundamental objects of study in space. The other fundamental object is the *plane*, which we study in detail in the next section. Many complex three dimensional objects are studied by approximating their surfaces with lines and planes.

## 2.5.4 Exercises

**Exercise Group.** Write the vector, parametric and symmetric equations of the lines described.

1. Passes through  $P = (2, -4, 1)$ , parallel to

$$\vec{d} = \begin{bmatrix} 9 \\ 2 \\ 5 \end{bmatrix}.$$

3. Passes through  $P = (2, 1, 5)$  and  $Q = (7, -2, 4)$ .

5. Passes through  $P = (0, 1, 2)$  and orthogonal to both

$$\vec{d}_1 = \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix} \text{ and } \vec{d}_2 = \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix}.$$

7.  $\ell$  is a line that passes through the intersection

$$\text{of } \vec{\ell}_1(t) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix} \text{ and}$$

$$\vec{\ell}_2(t) = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \text{ and is orthogonal to both lines.}$$

9. Passes through  $P = (1, 1)$ , parallel to  $\vec{d} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ .

2.  $\ell$  is a line that passes through  $P = (6, 1, 7)$ ,

$$\text{parallel to } \vec{d} = \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix}.$$

4.  $\ell$  is a line that passes through  $P = (1, -2, 3)$  and  $Q = (5, 5, 5)$ .

6.  $\ell$  is a line that passes through  $P = (5, 1, 9)$  and is orthogonal to both  $\vec{d}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$  and  $\vec{d}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ .

8.  $\ell$  is a line that passes through the intersection

$$\text{of } \vec{\ell}_1(t) = \begin{cases} x = t \\ y = -2 + 2t \\ z = 1 + t \end{cases} \text{ and}$$

$$\vec{\ell}_2(t) = \begin{cases} x = 2 + t \\ y = 2 - t \\ z = 3 + 2t \end{cases}, \text{ and is orthogonal to both lines.}$$

10.  $\ell$  is a line that passes through  $P = (-2, 5)$ , parallel to  $\vec{d} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

**Exercise Group.** Determine if the described lines are the same line, parallel lines, intersecting or skew lines. If intersecting, give the point of intersection.

11.  $\vec{\ell}_1(t) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$  and

$$\vec{\ell}_2(t) = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} + t \begin{bmatrix} -4 \\ 2 \\ -2 \end{bmatrix}.$$

13.  $\vec{\ell}_1(t) = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix},$

$$\vec{\ell}_2(t) = \begin{bmatrix} -3 \\ 3 \\ -3 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}.$$

15.  $\vec{\ell}_1(t) = \begin{cases} x = 1 + 2t \\ y = 3 - 2t \\ z = t \end{cases}$

$$\vec{\ell}_2(t) = \begin{cases} x = 3 - t \\ y = 3 + 5t \\ z = 2 + 7t \end{cases}.$$

12.  $\vec{\ell}_1(t) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$  and

$$\vec{\ell}_2(t) = \begin{bmatrix} 14 \\ 5 \\ 9 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

14.  $\vec{\ell}_1(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$  and

$$\vec{\ell}_2(t) = \begin{bmatrix} 7 \\ 3 \\ 7 \end{bmatrix} + t \begin{bmatrix} 6 \\ 2 \\ 6 \end{bmatrix}.$$

16.  $\vec{\ell}_1(t) = \begin{cases} x = 1.1 + 0.6t \\ y = 3.77 + 0.9t \\ z = -2.3 + 1.5t \end{cases}$  and

$$\vec{\ell}_2(t) = \begin{cases} x = 3.11 + 3.4t \\ y = 2 + 5.1t \\ z = 2.5 + 8.5t \end{cases}.$$



$$17. \quad \ell_1 = \begin{cases} x = 0.2 + 0.6t \\ y = 1.33 - 0.45t \\ z = -4.2 + 1.05t \end{cases} \text{ and } \ell_2 = \begin{cases} x = 0.86 + 9.2t \\ y = 0.835 - 6.9t \\ z = -3.045 + 16.1t \end{cases}$$

$$18. \quad \vec{\ell}_1(t) = \begin{cases} x = 0.1 + 1.1t \\ y = 2.9 - 1.5t \\ z = 3.2 + 1.6t \end{cases} \text{ and } \vec{\ell}_2(t) = \begin{cases} x = 4 - 2.1t \\ y = 1.8 + 7.2t \\ z = 3.1 + 1.1t \end{cases}$$

**Exercise Group.** Find the distance from the point to the line.

$$19. \quad Q = (1, 1, 1), \vec{\ell}(t) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$21. \quad Q = (0, 3), \vec{\ell}(t) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

20. Find the distance from the point  $Q = (2, 5, 6)$

$$\text{to the line } \vec{\ell}(t) = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

22. Find the distance from the point  $Q = (1, 1)$  to

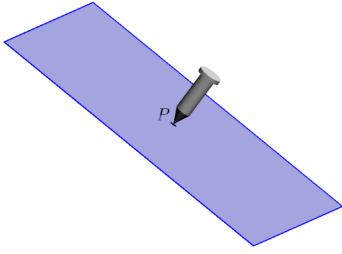
$$\text{the line } \vec{\ell}(t) = \begin{bmatrix} 4 \\ 5 \end{bmatrix} + t \begin{bmatrix} -4 \\ 3 \end{bmatrix}.$$

**Exercise Group.** Find the distance between the two lines.

$$23. \quad \vec{\ell}_1(t) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{\ell}_2(t) = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} + t \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}.$$

24. Find the distance between the line

$$\vec{\ell}_1(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and the line } \vec{\ell}_2(t) = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$



**Figure 2.6.1** Illustrating defining a plane with a sheet of cardboard and a nail

## 2.6 Planes

Any flat surface, such as a wall, table top or stiff piece of cardboard can be thought of as representing part of a plane. Consider a piece of cardboard with a point  $P$  marked on it. One can take a nail and stick it into the cardboard at  $P$  such that the nail is perpendicular to the cardboard; see Figure 2.6.1.

This nail provides a “handle” for the cardboard. Moving the cardboard around moves  $P$  to different locations in space. Tilting the nail (but keeping  $P$  fixed) tilts the cardboard. Both moving and tilting the cardboard defines a different plane in space. In fact, we can define a plane by: 1) the location of  $P$  in space, and 2) the direction of the nail.

The previous section showed that one can define a line given a point on the line and the direction of the line (usually given by a vector). One can make a similar statement about planes: we can define a plane in space given a point on the plane and the direction the plane “faces” (using the description above, the direction of the nail). Once again, the direction information will be supplied by a vector, called a *normal vector*, that is orthogonal to the plane.

What exactly does “orthogonal to the plane” mean? Choose any two points  $P$  and  $Q$  in the plane, and consider the vector  $\overrightarrow{PQ}$ . We say a vector  $\vec{n}$  is orthogonal to the plane if  $\vec{n}$  is perpendicular to  $\overrightarrow{PQ}$  for all choices of  $P$  and  $Q$ ; that is, if  $\vec{n} \cdot \overrightarrow{PQ} = 0$  for all  $P$  and  $Q$ .

This gives us way of writing an equation describing the plane. Let  $P = (x_0, y_0, z_0)$  be a point in the plane and let  $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$  be a normal vector to the plane. A point  $Q = (x, y, z)$  lies in the plane defined by  $P$  and  $\vec{n}$  if, and only if,  $\overrightarrow{PQ}$  is orthogonal to  $\vec{n}$ . Knowing  $\overrightarrow{PQ} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$ , consider:

$$\begin{aligned} \overrightarrow{PQ} \cdot \vec{n} &= 0 \\ \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0. \end{aligned} \tag{2.6.1}$$

Equation (2.6.1) defines an *implicit* function describing the plane. More algebra produces:

$$ax + by + cz = ax_0 + by_0 + cz_0.$$

The right hand side is just a number, so we replace it with  $d$ :

$$ax + by + cz = d. \tag{2.6.2}$$

As long as  $c \neq 0$ , we can solve for  $z$ :

$$z = \frac{1}{c}(d - ax - by). \tag{2.6.3}$$

Equation (2.6.3) is especially useful as many computer programs can graph functions in this form. Equations (2.6.1) and (2.6.2) have specific names, given next.

**Definition 2.6.2 Equations of a Plane in Standard and General Forms.**

The plane passing through the point  $P = (x_0, y_0, z_0)$  with normal vector  $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$  can be described by an equation with **standard form**

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0;$$

the equation's **general form** is

$$ax + by + cz = d.$$

A key to remember throughout this section is this: to find the equation of a plane, we need a point and a normal vector. We will give several examples of finding the equation of a plane, and in each one different types of information are given. In each case, we need to use the given information to find a point on the plane and a normal vector.

**Example 2.6.3 Finding the equation of a plane.**

Write the equation of the plane that passes through the points  $P = (1, 1, 0)$ ,  $Q = (1, 2, -1)$  and  $R = (0, 1, 2)$  in standard form.

**Solution.** We need a vector  $\vec{n}$  that is orthogonal to the plane. Since  $P$ ,  $Q$  and  $R$  are in the plane, so are the vectors  $\vec{PQ}$  and  $\vec{PR}$ ;  $\vec{PQ} \times \vec{PR}$  is orthogonal to  $\vec{PQ}$  and  $\vec{PR}$  and hence the plane itself.

It is straightforward to compute  $\vec{n} = \vec{PQ} \times \vec{PR} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ . We can use

any point we wish in the plane (any of  $P$ ,  $Q$  or  $R$  will do) and we arbitrarily choose  $P$ . Following Definition 2.6.2, the equation of the plane in standard form is

$$2(x - 1) + (y - 1) + z = 0.$$

The plane is sketched in Figure 2.6.4.

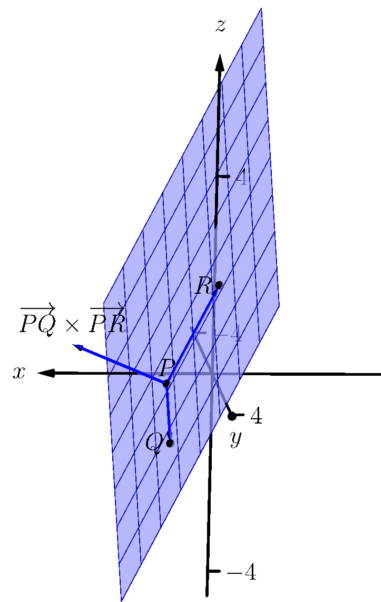
We have just demonstrated the fact that any three non-collinear points define a plane. (This is why a three-legged stool does not “rock;” its three feet always lie in a plane. A four-legged stool will rock unless all four feet lie in the same plane.)

**Example 2.6.5 Finding the equation of a plane.**

Verify that lines  $\ell_1$  and  $\ell_2$ , whose parametric equations are given below, intersect, then give the equation of the plane that contains these two lines in general form.

$$\begin{array}{ll} \ell_1 : x = -5 + 2s & x = 2 + 3t \\ y = 1 + s & y = 1 - 2t \\ z = -4 + 2s & z = 1 + t \end{array}$$

**Solution.** The lines clearly are not parallel. If they do not intersect, they are skew, meaning there is not a plane that contains them both. If



**Figure 2.6.4** Sketching the plane in Example 2.6.3

they do intersect, there is such a plane.

To find their point of intersection, we set the  $x$ ,  $y$  and  $z$  equations equal to each other and solve for  $s$  and  $t$ :

$$\begin{aligned} -5 + 2s &= 2 + 3t \\ 1 + s &= 1 - 2t \Rightarrow s = 2, t = -1. \\ -4 + 2s &= 1 + t \end{aligned}$$

When  $s = 2$  and  $t = -1$ , the lines intersect at the point  $P = (-1, 3, 0)$ .

Let  $\vec{d}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$  and  $\vec{d}_2 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$  be the directions of lines  $\ell_1$  and  $\ell_2$ ,

respectively. A normal vector to the plane containing these two lines will also be orthogonal to  $\vec{d}_1$  and  $\vec{d}_2$ . Thus we find a normal vector  $\vec{n}$  by computing  $\vec{n} = \vec{d}_1 \times \vec{d}_2 = \begin{bmatrix} 5 \\ 4 - 7 \end{bmatrix}$ .

We can pick any point in the plane with which to write our equation; each line gives us infinite choices of points. We choose  $P$ , the point of intersection. We follow Definition 2.6.2 to write the plane's equation in general form:

$$\begin{aligned} 5(x + 1) + 4(y - 3) - 7z &= 0 \\ 5x + 5 + 4y - 12 - 7z &= 0 \\ 5x + 4y - 7z &= 7. \end{aligned}$$

The plane's equation in general form is  $5x + 4y - 7z = 7$ ; it is sketched in Figure 2.6.6.

The two previous examples hint at an alternative method for describing a plane in  $\mathbb{R}^3$ : instead of providing a single direction orthogonal to the plane (given by the normal vector), we can give two directions that are *parallel* to the plane, such as the vectors  $\vec{PQ}$  and  $\vec{PR}$  in Figure 2.6.4, or the direction vectors  $\vec{d}_1$  and  $\vec{d}_2$  to the lines in Figure 2.6.6.

Suppose  $(x, y, z)$  is a point on the plane  $5x + 4y - 7z = 7$  from Example 2.6.5. We can treat the point  $(-1, 3, 0)$  where the lines  $\ell_1$  and  $\ell_2$  intersect as our point of reference on the plane. From this point, we can reach the point  $(x, y, z)$  by first travelling some distance in the direction of  $\vec{d}_1$  (parallel to  $\ell_1$ ), and then some distance in the direction of  $\vec{d}_2$  (parallel to  $\ell_2$ ). We can express this mathematically as follows:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} + s\vec{d}_1 + t\vec{d}_2 \quad (2.6.4)$$

$$= \begin{bmatrix} -1 + 2s + 3t \\ 3 + s - 2t \\ 2s + t \end{bmatrix}. \quad (2.6.5)$$

Equation (2.6.4) can be viewed as a two-dimensional analogue of the vector equation of a line given in the previous section. It tells us that to get from the origin  $(0, 0, 0)$  to the point  $(x, y, z)$  on the plane, we should first travel to the point  $(-1, 3, 0)$  on the plane, and then move parallel to the lines  $\ell_1$  and  $\ell_2$  until we reach our point. This vector equation for a plane is not particularly

We can think of the point  $P = (-1, 3, 0)$  in Example 2.6.5 as defining a point of “origin” on the plane, and even though they are not perpendicular, we can think of the lines  $\ell_1$  and  $\ell_2$  as a pair of coordinate axes on the plane. Any other point on the plane can be located with respect to these two lines. (Any two non-parallel lines define a coordinate system in a plane; perpendicular lines just happen to be more convenient than other options.)

useful in Science or Engineering applications, but it is useful mathematically. In particular, if we wanted to describe a two-dimensional plane in  $\mathbb{R}^4$  (or any higher-dimensional space), we would have to resort to this method. (Keeping this method for describing a plane in mind will also help us to access some geometric intuition when we discuss span and linear independence later in the text.)

### Example 2.6.7 Finding the equation of a plane.

Give the equation, in standard form, of the plane that passes through the point  $P = (-1, 0, 1)$  and is orthogonal to the line with vector equation  $\vec{\ell}(t) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ .

**Solution.** As the plane is to be orthogonal to the line, the plane must be orthogonal to the direction of the line given by  $\vec{d} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ . We use this as our normal vector. Thus the plane's equation, in standard form, is

$$(x + 1) + 2y + 2(z - 1) = 0.$$

The line and plane are sketched in Figure 2.6.8.

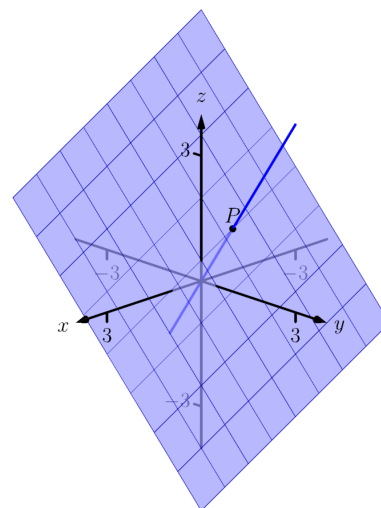


Figure 2.6.8 The line and plane in Example 2.6.7

### Example 2.6.9 Finding the intersection of two planes.

Give the parametric equations of the line that is the intersection of the planes  $p_1$  and  $p_2$ , where:

$$p_1 : x - (y - 2) + (z - 1) = 0$$

$$p_2 : -2(x - 2) + (y + 1) + (z - 3) = 0$$

**Solution.** To find an equation of a line, we need a point on the line and the direction of the line.

We can find a point on the line by solving each equation of the planes for  $z$ :

$$p_1 : z = -x + y - 1$$

$$p_2 : z = 2x - y - 2$$

We can now set these two equations equal to each other (i.e., we are finding values of  $x$  and  $y$  where the planes have the same  $z$  value):

$$-x + y - 1 = 2x - y - 2$$

$$2y = 3x - 1$$

$$y = \frac{1}{2}(3x - 1)$$

We can choose any value for  $x$ ; we choose  $x = 1$ . This determines that  $y = 1$ . We can now use the equations of either plane to find  $z$ : when  $x = 1$  and  $y = 1$ ,  $z = -1$  on both planes. We have found a point  $P$  on the line:  $P = (1, 1, -1)$ .

We now need the direction of the line. Since the line lies in each plane, its direction is orthogonal to a normal vector for each plane. Considering the equations for  $p_1$  and  $p_2$ , we can quickly determine their

The method used to solve for the point  $P$  in Example 2.6.9 would not be considered best practice in a linear algebra course, but we have not yet developed systematic methods for solving systems of equations.

Another option would be to pick a value for one coordinate (like the choice  $x = 1$ ), and then solve the remaining set of two equations in two variables ( $y$  and  $z$ ).

Later we will see that the general algorithm for solving systems of equations leads directly to parametric equations for the line of intersection of the two planes.

normal vectors. For  $p_1$ ,  $\vec{n}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$  and for  $p_2$ ,  $\vec{n}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ . A

direction orthogonal to both of these directions is their cross product:

$$\vec{d} = \vec{n}_1 \times \vec{n}_2 = \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix}.$$

The parametric equations of the line through  $P = (1, 1, -1)$  in the direction of  $d = \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix}$  is:

$$\ell : x = -2t + 1, y = -3t + 1, z = -t - 1.$$

The planes and line are graphed in Figure 2.6.10.

### Example 2.6.11 Finding the intersection of a plane and a line.

Find the point of intersection, if any, of the line  $\ell(t) = \begin{bmatrix} 3 \\ -3 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$  and the plane with equation in general form  $2x + y + z = 4$ .

**Solution.** The equation of the plane shows that the vector  $\vec{n} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$  is a normal vector to the plane, and the equation of the line shows that the line moves parallel to  $\vec{d} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ . Since these are not orthogonal, we know there is a point of intersection. (If there were orthogonal, it would mean that the plane and line were parallel to each other, either never intersecting or the line was in the plane itself.)

To find the point of intersection, we need to find a  $t$  value such that  $\ell(t)$  satisfies the equation of the plane. Rewriting the equation of the line with parametric equations will help:

$$\ell(t) = \begin{cases} x = 3 - t \\ y = -3 + 2t \\ z = -1 + t \end{cases}.$$

Replacing  $x, y$  and  $z$  in the equation of the plane with the expressions containing  $t$  found in the equation of the line allows us to determine a  $t$  value that indicates the point of intersection:

$$\begin{aligned} 2x + y + z &= 4 \\ 2(3 - t) + (-3 + 2t) + (-1 + t) &= 4 \\ t &= 2. \end{aligned}$$

When  $t = 2$ , the point on the line satisfies the equation of the plane;

that point is  $\ell(2) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ . Thus the point  $(1, 1, 1)$  is the point of intersection between the plane and the line, illustrated in Figure 2.6.12.

### 2.6.1 Distances

Just as it was useful to find distances between points and lines in the previous section, it is also often necessary to find the distance from a point to a plane.

Consider Figure 2.6.13, where a plane with normal vector  $\vec{n}$  is sketched containing a point  $P$  and a point  $Q$ , not on the plane, is given. We measure the distance from  $Q$  to the plane by measuring the length of the projection of  $\vec{PQ}$  onto  $\vec{n}$ . That is, we want:

$$\left\| \text{proj}_{\vec{n}} \vec{PQ} \right\| = \left\| \frac{\vec{n} \cdot \vec{PQ}}{\|\vec{n}\|^2} \vec{n} \right\| = \frac{|\vec{n} \cdot \vec{PQ}|}{\|\vec{n}\|} \quad (2.6.6)$$

Equation (2.6.6) is important as it does more than just give the distance between a point and a plane. We will see how it allows us to find several other distances as well: the distance between parallel planes and the distance from a line and a plane. Because Equation (2.6.6) is important, we restate it as a Key Idea.

#### Example 2.6.14 Distance between a point and a plane.

Find the distance between the point  $Q = (2, 1, 4)$  and the plane with equation  $2x - 5y + 6z = 9$ .

**Solution.** Using the equation of the plane, we find the normal vector

$\vec{n} = \begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix}$ . To find a point on the plane, we can let  $x$  and  $y$  be anything

we choose, then let  $z$  be whatever satisfies the equation. Letting  $x$  and  $y$  be 0 seems simple; this makes  $z = 1.5$ . Thus we let  $P = \begin{bmatrix} 0 \\ 0 \\ 1.5 \end{bmatrix}$ , and

$$\vec{PQ} = \begin{bmatrix} 2 \\ 1 \\ 2.5 \end{bmatrix}.$$

The distance  $h$  from  $Q$  to the plane is given by the length of the projection of  $\vec{PQ}$  onto  $\vec{n}$ :

$$\begin{aligned} h &= \frac{|\vec{n} \cdot \vec{PQ}|}{\|\vec{n}\|} \\ &= \frac{\left| \begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 2.5 \end{bmatrix} \right|}{\left\| \begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix} \right\|} \end{aligned}$$

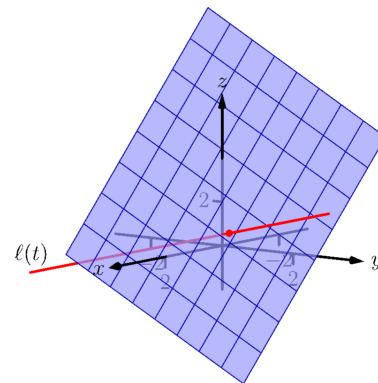


Figure 2.6.12 Illustrating the intersection of a line and a plane in Example 2.6.11

**Note:** Equation (2.6.6) is useful as it does more than just give the distance between a point and a plane. We will see how it allows us to find several other distances as well: the distance between parallel planes and the distance from a line to a plane.

However, as with the distance problems in the previous section, learning to follow the steps in Key Idea 2.5.14 will pay off more in the long run than memorizing a formula. Here, our key steps are to draw a diagram such as Figure 2.6.13, which doesn't need to be accurate, but does need to contain all the information needed to construct the projection whose length gives us the desired distance.

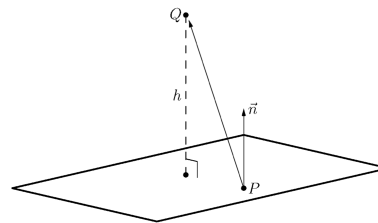


Figure 2.6.13 Illustrating finding the distance from a point to a plane

$$= \frac{|14|}{\sqrt{65}} \\ \approx 1.74.$$

Although it was not requested in Example 2.6.14, note that we can also find the point  $R$  on the plane that is closest to  $Q$ . The desired point must be such that  $\overrightarrow{RQ} = \text{proj}_{\vec{n}} \overrightarrow{PQ}$ . Since we know the point  $Q$  and the vector  $\overrightarrow{RQ}$ , we can find the point  $R$ : since  $\overrightarrow{RQ} = \overrightarrow{OQ} - \overrightarrow{OR}$ , we find that

$$\begin{aligned} \overrightarrow{OR} &= \overrightarrow{OQ} - \overrightarrow{RQ} \\ &= \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} - \frac{14}{65} \begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix} \\ &= \frac{1}{65} \begin{bmatrix} 102 \\ 135 \\ 176 \end{bmatrix}. \end{aligned}$$

The desired point  $R$  thus has coordinates  $\left(\frac{102}{65}, \frac{135}{65}, \frac{176}{65}\right)$ . To make sure that we haven't made any mistakes, let's make sure that this point is indeed on the plane. We have

$$2\left(\frac{102}{65}\right) - 5\left(\frac{135}{65}\right) + 6\left(\frac{176}{65}\right) = \frac{1}{65}(204 - 675 + 1056) = \frac{585}{65} = 9,$$

as expected.

### Example 2.6.15 Distance between a line and a plane.

Let  $\ell$  be the line with vector equation

$$\vec{\ell}(t) = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix},$$

and let  $p$  be the plane with equation  $x - 2y + z = 4$ . Verify that the line  $\ell$  is parallel to the plane  $p$ , and find the distance between them.

**Solution.** From the vector equation for  $\ell$  we have the direction vector

$$\vec{d} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \text{ and from the equation for } p \text{ we can read off the normal}$$

$$\text{vector } \vec{n} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}. \text{ Since}$$

$$\vec{d} \cdot \vec{n} = 3(1) + 1(-2) - 1(1) = 0,$$

we know that  $\vec{d}$  is orthogonal to  $\vec{n}$ , and thus  $\ell$  is parallel to  $p$ . To find the distance from  $\ell$  to  $p$ , we first choose a point on each object. From the vector equation for  $\ell$  we have the point  $P = (2, 0, -4)$ , and setting  $y = z = 0$  in the equation for  $p$ , we get  $x = 4$  and the point  $Q = (4, 0, 0)$ .



From these two points we can construct the vector

$$\vec{v} = \overrightarrow{PQ} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

which begins on  $\ell$  and ends on  $p$ . The distance from  $\ell$  to  $p$  is then given by the normal component of  $\vec{v}$ : we have

$$h = \|\text{proj}_{\vec{n}} \vec{v}\| = \frac{|\vec{n} \cdot \vec{v}|}{\|\vec{n}\|} = \frac{9}{6} = \frac{3}{2}.$$

In the previous section we used Equation (2.5.3) to find the shortest distance between a pair of skew lines. Although we provided some discussion of how this formula was obtained, it's once again the case that memorizing such a formula is not as effective as understanding the process that leads to it. In the next example, we repeat Example 2.5.18, but this time we try to understand the problem using planes.

#### Example 2.6.16 Distance between skew lines.

Find the distance between the skew lines

$$\begin{aligned} \ell_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \\ \ell_2 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} + t \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}. \end{aligned}$$

**Solution.** We already found the distance between these two lines in Example 2.5.18 using Equation (2.5.3). Supposing that we forgot this formula, how would we proceed? The key is to realize that whenever we have a pair of skew lines, we also have a pair of parallel planes, each of which contains one of the lines. To see this, we first compute the cross

product of the direction vectors  $\vec{d}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$  and  $\vec{d}_2 = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$  for the two

lines. We find

$$\vec{n} = \vec{d}_1 \times \vec{d}_2 = \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix},$$

which is the same as the cross product we computed in Example 2.5.18. Since  $\vec{n}$  is orthogonal to  $\vec{d}_1$ , the plane through the point  $(1, 2, 0)$  with normal vector  $\vec{n}$  contains the line  $\ell_1$ . Similarly, the plane through  $(-2, 3, 5)$  with normal vector  $\vec{n}$  contains  $\ell_2$ . We now have our parallel planes.

The next step is to realize that at this point, the problem is no different from the one we solved in Example 2.6.14: the distance from  $\ell_1$  to  $\ell_2$  is the same as the distance between the parallel planes, and the distance between parallel planes is equal to the distance between the first plane, and any point on the second plane.

By definition, the point  $P_1 = (1, 2, 0)$  on  $\ell_1$  lies on the first plane, and the point  $P_2 = (-2, 3, 5)$  on  $\ell_2$  lies on the second plane. We compute

the vector  $\overrightarrow{P_1P_2} = \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}$ , and then find the projection of this vector onto  $\vec{n}$ , as in Example 2.6.14. We have

$$\begin{aligned} \text{proj}_{\vec{n}} \overrightarrow{P_1P_2} &= \left( \frac{\overrightarrow{P_1P_2} \cdot \vec{n}}{\|\vec{n}\|^2} \right) \vec{n} \\ &= \left( \frac{\begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix}}{(3^2 + 2^2 + 7^2)} \right) \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix} \\ &= \frac{42}{62} \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix}. \end{aligned}$$

The distance is then given by

$$\|\text{proj}_{\vec{n}} \overrightarrow{P_1P_2}\| = \frac{42}{\sqrt{62}},$$

as before.

Given two parallel planes, we can find the distance between these planes by letting  $P$  be a point on one plane and  $Q$  a point on the other. If  $\ell$  is a line parallel to a plane, we can use the Key Idea to find the distance between them as well: again, let  $P$  be a point in the plane and let  $Q$  be any point on the line. The Exercise section contains problems of these types.

These past two sections have not explored lines and planes in space as an exercise of mathematical curiosity. However, there are many, many applications of these fundamental concepts. Complex shapes can be modeled (or, *approximated*) using planes. For instance, part of the exterior of an aircraft may have a complex, yet smooth, shape, and engineers will want to know how air flows across this piece as well as how heat might build up due to air friction. Many equations that help determine air flow and heat dissipation are difficult to apply to arbitrary surfaces, but simple to apply to planes. By approximating a surface with millions of small planes one can more readily model the needed behavior.

## 2.6.2 Exercises

**Exercise Group.** In the following exercises, give any two points in the given plane.

1.  $2x - 4y + 7z = 2$
2. List any two points in the plane with equation  $3(x + 2) + 5(y - 9) - 4z = 0$ .
3.  $x = 2$
4. List any two points in the plane with equation  $4(y + 2) - (z - 6) = 0$ .

**Exercise Group.** In the following exercises, give the equation of the described plane in standard and general forms.

5. Passes through  $(2, 3, 4)$  and has normal vector  $\vec{n} = \begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix}$ .
6. A plane passes through  $(1, 3, 5)$  and has normal vector  $\vec{n} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$ .
7. Passes through the points  $(1, 2, 3)$ ,  $(3, -1, 4)$  and  $(1, 0, 1)$ .
8. A plane passes through the points  $(5, 3, 8)$ ,  $(6, 4, 9)$  and  $(3, 3, 3)$ .
9. Contains the intersecting lines  $\vec{\ell}_1(t) = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  and  $\vec{\ell}_2(t) = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$ .
10. A plane contains the intersecting lines  $\vec{\ell}_1(t) = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$  and  $\vec{\ell}_2(t) = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$ .
11. Contains the parallel lines  $\vec{\ell}_1(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  and  $\vec{\ell}_2(t) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ .
12. A plane contains the parallel lines  $\vec{\ell}_1(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$  and  $\vec{\ell}_2(t) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$ .
13. Contains the point  $(2, -6, 1)$  and the line  $\vec{\ell}(t) = \begin{cases} x = 2 + 5t \\ y = 2 + 2t \\ z = -1 + 2t \end{cases}$ .
14. A plane contains the point  $(5, 7, 3)$  and the line  $\vec{\ell}(t) = \begin{cases} x = t \\ y = t \\ z = t \end{cases}$ .
15. A plane contains the point  $(5, 7, 3)$  and is orthogonal to the line  $\vec{\ell}(t) = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ .
16. A plane contains the point  $(4, 1, 1)$  and is orthogonal to the line  $\begin{cases} x = 4 + 4t \\ y = 1 + t \\ z = 1 + t \end{cases}$ .
17. A plane contains the point  $(-4, 7, 2)$  and is parallel to the plane  $3(x - 2) + 8(y + 1) - 10z = 0$ .
18. A plane contains the point  $(1, 2, 3)$  and is parallel to the plane  $x = 5$ .

**Exercise Group.** In the following exercises, give the equation of the line that is the intersection of the given planes.

19.  $p_1 : 3(x - 2) + (y - 1) + 4z = 0$ , and  $p_2 : 2(x - 1) - 2(y + 3) + 6(z - 1) = 0$ .
20. Give the equation of the line (in vector form) that is the intersection of the planes  $5(x - 5) + 2(y + 2) + 4(z - 1) = 0$ , and  $3x - 4(y - 1) + 2(z - 1) = 0$ .

**Exercise Group.** Find the point of intersection between the line and the plane.

21.

$$\text{line: } \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{plane: } 5x - y - z = -3$$

23.

$$\text{line: } \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$$

$$\text{plane: } 3x - 2y - z = 4$$

22.

$$\bullet \text{ line: } \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\bullet \text{ plane: } 3x + y - 2z = 8$$

24.

$$\bullet \text{ line: } \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$$

$$\bullet \text{ plane: } 3x - 2y - z = -4$$

**Exercise Group.** Find the indicated distance.

25. The distance from the point  $(1, 2, 3)$  to the plane

$$3(x - 1) + (y - 2) + 5(z - 2) = 0.$$

27. The distance between the parallel planes

$$x + y + z = 0 \text{ and}$$

$$(x - 2) + (y - 3) + (z + 4) = 0$$

26. Find the distance from the point  $(2, 6, 2)$  to the plane  $2(x - 1) - y + 4(z + 1) = 0$ .

28. Find the distance between the parallel planes

$$2(x - 1) + 2(y + 1) + (z - 2) = 0 \text{ and}$$

$$2(x - 3) + 2(y - 1) + (z - 3) = 0.$$

29. Show why if the point  $Q$  lies in a plane, then the distance formula correctly gives the distance from the point to the plane as 0.

## 2.7 Span and Linear Independence

So far in this chapter, we have restricted our attention to vectors in two or three dimensions, where we are able to visualize things geometrically. Something that you may have noticed is that the algebraic rules for the addition and scalar multiplication of vectors are the same in both dimensions.

In this section, we consider this algebra of vectors abstractly, which will allow us to move beyond three dimensions to vectors with an arbitrary number of components.

Despite the fact that we cannot easily visualize them, higher-dimensional vectors frequently arise in applications where there are many variables involved. The fact that the rules of algebra remain the same mean that we can continue to manipulate these objects, even though we can no longer picture them.

### 2.7.1 The vector space $\mathbb{R}^n$

For each positive integer  $n$ , a column vector  $\vec{x}$  is formed by arranging  $n$  real

numbers  $x_1, x_2, \dots, x_n$  into a column  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ . In Chapter 4 we will see

that this is a special type of *matrix*; for now, we can think of a column vector as

an alternative to the notation  $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$  encountered earlier in this chapter. In

particular, we still refer to the numbers  $x_1, x_2, \dots, x_n$  in  $\vec{x}$  as the **components** of  $\vec{x}$ , and we define addition and scalar multiplication of column vectors in terms of their components, as we did for vectors in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ .

That is, given vectors  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$  and  $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ , and a scalar  $c$ , we define

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

and

$$c\vec{x} = c \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}.$$

With these operations, the set of all  $n \times 1$  column vectors provides an example of what is known as a **vector space**.

**Definition 2.7.1 The vector space  $\mathbb{R}^n$ .**

The space of all **column vectors** of real numbers is denoted by

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mid x_1, x_2, \dots, x_n \in \mathbb{R} \right\}.$$

As with the vectors in  $\mathbb{R}^2$  and  $\mathbb{R}^3$  we encountered in Chapter 2, we allow the notation  $\mathbb{R}^n$  to represent both the *space* of points  $(x_1, x_2, \dots, x_n)$ , and the set of vectors defined within that space. Since we can identify any point  $P$  with the position vector  $\overrightarrow{OP}$ , the difference between viewing  $\mathbb{R}^n$  as a set of points or as a set of vectors is primarily one of perspective.

When  $n \geq 4$  we can no longer visualize vectors in  $\mathbb{R}^n$  as we did in Chapter 2, but we can handle them algebraically exactly as we did before, and we can extend the definitions of Chapter 2 to apply to vectors in  $\mathbb{R}^n$ .

In particular, we can define the **length** of a vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

by

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2},$$

and the dot product of vectors  $\vec{x}, \vec{y} \in \mathbb{R}^n$  by

$$\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + \dots + x_ny_n.$$

Having defined the dot product, we can still declare two vectors  $\vec{x}$  and  $\vec{y}$  to be orthogonal if  $\vec{x} \cdot \vec{y} = 0$ , and define the angle between two vectors by requiring that the identity

$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos \theta$$

remain valid. Using these definitions, along with Theorem 4.1.10, we can see that all of the properties of vector operations given in Theorem 2.2.16 remain valid in  $\mathbb{R}^n$ .

**Theorem 2.7.2 Algebraic properties of  $\mathbb{R}^n$ .**

The following properties hold for the space  $\mathbb{R}^n$  of  $n \times 1$  column vectors:

1. If  $\vec{v}$  and  $\vec{w}$  are vectors in  $\mathbb{R}^n$ ,  $\vec{v} + \vec{w}$  is also a vector in  $\mathbb{R}^n$ .
2. For any vectors  $\vec{v}, \vec{w}$ ,  $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ .
3. For any vectors  $\vec{u}, \vec{v}, \vec{w}$ ,  $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ .
4. For any vector  $\vec{v}$ ,  $\vec{v} + \vec{0} = \vec{v}$ .
5. For any vector  $\vec{v}$ , we can define  $-\vec{v}$  such that  $\vec{v} + (-\vec{v}) = \vec{0}$ .
6. If  $k$  is a scalar and  $\vec{v}$  is a vector in  $\mathbb{R}^n$ , then  $k\vec{v}$  is also a vector in  $\mathbb{R}^n$ .

7. For any vector  $\vec{v}$ ,  $1 \cdot \vec{v} = \vec{v}$ .
8. For any scalars  $c, d$  and any vector  $\vec{v}$ ,  $c(d\vec{v}) = (cd)\vec{v}$ .
9. For any scalar  $c$  and vectors  $\vec{v}, \vec{w}$ ,  $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$ .
10. For any scalars  $c, d$  and vector  $\vec{v}$ ,  $(c + d)\vec{v} = c\vec{v} + d\vec{v}$ .

Then ten properties listed in Theorem 2.7.2 are known as the **vector space axioms**. Any set of objects satisfying these axioms is known as a **vector space**. There are many interesting examples of vector spaces other than  $\mathbb{R}^n$ , but we will not study vector spaces in general in this text.

Vector spaces are defined in general to be sets on which one can define addition and scalar multiplication satisfying the algebraic properties given in Theorem 2.7.2. Examples of vector spaces other than  $\mathbb{R}^n$  include the space of  $m \times n$  matrices (each choice of  $m$  and  $n$  gives a different space), and the space of all polynomial functions of a real variable.

## 2.7.2 Linear combinations and span

One of the key insights of linear algebra is that a space such as  $\mathbb{R}^n$ , which contains infinitely many objects, can be generated using the operations of addition and scalar multiplication from a finite set of basic objects. We saw in Chapter 2, for example, that every vector in  $\mathbb{R}^3$  can be written in terms of just three basic unit vectors  $\vec{i}, \vec{j}$ , and  $\vec{k}$ .

Since addition and scalar multiplication are the main operations of linear algebra, it's not too surprising (if a little unimaginative) that any combination of these operations is called a **linear combination**.

### Definition 2.7.3 Linear combination in $\mathbb{R}^n$ .

A **linear combination** in  $\mathbb{R}^n$  is any expression of the form

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k,$$

where  $c_1, c_2, \dots, c_k \in \mathbb{R}$  are scalars, and  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \in \mathbb{R}^n$  are vectors.

### Example 2.7.4 Forming linear combinations.

Let  $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} -4 \\ 6 \\ 3 \end{bmatrix}$ ,  $\vec{w} = \begin{bmatrix} 2 \\ 3 \\ 12 \end{bmatrix}$  be vectors in  $\mathbb{R}^3$ . Form the following linear combinations:

1.  $3\vec{u} - 4\vec{w}$
2.  $\vec{u} + \vec{v} - 2\vec{w}$
3.  $7\vec{v} + 3\vec{v}$
4.  $3\vec{u} + \vec{v} - \vec{w}$

### Solution.

1. To simplify the linear combination, we first take care of the scalar multiplication, and then perform the addition. (We choose to interpret this expression as  $3\vec{u} + (-4)\vec{w}$ , and multiply by  $-4$  in the first step, and add in the second step, rather than multiplying by 4

and then subtracting.)

$$3\vec{u} - 4\vec{w} = 3 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} - 4 \begin{bmatrix} 2 \\ 3 \\ 12 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix} + \begin{bmatrix} -8 \\ -12 \\ -48 \end{bmatrix} = \begin{bmatrix} -2 \\ -15 \\ -39 \end{bmatrix}.$$

2. We proceed as with the previous problem, this time performing the scalar multiplication of  $\vec{w}$  by  $-2$  in our heads:

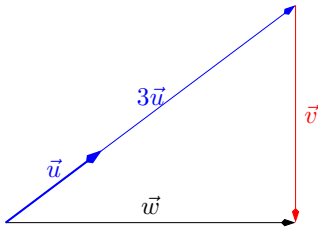
$$\vec{u} + \vec{v} - 2\vec{w} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} -4 \\ 6 \\ 3 \end{bmatrix} + \begin{bmatrix} -4 \\ -6 \\ -24 \end{bmatrix} = \begin{bmatrix} -6 \\ -1 \\ -18 \end{bmatrix}.$$

3. We find

$$7\vec{u} + 3\vec{v} = 7 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -4 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \\ 21 \end{bmatrix} + \begin{bmatrix} -12 \\ 18 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \\ 30 \end{bmatrix}.$$

4. For our last example, we compute

$$3\vec{u} + \vec{v} - \vec{w} = \begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix} + \begin{bmatrix} -4 \\ 6 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$



**Figure 2.7.5** Depicting the last linear combination in Example 2.7.4

Notice that in the last example above, our linear combination works out to be the zero vector. Let's think about this geometrically for a second: using the "tip-to-tail" method for adding vectors and beginning with the tail of  $3\vec{u}$  at the origin, if we add the vector  $\vec{v}$  at the tip of  $3\vec{u}$ , and then subtract  $\vec{w}$ , we end up back at the origin. The vectors  $3\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$  must therefore lie in the same plane, since they form three sides of a triangle, as depicted in Figure 2.7.5.

Viewed another way, notice that we can solve the equation  $3\vec{u} + \vec{v} - \vec{w} = \vec{0}$  for  $\vec{w}$ : we have

$$\vec{w} = 3\vec{u} + \vec{v}.$$

What this tells us is that when we're being asked to form linear combinations of the vectors  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$  in Example 2.7.4, then vector  $\vec{w}$  is redundant. Suppose the vector  $\vec{x}$  is an arbitrary linear combination of these vectors; that is,

$$\vec{x} = a\vec{u} + b\vec{v} + c\vec{w}$$

for some scalars  $a, b, c$ . If we plug in  $\vec{w} = 3\vec{u} + \vec{v}$ , then we get

$$\begin{aligned} \vec{x} &= a\vec{u} + b\vec{v} + c(3\vec{u} + \vec{v}) \\ &= a\vec{u} + b\vec{v} + 3c\vec{u} + c\vec{v} \quad (\text{distribute the scalar}) \\ &= (a + 3c)\vec{u} + (b + c)\vec{v} \quad (\text{collect terms}). \end{aligned}$$

Thus,  $\vec{x}$  has been written in terms of  $\vec{u}$  and  $\vec{v}$  only.

These ideas come up frequently enough in Linear Algebra that they have associated terminology. The definitions that follow seem innocent enough, but their importance to the theory of Linear Algebra cannot be understated.



**Definition 2.7.6 The span of a set of vectors.**

Let  $A = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$  be a set of vectors in  $\mathbb{R}^n$ . The **span** of the vectors in  $A$ , denoted  $\text{span}(A)$ , is the set  $S$  of all possible linear combinations of the vectors in  $A$ . That is,

$$S = \text{span}(A) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k \mid c_1, c_2, \dots, c_k \in \mathbb{R}\}.$$

**Example 2.7.7 Describing spans in  $\mathbb{R}^3$ .**

Let  $\vec{u}, \vec{v}, \vec{w}$  be as in Example 2.7.4. Describe the following spans:

1.  $\text{span}\{\vec{u}\}$
2.  $\text{span}\{\vec{u}, \vec{v}\}$
3.  $\text{span}\{\vec{u}, \vec{v}, \vec{w}\}$

**Solution.**

1. As a set, we have

$$\text{span}\{\vec{u}\} = \left\{ t \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \mid t \in \mathbb{R} \right\},$$

the set of all scalar multiples of the vector  $\vec{u}$ .

If we think back to Section 2.5, we can do a bit better with our description. The set  $\text{span}\{\vec{u}\}$  consists of all vectors  $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$  such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix},$$

which we recognize as the equation of a line through the origin in  $\mathbb{R}^3$  in the direction of the vector  $\vec{u}$ .

2. Again, as a set we can write

$$\text{span}\{\vec{u}, \vec{v}\} = \left\{ s \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} -4 \\ 6 \\ 3 \end{bmatrix} \mid s, t \in \mathbb{R} \right\},$$

so  $\text{span}\{\vec{u}, \vec{v}\}$  consists of all vectors of the form  $\begin{bmatrix} 2s - 4t \\ -s + 6t \\ 3s + 3t \end{bmatrix}$ ,

where  $s$  and  $t$  can be any real numbers. Again, with a bit of thought, we can come up with a geometric description of this set. Consider an arbitrary vector

$$\vec{x} = s\vec{u} + t\vec{v} \in \text{span}\{\vec{u}, \vec{v}\}.$$

Any such vector can be obtained by moving some distance (measured by the scalar  $s$ ) in the direction of  $\vec{u}$ , and then moving another distance (measured by the scalar  $t$ ) in the direction of  $\vec{v}$ . We now have two directions in which to move, and if we haven't forgotten what we learned in Section 2.6, this probably reminds us of the description of a plane.

To see that  $\text{span}\{\vec{u}, \vec{v}\}$  is indeed a plane, we compute

$$\vec{n} = \vec{u} \times \vec{v} = \begin{bmatrix} -21 \\ -18 \\ 8 \end{bmatrix},$$

which we know is orthogonal to both  $\vec{u}$  and  $\vec{v}$ . It follows from the properties of the dot product that for any other vector  $\vec{x} = s\vec{u} + t\vec{v}$  we have

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot (s\vec{u} + t\vec{v}) = s(\vec{n} \cdot \vec{u}) + t(\vec{n} \cdot \vec{v}) = s(0) + t(0) = 0,$$

so with  $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ , we have

$$-21x - 18y + 8z = 0,$$

which is the equation of a plane through the origin.

3. In the discussion following Example 2.7.4 we saw that any vector that can be written as a linear combination of  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$  can be written as a linear combination of  $\vec{u}$  and  $\vec{v}$  alone. Thus, the span of  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$  doesn't contain anything we didn't already have in the span of  $\vec{u}$  and  $\vec{v}$ ; that is,

$$\text{span}\{\vec{u}, \vec{v}, \vec{w}\} = \text{span}\{\vec{u}, \vec{v}\}.$$

For any plane through the origin, the sum of two vectors that lie in that plane (or indeed, any linear combination of vectors in the plane) is again a vector in that plane. We'll see shortly that this is the distinguishing characteristic of any *subspace* of  $\mathbb{R}^n$ .

### Example 2.7.8 Determining membership in a span.

Given the vectors  $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ , and  $\vec{w} = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$ , determine whether or not the following vectors belong to  $\text{span}\{\vec{u}, \vec{v}, \vec{w}\}$ :

1.  $\vec{x} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$

2.  $\vec{y} = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$

**Solution.** We do not yet have a general technique for solving problems of this type. Notice that the question “Does  $\vec{x}$  belong to the span of  $\{\vec{u}, \vec{v}, \vec{w}\}$ ?” is equivalent to the question, “Do there exist scalars  $a, b, c$  such that

$$a\vec{u} + b\vec{v} + c\vec{w} = \vec{x}?$$

” Answering this question amounts to solving a system of linear equa-

tions: if we plug in our vectors, we have

$$a \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} + c \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2a + 3b - 2c \\ -a + 2b + 5c \\ a + 5b + 3c \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$

By definition of the equality of vectors, this amounts to the system of equations

$$\begin{aligned} 2a + 3b - 2c &= 3 \\ -a + 2b + 5c &= 6 \\ a + 5b + 3c &= 9 \end{aligned}$$

We will develop systematic techniques for solving such systems in the next chapter. Until then, is there anything we can say? The very astute reader might notice that the vectors  $\vec{u}$ ,  $\vec{v}$ ,  $\vec{w}$  all have something in common: their third component is the sum of the first two:  $1 = 2 + (-1)$  for  $\vec{u}$ ,  $5 = 3 + 2$  for  $\vec{v}$ , and  $3 = -2 + 5$  for  $\vec{w}$ . Thus, all three vectors

are of the form  $\begin{bmatrix} x \\ y \\ x + y \end{bmatrix}$ . Now, notice what happens if we combine two

such vectors:

$$s \begin{bmatrix} a \\ b \\ a + b \end{bmatrix} + t \begin{bmatrix} c \\ d \\ c + d \end{bmatrix} = \begin{bmatrix} sa + tc \\ sb + td \\ s(a + b) + t(c + d) \end{bmatrix} = \begin{bmatrix} sa + tc \\ sb + td \\ (sa + tc) + (sb + td) \end{bmatrix},$$

which is another vector of the same form. The same will be true for combinations of three or more such vectors.

For the vector  $\vec{x}$ , we check that  $3 + 6 = 9$ , so  $\vec{x}$  has the correct form, and indeed (with a bit of “guess and check” work), we find that  $a = b = c = 1$  works, since  $\vec{u} + \vec{v} + \vec{w} = \vec{x}$ . Thus, we can conclude that

$$\vec{x} \in \text{span}\{\vec{u}, \vec{v}, \vec{w}\}.$$

For the vector  $\vec{y}$ , we add the first two components, getting  $4 + 1 = 5 \neq -3$ . Since the third component is not the sum of the first two, there is no way that  $\vec{y}$  could belong to the span of  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$ .

### 2.7.3 Linear independence

Notice in Example 2.7.7 that the span did not change when we added the vector  $\vec{w}$  to the set of spanning vectors. This was probably not too surprising, since we saw that  $\vec{w} = 3\vec{u} + \vec{v}$ , meaning that  $\vec{w}$  is a linear combination of  $\vec{u}$  and  $\vec{v}$ , and thus,

$$\vec{w} \in \text{span}\{\vec{u}, \vec{v}\}.$$

We don’t get anything new when we include the vector  $\vec{w}$  since it lies in the plane spanned by  $\vec{u}$  and  $\vec{v}$ . We say that the vector  $\vec{w}$  *depends* on  $\vec{u}$  and  $\vec{v}$ , in the same way that the total of a sum depends on the numbers being added. Since this dependence is defined in terms of linear combinations, we say that the vectors  $\vec{u}$ ,  $\vec{v}$ ,  $\vec{w}$  are **linearly dependent**.

In general, a set of vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$  is linearly dependent if one of the vectors can be written as a linear combination of the others. If this is impossible, we say that the vectors are **linearly independent**.

For reasons that become apparent as soon as one begins a discussion of basis and dimension (something we won't really cover in this text), we must consider any set containing the zero vector to be linearly dependent. If we use Equation (2.7.1) to define linear independence, this is included in the definition: if (for example)  $\vec{v}_1 = \vec{0}$ , then we can let  $c_1$  be any real number we want, and  $c_1\vec{v}_1 = \vec{0}$ , so it is possible to find a linear combination where not all of the scalars are zero.

The formal definition is as follows.

**Definition 2.7.9 Linear dependence.**

We say that a set of vectors

$$A = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$$

in  $\mathbb{R}^n$  is **linearly dependent** if  $\vec{0} \in A$ , or if one of the vectors  $\vec{v}_i$  in  $A$  can be written as a linear combination of the other vectors in  $A$ . If the set  $A$  is not linearly dependent, we say that it is **linearly independent**.

**Example 2.7.10 Determining linear independence.**

Determine whether or not following sets of vectors are linearly independent:

1. The vectors  $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ , and  $\vec{w} = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$  from Example 2.7.8.
2. The vectors

$$\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \vec{y} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}, \quad \text{and} \quad \vec{z} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

**Solution.** Like problems involving span, a general approach to answering questions like these about linear independence will have to wait until we develop methods for solving systems of equations in the next chapter. However, for these two sets of vectors, we can reason our way to an answer.

1. Here, we noticed that all three vectors satisfy the condition  $z = x + y$ , if we label their respective components as  $x$ ,  $y$ , and  $z$ . But this condition is simply the equation of a plane; namely,  $x + y - z = 0$ . Intuition tells us that any plane can be written as the span of two vectors, so we can expect that any one of the three vectors can be written in terms of the other two, and indeed, this is the case. With a bit of guesswork (or by asking a computer), we can determine that

$$\vec{w} = -\frac{19}{7}\vec{u} + \frac{8}{7}\vec{v},$$

showing that  $\vec{w}$  can be written as a linear combination of  $\vec{u}$  and  $\vec{v}$ , and thus, that our vectors are linearly dependent.

2. Here, we make the useful observation that two of our three vectors have zero as their third component. Since  $\vec{x}$  and  $\vec{z}$  have third component zero, it is impossible for  $\vec{y}$  to be written as a linear combination of  $\vec{x}$  and  $\vec{z}$ , since any such linear combination would still have a zero in the third component. To see that  $\vec{x}$  cannot be

written in terms of  $\vec{y}$  and  $\vec{z}$ , notice that for any  $a$  and  $b$ ,

$$a\vec{y} + b\vec{z} = \begin{bmatrix} -2a + b \\ 3a + b \\ a \end{bmatrix}.$$

If this is to equal  $\vec{x}$ , then we must have  $a = 0$ , giving us  $\vec{x} = b\vec{z}$ , but it's clear that  $\vec{x}$  is not a scalar multiple of  $\vec{z}$ . A similar argument shows that  $\vec{z}$  cannot be written in terms of  $\vec{x}$  and  $\vec{y}$ , and thus our vectors are linearly independent.

Another way to characterize linear independence is as follows: suppose we have a linear combination equal to the zero vector:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}. \quad (2.7.1)$$

This is always possible of course, since we can simply set each of the scalars equal to zero. The condition of linear independence tells us that if our vectors are independent, then this is the *only* way to obtain a linear combination equal to the zero vector.

To see why this rule works, suppose we can choose our scalars in Equation (2.7.1) so that at least one of them is non-zero. For simplicity, let's say  $c_1 \neq 0$ . Then we can rewrite Equation (2.7.1) as

$$c_1\vec{v}_1 = -c_2\vec{v}_2 - \cdots - c_k\vec{v}_k,$$

and since  $c_1 \neq 0$ , we can divide both sides by  $c_1$ , and we've written  $\vec{v}_1$  as a linear combination of the remaining vectors.

For example, from Example 2.7.10 we can conclude (with a bit of rearranging) that the vectors  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$  satisfy the relationship

$$19\vec{u} - 8\vec{v} + 7\vec{w} = \vec{0}.$$

Linear independence can be a difficult concept at first, but in three dimensions we can use Equation (2.7.1) to provide a visual interpretation on a case-by-case basis.

#### Key Idea 2.7.11 Linearly independent sets of vectors in $\mathbb{R}^3$ .

- Any set  $\{\vec{u}\}$  containing a single vector in  $\mathbb{R}^3$  is linearly dependent if  $\vec{u} = \vec{0}$ , and independent otherwise. (Here  $\text{eq-linear indep}$  becomes  $c\vec{u} = \vec{0}$ . If  $\vec{u} \neq \vec{0}$ , the only solution is to take  $c = 0$ .)
- Any set  $\{\vec{u}, \vec{v}\}$  containing two non-zero vectors in  $\mathbb{R}^3$  is linearly dependent if  $\vec{u}$  is parallel to  $\vec{v}$ , and independent otherwise. (In other words, two *dependent* vectors lie on the same line. Two independent vectors span a plane.)
- Any set  $\{\vec{u}, \vec{v}, \vec{w}\}$  of three non-zero vectors in  $\mathbb{R}^3$  is linearly dependent if all three vectors lie in the same plane, and independent otherwise.
- Any set of four or more vectors in  $\mathbb{R}^3$  is automatically linearly dependent.

This section introduced several new ideas. Some, like linear combinations, are straightforward. Others, like span and linear independence, take some getting used to. There remain two very obvious questions to address:

At this point in the text, we're not in a position to prove that any set of four vectors in  $\mathbb{R}^3$  (or more generally,  $k$  vectors in  $\mathbb{R}^n$ , where  $k > n$ ) is automatically independent. However, we'll soon see in Chapter 3 that in this case, the test for independence given by Equation (2.7.1) results in a "homogeneous" system of linear equations with more variables than equations, and that such a system is guaranteed to have non-trivial solutions.

1. How do we tell whether or not a given vector belongs to the span of a set of vectors?
2. How do we tell if a set of vectors is linearly independent?

It turns out that both questions lead to **systems of linear equations**. As we saw in Example 2.7.8, we are currently unable to systematically solve such problems. In Chapters Chapter 3 and Chapter 4 we will develop the techniques needed to systematically solve such systems, at which point we will be able to easily answer questions about linear independence and span.

To see how such systems arise, suppose we want to know whether or not

the vector  $\vec{w} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 0 \end{bmatrix} \in \mathbb{R}^4$  belongs to the set  $V = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ , where

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 4 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ -4 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -3 \\ 6 \\ 7 \\ 2 \end{bmatrix}.$$

By definition,  $V$  is the set of all possible linear combinations of the vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ , so saying that  $\vec{w} \in V$  is the same as saying that we can write  $\vec{w}$  as a linear combination of these vectors. Thus, what we want to know is whether or not there exist scalars  $x_1, x_2, x_3$  such that

$$\vec{w} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3.$$

Substituting in the values for our vectors, this gives

$$x_1 \begin{bmatrix} 0 \\ 2 \\ -1 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ -4 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 6 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 3x_2 - 3x_3 \\ 2x_1 + x_2 + 6x_3 \\ -x_1 + 7x_3 \\ 4x_1 - 4x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 0 \end{bmatrix}.$$

Since two vectors are equal if and only if each component is equal, the above vector equation leads to the following system of four equations:

$$\begin{aligned} 3x_2 - 3x_3 &= 2 \\ 2x_1 + x_2 + 6x_3 &= -1 \\ -x_1 + 7x_3 &= 3 \\ 4x_1 - 4x_2 + 2x_3 &= 0 \end{aligned}.$$

Thus, the question “Is the vector  $\vec{w}$  an element of  $V$ ?” is equivalent to the question “Is there a solution to the above system of equations?”

Questions about linear independence are similar, but not quite the same. With the above example involving span, what we wanted to know is “Does a solution exist?” With linear independence, it is not whether a solution exists that is in doubt, but whether or not that solution is *unique*. For example, suppose we wanted to know if the vectors in our span example above are linearly independent. We would start with the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0},$$

and ask whether or not there are any solutions other than  $x_1 = x_2 = x_3 = 0$ .

This vector equation leads to a system just like the one above, except that the numbers to the right of the  $=$  signs would all be zeros. The techniques needed to answer these and other questions will be developed beginning in Chapter 3.

## 2.7.4 Exercises

**Exercise Group.** Simplify the given linear combinations, where

$$\vec{u} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 4 \\ -5 \\ 0 \end{bmatrix}, \text{ and } \vec{w} = \begin{bmatrix} -3 \\ 2 \\ 0 \\ 7 \end{bmatrix}.$$

1.  $3\vec{u} - 2\vec{v}$
2.  $-2\vec{u} + 3\vec{v} + \vec{w}$
3.  $\vec{u} - 2\vec{v} + 5\vec{w}$
4.  $4\vec{v} - 3\vec{w}$

**Exercise Group.** Calculate the given quantity, where

$$\vec{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \\ 7 \end{bmatrix}, \vec{v} = \begin{bmatrix} -3 \\ 5 \\ 0 \\ -6 \\ 1 \end{bmatrix}, \text{ and } \vec{w} = \begin{bmatrix} 0 \\ -3 \\ 5 \\ 2 \\ -4 \end{bmatrix}.$$

5.  $\|\vec{v}\|$
6.  $\vec{u} \cdot \vec{v}$
7.  $\vec{w} \cdot (2\vec{u} - 3\vec{v})$
8.  $2(\vec{w} \cdot \vec{u}) - 3(\vec{w} \cdot \vec{v})$

**Exercise Group.** Determine if the given statement is true or false. Give a proof for any true statements, and give a counterexample for any false statements.

9. A subset of a linearly independent set is linearly independent.
10. A subset of a linearly dependent set is linearly dependent.
11. Any set of vectors that contains the zero vector is linearly dependent.
12. If the vector  $\vec{w}$  belongs to the span of the set  $\{\vec{v}_1, \dots, \vec{v}_k\}$ , then the set  $\{\vec{w}, \vec{v}_1, \dots, \vec{v}_k\}$  is linearly dependent.
13. If the set  $\{\vec{w}, \vec{v}_1, \dots, \vec{v}_k\}$  is linearly dependent, then  $\vec{w}$  belongs to the span of  $\{\vec{v}_1, \dots, \vec{v}_k\}$ .





## Chapter 3

# Systems of Linear Equations

In Section 2.7, we encountered the concepts of *span* and *linear independence*, and saw that these lead naturally to certain systems of equations. %, the *column space* of a matrix, and the *null space* of a matrix.

In each case we were able to explain the concept, but unable to compute any examples, since we lacked the machinery for solving the systems of equations that arose.

You have probably encountered simple systems of linear equations in high school; you can might be able to remember solving systems of equations where you had two or three equations in two or three unknowns, and you tried to find the value of the unknowns. In this chapter we will uncover some of the fundamental principles guiding the solution to such problems.

Solving such systems was a bit time consuming, but not terribly difficult. So why bother? We bother because, in addition to the theoretical applications mentioned above, there are many, many, *many* practical applications where systems of linear equations arise, from business and finance to engineering to computer graphics to understanding more mathematics. And not only are there many applications of systems of linear equations, on most occasions where these systems arise we are using far more than three variables. (Engineering applications, for instance, often require thousands of variables.) So getting a good understanding of how to solve these systems effectively is important.

Chapter 2 also presented us with a few situations where the ability to solve systems of equations would have come in handy, such as the derivation of the cross product formula, and when we wanted to compute the point of intersection of two lines, or the line of intersection of two planes.

### 3.1 Introduction to Linear Equations

We'll begin this section by examining a problem you probably already know how to solve.

#### Example 3.1.1 Counting marbles in a jar.

Suppose a jar contains red, blue and green marbles. You are told that there are a total of 30 marbles in the jar; there are twice as many red marbles as green ones; the number of blue marbles is the same as the sum of the red and green marbles. How many marbles of each colour are there?

**Solution.** We could attempt to solve this with some trial and error, and we'd probably get the correct answer without too much work. However, this won't lend itself towards learning a good technique for solving larger problems, so let's be more mathematical about it.

Let's let  $r$  represent the number of red marbles, and let  $b$  and  $g$  denote the number of blue and green marbles, respectively. We can use the given statements about the marbles in the jar to create some equations.

Since we know there are 30 marbles in the jar, we know that

$$r + b + g = 30. \quad (3.1.1)$$

Also, we are told that there are twice as many red marbles as green ones, so we know that

$$r = 2g. \quad (3.1.2)$$

Finally, we know that the number of blue marbles is the same as the sum of the red and green marbles, so we have

$$b = r + g. \quad (3.1.3)$$

From this stage, there isn't one "right" way of proceeding. Rather, there are many ways to use this information to find the solution. One way is to combine ideas from equations (3.1.2) and (3.1.3); in (3.1.3) replace  $r$  with  $2g$ . This gives us

$$b = 2g + g = 3g. \quad (3.1.4)$$

We can then combine equations (3.1.1), (3.1.2) and (3.1.4) by replacing  $r$  in (3.1.1) with  $2g$  as we did before, and replacing  $b$  with  $3g$  to get

$$r + b + g = 30 \quad (3.1.5)$$

$$2g + 3g + g = 30 \quad (3.1.6)$$

$$6g = 30 \quad (3.1.7)$$

$$g = 5. \quad (3.1.8)$$

We can now use equation (3.1.8) to find  $r$  and  $b$ ; we know from (3.1.2) that  $r = 2g = 10$  and then since  $r + b + g = 30$ , we easily find that  $b = 15$ .

Mathematicians often see solutions to given problems and then ask "What if...?" It's an annoying habit that we would do well to develop — we should learn to think like a mathematician. What are the right kinds of "what if" questions to ask? Here's another annoying habit of mathematicians: they often ask "wrong" questions. That is, they often ask questions and find that the answer isn't particularly interesting. But asking enough questions often leads to some good "right" questions. So don't be afraid of doing something "wrong;" we mathematicians do it all the time.

So what is a good question to ask after seeing Example 3.1.1? Here are two possible questions:

1. Did we really have to call the red balls " $r$ "? Could we call them " $q$ "?
2. What if we had 60 balls at the start instead of 30?

Let's look at the first question. Would the solution to our problem change if we called the red balls  $q$ ? Of course not. At the end, we'd find that  $q = 10$ , and we would know that this meant that we had 10 red balls.

Here's another "What if?" question: what if we had a total of 20 marbles in the jar? Could we still solve the problem? Chances are that you can still work out a solution to the system, but what does it mean? (In this case it's probably safe to assume that it doesn't make physical sense to deal with fractions of a marble.)

Now let's look at the second question. Suppose we had 60 balls, but the other relationships stayed the same. How would the situation and solution change? Let's compare the "original" equations to the "new" equations.

Original	New
$r + b + g = 30$	$r + b + g = 60$
$r = 2g$	$r = 2g$
$b = r + g$	$b = r + g$

By examining these equations, we see that nothing has changed except the first equation. It isn't too much of a stretch of the imagination to see that we would solve this new problem exactly the same way that we solved the original one, except that we'd have twice as many of each type of ball.

A conclusion from answering these two questions is this: it doesn't matter what we call our variables, and while changing constants in the equations changes the solution, they don't really change the *method* of how we solve these equations.

In fact, it is a great discovery to realize that all we care about are the *constants* and the *coefficients* of the equations. By systematically handling these, we can solve any set of linear equations in a very nice way. Before we go on, we must first define what a linear equation is.

#### Definition 3.1.2 Linear Equation.

A **linear equation** is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$$

where the  $x_i$  are variables (the unknowns), the  $a_i$  are coefficients, and  $c$  is a constant.

A *system of linear equations* is a set of linear equations that involve the same variables.

A *solution* to a system of linear equations is a set of values for the variables  $x_i$  such that each equation in the system is satisfied.

So in Example 3.1.1, when we answered "how many marbles of each colour are there?" we were also answering "find a solution to a certain system of linear equations."

The following are examples of linear equations:

$$2x + 3y - 7z = 29$$

$$x_1 + \frac{7}{2}x_2 + x_3 - x_4 + 17x_5 = \sqrt[3]{-10}$$

$$y_1 + 14^2y_4 + 4 = y_2 + 13 - y_1$$

$$\sqrt{7}r + \pi s + \frac{3t}{5} = \cos(45^\circ).$$

Notice that the coefficients and constants can be fractions and irrational numbers (like  $\pi$ ,  $\sqrt[3]{-10}$  and  $\cos(45^\circ)$ ). The variables only come in the form of  $a_ix_i$ ; that is, just one variable multiplied by a coefficient. (Note that  $\frac{3t}{5} = \frac{3}{5}t$ , just a variable multiplied by a coefficient.) Also, it doesn't really matter what side of the equation we put the variables and the constants, although most of the time we write them with the variables on the left and the constants on the right.

We would not regard the above collection of equations to constitute a system of equations, since each equation uses differently named variables. An example of a system of linear equations is

$$\begin{aligned}x_1 - x_2 + x_3 + x_4 &= 1 \\2x_1 + 3x_2 + x_4 &= 25 \\x_2 + x_3 &= 10.\end{aligned}$$

It is important to notice that not all equations used all of the variables (it is more accurate to say that the coefficients can be 0, so the last equation could have been written as  $0x_1 + x_2 + x_3 + 0x_4 = 10$ ). Also, just because we have four unknowns does not mean we have to have four equations. We could have had fewer, even just one, and we could have had more.

To get a better feel for what a linear equation is, we point out some examples of what are *not* linear equations.

$$\begin{aligned}2xy + z &= 1 \\5x^2 + 2y^5 &= 100 \\\frac{1}{x} + \sqrt{y} + 24z &= 3 \\\sin^2 x_1 + \cos^2 x_2 &= 29 \\2^{x_1} + \ln x_2 &= 13.\end{aligned}$$

The first example is not a linear equation since the variables  $x$  and  $y$  are multiplied together. The second is not a linear equation because the variables are raised to powers other than 1; that is also a problem in the third equation (remember that  $1/x = x^{-1}$  and  $\sqrt{x} = x^{1/2}$ ). Our variables cannot be the argument of function like  $\sin$ ,  $\cos$  or  $\ln$ , nor can our variables be raised as an exponent.

At this stage, we have yet to discuss how to efficiently find a solution to a system of linear equations. That is a goal for the upcoming sections. Right now we focus on identifying linear equations. It is also useful to “limber” up by solving a few systems of equations using any method we have at hand to refresh our memory about the basic process.

## 3.1.1 Exercises

**Exercise Group.** State whether or not the equation is linear.

- |                                   |                                       |
|-----------------------------------|---------------------------------------|
| 1. $x + y + z = 10$               | 2. $xy + yz + xz = 1$                 |
| 3. $-3x + 9 = 3y - 5z + x - 7$    | 4. $\sqrt{5}y + \pi x = -1$           |
| 5. $(x - 1)(x + 1) = 0$           | 6. $\sqrt{x_1^2 + x_2^2} = 25$        |
| 7. $x_1 + y + t = 1$              | 8. $\frac{1}{x} + 9 = 3 \cos(y) - 5z$ |
| 9. $\cos(15)y + \frac{x}{4} = -1$ | 10. $2^x + 2^y = 16$                  |

**Exercise Group.** Solve the system of linear equations.

- |  |  |
|--|--|
| 11. $\begin{array}{rcl} x & + & y = -1 \\ 2x & - & 3y = 8 \end{array}$                                 | 12. $\begin{array}{rcl} 2x & - & 3y = 3 \\ 3x & + & 6y = 8 \end{array}$                    |
| 13. $\begin{array}{rcl} x & - & y + z = 1 \\ 2x & + & 6y - z = -4 \\ 4x & - & 5y + 2z = 0 \end{array}$ | 14. $\begin{array}{rcl} x & + & y - z = 1 \\ 2x & + & y = 2 \\ & & y + 2z = 0 \end{array}$ |
15. A jar contains 100 marbles. We know there are twice as many green marbles as red; that the number of blue and yellow marbles together is the same as the number of green; and that three times the number of yellow marbles together with the red marbles gives the same numbers as the blue marbles. How many of each color of marble are in the jar?
16. A farmer looks out his window at his chickens and pigs. He tells his daughter that he sees 62 heads and 190 legs. How many chickens and pigs does the farmer have?
17. A lady buys 20 trinkets at a yard sale. The cost of each trinket is either \$0.30 or \$0.65. If she spends \$8.80, how many of each type of trinket does she buy?

## 3.2 Using Matrices To Solve Systems of Linear Equations

In Section 3.1 we solved a linear system using familiar techniques. Later, we commented that in the linear equations we formed, the most important information was the coefficients and the constants; the names of the variables really didn't matter. In Example 3.1.1 we had the following three equations:

$$\begin{aligned}r + b + g &= 30 \\r &= 2g \\b &= r + g.\end{aligned}$$

Let's rewrite these equations so that all variables are on the left of the equal sign and all constants are on the right. Also, for a bit more consistency, let's list the variables in alphabetical order in each equation. Therefore we can write the equations as

$$\begin{aligned}b + g + r &= 30 \\-2g + r &= 0 \\-b + g + r &= 0\end{aligned}\tag{3.2.1}$$

As we mentioned before, there isn't just one "right" way of finding the solution to this system of equations. Here is another way to do it, a way that is a bit different from our method in Section 3.1.

First, let's add the first and last equations together, and write the result as a new third equation. This gives us:

$$\begin{aligned}b + g + r &= 30 \\-2g + r &= 0 \\2g + 2r &= 30\end{aligned}$$

**Note:** when we multiplied the second equation by  $-\frac{1}{2}$  in our third step, we were proceeding according to a well-known algorithm (which we will soon explain) that is especially efficient when implemented on a computer. We should point out, however, that since humans tend to be less happy about working with fractions than computers, we might reasonably decide that our third step should be to first add the second equation to the third, so that the  $2g$  in the third equation is cancelled by the  $-2g$  in the second.

A nice feature of this is that the only equation with a  $b$  in it is the first equation. Now let's multiply the second equation by  $-\frac{1}{2}$ . This gives

$$\begin{aligned}b + g + r &= 30 \\g - 1/2r &= 0 \\2g + 2r &= 30\end{aligned}$$

Let's now do two steps in a row; our goal is to get rid of the  $g$ 's in the first and third equations. In order to remove the  $g$  in the first equation, let's multiply the second equation by  $-1$  and add that to the first equation, replacing the first equation with that sum. To remove the  $g$  in the third equation, let's multiply the second equation by  $-2$  and add that to the third equation, replacing the third equation. Our new system of equations now becomes

$$\begin{aligned}b + 3/2r &= 30 \\g - 1/2r &= 0 \\3r &= 30\end{aligned}$$

Clearly we can multiply the third equation by  $\frac{1}{3}$  and find that  $r = 10$ ; let's make this our new third equation, giving

$$\begin{aligned}b + 3/2r &= 30 \\g - 1/2r &= 0 \\r &= 10\end{aligned}$$

Now let's get rid of the  $r$ 's in the first and second equation. To remove the  $r$  in the first equation, let's multiply the third equation by  $-\frac{3}{2}$  and add the result

to the first equation, replacing the first equation with that sum. To remove the  $r$  in the second equation, we can multiply the third equation by  $\frac{1}{2}$  and add that to the second equation, replacing the second equation with that sum. This gives us:

$$\begin{array}{rcl} b & = & 15 \\ g & = & 5 \\ r & = & 10 \end{array}.$$

Clearly we have discovered the same result as when we solved this problem in Section 3.1.

Now again revisit the idea that all that really matters are the coefficients and the constants. There is nothing special about the letters  $b$ ,  $g$  and  $r$ ; we could have used  $x$ ,  $y$  and  $z$  or  $x_1$ ,  $x_2$  and  $x_3$ . And even then, since we wrote our equations so carefully, we really didn't need to write the variable names at all as long as we put things "in the right place."

Let's look again at our system of equations in Equation (3.2.1) and write the coefficients and the constants in a rectangular array. This time we won't ignore the zeros, but rather write them out.

$$\begin{array}{rrcrcl} b & + & g & + & r & = & 30 \\ 0b & - & 2g & + & r & = & 0 \\ -b & + & g & + & r & = & 0 \end{array} \Leftrightarrow \begin{bmatrix} 1 & 1 & 1 & 30 \\ 0 & -2 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix}.$$

Notice how even the equal signs are gone; we don't need them, for we know that the last *column* contains the coefficients.

The mathematical object we have created on the right is called a **matrix**. We will give a formal mathematical definition in Chapter 4, where we will see that matrices are interesting objects in their own right, with numerous interesting properties and applications. For now we can think of a matrix simply as a rectangular table of numbers that allows us to keep track of certain information; namely, the constants that define our system of linear equations. Note that each row in our matrix represents one of the equations, while each column (except for the last) represents a variable. We will often want to distinguish between the **coefficient matrix** of a system; that is, the matrix containing the coefficients (the numbers in front of the variables) of a system of linear equations, and the matrix that also includes the column of constants, as above. When we include the constants, we refer to the resulting matrix as an *augmented matrix*.

It is common (but not mandatory) to place a vertical line separating the final column of an augmented matrix (containing the constants) from the other columns (containing the coefficients). One advantage of doing so is that we can quickly recognize that we're dealing with an augmented matrix rather than a coefficient matrix. For example, the augmented matrix for the system Equation (3.2.1) would be written as seen below on the right:

$$\begin{bmatrix} 1 & 1 & 1 & 30 \\ 0 & -2 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix} \quad \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 0 & -2 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{array} \right]$$

Without the vertical line      With the vertical line

**Figure 3.2.1** Two ways of writing an augmented matrix

We can use augmented matrices to find solutions to linear equations by using essentially the same steps we used above. Every time we used the word "equation" above, substitute the word "row," as we show below. The comments explain how we get from the current set of equations (or matrix) to the one on the next line.

Let us look back at the steps we used to solve the system of equations at the beginning of this section. Clearly, the order in which we right down our equations does not affect the solution; in terms of our matrix, this means we can rearrange the rows as we please. Note, however, that we *cannot* rearrange the columns. To do so would mix up our variables, causing us to lose track of our system.

In addition to changing the order of our equations, we also saw that we can multiply both sides of any equation by a constant, and we can add a multiple of one equation to another. These three manipulations are known as the **elementary operations** used to solve a system of linear equations. We will see that with these three elementary operations, we can determine the solution (if any) to any linear system.

Each of these elementary operations corresponds to an operation performed on the corresponding augmented matrix known as an **elementary row operation**. We use a shorthand notation to describe these matrix operations; let  $R_1$ ,  $R_2$  represent “row 1” and “row 2,” respectively. We can write “add row 1 to row 3, and replace row 3 with that sum” as “ $R_1 + R_3 \rightarrow R_3$ .” The expression “ $R_1 \leftrightarrow R_2$ ” means “interchange row 1 and row 2.”

$$\begin{array}{rrcr} b & + & g & + & r & = & 30 \\ & & - & 2g & + & r & = & 0 \\ -b & + & g & + & r & = & 0 \end{array} \quad \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 0 & -2 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{array} \right]$$

Replace equation 3 with the sum of equations 1 and 3

Replace row 3 with the sum of rows 1 and 3.  
( $R_1 + R_3 \rightarrow R_3$ )

$$\begin{array}{rrcr} b & + & g & + & r & = & 30 \\ & & - & 2g & + & r & = & 0 \\ & & & 2g & + & 2r & = & 30 \end{array} \quad \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 0 & -2 & 1 & 0 \\ 0 & 2 & 2 & 30 \end{array} \right]$$

Multiply equation 2 by  $-\frac{1}{2}$

Multiply row 2 by  $-\frac{1}{2}$   
( $-\frac{1}{2}R_2 \rightarrow R_2$ )

$$\begin{array}{rrcr} b & + & g & + & r & = & 30 \\ & & g & + & -1/2r & = & 0 \\ & & 2g & + & 2r & = & 30 \end{array} \quad \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 2 & 2 & 30 \end{array} \right]$$

Replace equation 1 with the sum of  $-1$  times equation 2 plus equation 1;

Replace equation 3 with the sum of  $-2$  times equation 2 plus equation 3

Replace row 1 with the sum of  $-1$  times row 2 plus row 1  
( $-R_2 + R_1 \rightarrow R_1$ )

Replace row 3 with the sum of  $-2$  times row 2 plus row 3  
( $-2R_2 + R_3 \rightarrow R_3$ )

$$\begin{array}{rrcr} b & + & & 3/2r & = & 30 \\ & & g & - & 1/2r & = & 0 \\ & & & 3r & = & 30 \end{array} \quad \left[ \begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & 30 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 3 & 30 \end{array} \right]$$



Multiply equation 3 by  $\frac{1}{3}$

Multiply row 3 by  $\frac{1}{3}$   
 $(\frac{1}{3}R_3 \rightarrow R_3)$

$$\begin{array}{rcrcrcrcl} b & + & & 3/2r & = & 30 \\ & & g & - & 1/2r & = & 0 \\ & & & & r & = & 10 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & 30 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 10 \end{array} \right]$$

Replace equation 2 with the sum of  $\frac{1}{2}$  times equation 3 plus equation 2;

Replace equation 1 with the sum of  $-\frac{3}{2}$  times equation 3 plus equation 1

Replace row 2 with the sum of  $\frac{1}{2}$  times row 3 plus row 2  
 $(\frac{1}{2}R_3 + R_2 \rightarrow R_2)$

Replace row 1 with the sum of  $-\frac{3}{2}$  times row 3 plus row 1  
 $(-\frac{3}{2}R_3 + R_1 \rightarrow R_1)$

$$\begin{array}{rcrcrcrcl} b & & & & = & 15 \\ & & g & & = & 5 \\ & & & & r & = & 10 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 15 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 10 \end{array} \right]$$

The final matrix contains the same solution information as we have on the left in the form of equations. Recall that the first column of our matrices held the coefficients of the  $b$  variable; the second and third columns held the coefficients of the  $g$  and  $r$  variables, respectively. Therefore, the first row of the matrix can be interpreted as " $b + 0g + 0r = 15$ ," or more concisely, " $b = 15$ ."

Let's practice this manipulation again.

### Example 3.2.2 Solving a system using augmented matrices.

Find a solution to the following system of linear equations by simultaneously manipulating the equations and the corresponding augmented matrices.

$$\begin{array}{rcrcrcrcrcrcl} x_1 & + & x_2 & + & x_3 & = & 0 \\ 2x_1 & + & 2x_2 & + & x_3 & = & 0 \\ -1x_1 & + & x_2 & - & 2x_3 & = & 2 \end{array}$$

**Solution.** We'll first convert this system of equations into a matrix, then we'll proceed by manipulating the system of equations (and hence the matrix) to find a solution. Again, there is not just one "right" way of proceeding; we'll choose a method that is pretty efficient, but other methods certainly exist (and may be "better"! ). The method use here, though, is a good one, and it is the method that we will be learning in the future.

The given system and its corresponding augmented matrix are seen below.

Original system of equations

$$\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & + & x_3 & = & 0 \\ 2x_1 & + & 2x_2 & + & x_3 & = & 0 \\ -1x_1 & + & x_2 & - & 2x_3 & = & 2 \end{array}$$

Corresponding matrix

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 1 & 0 \\ -1 & 1 & -2 & 2 \end{array} \right]$$

We'll proceed by trying to get the  $x_1$  out of the second and third equation.

Replace equation 2 with the sum of  $-2$  times equation 1 plus equation 2;

Replace equation 3 with the sum of equation 1 and equation 3

$$\begin{array}{rrcr} x_1 & + & x_2 & + & x_3 & = & 0 \\ & & & & -x_3 & = & 0 \\ 2x_2 & - & x_3 & = & 2 \end{array}$$

Replace row 2 with the sum of  $-2$  times row 1 plus row 2

$$(-2R_1 + R_2 \rightarrow R_2)$$

Replace row 3 with the sum of row 1 and row 3

$$(R_1 + R_3 \rightarrow R_3)$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & 2 \end{array} \right]$$

Notice that the second equation no longer contains  $x_2$ . We'll exchange the order of the equations so that we can follow the convention of solving for the second variable in the second equation.

Interchange equations 2 and 3

$$\begin{array}{rrcr} x_1 & + & x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & x_3 & = & 2 \\ & & & & -x_3 & = & 0 \end{array}$$

Interchange rows 2 and 3

$$(R_2 \leftrightarrow R_3)$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & -1 & 0 \end{array} \right]$$

Multiply equation 2 by  $\frac{1}{2}$

$$\begin{array}{rrcr} x_1 & + & x_2 & + & x_3 & = & 0 \\ & & x_2 & - & \frac{1}{2}x_3 & = & 1 \\ & & & & -x_3 & = & 0 \end{array}$$

Multiply row 2 by  $\frac{1}{2}$   
( $\frac{1}{2}R_2 \rightarrow R_2$ )

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & -1 & 0 \end{array} \right]$$

Multiply equation 3 by  $-1$

$$\begin{array}{rrcr} x_1 & + & x_2 & + & x_3 & = & 0 \\ & & x_2 & - & \frac{1}{2}x_3 & = & 1 \\ & & & & x_3 & = & 0 \end{array}$$

Multiply row 3 by  $-1$   
( $-1R_3 \rightarrow R_3$ )

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Notice that the last equation (and also the last row of the matrix) show that  $x_3 = 0$ . Knowing this would allow us to simply eliminate the  $x_3$  from the first two equations. However, we will formally do this by manipulating the equations (and rows) as we have previously.

Replace equation 1 with the sum of  $-1$  times equation 3 plus equation 1;

Replace equation 2 with the sum of  $\frac{1}{2}$  times equation 3 plus equation 2

Replace row 1 with the sum of  $-1$  times row 3 plus row 1

$$(-R_3 + R_1 \rightarrow R_1)$$

Replace row 2 with the sum of  $\frac{1}{2}$  times row 3 plus row 2

$$(\frac{1}{2}R_3 + R_2 \rightarrow R_2)$$

$$\begin{array}{rcl} x_1 + x_2 & = & 0 \\ x_2 & = & 1 \\ x_3 & = & 0 \end{array} \quad \left[ \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Notice how the second equation shows that  $x_2 = 1$ . All that remains to do is to solve for  $x_1$ .

Replace equation 1 with the  
sum of  $-1$  times equation 2  
plus equation 1

Replace row 1 with the sum of  
 $-1$  times row 2 plus row 1  
( $-R_2 + R_1 \rightarrow R_1$ )

$$\begin{array}{rcl} x_1 & = & -1 \\ x_2 & = & 1 \\ x_3 & = & 0 \end{array} \quad \left[ \begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Obviously the equations on the left tell us that  $x_1 = -1$ ,  $x_2 = 1$  and  $x_3 = 0$ , and notice how the matrix on the right tells us the same information.

## 3.2.1 Exercises

**Exercise Group.** Convert the system of equations into an augmented matrix.

$$\begin{array}{rclcl} 1. & 3x & + & 4y & + & 5z & = & 7 \\ & -x & + & y & - & 3z & = & 1 \\ & 2x & - & 2y & + & 3z & = & 5 \end{array}$$

$$\begin{array}{rclcl} 3. & x_1 & + & 3x_2 & - & 4x_3 & + & 5x_4 & = & 17 \\ & & & -x_1 & + & 4x_3 & + & 8x_4 & = & 1 \\ & 2x_1 & + & 3x_2 & + & 4x_3 & + & 5x_4 & = & 6 \end{array}$$

$$\begin{array}{rclcl} 2. & 2x & + & 5y & - & 6z & = & 2 \\ & 9x & & & - & 8z & = & 10 \\ & -2x & + & 4y & + & z & = & -7 \end{array}$$

$$\begin{array}{rclcl} 4. & 3x_1 & - & 2x_2 & = & 4 \\ & 2x_1 & = & 3 \\ & -x_1 & + & 9x_2 & = & 8 \\ & 5x_1 & - & 7x_2 & = & 13 \end{array}$$

**Exercise Group.** Convert the given augmented matrix into a system of linear equations. Use the variables  $x_1, x_2$ , etc.

$$5. \quad \left[ \begin{array}{cc|c} 1 & 2 & 3 \\ -1 & 3 & 9 \end{array} \right]$$

$$7. \quad \left[ \begin{array}{cccc|c} 1 & 1 & -1 & -1 & 2 \\ 2 & 1 & 3 & 5 & 7 \end{array} \right]$$

$$9. \quad \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{array} \right]$$

$$6. \quad \left[ \begin{array}{cc|c} -3 & 4 & 7 \\ 0 & 1 & -2 \end{array} \right]$$

$$8. \quad \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{array} \right]$$

$$10. \quad \left[ \begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 7 & 2 \\ 0 & 1 & 3 & 2 & 0 & 5 \end{array} \right]$$

**Exercise Group.** Perform the given row operations on  $A$ , where

$$A = \begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -2 \\ 5 & 0 & 3 \end{bmatrix}.$$

$$11. \quad -1R_1 \rightarrow R_1$$

$$13. \quad R_1 + R_2 \rightarrow R_2$$

$$15. \quad \frac{1}{2}R_2 \rightarrow R_2$$

$$12. \quad R_2 \leftrightarrow R_3$$

$$14. \quad 2R_2 + R_3 \rightarrow R_3$$

$$16. \quad -\frac{5}{2}R_1 + R_3 \rightarrow R_3$$

**Exercise Group.** A matrix  $A$  is given below:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

For the given matrix  $B$ , give the row operation that transforms  $A$  into  $B$ .

$$17. \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$19. \quad B = \begin{bmatrix} 3 & 5 & 7 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$21. \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$18. \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$20. \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

**Exercise Group.** Rewrite the system of equations in matrix form. Find the solution to the linear system by simultaneously manipulating the equations and the matrix.

$$\begin{array}{rcl} \mathbf{22.} & x & + \quad y = 3 \\ & 2x & - \quad 3y = 1 \end{array}$$

$$\mathbf{24.} \quad \begin{array}{rcl} -2x & + & 3y = 2 \\ -x & + & y = 1 \end{array}$$

$$\mathbf{26.} \quad \begin{array}{rcl} -5x_1 & & + \quad 2x_3 = 14 \\ & x_2 & = 1 \\ -3x_1 & & + \quad x_3 = 8 \end{array}$$

$$\mathbf{23.} \quad \begin{array}{rcl} 2x & + & 4y = 10 \\ -x & + & y = 4 \end{array}$$

$$\mathbf{25.} \quad \begin{array}{rcl} 2x & + & 3y = 2 \\ -2x & + & 6y = 1 \end{array}$$

$$\mathbf{27.} \quad \begin{array}{rcl} & - & 5x_2 + 2x_3 = -11 \\ x_1 & & + 2x_3 = 15 \\ & - & 3x_2 + x_3 = -8 \end{array}$$

### 3.3 Elementary Row Operations and Gaussian Elimination

In our examples thus far, we have essentially used just three types of manipulations in order to find solutions to our systems of equations. These three manipulations are:

1. Add a scalar multiple of one equation to a second equation, and replace the second equation with that sum
2. Multiply one equation by a nonzero scalar
3. Swap the position of two equations in our list

We saw earlier how we could write all the information of a system of equations in a matrix, so it makes sense that we can perform similar operations on matrices (as we have done before). Again, simply replace the word “equation” above with the word “row.”

We didn’t justify our ability to manipulate our equations in the above three ways; it seems rather obvious that we should be able to do that. In that sense, these operations are “elementary.” These operations are *elementary* in another sense; they are *fundamental* — they form the basis for much of what we will do in matrix algebra. Since these operations are so important, we list them again here in the context of matrices.

#### Key Idea 3.3.1 Elementary Row Operations.

1. Add a scalar multiple of one row to another row, and replace the latter row with that sum
2. Multiply one row by a nonzero scalar
3. Swap the position of two rows

Given any system of linear equations, we can find a solution (if one exists) by using these three row operations. Elementary row operations give us a new linear system, but the solution to the new system is the same as the old. We can use these operations as much as we want and not change the solution. This brings to mind two good questions:

1. Since we can use these operations as much as we want, how do we know when to stop? (Where are we supposed to “go” with these operations?)
2. Is there an efficient way of using these operations? (How do we get “there” the fastest?)

We’ll answer the first question first. Most of the time (unless one prefers obfuscation to clarification) we will want to take our original matrix and, using the elementary row operations, put it into something called **reduced row echelon form**. This is our “destination,” for this form allows us to readily identify whether or not a solution exists, and in the case that it does, what that solution is.

In the previous section, when we manipulated matrices to find solutions, we were unwittingly putting the matrix into reduced row echelon form. However, not all solutions come in such a simple manner as we’ve seen so far. Putting a matrix into reduced row echelon form helps us identify all types of solutions.

We'll explore the topic of understanding what the reduced row echelon form of a matrix tells us in the following sections; in this section we focus on finding it.

**Definition 3.3.2 Reduced Row Echelon Form.**

A matrix is in *reduced row echelon form* if its entries satisfy the following conditions:

1. The first nonzero entry in each row is a 1 (called a *leading 1*).
2. Each leading 1 comes in a column to the right of the leading 1s in rows above it.
3. All rows of all 0s come at the bottom of the matrix.
4. If a column contains a leading 1, then all other entries in that column are 0.

A matrix that satisfies the first three conditions is said to be in *row echelon form*.

**Example 3.3.3 Determining if a matrix is in reduced row echelon form.**

Which of the following matrices is in reduced row echelon form?

(a)  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(f)  $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

(b)  $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

(c)  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(g)  $\begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(d)  $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(e)  $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

(h)  $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**Solution.** The matrices in (a), (b), (c), (d) and (g) are all in reduced row echelon form. Check to see that each satisfies the necessary conditions. If your instincts were wrong on some of these, correct your thinking accordingly.

The matrix in (e) is not in reduced row echelon form since the row of all zeros is not at the bottom. The matrix in (f) is not in reduced row echelon form since the first nonzero entries in rows 2 and 3 are not 1. Finally, the matrix in (h) is not in reduced row echelon form since the first entry in column 2 is not zero; the second 1 in column 2 is a leading one, hence all other entries in that column should be 0.

We end this example with a preview of what we'll learn in the future. Consider the matrix in (b). If this matrix came from the augmented matrix of a system of linear equations, then we can readily recognize that the solution of the system is  $x_1 = 1$  and  $x_2 = 2$ . Again, in previous

examples, when we found the solution to a linear system, we were unwittingly putting our matrices into reduced row echelon form.

We began this section discussing how we can manipulate the entries in a matrix with elementary row operations. This led to two questions, “Where do we go?” and “How do we get there quickly?” We’ve just answered the first question: most of the time we are “going to” reduced row echelon form. We now address the second question.

There is no one “right” way of using these operations to transform a matrix into reduced row echelon form. However, there is a general technique that works very well in that it is very efficient (so we don’t waste time on unnecessary steps). This technique is called *Gaussian elimination*. It is named in honour of the great mathematician Karl Friedrich Gauss.

While this technique isn’t very difficult to use, it is one of those things that is easier understood by watching it being used than explained as a series of steps. With this in mind, we will go through one more example highlighting important steps and then we’ll explain the procedure in detail.

#### Example 3.3.4 Using row operations to simplify an augmented matrix.

Put the augmented matrix of the following system of linear equations into reduced row echelon form.

$$\begin{array}{rrcr} -3x_1 & - & 3x_2 & + & 9x_3 & = & 12 \\ 2x_1 & + & 2x_2 & - & 4x_3 & = & -2 \\ & & -2x_2 & - & 4x_3 & = & -8 \end{array}$$

**Solution.** We start by converting the linear system into an augmented matrix.

$$\left[ \begin{array}{ccc|c} \boxed{-3} & -3 & 9 & 12 \\ 2 & 2 & -4 & -2 \\ 0 & -2 & -4 & -8 \end{array} \right].$$

Our next step is to change the entry in the box to a 1. To do this, let’s multiply row 1 by  $-\frac{1}{3}$ .

$$-\frac{1}{3}R_1 \rightarrow R_1 \quad \left[ \begin{array}{ccc|c} 1 & 1 & -3 & -4 \\ 2 & 2 & -4 & -2 \\ 0 & -2 & -4 & -8 \end{array} \right].$$

We have now created a *leading 1*; that is, the first entry in the first row is a 1. Our next step is to put zeros under this 1. To do this, we’ll use the elementary row operation given below.

$$-2R_1 + R_2 \rightarrow R_2 \quad \left[ \begin{array}{ccc|c} 1 & 1 & -3 & -4 \\ 0 & \boxed{0} & 2 & 6 \\ 0 & -2 & -4 & -8 \end{array} \right].$$

Once this is accomplished, we shift our focus from the leading one down one row, and to the right one column, to the position that is boxed. We again want to put a 1 in this position. We can use any elementary row operations, but we need to restrict ourselves to using only the second row and any rows below it. Probably the simplest thing we can do is



interchange rows 2 and 3, and then scale the new second row so that there is a 1 in the desired position.

$$\begin{array}{l}
 R_2 \leftrightarrow R_3 \\
 -\frac{1}{2}R_2 \rightarrow R_2
 \end{array}
 \quad
 \left[ \begin{array}{ccc|c}
 1 & 1 & -3 & -4 \\
 0 & \boxed{-2} & -4 & -8 \\
 0 & 0 & 2 & 6
 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c}
 1 & 1 & -3 & -4 \\
 0 & 1 & 2 & 4 \\
 0 & 0 & \boxed{2} & 6
 \end{array} \right].$$

We have now created another leading 1, this time in the second row. Our next desire is to put zeros underneath it, but this has already been accomplished by our previous steps. Therefore we again shift our attention to the right one column and down one row, to the next position put in the box. We want that to be a 1. A simple scaling will accomplish this.

$$\frac{1}{2}R_3 \rightarrow R_3
 \quad
 \left[ \begin{array}{ccc|c}
 1 & 1 & -3 & -4 \\
 0 & 1 & 2 & 4 \\
 0 & 0 & 1 & 3
 \end{array} \right]$$

This ends what we will refer to as the *forward steps*. Our next task is to use the elementary row operations and go back and put zeros above our leading 1s. This is referred to as the *backward steps*. These steps are given below.

$$\begin{array}{l}
 3R_3 + R_1 \rightarrow R_1 \\
 -2R_3 + R_2 \rightarrow R_2 \\
 -R_2 + R_1 \rightarrow R_1
 \end{array}
 \quad
 \left[ \begin{array}{ccc|c}
 1 & 1 & 0 & 5 \\
 0 & 1 & 0 & -2 \\
 0 & 0 & 1 & 3
 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c}
 1 & 0 & 0 & 7 \\
 0 & 1 & 0 & -2 \\
 0 & 0 & 1 & 3
 \end{array} \right].$$

It is now easy to read off the solution as  $x_1 = 7$ ,  $x_2 = -2$  and  $x_3 = 3$ .

We now formally explain the procedure used to find the solution above. As you read through the procedure, follow along with the example above so that the explanation makes more sense.

Technically, Gaussian elimination consists only of the “forward steps”; including the “backward steps” results in an algorithm known as *Gauss-Jordan elimination*. In Gaussian elimination, instead of the backward steps, one uses **back substitution**. At the end of the forward steps, we can immediately read off  $x_3 = 3$  from the third row. The second row corresponds to the equation  $x_2 + 2x_3 = 4$ , and plugging in  $x_3 = 3$  to this equation, we get  $x_2 + 6 = 4$ , so  $x_2 = -2$ . Finally, the first row gives the equation  $x_1 + x_2 - 3x_3 = -4$ , and plugging in  $x_2 = -2$  and  $x_3 = 3$ , we get  $x_1 - 2 - 9 = -4$ , so  $x_1 = 7$ . Using back substitution is slightly more efficient than including the backward steps, but once we get to examples with free parameters, you’ll probably find that it’s easier to go all the way to reduced row-echelon form.

**Forward Steps**

1. Working from left to right, consider the first column that isn't all zeros that hasn't already been worked on. Then working from top to bottom, consider the first row that hasn't been worked on.
2. If the entry in the row and column that we are considering is zero, interchange rows with a row below the current row so that that entry is nonzero. If all entries below are zero, we are done with this column; start again at step 1.
3. Multiply the current row by a scalar to make its first entry a 1 (a leading 1).
4. Repeatedly use Elementary Row Operation 1 to put zeros underneath the leading one.
5. Go back to step 1 and work on the new rows and columns until either all rows or columns have been worked on.

If the above steps have been followed properly, then the following should be true about the current state of the matrix:

1. The first nonzero entry in each row is a 1 (a leading 1).
2. The entries under each leading 1 are all zeros.
3. Each leading 1 is in a column to the right of the leading 1s above it.
4. All rows of all zeros come at the bottom of the matrix.

Note that this means we have just put a matrix into row echelon form. The next steps finish the conversion into *reduced* row echelon form. These next steps are referred to as the *backward* steps. These are much easier to state.

**Backward Steps**

1. Starting from the right and working left, use Elementary Row Operation 1 repeatedly to put zeros above each leading 1.

The basic method of Gaussian elimination is this: create leading ones and then use elementary row operations to put zeros above and below these leading ones. We can do this in any order we please, but by following the “Forward Steps” and “Backward Steps,” we make use of the presence of zeros to make the overall computations easier. This method is very efficient, so it gets its own name (which we’ve already been using).

**Definition 3.3.5 Gaussian Elimination.**

**Gaussian elimination** is the technique for finding the reduced row echelon form of a matrix using the above procedure. It can be abbreviated to:

1. Create a leading 1.
2. Use this leading 1 to put zeros underneath it.
3. Repeat the above steps until all possible rows have leading 1s.
4. Put zeros above these leading 1s.

Let's practice some more.

**Example 3.3.6 Using Gaussian elimination.**

Use Gaussian elimination to put the matrix  $A$  into reduced row echelon form, where

$$A = \begin{bmatrix} -2 & -4 & -2 & -10 & 0 \\ 2 & 4 & 1 & 9 & -2 \\ 3 & 6 & 1 & 13 & -4 \end{bmatrix}.$$

**Solution.** We start by wanting to make the entry in the first column and first row a 1 (a leading 1). To do this we'll scale the first row by a factor of  $-\frac{1}{2}$ .

$$-\frac{1}{2}R_1 \rightarrow R_1 \quad \begin{bmatrix} 1 & 2 & 1 & 5 & 0 \\ 2 & 4 & 1 & 9 & -2 \\ 3 & 6 & 1 & 13 & -4 \end{bmatrix}$$

Next we need to put zeros in the column below this newly formed leading 1.

$$\begin{array}{l} -2R_1 + R_2 \rightarrow R_2 \\ -3R_1 + R_3 \rightarrow R_3 \end{array} \quad \begin{bmatrix} 1 & 2 & 1 & 5 & 0 \\ 0 & \boxed{0} & -1 & -1 & -2 \\ 0 & 0 & -2 & -2 & -4 \end{bmatrix}$$

Our attention now shifts to the right one column and down one row to the position indicated by the box. We want to put a 1 in that position. Our only options are to either scale the current row or to interchange rows with a row below it. However, in this case neither of these options will accomplish our goal. Therefore, we shift our attention to the right one more column.

We want to put a 1 where there is a  $-1$ . A simple scaling will accomplish this; once done, we will put a 0 underneath this leading one.

$$\begin{array}{l} -R_2 \rightarrow R_2 \\ 2R_2 + R_3 \rightarrow R_3 \end{array} \quad \begin{bmatrix} 1 & 2 & 1 & 5 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & -2 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 5 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & \boxed{0} & 0 \end{bmatrix}$$

In Example 3.3.6, we create our first leading 1 by multiplying Row 1 by  $-\frac{1}{2}$ . On a computer, this is the most efficient option, and it's a good choice in this example, since every entry in Row 1 is even, so no fractions are introduced. Most people like to avoid fractions if they can, and in some cases that's impossible with this type of row operation. The other option we could have tried here is adding Row 3 to Row 1: since  $-2 + 3 = 1$ , the row operation  $R_1 \rightarrow R_1 + R_3$  also would have created a leading 1 for us.

Our attention now shifts over one more column and down one row to the position indicated by the box; we wish to make this a 1. Of course, there is no way to do this, so we are done with the forward steps.

Our next goal is to put a 0 above each of the leading 1s (in this case there is only one leading 1 to deal with).

$$-R_2 + R_1 \rightarrow R_1 \quad \begin{bmatrix} 1 & 2 & 0 & 4 & -2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This final matrix is in reduced row echelon form.

### Example 3.3.7 Gaussian elimination, again.

Put the matrix

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 1 & 1 \\ 3 & 3 & 2 & 1 \end{bmatrix}$$

into reduced row echelon form.

**Solution.** Here we will show all steps without explaining each one.

$$-2R_1 + R_2 \rightarrow R_2 \quad \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -3 & -1 & -5 \\ 0 & -3 & -1 & -8 \end{bmatrix}$$

$$-\frac{1}{3}R_2 \rightarrow R_2 \quad \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & -3 & -1 & -8 \end{bmatrix}$$

$$3R_2 + R_3 \rightarrow R_3 \quad \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$-\frac{1}{3}R_3 \rightarrow R_3 \quad \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} -3R_3 + R_1 &\rightarrow R_1 \\ -\frac{5}{3}R_3 + R_2 &\rightarrow R_2 \end{aligned} \quad \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$-2R_2 + R_1 \rightarrow R_1 \quad \begin{bmatrix} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The last matrix in the above example is in reduced row echelon form. If one thinks of the original matrix as representing the augmented matrix of a system of linear equations, this final result is interesting. What does it mean to have a leading one in the last column? We'll figure this out in the next section.

**Example 3.3.8 Using back substitution.**

Put the matrix  $A$  into reduced row echelon form, where

$$A = \begin{bmatrix} 2 & 1 & -1 & 4 \\ 1 & -1 & 2 & 12 \\ 2 & 2 & -1 & 9 \end{bmatrix}.$$

**Solution.** We'll again show the steps without explanation, although we will stop at the end of the forward steps and make a comment.

$$\begin{array}{ll} \frac{1}{2}R_1 \rightarrow R_1 & \begin{bmatrix} 1 & 1/2 & -1/2 & 2 \\ 1 & -1 & 2 & 12 \\ 2 & 2 & -1 & 9 \end{bmatrix} \\ -R_1 + R_2 \rightarrow R_2 & \begin{bmatrix} 1 & 1/2 & -1/2 & 2 \\ 0 & -3/2 & 5/2 & 10 \\ -2R_1 + R_3 \rightarrow R_3 & \begin{bmatrix} 1 & 1/2 & -1/2 & 2 \\ 0 & -3/2 & 5/2 & 10 \\ 0 & 1 & 0 & 5 \end{bmatrix} \\ -\frac{2}{3}R_2 \rightarrow R_2 & \begin{bmatrix} 1 & 1/2 & -1/2 & 2 \\ 0 & 1 & -5/3 & -20/3 \\ 0 & 1 & 0 & 5 \end{bmatrix} \\ -R_2 + R_3 \rightarrow R_3 & \begin{bmatrix} 1 & 1/2 & -1/2 & 2 \\ 0 & 1 & -5/3 & -20/3 \\ 0 & 0 & 5/3 & 35/3 \end{bmatrix} \\ \frac{3}{5}R_3 \rightarrow R_3 & \begin{bmatrix} 1 & 1/2 & -1/2 & 2 \\ 0 & 1 & -5/3 & -20/3 \\ 0 & 0 & 1 & 7 \end{bmatrix}. \end{array}$$

Let's take a break here and think about the state of our linear system at this moment. Converting back to linear equations, we now know

$$\begin{array}{rcl} x_1 + 1/2x_2 - 1/2x_3 & = & 2 \\ x_2 - 5/3x_3 & = & -20/3 \\ x_3 & = & 7 \end{array}.$$

Since we know that  $x_3 = 7$ , the second equation turns into

$$x_2 - (5/3)(7) = -20/3,$$

telling us that  $x_2 = 5$ .

Finally, knowing values for  $x_2$  and  $x_3$  lets us substitute in the first equation and find

$$x_1 + (1/2)(5) - (1/2)(7) = 2,$$

so  $x_1 = 3$ .

This process of substituting known values back into other equations is called *back substitution*. This process is essentially what happens when we perform the backward steps of Gaussian elimination. We make note of this below as we finish out finding the reduced row echelon form of our matrix.

$$\frac{5}{3}R_3 + R_2 \rightarrow R_2 \quad \begin{bmatrix} 1 & 1/2 & -1/2 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

(knowing  $x_3 = 7$  allows us to find  $x_2 = 5$ )

$$\begin{array}{l} \frac{1}{2}R_3 + R_1 \rightarrow R_1 \\ -\frac{1}{2}R_2 + R_1 \rightarrow R_1 \end{array} \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

(knowing  $x_2 = 5$  and  $x_3 = 7$  allows us to find  $x_1 = 3$ )

We did our operations slightly “out of order” in that we didn’t put the zeros above our leading 1 in the third column in the same step, highlighting how back substitution works.

In all of our practice, we’ve only encountered systems of linear equations with exactly one solution. Is this always going to be the case? Could we ever have systems with more than one solution? If so, how many solutions could there be? Could we have systems without a solution? These are some of the questions we’ll address in the next section.

## 3.3.1 Exercises

**Exercise Group.** State whether or not the given matrices are in reduced row echelon form. If it is not, state why.

1.

(a)  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b)  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c)  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(d)  $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

3.

(a)  $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(b)  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c)  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d)  $\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

2.

(a)  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b)  $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

(c)  $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(d)  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

4.

(a)  $\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$

(b)  $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c)  $\begin{bmatrix} 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(d)  $\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

**Exercise Group.** Use Gaussian Elimination to put the given matrix into reduced row echelon form.

5.  $\begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix}$

7.  $\begin{bmatrix} 4 & 12 \\ -2 & -6 \end{bmatrix}$

9.  $\begin{bmatrix} -1 & 1 & 4 \\ -2 & 1 & 1 \end{bmatrix}$

11.  $\begin{bmatrix} 3 & -3 & 6 \\ -1 & 1 & -2 \end{bmatrix}$

13.  $\begin{bmatrix} -2 & -4 & -8 \\ -2 & -3 & -5 \\ 2 & 3 & 6 \end{bmatrix}$

15.  $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ -1 & -3 & 0 \end{bmatrix}$

17.  $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & -1 & -1 & 1 \\ -1 & 1 & 1 & 0 \end{bmatrix}$

6.  $\begin{bmatrix} 2 & -2 \\ 3 & -2 \end{bmatrix}$

8.  $\begin{bmatrix} -5 & 7 \\ 10 & 14 \end{bmatrix}$

10.  $\begin{bmatrix} 7 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

12.  $\begin{bmatrix} 4 & 5 & -6 \\ -12 & -15 & 18 \end{bmatrix}$

14.  $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

16.  $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 6 & 9 \end{bmatrix}$

18.  $\begin{bmatrix} 2 & -1 & 1 & 5 \\ 3 & 1 & 6 & -1 \\ 3 & 0 & 5 & 0 \end{bmatrix}$

$$19. \begin{bmatrix} 1 & 1 & -1 & 7 \\ 2 & 1 & 0 & 10 \\ 3 & 2 & -1 & 17 \end{bmatrix}$$

$$21. \begin{bmatrix} 2 & 2 & 1 & 3 & 1 & 4 \\ 1 & 1 & 1 & 3 & 1 & 4 \end{bmatrix}$$

$$20. \begin{bmatrix} 4 & 1 & 8 & 15 \\ 1 & 1 & 2 & 7 \\ 3 & 1 & 5 & 11 \end{bmatrix}$$

$$22. \begin{bmatrix} 1 & -1 & 3 & 1 & -2 & 9 \\ 2 & -2 & 6 & 1 & -2 & 13 \end{bmatrix}$$



## 3.4 Existence and Uniqueness of Solutions

So far, whenever we have solved a system of linear equations, we have always found exactly one solution. This is not always the case; we will find in this section that some systems do not have a solution, and others have more than one.

We start with a very simple example. Consider the following linear system:

$$x - y = 0.$$

There are obviously infinitely many solutions to this system; as long as  $x = y$ , we have a solution. We can picture all of these solutions by thinking of the graph of the equation  $y = x$  on the traditional  $x, y$  coordinate plane.

Let's continue this visual aspect of considering solutions to linear systems. Consider the system

$$x + y = 2$$

$$x - y = 0.$$

Each of these equations can be viewed as lines in the coordinate plane, and since their slopes are different, we know they will intersect somewhere (see Figure 3.4.1(a)). In this example, they intersect at the point  $(1, 1)$  — that is, when  $x = 1$  and  $y = 1$ , both equations are satisfied and we have a solution to our linear system. Since this is the only place the two lines intersect, this is the only solution.

Now consider the linear system

$$x + y = 1$$

$$2x + 2y = 2.$$

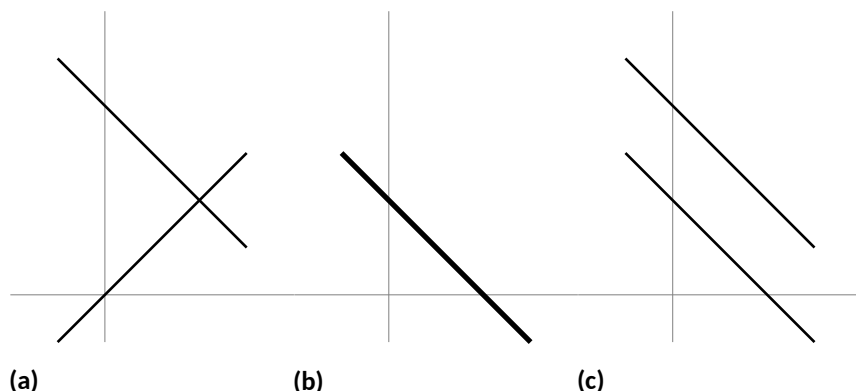
It is clear that while we have two equations, they are essentially the same equation; the second is just a multiple of the first. Therefore, when we graph the two equations, we are graphing the same line twice (see Figure 3.4.1(b); the thicker line is used to represent drawing the line twice). In this case, we have an infinite solution set, just as if we only had the one equation  $x + y = 1$ . We often write the solution as  $x = 1 - y$  to demonstrate that  $y$  can be any real number, and  $x$  is determined once we pick a value for  $y$ .

Finally, consider the linear system

$$x + y = 1$$

$$x + y = 2.$$

We should immediately spot a problem with this system; if the sum of  $x$  and  $y$  is 1, how can it also be 2? There is no solution to such a problem; this linear system has no solution. We can visualize this situation in Figure 3.4.1(c); the two lines are parallel and never intersect.



**Figure 3.4.1** The three possibilities for two linear equations with two unknowns

If we were to consider a linear system with three equations and two unknowns, we could visualize the solution by graphing the corresponding three lines. We can picture that perhaps all three lines would meet at one point, giving exactly 1 solution; perhaps all three equations describe the same line, giving an infinite number of solutions; perhaps we have different lines, but they do not all meet at the same point, giving no solution. We further visualize similar situations with, say, 20 equations with two variables.

On the other hand, if we increase the number of variables to three, then we know from Section 2.6 that the correct geometric visualization of the situation involves planes. A single equation such as

$$2x - 3y + 4z = 7$$

describes a plane in  $\mathbb{R}^3$  with normal vector  $\vec{n} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$ . (By inspection, we can also choose a point on the plane. Since  $3(-1) + 4(1) = 7$ , we know that the point  $(0, -1, 1)$  lies on the plane.)

If we have two equations in three unknowns, three possibilities arise. The second equation could be a multiple of the first, in which case we still have our original plane. The second equation could also describe a parallel plane; for example, we might have

$$\begin{array}{rrrrrcl} 2x & - & 3y & + & 4z & = & 7 \\ 2x & - & 3y & + & 4z & = & 3 \end{array}$$

In this case we know that there is no possible solution to the system, since there can be no point in common to these two parallel planes. The last possibility is that the planes are non-parallel, in which case they intersect in a line. For example, the system

$$\begin{array}{rrrrrcl} 2x & - & 3y & + & 4z & = & 7 \\ -x & + & 3y & - & z & = & 1 \end{array}$$

describes a pair of planes that intersect in the line (exercise)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -9 \\ -2 \\ 1 \end{bmatrix}.$$

From here, we can go on to consider three or more equations in three variables; there might be no solution, or the planes might intersect in a single point, or along a common line, or they might even all describe the same plane.

While it becomes harder to visualize when we add variables, no matter how many equations and variables we have, solutions to linear equations always come in one of three forms: exactly one solution, infinitely many solutions, or no solution. This is a fact that we will not prove here, but it deserves to be stated.

#### Theorem 3.4.2 Solution Forms of Linear Systems.

*Every linear system of equations has exactly one solution, infinitely many solutions, or no solution.*

This leads us to a definition. Here we don't differentiate between having one solution and infinitely many solutions, but rather just whether or not a solution exists.

**Definition 3.4.3 Consistent and Inconsistent Linear Systems.**

A system of linear equations is **consistent** if it has a solution (perhaps more than one). A linear system is **inconsistent** if it does not have a solution.

How can we tell what kind of solution (if one exists) a given system of linear equations has? The answer to this question lies with properly understanding the reduced row echelon form of a matrix. To discover what the solution is to a linear system, we first put the matrix into reduced row echelon form and then interpret that form properly.

Before we start with a simple example, let us make a note about finding the reduced row echelon form of a matrix.

As a general rule, when we are learning a new technique, it is best to not use technology to aid us. This helps us learn not only the technique but some of its “inner workings.” We can then use technology once we have mastered the technique and are now learning how to use it to solve problems.

From here on out, in our examples, when we need the reduced row echelon form of a matrix, we will not show the steps involved. Rather, we will give the initial matrix, then immediately give the reduced row echelon form of the matrix. We trust that the reader can verify the accuracy of this form by both performing the necessary steps by hand or utilizing some technology to do it for them.

Our first example explores officially a quick example used in the introduction of this section.

**Example 3.4.4 Solving a linear system.**

Find the solution to the linear system

$$\begin{array}{rcrcrcrcrl} x_1 & + & x_2 & = & 1 \\ 2x_1 & + & 2x_2 & = & 2 \end{array}.$$

**Solution.** Create the corresponding augmented matrix, and then put the matrix into reduced row echelon form.

$$\left[ \begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right].$$

Now convert the reduced matrix back into equations. In this case, we only have one equation,

$$x_1 + x_2 = 1,$$

or, equivalently,

$$\begin{array}{l} x_1 = 1 - x_2 \\ x_2 \text{ is a free parameter.} \end{array}$$

We have just introduced a new term, **free parameter**, or free variable, or simply, **parameter**. It is used to stress that idea that  $x_2$  can take on any value; we are “free” to choose any value for  $x_2$ . Once this value is chosen, the value of  $x_1$  is determined. We have infinitely many choices for the value of  $x_2$ , so therefore we have infinitely many solutions. The variable  $x_1$  is also known as a **leading variable**, since the column corresponding to  $x_1$  contains a leading 1 in the reduced row echelon form of our augmented matrix.

**Technology Note:** In the previous section, we learned how to find the reduced row echelon form of a matrix using Gaussian elimination — by hand. We need to know how to do this; understanding the process has benefits. However, actually executing the process by hand for every problem is not usually beneficial. In fact, with large systems, computing the reduced row echelon form by hand is effectively impossible. The ability to use the RREF to quickly read off the solution to a system is probably more important than being able to find the RREF in the first place. There are plenty of programs available for free online that will do the Gaussian elimination for you, but use these with caution: you probably won’t have access to a computer on the exams, so you’d better not become overly reliant on using software.

In other words, any free parameters in our system can be regarded as independent variables; the remaining variables are then seen as dependent variables, and the equations determined by the reduced row echelon form of our system tell us how the dependent variables depend on the parameters.

For example, if we set  $x_2 = 0$ , then  $x_1 = 1$ ; if we set  $x_2 = 5$ , then  $x_1 = -4$ . Common practice is to assign a new name to our free parameter;  $t$  is always a good choice. Doing so, we can write the solution to our system as follows:

$$\begin{aligned}x_1 &= 1 - t \\x_2 &= t,\end{aligned}$$

where  $t$  can be any real number. (Writing the solution in this way is meant to be reminiscent of the parametric equations for a line in Section 2.5.)

Let's try another example, one that uses more variables.

**Example 3.4.5 Solving another linear system.**

Find the solution to the linear system

$$\begin{array}{rclcrcl} & x_2 & - & x_3 & = & 3 \\ x_1 & & & + & 2x_3 & = & 2 \\ & -3x_2 & + & 3x_3 & = & -9 \end{array}$$

**Solution.** To find the solution, put the corresponding matrix into reduced row echelon form.

$$\left[ \begin{array}{ccc|c} 0 & 1 & -1 & 3 \\ 1 & 0 & 2 & 2 \\ 0 & -3 & 3 & -9 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Now convert this reduced matrix back into equations. We have

$$\begin{aligned}x_1 + 2x_3 &= 2 \\x_2 - x_3 &= 3\end{aligned}$$

or, equivalently,

$$\begin{aligned}x_1 &= 2 - 2t \\x_2 &= 3 + t \\x_3 &= t,\end{aligned}$$

where the parameter  $t$  can be any real number.

These two equations tell us that the values of  $x_1$  and  $x_2$  depend on what  $t = x_3$  is. As we saw before, there is no restriction on what  $t$  must be; it is “free” to take on the value of any real number. Once  $t$  is chosen, we obtain a solution  $(x_1, x_2, x_3)$  to our system. Since we have infinitely many choices for the value of  $t$ , we have infinitely many solutions.

As examples,  $x_1 = 2, x_2 = 3, x_3 = 0$  is one solution;  $x_1 = -2, x_2 = 5, x_3 = 2$  is another solution. Try plugging these values back into the original equations to verify that these indeed are solutions. (By the way, since infinitely many solutions exist, this system of equations is consistent.)

Examples such as these are known as **particular solutions** to our system of equations; they correspond to a **particular** choice for the parameter  $t$ . The solution written in terms of the parameter  $t$  is known as

the **general solution** to the system of equations. Note that we're not restricted to verifying particular solutions: we can go whole hog and plug the general solution into our system to make sure it works. For example, in the first equation, we have

$$x_2 - x_3 = (3 + t) - t = 3,$$

so the first equation is satisfied.

In the second equation, we have

$$x_1 + 2x_3 = (2 - 2t) + 2t = 2,$$

so the second equation is satisfied. The reader can just as easily verify that our solution works in the third equation as well.

In the two previous examples we have used the word “free” to describe certain variables. What exactly is a free variable? How do we recognize which variables are free and which are not?

Look back to the reduced matrix in Example 3.4.4. Notice that there is only one leading 1 in that matrix, and that leading 1 corresponded to the  $x_1$  variable. That told us that  $x_1$  was *not* a free variable; since  $x_2$  *did not* correspond to a leading 1, it was a free variable.

Look also at the reduced matrix in Example 3.4.5. There were two leading 1s in that matrix; one corresponded to  $x_1$  and the other to  $x_2$ . This meant that  $x_1$  and  $x_2$  were not free variables; since there was not a leading 1 that corresponded to  $x_3$ , it was a free variable.

We formally define this and a few other terms in this following definition.

#### Definition 3.4.6 Dependent and Independent Variables.

Consider the reduced row echelon form of an augmented matrix of a linear system of equations. Then:

- A variable that corresponds to a leading 1 is a **leading**, or **dependent**, variable
- A variable that does not correspond to a leading 1 is a **non-leading**, or **free**, or **independent**, variable. Free variables are also known as **parameters**.

These definitions help us understand when a consistent system of linear equations will have infinitely many solutions. If there are no free variables, then there is exactly one solution; if there are any free variables, there are infinitely many solutions.

#### Key Idea 3.4.7 Consistent Solution Types.

A consistent linear system of equations will have exactly one solution if and only if there is a leading 1 for each variable in the system.

If a consistent linear system of equations has a free variable, it has infinitely many solutions.

If a consistent linear system has more variables than leading 1s, then the system will have infinitely many solutions.

A consistent linear system with more variables than equations will always have infinitely many solutions.

**Note:** Key Idea 3.4.7 applies only to *consistent* systems. If a system is *inconsistent*, then no solution exists and talking about free and basic variables is meaningless.

When a consistent system has only one solution, each equation that comes from the reduced row echelon form of the corresponding augmented matrix will contain exactly one variable. If the consistent system has infinitely many solutions, then there will be at least one equation coming from the reduced row echelon form that contains more than one variable. The “first” variable will be the basic (or dependent) variable; all others will be free variables.

We have now seen examples of consistent systems with exactly one solution and others with infinitely many solutions. How will we recognize that a system is inconsistent? Let’s find out through an example.

**Example 3.4.8 An inconsistent system.**

Find the solution to the linear system

$$\begin{array}{rrcr} x_1 & + & x_2 & + & x_3 & = & 1 \\ x_1 & + & 2x_2 & + & x_3 & = & 2 \\ 2x_1 & + & 3x_2 & + & 2x_3 & = & 0 \end{array}$$

**Solution.** We start by putting the corresponding matrix into reduced row echelon form.

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 0 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right].$$

Now let us take the reduced matrix and write out the corresponding equations. The first two rows give us the equations

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 &= 0. \end{aligned}$$

So far, so good. However the last row gives us the equation

$$0x_1 + 0x_2 + 0x_3 = 1,$$

or, more concisely,  $0 = 1$ . Obviously, this is not true; we have reached a contradiction. Therefore, no solution exists; this system is inconsistent.

In previous sections we have only encountered linear systems with unique solutions (exactly one solution). Now we have seen three more examples with different solution types. The first two examples in this section had infinitely many solutions, and the third had no solution. How can we tell if a system is inconsistent?

A linear system will be inconsistent only when it implies that 0 equals 1. We can tell if a linear system implies this by putting its corresponding augmented matrix into reduced row echelon form. If we have any row where all entries are 0 except for the entry in the last column, then the system implies  $0 = 1$ . More succinctly, if we have a leading 1 in the last column of an augmented matrix, then the linear system has no solution.

**Key Idea 3.4.9 Inconsistent Systems of Linear Equations.**

A system of linear equations is inconsistent if the reduced row echelon form of its corresponding augmented matrix has a leading 1 in the last column.

**Example 3.4.10 Verifying that a system is inconsistent.**

Confirm that the linear system

$$\begin{array}{rcrcrcrcl} & + & y & = & 0 \\ 2x & + & 2y & = & 4 \end{array}$$

has no solution.

**Solution.** We can verify that this system has no solution in two ways. First, let's just think about it. If  $x + y = 0$ , then it stands to reason, by multiplying both sides of this equation by 2, that  $2x + 2y = 0$ . However, the second equation of our system says that  $2x + 2y = 4$ . Since  $0 \neq 4$ , we have a contradiction and hence our system has no solution. (We cannot possibly pick values for  $x$  and  $y$  so that  $2x + 2y$  equals both 0 and 4.)

Now let us confirm this using the prescribed technique from above. The reduced row echelon form of the corresponding augmented matrix is

$$\left[ \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right].$$

We have a leading 1 in the last column, so therefore the system is inconsistent.

Let's summarize what we have learned up to this point. Consider the reduced row echelon form of the augmented matrix of a system of linear equations. (That sure seems like a mouthful in and of itself. However, it boils down to "look at the reduced form of the usual matrix.") If there is a leading 1 in the last column, the system has no solution. Otherwise, if there is a leading 1 for each variable, then there is exactly one solution; otherwise there are infinitely many solutions. (i.e., there are free variables.)

Systems with exactly one solution or no solution are the easiest to deal with; systems with infinitely many solutions are a bit harder to deal with. Therefore, we'll do a little more practice. First, a definition: if there are infinitely many solutions, what do we call one of those infinitely many solutions?

**Definition 3.4.11 Particular Solution.**

Consider a linear system of equations with infinitely many solutions. A **particular solution** is one solution out of the infinitely many set of possible solutions.

The easiest way to find a particular solution is to pick values for the free variables which then determines the values of the dependent variables. Again, more practice is called for.

**Example 3.4.12 Finding general and particular solutions.**

Give the general solution to a linear system whose augmented matrix in reduced row echelon form is

$$\left[ \begin{array}{cccc|c} 1 & -1 & 0 & 2 & 4 \\ 0 & 0 & 1 & -3 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right],$$

and give two particular solutions.

**Solution.** We can essentially ignore the third row; it does not divulge any information about the solution.

The first and second rows can be rewritten as the following equations:

$$\begin{aligned} x_1 - x_2 + 2x_4 &= 4 \\ x_3 - 3x_4 &= 7. \end{aligned}$$

Notice how the variables  $x_1$  and  $x_3$  correspond to the leading 1s of the given matrix. Therefore  $x_1$  and  $x_3$  are dependent variables; all other variables (in this case,  $x_2$  and  $x_4$ ) are free variables.

We generally write our solution with the dependent variables on the left and independent variables and constants on the right. It is also a good practice to acknowledge the fact that our free variables are, in fact, free. So our final solution would look something like

$$\begin{aligned} x_1 &= 4 + s - 2t \\ x_2 &= s \text{ is free} \\ x_3 &= 7 + 3t \\ x_4 &= t \text{ is free.} \end{aligned}$$

Note that in this case we have two free variables,  $x_2$  and  $x_4$ , so we introduce two parameters,  $s$  and  $t$ , in our general solution. While we would be correct in saying that this system has infinitely many solutions, we can be more precise about the nature of those solutions. Here, we say that we have a **two-parameter family of solutions**, to indicate the fact that two different parameters are required to describe the most general solution.

To find particular solutions, choose values for our free variables. There is no “right” way of doing this; we are “free” to choose whatever we wish.

By setting  $x_2 = 0 = x_4$ , we have the solution  $x_1 = 4$ ,  $x_2 = 0$ ,  $x_3 = 7$ ,  $x_4 = 0$ . By setting  $x_2 = 1$  and  $x_4 = -5$ , we have the solution  $x_1 = 15$ ,  $x_2 = 1$ ,  $x_3 = -8$ ,  $x_4 = -5$ . It is easier to read this when the variables are listed vertically, so we repeat these solutions:

One particular solution is:

$$\begin{aligned} x_1 &= 4 \\ x_2 &= 0 \\ x_3 &= 7 \\ x_4 &= 0. \end{aligned}$$

Another particular solution is:

$$\begin{aligned} x_1 &= 15 \\ x_2 &= 1 \\ x_3 &= -8 \\ x_4 &= -5. \end{aligned}$$

Why include the row of zeros in Example 3.4.12? Rows of zeros sometimes appear “unexpectedly” in matrices after they have been put in reduced row echelon form. When this happens, we do learn *something*; it means that at least one equation was a combination of some of the others.

**Note:** It’s useful to state the number of parameters in the general solution to a system with infinitely many solutions, since the number of parameters has geometric significance. A one-parameter family of solutions defines a *line*, while a two-parameter family of solutions defines a *plane*.



**Example 3.4.13 Finding general and particular solutions.**

Find the solution to a linear system whose augmented matrix in reduced row echelon form is

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 4 & 5 \end{array} \right]$$

and give two particular solutions.

**Solution.** Converting the two rows into equations we have

$$x_1 + 2x_4 = 3$$

$$x_2 + 4x_4 = 5.$$

We see that  $x_1$  and  $x_2$  are our dependent variables, for they correspond to the leading 1s. Therefore,  $x_3$  and  $x_4$  are independent variables. This situation feels a little unusual, for  $x_3$  doesn't appear in any of the equations above, but cannot overlook it; it is still a free variable since there is not a leading 1 that corresponds to it. We write our solution as:

$$x_1 = 3 - 2t$$

$$x_2 = 5 - 4t$$

$$x_3 = s \text{ is free}$$

$$x_4 = t \text{ is free.}$$

To find two particular solutions, we pick values for our free variables. Again, there is no "right" way of doing this (in fact, there are ... infinitely many ways of doing this) so we give only an example here.

One particular solution is:

$$x_1 = 3$$

$$x_2 = 5$$

$$x_3 = 1000$$

$$x_4 = 0.$$

Another particular solution is:

$$x_1 = 3 - 2\pi$$

$$x_2 = 5 - 4\pi$$

$$x_3 = e^2$$

$$x_4 = \pi.$$

(In the second particular solution we picked "unusual" values for  $x_3$  and  $x_4$  just to highlight the fact that we can.)

What kind of situation would lead to a column of all zeros? To have such a column, the original matrix needed to have a column of all zeros, meaning that while we acknowledged the existence of a certain variable, we never actually used it in any equation. In practical terms, we could respond by removing the corresponding column from the matrix and just keep in mind that that variable is free. In very large systems, it might be hard to determine whether or not a variable is actually used and one would not worry about it.

When we learn about eigenvectors and eigenvalues, we will see that under certain circumstances this situation arises. In those cases we leave the variable in the system just to remind ourselves that it is there.

**Example 3.4.14 Finding general and particular solutions.**

Find the solution to the linear system

$$x_1 + x_2 + x_3 = 5$$

$$x_1 - x_2 + x_3 = 3$$

and give two particular solutions.

**Solution.** The corresponding augmented matrix and its reduced row echelon form are given below.

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & -1 & 1 & 3 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 1 \end{array} \right]$$

Converting these two rows into equations, we have

$$x_1 + x_3 = 4$$

$$x_2 = 1$$

giving us the solution

$$x_1 = 4 - t$$

$$x_2 = 1$$

$$x_3 = t \text{ is free.}$$

Once again, we get a bit of an “unusual” solution; while  $x_2$  is a dependent variable, it does not depend on any free variable; instead, it is always 1. (We can think of it as depending on the value of 1.) By picking two values for  $x_3$ , we get two particular solutions.

One particular solution is:

$$x_1 = 4$$

$$x_2 = 1$$

$$x_3 = 0.$$

Another particular solution is:

$$x_1 = 3$$

$$x_2 = 1$$

$$x_3 = 1.$$

The constants and coefficients of a matrix work together to determine whether a given system of linear equations has one, infinitely many, or no solution. The concept will be fleshed out more in later chapters, but in short, the coefficients determine whether a matrix will have exactly one solution or not. In the “or not” case, the constants determine whether or not infinitely many solutions or no solution exists. (So if a given linear system has exactly one solution, it will always have exactly one solution even if the constants are changed.) Let’s look at an example to get an idea of how the values of constants and coefficients work together to determine the solution type.

#### Example 3.4.15 Solving a system with a variable coefficient.

For what values of  $k$  will the given system have exactly one solution, infinitely many solutions, or no solution?

$$x_1 + 2x_2 = 3$$

$$3x_1 + kx_2 = 9$$

**Solution.** We answer this question by forming the augmented matrix and starting the process of putting it into reduced row echelon form. Below we see the augmented matrix and one elementary row operation that starts the Gaussian elimination process.

$$\left[ \begin{array}{cc|c} 1 & 2 & 3 \\ 3 & k & 9 \end{array} \right] \xrightarrow{-3R_1 + R_2 \rightarrow R_2} \left[ \begin{array}{cc|c} 1 & 2 & 3 \\ 0 & k-6 & 0 \end{array} \right]$$

This is as far as we need to go. In looking at the second row, we see that if  $k = 6$ , then that row contains only zeros and  $x_2$  is a free variable; we have infinitely many solutions. If  $k \neq 6$ , then our next step would be to make that second row, second column entry a leading one. We don’t particularly care about the solution, only that we would have

exactly one as both  $x_1$  and  $x_2$  would correspond to a leading one and hence be dependent variables.

Our final analysis is then this. If  $k \neq 6$ , there is exactly one solution; if  $k = 6$ , there are infinitely many solutions. In this example, it is not possible to have no solutions.

As an extension of the previous example, consider the similar augmented matrix where the constant 9 is replaced with a 10. Performing the same elementary row operation gives

$$\left[ \begin{array}{cc|c} 1 & 2 & 3 \\ 3 & k & 10 \end{array} \right] \xrightarrow{-3R_1 + R_2 \rightarrow R_2} \left[ \begin{array}{cc|c} 1 & 2 & 3 \\ 0 & k-6 & 1 \end{array} \right].$$

As in the previous example, if  $k \neq 6$ , we can make the second row, second column entry a leading one and hence we have one solution. However, if  $k = 6$ , then our last row is  $[0 \ 0 \ 1]$ , meaning we have no solution.

We have been studying the solutions to linear systems mostly in an “academic” setting; we have been solving systems for the sake of solving systems. In the next section, we’ll look at situations which create linear systems that need solving (i.e., “word problems”).

## 3.4.1 Exercises

**Exercise Group.** Find the solution to the given linear system. If the system has infinitely many solutions, give 2 particular solutions. Then give a geometric description of the system and its solution in terms of points, lines, and planes.

$$\begin{array}{l} \mathbf{1.} \quad 2x_1 + 4x_2 = 2 \\ \quad \quad x_1 + 2x_2 = 1 \end{array}$$

$$\begin{array}{l} \mathbf{3.} \quad 2x_1 + x_2 + 2x_3 = 0 \\ \quad \quad x_1 + x_2 + 3x_3 = 1 \\ \quad \quad 3x_1 + 2x_2 + 5x_3 = 3 \end{array}$$

$$\begin{array}{l} \mathbf{5.} \quad -x_1 - x_2 + x_3 + x_4 = 0 \\ \quad \quad -2x_1 - 2x_2 + x_3 = -1 \end{array}$$

$$\begin{array}{l} \mathbf{7.} \quad x_1 + x_2 + 6x_3 + 9x_4 = 0 \\ \quad \quad -x_1 - x_3 - 2x_4 = -3 \end{array}$$

$$\begin{array}{l} \mathbf{9.} \quad x_1 + x_2 = 3 \\ \quad \quad 2x_1 + x_2 = 4 \end{array}$$

$$\begin{array}{l} \mathbf{11.} \quad -3x_1 + 7x_2 = -7 \\ \quad \quad 2x_1 - 8x_2 = 8 \end{array}$$

$$\begin{array}{l} \mathbf{13.} \quad x_1 + 2x_2 + 2x_3 = 1 \\ \quad \quad 2x_1 + x_2 + 3x_3 = 1 \\ \quad \quad 3x_1 + 3x_2 + 5x_3 = 2 \end{array}$$

$$\begin{array}{l} \mathbf{2.} \quad -2x_1 + 4x_2 + 4x_3 = 6 \\ \quad \quad x_1 - 3x_2 + 2x_3 = 1 \end{array}$$

$$\begin{array}{l} \mathbf{4.} \quad 2x_1 + 3x_2 = 1 \\ \quad \quad -2x_1 - 3x_2 = 1 \end{array}$$

$$\begin{array}{l} \mathbf{6.} \quad -x_1 + 2x_2 + 2x_3 = 2 \\ \quad \quad 2x_1 + 5x_2 + x_3 = 2 \end{array}$$

$$\begin{array}{l} \mathbf{8.} \quad -x_1 + 5x_2 = 3 \\ \quad \quad 2x_1 - 10x_2 = -6 \end{array}$$

$$\begin{array}{l} \mathbf{10.} \quad x_1 + 3x_2 + 3x_3 = 1 \\ \quad \quad 2x_1 - x_2 + 2x_3 = -1 \\ \quad \quad 4x_1 + 5x_2 + 8x_3 = 2 \end{array}$$

$$\begin{array}{l} \mathbf{12.} \quad x_1 + 2x_2 = 1 \\ \quad \quad -x_1 - 2x_2 = 5 \end{array}$$

$$\begin{array}{l} \mathbf{14.} \quad 2x_1 + 4x_2 + 6x_3 = 2 \\ \quad \quad 1x_1 + 2x_2 + 3x_3 = 1 \\ \quad \quad -3x_1 - 6x_2 - 9x_3 = -3 \end{array}$$

**Exercise Group.** State the values of  $k$  for which the given system will have exactly one solution, infinitely many solutions, or no solution.

$$\begin{array}{l} \mathbf{15.} \quad x_1 + 2x_2 = 1 \\ \quad \quad x_1 + kx_2 = 1 \end{array}$$

$$\begin{array}{l} \mathbf{17.} \quad x_1 + 2x_2 = 1 \\ \quad \quad 2x_1 + 4x_2 = k \end{array}$$

$$\begin{array}{l} \mathbf{16.} \quad x_1 + 2x_2 = 1 \\ \quad \quad x_1 + 3x_2 = k \end{array}$$

$$\begin{array}{l} \mathbf{18.} \quad x_1 + 2x_2 = 1 \\ \quad \quad x_1 + kx_2 = 2 \end{array}$$

## 3.5 Applications of Linear Systems

We've started this chapter by addressing the issue of finding the solution to a system of linear equations. In subsequent sections, we defined matrices to store linear equation information; we described how we can manipulate matrices without changing the solutions; we described how to efficiently manipulate matrices so that a working solution can be easily found.

We shouldn't lose sight of the fact that our work in the previous sections was aimed at finding solutions to systems of linear equations. In this section, we'll learn how to apply what we've learned to actually solve some problems.

Many, problems that are addressed by engineers, businesspeople, scientists and mathematicians can be solved by properly setting up systems of linear equations. In this section we highlight only a few of the wide variety of problems that matrix algebra can help us solve.

We start with a simple example.

### Example 3.5.1 Counting marbles, again.

A jar contains 100 blue, green, red and yellow marbles. There are twice as many yellow marbles as blue; there are 10 more blue marbles than red; the sum of the red and yellow marbles is the same as the sum of the blue and green. How many marbles of each color are there?

**Solution.** Let's call the number of blue balls  $b$ , and the number of the other balls  $g$ ,  $r$  and  $y$ , each representing the obvious. Since we know that we have 100 marbles, we have the equation

$$b + g + r + y = 100.$$

The next sentence in our problem statement allows us to create three more equations.

We are told that there are twice as many yellow marbles as blue. One of the following two equations is correct, based on this statement; which one is it?

$$2y = b \quad \text{or} \quad 2b = y$$

The first equation says that if we take the number of yellow marbles, then double it, we'll have the number of blue marbles. That is not what we were told. The second equation states that if we take the number of blue marbles, then double it, we'll have the number of yellow marbles. This is what we were told.

The next statement of "there are 10 more blue marbles as red" can be written as either

$$b = r + 10 \quad \text{or} \quad r = b + 10.$$

Which is it?

The first equation says that if we take the number of red marbles, then add 10, we'll have the number of blue marbles. This is what we were told. The next equation is wrong; it implies there are more red marbles than blue.

The final statement tells us that the sum of the red and yellow marbles is the same as the sum of the blue and green marbles, giving us the equation

$$r + y = b + g.$$

We have four equations; altogether, they are

$$b + g + r + y = 100$$

$$2b = y$$

$$b = r + 10$$

$$r + y = b + g.$$

We want to write these equations in a standard way, with all the unknowns on the left and the constants on the right. Let us also write them so that the variables appear in the same order in each equation (we'll use alphabetical order to make it simple). We now have

$$b + g + r + y = 100$$

$$2b - y = 0$$

$$b - r = 10$$

$$-b - g + r + y = 0.$$

To find the solution, let's form the appropriate augmented matrix and put it into reduced row echelon form. We do so here, without showing the steps.

$$\left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 100 \\ 2 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 10 \\ -1 & -1 & 1 & 1 & 0 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & 30 \\ 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & 40 \end{array} \right].$$

We interpret from the reduced row echelon form of the matrix that we have 20 blue, 30 green, 10 red and 40 yellow marbles.

Even if you had a bit of difficulty with the previous example, in reality, this type of problem is pretty simple. The unknowns were easy to identify, the equations were pretty straightforward to write (maybe a bit tricky for some), and only the necessary information was given.

Most problems that we face in the world do not approach us in this way; most problems do not approach us in the form of "Here is an equation. Solve it." Rather, most problems come in the form of:

Here is a problem. I want the solution. To help, here is lots of information. It may be just enough; it may be too much; it may not be enough. You figure out what you need; just give me the solution.

Faced with this type of problem, how do we proceed? Like much of what we've done in the past, there isn't just one "right" way. However, there are a few steps that can guide us. You don't have to follow these steps, "step by step," but if you find that you are having difficulty solving a problem, working through these steps may help. (Note: while the principles outlined here will help one solve any type of problem, these steps are written specifically for solving problems that involve only linear equations.)

#### Key Idea 3.5.2 Mathematical Problem Solving.

1. Understand the problem. What exactly is being asked?
2. Identify the unknowns. What are you trying to find? What units are involved?
3. Give names to your unknowns (these are your *variables*).

4. Use the information given to write as many equations as you can that involve these variables.
5. Use the equations to form an augmented matrix; use Gaussian elimination to put the matrix into reduced row echelon form.
6. Interpret the reduced row echelon form of the matrix to identify the solution.
7. Ensure the solution makes sense in the context of the problem.

Having identified some steps, let us put them into practice with some examples.

### Example 3.5.3 Arranging seating.

A concert hall has seating arranged in three sections. As part of a special promotion, guests will receive two of three prizes. Guests seated in the first and second sections will receive Prize A, guests seated in the second and third sections will receive Prize B, and guests seated in the first and third sections will receive Prize C. Concert promoters told the concert hall managers of their plans, and asked how many seats were in each section. (The promoters want to store prizes for each section separately for easier distribution.) The managers, thinking they were being helpful, told the promoters they would need 105 A prizes, 103 B prizes, and 88 C prizes, and have since been unavailable for further help. How many seats are in each section?

**Solution.** Before we rush in and start making equations, we should be clear about what is being asked. The final sentence asks: “How many seats are in each section?” This tells us what our unknowns should be: we should name our unknowns for the number of seats in each section. Let  $x_1$ ,  $x_2$  and  $x_3$  denote the number of seats in the first, second and third sections, respectively. This covers the first two steps of our general problem solving technique.

(It is tempting, perhaps, to name our variables for the number of prizes given away. However, when we think more about this, we realize that we already know this — that information is given to us. Rather, we should name our variables for the things we don’t know.)

Having our unknowns identified and variables named, we now proceed to forming equations from the information given. Knowing that Prize A goes to guests in the first and second sections and that we’ll need 105 of these prizes tells us

$$x_1 + x_2 = 105.$$

Proceeding in a similar fashion, we get two more equations,

$$x_2 + x_3 = 103 \quad \text{and} \quad x_1 + x_3 = 88.$$

Thus our linear system is

$$\begin{aligned} x_1 + x_2 &= 105 \\ x_2 + x_3 &= 103, \\ x_1 + x_3 &= 88 \end{aligned}$$

and the corresponding augmented matrix is

$$\left[ \begin{array}{ccc|c} 1 & 1 & 0 & 105 \\ 0 & 1 & 1 & 103 \\ 1 & 0 & 1 & 88 \end{array} \right].$$

To solve our system, let's put this matrix into reduced row echelon form.

$$\left[ \begin{array}{ccc|c} 1 & 1 & 0 & 105 \\ 0 & 1 & 1 & 103 \\ 1 & 0 & 1 & 88 \end{array} \right] \xrightarrow{\text{ref}} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 45 \\ 0 & 1 & 0 & 60 \\ 0 & 0 & 1 & 43 \end{array} \right].$$

We can now read off our solution. The first section has 45 seats, the second has 60 seats, and the third has 43 seats.

#### Example 3.5.4 Determining river speed.

A lady takes a 2-mile motorized boat trip down the Highwater River, knowing the trip will take 30 minutes. She asks the boat pilot "How fast does this river flow?" He replies "I have no idea, lady. I just drive the boat."

She thinks for a moment, then asks "How long does the return trip take?" He replies "The same; half an hour." She follows up with the statement, "Since both legs take the same time, you must not drive the boat at the same speed."

"Naw," the pilot said. "While I really don't know exactly how fast I go, I do know that since we don't carry any tourists, I drive the boat twice as fast."

The lady walks away satisfied; she knows how fast the river flows. (How fast *does* it flow?)

**Solution.** This problem forces us to think about what information is given and how to use it to find what we want to know. In fact, to find the solution, we'll find out extra information that we weren't asked for!

We are asked to find how fast the river is moving (step 1). To find this, we should recognize that, in some sense, there are three speeds at work in the boat trips: the speed of the river (which we want to find), the speed of the boat, and the speed that they actually travel at.

We know that each leg of the trip takes half an hour; if it takes half an hour to cover 2 miles, then they must be travelling at 4 mph, each way.

The other two speeds are unknowns, but they are related to the overall speeds. Let's call the speed of the river  $r$  and the speed of the boat  $b$ . (And we should be careful. From the conversation, we know that the boat travels at two different speeds. So we'll say that  $b$  represents the speed of the boat when it travels downstream, so  $2b$  represents the speed of the boat when it travels upstream.) Let's let our speed be measured in the units of miles/hour (mph) as we used above (steps 2 and 3).

What is the rate of the people on the boat? When they are travelling downstream, their rate is the sum of the water speed and the boat speed. Since their overall speed is 4 mph, we have the equation  $r + b = 4$ .



When the boat returns going against the current, its overall speed is the rate of the boat minus the rate of the river (since the river is working against the boat). The overall trip is still taken at 4 mph, so we have the equation  $2b - r = 4$ . (Recall: the boat is travelling twice as fast as before.)

The corresponding augmented matrix is

$$\left[ \begin{array}{cc|c} 1 & 1 & 4 \\ 2 & -1 & 4 \end{array} \right].$$

Note that we decided to let the first column hold the coefficients of  $b$ .

Putting this matrix in reduced row echelon form gives us:

$$\left[ \begin{array}{cc|c} 1 & 1 & 4 \\ 2 & -1 & 4 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{cc|c} 1 & 0 & 8/3 \\ 0 & 1 & 4/3 \end{array} \right].$$

We finish by interpreting this solution: the speed of the boat (going downstream) is  $8/3$  mph, or  $2.\bar{6}$  mph, and the speed of the river is  $4/3$  mph, or  $1.\bar{3}$  mph. All we really wanted to know was the speed of the river, at about 1.3 mph.

### Example 3.5.5 Fitting a quadratic curve.

Find the equation of the quadratic function that goes through the points  $(-1, 6)$ ,  $(1, 2)$  and  $(2, 3)$ .

**Solution.** This may not seem like a “linear” problem since we are talking about a quadratic function, but closer examination will show that it really is.

We normally write quadratic functions as  $y = ax^2 + bx + c$  where  $a$ ,  $b$  and  $c$  are the coefficients; in this case, they are our unknowns. We have three points; consider the point  $(-1, 6)$ . This tells us directly that if  $x = -1$ , then  $y = 6$ . Therefore we know that  $6 = a(-1)^2 + b(-1) + c$ . Writing this in a more standard form, we have the linear equation

$$a - b + c = 6.$$

The second point tells us that  $a(1)^2 + b(1) + c = 2$ , which we can simplify as  $a + b + c = 2$ , and the last point tells us  $a(2)^2 + b(2) + c = 3$ , or  $4a + 2b + c = 3$ . Thus our linear system is

$$\begin{aligned} a - b + c &= 6 \\ a + b + c &= 2 \\ 4a + 2b + c &= 3. \end{aligned}$$

Again, to solve our system, we find the reduced row echelon form of the corresponding augmented matrix. We don't show the steps here, just the final result.

$$\left[ \begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & 3 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

This tells us that  $a = 1$ ,  $b = -2$  and  $c = 3$ , giving us the quadratic function  $y = x^2 - 2x + 3$ .

One thing interesting about the previous example is that it confirms for us something that we may have known for a while (but didn't know *why* it was true). Why do we need two points to find the equation of the line? Because in the equation of the a line, we have two unknowns, and hence we'll need two equations to find values for these unknowns.

A quadratic has three unknowns (the coefficients of the  $x^2$  term and the  $x$  term, and the constant). Therefore we'll need three equations, and therefore we'll need three points.

What happens if we try to find the quadratic function that goes through 3 points that are all on the same line? The fast answer is that you'll get the equation of a line; there isn't a quadratic function that goes through 3 colinear points. Try it and see! (Pick easy points, like  $(0, 0)$ ,  $(1, 1)$  and  $(2, 2)$ . You'll find that the coefficient of the  $x^2$  term is 0.)

Of course, we can do the same type of thing to find polynomials that go through 4, 5, etc., points. In general, if you are given  $n + 1$  points, a polynomial that goes through all  $n + 1$  points will have degree at most  $n$ .

### Example 3.5.6 A money counting problem.

A woman has 32 \$1, \$5, and \$10 bills in her purse, giving her a total of \$100. How many bills of each denomination does she have?

**Solution.** Let's name our unknowns  $x$ ,  $y$  and  $z$  for our ones, fives and tens, respectively (it is tempting to call them  $o$ ,  $f$  and  $t$ , but  $o$  looks too much like 0). We know that there are a total of 32 bills, so we have the equation

$$x + y + z = 32.$$

We also know that we have \$100, so we have the equation

$$x + 5y + 10z = 100.$$

We have three unknowns but only two equations, so we know that we cannot expect a unique solution. Let's try to solve this system anyway and see what we get.

Putting the system into a matrix and then finding the reduced row echelon form, we have

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 32 \\ 1 & 5 & 10 & 100 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{ccc|c} 1 & 0 & -\frac{5}{4} & 15 \\ 0 & 1 & \frac{9}{4} & 17 \end{array} \right].$$

Reading from our reduced matrix, we have the infinite solution set

$$\begin{aligned} x &= 15 + \frac{5}{4}z \\ y &= 17 - \frac{9}{4}z \\ z &\text{ is free.} \end{aligned}$$

While we do have infinitely many solutions, most of these solutions really don't make sense in the context of this problem. (Setting  $z = \frac{1}{2}$  doesn't make sense, for having half a ten dollar bill doesn't give us \$5. Likewise, having  $z = 8$  doesn't make sense, for then we'd have " $-1$ " \$5 bills.) So we must make sure that our choice of  $z$  doesn't give us fractions of bills or negative amounts of bills.

To avoid fractions,  $z$  must be a multiple of 4 ( $-4, 0, 4, 8, \dots$ ). Of course,  $z \geq 0$ , for a negative number wouldn't make sense. If  $z = 0$ ,

then we have 15 one dollar bills and 17 five dollar bills, giving us \$100. If  $z = 4$ , then we have  $x = 20$  and  $y = 8$ . We already mentioned that  $z = 8$  doesn't make sense, nor does any value of  $z$  where  $z \geq 8$ .

So it seems that we have two answers; one with  $z = 0$  and one with  $z = 4$ . Of course, by the statement of the problem, we are led to believe that the lady has at least one \$10 bill, so probably the "best" answer is that we have 20 \$1 bills, 8 \$5 bills and 4 \$10 bills. The real point of this example, though, is to address how infinitely many solutions may appear in a real world situation, and how surprising things may result.

### Example 3.5.7 Recreating a football score.

In a football game, teams can score points through touchdowns worth 6 points, extra points (that follow touchdowns) worth 1 point, two point conversions (that also follow touchdowns) worth 2 points and field goals, worth 3 points. You are told that in a football game, the two competing teams scored on 7 occasions, giving a total score of 24 points. Each touchdown was followed by either a successful extra point or two point conversion. In what ways were these points scored?

**Solution.** The question asks how the points were scored; we can interpret this as asking how many touchdowns, extra points, two point conversions and field goals were scored. We'll need to assign variable names to our unknowns; let  $t$  represent the number of touchdowns scored; let  $x$  represent the number of extra points scored, let  $w$  represent the number of two point conversions, and let  $f$  represent the number of field goals scored.

Now we address the issue of writing equations with these variables using the given information. Since we have a total of 7 scoring occasions, we know that

$$t + x + w + f = 7.$$

The total points scored is 24; considering the value of each type of scoring opportunity, we can write the equation

$$6t + x + 2w + 3f = 24.$$

Finally, we know that each touchdown was followed by a successful extra point or two point conversion. This is subtle, but it tells us that the number of touchdowns is equal to the sum of extra points and two point conversions. In other words,

$$t = x + w.$$

To solve our problem, we put these equations into a matrix and put the matrix into reduced row echelon form. Doing so, we find

$$\left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 6 & 1 & 2 & 3 & 24 \\ 1 & -1 & -1 & 0 & 0 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0.5 & 3.5 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & -0.5 & -0.5 \end{array} \right].$$

Therefore, we know that

$$\begin{aligned} t &= 3.5 - 0.5f \\ x &= 4 - f \end{aligned}$$

$$w = -0.5 + 0.5f.$$

We recognize that this means there are “infinitely many solutions,” but of course most of these will not make sense in the context of a real football game. We must apply some logic to make sense of the situation.

Progressing in no particular order, consider the second equation,  $x = 4 - f$ . In order for us to have a positive number of extra points, we must have  $f \leq 4$ . (And of course, we need  $f \geq 0$ , too.) Therefore, right away we know we have a total of only 5 possibilities, where  $f = 0, 1, 2, 3$  or  $4$ .

From the first and third equations, we see that if  $f$  is an even number, then  $t$  and  $w$  will both be fractions (for instance, if  $f = 0$ , then  $t = 3.5$ ) which does not make sense. Therefore, we are down to two possible solutions,  $f = 1$  and  $f = 3$ .

If  $f = 1$ , we have 3 touchdowns, 3 extra points, no two point conversions, and (of course), 1 field goal. (Check to make sure that gives 24 points!) If  $f = 3$ , then we 2 touchdowns, 1 extra point, 1 two point conversion, and (of course) 3 field goals. Again, check to make sure this gives us 24 points. Also, we should check each solution to make sure that we have a total of 7 scoring occasions and that each touchdown could be followed by an extra point or a two point conversion.

We have seen a variety of applications of systems of linear equations. We would do well to remind ourselves of the ways in which solutions to linear systems come: there can be exactly one solution, infinitely many solutions, or no solutions. While we did see a few examples where it seemed like we had only 2 solutions, this was because we were restricting our solutions to “make sense” within a certain context.

We should also remind ourselves that linear equations are immensely important. The examples we considered here ask fundamentally simple questions like “How fast is the water moving?” or “What is the quadratic function that goes through these three points?” or “How were points in a football game scored?” The real “important” situations ask much more difficult questions that often require *thousands* of equations! (Gauss began the systematic study of solving systems of linear equations while trying to predict the next sighting of a comet; he needed to solve a system of linear equations that had 17 unknowns. Today, this a relatively easy situation to handle with the help of computers, but to do it by hand is a real pain.) Once we understand the fundamentals of solving systems of equations, we can move on to looking at solving bigger systems of equations; this text focuses on getting us to understand the fundamentals.

**Span and linear independence.** We mentioned in Section 2.7 that the techniques of this chapter were necessary to answer questions about vectors, including those involving span and linear independence. We complete this section by including a few examples involving these concepts.

**Example 3.5.8 Determining which vectors belong to a span.**

Let  $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}$ , and  $\vec{v}_3 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ . Determine which of the following vectors can be written as a linear combination of the

vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ .

$$1. \vec{a} = \begin{bmatrix} 2 \\ -14 \\ -8 \end{bmatrix} \quad 2. \vec{b} = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} \quad 3. \vec{c} = \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

**Solution.** Put another way, this problem is asking us which, if any, of the vectors  $\vec{a}, \vec{b}, \vec{c}$  belong to the span of the vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ . Recall that

$$\vec{a} \in \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$$

if and only if there exist scalars  $x_1, x_2, x_3$  such that

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{a}.$$

Plugging in values for our vectors, we are trying to solve

$$x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -14 \\ -8 \end{bmatrix}.$$

If we combine the vectors on the left-hand side above into a single vector and equate coefficients, we obtain the system

$$\begin{array}{rrcrcl} x_1 & + & 3x_2 & + & 3x_3 & = & 2 \\ -2x_1 & + & 4x_2 & - & x_3 & = & -14 \\ & & 8x_2 & + & 4x_3 & = & -8 \end{array}$$

We now what to do from here: we set up our augmented matrix and reduce:

$$\left[ \begin{array}{ccc|c} 1 & 3 & 3 & 2 \\ -2 & 4 & -1 & -14 \\ 0 & 8 & 4 & -8 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & 5 \\ 0 & 1 & \frac{1}{2} & -1 \\ 0 & 0 & 0 & 0 \end{array} \right] ..$$

The reduced row echelon form of our matrix tells us that there are infinitely many solutions; in particular, there *is* a solution. Our general solution is

$$\begin{aligned} x_1 &= 5 - \frac{3}{2}t \\ x_2 &= -1 - \frac{1}{2}t \\ x_3 &= t, \end{aligned}$$

but any particular solution will do. Taking  $t = 0$ , we get  $x_1 = 5, x_2 = -1$ , and  $x_3 = 0$ , telling us that

$$\vec{a} = 5\vec{v}_1 - \vec{v}_2 + 0\vec{v}_3.$$

The reader is encouraged to verify that this is indeed the case.

For the vector  $\vec{b}$ , we proceed as above. It should be clear that the system of equations we obtain differs from the one above only in the constants on the right-hand side (since these are defined by the vector  $\vec{b}$ ),

so we can proceed by replacing the right-hand column in our augmented matrix above by the vector  $\vec{b}$ , and reducing:

$$\left[ \begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ -2 & 4 & -1 & 0 \\ 0 & 8 & 4 & -4 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & \frac{2}{5} \\ 0 & 1 & \frac{1}{2} & \frac{1}{5} \\ 0 & 0 & 0 & -\frac{7}{10} \end{array} \right].$$

This time, we see that the third row in the reduced row echelon form of our augmented matrix corresponds to the impossible equation  $0 = -\frac{7}{10}$ , so no solution exists. This means that it is impossible to find scalars  $x_1, x_2, x_3$  such that

$$\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3,$$

and therefore the vector  $\vec{b}$  *cannot* be written as a linear combination of the vectors  $\vec{v}_1, \vec{v}_2$ , and  $\vec{v}_3$ .

The solution for the vector  $\vec{c}$  is similar. We replace the right-hand column in our augmented matrix by  $\vec{c}$  and reduce:

$$\left[ \begin{array}{ccc|c} 1 & 3 & 3 & -4 \\ -2 & 4 & -1 & 3 \\ 0 & 8 & 4 & 1 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & -\frac{5}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{5}{8} \end{array} \right].$$

Again, the third row in the reduced row echelon form of our augmented matrix tells us that our system of equations is inconsistent, and therefore, the vector  $\vec{c}$  does *not* belong to the span of the vectors  $\vec{v}_1, \vec{v}_2$ , and  $\vec{v}_3$ .

Let us make a couple of observations before proceeding to the next example. First, let us write out the result for the vector  $\vec{a}$  using the general solution, written in terms of the parameter  $t$ , and manipulate things a bit:

$$\begin{aligned} \vec{a} &= x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 \\ &= \left(5 - \frac{3}{2}t\right)\vec{v}_1 + \left(-1 - \frac{1}{2}t\right)\vec{v}_3 + t\vec{v}_3 \\ &= 5\vec{v}_1 - \vec{v}_2 + t\left(\vec{v}_3 - \frac{3}{2}\vec{v}_1 - \frac{1}{2}\vec{v}_2\right). \end{aligned}$$

**Note:** In Section 3.6, we'll start writing our solutions to linear systems in vector form, and doing so gives us additional insight into what's going on when there are infinitely many solutions. We see something of a preview in Example 3.5.8. The parameter  $t$  resulted because the vectors  $\vec{v}_1, \vec{v}_2$ , and  $\vec{v}_3$  were not linearly independent: we don't really need the variable  $x_3$  since the vector  $\vec{v}_3$  is redundant. In general, we will end up with one or more parameters in a consistent system whenever the columns of the coefficient matrix are not linearly independent.

Setting  $t = 0$  gives us the particular solution we chose. What happens if we choose other values of  $t$ ? Well, nothing happens to the left-hand side of the equation above, so it must be that nothing happens on the right either. How can this be? Look at the vector in parentheses: we have

$$\vec{v}_3 - \frac{3}{2}\vec{v}_1 - \frac{1}{2}\vec{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}!$$

Of course nothing happens: choosing different values for  $t$  simply means we're adding different multiples of the zero vector to our particular solution!

The other thing to note, now that we've done a few examples, is that once we have our vectors, we can jump straight to our augmented matrix, since the columns of this matrix are simply the vectors we started with. It's important to do it once the long way, however, so that we understand where everything is coming from. (Otherwise, you'll find yourself staring at the solution to a system, and having no idea what that solution is telling you!)

On the subject of saving time, if you went through the row operations for each of the vectors above (if you didn't, maybe you should: working through the examples is good practice if you haven't yet got the hang of row operations), you probably noticed that you used the exact same row operations for each of the three vectors. This is because the row operations in an augmented matrix  $[A|\vec{b}]$  are dictated by the matrix  $A$ , and not by the vector  $\vec{b}$ . With this observation in place, we can speed things up considerably by solving the problem for all three vectors at once. We simply set up and reduce an augmented matrix with *three* columns on the right, like so:

$$\left[ \begin{array}{ccc|ccc} 1 & 3 & 3 & 2 & 1 & -4 \\ -2 & 4 & -1 & -14 & 0 & 3 \\ 0 & 8 & 4 & -8 & -4 & 1 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & 5 & \frac{2}{5} & -\frac{5}{2} \\ 0 & 1 & \frac{1}{2} & -1 & \frac{1}{5} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{7}{10} & \frac{5}{8} \end{array} \right].$$

If we then consider each column on the right of the reduced row echelon form separately, we obtain the same results as before.

We conclude this section with a pair of examples involving linear independence.

#### Example 3.5.9 Determining linear independence.

Determine whether or not the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

are linearly independent.

**Solution.** We test for linear independence by setting up and solving Equation (2.7.1): we suppose that there exist scalars  $x_1, x_2, x_3$  such that

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0},$$

and determine what values of  $x_1, x_2, x_3$  are possible. Putting in our vectors we have

$$x_1 \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + 2x_3 \\ -3x_1 + 2x_2 \\ 4x_1 - 3x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

leading to the system of linear equations

$$\begin{array}{rrcrcl} x_1 & - & 2x_2 & + & 2x_3 & = & 0 \\ -3x_1 & + & 2x_2 & & & = & 0 \\ 4x_1 & - & 3x_2 & - & x_3 & = & 0 \end{array}$$

As with the examples involving span above, we end up with a system of linear equations. As we'll see in Section 3.6, this is an example of a *homogeneous* system of equations. Any problem involving linear independence results in such a system, and as you've probably already noticed, we don't need to worry about whether or not the system is consistent. The so-called *trivial solution*  $x_1 = x_2 = x_3 = 0$  is always a possibility. With linear independence, we don't care about whether a solution exists; what we want to know is whether or not that solution is *unique*.

Since a homogeneous system is, after all, a linear system, we know how to proceed: set up the corresponding augmented matrix and reduce.

$$\left[ \begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ -3 & 1 & 0 & 0 \\ 4 & -3 & -1 & 0 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right].$$

From the reduced row echelon form of the augmented matrix, we can see that our system has the unique solution  $x_1 = x_2 = x_3 = 0$ . Since this is the *only* possible solution, we can conclude that our vectors are linearly independent.

### Example 3.5.10 Determining linear independence.

Determine whether the vectors

$$\vec{w}_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \vec{w}_3 = \begin{bmatrix} 1 \\ 12 \\ 8 \end{bmatrix}$$

in  $\mathbb{R}^3$  are linearly independent.

**Solution.** As with Example 3.5.9, we know that Equation (2.7.1) leads to a system of linear equations with zeros on the right-hand sides, and the vectors  $\vec{w}_1, \vec{w}_2, \vec{w}_3$  make up the columns on the left of the corresponding augmented matrix. We find

$$\left[ \begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ -3 & 0 & 12 & 0 \\ 1 & 4 & 8 & 0 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

In this case, lack of leading 1 in the  $x_3$  column of the reduced row echelon form of the augmented matrix tells us that we should expect infinitely many solutions. Setting  $x_3 = t$  as a free parameter, we have

$$\begin{aligned} x_1 &= 4t \\ x_2 &= -3t \\ x_3 &= t. \end{aligned}$$

Choosing any value of  $t \neq 0$  gives us a non-trivial solution. For example, setting  $t = 1$  gives us  $4\vec{w}_1 - 3\vec{w}_2 + \vec{w}_3 = \vec{0}$ , so  $\vec{w}_3$  can be written as the linear combination

$$\vec{w}_3 = -4\vec{w}_1 + 3\vec{w}_2,$$

which shows that the given vectors are *not* linearly independent.

In these last examples we've started to explore the connection between vectors and systems of linear equations. We'll continue to do so in the next section, where we begin to write our solutions in vector form.



## 3.5.1 Exercises

**Exercise Group.** Find the polynomial with the smallest degree that goes through the given points.

1.  $(1, 3)$  and  $(3, 15)$
2.  $(-2, 14)$  and  $(3, 4)$
3.  $(1, 5)$ ,  $(-1, 3)$  and  $(3, -1)$
4.  $(-4, -3)$ ,  $(0, 1)$  and  $(1, 4.5)$
5.  $(-2, 15)$ ,  $(-1, 4)$ ,  $(1, 0)$  and  $(2, -5)$
6.  $(-2, -7)$ ,  $(1, 2)$ ,  $(2, 9)$  and  $(3, 28)$
7.  $(-1, -8)$ ,  $(1, -2)$  and  $(3, 4)$
8.  $(-3, 3)$ ,  $(1, 3)$  and  $(2, 3)$
9.  $(-3, 10)$ ,  $(-1, 2)$ ,  $(1, 2)$  and  $(2, 5)$
10.  $(0, 1)$ ,  $(-3, -3.5)$ ,  $(-2, -2)$  and  $(4, 7)$
11. The general exponential function has the form  $f(x) = ae^{bx}$ , where  $a$  and  $b$  are constants and  $e$  is Euler's constant ( $\approx 2.718$ ). We want to find the equation of the exponential function that goes through the points  $(1, 2)$  and  $(2, 4)$ .
  - (a) Show why we cannot simply substitute in values for  $x$  and  $y$  in  $y = ae^{bx}$  and solve using the techniques we used for polynomials.
  - (b) Show how the equality  $y = ae^{bx}$  leads us to the linear equation  $\ln y = \ln a + bx$ .
  - (c) Use the techniques we developed to solve for the unknowns  $\ln a$  and  $b$ .
  - (d) Knowing  $\ln a$ , find  $a$ ; find the exponential function  $f(x) = ae^{bx}$  that goes through the points  $(1, 2)$  and  $(2, 4)$ .
12. In a football game, 24 points are scored from 8 scoring occasions. The number of successful extra point kicks is equal to the number of successful two point conversions. Find all ways in which the points may have been scored in this game.
13. In a football game, 29 points are scored from 8 scoring occasions. There are 2 more successful extra point kicks than successful two point conversions. Find all ways in which the points may have been scored in this game.
14. In a basketball game, where points are scored either by a 3 point shot, a 2 point shot or a 1 point free throw, 80 points were scored from 30 successful shots. Find all ways in which the points may have been scored in this game.
15. In a basketball game, where points are scored either by a 3 point shot, a 2 point shot or a 1 point free throw, 110 points were scored from 70 successful shots. Find all ways in which the points may have been scored in this game.
16. Describe the equations of the linear functions that go through the point  $(1, 3)$ . Give 2 examples.
17. Describe the equations of the linear functions that go through the point  $(2, 5)$ . Give 2 examples.
18. Describe the equations of the quadratic functions that go through the points  $(2, -1)$  and  $(1, 0)$ . Give 2 examples.
19. Describe the equations of the quadratic functions that go through the points  $(-1, 3)$  and  $(2, 6)$ . Give 2 examples.

**Exercise Group.** Determine whether or not the vector  $\vec{x}$  belongs to  $\text{span}\{\vec{v}_1, \vec{v}_2\}$ .

$$20. \quad \vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}, \vec{x} = \begin{bmatrix} -5 \\ 6 \\ 10 \end{bmatrix}$$

$$21. \quad \vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}, \vec{x} = \begin{bmatrix} 8 \\ -4 \\ -2 \end{bmatrix}$$

$$22. \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 5 \\ -3 \end{bmatrix}, \vec{x} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$$

$$23. \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 5 \\ -3 \end{bmatrix}, \vec{x} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$$

**Exercise Group.** Determine whether or not the given vectors are linearly independent.

24.  $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 5 \\ -3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$

25.  $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

26.  $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

27.  $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -3 \\ 2 \\ 7 \end{bmatrix}$

## 3.6 Vector Solutions to Linear Systems

So far in this chapter, we've learned a systematic method for solving systems of linear equations. Some of the applied examples we considered in the previous section led naturally to systems of equations, and had solutions that were best interpreted in that context. Other examples, such as those involving span and linear independence, were stated in terms of vectors. In this section, we discuss how to write a system of linear equations in terms of vectors and matrices, and express solutions as vectors.

Expressing the solutions of linear systems in terms of vectors will give us additional insight into the behaviour of those systems, and provides a stepping-off point for the study of the algebra of matrices.

We have often relied on previous algebra experience to help us understand linear algebra concepts. We do that again here. Consider the equation  $ax = b$ , where  $a = 3$  and  $b = 6$ . If we asked one to "solve for  $x$ ," what exactly would we be asking? We would want to find a number, which we call  $x$ , where  $a$  times  $x$  gives  $b$ ; in this case, it is a number, when multiplied by 3, returns 6. As long as  $a \neq 0$  (what if  $a$  does equal zero?), we know that we can multiply both sides of the equation by  $\frac{1}{a}$  to get  $x = \frac{1}{a}(b) = \frac{b}{a}$ .

Consider a general system of linear equations, of the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \quad (3.6.1)$$

Notice that the information we place into our augmented matrix can be divided into two pieces: the coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and the column vector

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Moreover, instead of writing our solution as a list ( $x_1 =, x_2 = \dots$ ), we can arrange our variables into a vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

and express our solution as a single vector rather than a list of numbers.

To create an analogy with the single variable equation  $ax = b$ , we ask: Is there a way to define the product  $A\vec{x}$  of a matrix and a column vector in such a way that the system (3.6.1) can be written in the form

$$A\vec{x} = \vec{b}?$$

Fortunately for us, the answer is yes! Even better, we'll see that the definition we give here turns out to be a special case (and motivating example) for the general definition of matrix multiplication given in Section 4.2.

The definition of the product  $A\vec{x}$  is straightforward. We want the result to be a column vector of size  $m$  (that is, with  $m$  entries), so that we can set it equal to the column vector  $\vec{b}$ . Furthermore, each entry in  $\vec{b}$  is the right-hand side of an equation in (3.6.1), so we want the corresponding entry in  $A\vec{x}$  to be the left-hand side. We are immediately forced to adopt the following rule.

**Definition 3.6.1** The product  $A\vec{x}$ .

The product of the matrix  $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$  and the vector  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$  is given by

$$A\vec{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}.$$

A few remarks about Definition 3.6.1 are needed here. First, note that the number of columns in the matrix  $A$  matches the number of entries in the vector  $\vec{x}$ , and the number of rows in  $A$  matches the number of entries in the vector  $\vec{b}$ . Moreover,

The  $i$ th entry in the vector  $A\vec{x}$  is obtained by forming the **dot product** of row  $i$  in the matrix  $A$  (viewed as a vector in  $\mathbb{R}^n$ ) with the vector  $\vec{x}$ .

That is, since each row of  $A$  has  $n$  entries, as does the vector  $\vec{x}$ , we can form the dot product of  $\vec{x}$  with any of the rows of  $A$ . Each such dot product forms the corresponding entry in the vector  $A\vec{x}$ .

Second, the right-hand side of the equation defining  $A\vec{x}$  in Definition 3.6.1 can be rewritten as follows:

$$A\vec{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

That is, the product  $A\vec{x}$  is a linear combination of the columns of  $A$ , with the entries of  $\vec{x}$  as coefficients.

Or equivalently, using the language of Section 2.7, the vector  $A\vec{x}$  belongs to the **span** of the columns of  $A$ . This will be a useful observation when solving questions about span.

To help understand what this is asking, we'll consider an example. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

(We don't know what  $\vec{x}$  is, so we have to represent its entries with the variables  $x_1$ ,  $x_2$  and  $x_3$ .) Let's "solve for  $\vec{x}$ ," given the equation  $A\vec{x} = \vec{b}$ .

We multiply out the left hand side of this equation according to Definition 3.6.1. The first row of  $A$  is the row vector  $\vec{a}_1 = [1 \ 1 \ 1]$ . Treating this as a column vector, we form the dot product with  $\vec{x}$ , giving us

$$\vec{a}_1 \cdot \vec{x} = x_1 + x_2 + x_3.$$

Similarly, letting  $\vec{a}_2$  and  $\vec{a}_3$  denote the second and third rows of  $A$ , respectively, we find

$$\begin{aligned} \vec{a}_2 \cdot \vec{x} &= x_1 - x_2 + 2x_3 \\ \vec{a}_3 \cdot \vec{x} &= 2x_1 + x_3. \end{aligned}$$

Putting things together, we find that

$$A\vec{x} = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 - x_2 + 2x_3 \\ 2x_1 + x_3 \end{bmatrix}.$$

Be sure to note that the product is just a vector; it has just one column.

If you prefer to think in terms of the columns of  $A$ , we can use the observation above, that  $A\vec{x}$  is a linear combination of the columns of  $A$ . This gives us

$$A\vec{x} = x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ 2x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ -x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} x_3 \\ 2x_3 \\ x_3 \end{bmatrix}.$$

When we add these three columns together, we get the same result as above.

Setting  $A\vec{x}$  equal to  $\vec{b}$ , we have

$$\begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 - x_2 + 2x_3 \\ 2x_1 + x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}.$$

Since two vectors are equal only when their corresponding entries are equal, we know

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_1 - x_2 + 2x_3 &= -3 \\ 2x_1 + x_3 &= 1. \end{aligned}$$

This should look familiar; it is a system of linear equations! Given the matrix-vector equation  $A\vec{x} = \vec{b}$ , we can recognize  $A$  as the coefficient matrix from a linear system and  $\vec{b}$  as the vector of the constants from the linear system. Given a system of equations, rewriting it in matrix form is equally straightforward.

To solve a matrix-vector equation (and the corresponding linear system), we simply augment the matrix  $A$  with the vector  $\vec{b}$ , put this matrix into reduced row echelon form, and interpret the results.

We convert the above linear system into an augmented matrix and find the reduced row echelon form:

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & -3 \\ 2 & 0 & 1 & 1 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array} \right].$$

This tells us that  $x_1 = 1$ ,  $x_2 = 2$  and  $x_3 = -1$ , so

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

We should check our work; multiply out  $A\vec{x}$  and verify that we indeed get  $\vec{b}$ :

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \text{ does equal } \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}.$$

### Example 3.6.2 Solving a matrix equation.

Solve the equation  $A\vec{x} = \vec{b}$  for  $\vec{x}$  where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}.$$

**Solution.** The solution is rather straightforward, even though we did a lot of work before to find the answer. Form the augmented matrix  $[A \ \vec{b}]$  and interpret its reduced row echelon form.

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ -1 & 2 & 1 & -1 \\ 1 & 1 & 0 & 2 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right].$$

In previous sections we were fine stating that the result as

$$x_1 = 2, \quad x_2 = 0, \quad x_3 = 1,$$

but we were asked to find  $\vec{x}$ ; therefore, we state the solution as

$$\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

This probably seems all well and good. While asking one to solve the equation  $A\vec{x} = \vec{b}$  for  $\vec{x}$  seems like a new problem, in reality it is just asking that we solve a system of linear equations. Our variables  $x_1$ , etc., appear not individually but as the entries of our vector  $\vec{x}$ . We are simply writing an old problem in a new way.

In line with this new way of writing the problem, we have a new way of writing the solution. Instead of listing, individually, the values of the unknowns, we simply list them as the elements of our vector  $\vec{x}$ .

These are important ideas, so we state the basic principle once more: solving the equation  $A\vec{x} = \vec{b}$  for  $\vec{x}$  is the same thing as solving a linear system of

equations. Equivalently, any system of linear equations can be written in the form  $A\vec{x} = \vec{b}$  for some matrix  $A$  and vector  $\vec{b}$ .

Since these ideas are equivalent, we'll refer to  $A\vec{x} = \vec{b}$  both as a matrix-vector equation and as a system of linear equations: they are the same thing.

We've seen two examples illustrating this idea so far, and in both cases the linear system had exactly one solution. We know from Theorem 3.4.2 that any linear system has either one solution, infinitely many solutions, or no solution. So how does our new method of writing a solution work with infinitely many solutions and no solutions?

Certainly, if  $A\vec{x} = \vec{b}$  has no solution, we simply say that the linear system has no solution. There isn't anything special to write. So the only other option to consider is the case where we have infinitely many solutions. We'll learn how to handle these situations through examples.

### Example 3.6.3 Finding the vector solution to a linear system.

Solve the linear system  $A\vec{x} = \vec{0}$  for  $\vec{x}$  and write the solution in vector form, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

**Solution.** To solve this system, put the augmented matrix into reduced row echelon form, which we do below.

$$\left[ \begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array} \right] \xrightarrow{\text{ref}} \left[ \begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

We interpret the reduced row echelon form of this matrix to write the solution as

$$\begin{aligned} x_1 &= -2t \\ x_2 &= t \text{ is free.} \end{aligned}$$

We are not done; we need to write the solution in vector form, for our solution is the vector  $\vec{x}$ . Recall that

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

From above we know that  $x_1 = -2x_2$ , where  $x_2 = t$ , so we replace the  $x_1$  in  $\vec{x}$  with  $-2t$  and replace  $x_2$  by  $t$ . This gives our solution as

$$\vec{x} = \begin{bmatrix} -2t \\ t \end{bmatrix}.$$

Note that we can now pull the  $t$  out of the vector (it is just a scalar) and write  $\vec{x}$  as

$$\vec{x} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

For convenience, if we set

$$\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

then our solution can be simply written as

$$\vec{x} = t\vec{v}.$$

**Note:** Our convention is always that  $\vec{0}$  denotes the vector of the appropriate size whose entries are all zero, so we didn't really need to specify that  $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ , but we did just to eliminate any uncertainty.

Recall that since our system was consistent and had a free variable, we have infinitely many solutions. This form of the solution highlights this fact; pick any value for  $t$  and we get a different solution.

For instance, by setting  $t = -1, 0$ , and  $5$ , we get the solutions

$$\vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} -10 \\ 5 \end{bmatrix},$$

respectively.

We should check our work; multiply each of the above vectors by  $A$  to see if we indeed get  $\vec{0}$ . Or, we can save ourselves some time and check the general solution. We have

$$A\vec{x} = A(t\vec{v}) = t(A\vec{v}) = t \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for every value of  $t$ .

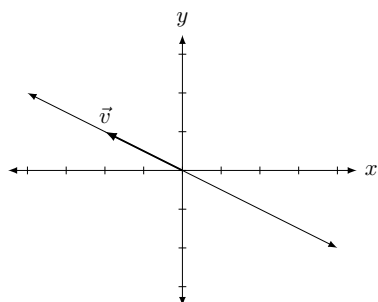
We have officially solved this problem; we have found the solution to  $A\vec{x} = \vec{0}$  and written it properly. One final thing we will do here is *graph* the solution, using our skills learned in the previous section.

Our solution is

$$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

This means that any scalar multiple of the vector  $\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$  is a solution; we know how to sketch the scalar multiples of  $\vec{v}$ . This is done in Figure 3.6.4.

Here vector  $\vec{v}$  is drawn as well as the line that goes through the origin in the direction of  $\vec{v}$ . Any vector along this line is a solution. So in some sense, we can say that the solution to  $A\vec{x} = \vec{0}$  is *a line*.



**Figure 3.6.4** The solution, as a line, to  $A\vec{x} = \vec{0}$  in Example 3.6.3

A few comments are in order here. First, matrix equations (or the corresponding system of linear equations) such as the above where the vector on the right-hand side is the zero vector form a special case that is important enough to have its own name: these are known as **homogeneous** systems of equations. The formal definition is as follows.

#### Definition 3.6.5 Homogeneous Linear System of Equations.

A system of linear equations is *homogeneous* if the constant term in each equation is zero.

Note: a homogeneous system of equations can be written in vector form as  $A\vec{x} = \vec{0}$ .

The term *homogeneous* comes from two Greek words; *homo* meaning “same” and *genus* meaning “type.” A homogeneous system of equations is a system in which each equation is of the same type — all constants are 0. In mathematics, objects that are considered homogeneous often have some sort of *scale invariance*, and that is the case here: if  $\vec{x}$  is a solution to  $A\vec{x} = \vec{0}$ , then so is  $c\vec{x}$  for any scalar  $c$ . (Do you see why?)

Notice that the line  $\vec{x} = t\vec{v}$  in the solution of Example 3.6.3 above passes through the origin. This is an important characteristic of homogeneous systems: since  $A\vec{0} = \vec{0}$  for any matrix  $A$ , we always have (at least) the solution  $\vec{x} = \vec{0}$ .



(We'll have more to say about this below.)

**Example 3.6.6 Determining the solution of a homogeneous system.**

Determine the solution to the system  $A\vec{x} = \vec{0}$ , where

$$A = \begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix}.$$

**Solution.** We proceed exactly as we did in Example 3.6.3, by forming the proper augmented matrix and putting it into reduced row echelon form, which we do below.

$$\left[ \begin{array}{cc|c} 2 & -3 & 0 \\ -2 & 3 & 0 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{cc|c} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

We interpret the reduced row echelon form of this matrix to find that

$$\begin{aligned} x_1 &= 3/2t \\ x_2 &= t \text{ is free.} \end{aligned}$$

As before, we can say that  $A\vec{x} = \vec{0}$  provided that

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3/2t \\ t \end{bmatrix} = t \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}.$$

If we set

$$\vec{v} = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix},$$

then our solution can be written as

$$\left\{ t\vec{v} \mid t \in \mathbb{R} \text{ and } \vec{v} = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \right\}.$$

Again, we have infinitely many solutions to the equation  $A\vec{x} = \vec{0}$ ; any choice of  $x_2$  gives us one of these solutions. For instance, picking  $x_2 = 2$  gives the solution

$$\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

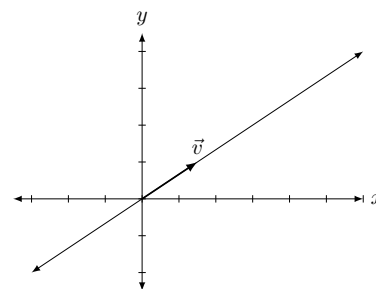
This is a particularly nice solution, since there are no fractions! In fact, since the parameter  $t$  can take on any *real* value, there is nothing preventing us from defining a new parameter  $s = t/2$ , and then

$$\vec{x} = t \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} = t \left( \frac{1}{2} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) = \frac{t}{2} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = s \begin{bmatrix} 3 \\ 2 \end{bmatrix} = s\vec{w},$$

where  $\vec{w} = 2\vec{v}$ .

As in the previous example, our solutions are multiples of a vector, and hence we can graph this, as done in Figure 3.6.7.

In the last two examples, we saw that the general solution could be written in the form  $\vec{x} = t\vec{v}$  for a vector  $\vec{v}$  such that  $A\vec{v} = \vec{0}$ . Such vectors are known as the **basic solutions** to a homogeneous linear system.



**Figure 3.6.7** The solution, as a line, to  $A\vec{x} = \vec{0}$  in Example 3.6.6

**Definition 3.6.8 Basic solution.**

Let  $A\vec{x} = \vec{0}$  be a homogeneous linear system of equations with infinitely many solutions and free variables

$$x_{i_1} = t_1, x_{i_2} = t_2, \dots, x_{i_k} = t_k.$$

The **basic** solutions to the system  $A\vec{x} = \vec{0}$  are the vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$  such that the general solution to the system is given by

$$\vec{x} = t_1\vec{v}_1 + t_2\vec{v}_2 + \dots + t_k\vec{v}_k.$$

To help clarify Definition 3.6.8, let's do one more example where we have more than one basic solution.

**Example 3.6.9 A homogeneous system with two basic solutions.**

Find the general solution to the homogeneous system  $A\vec{x} = \vec{0}$ , where

$$A = \begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & -1 & 5 & 2 \\ -2 & -6 & -10 & 12 \end{bmatrix}.$$

**Solution.** As usual, to find the basic solutions, we set up the augmented matrix of the system and reduce:

$$\left[ \begin{array}{cccc|c} 1 & -2 & 0 & 4 & 0 \\ 3 & -1 & 5 & 2 & 0 \\ -2 & -6 & -10 & 12 & 0 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

From the reduced row echelon form of the augmented matrix, we can read off the following general solution:

$$\begin{aligned} x_1 &= -2s \\ x_2 &= -s + 2t \\ x_3 &= s \text{ is free} \\ x_4 &= t \text{ is free.} \end{aligned}$$

In this case, we have two parameters, so we expect two basic solutions. To find these, we write our solution in vector form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s \\ -s + 2t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

From the above, we see that the general solution can be written as  $\vec{x} = s\vec{v} + t\vec{w}$ , where

$$\vec{v} = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{w} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

are the basic solutions to  $A\vec{x} = \vec{0}$ .

Let's practice finding vector solutions again; this time, we won't solve a system of the form  $A\vec{x} = \vec{0}$ , but instead  $A\vec{x} = \vec{b}$ , for some vector  $\vec{b} \neq \vec{0}$ . Such systems are known (unsurprisingly) as **non-homogeneous** systems.

**Example 3.6.10 A non-homogeneous linear system.**

Solve the linear system  $A\vec{x} = \vec{b}$ , where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

**Solution.** (Note that this is the same matrix  $A$  that we used in Example 3.6.3. This will be important later.)

Our methodology is the same as before; we form the augmented matrix and put it into reduced row echelon form.

$$\left[ \begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & 6 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array} \right]$$

Interpreting this reduced row echelon form, we find that

$$\begin{aligned} x_1 &= 3 - 2t \\ x_2 &= t \text{ is free.} \end{aligned}$$

Putting things into vector form, we have

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 - 2t \\ t \end{bmatrix}.$$

This solution is different than what we've seen in the past two examples; we can't simply pull out a  $t$  since there is a 3 in the first entry. Using the properties of matrix addition, we can "pull apart" this vector and write it as the sum of two vectors: one which contains only constants, and one that contains only terms involving the parameter  $t$ . We do this below.

$$\begin{aligned} \vec{x} &= \begin{bmatrix} 3 - 2t \\ t \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2t \\ t \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \end{bmatrix}. \end{aligned}$$

Once again, let's give names to the different component vectors of this solution (we are getting near the explanation of why we are doing this). Let

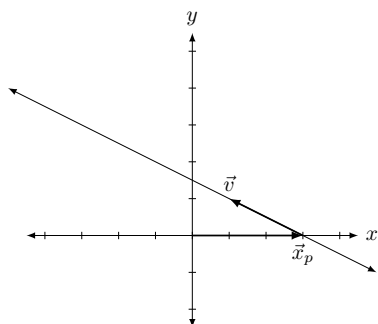
$$\vec{x}_p = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

We can then write our solution in the form

$$\vec{x} = \vec{x}_p + t\vec{v}.$$

We still have infinitely many solutions; by picking a value for  $t$  we get one of these solutions. For instance, by letting  $t = -1, 0$ , or  $2$ , we get the solutions

$$\begin{bmatrix} 5 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$



**Figure 3.6.11** The solution, as a line, to  $A\vec{x} = \vec{b}$  in Example 3.6.10

**Note:** Visually, the solutions in Examples Example 3.6.3 and Example 3.6.10 were both lines; from our experience with Section 2.5 we know that this makes sense, since the solutions involved a single parameter  $t$ . The reader can similarly expect that a solution involving two parameters can be visualized as a plane. Such was the case in Example 3.6.9, except that here, our solution is a plane in  $\mathbb{R}^4$ , making it somewhat harder to visualize.

We have officially solved the problem; we have solved the equation  $A\vec{x} = \vec{b}$  for  $\vec{x}$  and have written the solution in vector form. As an additional visual aid, we will graph this solution.

Each vector in the solution can be written as the sum of two vectors:  $\vec{x}_p$  and a multiple of  $\vec{v}$ . In Figure 3.6.11,  $\vec{x}_p$  is graphed and  $\vec{v}$  is graphed with its origin starting at the tip of  $\vec{x}_p$ . Finally, a line is drawn in the direction of  $\vec{v}$  from the tip of  $\vec{x}_p$ ; any vector pointing to any point on this line is a solution to  $A\vec{x} = \vec{b}$ .

Notice that in this case our line does *not* pass through the origin, so the set of solutions is not a subspace. On the other hand, every solution to the system  $A\vec{x} = \vec{b}$  can be obtained by adding the vector  $\vec{x}_p$  to an element of the set of solutions to the homogeneous system  $A\vec{x} = \vec{0}$ . We'll elaborate on this shortly.

The previous examples illustrate some important concepts. One is that (at least, when  $\vec{x} \in \mathbb{R}^2$  or  $\mathbb{R}^3$ ) we can visualize the solution to a system of linear equations. Before, when we had infinitely many solutions, we knew we could arbitrarily pick values for our free variables and get different solutions. We knew this to be true, and we even practised it, but the result was not very “tangible.” Now, we can view our solution as a vector; by picking different values for our free variables, we see this as multiplying certain important vectors by a scalar which gives a different solution.

Another important concept that these examples demonstrate comes from the fact that Examples 3 and 10 were only “slightly different” and hence had only “slightly different” answers. Both solutions had

$$t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

in them; in Example 3.6.10 the solution also had another vector added to this. The addition of the vector  $\vec{x}_p$  in Example 3.6.10 is needed to account for the fact that we were dealing with a non-homogeneous system of linear equations.

Recall that for a homogeneous system of linear equations, we know that  $\vec{x} = \vec{0}$  will be a solution, since no matter what the matrix  $A$  is, we can be certain that  $A\vec{0} = \vec{0}$ . This fact is important; the zero vector is *always* a solution to a homogeneous linear system. Therefore a homogeneous system is always consistent; we need only to determine whether we have exactly one solution (just  $\vec{0}$ ) or infinitely many solutions. This idea is important, so we give it its own box.

#### Key Idea 3.6.12 Homogeneous Systems and Consistency.

All homogeneous linear systems are consistent.

How do we determine if we have exactly one or infinitely many solutions? Recall Key Idea 3.4.7: if the solution has any free variables, then it will have infinitely many solutions. How can we tell if the system has free variables? Form the augmented matrix  $[A \quad \vec{0}]$ , put it into reduced row echelon form, and interpret the result.

It may seem that we've brought up a new question, “When does  $A\vec{x} = \vec{0}$  have exactly one or infinitely many solutions?” only to answer with “Look at the reduced row echelon form of  $A$  and interpret the results, just as always.” Why bring up a new question if the answer is an old one?

While the new question has an old solution, it does lead to a great idea. Let's

refresh our memory; earlier we solved two linear systems,

$$A\vec{x} = \vec{0} \quad \text{and} \quad A\vec{x} = \vec{b}$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

The solution to the first system of equations,  $A\vec{x} = \vec{0}$ , is

$$\vec{x} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix} = t\vec{v}$$

and the solution to the second set of equations,  $A\vec{x} = \vec{b}$ , is

$$\vec{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \vec{x}_p + t\vec{v},$$

for all  $t \in \mathbb{R}$ , where  $\vec{x}_p = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ .

To see why the general solution to  $A\vec{x} = \vec{b}$  works, note that

$$A\vec{x}_p = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \vec{b},$$

so  $\vec{x}_p$  is a solution. (The subscript  $p$  of “ $\vec{x}_p$ ” is used to denote that this vector is a **particular** solution: see Definition 3.4.11.) What about the general solution  $\vec{x} = \vec{x}_p + t\vec{v}$ ? Recalling that  $A\vec{v} = \vec{0}$ , we have

$$\begin{aligned} A\vec{x} &= A(\vec{x}_p + t\vec{v}) = A\vec{x}_p + t(A\vec{v}) \\ &= \vec{b} + t(\vec{0}) = \vec{b}, \end{aligned}$$

for any value of  $t$ , so there are infinitely many solutions to our system, one for each  $t \in \mathbb{R}$ . The whole point is that  $\vec{x}_p$  itself is a solution to  $A\vec{x} = \vec{b}$ , and we could find more solutions by adding vectors “that go to zero” when multiplied by  $A$ .

So we wonder: does this mean that  $A\vec{x} = \vec{b}$  will have infinitely many solutions? After all, if  $\vec{x}_p$  and  $\vec{x}_p + \vec{v}$  are both solutions, don't we have infinitely many solutions?

No. If  $A\vec{x} = \vec{0}$  has exactly one solution, then  $\vec{v} = \vec{0}$ , and  $\vec{x}_p = \vec{x}_p + \vec{v}$ ; we only have one solution.

So here is the culmination of all of our fun that started a few pages back. If  $\vec{v}$  is a solution to  $A\vec{x} = \vec{0}$  and  $\vec{x}_p$  is a solution to  $A\vec{x} = \vec{b}$ , then  $\vec{x}_p + \vec{v}$  is also a solution to  $A\vec{x} = \vec{b}$ . If  $A\vec{x} = \vec{0}$  has infinitely many solutions, so does  $A\vec{x} = \vec{b}$ ; if  $A\vec{x} = \vec{0}$  has only one solution, so does  $A\vec{x} = \vec{b}$ . This culminating idea is of course important enough to be stated again.

#### Key Idea 3.6.13 Solutions of Consistent Systems.

Let  $A\vec{x} = \vec{b}$  be a consistent system of linear equations.

1. If  $A\vec{x} = \vec{0}$  has exactly one solution ( $\vec{x} = \vec{0}$ ), then  $A\vec{x} = \vec{b}$  has exactly one solution.
2. If  $A\vec{x} = \vec{0}$  has infinitely many solutions, then  $A\vec{x} = \vec{b}$  has infinitely many solutions.

A key word in the above statement is *consistent*. If  $A\vec{x} = \vec{b}$  is inconsistent (the linear system has no solution), then it doesn't matter how many solutions  $A\vec{x} = \vec{0}$  has;  $A\vec{x} = \vec{b}$  has no solution.

We can elaborate on Key Idea 3.6.13 above, as well as Key Idea 3.4.7 from Section 3.4 by introducing one more piece of important terminology. By now it is probably clear that the leading 1s in the reduced row echelon form of a matrix play a key role in understanding the system. In fact, it turns out that we can describe all of the different possibilities for a linear system in terms of one number: the number of leading 1s in the reduced row echelon form of a matrix.

**Definition 3.6.14 The rank of a matrix.**

The **rank** of a matrix  $A$  is denoted by  $\text{rank}(A)$  and defined as the number of leading 1s in the reduced row echelon form of  $A$ .

Although we do not prove it in this textbook, the reduced row echelon form of any matrix is *unique*; it follows from this fact that the rank of a matrix is a well-defined number. The importance of rank is outlined in the following result.

**Theorem 3.6.15 Rank and solution types.**

Let  $A$  be an  $m \times n$  matrix. For any linear system  $A\vec{x} = \vec{b}$  in  $n$  variables, we have the following possibilities:

1. If  $\text{rank}(A) < \text{rank} \begin{bmatrix} A & | & \vec{b} \end{bmatrix}$ , then the system  $A\vec{x} = \vec{b}$  is inconsistent.
2. If  $\text{rank}(A) = \text{rank} \begin{bmatrix} A & | & \vec{b} \end{bmatrix} = n$  (where  $n$  is the number of variables), then the system  $A\vec{x} = \vec{b}$  has a unique solution.
3. If  $\text{rank}(A) = \text{rank} \begin{bmatrix} A & | & \vec{b} \end{bmatrix} < n$ , then the system  $A\vec{x} = \vec{b}$  has infinitely solutions. Moreover, the general solution to  $A\vec{x} = \vec{b}$  will involve  $k$  parameters, where

$$k = n - \text{rank}(A).$$

To understand Item 1 above, note that if

$$\text{rank}(A) < \text{rank} \begin{bmatrix} A & | & \vec{b} \end{bmatrix},$$

then there must be a leading 1 in the right-hand column of the reduced row echelon form of  $\begin{bmatrix} A & | & \vec{b} \end{bmatrix}$ , meaning that we have a row of the form

$$[0 \ 0 \ \dots \ 0 \ 1],$$

which is exactly what we expect in a system with no solutions.

Items 2 and 3 in Theorem 3.6.15 simply give another way of stating the fact that the free variables are those variables that do *not* have a leading 1 in their column. This seems like an obvious fact, but it is very important. We will see in Section 5.4 that this observation leads to a major theorem, sometimes known as the Fundamental Theorem of Linear Transformations: see Theorem 5.4.12.

Let us explore this result with a series of examples.

**Example 3.6.16** Using matrices and vectors to solve a system of equations.

Rewrite the linear system

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 2x_4 + 7x_5 &= 2 \\ 3x_1 + 4x_2 + 5x_3 + 2x_4 + 3x_5 &= -4 \end{aligned}$$

as a matrix-vector equation, solve the system using vector notation, and give the solution to the related homogeneous equations.

**Solution.** Rewriting the linear system in the form of  $A\vec{x} = \vec{b}$ , we have that

$$A = \begin{bmatrix} 1 & 2 & -3 & 2 & 7 \\ 3 & 4 & 5 & 2 & 3 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}.$$

To solve the system, we put the associated augmented matrix into reduced row echelon form and interpret the results.

$$\left[ \begin{array}{ccccc|c} 1 & 2 & -3 & 2 & 7 & 2 \\ 3 & 4 & 5 & 2 & 3 & -4 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{ccccc|c} 1 & 0 & 11 & -2 & -11 & -8 \\ 0 & 1 & -7 & 2 & 9 & 5 \end{array} \right]$$

$$x_1 = -8 - 11r + 2s + 11t$$

$$x_2 = 5 + 7r - 2s - 9t$$

$$x_3 = r \text{ is free}$$

$$x_4 = s \text{ is free}$$

$$x_5 = t \text{ is free.}$$

We use this information to write  $\vec{x}$ , again pulling it apart. Since we have three free variables and also constants, we'll need to pull  $\vec{x}$  apart into four separate vectors.

$$\begin{aligned} \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} &= \begin{bmatrix} -8 - 11r + 2s + 11t \\ 5 + 7r - 2s - 9t \\ r \\ s \\ t \end{bmatrix} \\ &= \begin{bmatrix} -8 \\ 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -11r \\ 7r \\ r \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2s \\ -2s \\ 0 \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 11t \\ -9t \\ 0 \\ 0 \\ t \end{bmatrix} \\ &= \begin{bmatrix} -8 \\ 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -11 \\ 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 11 \\ -9 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

$$= \underbrace{\vec{x}_p}_{\text{particular solution}} + \underbrace{r\vec{u} + s\vec{v} + t\vec{w}}_{\text{solution to homogeneous equations } A\vec{x}=\vec{0}}.$$

So  $\vec{x}_p$  is a particular solution;  $A\vec{x}_p = \vec{b}$ . (Multiply it out to verify that this is true.) The other vectors,  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$ , that are multiplied by our free variables  $x_3 = r$ ,  $x_4 = s$  and  $x_5 = t$ , are each solutions to the homogeneous equations,  $A\vec{x} = \vec{0}$ . Any linear combination of these three vectors, i.e., any vector found by choosing values for  $r$ ,  $s$  and  $t$  in  $r\vec{u} + s\vec{v} + t\vec{w}$  is a solution to  $A\vec{x} = \vec{0}$ .

### Example 3.6.17 Finding vector solutions.

Let

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

Find the solutions to  $A\vec{x} = \vec{b}$  and  $A\vec{x} = \vec{0}$ .

**Solution.** We go through the familiar work of finding the reduced row echelon form of the appropriate augmented matrix and interpreting the solution.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$x_1 = -1$$

$$x_2 = 2.$$

Thus

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

This may strike us as a bit odd; we are used to having lots of different vectors in the solution. However, in this case, the linear system  $A\vec{x} = \vec{b}$  has exactly one solution, and we've found it. What is the solution to  $A\vec{x} = \vec{0}$ ? Since we've only found one solution to  $A\vec{x} = \vec{b}$ , we can conclude from Key Idea 3.6.13 the related homogeneous equations  $A\vec{x} = \vec{0}$  have only one solution, namely  $\vec{x} = \vec{0}$ . We can write our solution vector  $\vec{x}$  in a form similar to our previous examples to highlight this:

$$\begin{aligned} \vec{x} &= \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= \underbrace{\vec{x}_p}_{\text{particular solution}} + \underbrace{\vec{0}}_{\text{solution to } A\vec{x}=\vec{0}}. \end{aligned}$$

Again, in light of Theorem 3.6.15, this should not be too surprising. The reduced row echelon form of  $A$  is  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ , so the rank of  $A$  is 2, and there are 2 variables in our system, so we expect  $2 - 2 = 0$  parameters in our general solution.



**Example 3.6.18 Further vector solutions.**

Let

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Find the solutions to  $A\vec{x} = \vec{b}$  and  $A\vec{x} = \vec{0}$ .

**Solution.** To solve  $A\vec{x} = \vec{b}$ , we put the appropriate augmented matrix into reduced row echelon form and interpret the results.

$$\left[ \begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 1 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

We immediately have a problem; we see that the second row tells us that  $0x_1 + 0x_2 = 1$ , the sign that our system does not have a solution. Thus  $A\vec{x} = \vec{b}$  has no solution. Of course, this does not mean that  $A\vec{x} = \vec{0}$  has no solution; it always has a solution.

To find the solution to  $A\vec{x} = \vec{0}$ , we interpret the reduced row echelon form of the appropriate augmented matrix.

$$\left[ \begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$x_1 = -x_2$$

$x_2$  is free.

Thus,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = x_2 \vec{u}.$$

We have no solution to  $A\vec{x} = \vec{b}$ , but infinitely many solutions to  $A\vec{x} = \vec{0}$ .

The previous example may seem to violate the principle of Key Idea 3.6.13. After all, it seems that having infinitely many solutions to  $A\vec{x} = \vec{0}$  should imply infinitely many solutions to  $A\vec{x} = \vec{b}$ . However, we remind ourselves of the key word in the idea that we observed before: *consistent*. If  $A\vec{x} = \vec{b}$  is consistent and  $A\vec{x} = \vec{0}$  has infinitely many solutions, then so will  $A\vec{x} = \vec{b}$ . But if  $A\vec{x} = \vec{b}$  is not consistent, it does not matter how many solutions  $A\vec{x} = \vec{0}$  has;  $A\vec{x} = \vec{b}$  is still inconsistent.

In this chapter, we developed a systematic method for solving systems of linear equations. A key tool in this method was the augmented matrix corresponding to a given system. In this final section, we've seen that further insight into the structure of solutions can be gained by considering our systems in terms of matrices and vectors.

In the next chapter, we will begin the study of matrices as objects unto themselves. We will see that they can be added and multiplied by scalars in exactly the same way as vectors, and in addition to this, matrices of the correct sizes can be *multiplied* in a way that reproduces Definition 3.6.1 above as a special case.

One question that may have occurred to you as you worked through this section is the following: in the one-variable linear equation  $ax = b$ , we know that as long as  $a \neq 0$ , we can divide both sides by  $a$ , giving us the solution  $x = b/a$ . Now, given the matrix equation  $A\vec{x} = \vec{b}$ , is there some equivalent means of "dividing by  $A$ " to obtain the solution  $\vec{x}$ ? The short answer is no. Indeed, there

As previously noted, the fact that  $A\vec{x} = \vec{b}$  has no solution in Example 3.6.18 simply indicates the fact that  $\vec{b}$  is not in the column space of  $A$ . Since the rank of  $A$  is equal to one, we know that  $\text{col}(A)$  is spanned by the single vector  $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ . Thus, we can only expect  $A\vec{x} = \vec{b}$  to have a solution if  $\vec{b}$  is a scalar multiple of  $\vec{v}$ .

is no such thing as matrix division; the algebraic rules for matrix multiplication are much more complicated than they are for numbers. (In particular, we'll see that for matrices,  $AB$  is usually *not* the same thing as  $BA$ !)

The slightly longer answer to our question might be phrased as "Sometimes. Well, sort of." To obtain the correct (and much longer) answer, we will be led in the next chapter to the definition of the *inverse* of a matrix.

## 3.6.1 Exercises

**Exercise Group.** A matrix  $A$  and vectors  $\vec{b}$ ,  $\vec{u}$  and  $\vec{v}$  are given. Verify that  $\vec{u}$  and  $\vec{v}$  are both solutions to the equation  $A\vec{x} = \vec{b}$ ; that is, show that  $A\vec{u} = A\vec{v} = \vec{b}$ .

$$1. \quad A = \begin{bmatrix} 1 & -2 & -3 & 6 \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \\ \vec{v} = \begin{bmatrix} -10 \\ -5 \end{bmatrix}$$

$$2. \quad A = \begin{bmatrix} 1 & 0 & 2 & 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \vec{u} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \\ \vec{v} = \begin{bmatrix} 0 \\ 59 \end{bmatrix}$$

$$3. \quad A = \begin{bmatrix} 1 & 0 & 2 & 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} -3 \\ -6 \end{bmatrix}, \vec{u} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \\ \vec{v} = \begin{bmatrix} -3 \\ 59 \end{bmatrix}$$

$$4. \quad A = \begin{bmatrix} 1 & -2 & -3 & 6 \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}, \vec{u} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \\ \vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$5. \quad A = \begin{bmatrix} 0 & -3 & -1 & -3 & -4 & 2 & -3 & 5 \end{bmatrix}, \\ \vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \vec{u} = \begin{bmatrix} 11 \\ 4 \\ -12 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 9 \\ -12 \\ 0 \\ 12 \end{bmatrix}$$

$$6. \quad A = \begin{bmatrix} 0 & -3 & -1 & -3 & -4 & 2 & -3 & 5 \end{bmatrix}, \\ \vec{b} = \begin{bmatrix} 48 \\ 36 \end{bmatrix}, \vec{u} = \begin{bmatrix} -17 \\ -16 \\ 0 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} -8 \\ -28 \\ 0 \\ 12 \end{bmatrix}$$

**Exercise Group.** A matrix  $A$  and vectors  $\vec{b}$ ,  $\vec{u}$  and  $\vec{v}$  are given. Verify that  $A\vec{u} = \vec{0}$ ,  $A\vec{v} = \vec{b}$  and  $A(\vec{u} + \vec{v}) = \vec{b}$ .

$$7. \quad A = \begin{bmatrix} 2 & -2 & -1 & -1 & 1 & -1 \\ -2 & 2 & & -1 & & \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$8. \quad A = \begin{bmatrix} 1 & -1 & 3 & 3 & -3 & -3 \\ -1 & 1 & 1 & & & \end{bmatrix}, \vec{b} = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}, \vec{u} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

$$9. \quad A = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & -3 \\ 3 & 1 & -3 & & & \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix}, \vec{u} = \begin{bmatrix} 0 \\ 6 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

**Exercise Group.** A matrix  $A$  and vector are given.

(a) Solve the equation  $A\vec{x} = \vec{0}$ .

(b) Solve the equation  $A\vec{x} = \vec{b}$ .

In each of the above, be sure to write your answer in vector format. Also, when possible, give 2 particular solutions to each equation.

$$10. \quad A = \begin{bmatrix} 0 & 2 & -1 & 3 \end{bmatrix}, \vec{b} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$11. \quad A = \begin{bmatrix} -4 & -1 & -3 & -2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$12. \quad A = \begin{bmatrix} 1 & -2 & 0 & 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$$

$$13. \quad A = \begin{bmatrix} 1 & 0 & 5 & -4 \end{bmatrix}, \vec{b} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$14. \quad A = \begin{bmatrix} 2 & -3 & -4 & 6 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$15. \quad A = \begin{bmatrix} -4 & 3 & 2 & -4 & 5 & 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$$

$$16. \quad A = \begin{bmatrix} 1 & 5 & -2 & 1 & 4 & 5 \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$17. \quad A = \begin{bmatrix} -1 & -2 & -2 & 3 & 4 & -2 \end{bmatrix}, \vec{b} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$$

$$18. \quad A = \begin{bmatrix} 2 & 2 & 2 & 5 & 5 & -3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

$$19. \quad A = \begin{bmatrix} 1 & 5 & -4 & -1 & 1 & 0 & -2 & 1 \end{bmatrix}, \\ \vec{b} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$20. \quad A = \begin{bmatrix} -4 & 2 & -5 & 4 & 0 & 1 & -1 & 5 \end{bmatrix}, \\ \vec{b} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$$

$$22. \quad A = \begin{bmatrix} 3 & 0 & -2 & -4 & 5 & 2 & 3 & 2 & 0 & 2 \\ -5 & 0 & 4 & 0 & 5 & & & & & \end{bmatrix}, \\ \vec{b} = \begin{bmatrix} -1 \\ -5 \\ 4 \end{bmatrix}$$

$$21. \quad A = \begin{bmatrix} 0 & 0 & 2 & 1 & 4 & -2 & -1 & -4 & -1 & 5 \end{bmatrix}, \\ \vec{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$23. \quad A = \begin{bmatrix} -1 & 3 & 1 & -3 & 4 & 3 & -3 & -1 & 1 & -4 \\ -2 & 3 & -2 & -3 & 1 & & & & & \end{bmatrix}, \\ \vec{b} = \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix}$$

**Exercise Group.** A matrix  $A$  and vector are given. Solve the equation  $A\vec{x} = \vec{b}$ , write the solution in vector format, and sketch the solution as the appropriate line on the Cartesian plane.

$$24. \quad A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$26. \quad A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}, \vec{b} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$$

$$25. \quad A = \begin{bmatrix} 2 & -5 \\ -4 & -10 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$27. \quad A = \begin{bmatrix} 2 & -5 \\ -4 & -10 \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

## Chapter 4

# Matrix Algebra

In the last chapter we learned how to solve systems of equations, and along the way, we saw that a key tool for solving systems efficiently was the use of **matrices**. In this chapter, we will finally give a proper definition of what, exactly, a matrix is, after which we will proceed to develop the algebraic properties of matrices, just as we did for vectors in Chapter 2.

A fundamental topic of mathematics is arithmetic; adding, subtracting, multiplying and dividing numbers. After learning how to do this, most of us went on to learn how to add, subtract, multiply and divide “ $x$ ”. We are comfortable with expressions such as

$$x + 3x - x \cdot x^2 + x^5 \cdot x^{-1}$$

and know that we can “simplify” this to

$$4x - x^3 + x^4.$$

This chapter deals with the idea of doing similar operations, but instead of an unknown number  $x$ , we will be using a matrix  $A$ . So what exactly does the expression

$$A + 3A - A \cdot A^2 + A^5 \cdot A^{-1}$$

mean? Before we can do anything, we need to actually define what a matrix is! Once we’ve taken care of that, we are going to need to learn to define what matrix addition, scalar multiplication, matrix multiplication and matrix inversion are. We will learn just that, plus some more good stuff, in this chapter.

### 4.1 Matrix Addition and Scalar Multiplication

As mentioned above, a **matrix** is a construction that allows us to organize information in a tabular form. For example, we may be interested in the following (made up) data involving crop yields on several Southern Alberta farms, given in tabular form:

**Table 4.1.1 2014 crop yields, in metric tonnes**

	Corn	Potatoes	Soybeans	Wheat
Farm A	48	18	92	0
Farm B	0	0	73	152
Farm C	34	203	0	88

Someone in charge of compiling data on farms and crops probably already has a program or spreadsheet set up with all of the farms and crops pre-defined; what they are interested in are the numbers giving the crop yields. Thus, when

Matrices work well in situations such as our farming example where the data we need to organize falls into two categories, in this case, farms and crops. What if our data depends on three or more categories? (Perhaps we also want to track type of fertilizer?) It turns out that along with matrices, mathematicians have invented higher-dimensional arrays called tensors, that can handle almost any data organization scenario you can think of. Tensors are usually encountered in more advanced courses in abstract algebra.

they enter their data into the computer, they are probably more interested in the array

$$A = \begin{bmatrix} 48 & 18 & 92 & 0 \\ 0 & 0 & 73 & 152 \\ 34 & 203 & 0 & 88 \end{bmatrix}. \quad (4.1.1)$$

As long as we're consistent and always assign each farm to the same row, and each crop to the same column, we can dispense with the labels and work directly with the data.

The array above is our first example of a matrix, which we now define. The definition of matrix is remarkable only in how unremarkable it seems — it is simply a way of organizing information (usually numbers) into an array.

#### Definition 4.1.2 Matrix.

A **matrix** is a rectangular array of numbers.

The horizontal lines of numbers form **rows** and the vertical lines of numbers form **columns**. A matrix with  $m$  rows and  $n$  columns is said to be an  $m \times n$  matrix ("an  $m$  by  $n$  matrix", or a matrix of **size**  $m \times n$ ).

The entries of an  $m \times n$  matrix are indexed as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

That is,  $a_{32}$  means "the number in the third row and second column." To save space, we will sometimes use the shorthand notation  $A = [a_{ij}]$  to denote a matrix  $A$  with entries  $a_{ij}$ . If we need to specify the size, we can also write  $A = [a_{ij}]_{m \times n}$ .

Two special types of matrix are worth noting: those with a single row or column. Such matrices are known as **vectors**.

#### Definition 4.1.3 Row and column vectors.

A **row vector** is a  $1 \times n$  matrix of the form

$$R = [r_1 \quad r_2 \quad \cdots \quad r_n].$$

A **column vector** is a  $m \times 1$  matrix of the form

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}.$$

In particular, we can obtain row or column vectors by isolating any row or column of a given  $m \times n$  matrix. For example, given our matrix of crop data in Equation (4.1.1), we might be interested only in Farm B, in which case we would want the row vector

$$R_B = [0 \quad 0 \quad 73 \quad 152].$$

Similarly, we might only be interested in yields for soybean crops, in which case the column vector

$$C_{\text{soy}} = \begin{bmatrix} 92 \\ 73 \\ 0 \end{bmatrix}$$

is our object of interest.

Continuing with our farming example, suppose that in addition to the matrix  $A$  of 2014 crop yields above we also have data for 2015 crop yields given by

$$B = \begin{bmatrix} 41 & 25 & 15 & 20 \\ 0 & 72 & 0 & 165 \\ 47 & 193 & 0 & 77 \end{bmatrix}.$$

We might be interested in quantities such as the total crop yields over two years. If we arrange these totals into a matrix  $T$ , it seems like it should be reasonable to define matrix addition in such a way that we can write

$$T = A + B.$$

This leads to two questions. First, how do we define matrix addition in order to ensure this outcome? Second, and perhaps more fundamentally, what do we mean by “=” in the context of matrices? Let us tackle the second question first.

**Definition 4.1.4 Matrix Equality.**

Two  $m \times n$  matrices  $A$  and  $B$  are **equal** if their corresponding entries are equal.

Notice that our more formal definition specifies that if matrices are equal, they have the same size. This should make sense.

Now we move on to describing how to add two matrices together. To start off, take a wild stab: how do you think we should add our matrices of crop data above? Well, if we want the sum to represent the total yields for each crop, then it stands to reason that to add the two matrices, we add together each of the corresponding crop yields within them:

$$\begin{aligned} T &= A + B \\ &= \begin{bmatrix} 48 & 18 & 92 & 0 \\ 0 & 0 & 73 & 152 \\ 34 & 203 & 0 & 88 \end{bmatrix} + \begin{bmatrix} 41 & 25 & 15 & 20 \\ 0 & 72 & 0 & 165 \\ 47 & 193 & 0 & 77 \end{bmatrix} \\ &= \begin{bmatrix} 48 + 41 & 18 + 25 & 92 + 15 & 0 + 20 \\ 0 + 0 & 0 + 72 & 73 + 0 & 152 + 165 \\ 34 + 47 & 203 + 193 & 0 + 0 & 88 + 77 \end{bmatrix} \\ &= \begin{bmatrix} 89 & 43 & 107 & 20 \\ 0 & 72 & 73 & 317 \\ 81 & 396 & 0 & 165 \end{bmatrix}. \end{aligned}$$

So to add the two matrices, we added their corresponding entries. This is exactly how we define matrix addition in general:

**Definition 4.1.5 Matrix Addition.**

Let  $A = [a_{ij}]$  and  $B = [b_{ij}]$  be  $m \times n$  matrices. The **sum** of  $A$  and  $B$ , denoted  $A + B$ , is

$$\begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}.$$

For another example, suppose we wanted to know the total production for each farm in 2014, assuming that our matrices represent all of the crops each farm produces. This would be obtained by simply calculating the total for each row in the matrix  $A$ . Another way to accomplish the same task is as follows: let

$$C_1 = \begin{bmatrix} 48 \\ 0 \\ 34 \end{bmatrix}, C_2 = \begin{bmatrix} 18 \\ 0 \\ 203 \end{bmatrix}, C_3 = \begin{bmatrix} 92 \\ 73 \\ 0 \end{bmatrix}, C_4 = \begin{bmatrix} 0 \\ 152 \\ 88 \end{bmatrix}$$

denote the columns of the matrix  $A$ ; note that each column represents the yields across all three farms for each crop.

The total yield for each farm can then be calculated according to

$$Y = C_1 + C_2 + C_3 + C_4$$

$$\begin{bmatrix} 48 \\ 0 \\ 34 \end{bmatrix} + \begin{bmatrix} 18 \\ 0 \\ 203 \end{bmatrix} + \begin{bmatrix} 92 \\ 73 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 152 \\ 88 \end{bmatrix} = \begin{bmatrix} 158 \\ 225 \\ 325 \end{bmatrix}.$$

The column vector  $Y$  then contains the total yields for each farm. Notice that although we only defined the sum of two matrices in Definition 4.1.5, it makes sense to add any number of matrices, and there is no need to add parentheses.

Recall from Chapter 2 that we defined the multiplication of a vector  $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

by a scalar  $t \in \mathbb{R}$  by

$$t\vec{v} = t \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} ta \\ tb \\ tc \end{bmatrix}.$$

It stands to reason that scalar multiplication should be defined in exactly the same way for row and column vectors; after all, in most cases these are just different ways of writing down the same mathematical object. Thus, in order to multiply a row or column vector by a scalar, we should multiply each entry in the vector by that scalar. From here, it's not too much of a stretch to conclude that the same definition is reasonable for matrices in general.



**Definition 4.1.6 Scalar Multiplication.**

Let  $A = [a_{ij}]$  be an  $m \times n$  matrix and let  $k$  be a scalar. The **scalar multiplication**  $A$  by  $k$ , denoted  $kA$ , is defined by

$$\begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}.$$

Referring one last time to our farming data, we could imagine that a fertilizer company is advertising a new product they claim will increase crop yields by 30%. Increasing the yield of each crop by 30% amounts to multiplying each of the entries in the matrices  $A$  or  $B$  by a factor of 1.3; according to Definition 4.1.6, this is the same as forming the scalar multiples  $1.3A$  and  $1.3B$ .

Since we have two years' worth of crop data, we could also ask for the *average* yield for each crop on each farm. For example, Farm A produced 48 tonnes of corn in 2014, and 41 tonnes of corn in 2015. The two-year average for corn on Farm A is thus

$$\frac{48 + 41}{2} = \frac{89}{2} = 44.5 \text{ tonnes.}$$

Notice that we can obtain the average for each entry by dividing each entry in  $T = A + B$  by 2, which is the same thing as multiplying by  $\frac{1}{2}$ . Our matrix of averages is

$$\frac{1}{2}T = \frac{1}{2} \begin{bmatrix} 89 & 43 & 107 & 20 \\ 0 & 72 & 73 & 317 \\ 81 & 396 & 0 & 165 \end{bmatrix} = \begin{bmatrix} 89/2 & 43/2 & 107/2 & 10 \\ 0 & 36 & 73/2 & 317/2 \\ 81/2 & 198 & 0 & 165/2 \end{bmatrix}.$$

Finally, notice also that since

$$\frac{48 + 41}{2} = \frac{1}{2}(48) + \frac{1}{2}(41),$$

with similar considerations for the other entries, we also could have obtained the average by first dividing  $A$  and  $B$  by  $\frac{1}{2}$ , and then adding the result. That is,

$$\frac{1}{2}T = \frac{1}{2}(A + B) = \frac{1}{2}A + \frac{1}{2}B.$$

We'll see shortly that this result is due to a general property of matrix arithmetic, called the *distributive property*.

Expressions such as  $\frac{1}{2}A + \frac{1}{2}B$  that use both addition and scalar multiplication together occur frequently in Linear Algebra, and are known as **linear combinations**. In general, a linear combination can be formed from any number of matrices, as long as they're all of the same size.

**Definition 4.1.7 Linear combination.**

Given  $m \times n$  matrices  $A_1, A_2, \dots, A_k$ , a **linear combination** of these matrices is any expression of the form

$$B = c_1A_1 + c_2A_2 + \cdots + c_kA_k,$$

where  $c_1, c_2, \dots, c_k$  are scalars.

**Note:** In Linear Algebra, a **scalar** is simply a number. For most of this text, the scalars we consider are real numbers, although one can also use complex numbers as scalars. (Complex numbers were introduced in Section 1.4.) As we saw in Chapter 2, multiplying a vector by a scalar *scales* the length of that vector.

It is time to forget our farm data and move into some abstract computational examples to make sure we have the hang of these new operations.

**Example 4.1.8 Matrix addition and scalar multiplication.**

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 5 & 5 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 2 \\ -1 & 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 9 & 8 & 7 \end{bmatrix}.$$

Simplify the following matrix expressions.

- |            |               |
|------------|---------------|
| 1. $A + B$ | 5. $-3A + 2B$ |
| 2. $B + A$ | 6. $A - A$    |
| 3. $A - B$ | 7. $5A + 5B$  |
| 4. $A + C$ | 8. $5(A + B)$ |

**Solution.**

1. Adding the corresponding entries of  $A$  and  $B$ , we have

$$A + B = \begin{bmatrix} 1+2 & 2+4 & 3+6 \\ -1+1 & 2+2 & 1+2 \\ 5-1 & 5+0 & 5+4 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 4 & 3 \\ 4 & 5 & 9 \end{bmatrix}.$$

2. To compute  $B + A$ , we again add the corresponding entries, but in the opposite order:

$$B + A = \begin{bmatrix} 2+1 & 4+2 & 6+3 \\ 1-1 & 2+2 & 2+1 \\ -1+5 & 0+5 & 4+5 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 4 & 3 \\ 4 & 5 & 9 \end{bmatrix}.$$

3. In this case, we have to subtract the entries of  $B$  from those of  $A$ . This gives us

$$A - B = \begin{bmatrix} -1 & -2 & -3 \\ -2 & 0 & -1 \\ 6 & 5 & 1 \end{bmatrix}.$$

4.  $A + C$  is not defined. If we look at our definition of matrix addition, we see that the two matrices need to be the same size. Since  $A$  and  $C$  have different dimensions, we don't even try to create something as an addition; we simply say that the sum is not defined.

5. To compute this linear combination we first carry out the scalar multiplication, followed by the addition:

$$-3A + 2B = -3 \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 5 & 5 & 5 \end{bmatrix} + 2 \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 2 \\ -1 & 0 & 4 \end{bmatrix}$$

$$\begin{aligned}
 &= \begin{bmatrix} -3 & -6 & -9 \\ 3 & -6 & -3 \\ -15 & -15 & -15 \end{bmatrix} + \begin{bmatrix} 4 & 8 & 12 \\ 2 & 4 & 4 \\ -2 & 0 & 8 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 2 & 3 \\ 5 & -2 & 1 \\ -17 & -15 & -7 \end{bmatrix}.
 \end{aligned}$$

6. Subtracting each entry of  $A$  from itself, we get

$$A - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

7. Carrying out the scalar multiplication followed by the addition, we find

$$5A + 5B = \begin{bmatrix} 5 & 10 & 15 \\ -5 & 10 & 5 \\ 25 & 25 & 25 \end{bmatrix} + \begin{bmatrix} 10 & 20 & 30 \\ 5 & 10 & 10 \\ -5 & 0 & 20 \end{bmatrix} = \begin{bmatrix} 15 & 30 & 45 \\ 0 & 20 & 15 \\ 20 & 25 & 45 \end{bmatrix}.$$

8. In this case, we first perform the addition, followed by the scalar multiplication. We obtain:

$$\begin{aligned}
 5 \left( \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 5 & 5 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 2 \\ -1 & 0 & 4 \end{bmatrix} \right) &= 5 \cdot \begin{bmatrix} 3 & 6 & 9 \\ 0 & 4 & 3 \\ 4 & 5 & 9 \end{bmatrix} \\
 &= \begin{bmatrix} 15 & 30 & 45 \\ 0 & 20 & 15 \\ 20 & 25 & 45 \end{bmatrix}.
 \end{aligned}$$

Our example raised a few interesting points. Notice how  $A + B = B + A$ . We probably aren't surprised by this, since we know that when dealing with numbers,  $a + b = b + a$ . Also, notice that  $5A + 5B = 5(A + B)$ . In our example, we were careful to compute each of these expressions following the proper order of operations; knowing these are equal allows us to compute similar expressions in the most convenient way.

Another interesting thing that came from our previous example is that

$$A - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It seems like this should be a special matrix; after all, every entry is 0 and 0 is a special number.

In fact, this is a special matrix. We define  $\mathbf{0}$ , which we read as “the zero matrix,” to be the matrix of all zeros. We should be careful; this previous “definition” is a bit ambiguous, for we have not stated what size the zero matrix should be. Is  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  the zero matrix? How about  $\begin{bmatrix} 0 & 0 \end{bmatrix}$ ?

Let's not get bogged down in semantics. If we ever see  $\mathbf{0}$  in an expression, we will usually know right away what size  $\mathbf{0}$  should be; it will be the size that allows the expression to make sense. If  $A$  is a  $3 \times 5$  matrix, and we write  $A + \mathbf{0}$ ,

we'll simply assume that  $\mathbf{0}$  is also a  $3 \times 5$  matrix. If we are ever in doubt, we can add a subscript; for instance,  $\mathbf{0}_{2 \times 7}$  is the  $2 \times 7$  matrix of all zeros.

Since the zero matrix is an important concept, we give it its own definition box.

**Definition 4.1.9 The Zero Matrix.**

The  $m \times n$  matrix of all zeros, denoted  $\mathbf{0}_{m \times n}$ , is the **zero matrix**.

When the dimensions of the zero matrix are clear from the context, the subscript is generally omitted.

We use the bold face to distinguish the zero matrix,  $\mathbf{0}$ , from the number zero, 0.

The following presents some of the properties of matrix addition and scalar multiplication that we discovered above, plus a few more.

**Theorem 4.1.10 Properties of Matrix Addition and Scalar Multiplication.**

The following equalities hold for all  $m \times n$  matrices  $A$ ,  $B$  and  $C$  and scalars  $k$ .

1.  $A + B = B + A$  (Commutative Property)
2.  $(A + B) + C = A + (B + C)$  (Associative Property)
3.  $k(A + B) = kA + kB$  (Scalar Multiplication Distributive Property)
4.  $kA = Ak$
5.  $A + \mathbf{0} = \mathbf{0} + A = A$  (Additive Identity)
6.  $0A = \mathbf{0}$

Be sure that this last property makes sense; it says that if we multiply any matrix by the *number* 0, the result is the *zero matrix*, or  $\mathbf{0}$ . (You now have more than one kind of zero to keep track of!)

It's important to understand that since matrix addition and scalar multiplication are defined in terms of the entries of our matrices, the properties in Theorem 4.1.10 follow directly from the properties of real number arithmetic in Section 1.2. For example to prove item 1 above, let  $A = [a_{ij}]$  and  $B = [b_{ij}]$  be  $m \times n$  matrices. We then have

$$\begin{aligned}
 A + B &= [a_{ij}] + [b_{ij}] \\
 &= [a_{ij} + b_{ij}] \quad (\text{by Definition 4.1.5}) \\
 &= [b_{ij} + a_{ij}] \quad (\text{since addition of real numbers commutes}) \\
 &= [b_{ij}] + [a_{ij}] \quad (\text{by Definition 4.1.5}) \\
 &= B + A.
 \end{aligned}$$

Similarly, the distributive property in item 3 is valid since

$$\begin{aligned}
 k(A + B) &= k([a_{ij}] + [b_{ij}]) \\
 &= k[a_{ij} + b_{ij}] \quad (\text{definition of matrix addition}) \\
 &= [k(a_{ij} + b_{ij})] \quad (\text{definition of scalar multiplication}) \\
 &= [k \cdot a_{ij} + k \cdot b_{ij}] \quad (\text{distributive property of real numbers}) \\
 &= [k \cdot a_{ij}] + [k \cdot b_{ij}] \quad (\text{definition of matrix addition}) \\
 &= k[a_{ij}] + k[b_{ij}] \quad (\text{definition of scalar multiplication})
 \end{aligned}$$

$$= kA + kB.$$

The verification of the remaining properties in Theorem 4.1.10 is similar, and left as an exercise for the reader.

We began this section with the concept of matrix equality. Let's put our matrix addition properties to use and solve a matrix equation.

**Example 4.1.11 Solving a matrix equation.**

Let

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 6 \end{bmatrix}.$$

Find the matrix  $X$  such that

$$2A + 3X = -4A.$$

**Solution.** We can use basic algebra techniques to manipulate this equation for  $X$  first, let's subtract  $2A$  from both sides. This gives us

$$3X = -6A.$$

Now divide both sides by 3 to get

$$X = -2A.$$

Now we just need to compute  $-2A$ ; we find that

$$X = \begin{bmatrix} -4 & 2 \\ -6 & -12 \end{bmatrix}.$$

Our matrix properties identified  $\mathbf{0}$  as the Additive Identity; i.e., if you add  $\mathbf{0}$  to any matrix  $A$ , you simply get  $A$ . This is similar in notion to the fact that for all numbers  $a$ ,  $a + 0 = a$ . A *Multiplicative Identity* would be a matrix  $I$  where  $I \times A = A$  for all matrices  $A$ . (What would such a matrix look like? A matrix of all 1s, perhaps?) However, in order for this to make sense, we'll need to learn to multiply matrices together, which we'll do in the next section.

### 4.1.1 Exercises

**Exercise Group.** For the matrices  $A$  and  $B$  given below, simplify the given expression.

$$A = \begin{bmatrix} 1 & -1 \\ 7 & 4 \end{bmatrix} \quad B = \begin{bmatrix} -3 & 2 \\ 5 & 9 \end{bmatrix}.$$

- |                   |                          |
|-------------------|--------------------------|
| 1. $A + B$        | 2. $2A - 3B$             |
| 3. $3A - A$       | 4. $4B - 2A$             |
| 5. $3(A - B) + B$ | 6. $2(A - B) - (A - 3B)$ |

**Exercise Group.** For the matrices  $A$  and  $B$  given below, simplify the given expression.

$$A = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad B = \begin{bmatrix} -2 \\ 4 \end{bmatrix}.$$

- |               |                    |
|---------------|--------------------|
| 7. $4B - 2A$  | 8. $-2A + 3A$      |
| 9. $-2A - 3A$ | 10. $-B + 3B - 2B$ |

**Exercise Group.** For the matrices  $A$  and  $B$  given below, find  $X$  that satisfies the given equation.

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 7 \\ 3 & -4 \end{bmatrix}$$

- |                     |                             |
|---------------------|-----------------------------|
| 11. $2A + X = B$    | 12. $A - X = 3B$            |
| 13. $3A + 2X = -1B$ | 14. $A - \frac{1}{2}X = -B$ |

**Exercise Group.** Find values for the scalars  $a$  and  $b$  that satisfy the given equation.

- |   |   |
|---|---|
| 15. $a \begin{bmatrix} 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$               | 16. $a \begin{bmatrix} -3 \\ 1 \end{bmatrix} + b \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$                 |
| 17. $a \begin{bmatrix} 4 \\ -2 \end{bmatrix} + b \begin{bmatrix} -6 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$            | 18. $a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$                 |
| 19. $a \begin{bmatrix} 1 \\ 3 \end{bmatrix} + b \begin{bmatrix} -3 \\ -9 \end{bmatrix} = \begin{bmatrix} 4 \\ -12 \end{bmatrix}$            | 20. $a \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$ |
| 21. $a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$ |   |

22. Complete the proof of Theorem 4.1.10.

## 4.2 Matrix Multiplication

In the previous section we found that the definition of matrix addition was very intuitive, and we ended that section discussing the fact that eventually we'd like to know what it means to multiply matrices together.

In the spirit of the last section, take another wild stab: what do you think

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$$

means?

You are likely to have guessed

$$\begin{bmatrix} 1 & -2 \\ 6 & 8 \end{bmatrix}$$

but this is, in fact, *not* right. (You *could* define multiplication this way; you'll even find that it satisfies plenty of nice properties. Unfortunately, nice properties don't make up for the fact that this definition just isn't useful.) The actual answer is

$$\begin{bmatrix} 5 & 3 \\ 11 & 5 \end{bmatrix}.$$

If you can look at this one example and suddenly understand exactly how matrix multiplication works, then you are probably smarter than the author. While matrix multiplication isn't hard, it isn't nearly as intuitive as matrix addition is.

To further muddy the waters (before we clear them), consider

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & -1 \end{bmatrix}.$$

Our experience from the last section would lend us to believe that this is not defined, but our confidence is probably a bit shaken by now. In fact, this multiplication *is* defined, and it is

$$\begin{bmatrix} 5 & 3 & -2 \\ 11 & 5 & -4 \end{bmatrix}.$$

You may see some similarity in this answer to what we got before, but again, probably not enough to really figure things out.

Before diving in to the general definition of matrix multiplication, let's start simple, with row and column vectors. Recall from Definition 4.1.3 in Section 4.1 that a **row vector** is a  $1 \times n$  matrix of the form  $\vec{a} = [a_1 \ a_2 \ \cdots \ a_n]$ , and a

**column vector** is an  $m \times 1$  matrix of the form  $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ .

### Definition 4.2.1 Multiplying a row vector by a column vector.

Let  $\vec{u}$  be an  $1 \times n$  row vector with entries  $u_1, u_2, \dots, u_n$  and let  $\vec{v}$  be an  $n \times 1$  column vector with entries  $v_1, v_2, \dots, v_n$ . The **product of  $\vec{u}$  and  $\vec{v}$** , denoted  $\vec{u} \cdot \vec{v}$  or  $\vec{u}\vec{v}$ , is

$$\vec{u}\vec{v} = \sum_{i=1}^n u_i v_i = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

In this text, row vectors are only used in this section when we discuss matrix multiplication, whereas we'll make extensive use of column vectors. Other texts make great use of row vectors, but little use of column vectors. It is a matter of preference and tradition: "most" texts use column vectors more. In some more advanced textbooks, row vectors are considered to be "dual" to column vectors. Abstractly, a **dual vector** is an object that eats a vector and spits out a number. Here, we see that the way a row vector eats a column vector and produces a number is via multiplication.

Notice that this is essentially the same as the definition of the **dot product** given at the beginning of Section 2.7. There are two key points to notice about the product defined in Definition 4.2.1:

1. In order for the product  $\vec{u}\vec{v}$  to be defined,  $\vec{u}$  and  $\vec{v}$  need to have the *same number of entries*.
2. To multiply  $\vec{u}$  and  $\vec{v}$ , we *multiply the corresponding entries*, and then **add up** the resulting values.

**Example 4.2.2 Multiplying row and column vectors.**

Let

$$\vec{u} = [1 \quad 2 \quad 3], \vec{v} = [2 \quad 0 \quad 1 \quad -1], \vec{x} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}, \vec{y} = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 0 \end{bmatrix}.$$

Find the following products.

- |                     |                     |                     |
|---------------------|---------------------|---------------------|
| 1. $\vec{u}\vec{x}$ | 3. $\vec{u}\vec{y}$ | 5. $\vec{x}\vec{u}$ |
| 2. $\vec{v}\vec{y}$ | 4. $\vec{u}\vec{v}$ |                     |

**Solution.**

$$1. \vec{u}\vec{x} = [1 \quad 2 \quad 3] \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix} = 1(-2) + 2(4) + 3(3) = 15$$

$$2. \vec{v}\vec{y} = [2 \quad 0 \quad 1 \quad -1] \begin{bmatrix} 1 \\ 2 \\ 5 \\ 0 \end{bmatrix} = 2(1) + 0(2) + 1(5) - 1(0) = 7$$

3.  $\vec{u}\vec{y}$  is not defined; Definition 4.2.1 specifies that in order to multiply a row vector and column vector, they must have the same number of entries.
4.  $\vec{u}\vec{v}$  is not defined; we only know how to multiply row vectors by column vectors. We haven't defined how to multiply two row vectors (in general, it can't be done).
5. The product  $\vec{x}\vec{u}$  is defined, but we don't know how to do it yet. Right now, we only know how to multiply a row vector times a column vector; we don't know how to multiply a column vector times a row vector. (That's right:  $\vec{u}\vec{x} \neq \vec{x}\vec{u}$ !)

Now that we understand how to multiply a row vector by a column vector, we are ready to define matrix multiplication.



**Definition 4.2.3 Matrix Multiplication.**

Let  $A$  be an  $m \times r$  matrix, and let  $B$  be an  $r \times n$  matrix. The **matrix product** of  $A$  and  $B$ , denoted  $A \cdot B$ , or simply  $AB$ , is the  $m \times n$  matrix  $M$  whose entry in the  $i$ th row and  $j$ th column is the product of the  $i$ th row of  $A$  and the  $j$ th column of  $B$ .

It may help to illustrate it in this way. Let matrix  $A$  have rows  $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_m$  and let  $B$  have columns  $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n$ . Thus  $A$  looks like

$$\begin{bmatrix} - & \vec{a}_1 & - \\ - & \vec{a}_2 & - \\ & \vdots & \\ - & \vec{a}_m & - \end{bmatrix},$$

where the “—” symbols just serve as reminders that the  $\vec{a}_i$  represent rows, and  $B$  looks like

$$\begin{bmatrix} | & | & \dots & | \\ \vec{b}_1 & \vec{b}_2 & \dots & \vec{b}_n \\ | & | & & | \end{bmatrix},$$

where again, the “|” symbols just remind us that the  $\vec{b}_i$  represent column vectors. Then

$$AB = \begin{bmatrix} \vec{a}_1 \vec{b}_1 & \vec{a}_1 \vec{b}_2 & \dots & \vec{a}_1 \vec{b}_n \\ \vec{a}_2 \vec{b}_1 & \vec{a}_2 \vec{b}_2 & \dots & \vec{a}_2 \vec{b}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{a}_m \vec{b}_1 & \vec{a}_m \vec{b}_2 & \dots & \vec{a}_m \vec{b}_n \end{bmatrix}.$$

Two quick notes about this definition. First, notice that in order to multiply  $A$  and  $B$ , the number of *columns* of  $A$  must be the same as the number of *rows* of  $B$  (we refer to these as the “inner dimensions”). Secondly, the resulting matrix has the same number of *rows* as  $A$  and the same number of *columns* as  $B$  (we refer to these as the “outer dimensions”).

$$\begin{array}{c} \text{final dimensions are the outer dimensions} \\ \underbrace{(m \times r) \times (r \times n)} \\ \text{these inner dimensions must match} \end{array}$$

Of course, this will make much more sense when we see an example.

**Example 4.2.4 A more general matrix product.**

Revisit the matrix product we saw at the beginning of this section; multiply

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & -1 \end{bmatrix}.$$

**Solution.** Let’s call our first matrix  $A$  and the second  $B$ . We should first check to see that we can actually perform this multiplication. Matrix  $A$  is  $2 \times 2$  and  $B$  is  $2 \times 3$ . The “inner” dimensions match up, so we can compute the product; the “outer” dimensions tell us that the product will be  $2 \times 3$ . Let

$$AB = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{bmatrix}.$$

Let's find the value of each of the entries.

The entry  $m_{11}$  is in the first row and first column; therefore to find its value, we need to multiply the first row of  $A$  by the first column of  $B$ . Thus

$$m_{11} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1(1) + 2(2) = 5.$$

So now we know that

$$AB = \begin{bmatrix} 5 & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{bmatrix}.$$

Finishing out the first row, we have

$$m_{12} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 1(-1) + 2(2) = 3$$

using the first row of  $A$  and the second column of  $B$ , and

$$m_{13} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 1(0) + 2(-1) = -2$$

using the first row of  $A$  and the third column of  $B$ . Thus we have

$$AB = \begin{bmatrix} 5 & 3 & -2 \\ m_{21} & m_{22} & m_{23} \end{bmatrix}.$$

To compute the second row of  $AB$ , we multiply with the second row of  $A$ . We find

$$m_{21} = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 11,$$

$$m_{22} = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 5, \text{ and}$$

$$m_{23} = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -4.$$

Thus

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 3 & -2 \\ 11 & 5 & -4 \end{bmatrix}.$$

#### Example 4.2.5 Multiplying matrices.

Multiply

$$\begin{bmatrix} 1 & -1 \\ 5 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 6 & 7 & 9 \end{bmatrix}.$$

**Solution.** Let's first check to make sure this product is defined. Again calling the first matrix  $A$  and the second  $B$ , we see that  $A$  is a  $3 \times 2$  matrix and  $B$  is a  $2 \times 4$  matrix; the inner dimensions match so the product is

defined, and the product will be a  $3 \times 4$  matrix,

$$AB = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix}.$$

We will demonstrate how to compute some of the entries, then give the final answer. The reader can fill in the details of how each entry was computed.

$$m_{11} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -1$$

$$m_{13} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix} = -6$$

$$m_{23} = \begin{bmatrix} 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix} = 19$$

$$m_{24} = \begin{bmatrix} 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 9 \end{bmatrix} = 23$$

$$m_{32} = \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \end{bmatrix} = 16$$

$$m_{34} = \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 9 \end{bmatrix} = 25.$$

So far, we've computed this much of  $AB$ :

$$AB = \begin{bmatrix} -1 & m_{12} & -6 & m_{14} \\ m_{21} & m_{22} & 19 & 23 \\ m_{31} & 16 & m_{33} & 25 \end{bmatrix}.$$

The final product is

$$AB = \begin{bmatrix} -1 & -5 & -6 & -8 \\ 9 & 17 & 19 & 23 \\ 4 & 16 & 19 & 25 \end{bmatrix}.$$

#### Example 4.2.6 An undefined product.

Multiply, if possible,

$$\begin{bmatrix} 2 & 3 & 4 \\ 9 & 8 & 7 \end{bmatrix} \begin{bmatrix} 3 & 6 \\ 5 & -1 \end{bmatrix}.$$

**Solution.** Again, we'll call the first matrix  $A$  and the second  $B$ . Checking the dimensions of each matrix, we see that  $A$  is a  $2 \times 3$  matrix, whereas  $B$  is a  $2 \times 2$  matrix. The inner dimensions do not match, therefore this multiplication is not defined.

**Example 4.2.7 A vector product revisited.**

In Example 4.2.2, we were told that the product  $\vec{x}\vec{u}$  was defined, where

$$\vec{x} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix} \quad \text{and} \quad \vec{u} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix},$$

although we were not shown what that product was. Find  $\vec{x}\vec{u}$ .

**Solution.** Again, we need to check to make sure the dimensions work correctly (remember that even though we are referring to  $\vec{u}$  and  $\vec{x}$  as vectors, they are, in fact, just matrices).

The column vector  $\vec{x}$  has dimensions  $3 \times 1$ , whereas the row vector  $\vec{u}$  has dimensions  $1 \times 3$ . Since the inner dimensions match, the matrix product is defined; the outer dimensions tell us that the product will be a  $3 \times 3$  matrix, as shown below:

$$\vec{x}\vec{u} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}.$$

To compute the entry  $m_{11}$ , we multiply the first row of  $\vec{x}$  by the first column of  $\vec{u}$ . What is the first row of  $\vec{x}$ ? Simply the number  $-2$ . What is the first column of  $\vec{u}$ ? Just the number  $1$ . Thus  $m_{11} = -2$ . (This does seem odd, but through checking, you can see that we are indeed following the rules.)

What about the entry  $m_{12}$ ? Again, we multiply the first row of  $\vec{x}$  by the first column of  $\vec{u}$ ; that is, we multiply  $-2(2)$ . So  $m_{12} = -4$ .

What about  $m_{23}$ ? Multiply the second row of  $\vec{x}$  by the third column of  $\vec{u}$ ; multiply  $4(3)$ , so  $m_{23} = 12$ .

One final example:  $m_{31}$  comes from multiplying the third row of  $\vec{x}$ , which is  $3$ , by the first column of  $\vec{u}$ , which is  $1$ . Therefore  $m_{31} = 3$ .

So far we have computed

$$\vec{x}\vec{u} = \begin{bmatrix} -2 & -4 & m_{13} \\ m_{21} & m_{22} & 12 \\ 3 & m_{32} & m_{33} \end{bmatrix}.$$

After performing all 9 multiplications, we find

$$\vec{x}\vec{u} = \begin{bmatrix} -2 & -4 & -6 \\ 4 & 8 & 12 \\ 3 & 6 & 9 \end{bmatrix}.$$

In this last example, we saw a “nonstandard” multiplication (at least, it felt nonstandard). Studying the entries of this matrix, it seems that there are several different patterns that can be seen amongst the entries. (Remember that mathematicians like to look for patterns. Also remember that we often guess wrong at first; don’t be scared and try to identify some patterns.)

In Section 4.1, we identified the zero matrix  $\mathbf{0}$  that had a nice property in relation to matrix addition (i.e.,  $A + \mathbf{0} = A$  for any matrix  $A$ ). In the following example we’ll identify a matrix that works well with multiplication as well as some multiplicative properties. For instance, we’ve learned how  $1 \cdot A = A$ ; is there a *matrix* that acts like the number  $1$ ? That is, can we find a matrix  $X$  where  $X \cdot A = A$ ? (We made a guess in Section 4.1 that maybe a matrix of all  $1$ s would

work, but you can probably already see that this guess is doomed to failure.)

**Example 4.2.8 Computing matrix products.**

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -7 & 5 \\ -2 & -8 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find the following products.

1.  $AB$
2.  $BA$
3.  $A\mathbf{0}_{3 \times 4}$
4.  $AI$
5.  $IA$
6.  $I^2$
7.  $BC$
8.  $B^2$

**Solution.** We will find each product, but we leave the details of each computation to the reader.

$$1. AB = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -7 & 5 \\ -2 & -8 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 6 \\ 0 & 0 & 0 \\ -7 & -7 & -7 \end{bmatrix}$$

$$2. BA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -7 & 5 \\ -2 & -8 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -13 & 11 \\ 1 & -13 & 11 \\ 1 & -13 & 11 \end{bmatrix}$$

$$3. A\mathbf{0}_{3 \times 4} = \mathbf{0}_{3 \times 4}.$$

$$4. AI = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -7 & 5 \\ -2 & -8 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -7 & 5 \\ -2 & -8 & 3 \end{bmatrix}$$

$$5. IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -7 & 5 \\ -2 & -8 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -7 & 5 \\ -2 & -8 & 3 \end{bmatrix}$$

6. We haven't formally defined what  $I^2$  means, but we could probably make the reasonable guess that  $I^2 = I \cdot I$ . Thus

$$I^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$7. BC = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

$$8. B^2 = BB = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

This example is simply chock full of interesting ideas; it is almost hard to think about where to start.

- Interesting Idea #1** Notice that in our example,  $AB \neq BA$ ! When dealing with numbers, we were used to the idea that  $ab = ba$ . With matrices, multiplication is *not* commutative. (Of course, we can find special situations where it does work. In general, though, it doesn't.)
- Interesting Idea #2** Right before this example we wondered if there was a matrix that “acted like the number 1,” and guessed it may be a matrix of all 1s. However, we found out that such a matrix does not work in that way; in our example,  $AB \neq A$ . We did find that  $AI = IA = A$ . There is a Multiplicative Identity; it just isn't what we thought it would be. And just as  $1^2 = 1$ ,  $I^2 = I$ .
- Interesting Idea #3** When dealing with numbers, we are very familiar with the notion that “If  $ax = bx$ , then  $a = b$ .” (As long as  $x \neq 0$ .) Notice that, in our example,  $BB = BC$ , yet  $B \neq C$ . In general, just because  $AX = BX$ , we *cannot* conclude that  $A = B$ .

Matrix multiplication is turning out to be a very strange operation. We are very used to multiplying numbers, and we know a bunch of properties that hold when using this type of multiplication. When multiplying matrices, though, we probably find ourselves asking two questions, “What *does* work?” and “What *doesn't* work?” We'll answer these questions; first we'll do an example that demonstrates some of the things that do work.

#### Example 4.2.9 Exploring properties of matrix multiplication.

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Find the following:

1.  $A(B + C)$
2.  $AB + AC$
3.  $A(BC)$
4.  $(AB)C$

**Solution.** We'll compute each of these without showing all the intermediate steps. Keep in mind order of operations: things that appear inside of parentheses are computed first.

1.

$$\begin{aligned} A(B + C) &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \left( \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 4 \\ 17 & 10 \end{bmatrix} \end{aligned}$$

2.

$$\begin{aligned}
 AB + AC &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\
 &= \begin{bmatrix} 3 & -1 \\ 7 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 10 & 11 \end{bmatrix} \\
 &= \begin{bmatrix} 7 & 4 \\ 17 & 10 \end{bmatrix}
 \end{aligned}$$

3.

$$\begin{aligned}
 A(BC) &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \left( \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) \\
 &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 1 & -1 \end{bmatrix} \\
 &= \begin{bmatrix} 5 & 1 \\ 13 & 5 \end{bmatrix}
 \end{aligned}$$

4.

$$\begin{aligned}
 (AB)C &= \left( \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right) \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\
 &= \begin{bmatrix} 3 & -1 \\ 7 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\
 &= \begin{bmatrix} 5 & 1 \\ 13 & 5 \end{bmatrix}
 \end{aligned}$$

In looking at our example, we should notice two things. First, it looks like the “distributive property” holds; that is,  $A(B + C) = AB + AC$ . This is nice as many algebraic techniques we have learned about in the past (when doing “ordinary algebra”) will still work. Secondly, it looks like the “associative property” holds; that is,  $A(BC) = (AB)C$ . This is nice, for it tells us that when we are multiplying several matrices together, we don’t have to be particularly careful in what order we multiply certain pairs of matrices together.

In leading to an important theorem, let’s define a matrix we saw in an earlier example.

**Definition 4.2.10 Identity Matrix.**

The  $n \times n$  matrix with 1s on the diagonal and zeros elsewhere is the  $n \times n$  **identity matrix**, denoted  $I_n$ . When the context makes the dimension of the identity clear, the subscript is generally omitted.

Definition 4.2.10 uses a term we won’t define until Definition 6.1.4: *diagonal*. In short, a “diagonal matrix” is one in which the only nonzero entries are the “diagonal entries.” The examples given here and in the exercises should suffice until we meet the full definition later.

Note that while the zero matrix can come in all different shapes and sizes, the identity matrix is always a square matrix. We show a few identity matrices below.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In our examples above, we have seen examples of things that do and do not work. We should be careful about what examples *prove*, though. If someone were to claim that  $AB = BA$  is always true, one would only need to show them one example where they were false, and we would know the person was wrong. However, if someone claims that  $A(B + C) = AB + AC$  is always true, we can't prove this with just one example. We need something more powerful; we need a true proof.

In this text, we forgo most proofs. The reader should know, though, that when we state something in a theorem, there is a proof that backs up what we state. Our justification comes from something stronger than just examples.

Now we give the good news of what does work when dealing with matrix multiplication.

**Theorem 4.2.11 Properties of Matrix Multiplication.**

*Let  $A$ ,  $B$  and  $C$  be matrices whose sizes are such that the following operations make sense, and let  $k$  be a scalar. The following equalities hold:*

1.  $A(BC) = (AB)C$  (Associative Property)
2.  $A(B + C) = AB + AC$  and  $(B + C)A = BA + CA$  (Distributive Property)
3.  $k(AB) = (kA)B = A(kB)$
4.  $AI = IA = A$

The above box contains some very good news, and probably some very surprising news. Matrix multiplication probably seems to us like a very odd operation, so we probably wouldn't have been surprised if we were told that  $A(BC) \neq (AB)C$ . It is a very nice thing that the Associative Property does hold.

As we near the end of this section, we raise one more issue of notation. We define  $A^0 = I$ . If  $n$  is a positive integer, we define

$$A^n = \underbrace{A \cdot A \cdot \cdots \cdot A}_{n \text{ times}}.$$

With numbers, we are used to  $a^{-n} = \frac{1}{a^n}$ . Do negative exponents work with matrices, too? The answer is yes, sort of. We'll have to be careful, and we'll cover the topic in detail once we define the inverse of a matrix. For now, though, we recognize the fact that  $A^{-1} \neq \frac{1}{A}$ , for  $\frac{1}{A}$  makes no sense; we don't know how to "divide" by a matrix.

We end this section with a reminder of some of the things that do not work with matrix multiplication. The good news is that there are really only two things on this list.

1. Matrix multiplication is not commutative; that is,  $AB \neq BA$ .
2. In general, just because  $AX = BX$ , we cannot conclude that  $A = B$ .

The bad news is that these ideas pop up in many places where we don't expect them. For instance, we are used to

$$(a + b)^2 = a^2 + 2ab + b^2.$$

What about  $(A + B)^2$ ? All we'll say here is that

$$(A + B)^2 \neq A^2 + 2AB + B^2;$$

we leave it to the reader to figure out why.

With numbers, we can make sense of the expression  $\frac{a}{b}$ , because  $ab^{-1} = b^{-1}a$ . But the expression  $\frac{A}{B}$  makes no sense for matrices, even once we know how to define  $B^{-1}$ , because in general,  $B^{-1}A \neq AB^{-1}$ , and we don't know which of these two is meant by  $\frac{A}{B}$ .



## 4.2.1 Exercises

**Exercise Group.** Row and column vectors  $\vec{u}$  and  $\vec{v}$  are given. Find the product  $\vec{u}\vec{v}$ , where possible.

1.  $\vec{u} = [1 \quad -4] \quad \vec{v} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

2.  $\vec{u} = [2 \quad 3] \quad \vec{v} = [74]$

3.  $\vec{u} = [1 \quad -1] \quad \vec{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

4.  $\vec{u} = [0.6 \quad 0.8] \quad \vec{v} = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$

5.  $\vec{u} = [1 \quad 2 \quad -1] \quad \vec{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

6.  $\vec{u} = [3 \quad 2 \quad -2] \quad \vec{v} = \begin{bmatrix} -1 \\ 0 \\ 9 \end{bmatrix}$

7.  $\vec{u} = [8 \quad -4 \quad 3] \quad \vec{v} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$

8.  $\vec{u} = [-3 \quad 6 \quad 1] \quad \vec{v} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$

9.  $\vec{u} = [1 \quad 2 \quad 3 \quad 4] \quad \vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

10.  $\vec{u} = [6 \quad 2 \quad -1 \quad 2] \quad \vec{v} = \begin{bmatrix} 3 \\ 2 \\ 9 \\ 5 \end{bmatrix}$

11.  $\vec{u} = [1 \quad 2 \quad 3] \quad \vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

12.  $\vec{u} = [2 \quad -5] \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

**Exercise Group.** Matrices  $A$  and  $B$  are defined.

(a) Give the dimensions of  $A$  and  $B$ . If the dimensions properly match, give the dimensions of  $AB$  and  $BA$ .

(b) Find the products  $AB$  and  $BA$ , if possible.

13.  $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}$

14.  $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$

15.  $A = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 7 \\ 4 & 2 & 9 \end{bmatrix}$

16.  $A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -2 & -4 \end{bmatrix}, B = \begin{bmatrix} -2 & 0 \\ 3 & 8 \end{bmatrix}$

17.  $A = \begin{bmatrix} 9 & 4 & 3 \\ 9 & -5 & 9 \end{bmatrix}, B = \begin{bmatrix} -2 & 5 \\ -2 & -1 \end{bmatrix}$

18.  $A = \begin{bmatrix} -2 & -1 \\ 9 & -5 \\ 3 & -1 \end{bmatrix}, B = \begin{bmatrix} -5 & 6 & -4 \\ 0 & 6 & -3 \end{bmatrix}$

19.  $A = \begin{bmatrix} 2 & 6 \\ 6 & 2 \\ 5 & -1 \end{bmatrix}, B = \begin{bmatrix} -4 & 5 & 0 \\ -4 & 4 & -4 \end{bmatrix}$

20.  $A = \begin{bmatrix} -5 & 2 \\ -5 & -2 \\ -5 & -4 \end{bmatrix}, B = \begin{bmatrix} 0 & -5 & 6 \\ -5 & -3 & -1 \end{bmatrix}$

21.  $A = \begin{bmatrix} 8 & -2 \\ 4 & 5 \\ 2 & -5 \end{bmatrix}, B = \begin{bmatrix} -5 & 1 & -5 \\ 8 & 3 & -2 \end{bmatrix}$

22.  $A = \begin{bmatrix} 1 & 4 \\ 7 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & -5 & 5 \\ -2 & 1 & 3 & -5 \end{bmatrix}$

23.  $A = \begin{bmatrix} -1 & 5 \\ 6 & 7 \end{bmatrix}, B = \begin{bmatrix} 5 & -3 & -4 & -4 \\ -2 & -5 & -5 & -1 \end{bmatrix}$

24.  $A = \begin{bmatrix} -1 & 2 & 1 \\ -1 & 2 & -1 \\ 0 & 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix}$

25.  $A = \begin{bmatrix} -1 & 1 & 1 \\ -1 & -1 & -2 \\ 1 & 1 & -2 \end{bmatrix}, B = \begin{bmatrix} -2 & -2 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 2 \end{bmatrix}$

26.  $A = \begin{bmatrix} -4 & 3 & 3 \\ -5 & -1 & -5 \\ -5 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 5 & 0 \\ -5 & -4 & 3 \\ 5 & -4 & 3 \end{bmatrix}$

$$27. \quad A = \begin{bmatrix} -4 & -1 & 3 \\ 2 & -3 & 5 \\ 1 & 5 & 3 \end{bmatrix}, B = \begin{bmatrix} -2 & 4 & 3 \\ -1 & 1 & -1 \\ 4 & 0 & 2 \end{bmatrix}$$

**Exercise Group.** A diagonal matrix  $D$  and a matrix  $A$  are given. Find the products  $DA$  and  $AD$ , where possible.

$$28. \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$29. \quad D = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -3 & -3 & -3 \end{bmatrix}, A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$30. \quad D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}, A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

$$31. \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$32. \quad D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$33. \quad D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}, A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

**Exercise Group.** A matrix  $A$  and a vector  $\vec{x}$  are given. Find the product  $A\vec{x}$ .

$$34. \quad A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}, \vec{x} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$$

$$35. \quad A = \begin{bmatrix} -1 & 4 \\ 7 & 3 \end{bmatrix}, \vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$36. \quad A = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 1 & 1 \\ 3 & -1 & 2 \end{bmatrix}, \vec{x} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

$$37. \quad A = \begin{bmatrix} -2 & 0 & 3 \\ 1 & 1 & -2 \\ 4 & 2 & -1 \end{bmatrix}, \vec{x} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

$$38. \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$39. \quad A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$40. \quad \text{Let } A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \text{ Find } A^2 \text{ and } A^3.$$

$$41. \quad \text{Let } A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}. \text{ Find } A^2 \text{ and } A^3.$$

$$42. \quad \text{Let } A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}. \text{ Find } A^2 \text{ and } A^3.$$

$$43. \quad \text{Let } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \text{ Find } A^2 \text{ and } A^3.$$

$$44. \quad \text{Let } A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \text{ Find } A^2 \text{ and } A^3.$$

45. In the text we state that  $(A + B)^2 \neq A^2 + 2AB + B^2$ . We investigate that claim here.

(a) Let  $A = \begin{bmatrix} 5 & 3 \\ -3 & -2 \end{bmatrix}$  and let  $B = \begin{bmatrix} -5 & -5 & -2 & 1 \end{bmatrix}$ . Compute  $A + B$ .

(b) Find  $(A + B)^2$  by using your answer from (a).

(c) Compute  $A^2 + 2AB + B^2$ .

(d) Are the results from (a) and (b) the same?

(e) Carefully expand the expression

$$(A + B)^2 = (A + B)(A + B)$$

and show why this is not equal to  $A^2 + 2AB + B^2$ .

### 4.3 Solving Matrix Equations $AX = B$

We concluded the last chapter with a discussion about solving numerical equations like  $ax = b$  for  $x$ . We have seen how to solve equations of the form  $A\vec{x} = \vec{b}$  by identifying them as systems of linear equations. In this section we will learn how to solve the general matrix equation  $AX = B$  for  $X$ .

We will start by considering the best case scenario when solving  $A\vec{x} = \vec{b}$ ; that is, when  $A$  is square and we have exactly one solution. For instance, suppose we want to solve  $A\vec{x} = \vec{b}$  where

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We know how to solve this; put the appropriate matrix into reduced row echelon form and interpret the result.

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

We read from this that

$$\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Written in a more general form, we found our solution by forming the augmented matrix

$$\begin{bmatrix} A & \vec{b} \end{bmatrix}$$

and interpreting its reduced row echelon form:

$$\begin{bmatrix} A & \vec{b} \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} I & \vec{x} \end{bmatrix}.$$

Notice that when the reduced row echelon form of  $A$  is the identity matrix  $I$  we have exactly one solution. This, again, is the best case scenario.

We apply the same general technique to solving the matrix equation  $AX = B$  for  $X$ . We'll assume that  $A$  is a square matrix ( $B$  need not be) and we'll form the augmented matrix

$$\begin{bmatrix} A & B \end{bmatrix}.$$

Putting this matrix into reduced row echelon form will give us  $X$ , much like we found  $\vec{x}$  before.

$$\begin{bmatrix} A & B \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} I & X \end{bmatrix}.$$

As long as the reduced row echelon form of  $A$  is the identity matrix, this technique works great. After a few examples, we'll discuss why this technique works, and we'll also talk just a little bit about what happens when the reduced row echelon form of  $A$  is not the identity matrix.

First, some examples.

#### Example 4.3.1 Solving a matrix equation.

Solve the matrix equation  $AX = B$  where

$$A = \begin{bmatrix} 1 & -1 \\ 5 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -8 & -13 & 1 \\ 32 & -17 & 21 \end{bmatrix}.$$

**Solution.** To solve  $AX = B$  for  $X$ , we form the proper augmented

matrix, put it into reduced row echelon form, and interpret the result.

$$\begin{bmatrix} 1 & -1 & -8 & -13 & 1 \\ 5 & 3 & 32 & -17 & 21 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1 & -7 & 3 \\ 0 & 1 & 9 & 6 & 2 \end{bmatrix}$$

We read from the reduced row echelon form of the matrix that

$$X = \begin{bmatrix} 1 & -7 & 3 \\ 9 & 6 & 2 \end{bmatrix}.$$

We can easily check to see if our answer is correct by multiplying  $AX$ .

#### Example 4.3.2 Another matrix equation.

Solve the matrix equation  $AX = B$  where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 2 \\ 2 & -6 \\ 2 & -4 \end{bmatrix}.$$

**Solution.** To solve, let's again form the augmented matrix

$$[A \ B],$$

put it into reduced row echelon form, and interpret the result.

$$\begin{bmatrix} 1 & 0 & 2 & -1 & 2 \\ 0 & -1 & -2 & 2 & -6 \\ 2 & -1 & 0 & 2 & -4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

We see from this that

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ -1 & 1 \end{bmatrix}.$$

Why does this work? To see the answer, let's define five matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 1 & -1 & 5 \\ 1 & 1 & 6 \end{bmatrix}.$$

Notice that  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  are the first, second and third columns of  $X$ , respectively. Now consider this list of matrix products:  $A\vec{u}$ ,  $A\vec{v}$ ,  $A\vec{w}$  and  $AX$ .

$$\begin{aligned} A\vec{u} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 7 \end{bmatrix} \\ A\vec{w} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 17 \\ 39 \end{bmatrix} \\ A\vec{v} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
AX &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 5 \\ 1 & 1 & 6 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 1 & 17 \\ 7 & 1 & 39 \end{bmatrix}.
\end{aligned}$$

So again note that the columns of  $X$  are  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$ ; that is, we can write

$$X = [\vec{u} \quad \vec{v} \quad \vec{w}].$$

Notice also that the columns of  $AX$  are  $A\vec{u}$ ,  $A\vec{v}$  and  $A\vec{w}$ , respectively. Thus we can write

$$\begin{aligned}
AX &= A [\vec{u} \quad \vec{v} \quad \vec{w}] \\
&= [A\vec{u} \quad A\vec{v} \quad A\vec{w}] \\
&= \left[ \begin{bmatrix} 3 \\ 7 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 17 \\ 39 \end{bmatrix} \right] \\
&= \begin{bmatrix} 3 & 1 & 17 \\ 7 & 1 & 39 \end{bmatrix}.
\end{aligned}$$

This is exactly the same sort of thing we did in Section 3.5 when we had several vectors and we wanted to determine whether or not each of them belonged to a given span. Rather than perform the same set of row operations for each vector separately, we can do them all together. (See the discussion following Example 3.5.8.)

Thus, we are once again making use of the following fact:

The columns of a matrix product  $AX$  are  $A$  times the columns of  $X$ .

How does this help us solve the matrix equation  $AX = B$  for  $X$ ? Assume that  $A$  is a square matrix (that forces  $X$  and  $B$  to be the same size). We'll let  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$  denote the columns of the (unknown) matrix  $X$ , and we'll let  $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n$  denote the columns of  $B$ . We want to solve  $AX = B$  for  $X$ . That is, we want  $X$  where

$$AX = B \tag{4.3.1}$$

$$A [\vec{x}_1 \quad \vec{x}_2 \quad \dots \quad \vec{x}_n] = [\vec{b}_1 \quad \vec{b}_2 \quad \dots \quad \vec{b}_n] \tag{4.3.2}$$

$$[A\vec{x}_1 \quad A\vec{x}_2 \quad \dots \quad A\vec{x}_n] = [\vec{b}_1 \quad \vec{b}_2 \quad \dots \quad \vec{b}_n]. \tag{4.3.3}$$

If the matrix on the left hand side is equal to the matrix on the right, then their respective columns must be equal. This means we need to solve  $n$  equations:

$$\begin{aligned}
A\vec{x}_1 &= \vec{b}_1 \\
A\vec{x}_2 &= \vec{b}_2 \\
&\vdots \\
A\vec{x}_n &= \vec{b}_n.
\end{aligned}$$

We already know how to do this; this is what we learned in the previous section. Let's do this in a concrete example. In our above work we defined

matrices  $A$  and  $X$ , and looked at the product  $AX$ . Let's call the product  $B$ ; that is, set  $B = AX$ . Now, let's pretend that we don't know what  $X$  is, and let's try to find the matrix  $X$  that satisfies the equation  $AX = B$ . As a refresher, recall that

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 & 17 \\ 7 & 1 & 39 \end{bmatrix}.$$

Since  $A$  is a  $2 \times 2$  matrix and  $B$  is a  $2 \times 3$  matrix, what dimensions must  $X$  be in the equation  $AX = B$ ? The number of rows of  $X$  must match the number of columns of  $A$ ; the number of columns of  $X$  must match the number of columns of  $B$ . Therefore we know that  $X$  must be a  $2 \times 3$  matrix.

We'll call the three columns of  $X$   $\vec{x}_1$ ,  $\vec{x}_2$  and  $\vec{x}_3$ . Our previous explanation tells us that if  $AX = B$ , then:

$$\begin{aligned} AX &= B \\ A \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{bmatrix} &= \begin{bmatrix} 3 & 1 & 17 \\ 7 & 1 & 39 \end{bmatrix} \\ \begin{bmatrix} A\vec{x}_1 & A\vec{x}_2 & A\vec{x}_3 \end{bmatrix} &= \begin{bmatrix} 3 & 1 & 17 \\ 7 & 1 & 39 \end{bmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} A\vec{x}_1 &= \begin{bmatrix} 3 \\ 7 \end{bmatrix} \\ A\vec{x}_2 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ A\vec{x}_3 &= \begin{bmatrix} 17 \\ 39 \end{bmatrix}. \end{aligned}$$

To find  $\vec{x}_1$ , we form the proper augmented matrix and put it into reduced row echelon form and interpret the results.

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 7 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

This shows us that

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

To find  $\vec{x}_2$ , we again form an augmented matrix and interpret its reduced row echelon form.

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Thus

$$\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

which matches with what we already knew from above.

Before continuing on in this manner to find  $\vec{x}_3$ , we should stop and think. If the matrix vector equation  $A\vec{x} = \vec{b}$  is consistent, then the steps involved in putting

$$\begin{bmatrix} A & \vec{b} \end{bmatrix}$$

into reduced row echelon form depend only on  $A$ ; it does not matter what is. So when we put the two matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 7 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

from above into reduced row echelon form, we performed exactly the same steps! (In fact, those steps are:  $-3R_1 + R_2 \rightarrow R_2$ ;  $-\frac{1}{2}R_2 \rightarrow R_2$ ;  $-2R_2 + R_1 \rightarrow R_1$ .)

This is just as we noted after Example 3.5.8. Instead of solving for each column of  $X$  separately, performing the same steps to put the necessary matrices into reduced row echelon form three different times, why don't we just do it all at once? (Unless you enjoy doing unnecessary work.) Instead of individually putting

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 7 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 17 \\ 3 & 4 & 39 \end{bmatrix}$$

into reduced row echelon form, let's just put

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 17 \\ 3 & 4 & 7 & 1 & 39 \end{bmatrix}$$

into reduced row echelon form.

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 17 \\ 3 & 4 & 7 & 1 & 39 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1 & -1 & 5 \\ 0 & 1 & 1 & 1 & 6 \end{bmatrix}.$$

By looking at the last three columns, we see  $X$ :

$$X = \begin{bmatrix} 1 & -1 & 5 \\ 1 & 1 & 6 \end{bmatrix}.$$

In each of the examples we've considered so far, the reduced row echelon form  $R$  of the matrix  $A$  was equal to the  $n \times n$  identity matrix:  $R = I$ . It follows from Definition 3.6.14 in Section 3.6 that for an  $n \times n$  matrix  $A$ , we have  $R = I$  if and only if the rank of  $A$  is equal to  $n$ . At this point we should recall Theorem 3.6.15 from Section 3.6, and the discussion that followed. One of the things Theorem 3.6.15 tells us is that if  $A$  is an  $n \times n$  matrix and  $\text{rank}(A) = n$ , then the equation  $A\vec{x} = \vec{b}$  is **guaranteed** to have a unique solution, no matter what the vector  $\vec{b}$  is.

But what if  $\text{rank}(A) < n$ ? In this case, the reduced row echelon form of  $A$  is an  $n \times n$  matrix  $R$  with at least row of zeros on the bottom. Our experience with solving systems of the form  $A\vec{x} = \vec{b}$  tells us that in this case, the matrix equation  $AX = B$  may have infinitely many solutions, or no solution at all. Let us consider an example.

#### Example 4.3.3 Solving $AX = B$ when $\text{rank}(A) < n$ .

Solve the matrix equations  $AX = B$  and  $AX = C$ , where

$$A = \begin{bmatrix} 1 & 0 & -3 \\ -2 & 3 & 4 \\ 0 & 6 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 3 \\ 6 & 3 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 4 & -2 & -3 \\ -6 & 5 & 7 \\ 4 & 2 & 2 \end{bmatrix}.$$

**Solution.** We proceed as in the previous examples. For the equation



$AX = B$ , we have

$$\begin{aligned} [A \ B] &= \left[ \begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 2 & 0 \\ -2 & 3 & 4 & 1 & 1 & 3 \\ 0 & 6 & -4 & 6 & 3 & 0 \end{array} \right] \\ &\rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 2 & 0 \\ 0 & 3 & -2 & 3 & 5 & 3 \\ 0 & 0 & 0 & 0 & -7 & -6 \end{array} \right]. \end{aligned}$$

We stopped before reaching the reduced row echelon form, but there's no reason to continue: we already have a row of zeros on the left-hand side of the augmented matrix, and two non-zero entries in that same row, on the right. What this tells us is that it will be impossible to solve for the second and third columns of  $X$ ; thus, there is no solution in this case.

For the equation  $AX = C$ , we have

$$\begin{aligned} [A \ C] &= \left[ \begin{array}{ccc|ccc} 1 & 0 & -3 & 4 & -2 & -3 \\ -2 & 3 & 4 & -6 & 5 & 7 \\ 0 & 6 & -4 & 4 & 2 & 2 \end{array} \right] \\ &\xrightarrow{\text{rref}} \left[ \begin{array}{ccc|ccc} 1 & 0 & -3 & 4 & -2 & 3 \\ 0 & 1 & -2/3 & 2/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]. \end{aligned}$$

In this case, we *are* able to solve for each column of  $X$ , but in each case there are infinitely many possibilities: we find  $X = [\vec{x}_1 \ \vec{x}_2 \ \vec{x}_3]$ , where

$$\vec{x}_1 = \begin{bmatrix} 4 + 3r \\ \frac{2}{3} + \frac{2}{3}r \\ r \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -2 + 3s \\ \frac{1}{3} + \frac{2}{3}s \\ s \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} -3 + 3t \\ \frac{1}{3} + \frac{2}{3}t \\ t \end{bmatrix},$$

for parameters  $r, s, t$ . Any choice of values for each of these parameters provides us with a solution. For simple example, we can set all three parameters equal to zero, giving us

$$X = \begin{bmatrix} 4 & -2 & -3 \\ 2/3 & 1/3 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}.$$

It's easy to check that indeed,  $AX = C$  in this case.

In the previous example, we saw that the equation  $AX = B$  had no solution, while we were able to solve  $AX = C$ . How do we know which of these will be the case? Let's go back to Equation (4.3.2) above. From this equation, we can see that each column of  $B$  is of the form  $\vec{b}_i = A\vec{x}_i$  for some vector  $\vec{x}_i$ . Now recall from Section 2.7 that the set of vectors of the form  $A\vec{x}$  is precisely the column space of  $A$ . In the case of the matrix  $C$ , we can check that if we write

$$A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3] \text{ and } C = [\vec{c}_1 \ \vec{c}_2 \ \vec{c}_3],$$

then  $\vec{c}_1 = \vec{a}_1 - \vec{a}_2$ ,  $\vec{c}_2 = \vec{a}_1 + \vec{a}_2 + \vec{a}_3$ , and  $\vec{c}_3 = \vec{a}_2 + \vec{a}_3$ . Thus, all three columns of  $C$  are linear combinations of the columns of  $A$ , which is what allowed us to find a solution to  $AX = C$  even though the rank of  $A$  was less than  $n$ .

Now that we've justified the technique we've been using in this section to solve  $AX = B$  for  $X$ , we reinforce its importance by restating it as a Key Idea.

**Key Idea 4.3.4 Solving  $AX = B$ .**

Let  $A$  be an  $n \times n$  matrix, where the reduced row echelon form of  $A$  is  $I$ . To solve the matrix equation  $AX = B$  for  $X$ ,

1. Form the augmented matrix  $[A \ B]$ .
2. Put this matrix into reduced row echelon form.
  - (a) If it is of the form  $[I \ C]$ , for some matrix  $C$  that appears in the columns where  $B$  once was, then  $C = X$ .
  - (b) If it is of the form  $[R \ C]$ , where the matrix  $R$  (the reduced row echelon form of  $A$ ) has one or more rows of zeros, then there will be either no solution or infinitely many solutions, depending on whether or not the columns of  $B$  belong to the column space of  $A$ .

These simple steps cause us to ask certain questions. One of these we asked (and answered) above: What if  $A$  does not have maximum rank, so that the reduced row echelon form of  $A$  is not equal to  $I$ ? Second, we specify above that  $A$  should be a square matrix. What happens if  $A$  isn't square? Is a solution still possible? If you study what happens in Example 4.3.3 carefully, you can probably guess that a similar argument applies, by applying the ideas of Section 3.6 to each column of  $B$  individually.

These questions are good to ask, and we leave it to the reader to discover their answers. Instead of tackling these questions, we instead tackle the problem of "Why do we care about solving  $AX = B$ ?" The simple answer is that, for now, we only care about the special case when  $B = I$ . By solving  $AX = I$  for  $X$ , we find a matrix  $X$  that, when multiplied by  $A$ , gives the identity  $I$ . That will be very useful.

**4.3.1 Exercises**

**Exercise Group.** Matrices  $A$  and  $B$  are given. Solve the matrix equation  $AX = B$ .

1.  $A = \begin{bmatrix} -3 & -6 & 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 48 & -30 \\ 0 & -8 \end{bmatrix}$

2.  $A = \begin{bmatrix} 0 & -2 & 1 & 0 & 2 & 2 \\ 1 & 2 & -3 & & & \end{bmatrix}, B = I_3$

3.  $A = \begin{bmatrix} -4 & 1 & -1 & -2 \end{bmatrix},$   
 $B = \begin{bmatrix} -2 & -10 & 19 \\ 13 & 2 & -2 \end{bmatrix}$

4.  $A = \begin{bmatrix} 1 & -3 & -3 & 6 \end{bmatrix}, B = \begin{bmatrix} 12 & -10 \\ -27 & 27 \end{bmatrix}$

5.  $A = \begin{bmatrix} -1 & -2 & -2 & -3 \end{bmatrix}, B = \begin{bmatrix} 13 & 4 & 7 \\ 22 & 5 & 12 \end{bmatrix}$

6.  $A = \begin{bmatrix} 3 & 3 & 6 & 4 \end{bmatrix}, B = \begin{bmatrix} 15 & -39 \\ 16 & -66 \end{bmatrix}$

7.  $A = \begin{bmatrix} 4 & -1 & -7 & 5 \end{bmatrix}, B = \begin{bmatrix} 8 & -31 \\ -27 & 38 \end{bmatrix}$

8.  $A = \begin{bmatrix} 2 & 2 & 3 & 1 \end{bmatrix}, B = I_2$

9.  $A = \begin{bmatrix} -2 & 0 & 4 & -5 & -4 & 5 \\ -3 & 5 & -3 & & & \end{bmatrix},$   
 $B = \begin{bmatrix} -18 & 2 & -14 \\ -38 & 18 & -13 \\ 10 & 2 & -18 \end{bmatrix}$

10.  $A = \begin{bmatrix} -5 & -4 & -1 & 8 & -2 & -3 \\ 6 & 1 & -8 & & & \end{bmatrix},$   
 $B = \begin{bmatrix} -21 & -8 & -19 \\ 65 & -11 & -10 \\ 75 & -51 & 33 \end{bmatrix}$

11.  $A = \begin{bmatrix} -3 & 3 & -2 & 1 & -3 & 2 \\ -1 & -1 & 2 & & & \end{bmatrix}, B = I_3$

12.  $A = \begin{bmatrix} 1 & 0 & 3 & -1 \end{bmatrix}, B = I_2$

## 4.4 The Matrix Inverse

Once again we visit the old algebra equation,  $ax = b$ . How do we solve for  $x$ ? We know that, as long as  $a \neq 0$ ,

$$x = \frac{b}{a}, \text{ or, stated in another way, } x = a^{-1}b.$$

What is  $a^{-1}$ ? It is the number that, when multiplied by  $a$ , returns 1. That is,

$$a^{-1}a = 1.$$

Let us now think in terms of matrices. We have learned of the identity matrix  $I$  that “acts like the number 1.” That is, if  $A$  is a square matrix, then

$$IA = AI = A.$$

If we had a matrix, which we’ll call  $A^{-1}$ , where  $A^{-1}A = I$ , then by analogy to our algebra example above it seems like we might be able to solve the linear system  $A\vec{x} = \vec{b}$  for  $\vec{x}$  by multiplying both sides of the equation by  $A^{-1}$ . That is, perhaps

$$\vec{x} = A^{-1}\vec{b}.$$

There is no guarantee that such a matrix is going to exist for an arbitrary  $n \times n$  matrix  $A$ , but if it does, we say that  $A$  is *invertible*.

### Definition 4.4.1 Invertible Matrices and the Inverse of $A$ .

We say that an  $n \times n$  matrix  $A$  is **invertible** if there exists a matrix  $X$  such that

$$AX = XA = I_n.$$

When this is the case, we call the matrix  $X$  the **inverse** of  $A$  and write  $X = A^{-1}$ .

Of course, there is a lot of speculation here. We don’t know in general that such a matrix like  $A^{-1}$  exists. (And if it does, whether that matrix is *unique*, despite the use of the definite article in stating that  $X$  is “the” inverse of  $A$ .) However, we do know how to solve the matrix equation  $AX = B$ , so we can use that technique to solve the equation  $AX = I$  for  $X$ . This seems like it will get us close to what we want. Let’s practice this once and then study our results.

### Example 4.4.2 Solving $AX = I$ .

Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Find a matrix  $X$  such that  $AX = I$ .

**Solution.** We know how to solve this from the previous section: we form the proper augmented matrix, put it into reduced row echelon form and interpret the results.

$$\left[ \begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array} \right]$$

We read from our matrix that

$$X = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

Let's check our work:

$$\begin{aligned} AX &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

Sure enough, it works.

Looking at our previous example, we are tempted to jump in and call the matrix  $X$  that we found " $A^{-1}$ ." However, there are two obstacles in the way of us doing this.

First, we know that in general  $AB \neq BA$ . So while we found that  $AX = I$ , we can't automatically assume that  $XA = I$ .

Secondly, we have seen examples of matrices where  $AB = AC$ , but  $B \neq C$ . So just because  $AX = I$ , it is possible that another matrix  $Y$  exists where  $AY = I$ . If this is the case, using the notation  $A^{-1}$  would be misleading, since it could refer to more than one matrix.

These obstacles that we face are not insurmountable. The first obstacle was that we know that  $AX = I$  but didn't know that  $XA = I$ . That's easy enough to check, though. Let's look at  $A$  and  $X$  from our previous example.

$$\begin{aligned} XA &= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I. \end{aligned}$$

Perhaps this first obstacle isn't much of an obstacle after all. Of course, we only have one example where it worked, so this doesn't mean that it always works. We have good news, though: it always does work. The only "bad" news is that this is a bit harder to prove. For now, we will state it as theorem, but the proof will have to wait until later: see the proof of Theorem 4.5.1.

#### Theorem 4.4.3 Special Commuting Matrix Products.

Let  $A$  be an  $n \times n$  matrix.

1. If there is a matrix  $X$  such that  $AX = I_n$ , then  $XA = I_n$ .
2. If there is a matrix  $X$  such that  $XA = I_n$ , then  $AX = I_n$ .

The second obstacle is easier to address. We want to know if another matrix  $Y$  exists where  $AY = I = YA$ . Let's suppose that it does. Consider the expression  $XAY$ . Since matrix multiplication is associative, we can group this any way we choose. We could group this as  $(XA)Y$ ; this results in

$$\begin{aligned} (XA)Y &= IY \\ &= Y. \end{aligned}$$

We could also group  $XAY$  as  $X(AY)$ . This tells us

$$X(AY) = XI$$

**Note:** Theorem 4.4.3 only applies to *square* matrices. If  $A$  is an  $m \times n$  matrix, with  $m \neq n$ , it is sometimes possible to find an  $n \times m$  matrix  $B$  such that  $AB = I_m$  (in this case  $B$  is called a "right inverse" for  $A$ ), or an  $n \times m$  matrix  $C$  such that  $CA = I_n$  (a "left inverse" for  $A$ ). However, the only case where  $A$  has both a left *and* a right inverse is when  $m = n$ , in which case  $B = C$ . If  $m < n$ , only a right inverse is possible, while if  $m > n$ , only a left inverse is possible.

$$= X.$$

Combining the two ideas above, we see that  $X = XAY = Y$ ; that is,  $X = Y$ . We conclude that there is only one matrix  $X$  where  $XA = I = AX$ . (Even if we think we have two, we can do the above exercise and see that we really just have one.)

We have just proved the following theorem.

**Theorem 4.4.4 Uniqueness of Solutions to  $AX = I_n$ .**

*Let  $A$  be an  $n \times n$  matrix and let  $X$  be a matrix where  $AX = I_n$ . Then  $X$  is unique; it is the only matrix that satisfies this equation. In other words, if  $A$  is an  $n \times n$  matrix and  $AX = AY = I_n$ , then  $X = Y = A^{-1}$ .*

Thus, we were justified in Definition 4.4.1 in calling  $A^{-1}$  “the” inverse of  $A$  (rather than merely “an” inverse). Theorem 4.4.4 is incredibly important in practice. It tells us that if we are able to establish that *either*  $AX = I_n$  or  $XA = I_n$  for some matrix  $X$ , then we can immediately conclude two things: first, that  $A$  is invertible, and second, that  $A = A^{-1}$ . We put this observation to use in the next example.

**Example 4.4.5 Using Theorems Theorem 4.4.3 and Theorem 4.4.4.**

Suppose  $A$  is an  $n \times n$  matrix such that  $A^5 = I_n$ . Prove that  $A$  is invertible, and find an expression for  $A^{-1}$ .

**Solution.** Using Theorem 4.4.4, we can quickly kill two birds with one stone. Using properties of exponents (and the fact that  $5 = 1 + 4$ ), we have

$$A^5 = A \cdot (A \cdot A \cdot A \cdot A) = A(A^4) = I_n.$$

Thus, if we set  $X = A^4$ , then  $AX = I_n$ , so by Theorems Theorem 4.4.3 and Theorem 4.4.4,  $A$  is invertible, and  $A^{-1} = A^4$ .

At this point, it is natural to wonder which  $n \times n$  matrices will be invertible. Will any non-zero matrix do? (No.) Are such matrices a rare occurrence? (No.) As we proceed through this chapter and the next, we will see that there are many different conditions one can place on an  $n \times n$  matrix that are equivalent to the statement “The matrix  $A$  is invertible.” Before we begin our attempt to answer this question in general, let’s look at a particular example.

**Example 4.4.6 A non-invertible matrix.**

Find the inverse of  $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ .

**Solution.** By solving the equation  $AX = I$  for  $X$  will give us the inverse of  $A$ . Forming the appropriate augmented matrix and finding its reduced row echelon form gives us

$$\left[ \begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{cc|cc} 1 & 2 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{array} \right]$$

Yikes! We were expecting to find that the reduced row echelon form of this matrix would look like

$$\left[ I \quad A^{-1} \right].$$

However, we don't have the identity on the left hand side. Our conclusion:  $A$  is not invertible.

We have just seen that not all matrices are invertible. The attentive reader might have been able to spot the source of the trouble in the previous example: notice that the second row of  $A$  is a multiple of the first, so that the row operation  $R_2 - 2R_1 \rightarrow R_2$  created a row of zeros. Can you think what sort of condition would signal trouble for a general  $n \times n$  matrix? Here, we need to think back to our discussions of the various theoretical concepts we've encountered, such as rank, span, linear independence, and so on. Let us think of the rows of  $A$  as row vectors.

The elementary row operations that we perform on a matrix either rearrange these vectors, or create new vectors that are linear combinations of the old ones. The only way we end up with a row of zeros in the reduced row echelon form of  $A$  is if one of the rows of  $A$  can be written as a linear combination of the others; that is, if the rows of  $A$  are *linearly dependent*. We also know that if there is a row of zeros in the reduced row echelon form of  $A$ , then not every row contains a leading 1. Recalling that the rank of  $A$  is equal to the number of leading 1s in the reduced row echelon form of  $A$ , we have the following:

**Theorem 4.4.7 Inverses and rank.**

Let  $A$  be an  $n \times n$  matrix. Then the following statements are equivalent:

1. The matrix  $A$  is invertible.
2. The rank of  $A$  is equal to  $n$ .
3. The rows of  $A$  are linearly independent.
4. The columns of  $A$  are linearly independent.

The claim that “the following statements are equivalent” in Theorem 4.4.7 means that as soon as we know that one of the statements on the list is true, we can immediately conclude that the others are true as well. This is also the case if we know one of the statements is false. For example, if we know that  $\text{rank}(A) < n$ , then we can immediately conclude that  $A$  will not be invertible.

Let's sum up what we've learned so far. We've discovered that if a matrix has an inverse, it has only one. Therefore, we gave that special matrix a name, “the inverse.” Finally, we describe the most general way to find the inverse of a matrix, and a way to tell if it does not have one.

**Key Idea 4.4.8 Finding  $A^{-1}$ .**

Let  $A$  be an  $n \times n$  matrix. To find  $A^{-1}$ , put the augmented matrix

$$[A \quad I_n]$$

into reduced row echelon form. If the result is of the form

$$[I_n \quad X],$$

then  $A^{-1} = X$ . If not, (that is, if the first  $n$  columns of the reduced row echelon form are not  $I_n$ ), then  $A$  is not invertible.

Example 4.4.6 shows that not all square matrices (or even non-zero square matrices) are invertible, hence Definition 4.4.1 is necessary: why bother calling  $A$  “invertible” if every square matrix is? If everyone is special, then no one is. Then again, everyone is special.

Here's a useful exercise for the reader to consider: can you prove in general that for a  $2 \times 2$  matrix  $A$ , if one row of  $A$  is a multiple of the other, then the same is true of the columns? (We can see that this is the case in Example 4.4.6.) A fundamental theorem in linear algebra states that for a general  $m \times n$  matrix, the number of linearly independent rows is equal to the number of linearly independent columns.

The proof of this fact is rather technical, so we have not included it in this book. However, the rough idea is to follow the leading 1s: in the reduced row echelon form of  $A$ , each leading 1 occupies both a row and a column, and the rows (or columns) that end up with leading 1s are the ones that are linearly independent.

Let's try again.

**Example 4.4.9 Computing the inverse of a matrix.**

Find the inverse, if it exists, of  $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ .

**Solution.** We'll try to solve  $AX = I$  for  $X$  and see what happens.

$$\begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 1/5 & -2/5 & 1/5 \\ 0 & 0 & 1 & -3/10 & 1/10 & 1/5 \end{bmatrix}$$

We have a solution, so

$$A^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/5 & -2/5 & 1/5 \\ -3/10 & 1/10 & 1/5 \end{bmatrix}.$$

Multiply  $AA^{-1}$  to verify that it is indeed the inverse of  $A$ .

In general, given a matrix  $A$ , to find  $A^{-1}$  we need to form the augmented matrix  $[A \ I]$  and put it into reduced row echelon form and interpret the result. In the case of a  $2 \times 2$  matrix, though, there is a shortcut. We give the shortcut in terms of a theorem.

**Theorem 4.4.10 The Inverse of a  $2 \times 2$  Matrix.**

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then  $A$  is invertible if and only if  $ad - bc \neq 0$ .

If  $ad - bc \neq 0$ , then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

We don't prove Theorem 4.4.10 here, but it really isn't hard to do. Put the matrix

$$\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix}$$

into reduced row echelon form and you'll discover the result of the theorem. Alternatively, multiply  $A$  by what we propose is the inverse and see that we indeed get  $I$ . It turns out that Theorem 4.4.10 is a special case of a general formula for the inverse of a matrix in terms of *determinants*. We will discuss determinants in the next chapter. While useful in certain theoretical situations, the determinant formula for the inverse is only a shortcut in the  $2 \times 2$  case. For larger matrices, the method given in this section is computationally much faster.

We can't divide by 0, so if  $ad - bc = 0$ , we don't have an inverse. Recall Example 4.4.6, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

Here,  $ad - bc = 1(4) - 2(2) = 0$ , which is why  $A$  didn't have an inverse. Although this idea is simple, we should practice it.

**Example 4.4.11 Computing a  $2 \times 2$  inverse using Theorem 4.4.10.**

Use Theorem 4.4.10 to find the inverse of

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 9 \end{bmatrix}$$

if it exists.



**Solution.** Since  $ad - bc = 29 \neq 0$ ,  $A^{-1}$  exists. By Theorem 4.4.10,

$$\begin{aligned} A^{-1} &= \frac{1}{3(9) - 2(-1)} \begin{bmatrix} 9 & -2 \\ 1 & 3 \end{bmatrix} \\ &= \frac{1}{29} \begin{bmatrix} 9 & -2 \\ 1 & 3 \end{bmatrix}. \end{aligned}$$

We can leave our answer in this form, or we could “simplify” it as

$$A^{-1} = \frac{1}{29} \begin{bmatrix} 9 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 9/29 & -2/29 \\ 1/29 & 3/29 \end{bmatrix}.$$

We started this section out by speculating that just as we solved algebraic equations of the form  $ax = b$  by computing  $x = a^{-1}b$ , we might be able to solve matrix equations of the form  $A\vec{x} = \vec{b}$  by computing  $\vec{x} = A^{-1}\vec{b}$ . If  $A^{-1}$  does exist, then we can solve the equation  $A\vec{x} = \vec{b}$  this way. Consider:

$$\begin{aligned} A\vec{x} &= \vec{b} && \text{(original equation)} \\ A^{-1}A\vec{x} &= A^{-1}\vec{b} && \text{(multiply both sides on the left by } A^{-1}\text{)} \\ I\vec{x} &= A^{-1}\vec{b} && \text{(since } A^{-1}A = I\text{)} \\ \vec{x} &= A^{-1}\vec{b} && \text{(since } I\vec{x} = \vec{x}\text{).} \end{aligned}$$

Let’s step back and think about this for a moment. The only thing we know about the equation  $A\vec{x} = \vec{b}$  is that  $A$  is invertible. We also know that solutions to  $A\vec{x} = \vec{b}$  come in three forms: exactly one solution, infinitely many solutions, and no solution. We just showed that if  $A$  is invertible, then  $A\vec{x} = \vec{b}$  has *at least* one solution. We showed that by setting  $\vec{x}$  equal to  $A^{-1}\vec{b}$ , we have a solution. Is it possible that more solutions exist?

No. Suppose we are told that a known vector  $\vec{v}$  is a solution to the equation  $A\vec{x} = \vec{b}$ ; that is, we know that  $A\vec{v} = \vec{b}$ . We can repeat the above steps:

$$\begin{aligned} A\vec{v} &= \vec{b} \\ A^{-1}A\vec{v} &= A^{-1}\vec{b} \\ I\vec{v} &= A^{-1}\vec{b} \\ \vec{v} &= A^{-1}\vec{b}. \end{aligned}$$

This shows that *all* solutions to  $A\vec{x} = \vec{b}$  are exactly  $\vec{x} = A^{-1}\vec{b}$  when  $A$  is invertible. We have just proved the following theorem.

**Theorem 4.4.12 Invertible Matrices and Solutions to  $A\vec{x} = \vec{b}$ .**

Let  $A$  be an invertible  $n \times n$  matrix, and let  $\vec{b}$  be any  $n \times 1$  column vector. Then the equation  $A\vec{x} = \vec{b}$  has exactly one solution, namely

$$\vec{x} = A^{-1}\vec{b}.$$

A corollary to this theorem is: If  $A$  is not invertible, then  $A\vec{x} = \vec{b}$  does not have exactly one solution. It may have infinitely many solutions and it may have no solution, and we would need to examine the reduced row echelon form of the augmented matrix  $\begin{bmatrix} A & \vec{b} \end{bmatrix}$  to see which case applies.

We demonstrate our theorem with an example.

A *corollary* is an idea that follows directly from a theorem; typically, all the hard work is done in proving the theorem, and any resulting corollaries are easy consequences.

The method employed in Example 4.4.13 is useful in theory, but not in practice: the amount of work required to solve a system this way is significantly greater than the amount of work involved in Gaussian Elimination. The only scenario where you should consider using the inverse to solve a system (aside from being asked to do so on a test!) is when there are *several* systems you need to solve that all have the same coefficient matrix.

As odd as it may sound, *knowing* a matrix is invertible is useful; actually computing the inverse isn't. This is discussed at the end of the next section.

**Example 4.4.13 Using a matrix inverse to solve a system.**

Solve  $A\vec{x} = \vec{b}$  by computing  $\vec{x} = A^{-1}\vec{b}$ , where

$$A = \begin{bmatrix} 1 & 0 & -3 \\ -3 & -4 & 10 \\ 4 & -5 & -11 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} -15 \\ 57 \\ -46 \end{bmatrix}.$$

**Solution.** Without showing our steps, we compute

$$A^{-1} = \begin{bmatrix} 94 & 15 & -12 \\ 7 & 1 & -1 \\ 31 & 5 & -4 \end{bmatrix}.$$

We then find the solution to  $A\vec{x} = \vec{b}$  by computing  $A^{-1}\vec{b}$ :

$$\begin{aligned} \vec{x} &= A^{-1}\vec{b} \\ &= \begin{bmatrix} 94 & 15 & -12 \\ 7 & 1 & -1 \\ 31 & 5 & -4 \end{bmatrix} \begin{bmatrix} -15 \\ 57 \\ -46 \end{bmatrix} \\ &= \begin{bmatrix} -3 & -2 \\ 4 \end{bmatrix}. \end{aligned}$$

We can easily check our answer:

$$\begin{bmatrix} 1 & 0 & -3 \\ -3 & -4 & 10 \\ 4 & -5 & -11 \end{bmatrix} \begin{bmatrix} -3 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -15 \\ 57 \\ -46 \end{bmatrix}.$$

Knowing a matrix is invertible is incredibly useful. Among many other reasons, if you know  $A$  is invertible, then you know for sure that  $A\vec{x} = \vec{b}$  has a solution (as we just stated in Theorem 4.4.12). In the next section we'll demonstrate many different properties of invertible matrices, including stating several different ways in which we know that a matrix is invertible.

## 4.4.1 Exercises

**Exercise Group.** A matrix  $A$  is given. Find  $A^{-1}$  using Theorem 4.4.10, if it exists.

1. 
$$\begin{bmatrix} 1 & 5 \\ -5 & -24 \end{bmatrix}$$

2. 
$$\begin{bmatrix} 1 & -4 \\ 1 & -3 \end{bmatrix}$$

3. 
$$\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$$

4. 
$$\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$$

5. 
$$\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$$

6. 
$$\begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix}$$

7. 
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

8. 
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

**Exercise Group.** A matrix  $A$  is given. Find  $A^{-1}$  using Key Idea 4.4.8, if it exists.

9. 
$$\begin{bmatrix} -2 & 3 \\ 1 & 5 \end{bmatrix}$$

10. 
$$\begin{bmatrix} -5 & -2 \\ 9 & 2 \end{bmatrix}$$

11. 
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

12. 
$$\begin{bmatrix} 5 & 7 \\ 5/3 & 7/3 \end{bmatrix}$$

13. 
$$\begin{bmatrix} 25 & -10 & -4 \\ -18 & 7 & 3 \\ -6 & 2 & 1 \end{bmatrix}$$

14. 
$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & -7 \\ 20 & 7 & -48 \end{bmatrix}$$

15. 
$$\begin{bmatrix} -4 & 1 & 5 \\ -5 & 1 & 9 \\ -10 & 2 & 19 \end{bmatrix}$$

16. 
$$\begin{bmatrix} 1 & -5 & 0 \\ -2 & 15 & 4 \\ 4 & -19 & 1 \end{bmatrix}$$

17. 
$$\begin{bmatrix} 25 & -8 & 0 \\ -78 & 25 & 0 \\ 48 & -15 & 1 \end{bmatrix}$$

18. 
$$\begin{bmatrix} 1 & 0 & 0 \\ 7 & 5 & 8 \\ -2 & -2 & -3 \end{bmatrix}$$

19. 
$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

20. 
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

21. 
$$\begin{bmatrix} 2 & 3 & 4 \\ -3 & 6 & 9 \\ -1 & 9 & 13 \end{bmatrix}$$

22. 
$$\begin{bmatrix} 5 & -1 & 0 \\ 7 & 7 & 1 \\ -2 & -8 & -1 \end{bmatrix}$$

23. 
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -19 & -9 & 0 & 4 \\ 33 & 4 & 1 & -7 \\ 4 & 2 & 0 & -1 \end{bmatrix}$$

24. 
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 27 & 1 & 0 & 4 \\ 18 & 0 & 1 & 4 \\ 4 & 0 & 0 & 1 \end{bmatrix}$$

25. 
$$\begin{bmatrix} -15 & 45 & -3 & 4 \\ 55 & -164 & 15 & -15 \\ -215 & 640 & -62 & 59 \\ -4 & 12 & 0 & 1 \end{bmatrix}$$

26. 
$$\begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & -29 & -110 \\ 0 & -3 & -5 & -19 \end{bmatrix}$$

27. 
$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

28. 
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

**Exercise Group.** A matrix  $A$  and a vector are given. Solve the equation  $A\vec{x} = \vec{b}$  using Theorem 4.4.12.

$$29. \quad A = \begin{bmatrix} 1 & 2 & 12 \\ 0 & 1 & 6 \\ -3 & 0 & 1 \end{bmatrix}$$

$$31. \quad A = \begin{bmatrix} 5 & 0 & -2 \\ -8 & 1 & 5 \\ -2 & 0 & 1 \end{bmatrix}$$

$$33. \quad A = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 21 \\ 13 \end{bmatrix}$$

$$35. \quad A = \begin{bmatrix} 9 & 70 \\ -4 & -31 \end{bmatrix}$$

$$30. \quad A = \begin{bmatrix} 1 & 0 & -3 \\ 8 & -2 & -13 \\ 12 & -3 & -20 \end{bmatrix}$$

$$32. \quad A = \begin{bmatrix} 1 & -6 & 0 \\ 0 & 1 & 0 \\ 2 & -8 & 1 \end{bmatrix}$$

$$34. \quad A = \begin{bmatrix} 1 & -4 \\ 4 & -15 \end{bmatrix}$$

$$36. \quad A = \begin{bmatrix} 10 & -57 \\ 3 & -17 \end{bmatrix}, \vec{b} = \begin{bmatrix} -14 \\ -4 \end{bmatrix}$$

## 4.5 Properties of the Matrix Inverse

We ended the previous section by stating that invertible matrices are important. Since they are, in this section we study invertible matrices in two ways. First, we look at ways to tell whether or not a matrix is invertible, and second, we study properties of invertible matrices (that is, how they interact with other matrix operations).

In the last section we stated Theorem 4.4.7, in which we listed several properties of a matrix that equivalent to that matrix being invertible, including the fact that an  $n \times n$  matrix must have rank  $n$  in order to be invertible.

We begin this section by collecting additional properties that are equivalent to matrix a matrix being invertible. Some of these results were established in the previous section, but we state them again here for ease of reference.

### Theorem 4.5.1 Invertible Matrix Theorem.

Let  $A$  be an  $n \times n$  matrix. The following statements are equivalent.

- (a)  $A$  is invertible.
- (b) The equation  $A\vec{x} = \vec{0}$  has exactly one solution (namely,  $\vec{x} = \vec{0}$ ).
- (c) The reduced row echelon form of  $A$  is  $I$ .
- (d) The equation  $A\vec{x} = \vec{b}$  has exactly one solution for every  $n \times 1$  vector  $\vec{b}$ .
- (e) There exists a matrix  $C$  such that  $AC = I$ .
- (f) There exists a matrix  $B$  such that  $BA = I$ .

Note that the theorem uses the phrase “the following statements are equivalent.” Recall that when two or more statements are equivalent, it means that the truth of any one of them implies that the rest are also true; if any one of the statements is false, then they are all false. So, for example, if we determined that the equation  $A\vec{x} = \vec{0}$  had exactly one solution (and  $A$  was an  $n \times n$  matrix) then we would know that  $A$  was invertible, that  $A\vec{x} = \vec{b}$  had only one solution, that the reduced row echelon form of  $A$  was  $I$ , etc.

Let’s see exactly *why* all of these statements are equivalent. What we will do is establish the following chain of logic:

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (a).$$

Since this chain of implications circles back on itself, each of the statements implies the others. (For example, to show (e) implies (c), we start at (e) and circle around through (a) until we get to (c).)

- (a)  $\Rightarrow$  (b): Suppose  $A\vec{x} = \vec{0}$  for some vector  $\vec{x}$ . Since  $A$  is invertible, we can multiply both sides of this equation by  $A^{-1}$ , giving us

$$\vec{x} = I\vec{x} = (A^{-1}A)\vec{x} = A^{-1}(A\vec{x}) = A^{-1}\vec{0} = \vec{0}$$

Thus, the only possible solution to the system is  $\vec{x} = \vec{0}$ .

- (b)  $\Rightarrow$  (c): Suppose the only solution to  $A\vec{x} = \vec{0}$  is  $\vec{x} = \vec{0}$ . Why must the reduced row echelon form of  $A$  be equal to  $I$ ? If we let  $R$  denote the reduced row echelon form of  $A$ , we know that to solve  $A\vec{x} = \vec{0}$ , we form the augmented matrix  $[A \ \vec{0}]$  and reduce:

$$[A \ \vec{0}] \xrightarrow{\text{rref}} [R \ \vec{0}].$$

**Note:** Theorem 4.5.1 gives us several different ways of saying what is essentially the same thing (logically speaking). Theorems like this are very useful, since it’s often the case that, in a given situation, one of the conditions can easily be checked, allowing us to immediately obtain information that might be difficult (or impossible) to verify directly.

Also note: If we know that  $A$  is invertible, then we already know that there is a matrix  $B$  where  $BA = I$ . That is part of the definition of invertible. Part of the significance of Theorem 4.5.1 is that as soon as we find a matrix  $B$  such that  $AB = I$ , we immediately know that  $A$  is invertible, and that  $BA = I$  as well. We made this claim in Theorem 4.4.3, but did not provide a proof at the time.

If the  $R \neq I$ , then  $R$  has a row of zeros, and thus, so does  $[R \ \vec{0}]$ , in which case the system  $A\vec{x} = \vec{0}$  would have at least one parameter, and thus infinitely many solutions. Since we're assuming that  $A\vec{x} = \vec{0}$  has the unique solution  $\vec{x} = \vec{0}$ , it follows that  $R$  cannot have a row of zeros, and thus  $R = I$ .

- (c)  $\Rightarrow$  (d): Suppose that the reduced row echelon form of  $A$  is equal to  $I$ . It follows that when solving the system  $A\vec{x} = \vec{b}$ , we would have

$$\left[ A \ \vec{b} \right] \xrightarrow{\text{rref}} \left[ I \ \vec{c} \right]$$

for some column vector  $\vec{c}$ , and thus  $\vec{x} = \vec{c}$  is the unique solution to  $A\vec{x} = \vec{b}$ .

- (d)  $\Rightarrow$  (e): Suppose that  $A\vec{x} = \vec{b}$  has a unique solution for every column vector  $\vec{b}$ . Then in particular, we have a unique solution  $\vec{c}_j$  to the systems  $A\vec{x} = \vec{e}_j$  for  $j = 1, \dots, n$ , where  $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$  are the standard basis vectors in  $\mathbb{R}^n$  (and also the columns of  $I_n$ ). If we let

$$C = [\vec{c}_1 \ \vec{c}_2 \ \cdots \ \vec{c}_n],$$

then

$$AC = [A\vec{c}_1 \ A\vec{c}_2 \ \cdots \ A\vec{c}_n] = [\vec{e}_1 \ \vec{e}_2 \ \cdots \ \vec{e}_n] = I_n.$$

- (e)  $\Rightarrow$  (f): Suppose that  $AC = I_n$  for some  $n \times n$  matrix  $C$ . We claim that Property Item b holds for the matrix  $C$ . To see this, note that since  $AC = I_n$ , if  $C\vec{x} = \vec{0}$ , then

$$\vec{x} = I_n(\vec{x}) = (AC)\vec{x} = A(C\vec{x}) = A\vec{0} = \vec{0}.$$

Since Property Item b holds for the matrix  $C$ , it follows that Properties Item c, Item d, and Item e do as well, by what we've proven so far. Thus, there exists a matrix  $D$  such that  $CD = I_n$ . We can complete our proof by showing that  $D = A$ , and this is the case since (recalling that we've assumed  $AC = I_n$ )

$$A = A(I_n) = A(CD) = (AC)D = I_n D = D.$$

Thus, (f) holds. (Note that this argument finally establishes the truth of Theorem 4.4.3.)

- (f)  $\Rightarrow$  (a): Suppose that  $BA = I_n$  for some matrix  $B$ . Using the same argument we just gave, it then follows that  $AB = I_n$ , and if  $AB = BA = I_n$ , then by definition  $A$  is invertible and  $B = A^{-1}$ .

So we came up with a list of statements that are all *equivalent* to the statement " $A$  is invertible." Again, if we know that if any one of them is true (or false), then they are all true (or all false).

Theorem 4.5.1 states formally that if  $A$  is invertible, then  $A\vec{x} = \vec{b}$  has exactly one solution, namely  $A^{-1}\vec{b}$ . What if  $A$  is not invertible? What are the possibilities for solutions to  $A\vec{x} = \vec{b}$ ?

We know that  $A\vec{x} = \vec{b}$  *cannot* have exactly one solution; if it did, then by our theorem it would be invertible. Recalling that linear equations have either one solution, infinitely many solutions, or no solution, we are left with the latter options when  $A$  is not invertible. This idea is important and so we'll state it again as a Key Idea.

Think about how we, up to this point, determined the solution to  $A\vec{x} = \vec{b}$ . We set up the augmented matrix  $\left[ A \mid \vec{b} \right]$  and put it into reduced row echelon form. We know that getting the identity matrix on the left means that we had a unique solution (and not getting the identity means we either have no solution or infinitely many solutions). So getting  $I$  on the left means having a unique solution; having  $I$  on the left means that the reduced row echelon form of  $A$  is  $I$ , which we know is the same as  $A$  being invertible.

**Key Idea 4.5.2 Solutions to  $A\vec{x} = \vec{b}$  and the Invertibility of  $A$ .**

Consider the system of linear equations  $A\vec{x} = \vec{b}$ .

1. If  $A$  is invertible, then  $A\vec{x} = \vec{b}$  has exactly one solution, namely  $A^{-1}\vec{b}$ .
2. If  $A$  is not invertible, then  $A\vec{x} = \vec{b}$  has either infinitely many solutions or no solution.

In Theorem 4.5.1 we've come up with a list of ways in which we can tell whether or not a matrix is invertible. At the same time, we have come up with a list of properties of invertible matrices — things we know that are true about them. (For instance, if we know that  $A$  is invertible, then we know that  $A\vec{x} = \vec{b}$  has only one solution.)

We now go on to discover other properties of invertible matrices. Specifically, we want to find out how invertibility interacts with other matrix operations. For instance, if we know that  $A$  and  $B$  are invertible, what is the inverse of  $A + B$ ? What is the inverse of  $AB$ ? What is “the inverse of the inverse?” We'll explore these questions through an example.

**Example 4.5.3 Exploring properties of the inverse.**

Let

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}.$$

Find:

- |                |                    |
|----------------|--------------------|
| 1. $A^{-1}$    | 4. $(A^{-1})^{-1}$ |
| 2. $B^{-1}$    | 5. $(A + B)^{-1}$  |
| 3. $(AB)^{-1}$ | 6. $(5A)^{-1}$     |

In addition, try to find connections between each of the above.

**Solution.**

1. Computing  $A^{-1}$  is straightforward; we'll use Theorem 4.4.10.

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 \\ 0 & 1 \end{bmatrix}.$$

2. We compute  $B^{-1}$  in the same way as above.

$$B^{-1} = \frac{1}{-2} \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 \\ 1/2 & 1 \end{bmatrix}.$$

3. To compute  $(AB)^{-1}$ , we first compute  $AB$ :

$$AB = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 1 & 1 \end{bmatrix}.$$

We now apply Theorem 4.4.10 to find  $(AB)^{-1}$ .

$$(AB)^{-1} = \frac{1}{-6} \begin{bmatrix} 1 & -2 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} -1/6 & 1/3 \\ 1/6 & 2/3 \end{bmatrix}.$$

4. To compute  $(A^{-1})^{-1}$ , we simply apply Theorem 4.4.10 to  $A^{-1}$ :

$$(A^{-1})^{-1} = \frac{1}{1/3} \begin{bmatrix} 1 & 2/3 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}.$$

5. To compute  $(A + B)^{-1}$ , we first compute  $A + B$  then apply Theorem 4.4.10:

$$A + B = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}.$$

We notice immediately that the two rows of  $A + B$  are the same! Subtracting Row 1 from Row 2 would produce a row of zeros, so  $A + B$  has rank  $1 < 2$ , and therefore cannot be invertible.

6. To compute  $(5A)^{-1}$ , we compute  $5A$  and then apply Theorem 4.4.10.

$$(5A)^{-1} = \left( \begin{bmatrix} 15 & 10 \\ 0 & 5 \end{bmatrix} \right)^{-1} = \frac{1}{75} \begin{bmatrix} 5 & -10 \\ 0 & 15 \end{bmatrix} = \begin{bmatrix} 1/15 & -2/15 \\ 0 & 1/5 \end{bmatrix}.$$

We now look for connections between  $A^{-1}$ ,  $B^{-1}$ ,  $(AB)^{-1}$ ,  $(A^{-1})^{-1}$  and  $(A + B)^{-1}$ .

Is there some sort of relationship between  $(AB)^{-1}$  and  $A^{-1}$  and  $B^{-1}$ ? A first guess that seems plausible is  $(AB)^{-1} = A^{-1}B^{-1}$ . Is this true? Using our work from above, we have

$$A^{-1}B^{-1} = \begin{bmatrix} 1/3 & -2/3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & 0 \\ 1/2 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & -2/3 \\ 1/2 & 1 \end{bmatrix}.$$

Obviously, this is not equal to  $(AB)^{-1}$ . Before we do some further guessing, let's think about what the inverse of  $AB$  is supposed to do. The inverse — let's call it  $C$  — is supposed to be a matrix such that

$$(AB)C = C(AB) = I.$$

In examining the expression  $(AB)C$ , we see that we want  $B$  to somehow “cancel” with  $C$ . What “cancels”  $B$ ? An obvious answer is  $B^{-1}$ . This gives us a thought: perhaps we got the order of  $A^{-1}$  and  $B^{-1}$  wrong before. After all, we were hoping to find that

$$ABA^{-1}B^{-1} \stackrel{?}{=} I,$$

but algebraically speaking, it is hard to cancel out these terms. (Recall that matrix multiplication is not commutative:  $AB \neq BA$  in general.)

However, switching the order of  $A^{-1}$  and  $B^{-1}$  gives us some hope. Is  $(AB)^{-1} = B^{-1}A^{-1}$ ? Let's see.

$$\begin{aligned} (AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \text{ (regrouping by the associative property)} \\ &= AIA^{-1} \quad (BB^{-1} = I) \\ &= AA^{-1} \quad (AI = A) \\ &= I \quad (AA^{-1} = I). \end{aligned}$$



Since  $(AB)(B^{-1}A^{-1}) = I_n$ , we know immediately from Theorem 4.5.1 that  $(AB)^{-1} = B^{-1}A^{-1}$ . Note also that our argument above was completely general, so this result holds true for *any* pair of  $n \times n$  matrices  $A$  and  $B$ . Let's confirm this with our example matrices.

$$B^{-1}A^{-1} = \begin{bmatrix} -1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & -2/3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/6 & 1/3 \\ 1/6 & 2/3 \end{bmatrix} = (AB)^{-1}.$$

It worked!

Is there some sort of connection between  $(A^{-1})^{-1}$  and  $A$ ? The answer is pretty obvious: they are equal. The “inverse of the inverse” returns one to the original matrix.

Is there some sort of relationship between  $(A + B)^{-1}$ ,  $A^{-1}$  and  $B^{-1}$ ? Certainly, if we were forced to make a guess without working any examples, we would guess that

$$(A + B)^{-1} \stackrel{?}{=} A^{-1} + B^{-1}.$$

However, we saw that in our example, the matrix  $(A + B)$  isn't even invertible. This pretty much kills any hope of a connection.

Is there a connection between  $(5A)^{-1}$  and  $A^{-1}$ ? Consider:

$$\begin{aligned} (5A)^{-1} &= \begin{bmatrix} 1/15 & -2/15 \\ 0 & 1/5 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 1/3 & -2/3 \\ 0 & 1/5 \end{bmatrix} \\ &= \frac{1}{5} A^{-1}. \end{aligned}$$

Yes, there is a connection!

Let's summarize the results of this example. If  $A$  and  $B$  are both invertible matrices, then so is their product,  $AB$ . We demonstrated this with our example, and there is more to be said. Let's suppose that  $A$  and  $B$  are  $n \times n$  matrices, but we don't yet know if they are invertible. If  $AB$  is invertible, then each of  $A$  and  $B$  are; if  $AB$  is not invertible, then either  $A$  or  $B$  is not invertible.

In short, invertibility “works well” with matrix multiplication. However, we saw that it doesn't work well with matrix addition. Knowing that  $A$  and  $B$  are invertible does not help us find the inverse of  $(A + B)$ ; in fact, the latter matrix may not even be invertible.

Let's do one more example, then we'll summarize the results of this section in a theorem.

#### Example 4.5.4 Computing the inverse of a diagonal matrix.

Find the inverse of  $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -7 \end{bmatrix}$ .

**Solution.** We'll find  $A^{-1}$  using Key Idea 4.4.8.

$$\begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -7 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1/7 \end{bmatrix}$$

A natural question to ask at this point is whether or not the product of two non-invertible matrices could ever be invertible. The answer, not surprisingly, is “No.” To see this, suppose  $AB$  is invertible. Then we know that the only solution to the equation  $(AB)\vec{x} = \vec{0}$  is  $\vec{x} = \vec{0}$ . But if  $B$  is not invertible, then the equation  $B\vec{x} = \vec{0}$  does have a non-trivial solution  $\vec{x} \neq \vec{0}$ , and then

$$(AB)\vec{x} = A(B\vec{x}) = A\vec{0} = \vec{0},$$

implying that  $\vec{x}$  is a non-zero solution to  $(AB)\vec{x} = \vec{0}$ , and thus  $AB$  cannot be invertible.

The fact that invertibility works well with matrix multiplication should not come as a surprise. After all, saying that  $A$  is invertible makes a statement about the multiplicative properties of  $A$ . It says that I can multiply  $A$  with a special matrix to get  $I$ . Invertibility, in and of itself, says nothing about matrix addition, therefore we should not be too surprised that it doesn't work well with it.

We still haven't formally defined *diagonal*, but the definition is rather visual so we risk it. See Definition 6.1.4 for more details.

Yes, real people do solve linear equations in real life. Not just mathematicians, but economists, engineers, and scientists of all flavours regularly need to solve linear equations, and the matrices they use are often *huge*.

Most people see matrices at work without thinking about it. Digital pictures are simply “rectangular arrays” of numbers representing colours — they are matrices of colours. Many of the standard image processing operations involve matrix operations. The author's wife has a “7 megapixel” camera which creates pictures that are  $3072 \times 2304$  in size, giving over 7 million pixels, and that isn't even considered a “large” picture these days.

(Note that the previous sentence was written by Greg Hartman circa 2010, when 7 megapixels was pretty good, and required a standalone camera.)

Therefore

$$A^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -1/7 \end{bmatrix}.$$

The matrix  $A$  in the previous example is a *diagonal* matrix: the only nonzero entries of  $A$  lie on the *diagonal*. The relationship between  $A$  and  $A^{-1}$  in the above example seems pretty strong, and it holds true in general. We'll state this and summarize the results of this section with the following theorem.

#### Theorem 4.5.5 Properties of Invertible Matrices.

Let  $A$  and  $B$  be  $n \times n$  invertible matrices. Then:

1.  $AB$  is invertible;  $(AB)^{-1} = B^{-1}A^{-1}$ .
2.  $A^{-1}$  is invertible;  $(A^{-1})^{-1} = A$ .
3.  $nA$  is invertible for any nonzero scalar  $n$ ;  $(nA)^{-1} = \frac{1}{n}A^{-1}$ .
4. If  $A$  is a diagonal matrix, with diagonal entries  $d_1, d_2, \dots, d_n$ , where none of the diagonal entries are 0, then  $A^{-1}$  exists and is a diagonal matrix. Furthermore, the diagonal entries of  $A^{-1}$  are  $1/d_1, 1/d_2, \dots, 1/d_n$ .

Furthermore,

1. If a product  $AB$  is not invertible, then  $A$  or  $B$  is not invertible.
2. If  $A$  or  $B$  are not invertible, then  $AB$  is not invertible.

We end this section with a comment about solving systems of equations “in real life.” Solving a system  $A\vec{x} = \vec{b}$  by computing  $A^{-1}\vec{b}$  seems pretty slick, so it would make sense that this is the way it is normally done. However, in practice, this is rarely done. There are two main reasons why this is the case.

First, computing  $A^{-1}$  and  $A^{-1}\vec{b}$  is “expensive” in the sense that it takes up a lot of computing time. Certainly, our calculators have no trouble dealing with the  $3 \times 3$  cases we often consider in this textbook, but in real life the matrices being considered are very large (as in, hundreds of thousand rows and columns). Computing  $A^{-1}$  alone is rather impractical, and we waste a lot of time if we come to find out that  $A^{-1}$  does not exist. Even if we already know what  $A^{-1}$  is, computing  $A^{-1}\vec{b}$  is computationally expensive — Gaussian elimination is faster.

Secondly, computing  $A^{-1}$  using the method we've described often gives rise to numerical roundoff errors. Even though computers often do computations with an accuracy to more than 8 decimal places, after thousands of computations, rounding off can cause big errors. (A “small”  $1,000 \times 1,000$  matrix has 1,000,000 entries! That's a lot of places to have roundoff errors accumulate!) It is not unheard of to have a computer compute  $A^{-1}$  for a large matrix, and then immediately have it compute  $AA^{-1}$  and *not* get the identity matrix. (The result is usually very close, with the numbers on the diagonal close to 1 and the other entries near 0. But it isn't exactly the identity matrix.)

Therefore, in real life, solutions to  $A\vec{x} = \vec{b}$  are usually found using the methods we learned in Section 3.6. It turns out that even with all of our advances in mathematics, it is hard to beat the basic method that Gauss introduced a long time ago.

## 4.5.1 Exercises

**Exercise Group.** Matrices  $A$  and  $B$  are given. Compute  $(AB)^{-1}$  and  $B^{-1}A^{-1}$ .

1.  $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 5 \\ 2 & 5 \end{bmatrix}$

2.  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 7 & 1 \\ 2 & 1 \end{bmatrix}$

3.  $A = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix}$

4.  $A = \begin{bmatrix} 2 & 4 \\ 2 & 5 \end{bmatrix}, B = \begin{bmatrix} 2 & 2 \\ 6 & 5 \end{bmatrix}$

**Exercise Group.** A  $2 \times 2$  matrix  $A$  is given. Compute  $A^{-1}$  and  $(A^{-1})^{-1}$  using Theorem 4.4.10.

5.  $A = \begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix}$

6.  $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$

7.  $A = \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}$

8.  $A = \begin{bmatrix} 9 & 0 \\ 7 & 9 \end{bmatrix}$

9. Find  $2 \times 2$  matrices  $A$  and  $B$  that are each invertible, but  $A + B$  is not.

10. Create a random  $6 \times 6$  matrix  $A$ , then have a calculator or computer compute  $AA^{-1}$ . Was the identity matrix returned exactly? Comment on your results.

11. Use a calculator or computer to compute  $AA^{-1}$ , where  $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \\ 1 & 16 & 81 & 256 \end{bmatrix}$ . Was the identity matrix returned exactly? Comment on your results.

## 4.6 Elementary Matrices

In this section we introduce the concept of an *elementary matrix*. Elementary matrices are relatively simple objects, as their name suggests, but as we will see, they give us a simple method for understanding why our algorithm for computing the inverse of a matrix works. When we reach the discussion of determinants and their properties, we'll also see that elementary matrices provide a simple proof of an important result: the product rule for determinants.

### Definition 4.6.1 Elementary Matrix.

An **elementary matrix** is an  $n \times n$  matrix  $E$  that can be obtained from the identity matrix using a single row operation.

Note that by reversing the elementary row operation used to create an  $n \times n$  elementary matrix  $E$ , we could equally well say that  $E$  is a matrix of rank  $n$  that is one row operation away from its reduced row-echelon form (namely, the identity matrix).

The following are examples of elementary matrices, along with the elementary row operation used to obtain it from the identity:

$$E_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad R_1 \rightarrow R_1 + 2R_2$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$E_3 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \quad 4R_1 \rightarrow R_1$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \frac{1}{5}R_2 \rightarrow R_2$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 - 4R_3 \rightarrow R_2.$$

The main reason that elementary matrices are useful is that they give us a way of encoding (or more to the point, keeping track of) the elementary row operations used to define them. The primary result is the following:

### Theorem 4.6.2 Effect of Multiplication by an Elementary Matrix.

Let  $A$  be an  $n \times k$  matrix and suppose that the matrix  $B$  is obtained from  $A$  using an elementary row operation. Then  $B = EA$ , where  $E$  is the elementary matrix obtained from the identity matrix using the same row operation used to obtain  $B$  from  $A$ .

In other words, if use the notation  $X \xrightarrow{RO} Y$  to denote that a particular row operation is applied to the matrix  $X$  to obtain the matrix  $Y$ , if

$$A \xrightarrow{RO} B$$

and

$$I \xrightarrow{RO} E,$$

then  $B = EA$ .

**Example 4.6.3 Multiplication by an elementary matrix.**

Let  $A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 2 \\ 4 & 1 & -1 \end{bmatrix}$ . For each of the following elementary row

operations, construct the corresponding elementary matrix, and verify that multiplication by this matrix performs the desired row operation.

$$1. A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 2 \\ 4 & 1 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 4 & 1 & -1 \\ 0 & 3 & 2 \\ 2 & -1 & 1 \end{bmatrix} = B_1.$$

$$2. A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 2 \\ 4 & 1 & -1 \end{bmatrix} \xrightarrow{2R_1 \rightarrow R_1} \begin{bmatrix} 4 & -2 & 2 \\ 0 & 3 & 2 \\ 4 & 1 & -1 \end{bmatrix} = B_2$$

$$3. A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 2 \\ 4 & 1 & -1 \end{bmatrix} \xrightarrow{R_3 - 2R_1 \rightarrow R_3} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 2 \\ 0 & 3 & -3 \end{bmatrix} = B_3$$

**Solution.**

1. Applying the same row operation to the  $3 \times 3$  identity matrix gives

us the elementary matrix  $E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ , and we can easily

verify that

$$E_1 A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 2 \\ 4 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 3 & 2 \\ 2 & -1 & 1 \end{bmatrix} = B_1.$$

2. For the row operation  $2R_1 \rightarrow R_1$  the corresponding elementary

matrix is  $E_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , and we have

$$E_2 A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 2 \\ 4 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 2 \\ 0 & 3 & 2 \\ 4 & 1 & -1 \end{bmatrix}.$$

3. The elementary matrix corresponding to the row operation  $R_3 -$

$2R_1 \rightarrow R_3$  is  $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ , and

$$E_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 2 \\ 4 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 2 \\ 0 & 3 & -3 \end{bmatrix} = B_3.$$

**Inverses of Elementary Matrices.** Notice that every elementary row operation is reversible. If we apply a row operation of the type  $R_i \leftrightarrow R_j$  to a matrix, applying it again will return our matrix to its original state. (Swapping two rows and then swapping them back again results in no net change.) It follows that any “Type 1” elementary matrix obtained by a row operation of this type is its own inverse. In terms of row operations,

$$I \xrightarrow{R_i \leftrightarrow R_j} E \xrightarrow{R_i \leftrightarrow R_j} I.$$

But we know that applying the second row operation is the same as multiplying on the left by  $E$ ; therefore, we have  $E(E)E^2 = I$ . It follows from the definition of the inverse that  $E = E^{-1}$ .

Now let us consider the other two types of row operation. For a “Type 2” elementary matrix, obtained using a row operation of the type  $kR_i \rightarrow R_i$ , we multiply one of the rows of our matrix by a common constant  $k \neq 0$ . If we then multiply by  $\frac{1}{k}$ , we will be back to where we started.

Thus, if  $E$  is obtained from the identity using the row operation  $kR_i \rightarrow R_i$ , then  $E^{-1}$  is obtained from the identity using the row operation  $\frac{1}{k}R_i \rightarrow R_i$ . For any matrix  $A$  we have

$$A \xrightarrow{kR_i \rightarrow R_i} EA \xrightarrow{\frac{1}{k}R_i \rightarrow R_i} E^{-1}(EA) = A.$$

Finally, for a “Type 3” elementary matrix, obtained from the identity using a row operation of the type  $R_i + kR_j \rightarrow R_i$ , we add a multiple of one row to another. If we want to undo the affect of adding row  $j$  to row  $i$ , we simply subtract the same multiple of row  $j$  to row  $i$ . Thus,  $E^{-1}$  is obtained from the identity using the row operation  $R_i - kR_j \rightarrow R_i$ .

#### Example 4.6.4 Inverses of elementary matrices.

Determine the inverse of each of the elementary matrices below:

$$1. E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$2. E_2 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$3. E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{3}{7} \end{bmatrix}.$$

#### Solution.

1. The matrix  $E_1$  is a Type 1 elementary matrix, obtained by exchanging rows 1 and 2. Thus, we have  $E_1^{-1} = E_1$ . This is easily verified by checking that  $E_1^2 = I$ .

2. The matrix  $E_2$  is a Type 3 elementary matrix, obtained from the identity using the row operation  $R_1 - 3R_3 \rightarrow R_1$ . The opposite

row operation is  $R_1 + 3R_3 \rightarrow R_1$ ; thus,  $E_2^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

Again, we can easily check that  $E_2 E_2^{-1} = I$ .

3. The matrix  $E_3$  is a Type 2 elementary matrix, obtained from the identity matrix using the row operation  $\frac{3}{7}R_3 \rightarrow R_3$ . The opposite row operation is  $\frac{7}{3}R_3 \rightarrow R_3$ , since  $\frac{1}{3/7} = \frac{7}{3}$ . It follows that

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{7}{3} \end{bmatrix}.$$

**Elementary matrices and inverses.** Consider an  $n \times n$  matrix  $A$ . Suppose we wish to reduce  $A$  to its reduced row-echelon form (to solve a system of equations, perhaps, or determine the null space of  $A$ , etc.) To do so, we carry out a series of elementary row operations, say

$$A \xrightarrow{RO_1} A_1 \xrightarrow{RO_2} A_2 \xrightarrow{RO_3} \cdots \xrightarrow{RO_k} A_k = R,$$

where  $R$  is the reduced row-echelon form of  $A$ . Let  $E_1, E_2, \dots, E_k$  be the elementary matrices corresponding to the elementary row operations  $RO_1, RO_2, \dots, RO_k$ . Then we have

$$\begin{aligned} A_1 &= E_1 A \\ A_2 &= E_2 A_1 = (E_2 E_1) A \\ &\vdots \\ R &= A_k = E_k A_{k-1} = (E_k E_{k-1} \cdots E_2 E_1) A. \end{aligned}$$

Now, the reduced row-echelon form  $R$  might have one or more rows of zeros, depending on the rank of  $A$ , but let us focus for now on the case where  $\text{rank}(A) = n$ , in which case we know that  $R = I_n$ , the  $n \times n$  identity matrix.

In this case, we have (putting  $R = I$  in the last equality above):

$$(E_k \cdots E_2 E_1) A = I_n.$$

Letting  $B = E_k \cdots E_2 E_1$ , we have  $BA = I_n$ , and it follows from the Invertible Matrix Theorem that  $B = A^{-1}$ . We have the following theorem.

**Theorem 4.6.5 The inverse is a product of elementary matrices.**

Let  $A$  be an invertible  $n \times n$  matrix, and let  $E_1, E_2, \dots, E_k$  be the elementary matrices corresponding (in order) to the elementary row operations used to reduce  $A$  to the identity matrix. Then

$$A^{-1} = E_k \cdots E_2 E_1.$$

This result makes sense in the context of our algorithm for computing the inverse. Recall that to compute  $A^{-1}$ , we form the augmented matrix  $[A \mid I]$ , and apply elementary row operations until we reach the reduced row-echelon form  $[I \mid A^{-1}]$ .

Notice that at each step, applying an elementary row operation to an augmented matrix  $[M \mid N]$  is the same as multiplying both  $M$  and  $N$  by the corresponding elementary matrix. In terms of elementary matrices, our algorithm looks like the following:

$$[E \mid I] \xrightarrow{RO_1} [E_1 A \mid E_1 I]$$

$$\begin{array}{l}
\begin{array}{c} \xrightarrow{RO_2} \\ \vdots \\ \xrightarrow{RO_k} \end{array} \left[ \begin{array}{c|c} E_2(E_1 A) & E_2(E_1) \end{array} \right] \quad (\text{Note } E_1 I = E_1) \\
\vdots \\
\left[ \begin{array}{c|c} (E_k \cdots E_2 E_1) A & E_k \cdots E_2 E_1 \end{array} \right] = \left[ \begin{array}{c|c} I_n & A^{-1} \end{array} \right].
\end{array}$$

Since we have  $(E_k \cdots E_2 E_1)A = I_n$  on the left, it follows that we must have  $E_k \cdots E_2 E_1 = A^{-1}$  on the right.

We also have the following consequence of our above theorem (which we may view as an additional entry in our list of equivalent statements in the Invertible Matrix Theorem):

**Theorem 4.6.6 Invertible matrices are products of elementary matrices.**

*An  $n \times n$  matrix  $A$  is invertible if and only if it is a product of elementary matrices.*

To see why this result is true, recall from Theorem 4.5.5 that if  $A$  and  $B$  are invertible  $n \times n$  matrices, then so is  $AB$ , and  $(AB)^{-1} = B^{-1}A^{-1}$ . We can easily extend this result to products of three or more matrices. If  $A_1, A_2, \dots, A_k$  are all invertible  $n \times n$  matrices, then  $A_1 A_2 \cdots A_k$  is invertible, and

$$(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1} A_1^{-1}.$$

We know from our discussion above that every invertible matrix is invertible; therefore, if  $A = E_1 E_2 \cdots E_k$  is a product of elementary matrices, then  $A$  is invertible.

Conversely, suppose that  $A$  is invertible. From the previous theorem, we know that  $A^{-1}$  is a product of elementary matrices; namely,

$$A^{-1} = E_k \cdots E_2 E_1,$$

where  $E_1, E_2, \dots, E_k$  are the elementary matrices corresponding to the elementary row operations used to carry  $A$  to the identity matrix. Thus, we have

$$\begin{aligned}
A &= (A^{-1})^{-1} \\
&= (E_k \cdots E_2 E_1)^{-1} \\
&= E_1^{-1} E_2^{-1} \cdots E_k^{-1}.
\end{aligned}$$

Since the inverse of an elementary matrix is another elementary matrix, our result follows. Note that when we take the inverse of the product of elementary matrices, we must reverse the order of multiplication.

**Example 4.6.7 Writing an invertible matrix as a product of elementary matrices.**

Write the matrix  $A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 0 & 4 \\ 1 & -1 & 3 \end{bmatrix}$  as a product of elementary

matrices, if possible.

**Solution.** We use Gauss-Jordan elimination to carry the matrix  $A$  to its reduced row-echelon form. For each elementary row operation performed, we keep track of the corresponding elementary matrix and its



inverse.

$$\begin{array}{lll}
 \begin{bmatrix} 2 & -1 & 3 \\ -1 & 0 & 4 \\ 1 & -1 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 3 \\ -1 & 0 & 4 \\ 2 & -1 & 3 \end{bmatrix} & E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} & E_1^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
 \xrightarrow{R_2 + R_1 \rightarrow R_2} \begin{bmatrix} 1 & -1 & 3 \\ 0 & -1 & 7 \\ 2 & -1 & 3 \end{bmatrix} & E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 \xrightarrow{R_3 - 2R_1 \rightarrow R_3} \begin{bmatrix} 1 & -1 & 3 \\ 0 & -1 & 7 \\ 0 & 1 & -3 \end{bmatrix} & E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} & E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \\
 \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -3 \\ 0 & -1 & 7 \end{bmatrix} & E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
 \xrightarrow{R_3 + R_2 \rightarrow R_3} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix} & E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} & E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \\
 \xrightarrow{\frac{1}{4}R_3 \rightarrow R_3} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} & E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} & E_6^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \\
 \xrightarrow{R_2 + 3R_3 \rightarrow R_2} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & E_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} & E_7^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \\
 \xrightarrow{R_1 - 3R_3 \rightarrow R_1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & E_8 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & E_8^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 \xrightarrow{R_1 + R_2 \rightarrow R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & E_9 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & E_9^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
 \end{array}$$

We then have

$$A^{-1} = E_9 E_8 E_7 E_6 E_5 E_4 E_3 E_2 E_1$$

and

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} E_7^{-1} E_8^{-1} E_9^{-1}.$$

(Ideally we'd actually write out the matrices above, but since we needed nine steps to get  $A$  to the identity, limitations on space prevent us from doing so.)

### 4.6.1 Exercises

**Exercise Group.** Identify the row operation encoded by the given elementary matrix.

1.  $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

2.  $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

3.  $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4.  $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

**Exercise Group.** Write the elementary matrix of the given size that corresponds to the given row operation.

5.  $R_1 - 3R_2 \rightarrow R_1$  for a  $2 \times 2$  matrix.

6.  $R_3 + 2R_1 \rightarrow R_3$  for a  $3 \times 3$  matrix.

7.  $7R_2 \rightarrow R_2$  for a  $2 \times 2$  matrix.

8.  $R_1 \leftrightarrow R_2$  for a  $3 \times 5$  matrix.

**Exercise Group.** Find the inverse of the given elementary matrix, and state the row operation corresponding to the inverse. (Note: these are the same matrices as problems Exercises 1–4.)

9.  $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

10.  $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

11.  $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

12.  $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

**Exercise Group.** For the given matrix  $A$ ,

(a) write  $A^{-1}$  as a product of elementary matrices

(b) write  $A$  as a product of elementary matrices.

13.  $A = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$

14.  $A = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}$

15.  $A = \begin{bmatrix} 2 & 0 & 2 \\ 3 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

## Chapter 5

# Matrix Transformations

### 5.1 Matrix Transformations

We already looked at the basics of graphing vectors, and the arithmetic of multiplying matrices. In this section and the next, we will return to the geometric interpretation of vectors given in Chapter 2. Our goal in doing so is to obtain a visual understanding of the definition of matrix multiplication given in Section 4.2. Although the algebraic definition of matrix multiplication appears strange at first, we'll see that our definition of matrix multiplication allows us to use matrices to define **functions** that transform one vector into another, just as the functions you're familiar with from high school or Calculus transform one number into another. We can then visualize matrices and matrix multiplication in terms of their effect on vectors.

Given an  $m \times n$  matrix  $A$  we can define a function  $T$  that takes an  $n \times 1$  column vector  $\vec{x} \in \mathbb{R}^n$  as input, and produces an  $m \times 1$  column vector  $\vec{y} \in \mathbb{R}^m$  as output, according to the relationship

$$\vec{y} = T(\vec{x}) = A\vec{x}.$$

Such a function is known as a **matrix transformation**; it is an example of a more general class of functions between vector spaces known as **linear transformations**.

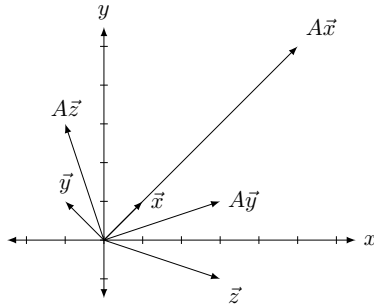
The graphical representation of vectors allows us to visualize matrix transformations (at least, in lower dimensions). This visualization plays a key role in applications such as computer graphics. We'll also see that the desire to define functions via matrix multiplication provides some justification for defining matrix multiplication the way we do.

#### 5.1.1 Matrix - Vector Multiplication

To simplify the discussion, and to make it easier for us to picture what's going on, we'll restrict ourselves (for now) to vectors in  $\mathbb{R}^2$ . We want to visualize the result of multiplying a vector by a matrix. In order to multiply a 2D vector by a matrix and get a 2D vector back, our matrix must be a square,  $2 \times 2$  matrix.

We'll start with an example. Given a matrix  $A$  and several vectors, we'll graph the vectors before and after they've been multiplied by  $A$  and see what we learn.

We can multiply a  $3 \times 2$  matrix by a 2D vector and get a 3D vector back, and this gives very interesting results. See Section 5.2.



**Figure 5.1.2** Multiplying vectors by a matrix in Example 5.1.1

### Example 5.1.1 Multiplying a vector by a matrix.

Let  $A$  be a matrix, and  $\vec{x}$ ,  $\vec{y}$ , and  $\vec{z}$  be vectors as given below.

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \vec{z} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Graph  $\vec{x}$ ,  $\vec{y}$  and  $\vec{z}$ , as well as  $A\vec{x}$ ,  $A\vec{y}$  and  $A\vec{z}$ .

**Solution.** It is straightforward to compute:

$$A\vec{x} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \quad A\vec{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \text{and} \quad A\vec{z} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

The vectors are sketched in Figure 5.1.2.

There are several things to notice. When each vector is multiplied by  $A$ , the result is a vector with a different length (in this example, always longer), and in two of the cases (for  $\vec{y}$  and  $\vec{z}$ ), the resulting vector points in a different direction.

This isn't surprising. In the previous section we learned about matrix multiplication, which is a strange and seemingly unpredictable operation. Would you expect to see some sort of immediately recognizable pattern appear from multiplying a matrix and a vector? (This is a rhetorical question; the expected answer is "No.") In fact, the surprising thing from the example is that  $\vec{x}$  and  $A\vec{x}$  point in the same direction! Why does the direction of  $\vec{x}$  not change after multiplication by  $A$ ? (We'll answer this in Section 7.1 when we learn about something called "eigenvectors.")

Different matrices act on vectors in different ways. (That's one reason we call them "different.") Some always increase the length of a vector through multiplication, others always decrease the length, others increase the length of some vectors and decrease the length of others, and others still don't change the length at all. A similar statement can be made about how matrices affect the direction of vectors through multiplication: some change every vector's direction, some change "most" vector's direction but leave some the same, and others still don't change the direction of any vector.

How do we set about studying how matrix multiplication affects vectors? We could just create lots of different matrices and lots of different vectors, multiply, then graph, but this would be a lot of work with very little useful result. It would be too hard to find a pattern of behaviour in this. (Remember, that's what mathematicians do. We look for patterns.)

Instead, we'll begin by using a technique we've employed often in the past. We have a "new" operation; let's explore how it behaves with "old" operations. Specifically, we know how to sketch vector addition. What happens when we throw matrix multiplication into the mix? Let's try an example.

### Example 5.1.3 Combining addition and matrix multiplication.

Let  $A$  be a matrix and  $\vec{x}$  and  $\vec{y}$  be vectors as given below.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Sketch  $\vec{x} + \vec{y}$ ,  $A\vec{x}$ ,  $A\vec{y}$ , and  $A(\vec{x} + \vec{y})$ .

**Solution.** It is pretty straightforward to compute:

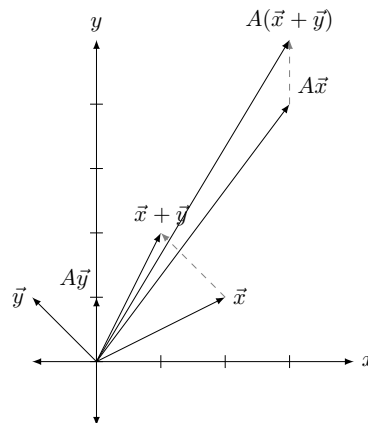
$$\vec{x} + \vec{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \quad A\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}; \quad A\vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A(\vec{x} + \vec{y}) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

In Figure 5.1.4, we have graphed the above vectors and have included dashed gray vectors to highlight the additive nature of  $\vec{x} + \vec{y}$  and  $A(\vec{x} + \vec{y})$ . Does anything strike you as interesting?

Let's not focus on things which don't matter right now: let's not focus on how long certain vectors became, nor necessarily how their direction changed. Rather, think about how matrix multiplication interacted with the vector addition.

In some sense, we started with three vectors,  $\vec{x}$ ,  $\vec{y}$ , and  $\vec{x} + \vec{y}$ . This last vector is special; it is the sum of the previous two. Now, multiply all three by  $A$ . What happens? We get three new vectors, but the significant thing is this: the last vector is still the sum of the previous two! (We emphasize this by drawing dotted vectors to represent part of the Parallelogram Law.)

Of course, we knew this already: we already knew that  $A\vec{x} + A\vec{y} = A(\vec{x} + \vec{y})$ , for this is just the Distributive Property of matrix multiplication given in Theorem 4.2.11. However, now we get to see this graphically.



**Figure 5.1.4** Vector addition and matrix multiplication in Example 5.1.3

Let's do one more example.

#### Example 5.1.5 Sketching the effect of matrix multiplication.

Let  $A$ ,  $\vec{x}$ ,  $\vec{y}$ , and  $\vec{z}$  be as given below.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \vec{z} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

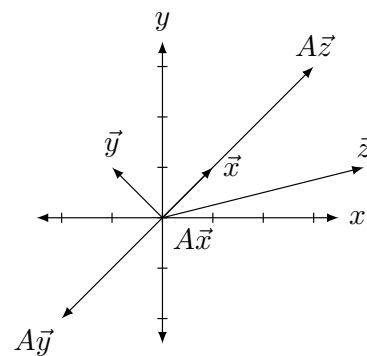
Graph  $\vec{x}$ ,  $\vec{y}$  and  $\vec{z}$ , as well as  $A\vec{x}$ ,  $A\vec{y}$  and  $A\vec{z}$ .

**Solution.** It is straightforward to compute:

$$A\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A\vec{y} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \quad \text{and} \quad A\vec{z} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

The vectors are sketched in Figure 5.1.6.

These results are interesting. While we won't explore them in great detail here, notice how  $\vec{x}$  got sent to the zero vector. Notice also that  $A\vec{x}$ ,  $A\vec{y}$  and  $A\vec{z}$  are all in a line (as well as  $\vec{x}$ !). Why is that? Are  $\vec{x}$ ,  $\vec{y}$  and  $\vec{z}$  just special vectors, or would any other vector get sent to the same line when multiplied by  $A$ ? (Don't just sit there, try it out!)



**Figure 5.1.6** Multiplying vectors by a matrix in Example 5.1.5

### 5.1.2 Transformations of the Cartesian Plane

We studied in Chapter 2 how to visualize vectors and how the matrix arithmetic operations of addition and scalar multiplication can be graphically represented for vectors. In the discussion above, we limited our visual understanding of matrix multiplication to graphing a vector, multiplying it by a matrix, then graphing the resulting vector. In rest of this section we'll explore these multiplication ideas in greater depth. Instead of multiplying individual vectors by a matrix  $A$ , we'll study what happens when we multiply every vector in the Cartesian plans

by  $A$ . (No, we won't do them one by one.)

Because of the Distributive Property, as illustrated in Example 5.1.3, we know that the Cartesian plane will be *transformed* in a very nice, predictable way. Straight lines will be transformed into other straight lines (and they won't become curvy, or jagged, or broken). Curved lines will be transformed into other curved lines (perhaps the curve will become "straight," but it won't become jagged or broken).

Example 5.1.3 has very significant implications. We usually think of the Cartesian plane as a set of points; we can adjust this thought just slightly and think of it as a set of vectors that point to each of these points. What happens to the Cartesian plane if we multiply *every vector* in the plane by the same matrix  $A$ ?

Checking every single vector in the plane isn't practical, so we have to look for other ways to visualize the effects of matrix multiplication. One way of studying how the whole Cartesian plane is affected by multiplication by a matrix  $A$  is to study how the *unit square* is affected. The unit square is the square with corners at the points  $(0, 0)$ ,  $(1, 0)$ ,  $(1, 1)$ , and  $(0, 1)$ . Each corner can be represented by the vector that points to it; multiply each of these vectors by  $A$  and we can get an idea of how  $A$  affects the whole Cartesian plane.

Let's try an example.

**Example 5.1.7 Visualizing a matrix transformation using vectors.**

Plot the vectors of the unit square before and after they have been multiplied by  $A$ , where

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}.$$

**Solution.** The four corners of the unit square can be represented by the vectors

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Multiplying each by  $A$  gives the vectors

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix},$$

respectively.

(Hint: one way of using your calculator to do this for you quickly is to make a  $2 \times 4$  matrix whose columns are each of these vectors. In this case, create a matrix

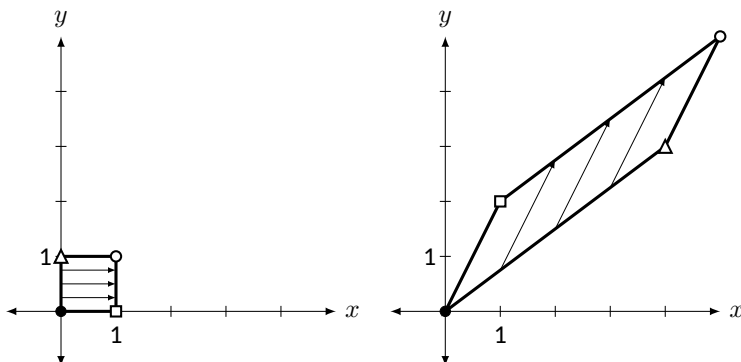
$$B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then multiply  $B$  by  $A$  and read off the transformed vectors from the respective columns:

$$AB = \begin{bmatrix} 0 & 1 & 5 & 4 \\ 0 & 2 & 5 & 3 \end{bmatrix}.$$

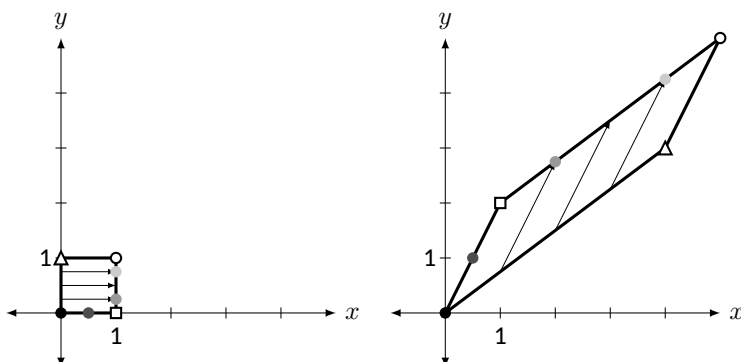
This saves time, especially if you do a similar procedure for multiple matrices  $A$ . Of course, we can save more time by skipping the first column; since it is the column of zeros, it will stay the column of zeros after multiplication by  $A$ .)

The unit square and its transformation are graphed in Figure 5.1.8, where the shaped vertices correspond to each other across the two graphs. Note how the square got turned into some sort of quadrilateral (it's actually a parallelogram). A really interesting thing is how the triangular and square vertices seem to have changed places — it is as though the square, in addition to being stretched out of shape, was flipped.



**Figure 5.1.8** Transforming the unit square by matrix multiplication in Example 5.1.7

To stress how “straight lines get transformed to straight lines,” consider Figure 5.1.9. Here, the unit square has some additional points drawn on it which correspond to the shaded dots on the transformed parallelogram. Note how relative distances are also preserved; the dot halfway between the black and square dots is transformed to a position along the line, halfway between the black and square dots.



**Figure 5.1.9** Emphasizing straight lines going to straight lines in Example 5.1.7

Much more can be said about this example. Before we delve into this, though, let's try one more example.

#### Example 5.1.10 Visualizing a matrix transformation using a region.

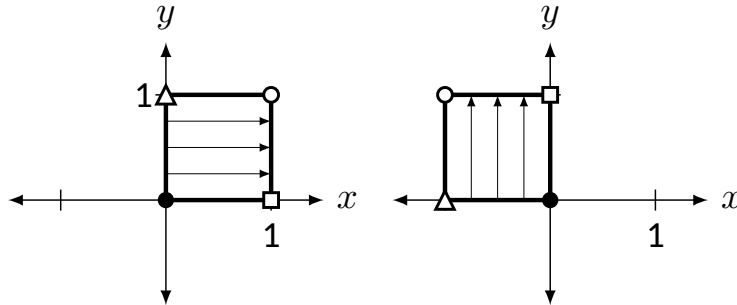
Plot the transformed unit square after it has been transformed by  $A$ , where

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

**Solution.** We'll put the vectors that correspond to each corner in a matrix  $B$  as before and then multiply it on the left by  $A$ . Doing so gives:

$$\begin{aligned} AB &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \end{aligned}$$

In Figure 5.1.11 the unit square is again drawn along with its transformation by  $A$ .



**Figure 5.1.11** Transforming the unit square by matrix multiplication in Example 5.1.10

Make note of how the square moved. It did not simply “slide” to the left; (mathematically, that is called a **translation**) nor did it “flip” across the  $y$  axis. Rather, it was **rotated** counterclockwise about the origin  $90^\circ$ . In a rotation, the shape of an object does not change; in our example, the square remained a square of the same size.

We have broached the topic of how the Cartesian plane can be transformed via multiplication by a  $2 \times 2$  matrix  $A$ . We have seen a few examples so far, and our intuition as to how the plane is changed has been informed only by seeing how the unit square changes. Let's explore this further by investigating two questions:

1. Suppose we want to transform the Cartesian plane in a known way (for instance, we may want to rotate the plane counterclockwise  $180^\circ$ ). How do we find the matrix (if one even exists) which performs this transformation?
2. How does knowing how the unit square is transformed really help in understanding how the entire plane is transformed?

These questions are closely related, and as we answer one, we will help answer the other.

To get started with the first question, look back at Example 5.1.7 and Example 5.1.10 and consider again how the unit square was transformed. In particular, is there any correlation between where the vertices ended up and the matrix  $A$ ?

If you are just reading on, and haven't actually gone back and looked at the examples, go back now and try to make some sort of connection. Otherwise -- you may have noted some of the following things:

1. The zero vector ( $\vec{0}$ , the “black” corner) never moved. That makes sense, though;  $A\vec{0} = \vec{0}$ .



2. The “square” corner, i.e., the corner corresponding to the vector  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , is always transformed to the vector in the first column of  $A$ !
3. Likewise, the “triangular” corner, i.e., the corner corresponding to the vector  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , is always transformed to the vector in the second column of  $A$ ! (This is less of a surprise, given the result of the previous point.)
4. The “white dot” corner is always transformed to the *sum* of the two column vectors of  $A$ . (This observation is a bit more obscure than the first three. It follows from the fact that this corner of the unit square is the “sum” of the other two nonzero corners.)

Let’s now take the time to understand these four points. The first point should be clear;  $\vec{0}$  will always be transformed to  $\vec{0}$  via matrix multiplication. (Hence the hint in the middle of Example 5.1.7, where we are told that we can ignore entering in the column of zeros in the matrix  $B$ .)

We can understand the second and third points simultaneously. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

What are  $A\vec{e}_1$  and  $A\vec{e}_2$ ?

$$\begin{aligned} A\vec{e}_1 &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} a \\ c \end{bmatrix} \\ A\vec{e}_2 &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} b \\ d \end{bmatrix}. \end{aligned}$$

So by mere mechanics of matrix multiplication, the square corner  $\vec{e}_1$  is transformed to the first column of  $A$ , and the triangular corner  $\vec{e}_2$  is transformed to the second column of  $A$ . A similar argument demonstrates why the white dot corner is transformed to the sum of the columns of  $A$ . (Another way of looking at all of this is to consider what  $A \cdot I$  is: of course, it is just  $A$ . What are the columns of  $I$ ? Just  $\vec{e}_1$  and  $\vec{e}_2$ .)

Revisit now the question “How do we find the matrix that performs a given transformation on the Cartesian plane?” The answer follows from what we just did. Think about the given transformation and how it would transform the corners of the unit square. Make the first column of  $A$  the vector where  $\vec{e}_1$  goes, and make the second column of  $A$  the vector where  $\vec{e}_2$  goes.

Let’s practice this in the context of an example.

#### Example 5.1.12 Determining a matrix transformation.

Find the matrix  $A$  that flips the Cartesian plane about the  $x$  axis and then stretches the plane horizontally by a factor of two.

**Solution.** We first consider  $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . Where does this corner go to under the given transformation? Flipping the plane across the  $x$  axis does

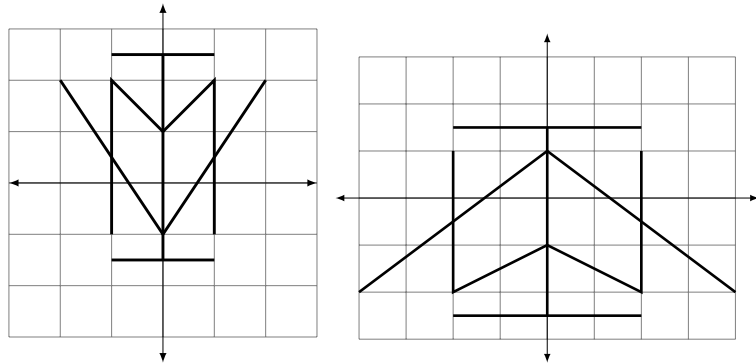
not change  $\vec{e}_1$  at all; stretching the plane sends  $\vec{e}_1$  to  $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ . Therefore, the first column of  $A$  is  $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ .

Now consider  $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . Flipping the plane about the  $x$  axis sends  $\vec{e}_2$  to the vector  $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ ; subsequently stretching the plane horizontally does not affect this vector. Therefore the second column of  $A$  is  $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ .

Putting this together gives

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

To help visualize this, consider Figure 5.1.13 where a shape is transformed under this matrix. Notice how it is turned upside down and is stretched horizontally by a factor of two. (The gridlines are given as a visual aid.)



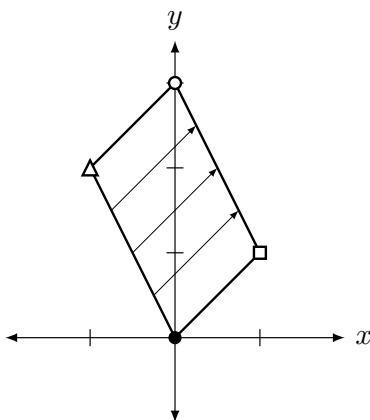
**Figure 5.1.13** Transforming the Cartesian plane in Example 5.1.12

A while ago we asked two questions. The first was “How do we find the matrix that performs a given transformation?” We have just answered that question (although we will do more to explore it in the future). The second question was “How does knowing how the unit square is transformed really help us understand how the entire plane is transformed?”

Consider Figure 5.1.14 where the unit square (with vertices marked with shapes as before) is shown transformed under an unknown matrix. How does this help us understand how the whole Cartesian plane is transformed? For instance, how can we use this picture to figure out how the point  $(2, 3)$  will be transformed?

There are two ways to consider the solution to this question. First, we know now how to compute the transformation matrix; the new position of  $\vec{e}_1$  is the first column of  $A$ , and the new position of  $\vec{e}_2$  is the second column of  $A$ . Therefore, by looking at the figure, we can deduce that

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}.$$



**Figure 5.1.14** The unit square under an unknown transformation.

To find where the point  $(2, 3)$  is sent, simply multiply

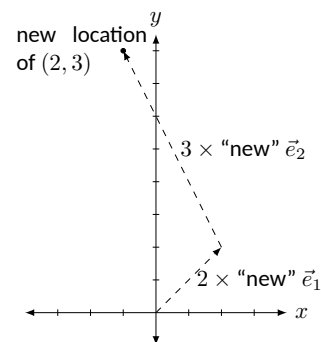
$$\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \end{bmatrix}.$$

There is another way of doing this which isn't as computational — it doesn't involve computing the transformation matrix. Consider the following equalities:

$$\begin{aligned} \begin{bmatrix} 2 \\ 3 \end{bmatrix} &= \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \\ &= 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= 2\vec{e}_1 + 3\vec{e}_2 \end{aligned}$$

This last equality states something that is somewhat obvious: to arrive at the vector  $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ , one needs to go 2 units in the  $\vec{e}_1$  direction and 3 units in the  $\vec{e}_2$  direction. To find where the point  $(2, 3)$  is transformed, one needs to go 2 units in the *new*  $\vec{e}_1$  direction and 3 units in the *new*  $\vec{e}_2$  direction. This is demonstrated in Figure 5.1.15.

We are coming to grips with how matrix transformations work. We asked two basic questions: “How do we find the matrix for a given transformation?” and “How do we understand the transformation without the matrix?”, and we've answered each accompanied by one example. Let's do another example that demonstrates both techniques at once.



**Figure 5.1.15** Finding the new location of the point  $(2, 3)$

#### Example 5.1.16 Determining and analyzing a matrix transformation.

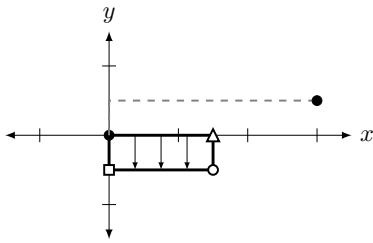
First, find the matrix  $A$  that transforms the Cartesian plane by stretching it vertically by a factor of 1.5, then stretches it horizontally by a factor of 0.5, then rotates it clockwise about the origin  $90^\circ$ . Secondly, using the new locations of  $\vec{e}_1$  and  $\vec{e}_2$ , find the transformed location of the point  $(-1, 2)$ .

**Solution.** To find  $A$ , first consider the new location of  $\vec{e}_1$ . Stretching the plane vertically does not affect  $\vec{e}_1$ ; stretching the plane horizontally by a factor of 0.5 changes  $\vec{e}_1$  to  $\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$ , and then rotating it  $90^\circ$  about the origin moves it to  $\begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$ . This is the first column of  $A$ .

Now consider the new location of  $\vec{e}_2$ . Stretching the plane vertically changes it to  $\begin{bmatrix} 0 \\ 3/2 \end{bmatrix}$ ; stretching horizontally does not affect it, and rotating  $90^\circ$  moves it to  $\begin{bmatrix} 3/2 \\ 0 \end{bmatrix}$ . This is then the second column of  $A$ . This gives

$$A = \begin{bmatrix} 0 & 3/2 \\ -1/2 & 0 \end{bmatrix}.$$

Where does the point  $(-1, 2)$  get sent to? The corresponding vector  $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$  is found by going  $-1$  units in the  $\vec{e}_1$  direction and 2 units in the  $\vec{e}_2$  direction. Therefore, the transformation will send the vector to  $-1$  units in the new  $\vec{e}_1$  direction and 2 units in the new  $\vec{e}_2$  direction. This is

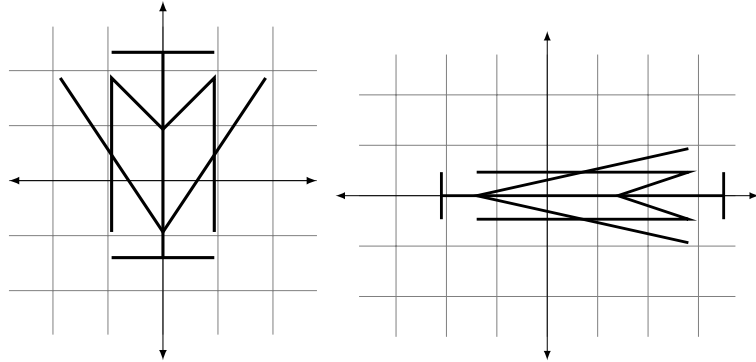


**Figure 5.1.17** Understanding the transformation in Example 5.1.16

sketched in Figure 5.1.17, along with the transformed unit square. We can also check this multiplicatively:

$$\begin{bmatrix} 0 & 3/2 \\ -1/2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1/2 \end{bmatrix}.$$

Figure 5.1.18 shows the effects of the transformation on another shape.



**Figure 5.1.18** Transforming the Cartesian plane in Example 5.1.16

Right now we are focusing on transforming the Cartesian plane — we are making 2D transformations. Knowing how to do this provides a foundation for transforming 3D space, which, among other things, is very important when producing 3D computer graphics. Basic shapes can be drawn and then rotated, stretched, and/or moved to other regions of space. This also allows for things like “moving the camera view.” Of course, algebraically, there is nothing stopping us from working with transformations of vectors in  $\mathbb{R}^n$  for *any* value of  $n$ . The limitation to two and three dimensions is strictly one of visualization.

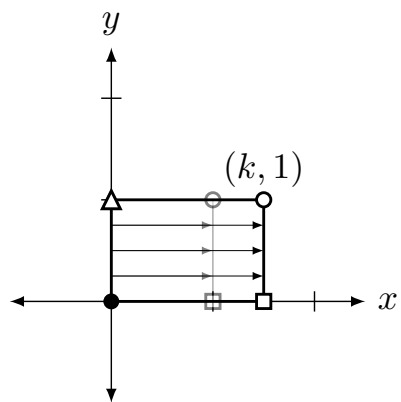
What kinds of transformations are possible? We have already seen some of the things that are possible: rotations, stretches, and flips. We have also mentioned some things that are not possible. For instance, we stated that straight lines always get transformed to straight lines. Therefore, we cannot transform the unit square into a circle using a matrix.

Let’s look at some common transformations of the Cartesian plane and the matrices that perform these operations. In the following figures, a transformation matrix will be given alongside a picture of the transformed unit square. (The original unit square is drawn lightly as well to serve as a reference.)

## 5.1.3 2D Matrix Transformations

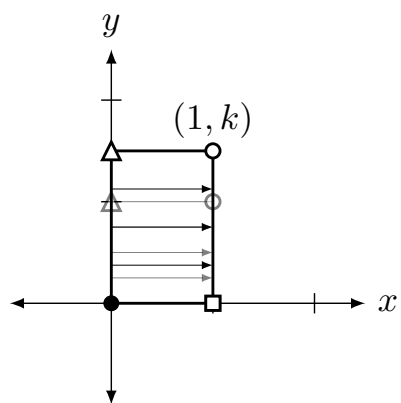
**Horizontal stretch** by a factor of  $k$ .

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$



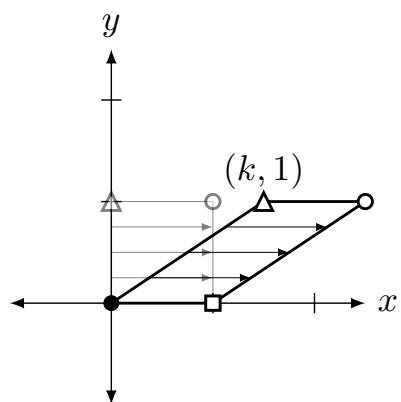
**Vertical stretch** by a factor of  $k$ .

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$



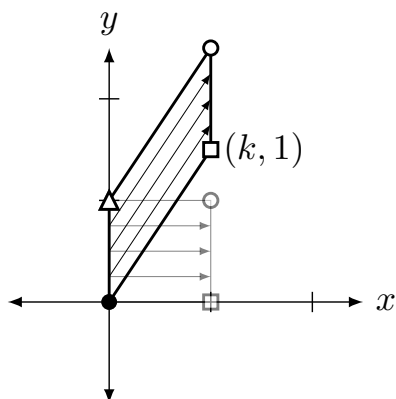
**Horizontal shear** by a factor of  $k$ .

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$



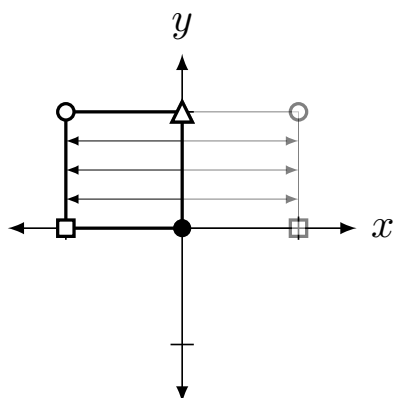
**Vertical shear** by a factor of  $k$ .

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$



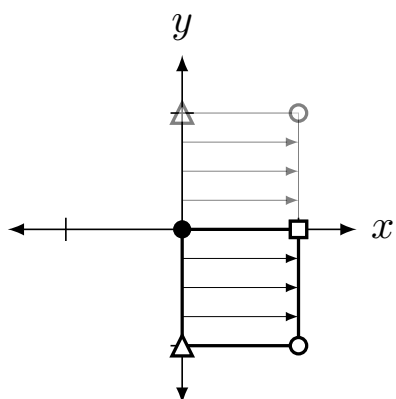
**Horizontal reflection** across the  $y$  axis.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



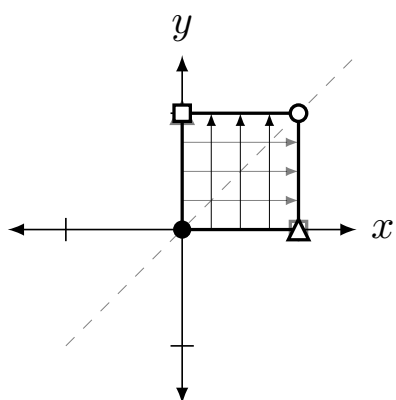
**Vertical reflection** across the  $x$  axis.

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



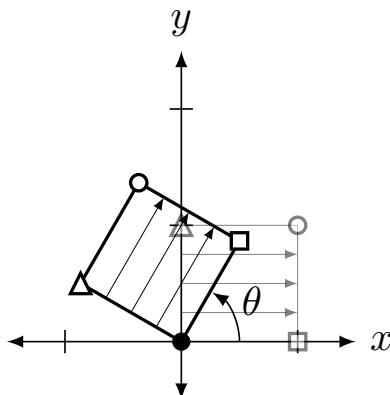
**Diagonal reflection** across the line  $y = x$ .

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



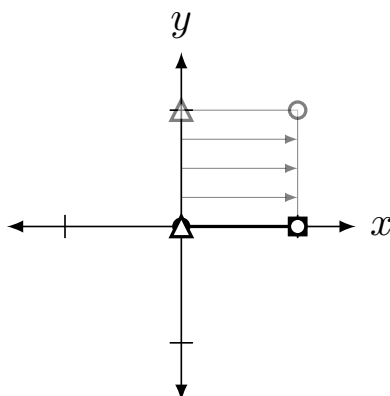
**Rotation** around the origin by an angle of  $\theta$ .

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$



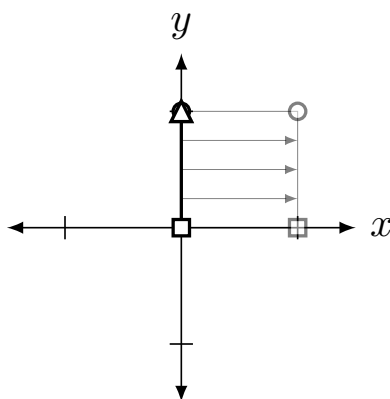
**Projection** onto the  $x$  axis. (Note how the square is “squashed” down onto the  $x$ -axis.)

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



**Projection** onto the  $y$  axis. (Note how the square is “squashed” over onto the  $y$ -axis.)

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



Now that we have seen a healthy list of transformations that we can perform on the Cartesian plane, let's practice a few more times creating the matrix that gives the desired transformation. In the following example, we develop our understanding one more critical step.

**Example 5.1.19 Determining the matrix of a transformation.**

Find the matrix  $A$  that transforms the Cartesian plane by performing the following operations in order:

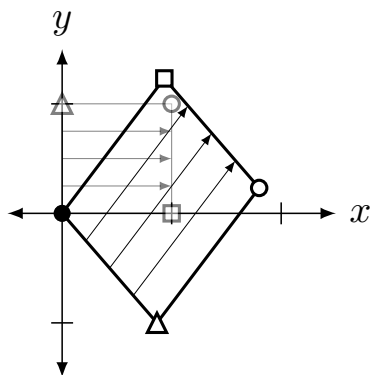
**Note:** recall from Section 4.2 that matrix multiplication is *not* commutative. Since  $A_1 A_2 \neq A_2 A_1$  in general, we should expect that the transformation  $A_2(A_1 \vec{x}) = (A_2 A_1) \vec{x}$  is not the same as the transformation  $A_1(A_2 \vec{x}) = (A_1 A_2) \vec{x}$ . Interpreted as transformations of the plane, this makes sense visually: for example, in most cases, a reflection followed by a rotation will not produce the same result as performing the rotation first, followed by the reflection. The reader is encouraged to experiment with a few examples to see what happens when the order is reversed.

To perform the final two operations, we note that

$$A_3 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

perform the horizontal stretch and diagonal reflection, respectively. Thus to perform all of the operations “at once,” we need to multiply by

$$\begin{aligned} A &= A_4 A_3 A_2 A_1 \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (\sqrt{3}+2)/4 & \sqrt{3}/2 \\ (2\sqrt{3}-1)/2 & -1 \end{bmatrix} \\ &\approx \begin{bmatrix} 0.933 & 0.866 \\ 1.232 & -1 \end{bmatrix}. \end{aligned}$$



**Figure 5.1.20** The transformed unit square in Example 5.1.19

1. Vertical shear by a factor of 0.5
2. Counterclockwise rotation about the origin by an angle of  $\theta = 30^\circ$
3. Horizontal stretch by a factor of 2
4. Diagonal reflection across the line  $y = x$

**Solution.** Wow! We already know how to do this — sort of. We know we can find the columns of  $A$  by tracing where  $\vec{e}_1$  and  $\vec{e}_2$  end up, but this also seems difficult. There is so much that is going on. Fortunately, we can accomplish what we need without much difficulty by being systematic.

First, let’s perform the vertical shear. The matrix that performs this is

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix}.$$

After that, we want to rotate everything clockwise by  $30^\circ$ . To do this, we use

$$A_2 = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}.$$

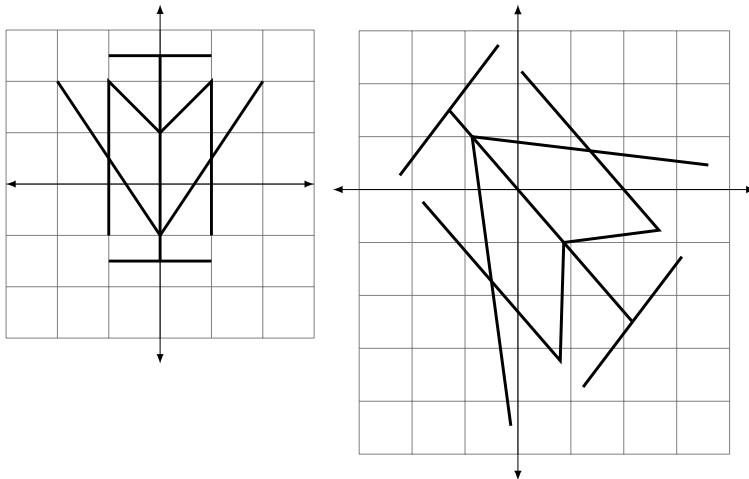
In order to do both of these operations, in order, we multiply  $A_2 A_1$ .

Let’s consider this closely. Suppose I want to know where a vector  $\vec{x}$  ends up. We claim we can find the answer by multiplying  $A\vec{x}$ . Why does this work? Consider:

$$\begin{aligned} A\vec{x} &= A_4 A_3 A_2 A_1 \vec{x} \\ &= A_4 A_3 A_2 (A_1 \vec{x}) && \text{(performs the vertical shear)} \\ &= A_4 A_3 (A_2 \vec{x}_1) && \text{(performs the rotation)} \\ &= A_4 (A_3 \vec{x}_2) && \text{(performs the horizontal stretch)} \\ &= A_4 \vec{x}_3 && \text{(performs the diagonal reflection)} \\ &= \vec{x}_4 && \text{(the result of transforming } \vec{x}) \end{aligned}$$

Most readers are not able to visualize exactly what the given list of operations does to the Cartesian plane. In Figure 5.1.20 we sketch the transformed unit square; in Figure 5.1.21 we sketch a shape and its transformation.





**Figure 5.1.21** A transformed shape in Example 5.1.19

Once we know what matrices perform the basic transformations, (or know where to find them) performing complex transformations on the Cartesian plane really isn't that . . . complex. It boils down to multiplying by a series of matrices.

We've shown many examples of transformations that we can do, and we've mentioned just a few that we can't — for instance, we can't turn a square into a circle. Why not? Why is it that straight lines get sent to straight lines?

All these questions require us to think like mathematicians — we are being asked to study the *properties* of an object we just learned about and their connections to things we've already learned. We'll do all this (and more!) in the following section.

## 5.1.4 Exercises

**Exercise Group.** A matrix  $A$  is given. Sketch  $\vec{x}$ ,  $\vec{y}$ ,  $A\vec{x}$ , and  $A\vec{y}$  on the same Cartesian axes, where

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \vec{y} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

1.  $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

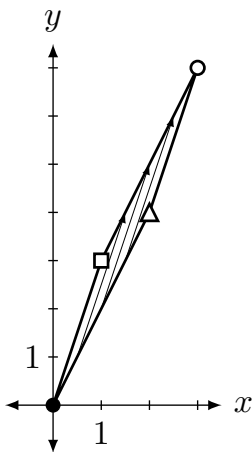
2.  $A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$

3.  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

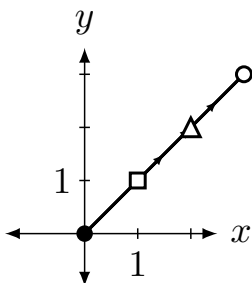
4.  $A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$

**Exercise Group.** A sketch of transformed unit square is given. Find the matrix  $A$  that performs this transformation.

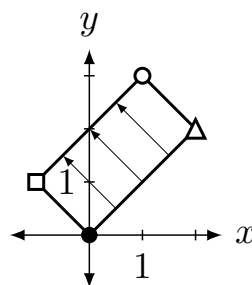
5.



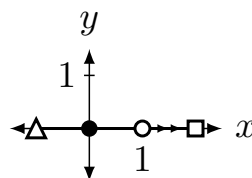
7.



6.



8.



**Exercise Group.** A list of transformations is given. Find the matrix  $A$  that performs those transformations, in order, on the Cartesian plane.

9.

- (a) vertical shear by a factor of 2
- (b) horizontal shear by a factor of 2

11.

- (a) horizontal stretch by a factor of 3
- (b) reflection across the line  $y = x$

10.

- (a) horizontal shear by a factor of 2
- (b) vertical shear by a factor of 2

12.

- (a) counterclockwise rotation by an angle of  $45^\circ$
- (b) vertical stretch by a factor of  $1/2$

13.

- (a) clockwise rotation by an angle of  $90^\circ$
- (b) horizontal reflection across the  $y$  axis
- (c) vertical shear by a factor of 1

14.

- (a) vertical reflection across the  $x$  axis
- (b) horizontal reflection across the  $y$  axis
- (c) diagonal reflection across the line  $y = x$

**Exercise Group.** Two sets of transformations are given. Sketch the transformed unit square under each set of transformations. Are the transformations the same? Explain why/why not.

15.

- (a) a horizontal reflection across the  $y$  axis, followed by a vertical reflection across the  $x$  axis, compared to
- (b) a counterclockwise rotation of  $180^\circ$

17.

- (a) a horizontal stretch by a factor of  $1/2$  followed by a vertical stretch by a factor of 3, compared to
- (b) the same operations but in opposite order

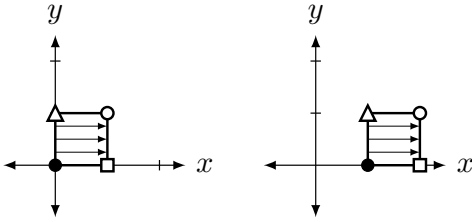
16.

- (a) a horizontal stretch by a factor of 2 followed by a reflection across the line  $y = x$ , compared to
- (b) a vertical stretch by a factor of 2

18.

- (a) a reflection across the line  $y = x$  followed by a reflection across the  $x$  axis, compared to
- (b) a reflection across the  $y$  axis, followed by a reflection across the line  $y = x$ .

## 5.2 Properties of Linear Transformations

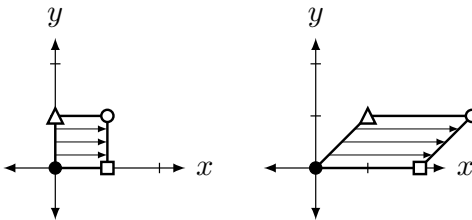


**Figure 5.2.1** Translating the unit square one unit to the right

In the previous section we discussed standard transformations of the Cartesian plane — rotations, reflections, etc.. As a motivational example for this section's study, let's consider another transformation. Let's find the matrix that moves the unit square one unit to the right (see Figure 5.2.1). This is called a **translation**.

Our work from the previous section allows us to find the matrix quickly. By looking at the picture, it is easy to see that  $\vec{e}_1$  is moved to  $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$  and  $\vec{e}_2$  is moved to  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Therefore, the transformation matrix should be

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$



**Figure 5.2.2** Actual transformation of the unit square by matrix  $A$

However, look at Figure 5.2.2 where the unit square is drawn after being transformed by  $A$ . It is clear that we did not get the desired result; the unit square was not translated, but rather stretched/sheared in some way.

What did we do wrong? We will answer this question, but first we need to develop a few thoughts and vocabulary terms.

We've been using the term "transformation" to describe how we've changed vectors. In fact, "transformation" is synonymous to "function." We are used to functions like  $f(x) = x^2$ , where the input is a number and the output is another number. In the previous section, we learned about transformations (functions) where the input was a vector and the output was another vector. If  $A$  is a "transformation matrix," then we could create a function of the form  $T(\vec{x}) = A\vec{x}$ . That is, a vector  $\vec{x}$  is the input, and the output is  $\vec{x}$  multiplied by  $A$ .

When we defined  $f(x) = x^2$  above, we let the reader assume that the input was indeed a number. If we wanted to be complete, we should have stated

$$f : \mathbb{R} \rightarrow \mathbb{R} \quad \text{where} \quad f(x) = x^2.$$

The first part of that line told us that the input was a real number (that was the first  $\mathbb{R}$ ) and the output was also a real number (the second  $\mathbb{R}$ ).

To define a transformation where a 2D vector is transformed into another 2D vector via multiplication by a  $2 \times 2$  matrix  $A$ , we should write

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{where} \quad T(\vec{x}) = A\vec{x}.$$

Here, the first  $\mathbb{R}^2$  means that we are using 2D vectors as our input, and the second  $\mathbb{R}^2$  means that a 2D vector is the output.

Consider a quick example:

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \quad \text{where} \quad T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 \\ 2x_1 \\ x_1x_2 \end{bmatrix}.$$

Notice that this takes 2D vectors as input and returns 3D vectors as output. For instance,

$$T\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 9 \\ 6 \\ -6 \end{bmatrix}.$$

We now define a special type of transformation (function).

We use  $T$  instead of  $f$  to define the function  $T(\vec{x}) = A\vec{x}$  to help differentiate it from "regular" functions. "Normally" functions are defined using lower case letters when the input is a number; when the input is a vector, we use upper case letters. (It also appears to be tradition to use the letter  $T$  to describe linear transformations, and mathematicians are suckers for tradition.)

**Definition 5.2.3 Linear Transformation.**

A transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a **linear transformation** if it satisfies the following two properties:

1.  $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$  for all vectors  $\vec{x}$  and  $\vec{y}$ , and
2.  $T(k\vec{x}) = kT(\vec{x})$  for all vectors  $\vec{x}$  and all scalars  $k$ .

If  $T$  is a linear transformation, it is often said that “ $T$  is **linear**.”

Note that the two defining properties of a linear transformation, when combined, tell us that linear transformations “map linear combinations to linear combinations.” That is, if we know the values of  $T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_k)$ , and we’re given  $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$ , then

$$\begin{aligned} T(\vec{v}) &= T(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k) \\ &= T(c_1\vec{v}_1) + T(c_2\vec{v}_2) + \dots + T(c_k\vec{v}_k) \\ &= c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \dots + c_kT(\vec{v}_k). \end{aligned}$$

**Example 5.2.4 Computing a linear transformation from given values.**

Given that  $T : \mathbb{R}_3 \rightarrow \mathbb{R}_2$  is a linear transformation such that

$$T(\vec{u}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, T(\vec{v}) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \text{ and } T(\vec{w}) = \begin{bmatrix} 1 \\ -4 \end{bmatrix},$$

compute the value of  $T(2\vec{u} - 3\vec{v} + \vec{w})$ .

**Solution.** As argued above, we have

$$\begin{aligned} T(2\vec{u} - 3\vec{v} + \vec{w}) &= 2T(\vec{u}) - 3T(\vec{v}) + T(\vec{w}) \\ &= 2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -4 \end{bmatrix} \\ &= \begin{bmatrix} -10 \\ 0 \end{bmatrix}. \end{aligned}$$

**Example 5.2.5 Computing a linear transformation from given values.**

Suppose  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  is a linear transformation such that

$$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}.$$

Compute the value of  $T\left(\begin{bmatrix} 1 \\ 11 \end{bmatrix}\right)$ .

**Solution.** This problem takes some more work, since we first need to figure out how to write the vector  $\begin{bmatrix} 1 \\ 11 \end{bmatrix}$  as a linear combination of the

vectors  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ . That is, we need to find scalars  $a$  and  $b$  such that

$$a \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

Writing the left-hand side as the single vector  $\begin{bmatrix} 2a - b \\ a + 3b \end{bmatrix}$ , we see that this amounts to solving the pair of equations

$$\begin{array}{rcrcrcrcl} 2a & - & b & = & 1 \\ a & + & 3b & = & 11 \end{array}.$$

If we multiply the first equation by 3 and add it to the second, we get  $7a + 0b = 14$ , so  $a = 2$ . Plugging this value back into either equation gives us  $b = 3$ , so we have

$$\begin{aligned} T\left(\begin{bmatrix} 1 \\ 11 \end{bmatrix}\right) &= T\left(2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 3 \end{bmatrix}\right) \\ &= 2T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) + 3T\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right) \\ &= 2 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 9 \end{bmatrix}. \end{aligned}$$

Notice that in Example 5.2.5, in order to make use of the properties of our linear transformation, we had to first solve a system of linear equations. With two equations in two unknowns, it's not too hard to come up with the answer. As the size and number of the vectors involved increases, such problems cannot be tackled without a systematic method for solving systems of linear equations. Fortunately, we will be introducing just such a method in Chapter 3.

The previous two examples show us what we can do if we know in advance that our transformation is linear. The next two examples show us how to determine if a given transformation is indeed a linear transformation.

### Example 5.2.6 Identifying linear transformations.

Determine whether or not the transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  is a linear transformation, where

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 \\ 2x_1 \\ x_1x_2 \end{bmatrix}.$$

**Solution.** We'll arbitrarily pick two vectors  $\vec{x}$  and  $\vec{y}$ :

$$\vec{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

Let's check to see if  $T$  is linear by using the definition.

Is  $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ ? First, compute  $\vec{x} + \vec{y}$ :

$$\vec{x} + \vec{y} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

Now compute  $T(\vec{x})$ ,  $T(\vec{y})$ , and  $T(\vec{x} + \vec{y})$ :

$$T(\vec{x}) = T\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 9 \\ 6 \\ -6 \end{bmatrix}$$

$$T(\vec{y}) = T\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$T(\vec{x} + \vec{y}) = T\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 16 \\ 8 \\ 12 \end{bmatrix}.$$

Is  $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ ?

$$\begin{bmatrix} 9 \\ 6 \\ -6 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \neq \begin{bmatrix} 16 \\ 8 \\ 12 \end{bmatrix}.$$

Therefore,  $T$  is **not** a linear transformation.

So we have an example of something that *doesn't* work. Let's try an example where things *do* work.

#### Example 5.2.7 Identifying linear transformations.

Determine whether or not the transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a linear transformation, where  $T(\vec{x}) = A\vec{x}$  and

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

**Solution.** Let's start by again choosing a couple of vectors and seeing what happens. Let's choose the same  $\vec{x}$  and  $\vec{y}$  from Example 5.2.6.

$$\vec{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

If the linearity properties hold for these vectors, then *maybe* it is actually linear (and we'll do more work).

Is  $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ ? Recall:

$$\vec{x} + \vec{y} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

Now compute  $T(\vec{x})$ ,  $T(\vec{y})$ , and  $T(\vec{x} + \vec{y})$ :

$$T(\vec{x}) = T\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$T(\vec{y}) = T\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 11 \\ 23 \end{bmatrix}$$

$$T(\vec{x} + \vec{y}) = T\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 10 \\ 24 \end{bmatrix}.$$

In Example 5.2.7, it's important to remember the following principle of logic: to show that something doesn't work, we just need to show one case where it fails, which we did in Example 5.2.6. To show that something *always* works, we need to show it works for *all* cases — simply showing it works for a few cases isn't enough. (An example is not a proof.) However, doing so can be helpful in understanding the situation better, and can give us clues as to how to construct a general proof.

Is  $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ ?

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 11 \\ 23 \end{bmatrix} = \begin{bmatrix} 10 \\ 24 \end{bmatrix} \quad !.$$

So far, so good:  $T(\vec{x} + \vec{y})$  is equal to  $T(\vec{x}) + T(\vec{y})$ .

Is  $T(k\vec{x}) = kT(\vec{x})$ ? Let's pick  $k = 7$  (or whatever value you prefer), and use  $\vec{x}$  as before.

$$\begin{aligned} T(7\vec{x}) &= T\left(\begin{bmatrix} 21 \\ -14 \end{bmatrix}\right) \\ &= \begin{bmatrix} -7 \\ 7 \end{bmatrix} \\ &= 7 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= 7 \cdot T(\vec{x}) \quad ! \end{aligned}$$

So far it *seems* that  $T$  is indeed linear, for it worked in one example with arbitrarily chosen vectors and scalar. Now we need to try to show it is always true.

Consider  $T(\vec{x} + \vec{y})$ . By the definition of  $T$ , we have

$$T(\vec{x} + \vec{y}) = A(\vec{x} + \vec{y}).$$

By Theorem 4.2.11, part Item 2 we state that the Distributive Property holds for matrix multiplication. (Recall that a vector is just a special type of matrix, so this theorem applies to matrix-vector multiplication as well.) So  $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$ . Recognize now that this last part is just  $T(\vec{x}) + T(\vec{y})$ ! We repeat the above steps, all together:

$$\begin{aligned} T(\vec{x} + \vec{y}) &= A(\vec{x} + \vec{y}) \quad (\text{by the definition of } T \text{ in this example}) \\ &= A\vec{x} + A\vec{y} \quad (\text{by the Distributive Property}) \\ &= T(\vec{x}) + T(\vec{y}) \quad (\text{again, by the definition of } T). \end{aligned}$$

Therefore, no matter what  $\vec{x}$  and  $\vec{y}$  are chosen,  $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ . Thus the first part of the linearity definition is satisfied.

The second part is satisfied in a similar fashion. Let  $k$  be a scalar, and consider:

$$\begin{aligned} T(k\vec{x}) &= A(k\vec{x}) \quad (\text{by the definition of } T \text{ in this example}) \\ &= kA\vec{x} \quad (\text{by Theorem 4.2.11 part Item 3}) \\ &= kT(\vec{x}) \quad (\text{again, by the definition of } T) \end{aligned}$$

Since  $T$  satisfies both parts of the definition, we conclude that  $T$  is a linear transformation.

In the previous two examples of transformations, we saw one transformation that was not linear and one that was. One might wonder “Why is linearity important?”, which we’ll address shortly.

First, consider how we proved the transformation in Example 5.2.7 was linear. We defined  $T$  by matrix multiplication, that is,  $T(\vec{x}) = A\vec{x}$ . We proved  $T$  was linear using properties of matrix multiplication — we *never considered the specific values of  $A$* ! That is, we didn’t just choose a good matrix for  $T$ ; any



matrix  $A$  would have worked. This leads us to an important theorem. The first part we have essentially just proved; the second part we won't prove, although its truth is very powerful.

**Theorem 5.2.8 Matrices and Linear Transformations.**

1. Define  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  by  $T(\vec{x}) = A\vec{x}$ , where  $A$  is an  $m \times n$  matrix. Then  $T$  is a linear transformation.
2. Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be any linear transformation. Then there exists a unique  $m \times n$  matrix  $A$  such that  $T(\vec{x}) = A\vec{x}$ .

The second part of the theorem says that *all* linear transformations can be described using matrix multiplication. Given *any* linear transformation, there is a matrix that completely defines that transformation. This important matrix gets its own name.

**Definition 5.2.9 Standard Matrix of a Linear Transformation.**

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. By Theorem 5.2.8, there is a matrix  $A$  such that  $T(\vec{x}) = A\vec{x}$ . This matrix  $A$  is called the **standard matrix of the linear transformation**  $T$ , and is denoted  $[T]$ .

While exploring all of the ramifications of Theorem 5.2.8 is outside the scope of this text, let it suffice to say that since 1) linear transformations are very, very important in economics, science, engineering and mathematics, and 2) the theory of matrices is well developed and easy to implement by hand and on computers, then 3) it is great news that these two concepts go hand in hand.

We have already used the second part of this theorem in a small way. In the previous section we looked at transformations graphically and found the matrices that produced them. At the time, we didn't realize that these transformations were linear, but indeed they were.

This brings us back to the motivating example with which we started this section. We tried to find the matrix that translated the unit square one unit to the right. Our attempt failed, and we have yet to determine why. Given our link between matrices and linear transformations, the answer is likely "the translation transformation is not a linear transformation." While that is a true statement, it doesn't really explain things all that well. Is there some way we could have recognized that this transformation wasn't linear? (That is, apart from applying the definition directly?)

Yes, there is. Consider the second part of the linear transformation definition. It states that  $T(k\vec{x}) = kT(\vec{x})$  for all scalars  $k$ . If we let  $k = 0$ , we have  $T(0\vec{x}) = 0 \cdot T(\vec{x})$ , or more simply,  $T(\vec{0}) = \vec{0}$ . That is, if  $T$  is to be a linear transformation, it must send the zero vector to the zero vector.

This is a quick way to see that the translation transformation fails to be linear. By shifting the unit square to the right one unit, the corner at the point  $(0, 0)$  was sent to the point  $(1, 0)$ , i.e.,

the vector  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$  was sent to the vector  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

This property relating to  $\vec{0}$  is important, so we highlight it here.

The matrix-like brackets around  $T$  are intended to suggest that the standard matrix  $A$  is a matrix representative of the given transformation.

The idea that linear transformations “send zero to zero” has an interesting relation to terminology. The reader is likely familiar with functions like  $f(x) = 2x + 3$  and would likely refer to this as a “linear function.” However,  $f(0) \neq 0$ , so  $f$  is not “linear” by our new definition of linear. We erroneously call  $f$  “linear” since its graph produces a line, though we should be careful to instead state that “the graph of  $f$  is a line.”

In the context of calculus are only linear transformations if  $b = 0$ . In the case of maps  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , we can similarly consider functions of the form  $T(\vec{x}) = A\vec{x} + \vec{b}$ ; such functions are again useful in areas such as vector calculus, but unless  $\vec{b} = \vec{0}$ , they are not linear transformations, since  $T(\vec{0}) = \vec{b}$ .

Mathematicians do, of course, have a word for such maps: a function of the form  $T(\vec{x}) = A\vec{x} + \vec{b}$  is known as an **affine transformation**.

Recall that in Example 5.2.5, we had to solve a system of equations to determine how to write our given vector in terms of two other vectors for which the value of a transformation is known. The advantage of knowing the values of a transformation on the standard unit vectors is that we know immediately how to write any other vector in terms of them.

### Key Idea 5.2.10 Linear Transformations and $\vec{0}$ .

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. Then:

$$T(\vec{0}_n) = \vec{0}_m.$$

That is, the zero vector in  $\mathbb{R}^n$  gets sent to the zero vector in  $\mathbb{R}^m$ .

**The Standard Matrix of a Linear Transformation.** It is often the case that while one can describe a linear transformation, one doesn't know what matrix performs that transformation (i.e., one doesn't know the standard matrix of that linear transformation). How do we systematically find it? We'll need a new definition.

### Definition 5.2.11 Standard Unit Vectors.

In  $\mathbb{R}^n$ , the **standard unit vectors**  $\vec{e}_i$  are the vectors with a 1 in the  $i$ th entry and 0s everywhere else.

We've already seen these vectors in the previous section. In  $\mathbb{R}^2$ , we identified

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In  $\mathbb{R}^4$ , there are 4 standard unit vectors:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

How do these vectors help us find the standard matrix of a linear transformation? Recall again our work in the previous section. There, we practised looking at the transformed unit square and deducing the standard transformation matrix  $A$ . We did this by making the first column of  $A$  the vector where  $\vec{e}_1$  ended up and making the second column of  $A$  the vector where  $\vec{e}_2$  ended up. One could represent this with:

$$A = [T(\vec{e}_1) \quad T(\vec{e}_2)] = [T].$$

That is,  $T(\vec{e}_1)$  is the vector where  $\vec{e}_1$  ends up, and  $T(\vec{e}_2)$  is the vector where  $\vec{e}_2$  ends up.

The same holds true in general. Given a linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , the standard matrix of  $T$  is the matrix whose  $i$ th column is the vector where  $\vec{e}_i$  ends up. To see that this is the case, note that any vector  $\vec{x} \in \mathbb{R}^n$  can be written as

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n,$$

and by Definition 5.2.3, we have

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n) \\ &= x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \cdots + x_nT(\vec{e}_n) \\ &= [T(\vec{e}_1) \quad T(\vec{e}_2) \quad \cdots \quad T(\vec{e}_n)] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \end{aligned}$$

$$= A\vec{x},$$

where  $A = [T(\vec{e}_1) \ T(\vec{e}_2) \ \cdots \ T(\vec{e}_n)]$  is the  $m \times n$  matrix whose columns are given by the vectors  $T(\vec{e}_i)$ , for  $i = 1, 2, \dots, n$ . Thus, we have the following theorem.

**Theorem 5.2.12 The Standard Matrix of a Linear Transformation.**

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. Then  $[T]$  is the  $m \times n$  matrix:

$$[T] = [T(\vec{e}_1) \ T(\vec{e}_2) \ \cdots \ T(\vec{e}_n)].$$

Let's practice this theorem in an example.

**Example 5.2.13 Computing the matrix of a linear transformation.**

Define  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  to be the linear transformation where

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 + x_2 \\ 3x_1 - x_3 \\ 2x_2 + 5x_3 \\ 4x_1 + 3x_2 + 2x_3 \end{bmatrix}.$$

Find  $[T]$ .

**Solution.**  $T$  takes vectors from  $\mathbb{R}^3$  into  $\mathbb{R}^4$ , so  $[T]$  is going to be a  $4 \times 3$  matrix. Note that

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We find the columns of  $[T]$  by finding where  $\vec{e}_1$ ,  $\vec{e}_2$  and  $\vec{e}_3$  are sent, that is, we find  $T(\vec{e}_1)$ ,  $T(\vec{e}_2)$  and  $T(\vec{e}_3)$ .

$$T(\vec{e}_1) = T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 4 \end{bmatrix}$$

$$T(\vec{e}_2) = T \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$$

$$T(\vec{e}_3) = T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ -1 \\ 5 \\ 2 \end{bmatrix}$$

Thus

$$[T] = A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & -1 \\ 0 & 2 & 5 \\ 4 & 3 & 2 \end{bmatrix}.$$

Let's check this. Consider the vector

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Strictly from the original definition, we can compute that

$$T(\vec{x}) = T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1+2 \\ 3-3 \\ 4+15 \\ 4+6+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 19 \\ 16 \end{bmatrix}.$$

Now compute  $T(\vec{x})$  by computing  $[T]\vec{x} = A\vec{x}$ .

$$A\vec{x} = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & -1 \\ 0 & 2 & 5 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 19 \\ 16 \end{bmatrix}.$$

They match! (Of course they do. That was the whole point.)

Let's do another example, one that is more application oriented.

#### Example 5.2.14 An application to baseball.

A baseball team manager has collected basic data concerning his hitters. He has the number of singles, doubles, triples, and home runs they have hit over the past year. For each player, he wants two more pieces of information: the total number of hits and the total number of bases.

Using the techniques developed in this section, devise a method for the manager to accomplish his goal.

**Solution.** If the manager only wants to compute this for a few players, then he could do it by hand fairly easily. After all:

total # hits = # of singles + # of doubles + # of triples + # of home runs,

and

total # bases = # of singles + 2 × # of doubles + 3 × # of triples + 4 × # of home runs.

However, if he has a lot of players to do this for, he would likely want a way to automate the work. One way of approaching the problem starts with recognizing that he wants to input four numbers into a function (i.e., the number of singles, doubles, etc.) and he wants two numbers as output (i.e., number of hits and bases). Thus he wants a transformation  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^2$  where each vector in  $\mathbb{R}^4$  can be interpreted as

$$\begin{bmatrix} \text{\# of singles} \\ \text{\# of doubles} \\ \text{\# of triples} \\ \text{\# of home runs} \end{bmatrix},$$

and each vector in  $\mathbb{R}^2$  can be interpreted as

$$\begin{bmatrix} \# \text{ of hits} \\ \# \text{ of bases} \end{bmatrix}.$$

To find  $[T]$ , he computes  $T(\vec{e}_1)$ ,  $T(\vec{e}_2)$ ,  $T(\vec{e}_3)$  and  $T(\vec{e}_4)$ .

$$T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$T(\vec{e}_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$T(\vec{e}_4) = T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

(What do these calculations mean? For example, finding  $T(\vec{e}_3) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

means that one triple counts as 1 hit and 3 bases.)

Thus our transformation matrix  $[T]$  is

$$[T] = A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

As an example, consider a player who had 102 singles, 30 doubles, 8 triples and 14 home runs. By using  $A$ , we find that

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 102 \\ 30 \\ 8 \\ 14 \end{bmatrix} = \begin{bmatrix} 154 \\ 242 \end{bmatrix},$$

meaning the player had 154 hits and 242 total bases.

A question that we should ask concerning the previous example is “How do we know that the function the manager used was actually a linear transformation? After all, we were wrong before — the translation example at the beginning of this section had us fooled at first.”

This is a good point; the answer is fairly easy. Recall from Example 5.2.6 the transformation

$$T_{6.2.3}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 \\ 2x_1 \\ x_1x_2 \end{bmatrix}$$

and from Example 5.2.13

$$T_{6.2.5} \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 + x_2 \\ 3x_1 - x_3 \\ 2x_2 + 5x_3 \\ 4x_1 + 3x_2 + 2x_3 \end{bmatrix},$$

where we use the subscripts for  $T$  to remind us which example they came from.

We found that  $T_{6.2.3}$  was not a linear transformation, but stated that  $T_{6.2.5}$  was (although we didn't prove this). What made the difference?

Look at the entries of  $T_{6.2.3}(\vec{x})$  and  $T_{6.2.5}(\vec{x})$ .  $T_{6.2.3}$  contains entries where a variable is squared, and where 2 variables are multiplied together — these prevent  $T_{6.2.3}$  from being linear. On the other hand, the entries of  $T_{6.2.5}$  are all of the form  $a_1x_1 + \cdots + a_nx_n$ ; that is, they are just sums of the variables multiplied by coefficients.  $T$  is linear if and only if the entries of  $T(\vec{x})$  are of this form. (Hence linear transformations are related to linear equations, as defined in Section 3.1.) This idea is important.

**Key Idea 5.2.15 Conditions on Linear Transformations.**

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a transformation and consider the entries of

$$T(\vec{x}) = T \left( \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right).$$

$T$  is linear if and only if each entry of  $T(\vec{x})$  is of the form  $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ .

Going back to our baseball example, the manager could have defined his transformation as

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = \begin{bmatrix} x_1 + x_2 + x_3 + x_4 \\ x_1 + 2x_2 + 3x_3 + 4x_4 \end{bmatrix}.$$

Since that fits the model shown in Key Idea 5.2.15, the transformation  $T$  is indeed linear and hence we can find a matrix  $[T]$  that represents it.

Let's practice this concept further in an example.

**Example 5.2.16 Using Key Idea 5.2.15 to identify linear transformations.**

Using Key Idea 5.2.15, determine whether or not each of the following transformations is linear.

$$\begin{aligned} T_1 \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) &= \begin{bmatrix} x_1 + 1 \\ x_2 \end{bmatrix} & T_2 \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) &= \begin{bmatrix} x_1/x_2 \\ \sqrt{x_2} \end{bmatrix} \\ T_3 \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) &= \begin{bmatrix} \sqrt{7}x_1 - x_2 \\ \pi x_2 \end{bmatrix} & & \end{aligned}$$

**Solution.**  $T_1$  is not linear! This may come as a surprise, but we are not allowed to add constants to the variables. By thinking about this, we can

see that this transformation is trying to accomplish the translation that got us started in this section — it adds 1 to all the  $x$  values and leaves the  $y$  values alone, shifting everything to the right one unit. However, this is not linear; again, notice how  $\vec{0}$  does not get mapped to  $\vec{0}$ .

$T_2$  is also not linear. We cannot divide variables, nor can we put variables inside the square root function (among other other things; again, see Section 3.1). This means that the baseball manager would not be able to use matrices to compute a batting average, which is (number of hits)/(number of at bats).

$T_3$  is linear. Recall that the coefficients  $\sqrt{7}$  and  $\pi$  are just numbers.

In the next section we introduce the concept of a **subspace**. This subject is closely related to the concepts of span and linear independence covered in Section 2.7. We introduce it here in preparation for our final section, where we will define two important subspaces associated to a matrix transformation: the null space and the column space.

### 5.2.1 Exercises

**Exercise Group.** A transformation  $T$  is given. Determine whether or not  $T$  is linear; if not, state why not.

$$1. \quad T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + x_2 \\ 3x_1 - x_2 \end{bmatrix}$$

$$2. \quad T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + x_2^2 \\ x_1 - x_2 \end{bmatrix}$$

$$3. \quad T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 1 \\ x_2 + 1 \end{bmatrix}$$

$$4. \quad T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$5. \quad T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

**Exercise Group.** A linear transformation  $T$  is given. Find  $[T]$ .

$$6. \quad T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 - 5x_2 \\ 2x_2 \end{bmatrix}$$

$$7. \quad T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = [x_1 + 2x_2 + 3x_3 + 4x_4]$$

$$8. \quad T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$9. \quad T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$$

$$10. \quad T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 2x_2 - 3x_3 \\ 0 \\ x_1 + 4x_3 \\ 5x_2 + x_3 \end{bmatrix}$$

$$11. \quad T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 3x_3 \\ x_1 - x_3 \\ x_1 + x_3 \end{bmatrix}$$



## 5.3 Subspaces of $\mathbb{R}^n$

One of the things we noted in Example 2.7.7 was that since  $\vec{w}$  belonged to  $\text{span}\{\vec{u}, \vec{v}\}$ , adding  $\vec{w}$  to any vector in  $\text{span}\{\vec{u}, \vec{v}\}$  resulted in another vector in  $\text{span}\{\vec{u}, \vec{v}\}$ . This leads to the notion of a **subspace**, another one of the key concepts in linear algebra.

What sets subspaces apart from other subsets of  $\mathbb{R}^n$  is the requirement that all of the properties listed in Theorem 2.7.2 remain valid when applied to vectors from that subspace. We will not prove it here, but it suffices that the subspace be *closed* under the operations of addition and scalar multiplication.

### Definition 5.3.1 Subspace of $\mathbb{R}^n$ .

A non-empty subset  $V \subseteq \mathbb{R}^n$  is called a **subspace** of  $\mathbb{R}^n$ , provided that the following conditions hold:

1. For any vectors  $\vec{u}, \vec{v} \in V$ ,  $\vec{u} + \vec{v} \in V$
2. For any vector  $\vec{v} \in V$  and scalar  $c \in \mathbb{R}$ ,  $c\vec{v} \in V$ .

It follows from Definition 5.3.1 that any linear combination of vectors in a subspace  $V$  is again an element of that subspace. One other important consequence of Definition 5.3.1 must be noted here: since any subspace  $V$  is closed under scalar multiplication by *any* scalar, and since  $0 \cdot \vec{v} = \vec{0}$  for any vector  $\vec{v}$ , *every subspace contains the zero vector*. This often provides an easy test when we want to rule out the possibility that a subset is a subspace.

### Example 5.3.2 Identifying subspaces.

Determine which of the following subsets of  $\mathbb{R}^3$  are subspaces:

$$1. R = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid 2x - 4y + 3z = 0 \right\}$$

$$2. S = \left\{ \begin{bmatrix} x \\ 4x \\ 3 \end{bmatrix} \mid x \in \mathbb{R} \right\}$$

#### Solution.

1. From our work in Section 2.6, we notice right away that the set

$R$  describes a plane with normal vector given by  $\vec{n} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$ .

Moreover, this particular plane passes through the origin, since  $2(0) - 4(0) + 3(0) = 0$ , telling us that  $\vec{0} \in R$ .

It turns out that any plane through the origin is a subspace, and  $R$  is no exception, but let's verify this directly using Definition 5.3.1.

Suppose

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

are vectors in  $R$ , so that  $2u_1 - 4u_2 + 3u_3 = 0$  and  $2v_1 - 4v_2 + 3v_3 =$

Another way of phrasing Definition 5.3.1 is to say that  $V$  is a subspace if any linear combination of vectors in  $V$  is another vector in  $V$ ; that is,  $V$  is closed under taking linear combinations.

A subspace of  $\mathbb{R}^n$  is a subset that looks like a copy of  $\mathbb{R}^m$ , where  $m \leq n$ . The visual examples you should keep in mind are lines (which look like copies of  $\mathbb{R}$ ) and planes (which look like copies of  $\mathbb{R}^2$ ) in  $\mathbb{R}^3$ . However, not all lines and planes will do: as we will see, only those lines and planes that pass through the origin form subspaces.

0. For the vector

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix},$$

we find

$$\begin{aligned} 2(u_1 + v_1) - 4(u_2 + v_2) + 3(u_3 + v_3) &= (2u_1 - 4u_2 + 3u_3) + (2v_1 - 4v_2 + 3v_3) \\ &= 0 + 0 = 0, \end{aligned}$$

which shows that  $\vec{u} + \vec{v} \in R$ . Similarly, for any scalar  $c$ ,

$$2(cu_1) - 4(cu_2) + 3(cu_3) = c(2u_1 - 4u_2 + 3u_3) = 0,$$

verifying that  $c\vec{u} \in R$ . Since  $R$  is closed under both addition and scalar multiplication, it is a subspace.

2. For the subset  $S$ , we immediately notice that the third component must always equal 3; therefore, it is impossible for the zero vector to belong to  $S$ , and thus  $S$  is not a subspace.

To make sure we've got the hang of things, we'll try a couple more examples.

### Example 5.3.3 Identifying subspaces.

Determine which of the following subsets of  $\mathbb{R}^3$  are subspaces:

$$1. T = \left\{ \begin{bmatrix} 3a - 2b \\ a + b \\ a - 4b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

$$2. U = \left\{ \begin{bmatrix} x + y \\ 3xy \\ x^2 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$$

**Solution.**

$$1. \text{ Suppose we have two vectors } \vec{v} = \begin{bmatrix} 3a - 2b \\ a + b \\ a - 4b \end{bmatrix} \text{ and } \vec{w} =$$

$$\begin{bmatrix} 3c - 2d \\ c + d \\ a - 4b \end{bmatrix} \text{ in the subset } T. \text{ Then we find}$$

$$\begin{aligned} \vec{v} + \vec{w} &= \begin{bmatrix} 3a - 2b \\ a + b \\ a - 4b \end{bmatrix} + \begin{bmatrix} 3c - 2d \\ c + d \\ a - 4b \end{bmatrix} = \begin{bmatrix} (3a - 2b) + (3c - 2d) \\ (a + b) + (c + d) \\ (a - 4b) + (c - 4d) \end{bmatrix} \\ &= \begin{bmatrix} 3(a + c) - 2(b + d) \\ (a + c) + (b + d) \\ (a + c) - 4(b + d) \end{bmatrix}. \end{aligned}$$

Since  $a + c$  and  $b + d$  are again real numbers, we see that  $\vec{v} + \vec{w}$  fits the definition of  $T$ , so  $\vec{v} + \vec{w} \in T$ .

Next, if  $k \in \mathbb{R}$  is a scalar, we have

$$k\vec{v} = k \begin{bmatrix} 3a - 2b \\ a + b \\ a - 4b \end{bmatrix} = \begin{bmatrix} k(3a - 2b) \\ k(a + b) \\ k(a - 4b) \end{bmatrix} = \begin{bmatrix} 3(ka) - 2(kb) \\ (ka) + (kb) \\ (ka) - 4(kb) \end{bmatrix},$$

which again fits the pattern for vectors in  $T$ , so  $k\vec{v} \in T$ . From Definition 5.3.1, we can conclude that  $T$  is a subspace.

2. You probably recall from your high school mathematics that a function such as  $f(x) = ax + b$  is considered *linear*, since its graph is a straight line. Functions like  $f(x) = x^2$  are considered non-linear, since their graphs are curved. One of the morals a student does well to learn quickly in *linear algebra* is that any expressions involving non-linear functions of any variables present are not going to play well with the rules of linear algebra.

For the subset  $U$ , the expressions  $3xy$  and  $x^2$  tip us off that we are probably not dealing with a subspace here. The easiest way to make sure of this is to check the rules in Definition 5.3.1 using *specific* vectors.

If we let  $x = 1$  and  $y = 2$  in the definition of the set  $U$ , we get the vector

$$\vec{v} = \begin{bmatrix} 1 + 2 \\ 3(1)(2) \\ 1^2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix}.$$

Now consider the vector  $4\vec{v}$ . We have

$$4\vec{v} = 4 \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 24 \\ 4 \end{bmatrix}.$$

Looking at the definition of the set  $U$ , we know that if  $4\vec{v} \in U$ , then the third component of  $\vec{v}$  tells us  $x^2 = 4$ , so  $x = \pm 2$ . Now, let's look at the other two components. If  $x = 2$ , we must have

$$2 + y = 12 \text{ and } 3(2)(y) = 6.$$

The first equation tells us that  $y = 10$ , while the second requires  $y = 1$ . Since  $10 \neq 1$ , this is impossible. Similarly, if  $x = -2$ , then we would have to have  $y = 14$  looking at the first component, and  $y = -1$  from the second. Since this is again impossible, it must be the case that  $4\vec{v} \notin U$ . Since  $U$  is not closed under scalar multiplication,  $U$  is not a subspace.

**Note:** When we want to show that a subset is a subspace, we have to verify that Definition 5.3.1 is satisfied for *all* possible vectors in that set. Since we generally have infinitely many vectors to deal with, our verification is going to require us to give a general argument, using variables instead of numbers.

On the other hand, if we want to show that a subset is *not* a subspace, we just have to show that Definition 5.3.1 fails for one or two *specific* vectors. A choice of vector(s) for which the definition fails is called a **counterexample**. When constructing a counterexample it's a good idea to choose small numbers to keep the arithmetic simple.

After seeing a few examples (and a few exercises), the reader can probably develop some intuition for identifying subspaces. To make sure we don't become too reliant on intuition, however, we'll give one more example with two very similar-looking sets, only one of which is a subspace.

**Example 5.3.4 Identifying subspaces.**

Determine which of the following subsets of  $\mathbb{R}^3$  are subspaces:

$$1. V = \left\{ \begin{bmatrix} u + 2v \\ v + 4 \\ u - 2 \end{bmatrix} \mid u, v \in \mathbb{R} \right\}$$

$$2. W = \left\{ \begin{bmatrix} 2u + v \\ v + 4 \\ u - 2 \end{bmatrix} \mid u, v \in \mathbb{R} \right\}$$

**Solution.**

1. The expressions  $v + 4$  and  $u - 3$  in the definition of  $V$  look like the sort of linear functions we see in high school, but we need to keep in mind that in linear algebra the zero vector has an important role in making sure the algebra works properly. In Linear Algebra, among all functions of the form  $f(x) = mx + b$ , only those with  $b = 0$  are considered “linear”: these are the functions whose graphs are lines through the origin.

The first thing we might check is whether or not  $\vec{0} \in V$ . If we want

$$\begin{bmatrix} u + 2v \\ v + 4 \\ u - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

then clearly we need  $v = -4$  and  $u = 2$  from the second and third components, but  $2 + 2(-4) = -6 \neq 0$ , so there is no way to obtain the zero vector as an element of  $V$ , telling us that  $V$  is not a subspace.

2. The subset  $W$  looks a lot like the subset  $V$ , so our instinct is probably telling us that  $W$  is not a subspace, either. To know for sure, the first thing we might check is whether or not  $\vec{0} \in W$ . In this case, we see that  $\vec{0}$  is indeed in there. Setting  $u = 2$  and  $v = -4$ , we get the vector

$$\begin{bmatrix} 2(2) + (-4) \\ -4 + 4 \\ 2 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0},$$

so  $\vec{0} \in W$ . Let's try the addition test. Setting  $u = 1$  and  $v = 0$  gives us the vector

$$\vec{v} = \begin{bmatrix} 2(1) + 0 \\ 0 + 4 \\ 1 - 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} \in W.$$

Setting  $u = 0$  and  $v = 1$  gives us the vector

$$\vec{w} = \begin{bmatrix} 2(0) + 1 \\ 1 + 4 \\ 0 - 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} \in W.$$

We now check to see whether or not  $\vec{v} + \vec{w} \in W$ . We have

$$\vec{v} + \vec{w} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ -3 \end{bmatrix}.$$

If this is an element of  $W$ , then we must have  $v + 4 = 9$  for some  $v \in \mathbb{R}$  (looking at the second component) and  $u - 2 = -3$  for some  $u \in \mathbb{R}$  (looking at the third component), so  $u = -1$  and  $v = 5$ . Putting these values into the first component, we need to have  $2(-1) + 5 = 3$ , which is true! Does this mean  $W$  is a subspace? Not so fast: we only checked addition for one pair of vectors, and we haven't checked scalar multiplication.

If we try a few more examples (the reader is encouraged to do so), we find that things keep working out, so we begin to suspect that maybe  $W$  really is a subspace. The only way to know for sure is to attempt to verify Definition 5.3.1 with a general proof. Suppose

$$\vec{v} = \begin{bmatrix} 2a + b \\ b + 4 \\ a - 2 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} 2c + d \\ d + 4 \\ c - 2 \end{bmatrix}$$

are arbitrary elements of  $W$ . Adding these vectors, we get

$$\vec{v} + \vec{w} = \begin{bmatrix} 2(a + c) + (b + d) \\ (b + d) + 8 \\ (a + c) - 4 \end{bmatrix},$$

which certainly doesn't look like an element of  $W$ ; the constants are all wrong! We have an 8 in the second component instead of a 4, and a  $-4$  in the third component instead of a  $-2$ . (This is why constant terms in the definition of a subset are generally problematic.)

However, with a bit of sleight of hand, things are not as bad as they seem. Let's write the second component as  $(b + d + 4) + 4$ , and the third as  $(a + c - 2) - 2$ , and let  $v = b + d + 4$ , and  $u = a + c - 2$ . If  $\vec{v} + \vec{w}$  is an element of  $W$ , then we're going to need

$$2(a + c) + (b + d) = 2u + v,$$

so that  $\vec{v} + \vec{w} = \begin{bmatrix} 2u + v \\ v + 4 \\ u - 2 \end{bmatrix}$  fits the definition of  $W$ . Is this true?

Let's check:

$$2u + v = 2(a + c - 2) + (b + d + 4) = 2(a + c) - 4 + (b + d) + 4 = 2(a + c) + (b + d).$$

The extra constants cancel, so addition works! Similarly, we find

$$k\vec{v} = \begin{bmatrix} k(2a + b) \\ k(b + 4) \\ k(a - 2) \end{bmatrix}$$

$$\begin{aligned}
&= \begin{bmatrix} 2(ka) + (kb) \\ (kb) + 4k \\ (ka) - 2k \end{bmatrix} \\
&= \begin{bmatrix} 2(ka) + (kb) \\ (kb + 4k - 4) + 4 \\ (ka - 2k + 2) - 2 \end{bmatrix} \\
&= \begin{bmatrix} 2(ka - 2k + 2) + (kb + 4k - 4) \\ (kb + 4k - 4) + 4 \\ (ka - 2k + 2) - 2 \end{bmatrix},
\end{aligned}$$

which fits the definition of  $W$ . Note that in the last equality, things cancel out again:

$$2(ka - 2k + 2) + (kb + 4k - 4) = 2(ka) - 4k + 4 + (kb) + 4k - 4 = 2(ka) + (kb).$$

Whew! That wasn't so straightforward. Could we have made our lives a little bit easier? (The answer to this rhetorical question is almost always yes.)

We know that the potential trouble here came from the constant terms, so one option we have is to try burying them. Given the element

$$\vec{v} = \begin{bmatrix} 2u + v \\ v + 4 \\ u - 2 \end{bmatrix} \in W,$$

we're under no obligation to stick with the variables  $u$  and  $v$ . Let's try to simplify a bit: if we let  $x = u - 2$  (so  $u = x + 2$ ) and  $y = v + 4$  (so  $v = y - 4$ ), then

$$2u + v = 2(x + 2) + (y - 4) = 2x + 4 + y - 4 = 2x + y,$$

and thus we can write  $\vec{v} = \begin{bmatrix} 2x + y \\ y \\ x \end{bmatrix}$ , with no more constant

terms. In this form it's much easier to verify that  $W$  is a subspace.

Let's take a second look at the subspaces  $T$  and  $W$  from Example 5.3.3 and

Example 5.3.4. Given an element  $\vec{v} = \begin{bmatrix} 3a - 2b \\ a + b \\ a - 4b \end{bmatrix}$  of  $T$ , we note that

$$\vec{v} = \begin{bmatrix} 3a - 2b \\ a + b \\ a - 4b \end{bmatrix} = \begin{bmatrix} 3a \\ a \\ a \end{bmatrix} + \begin{bmatrix} -2b \\ b \\ -4b \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix};$$

in other words,  $T$  can be written as the span of the vectors

$$\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix}.$$

Similarly, if we write  $\vec{w} = \begin{bmatrix} 2x + y \\ y \\ x \end{bmatrix} \in W$  for an arbitrary element of  $W$

(using our change of variables), we have

$$\vec{w} = \begin{bmatrix} 2x \\ 0 \\ x \end{bmatrix} + \begin{bmatrix} y \\ y \\ 0 \end{bmatrix} = x \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

so the subspace  $W$  again can be rewritten as

$$W = \text{span} \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

In fact, although we will not prove it in this textbook, every subspace of  $\mathbb{R}^n$  can be written as the span of some finite set of vectors. We can, however, prove that every span is a subspace.

**Theorem 5.3.5 Every span is a subspace.**

*Let  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$  be vectors in  $\mathbb{R}^n$ . Then  $V = \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$  is a subspace of  $\mathbb{R}^n$ .*

To see that this is true, recall that any element of a span is by definition a linear combination. Given two arbitrary elements

$$\begin{aligned} \vec{a} &= a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k \\ \vec{b} &= b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_k\vec{v}_k \end{aligned}$$

of  $V$ , we note that

$$\begin{aligned} \vec{a} + \vec{b} &= (a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k) + (b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_k\vec{v}_k) \\ &= (a_1\vec{v}_1 + b_1\vec{v}_1) + (a_2\vec{v}_2 + b_2\vec{v}_2) + \cdots + (a_k\vec{v}_k + b_k\vec{v}_k) \\ &= (a_1 + b_1)\vec{v}_1 + (a_2 + b_2)\vec{v}_2 + \cdots + (a_k + b_k)\vec{v}_k, \end{aligned}$$

so that  $\vec{a} + \vec{b}$  can be written as a linear combination of the vectors  $\vec{v}_i$  and therefore belongs to  $V$ .

Similarly, for any scalar  $c$ , we have

$$\begin{aligned} c\vec{a} &= c(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k) \\ &= (ca_1)\vec{v}_1 + (ca_2)\vec{v}_2 + \cdots + (ca_k)\vec{v}_k, \end{aligned}$$

so that  $c\vec{a}$  is an element of  $V$  as well. Thus, by Definition 5.3.1, we know that  $V$  is a subspace of  $\mathbb{R}^n$ .

We conclude with a discussion of how Theorem 5.3.5 and the concept of linear independence allows us to give a complete description of the possible subspaces of  $\mathbb{R}^n$ . To begin with, we have the simplest possible subspace, the **trivial subspace**

$$V_0 = \{\vec{0}\}.$$

If a subspace  $V$  has at least one non-zero vector, let's say  $\vec{v} \in V$ , then by definition it must contain every scalar multiple of that vector. Thus, the next simplest type of subspace is given as the span of a single, non-zero vector:

$$V_1 = \{t\vec{v} \mid t \in \mathbb{R} \text{ and } \vec{v} \neq \vec{0}\}.$$

The trivial subspace appears at first glance to be an exception to the rule that “every subspace is a span”, but we can consider it to be the span of the zero vector. In more advanced textbooks where the concept of a basis is discussed, the trivial subspace is often considered to be the “span” of the empty set. Since the empty set contains zero vectors, the trivial subspace is said to be zero-dimensional.

Of course, there are infinitely many possibilities for  $\vec{v}$ , but each choice of  $\vec{v} \neq \vec{0}$  leads to a subspace that looks and acts “the same”. As discussed earlier, we can picture a subspace of this type as a line through the origin.

Next, we could consider a subspace  $V_2 = \text{span}\{\vec{v}, \vec{w}\}$ , with  $\vec{v}, \vec{w} \neq \vec{0}$ . There are two possibilities. One is that  $\vec{v}$  and  $\vec{w}$  are parallel, so that the set  $\{\vec{v}, \vec{w}\}$  is linearly dependent. In this case we can write  $\vec{w} = k\vec{v}$  for some scalar  $k$ , and for any scalars  $a$  and  $b$ ,

$$a\vec{v} + b\vec{w} = a\vec{v} + b(k\vec{v}) = (a + bk)\vec{v},$$

so our subspace  $V_2$  is really of the same type as  $V_1$ . If, however, the vectors  $\vec{v}$  and  $\vec{w}$  are linearly independent, then adding the vector  $\vec{w}$  gives us a second direction to work with, and  $V_2$  becomes an object that is strictly larger than  $V_1$ . In this case, the visualization is that of a *plane* through the origin.

Depending on the size of  $n$ , this argument continues. If we add a third vector  $\vec{u}$  that is already in the span of  $\vec{v}$  and  $\vec{w}$ , then the set  $\{\vec{u}, \vec{v}, \vec{w}\}$  is linearly dependent, and the span of this set is the same as what we already had. If, however,  $\vec{u} \notin \text{span}\{\vec{v}, \vec{w}\}$ , then  $\{\vec{u}, \vec{v}, \vec{w}\}$  is linearly independent, and

$$V_3 = \text{span}\{\vec{u}, \vec{v}, \vec{w}\}$$

is a strictly larger subspace than  $\text{span}\{\vec{v}, \vec{w}\}$ . We could then look for a fourth vector, and so on. However, in the familiar case of  $\mathbb{R}^3$ , the process stops at 3.

#### Key Idea 5.3.6 Subspaces of $\mathbb{R}^3$ .

There are four different types of subspaces in  $\mathbb{R}^3$ :

- The trivial subspace,  $V_0 = \{\vec{0}\}$ . (Zero dimensional)
- Lines through the origin, of the form

$$V_1 = \text{span}\{\vec{v}\},$$

where  $\vec{v} \neq \vec{0}$ . (One dimensional)

- Planes through the origin, of the form

$$V_2 = \text{span}\{\vec{v}, \vec{w}\},$$

where the vectors  $\vec{v}, \vec{w}$  are linearly independent. (Two dimensional)

- The complete space  $V_3 = \mathbb{R}^3$ . (Three dimensional)

Notice the reference to dimension in Key Idea 5.3.6. In  $\mathbb{R}^3$ , we can rely on our intuitive (geometric) understanding of the concept of dimension. A complete understanding of the concept of dimension will have to wait until a second course in linear algebra; however, using the concepts in this section, we can make the following definition.

#### Definition 5.3.7 Dimension of a subspace.

The **dimension** of a subspace  $V \subseteq \mathbb{R}^n$  is the *smallest* number of vectors needed to span  $V$ .



One could also define dimension as the *largest* number of linearly independent vectors one can choose from a subspace. If  $B = \{\vec{v}_1, \dots, \vec{v}_k\}$  is a set of vectors in a subspace  $V$  such that

1.  $V = \text{span}(B)$ , and
2.  $B$  is linearly independent,

then we say  $B$  is a **basis** for  $V$ . For example, the set  $\{\vec{i}, \vec{j}, \vec{k}\}$  is a basis for  $\mathbb{R}^3$ . There are many possible bases for a subspace, but one can prove that the number of vectors in any basis is the same. Once this fact is established, we could alternatively define dimension as the number of vectors in any basis.

### 5.3.1 Exercises

**Exercise Group.** Determine if the given subset of  $\mathbb{R}^2$  is a subspace. Support your conclusion with a proof or counterexample.

1.  $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid 2x - 3y = 4 \right\}$

2.  $T = \left\{ \begin{bmatrix} x + y \\ x - y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

3.  $U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid y = 2x \right\}$

4.  $V = \left\{ \begin{bmatrix} x \\ x^2 \end{bmatrix} \mid x \in \mathbb{R} \right\}$

## 5.4 Null Space and Column Space

We close this chapter with some discussion of a theoretical nature. First, we will attempt to gain some additional insight into the (initially mysterious) definition of matrix multiplication by revisiting it from the point of view of linear transformations. We'll then introduce two fundamental subspaces associated with a matrix transformation.

### 5.4.1 Matrix multiplication as function composition

Recall that one of the ways we can obtain new functions from old ones is via **function composition**. Given two functions  $f(x)$  and  $g(x)$  (where  $x \in \mathbb{R}$ ), we can form the compositions

$$(f \circ g)(x) = f(g(x)) \text{ and } (g \circ f)(x) = g(f(x)),$$

as long as we meet certain conditions on the compatibility of the domains and ranges of the two functions.

If you paid attention in high school, you probably also remember that the order of composition matters: in general,

$$(f \circ g)(x) \neq (g \circ f)(x).$$

For example, if  $f(x) = 2x + 1$  and  $g(x) = x^2$ , then

$$(f \circ g)(x) = f(g(x)) = 2g(x) + 1 = 2x^2 + 1,$$

while

$$(g \circ f)(x) = g(f(x)) = (f(x))^2 = (2x + 1)^2 = 4x^2 + 4x + 1.$$

In this example, both functions are defined from  $\mathbb{R}$  to  $\mathbb{R}$ , and neither is a linear transformation in the sense of this section. In fact, if  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfies Definition 5.2.3, then we must have  $f(x) = ax$  for some real number  $a$ . If  $g(x) = bx$  is another linear transformation from  $\mathbb{R}$  to  $\mathbb{R}$ , notice that we have

$$(f \circ g)(x) = f(g(x)) = a(g(x)) = a(bx) = (ab)x.$$

Thus, to compose two linear transformations from  $\mathbb{R}$  to  $\mathbb{R}$ , we simply multiply the constants used to define the transformations.

Now, what about a general linear transformation  $S : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ? We know that any such transformation is a matrix transformation: we must have

$$S(\vec{x}) = A\vec{x}$$

for any  $\vec{x} \in \mathbb{R}^n$ , where  $A$  is an  $m \times n$  matrix. Since we're multiplying an  $m \times n$  matrix by an  $n \times 1$  matrix, the rules of matrix multiplication ensure that the output  $\vec{y} = A\vec{x}$  is an element of  $\mathbb{R}^m$ .

Suppose now that we want to define the composition  $(S \circ T)(\vec{x})$  for some other linear transformation  $T$ . Recall the following rule of function composition:

In order for the composition  $S \circ T$  to be defined, the *range* of  $T$  must be contained in the domain of  $S$ .

That is, since  $S \circ T$  is defined by  $(S \circ T)(\vec{x}) = S(T(\vec{x}))$ , the vector  $T(\vec{x})$  (which by definition is in the range of  $T$ ) must belong to the domain of  $S$ . This means that we must have  $T(\vec{x}) \in \mathbb{R}^n$ , so we have

$$T : \mathbb{R}^k \rightarrow \mathbb{R}^n$$

for some natural number  $k$ . On the other hand, we know that if  $T$  is a linear transformation, then  $T$  is defined by matrix multiplication:  $T(\vec{x}) = B\vec{x}$  for some  $n \times k$  matrix  $B$ .

Let us now recall one of the rules of matrix multiplication:

For the matrix product  $AB$  to be defined, the number of columns of  $A$  must equal the number of rows of  $B$ .

That is, if  $A$  is an  $m \times n$  matrix, then  $B$  must be an  $n \times k$  matrix for some  $k$ . But this is the same conclusion as above! What is the connection? Well, if we follow the rules for function composition, if  $T(\vec{x}) = B\vec{x}$  and  $S(\vec{y}) = A\vec{y}$ , we must have

$$(S \circ T)(\vec{x}) = S(T(\vec{x})) = A(T(\vec{x})) = A(B\vec{x}) = (AB)\vec{x},$$

where the last equality is due to the associative property of matrix multiplication from Theorem 4.2.11. Thus, we see that:

*Composition of linear transformations is the same as multiplication of the corresponding matrices!*

Looking at things from the point of view of matrix transformations gives us two insights on the nature of matrix multiplication:

1. When  $A$  and  $B$  are both  $n \times n$  matrices, the transformations  $S(\vec{x}) = A\vec{x}$  and  $T(\vec{x}) = B\vec{x}$  are both maps from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ , and we can define both

$$(S \circ T)(\vec{x}) = (AB)\vec{x}$$

and

$$(T \circ S)(\vec{x}) = (BA)\vec{x}.$$

Our experience with functions teaches us that most of the time,  $S \circ T \neq T \circ S$ , so of course it makes sense that  $AB \neq BA$  in general!

2. The fact that  $AB$  is defined only when the number of columns of  $A$  matches the number of rows of  $B$  is simply a consequence of the fact that  $S \circ T$  is only defined if the range of  $T$  is a subset of the domain of  $A$ .

What about the “row times column” rule for determining the entries of  $AB$ ? Let’s look at how things work in 2D. Suppose we’ve defined linear transformations

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{ and}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = B\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

If we write  $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix}$ , where  $u = b_{11}x + b_{12}y$  and  $v = b_{21}x + b_{22}y$ , then we have

$$\begin{aligned} (S \circ T)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= S\left(T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) = S\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) \\ &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \\ &= \begin{bmatrix} a_{11}u + a_{12}v \\ a_{21}u + a_{22}v \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} a_{11}(b_{11}x + b_{12}y) + a_{12}(b_{21}x + b_{22}y) \\ a_{21}(b_{11}x + b_{12}y) + a_{22}(b_{21}x + b_{22}y) \end{bmatrix} \\
&= \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21})x + (a_{11}b_{12} + a_{12}b_{22})y \\ (a_{21}b_{11} + a_{22}b_{21})x + (a_{21}b_{12} + a_{22}b_{22})y \end{bmatrix} \\
&= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.
\end{aligned}$$

But we also argued above that we should have

$$(S \circ T) \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = (AB) \begin{bmatrix} x \\ y \end{bmatrix},$$

from which we're forced to conclude that

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix},$$

and this is exactly the rule for multiplying two  $2 \times 2$  matrices! Of course, we can repeat the above argument for the general case where  $A$  is  $m \times n$  and  $B$  is  $n \times k$ , but you can probably guess that the algebra gets a bit messy on that one, so we'll spare you the details.

### 5.4.2 Column space

When we discussed the composition of linear transformations above, we briefly mentioned that this involves the consideration of the **range**. Recall that the range of a function is the set of all possible outputs when every input in the domain is considered. For example, with the function  $f(x) = x^2$ , where  $x$  can be any real number, the range is the set of all real numbers  $y \geq 0$ . (If  $y = x^2$  and  $x$  is real, then  $y$  can't be negative.)

If we're given a linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , we might want to know what sort of vectors  $\vec{y} \in \mathbb{R}^m$  can be obtained from  $T$ . Consider Examples Example 5.1.3 and Example 5.1.5 from way back at the beginning of the section. In Example 5.1.3, the vectors  $A\vec{x}$  and  $A\vec{y}$  were non-parallel, and therefore independent. It follows that for any other vector  $\vec{z} \in \mathbb{R}^2$ , we can find scalars  $a$  and  $b$  such that

$$\vec{z} = a(A\vec{x}) + b(A\vec{y}) = A(a\vec{x}) + A(b\vec{y}) = A(a\vec{x} + b\vec{y}),$$

so every vector in  $\mathbb{R}^2$  can be written as the output of the transformation  $T(\vec{x}) = A\vec{x}$ . On the other hand, using the matrix  $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$  in Example 5.1.5, for

any vector  $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ , we have

$$A \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a - b \\ a - b \end{bmatrix} = (a - b) \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so the only vectors in the range of  $T(\vec{x}) = A\vec{x}$  are those parallel to the vector  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

Next, we're going to consider a general matrix transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  given by  $T(\vec{x}) = A\vec{x}$ , but we'll play around with the multiplication a little bit. By

definition (and a bit of manipulation), we have

$$\begin{aligned}
 T(\vec{x}) = A\vec{x} &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
 &= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} \\
 &= \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{m1}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \\ \vdots \\ a_{m2}x_2 \end{bmatrix} + \cdots + \begin{bmatrix} a_{1n}x_n \\ a_{2n}x_n \\ \vdots \\ a_{mn}x_n \end{bmatrix} \\
 &= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.
 \end{aligned}$$

**Note:** Although we didn't say so at the time, we already encountered this rule for multiplying a vector by a matrix in the argument we gave in support of Theorem 5.2.12. Some textbooks actually use this observation to give an alternative definition of matrix multiplication. Once we know how the product  $A\vec{x}$  is defined for an  $m \times n$  matrix  $A$  and  $n \times 1$  vector  $\vec{x}$ , we can define  $AB$  for an  $n \times p$  matrix  $B$  as follows: first, we write

$$B = [\vec{b}_1 \quad \vec{b}_2 \quad \cdots \quad \vec{b}_p],$$

where the  $n \times 1$  vectors  $\vec{b}_1, \dots, \vec{b}_p$  are the columns of  $B$ . We then define

$$\begin{aligned}
 AB &= A [\vec{b}_1 \quad \vec{b}_2 \quad \cdots \quad \vec{b}_p] \\
 &= [A\vec{b}_1 \quad A\vec{b}_2 \quad \cdots \quad A\vec{b}_p].
 \end{aligned}$$

It's a good exercise to verify (with a few examples) that this definition of the product  $AB$  is the same as the "row times column" definition we gave earlier.

Thus, whenever we multiply a vector by a matrix, *the result is a linear combination of the columns of  $A$ !* If we think of each column of  $A$  as a column vector in  $\mathbb{R}^m$ , we can make the following definition:

**Definition 5.4.1 The column space of a matrix.**

The **column space** of an  $m \times n$  matrix  $A$  is the subspace of  $\mathbb{R}^m$  spanned by the columns of  $A$ :

$$\text{col}(A) = \text{span} \left\{ \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\}.$$

From the discussion above, we can make two conclusions. First, if  $T(\vec{x}) = A\vec{x}$  is a linear transformation, we have

$$\text{range}(T) = \text{col}(A).$$

Second, as mentioned in Definition 5.4.1, since the range of  $T$  can be written as a span, it is automatically a subspace of  $\mathbb{R}^m$  according to Theorem 5.3.5. The range of a linear transformation is one of the more important examples of a subspace.

To give a more useful description of the column space, we rely Theorem 5.4.2 below, whose proof is too technical for this text. To help with the statement of this theorem, we first introduce one more bit of terminology. We will call a column of a matrix  $A$  a **pivot column** if the corresponding column in the reduced row echelon form of  $A$  contains a leading 1.

**Theorem 5.4.2 Basis for the column space of a matrix.**

*A basis for the column space of an  $m \times n$  matrix  $A$  is given by the set of pivot columns of  $A$ .*

We will illustrate Theorem 5.4.2 with an example. It's important to note that while we need to find the reduced row echelon form of  $A$  in order to find the pivot columns, the columns we want are those of the *original* matrix  $A$ , not its ref.

**Example 5.4.3 Finding a basis for the column space.**

Determine a basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & -3 \\ 2 & -1 & 0 & 4 \\ -1 & 1 & 3 & 0 \end{bmatrix}.$$

**Solution.** We begin by computing the reduced row echelon form  $R$  of  $A$ . We find

$$R = \begin{bmatrix} 1 & 0 & 0 & -17 \\ 0 & 1 & 0 & -38 \\ 0 & 0 & 1 & 7 \end{bmatrix},$$

and note that  $R$  has leading 1s in columns 1, 2, and 3. It follows that

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \right\}$$

is a basis for  $\text{col}(A)$ .

Let's make a few observations about the previous example. Notice that we have three leading 1s, so  $\text{rank}(A) = 3$ . In particular, there is a leading 1 in each row, so we're guaranteed that the system  $A\vec{x} = \vec{b}$  is consistent, no matter what the vector  $\vec{b}$  is. Since the number of pivot columns of  $A$  is equal to the number of leading 1s, we obtain the following result.

**Theorem 5.4.4 Dimension of the column space.**

*The dimension of the column space of a matrix  $A$  (or equivalently, the dimension of range of the matrix transformation defined by  $A$ ) is equal to the rank of  $A$ .*

To see why this result can be useful, notice that in our previous example, the matrix transformation  $T(\vec{x}) = A\vec{x}$  determines a linear transformation  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ . Notice that there are three vectors in the basis for  $\text{col}(A)$ ; this means that the column space of  $A$  (and thus, the range of  $T$ ) is three-dimensional, and therefore the range of  $T$  is *all* of  $\mathbb{R}^3$ , and thus, no matter what vector  $\vec{b} \in \mathbb{R}^3$  we choose, we're guaranteed to be able to find a vector  $\vec{x} \in \mathbb{R}^4$  such that  $A\vec{x} = \vec{b}$ .

The key observation here is that the question "Does  $A\vec{x} = \vec{b}$  have a solution?" is equivalent to the question "Does the vector  $\vec{b}$  belong to  $\text{col}(A)$ ?" Unfortunately, while we may gain some insight from noticing that these questions are the same, we are no further ahead when it comes to answering them. What-

ever version we prefer, the only way to get an answer is to compute the reduced row echelon form of  $\left[ A \mid \vec{b} \right]$ .

Suppose we repeated Example 5.4.3 using the matrix  $A$  from Example 3.6.9. Both cases involved a matrix of size  $3 \times 4$ , but the matrix from Example 3.6.9 had rank 2, so the column space of  $A$  is only two-dimensional. In this case, the system  $A\vec{x} = \vec{b}$  will be consistent if  $\vec{b}$  belongs to the span of the first two columns of  $A$ , and inconsistent otherwise.

Reading off the first two columns of  $A$ , we find that

$$\text{col}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ -6 \end{bmatrix} \right\}.$$

We know that this is a plane through the origin in  $\mathbb{R}^3$ , but how do we quickly determine what vectors belong to this plane? There's an easy way and a hard way. The easy way is to compute the cross product, as we did in many of the problems from Section 2.6. We find

$$\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \times \begin{bmatrix} -2 \\ -1 \\ -6 \end{bmatrix} = \begin{bmatrix} -20 \\ 10 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix} = 5\vec{n},$$

where we've chosen to factor out the scalar multiple of 5 to simplify our normal vector.

From this we know that a vector

$$\vec{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

belongs to the column space of  $A$  if and only if

$$-4a + 2b + c = 0,$$

using the scalar form for the equation of a plane in  $\mathbb{R}^3$ . Having done it the easy way, let us do things once more the hard way. (Why do it the hard way if the easy way works? Because if we're in any other case than a two-dimensional subspace of  $\mathbb{R}^3$ , the hard way is the only option we have!) The hard way is to

solve the equation  $A\vec{x} = \vec{b}$  for an *arbitrary* vector  $\vec{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ . As with the previous

examples, we set up the augmented matrix and reduce:

$$\left[ \begin{array}{cccc|c} 1 & -2 & 0 & 4 & a \\ 3 & -1 & 5 & 2 & b \\ -2 & -6 & -10 & 12 & c \end{array} \right] \longrightarrow \left[ \begin{array}{cccc|c} 1 & -2 & 0 & 4 & a \\ 0 & 1 & 1 & -2 & (b-3a)/5 \\ 0 & 0 & 0 & 0 & (c-4a+2b)/10 \end{array} \right].$$

We stopped before getting all the way to the reduced row echelon form, but we're far enough along to realize that the only way our system can be consistent is if the last entry in the third row is equal to zero. This gives us the condition

$$\frac{c - 4a + 2b}{10} = 0,$$

which (after multiplying both sides by 10) is exactly the same as what we found using the cross product.



### 5.4.3 Null space

The other important example is the **null space** of a matrix. The null space of an  $m \times n$  matrix  $A$  is simply the set of all those vectors  $\vec{x} \in \mathbb{R}^n$  such that  $A\vec{x} = \vec{0}$ .

**Definition 5.4.5 The null space of a matrix.**

The **null space** of an  $m \times n$  matrix  $A$  is denoted by  $\text{null}(A)$ , and defined by

$$\text{null}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}.$$

For example, we saw in Example 5.1.5 that the vector  $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  belongs to the null space of the matrix  $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ , since

$$A\vec{x} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Given a general  $m \times n$  matrix  $A$ , we know from Section 3.6 that determining the null space amounts to simply solving a homogeneous system of linear equations. Let us see how this works in an example.

**Example 5.4.6 Determining the null space of a matrix.**

Determine the null space of the matrix

$$A = \begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix}.$$

**Solution.** Since the null space of  $A$  is equal to the set of all solutions  $\vec{x}$  to the matrix equation  $A\vec{x} = \vec{0}$ , we proceed by forming the proper augmented matrix and putting it into reduced row echelon form, which we do below.

$$\left[ \begin{array}{cc|c} 2 & -3 & 0 \\ -2 & 3 & 0 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{cc|c} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

We interpret the reduced row echelon form of this matrix to find that

$$\begin{aligned} x_1 &= 3/2t \\ x_2 &= t \text{ is free.} \end{aligned}$$

We can say that  $\vec{x} \in \text{null}(A)$  provided that

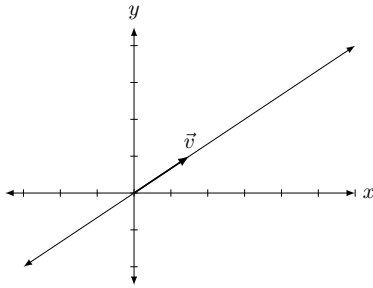
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3/2t \\ t \end{bmatrix} = t \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}.$$

If we set

$$\vec{v} = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix},$$

then we can write our solution as

$$\text{null}(A) = \left\{ t\vec{v} \mid t \in \mathbb{R} \text{ and } \vec{v} = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \right\}.$$



**Figure 5.4.7** The solution, as a line, to  $A\vec{x} = \vec{0}$  in Example 5.4.6

We see that the null space of  $A$  contains infinitely many solutions to the equation  $A\vec{x} = \vec{0}$ ; any choice of  $x_2$  gives us one of these solutions. For instance, picking  $x_2 = 2$  gives the solution

$$\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

This is a particularly nice solution, since there are no fractions! In fact, since the parameter  $t$  can take on any *real* value, there is nothing preventing us from defining a new parameter  $s = t/2$ , and then

$$\vec{x} = t \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} = t \left( \frac{1}{2} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) = \frac{t}{2} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = s \begin{bmatrix} 3 \\ 2 \end{bmatrix} = s\vec{w},$$

where  $\vec{w} = 2\vec{v}$ .

Our solutions are multiples of a vector, and hence we can graph this, as done in Figure 5.4.7.

In Example 5.4.6, we saw that the solution is a line through the origin, and thus, we can conclude that  $\text{null}(A)$  is a subspace! In fact, this is no coincidence: it is guaranteed by our next theorem.

**Theorem 5.4.8 The null space of a matrix is a subspace.**

*For any  $m \times n$  matrix  $A$ ,  $\text{null}(A)$  is a subspace of  $\mathbb{R}^n$ .*

The proof of this theorem is simple. Suppose  $\vec{x}, \vec{y} \in \text{null}(A)$ . By definition, this means  $A\vec{x} = \vec{0}$  and  $A\vec{y} = \vec{0}$ . Using the properties of matrix multiplication, we have

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0},$$

so  $\vec{x} + \vec{y} \in \text{null}(A)$ , and

$$A(c\vec{x}) = c(A\vec{x}) = c\vec{0} = \vec{0},$$

so  $c\vec{x} \in \text{null}(A)$ . It follows from the definition of a subspace that  $\text{null}(A)$  is a subspace of  $\mathbb{R}^n$ .

In Section 2.7 we discussed the fact that whenever we have a subspace of  $\mathbb{R}^n$ , it can be useful to determine a basis for our subspace. Recall from Definition 3.6.8 that the general solution to a homogeneous system of linear equations can be written in terms of certain **basic solutions**. In the context of null spaces, these basic solutions are just such a basis.

Although we will not prove it here, the basic solutions to a homogeneous system are always linearly independent. Moreover, it follows from the definition of  $\text{null}(A)$  that any  $\vec{x} \in \text{null}(A)$  can be written as a linear combination of the basic solutions. In the language of Section 2.7, the basic solutions to a homogeneous system  $A\vec{x} = \vec{0}$  form a **basis** for the null space of  $A$ . This is an important point to remember, so we emphasize it in the following Key Idea.

**Key Idea 5.4.9 Basis for the null space of a matrix.**

The basic solutions to the homogeneous system  $A\vec{x} = \vec{0}$  form a basis for the null space of  $A$ . That is, if  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$  are the basic solutions

to  $A\vec{x} = \vec{0}$ , then

$$\text{null}(A) = \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}.$$

To illustrate Key Idea 5.4.9, let's revisit an example from Section 3.6 using the language of null space and basis.

**Example 5.4.10 A two-dimensional null space.**

Find a basis for the null space of  $A$ , where

$$A = \begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & -1 & 5 & 2 \\ -2 & -6 & -10 & 12 \end{bmatrix}.$$

**Solution.** Again, determining  $\text{null}(A)$  is the same as solving the homogeneous system  $A\vec{x} = \vec{0}$ , and by Key Idea 5.4.9, a basis for  $\text{null}(A)$  is given by the basic solutions to this system. As usual, to find the basic solutions, we set up the augmented matrix of the system and reduce:

$$\left[ \begin{array}{cccc|c} 1 & -2 & 0 & 4 & 0 \\ 3 & -1 & 5 & 2 & 0 \\ -2 & -6 & -10 & 12 & 0 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

From the reduced row echelon form of the augmented matrix, we can read off the following general solution:

$$\begin{aligned} x_1 &= -2s \\ x_2 &= -s + 2t \\ x_3 &= s \text{ is free} \\ x_4 &= t \text{ is free.} \end{aligned}$$

In this case, we have two parameters, so we expect two basic solutions. To find these, we write our solution in vector form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s \\ -s + 2t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

From the above, we see that the general solution can be written as  $\vec{x} = s\vec{v} + t\vec{w}$ , where

$$\vec{v} = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{w} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

are the basic solutions to  $A\vec{x} = \vec{0}$ . Since the null space of  $A$  is equal to the set of solutions to  $A\vec{x} = \vec{0}$ , and since every solution to  $A\vec{x} = \vec{0}$  can be written in terms of  $\vec{v}$  and  $\vec{w}$ , it follows that

$$\text{null}(A) = \text{span}\{\vec{v}, \vec{w}\},$$

and that  $\{\vec{v}, \vec{w}\}$  is a basis for  $\text{null}(A)$ .

Another reason the null space is interesting is that it lets us determine whether or not a linear transformation is **one-to-one**. Suppose  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear transformation defined by  $T(\vec{x}) = A\vec{x}$ . We know that  $T(\vec{0}) = \vec{0}$ , so  $\vec{0} \in \text{null}(A)$  (as it must be, since  $\text{null}(A)$  is a subspace). If we have any non-zero vector  $\vec{v} \in \text{null}(A)$ , then  $T$  cannot be one-to-one, since we'd have

$$T(\vec{v}) = A\vec{v} = \vec{0} = T(\vec{0}).$$

Thus, if  $\text{null}(A) \neq \{\vec{0}\}$ , then  $T$  is not one-to-one. On the other hand, suppose  $\text{null}(A) = \{\vec{0}\}$ , and that  $T(\vec{x}) = T(\vec{y})$  for vectors  $\vec{x}, \vec{y} \in \mathbb{R}^n$ . Then we have

$$\vec{0} = T(\vec{x}) - T(\vec{y}) = T(\vec{x} - \vec{y}) = A(\vec{x} - \vec{y}),$$

so that  $\vec{x} - \vec{y} \in \text{null}(A) = \{\vec{0}\}$ , which means that  $\vec{x} - \vec{y} = \vec{0}$ , and thus  $\vec{x} = \vec{y}$ . We have proved the following:

**Theorem 5.4.11 Null space and one-to-one transformations.**

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be defined by  $T(\vec{x}) = A\vec{x}$  for some  $m \times n$  matrix  $A$ . Then  $T$  is one-to-one if and only if  $\text{null}(A) = \{\vec{0}\}$ .

The final result we'll state provides an interesting (and powerful) relationship between the null and column spaces.

**Theorem 5.4.12 The Fundamental Theorem of Linear Transformations.**

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation defined by  $T(\vec{x}) = A\vec{x}$  for some  $m \times n$  matrix  $A$ . Then

$$\dim \text{null}(A) + \dim \text{col}(A) = n.$$

This result is sometimes known as the “rank-nullity theorem”; it gives the relationship between the **rank** of a matrix  $A$ , which is equal to the dimension of its column space, and the **nullity** of  $A$ , which is defined to be the dimension of its null space.

A formal proof of this result is beyond the scope of this course, but the intuition we've gained from solving systems should make it plausible. Recall that Item 3 in Theorem 3.6.15 on the relationship between the rank of a matrix and types of solutions gives us the equation

$$k + \text{rank}(A) = n,$$

where  $k$  is the number of parameters in the general solution of  $A\vec{x} = \vec{b}$ . Now, we know from Definition 3.6.8 that the number of parameters in the general solution to  $A\vec{x} = \vec{b}$  is equal to the number of basic solutions to the system  $A\vec{x} = \vec{0}$ , and that the basic solutions to  $A\vec{x} = \vec{0}$  form a basis for  $\text{null}(A)$ . From this, we can conclude that

$$k = \dim \text{null}(A).$$

We also claimed in Theorem 5.4.4 above that the rank of  $A$  is equal to the dimension of its column space. Putting these facts together, we can see why the rank-nullity theorem must hold. Let's confirm that the result holds in one more example.

Recall that a function  $f$  is one-to-one if no two inputs give the same output. In other words, if  $f$  is one-to-one, then whenever  $f(a) = f(b)$ , we can conclude that  $a = b$ .

**Example 5.4.13 Null space and column space.**

Let

$$A = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 4 & 2 & 4 & 6 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}.$$

Determine:

1. The null space of  $A$ .
2. Whether or not the vector  $\vec{b}$  belongs to the column space of  $A$ .

**Solution.** We'll tackle the null space first. We form the augmented matrix for the system  $A\vec{x} = \vec{0}$ , put it into reduced row echelon form, and interpret the result.

$$\left[ \begin{array}{cccc|c} 1 & -1 & 1 & 3 & 0 \\ 4 & 2 & 4 & 6 & 0 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array} \right]$$

$$x_1 = -x_3 - 2x_4$$

$$x_2 = x_4$$

$$x_3 = s \text{ is free}$$

$$x_4 = t \text{ is free.}$$

We now obtain our vector solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s - 2t \\ t \\ s \\ t \end{bmatrix}.$$

Finally, we “pull apart” this vector into two vectors, one with the “ $s$  stuff” and one with the “ $t$  stuff.”

$$\begin{aligned} \vec{x} &= \begin{bmatrix} -x_3 - 2x_4 \\ x_4 \\ x_3 \\ x_4 \end{bmatrix} \\ &= \begin{bmatrix} -x_3 \\ 0 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_4 \\ x_4 \\ 0 \\ x_4 \end{bmatrix} \\ &= x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \\ &= x_3 \vec{u} + x_4 \vec{v}. \end{aligned}$$

We use  $\vec{u}$  and  $\vec{v}$  simply to give these vectors names (and save some space). In terms of these names, we can write

$$\text{null}(A) = \text{span}\{\vec{u}, \vec{v}\}.$$

It is easy to confirm that both  $\vec{u}$  and  $\vec{v}$  are solutions to the linear system  $A\vec{x} = \vec{0}$ . (Just multiply  $A\vec{u}$  and  $A\vec{v}$  and see that both are zero.) Since both are solutions to a homogeneous system of linear equations, any linear combination of  $\vec{u}$  and  $\vec{v}$  will be a solution, too, so the vectors  $\vec{u}$  and  $\vec{v}$  form a basis for the null space of  $A$ .

Now let's tackle the column space. Determining whether or not  $\vec{b}$  belongs to the column space is the same as solving the system  $A\vec{x} = \vec{b}$ . Once again we put the associated augmented matrix into reduced row echelon form and interpret the results.

$$\left[ \begin{array}{cccc|c} 1 & -1 & 1 & 3 & 1 \\ 4 & 2 & 4 & 6 & 10 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{cccc|c} 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & -1 & 1 \end{array} \right]$$

$$x_1 = 2 - s - 2t$$

$$x_2 = 1 + t$$

$$x_3 = s \text{ is free}$$

$$x_4 = t \text{ is free.}$$

Since our system is consistent, we can conclude that  $\vec{b} \in \text{col}(A)$ . Let us expand on this result a bit.

Writing this solution in vector form gives

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 - s - 2t \\ 1 + t \\ s \\ t \end{bmatrix}.$$

Again, we pull apart this vector, but this time we break it into three vectors: one with “ $s$ ” stuff, one with “ $t$ ” stuff, and one with just constants.

$$\begin{aligned} \vec{x} &= \begin{bmatrix} 2 - s - 2t \\ 1 + t \\ s \\ t \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -2t \\ t \\ 0 \\ t \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{particular solution}} + \underbrace{s\vec{u} + t\vec{v}}_{\text{solution to homogeneous system}}. \end{aligned}$$

Note that  $A\vec{x}_p = \vec{b}$ ; by itself,  $\vec{x}_p$  is a solution. The fact that we have at least one vector  $\vec{x}_p$  such that  $A\vec{x}_p = \vec{b}$  tells us that  $\vec{b}$  belongs to the range of the transformation  $T(\vec{x}) = A\vec{x}$ . The fact that there is more than one solution corresponds to the fact that the null space of  $A$  is non-trivial.

Why don't we graph this solution as we did in the past? Before we had only two variables, meaning the solution could be graphed in 2D. Here we have four variables, meaning that our solution "lives" in 4D. You *can* draw this on paper, but it is *very* confusing.

For further verification of Theorem 5.4.12, the reader is encouraged to revisit the examples of Section 3.6 and re-interpret them in the context of null space and column space.

### 5.4.4 Exercises

**Exercise Group.** Determine a basis for the null space and column space of the given matrix, and verify Theorem 5.4.12.

1.  $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & -1 & 6 \\ -3 & -6 & 2 & -4 \end{bmatrix}$

2.  $A = \begin{bmatrix} 3 & 2 & 0 & -5 & 1 & 6 & -4 & 3 \\ 2 & 8 & -4 & -2 & & & & \end{bmatrix}$

3.  $A = \begin{bmatrix} 1 & -2 & 12 & -4 & 6 \\ 0 & -8 & 8 & & \end{bmatrix}$

4.  $A = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 2 & 5 \\ 3 & -4 & 2 \end{bmatrix}$

5.  $A = \begin{bmatrix} 2 & 0 & 3 & 5 \\ 1 & -1 & 4 & 2 \\ 0 & 3 & -6 & 2 \end{bmatrix}$

6.  $A = \begin{bmatrix} 2 & 3 & -11 & 5 & 2 \\ 1 & 8 & 1 & & \end{bmatrix}$

7.  $A = \begin{bmatrix} 1 & -2 & 4 & 7 \\ 2 & -4 & 3 & 6 \\ -1 & 2 & 1 & 1 \end{bmatrix}$

8.  $A = \begin{bmatrix} 1 & -3 & 5 \\ 2 & -1 & 2 \\ -3 & 0 & 2 \end{bmatrix}$



## Chapter 6

# Operations on Matrices

In Chapter 4 we learned about matrix arithmetic: adding, subtracting, and multiplying matrices, finding inverses, and multiplying by scalars. In this chapter we learn about some operations that we perform on matrices. We can think of them as functions: you input a matrix, and you get something back. One of these operations, the transpose, will return another matrix. With the other operations, the trace and the determinant, we input matrices and get numbers in return, an idea that is different than what we have seen before.

### 6.1 The Matrix Transpose

This section introduces the **transpose** of a matrix: a simple, but useful operation. We jump right in with a definition.

#### Definition 6.1.1 Transpose.

Let  $A$  be an  $m \times n$  matrix. The **transpose** of  $A$ , denoted  $A^T$ , is the  $n \times m$  matrix whose columns are the respective rows of  $A$ .

If we write  $A = [a_{ij}]$  to emphasize the entries of  $A$ , then the transpose of  $A$  is the matrix  $A^T = [a_{ij}^T]$  where  $a_{ij}^T = a_{ji}$ ; that is, the  $(i, j)$ -entry of  $A^T$  is the  $(j, i)$ -entry of  $A$ . Examples will make this definition clear.

#### Example 6.1.2 Taking the transpose of a matrix.

Find the transpose of  $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ .

**Solution.** Note that  $A$  is a  $2 \times 3$  matrix, so  $A^T$  will be a  $3 \times 2$  matrix. By the definition, the first column of  $A^T$  is the first row of  $A$ ; the second column of  $A^T$  is the second row of  $A$ . Therefore,

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

**Example 6.1.3 Computing transposes.**

Find the transpose of the following matrices.

$$A = \begin{bmatrix} 7 & 2 & 9 & 1 \\ 2 & -1 & 3 & 0 \\ -5 & 3 & 0 & 11 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 10 & -2 \\ 3 & -5 & 7 \\ 4 & 2 & -3 \end{bmatrix} \quad C = [1 \quad -1 \quad 7 \quad 8 \quad 3].$$

**Solution.** We find each transpose using the definition without explanation. Make note of the dimensions of the original matrix and the dimensions of its transpose.

$$A^T = \begin{bmatrix} 7 & 2 & -5 \\ 2 & -1 & 3 \\ 9 & 3 & 0 \\ 1 & 0 & 11 \end{bmatrix} \quad B^T = \begin{bmatrix} 1 & 3 & 4 \\ 10 & -5 & 2 \\ -2 & 7 & -3 \end{bmatrix} \quad C^T = \begin{bmatrix} 1 \\ -1 \\ 7 \\ 8 \\ 3 \end{bmatrix}.$$

Notice that with matrix  $B$ , when we took the transpose, the *diagonal* did not change. We can see what the diagonal is below where we rewrite  $B$  and  $B^T$  with the diagonal in bold. We'll follow this by a definition of what we mean by "the diagonal of a matrix," along with a few other related definitions.

$$B = \begin{bmatrix} \mathbf{1} & 10 & -2 \\ 3 & \mathbf{-5} & 7 \\ 4 & 2 & \mathbf{-3} \end{bmatrix} \quad B^T = \begin{bmatrix} \mathbf{1} & 3 & 4 \\ 10 & \mathbf{-5} & 2 \\ -2 & 7 & \mathbf{-3} \end{bmatrix}.$$

It is probably pretty clear why we call those entries "the diagonal." Here is the formal definition.

**Definition 6.1.4 The Diagonal, a Diagonal Matrix, Triangular Matrices.**

Let  $A$  be an  $m \times n$  matrix. The **diagonal** of  $A$  consists of the entries  $a_{11}, a_{22}, \dots$  of  $A$ .

A **diagonal matrix** is an  $n \times n$  matrix in which the only nonzero entries lie on the diagonal.

An **upper (lower) triangular** matrix is a matrix in which any nonzero entries lie on or above (below) the diagonal.

**Example 6.1.5 Classifying matrices.**

Consider the matrices  $A$ ,  $B$ ,  $C$  and  $I_4$ , as well as their transposes, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}.$$

Identify the diagonal of each matrix, and state whether each matrix is diagonal, upper triangular, lower triangular, or none of the above.

**Solution.** We first compute the transpose of each matrix.

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix} \quad B^T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad C^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 3 & 5 & 6 & 0 \end{bmatrix}.$$

Note that  $I_4^T = I_4$ .

The diagonals of  $A$  and  $A^T$  are the same, consisting of the entries 1, 4 and 6. The diagonals of  $B$  and  $B^T$  are also the same, consisting of the entries 3, 7 and  $-1$ . Finally, the diagonals of  $C$  and  $C^T$  are the same, consisting of the entries 1, 4 and 6.

The matrix  $A$  is upper triangular; the only nonzero entries lie on or above the diagonal. Likewise,  $A^T$  is lower triangular.

The matrix  $B$  is diagonal. By their definitions, we can also see that  $B$  is both upper and lower triangular. Likewise,  $I_4$  is diagonal, as well as upper and lower triangular.

Finally,  $C$  is upper triangular, with  $C^T$  being lower triangular.

Make note of the definitions of diagonal and triangular matrices. We specify that a diagonal matrix must be square, but triangular matrices don't have to be. ("Most" of the time, however, the ones we study are.) Also, as we mentioned before in the example, by definition a diagonal matrix is also both upper and lower triangular. Finally, notice that by definition, the transpose of an upper triangular matrix is a lower triangular matrix, and vice-versa.

There are many questions to probe concerning the transpose operations. The first set of questions we'll investigate involve the matrix arithmetic we learned from last chapter. We do this investigation by way of examples, and then summarize what we have learned at the end.

#### Example 6.1.6 Adding transposed matrices.

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix}.$$

Find  $A^T + B^T$  and  $(A + B)^T$ .

**Solution.** We note that

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \text{ and } B^T = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 1 & 0 \end{bmatrix}.$$

Therefore

$$\begin{aligned} A^T + B^T &= \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 7 \\ 4 & 4 \\ 4 & 6 \end{bmatrix}. \end{aligned}$$

Also,

$$(A + B)^T = \left( \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix} \right)^T$$

Remember, this is what mathematicians do. We learn something new, and then we ask lots of questions about it. Often the first questions we ask are along the lines of "How does this new thing relate to the old things I already know about?"

$$\begin{aligned}
 &= \left( \begin{bmatrix} 2 & 4 & 4 \\ 7 & 4 & 6 \end{bmatrix} \right)^T \\
 &= \begin{bmatrix} 2 & 7 \\ 4 & 4 \\ 4 & 6 \end{bmatrix}.
 \end{aligned}$$

It looks like “the sum of the transposes is the transpose of the sum.” (This is kind of fun to say, especially when said fast. Regardless of how fast we say it, we should think about this statement. The “is” represents “equals.” The stuff before “is” equals the stuff afterwards.) This should lead us to wonder how the transpose works with multiplication.

**Example 6.1.7 Multiplying transposed matrices.**

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find  $(AB)^T$ ,  $A^T B^T$  and  $B^T A^T$ .

**Solution.** We first note that

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \text{ and } B^T = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}.$$

Find  $(AB)^T$ :

$$\begin{aligned}
 (AB)^T &= \left( \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \right)^T \\
 &= \left( \begin{bmatrix} 3 & 2 & 1 \\ 7 & 6 & 1 \end{bmatrix} \right)^T \\
 &= \begin{bmatrix} 3 & 7 \\ 2 & 6 \\ 1 & 1 \end{bmatrix}.
 \end{aligned}$$

Now find  $A^T B^T$ :

$$\begin{aligned}
 A^T B^T &= \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} \\
 &= \text{Not defined!}.
 \end{aligned}$$

So we can't compute  $A^T B^T$ . Let's finish by computing  $B^T A^T$ :

$$\begin{aligned}
 B^T A^T &= \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \\
 &= \begin{bmatrix} 3 & 7 \\ 2 & 6 \\ 1 & 1 \end{bmatrix}.
 \end{aligned}$$

We may have suspected that  $(AB)^T = A^T B^T$ . We saw that this wasn't the case, though — and not only was it not equal, the second product wasn't even defined! Oddly enough, though, we saw that  $(AB)^T = B^T A^T$ . (Then again, maybe this isn't all that "odd." It is reminiscent of the fact that, when invertible,  $(AB)^{-1} = B^{-1} A^{-1}$ .) To help understand why this is true, look back at the work above and confirm the steps of each multiplication.

We have one more arithmetic operation to look at: the inverse.

**Example 6.1.8 Inverting a transposed matrix.**

Let

$$A = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}.$$

Find  $(A^{-1})^T$  and  $(A^T)^{-1}$ .

**Solution.** We first find  $A^{-1}$  and  $A^T$ :

$$A^{-1} = \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}.$$

Finding  $(A^{-1})^T$ :

$$\begin{aligned} (A^{-1})^T &= \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix}^T \\ &= \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}. \end{aligned}$$

Finding  $(A^T)^{-1}$ :

$$\begin{aligned} (A^T)^{-1} &= \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}. \end{aligned}$$

It seems that "the inverse of the transpose is the transpose of the inverse." (Again, we should think about this statement. The part before "is" states that we take the transpose of a matrix, then find the inverse. The part after "is" states that we find the inverse of the matrix, then take the transpose. Since these two statements are linked by an "is," they are equal.)

We have just looked at some examples of how the transpose operation interacts with matrix arithmetic operations. (These examples don't *prove* anything, other than it worked in specific examples.) We now give a theorem that tells us that what we saw wasn't a coincidence, but rather is always true.

**Theorem 6.1.9 Properties of the Matrix Transpose.**

Let  $A$  and  $B$  be matrices where the following operations are defined. Then:

1.  $(A + B)^T = A^T + B^T$  and  $(A - B)^T = A^T - B^T$

2.  $(kA)^T = kA^T$
3.  $(AB)^T = B^T A^T$
4.  $(A^{-1})^T = (A^T)^{-1}$
5.  $(A^T)^T = A$

We included in the theorem two ideas we didn't discuss already. First, that  $(kA)^T = kA^T$ . This is probably obvious. It doesn't matter when you multiply a matrix by a scalar when dealing with transposes.

The second "new" item is that  $(A^T)^T = A$ . That is, if we take the transpose of a matrix, then take its transpose again, what do we have? The original matrix.

Now that we know some properties of the transpose operation, we are tempted to play around with it and see what happens. For instance, if  $A$  is an  $m \times n$  matrix, we know that  $A^T$  is an  $n \times m$  matrix. So no matter what matrix  $A$  we start with, we can always perform the multiplication  $AA^T$  (and also  $A^T A$ ) and the result is a square matrix!

Another thing to ask ourselves as we "play around" with the transpose: suppose  $A$  is a square matrix. Is there anything special about  $A + A^T$ ? The following example has us try out these ideas.

**Example 6.1.10** The matrices  $AA^T$ ,  $A + A^T$ , and  $A - A^T$ .

Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 2 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find  $AA^T$ ,  $A + A^T$  and  $A - A^T$ .

**Solution.** Finding  $AA^T$ :

$$\begin{aligned} AA^T &= \begin{bmatrix} 2 & 1 & 3 \\ 2 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 14 & 6 & 5 \\ 6 & 4 & 3 \\ 5 & 3 & 2 \end{bmatrix}. \end{aligned}$$

Finding  $A + A^T$ :

$$\begin{aligned} A + A^T &= \begin{bmatrix} 2 & 1 & 3 \\ 2 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 3 & 4 \\ 3 & -2 & 1 \\ 4 & 1 & 2 \end{bmatrix}. \end{aligned}$$

Finding  $A - A^T$ :

$$A - A^T = \begin{bmatrix} 2 & 1 & 3 \\ 2 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}.$$

Let's look at the matrices we've formed in this example. First, consider  $AA^T$ . Something seems to be nice about this matrix — look at the location of the 6's, the 5's and the 3's. More precisely, let's look at the transpose of  $AA^T$ . We should notice that if we take the transpose of this matrix, we have the very same matrix. That is,

$$\left( \begin{bmatrix} 14 & 6 & 5 \\ 6 & 4 & 3 \\ 5 & 3 & 2 \end{bmatrix} \right)^T = \begin{bmatrix} 14 & 6 & 5 \\ 6 & 4 & 3 \\ 5 & 3 & 2 \end{bmatrix}!$$

We'll formally define this in a moment, but a matrix that is equal to its transpose is called *symmetric*.

Look at the next part of the example; what do we notice about  $A + A^T$ ? We should see that it, too, is symmetric. Finally, consider the last part of the example: do we notice anything about  $A - A^T$ ?

We should immediately notice that it is not symmetric, although it does seem "close." Instead of it being equal to its transpose, we notice that this matrix is the *opposite* of its transpose. We call this type of matrix *skew symmetric*. (Some mathematicians use the term *antisymmetric*.) We formally define these matrices here.

**Definition 6.1.11 Symmetric and Skew Symmetric Matrices.**

A matrix  $A$  is *symmetric* if  $A^T = A$ .

A matrix  $A$  is *skew symmetric* if  $A^T = -A$ .

Note that in order for a matrix to be either symmetric or skew symmetric, it must be square.

So why was  $AA^T$  symmetric in our previous example? Did we just luck out? (Of course not.) Let's take the transpose of  $AA^T$  and see what happens.

$$\begin{aligned} (AA^T)^T &= (A^T)^T(A)^T \quad (\text{transpose multiplication rule}) \\ &= AA^T \quad ((A^T)^T = A). \end{aligned}$$

We have just *proved* that no matter what matrix  $A$  we start with, the matrix  $AA^T$  will be symmetric. Nothing in our string of equalities even demanded that  $A$  be a square matrix; it is always true.

We can do a similar proof to show that as long as  $A$  is square,  $A + A^T$  is a symmetric matrix. (Why do we say that  $A$  has to be square?) We'll instead show here that if  $A$  is a square matrix, then  $A - A^T$  is skew symmetric.

$$\begin{aligned} (A - A^T)^T &= A^T - (A^T)^T \quad (\text{transpose subtraction rule}) \\ &= A^T - A \\ &= -(A - A^T). \end{aligned}$$

So we took the transpose of  $A - A^T$  and we got  $-(A - A^T)$ ; this is the definition of being skew symmetric.

We'll take what we learned from Example 6.1.10 and put it in a box. (We've already proved most of this is true; the rest we leave to solve in the Exercises.)

**Theorem 6.1.12 Symmetric and Skew Symmetric Matrices.**

1. Given any matrix  $A$ , the matrices  $AA^T$  and  $A^T A$  are symmetric.
2. Let  $A$  be a square matrix. The matrix  $A + A^T$  is symmetric.
3. Let  $A$  be a square matrix. The matrix  $A - A^T$  is skew symmetric.

Why do we care about the transpose of a matrix? Why do we care about symmetric matrices?

There are two answers that each answer both of these questions. First, we are interested in the transpose of a matrix and symmetric matrices because they are interesting. One particularly interesting thing about symmetric and skew symmetric matrices is this: consider the sum of  $(A + A^T)$  and  $(A - A^T)$ :

$$(A + A^T) + (A - A^T) = 2A.$$

This gives us an idea — if we were to multiply both sides of this equation by  $\frac{1}{2}$ , then the right hand side would just be  $A$ . This means that

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew symmetric}}.$$

That is, any matrix  $A$  can be written as the sum of a symmetric and skew symmetric matrix. That's interesting.

The second reason we care about them is that they are very useful and important in various areas of mathematics. The transpose of a matrix turns out to be an important operation; symmetric matrices have many nice properties that make solving certain types of problems possible.

Most of this text focuses on the preliminaries of matrix algebra, and the actual uses are beyond our current scope. One easy to describe example is curve fitting. Suppose we are given a large set of data points that, when plotted, look roughly quadratic. How do we find the quadratic that “best fits” this data? The solution can be found using matrix algebra, and specifically a matrix called the *pseudoinverse*. If  $A$  is a matrix, the pseudoinverse of  $A$  is the matrix  $A^\dagger = (A^T A)^{-1} A^T$  (assuming that the inverse exists). We aren't going to worry about what all the above means; just notice that it has a cool sounding name and the transpose appears twice.

In the next section we'll learn about the trace, another operation that can be performed on a matrix that is relatively simple to compute but can lead to some deep results.



## 6.1.1 Exercises

**Exercise Group.** A matrix  $A$  is given. Find  $A^T$ ; make note if  $A$  is upper/lower triangular, diagonal, symmetric and/or skew symmetric.

1. 
$$\begin{bmatrix} 0 & -6 & 1 \\ 6 & 0 & 4 \\ -1 & -4 & 0 \end{bmatrix}$$

3. 
$$\begin{bmatrix} -9 & 4 & 10 \\ 6 & -3 & -7 \\ -8 & 1 & -1 \end{bmatrix}$$

5. 
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

7. 
$$\begin{bmatrix} 6 & -4 & -5 \\ -4 & 0 & 2 \\ -5 & 2 & -2 \end{bmatrix}$$

9. 
$$\begin{bmatrix} -7 & -8 & 2 & -3 \end{bmatrix}$$

11. 
$$\begin{bmatrix} 13 & -3 \\ -3 & 1 \end{bmatrix}$$

13. 
$$\begin{bmatrix} 4 & -7 & -4 & -9 \\ -9 & 6 & 3 & -9 \end{bmatrix}$$

15. 
$$\begin{bmatrix} 2 & -5 & -3 \\ 5 & 5 & -6 \\ 7 & -4 & -10 \end{bmatrix}$$

17. 
$$\begin{bmatrix} -3 & -4 & -5 \\ 0 & -3 & 5 \\ 0 & 0 & -3 \end{bmatrix}$$

19. 
$$\begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & 3 \\ -2 & 3 & 6 \end{bmatrix}$$

21. 
$$\begin{bmatrix} -5 & -9 \\ 3 & 1 \\ -10 & -8 \end{bmatrix}$$

23. 
$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -4 & 0 \end{bmatrix}$$

2. 
$$\begin{bmatrix} 0 & 3 & -2 \\ 3 & -4 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

4. 
$$\begin{bmatrix} -2 & 10 \\ 1 & -7 \\ 9 & -2 \end{bmatrix}$$

6. 
$$\begin{bmatrix} 4 & 0 & 0 \\ -2 & -7 & 0 \\ 4 & -2 & 5 \end{bmatrix}$$

8. 
$$\begin{bmatrix} 4 & 2 & -9 \\ 5 & -4 & -10 \\ -6 & 6 & 9 \end{bmatrix}$$

10. 
$$\begin{bmatrix} 3 & 1 \\ -7 & 8 \end{bmatrix}$$

12. 
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

14. 
$$\begin{bmatrix} 4 & -5 & 2 \\ 1 & 5 & 9 \\ 9 & 2 & 3 \end{bmatrix}$$

16. 
$$\begin{bmatrix} -7 & 4 \\ 4 & -6 \end{bmatrix}$$

18. 
$$\begin{bmatrix} -9 & 8 & 2 & -7 \end{bmatrix}$$

20. 
$$\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$$

22. 
$$\begin{bmatrix} 6 & -7 & 2 & 6 \\ 0 & -8 & -1 & 0 \\ 0 & 0 & 1 & -7 \end{bmatrix}$$

24. 
$$\begin{bmatrix} 3 & -10 & 0 & 6 \\ -10 & -2 & -3 & 1 \end{bmatrix}$$

## 6.2 The Matrix Trace

In the previous section, we learned about an operation we can perform on matrices, namely the transpose. Given a matrix  $A$ , we can “find the transpose of  $A$ ,” which is another matrix. In this section we learn about a new operation called the *trace*. It is a different type of operation than the transpose. Given a matrix  $A$ , we can “find the trace of  $A$ ,” which is not a matrix but rather a number. We formally define it here.

### Definition 6.2.1 The Trace.

Let  $A$  be an  $n \times n$  matrix. The **trace** of  $A$ , denoted  $\text{tr}(A)$ , is the sum of the diagonal elements of  $A$ . That is,

$$\text{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

This seems like a simple definition, and it really is. Just to make sure it is clear, let's practice.

### Example 6.2.2 Computing the trace of a matrix.

Find the trace of  $A$ ,  $B$ ,  $C$  and  $I_4$ , where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 8 & 1 \\ -2 & 7 & -5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

**Solution.** To find the trace of  $A$ , note that the diagonal elements of  $A$  are 1 and 4. Therefore,  $\text{tr}(A) = 1 + 4 = 5$ .

We see that the diagonal elements of  $B$  are 1, 8 and -5, so  $\text{tr}(B) = 1 + 8 - 5 = 4$ .

The matrix  $C$  is not a square matrix, and our definition states that we must start with a square matrix. Therefore  $\text{tr}(C)$  is not defined.

Finally, the diagonal of  $I_4$  consists of four 1s. Therefore  $\text{tr}(I_4) = 4$ .

Now that we have defined the trace of a matrix, we should think like mathematicians and ask some questions. The first questions that should pop into our minds should be along the lines of “How does the trace work with other matrix operations?” (Recall that we asked a similar question once we learned about the transpose.) We should think about how the trace works with matrix addition, scalar multiplication, matrix multiplication, matrix inverses, and the transpose.

We'll give a theorem that will formally tell us what is true in a moment, but first let's play with two sample matrices and see if we can see what will happen. Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 2 & -1 \end{bmatrix}.$$

It should be clear that  $\text{tr}(A) = 5$  and  $\text{tr}(B) = 3$ . What is  $\text{tr}(A + B)$ ?

$$\text{tr}(A + B) = \text{tr} \left( \begin{bmatrix} 2 & 1 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 2 & -1 \end{bmatrix} \right)$$

$$\begin{aligned}
 &= \text{tr} \left( \begin{bmatrix} 4 & 1 & 4 \\ 1 & 2 & -1 \\ 3 & 1 & 2 \end{bmatrix} \right) \\
 &= 8.
 \end{aligned}$$

So we notice that  $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$ . This probably isn't a coincidence.

How does the trace work with scalar multiplication? If we multiply  $A$  by 4, then the diagonal elements will be 8, 0 and 12, so  $\text{tr}(4A) = 20$ . Is it a coincidence that this is 4 times the trace of  $A$ ?

Let's move on to matrix multiplication. How will the trace of  $AB$  relate to the traces of  $A$  and  $B$ ? Let's see:

$$\begin{aligned}
 \text{tr}(AB) &= \text{tr} \left( \begin{bmatrix} 2 & 1 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 2 & -1 \end{bmatrix} \right) \\
 &= \text{tr} \left( \begin{bmatrix} 3 & 8 & -1 \\ 4 & -2 & 3 \\ 7 & 4 & 0 \end{bmatrix} \right) \\
 &= 1.
 \end{aligned}$$

It isn't exactly clear what the relationship is among  $\text{tr}(A)$ ,  $\text{tr}(B)$  and  $\text{tr}(AB)$ . Before moving on, let's find  $\text{tr}(BA)$ :

$$\begin{aligned}
 \text{tr}(BA) &= \text{tr} \left( \begin{bmatrix} 2 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 3 \end{bmatrix} \right) \\
 &= \text{tr} \left( \begin{bmatrix} 7 & 1 & 9 \\ 2 & -1 & -5 \\ 1 & 1 & -5 \end{bmatrix} \right) \\
 &= 1.
 \end{aligned}$$

We notice that  $\text{tr}(AB) = \text{tr}(BA)$ . Is this coincidental?

How are the traces of  $A$  and  $A^{-1}$  related? We compute  $A^{-1}$  and find that

$$A^{-1} = \begin{bmatrix} 1/17 & 6/17 & 1/17 \\ 9/17 & 3/17 & -8/17 \\ 2/17 & -5/17 & 2/17 \end{bmatrix}.$$

Therefore  $\text{tr}(A^{-1}) = 6/17$ . Again, the relationship isn't clear.

Finally, let's see how the trace is related to the transpose. We actually don't have to formally compute anything. Recall from the previous section that the diagonals of  $A$  and  $A^T$  are identical; therefore,  $\text{tr}(A) = \text{tr}(A^T)$ . That, we know for sure, isn't a coincidence.

We now formally state what equalities are true when considering the interaction of the trace with other matrix operations.

### Theorem 6.2.3 Properties of the Matrix Trace.

Let  $A$  and  $B$  be  $n \times n$  matrices. Then:

1.  $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$
2.  $\text{tr}(A - B) = \text{tr}(A) - \text{tr}(B)$

This example brings to light many interesting ideas that we'll flesh out just a little bit here.

1. Notice that the elements of  $A$  are 1, -2, 1 and 1. Add the squares of these numbers:  $1^2 + (-2)^2 + 1^2 + 1^2 = 7 = \text{tr}(A^T A)$ .

Notice that the elements of  $B$  are 6, 7, 11 and -4. Add the squares of these numbers:  $6^2 + 7^2 + 11^2 + (-4)^2 = 222 = \text{tr}(B^T B)$ .

Can you see why this is true? When looking at multiplying  $A^T A$ , focus only on where the elements on the diagonal come from since they are the only ones that matter when taking the trace.

2. You can confirm on your own that regardless of the dimensions of  $A$ ,  $\text{tr}(A^T A) = \text{tr}(AA^T)$ . To see why this is true, consider the previous point. (Recall also that  $A^T A$  and  $AA^T$  are always square, regardless of the dimensions of  $A$ .)
3. Mathematicians are actually more interested in  $\sqrt{\text{tr}(A^T A)}$  than just  $\text{tr}(A^T A)$ . The reason for this is a bit complicated; the short answer is that "it works better." The reason "it works better" is related to the Pythagorean Theorem, all of all things. If we know that the legs of a right triangle have length  $a$  and  $b$ , we are more interested in  $\sqrt{a^2 + b^2}$  than just  $a^2 + b^2$ . Of course, this explanation raises more questions than it answers; our goal here is just to whet your appetite and get you to do some more reading. A Numerical Linear Algebra book would be a good place to start.

$$3. \operatorname{tr}(kA) = k \cdot \operatorname{tr}(A)$$

$$4. \operatorname{tr}(AB) = \operatorname{tr}(BA)$$

$$5. \operatorname{tr}(A^T) = \operatorname{tr}(A)$$

One of the key things to note here is what this theorem does *not* say. It says nothing about how the trace relates to matrix multiplication; that is, we can't figure out what  $\operatorname{tr}(AB)$  is just by knowing what  $\operatorname{tr}(A)$  and  $\operatorname{tr}(B)$  are. The theorem also says nothing about how the trace relates to inverses. The reason for the silence in these areas is that there simply is not a relationship.

We end this section by again wondering why anyone would care about the trace of matrix. One reason mathematicians are interested in it is that it can give a measurement of the “size” of a matrix.

Consider the following  $2 \times 2$  matrices:

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & 7 \\ 11 & -4 \end{bmatrix}.$$

These matrices have the same trace, yet  $B$  clearly has bigger elements in it. So how can we use the trace to determine a “size” of these matrices? We can consider  $\operatorname{tr}(A^T A)$  and  $\operatorname{tr}(B^T B)$ .

$$\begin{aligned} \operatorname{tr}(A^T A) &= \operatorname{tr} \left( \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \right) \\ &= \operatorname{tr} \left( \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix} \right) \\ &= 7 \\ \operatorname{tr}(B^T B) &= \operatorname{tr} \left( \begin{bmatrix} 6 & 11 \\ 7 & -4 \end{bmatrix} \begin{bmatrix} 6 & 7 \\ 11 & -4 \end{bmatrix} \right) \\ &= \operatorname{tr} \left( \begin{bmatrix} 157 & -2 \\ -2 & 65 \end{bmatrix} \right) \\ &= 222. \end{aligned}$$

Our concern is not how to interpret what this “size” measurement means, but rather to demonstrate that the trace (along with the transpose) can be used to give (perhaps useful) information about a matrix.

There are many different measurements of a matrix size. In this text, we just refer to its dimensions. Some measurements of size refer the magnitude of the elements in the matrix. The next section describes yet another measurement of matrix size.

## 6.2.1 Exercises

**Exercise Group.** Find the trace of the given matrix.

1. 
$$\begin{bmatrix} 6 & 5 \\ 2 & 10 \\ 3 & 3 \end{bmatrix}$$

3. 
$$\begin{bmatrix} 7 & 5 \\ -5 & -4 \end{bmatrix}$$

5. 
$$\begin{bmatrix} -4 & 1 & 1 \\ -2 & 0 & 0 \\ -1 & -2 & -5 \end{bmatrix}$$

7. 
$$\begin{bmatrix} 0 & -3 & 1 \\ 5 & -5 & 5 \\ -4 & 1 & 0 \end{bmatrix}$$

9. 
$$\begin{bmatrix} -3 & -10 \\ -6 & 4 \end{bmatrix}$$

11. 
$$\begin{bmatrix} 2 & 6 & 4 \\ -1 & 8 & -10 \end{bmatrix}$$

13. A matrix  $A$  that is skew symmetric.

15.  $I_n$

2. 
$$\begin{bmatrix} 1 & -5 \\ 9 & 5 \end{bmatrix}$$

4. 
$$\begin{bmatrix} -10 & 6 & -7 & -9 \\ -2 & 1 & 6 & -9 \\ 0 & 4 & -4 & 0 \\ -3 & -9 & 3 & -10 \end{bmatrix}$$

6. 
$$\begin{bmatrix} 5 & 2 & 2 & 2 \\ -7 & 4 & -7 & -3 \\ 9 & -9 & -7 & 2 \\ -4 & 8 & -8 & -2 \end{bmatrix}$$

8. 
$$\begin{bmatrix} -2 & -3 & 5 \\ 5 & 2 & 0 \\ -1 & -3 & 1 \end{bmatrix}$$

10. 
$$\begin{bmatrix} 4 & 2 & -1 \\ -4 & 1 & 4 \\ 0 & -5 & 5 \end{bmatrix}$$

12. 
$$\begin{bmatrix} -6 & 0 \\ -10 & 9 \end{bmatrix}$$

14.  $I_4$

**Exercise Group.** Verify Theorem 6.2.3 by:(a) Showing that  $\text{tr}(A) + \text{tr}(B) = \text{tr}(A + B)$ (b) Showing that  $\text{tr}(AB) = \text{tr}(BA)$ 

16. 
$$A = \begin{bmatrix} -8 & -10 & 10 \\ 10 & 5 & -6 \\ -10 & 1 & 3 \end{bmatrix},$$
$$B = \begin{bmatrix} -10 & -4 & -3 \\ -4 & -5 & 4 \\ 3 & 7 & 3 \end{bmatrix}$$

18. 
$$A = \begin{bmatrix} 0 & -8 \\ 1 & 8 \end{bmatrix}, B = \begin{bmatrix} -4 & 5 \\ -4 & 2 \end{bmatrix}$$

17. 
$$A = \begin{bmatrix} -10 & 7 & 5 \\ 7 & 7 & -5 \\ 8 & -9 & 2 \end{bmatrix},$$
$$B = \begin{bmatrix} -3 & -4 & 9 \\ 4 & -1 & -9 \\ -7 & -8 & 10 \end{bmatrix}$$

19. 
$$A = \begin{bmatrix} 1 & -1 \\ 9 & -6 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ -6 & 3 \end{bmatrix}$$

## 6.3 The Determinant

In this chapter so far we've learned about the transpose (an operation on a matrix that returns another matrix) and the trace (an operation on a square matrix that returns a number). In this section we'll learn another operation on square matrices that returns a number, called the **determinant**. The determinant of an  $n \times n$  matrix  $A$  is a number, denoted  $\det(A)$ , that is determined by  $A$ . We had a brief encounter with determinants in Section 2.4, where we saw that determinants of  $2 \times 2$  and  $3 \times 3$  matrices are related to the cross product of vectors in  $\mathbb{R}^3$ .

The determinant is kind of a tricky thing to define. Once you know and understand it, it isn't that hard, but getting started is a bit complicated. (It's similar to learning to ride a bike. The riding itself isn't hard, it is getting started that's difficult.) Unlike many mathematical quantities, we do not give a single formula to define the determinant of a matrix. Instead, we define the determinant *recursively*: first, we'll explain how to compute the determinant of a  $2 \times 2$  matrix. Then, we'll explain how to compute the determinant of a  $3 \times 3$  matrix in terms of  $2 \times 2$  determinants, and so on. Let's get started, and define the determinant for  $2 \times 2$  matrices.

**Note:** we should point out that it's not that you *can't* give a formula for the determinant; you can, but it's complicated. To write down a formula, we'd have to use summation notation, and talk about *permutations* of the indices of the entries, and... Let's just say that if you really want to see a formula, you can find one with a Google search.

### Definition 6.3.1 Determinant of $2 \times 2$ Matrices.

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The *determinant* of  $A$ , denoted by

$$\det(A) \text{ or } \begin{vmatrix} a & b \\ c & d \end{vmatrix},$$

is  $ad - bc$ .

We've seen the expression  $ad - bc$  before. In Section 4.4, we saw that a  $2 \times 2$  matrix  $A$  has inverse

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

as long as  $ad - bc \neq 0$ ; otherwise, the inverse does not exist. We can rephrase the above statement now: If  $\det(A) \neq 0$ , then

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

A brief word about the notation: notice that we can refer to the determinant by using what *looks like* absolute value bars around the entries of a matrix. We discussed at the end of the last section the idea of measuring the "size" of a matrix, and mentioned that there are many different ways to measure size. The determinant is one such way. Just as the absolute value of a number measures its size (and ignores its sign), the determinant of a matrix is a measurement of the size of the matrix. (Be careful, though:  $\det(A)$  can be negative!)

**Note:** We saw in Section 5.1 that a matrix transformation  $T(\vec{x}) = A\vec{x}$ , where  $A$  is a  $2 \times 2$  matrix, can be used to define a transformation of the Cartesian plane. We saw that we can understand the effect of such a transformation by its effect on the unit square, and for this, it is enough to compute  $T(\vec{e}_1)$  and  $T(\vec{e}_2)$ . It's easy to see that these two vectors are the columns of  $A$ :

$$[T(\vec{e}_1) \quad T(\vec{e}_2)] = [A\vec{e}_1 \quad A\vec{e}_2]$$

$$\begin{aligned}
 &= A [\vec{e}_1 \quad \vec{e}_2] \\
 &= AI_2 = A.
 \end{aligned}$$

Now, it's possible that one of these vectors is zero, or that the two vectors are parallel, but if not, the vectors  $T(\vec{e}_1)$  and  $T(\vec{e}_2)$  determine a parallelogram. With a bit of work (see Example 2.4.9 in Section 2.4), we can show that the area of this parallelogram is given precisely by  $\det(A)$ . You may also recall that we already encountered the  $3 \times 3$  determinant once, in Section 2.4, where it was used to determine the volume of a *parallelepiped*: see the discussion following Example 2.4.14. In general,  $\det(A)$  computes the effect of the transformation  $T(\vec{x}) = A\vec{x}$  on area, and volume (in three dimensions or higher).

**Example 6.3.2 Computing  $2 \times 2$  determinants.**

Find the determinant of  $A$ ,  $B$  and  $C$  where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 \\ 2 & 7 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}.$$

**Solution.** Finding the determinant of  $A$ :

$$\begin{aligned}
 \det(A) &= \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \\
 &= 1(4) - 2(3) \\
 &= -2.
 \end{aligned}$$

Similar computations show that  $\det(B) = 3(7) - (-1)(2) = 23$  and  $\det(C) = 1(6) - (-3)(-2) = 0$ .

Finding the determinant of a  $2 \times 2$  matrix is pretty straightforward. It is natural to ask next “How do we compute the determinant of matrices that are not  $2 \times 2$ ?” We first need to define some terms.

**Definition 6.3.3 Matrix Minor, Cofactor.**

Let  $A$  be an  $n \times n$  matrix. The  $(i, j)$ -minor of  $A$ , denoted  $A_{ij}$ , is the determinant of the  $(n-1) \times (n-1)$  matrix formed by deleting the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column of  $A$ .

The  $(i, j)$ -cofactor of  $A$  is the number

$$C_{ij} = (-1)^{i+j} A_{ij}.$$

Notice that this definition makes reference to taking the determinant of a matrix, while we haven't yet defined what the determinant is beyond  $2 \times 2$  matrices. We recognize this problem, and we'll see how far we can go before it becomes an issue.

**Note:** if it is necessary for clarity, we may write  $A_{i,j}$  and  $C_{i,j}$  for minors and cofactors; usually this is necessary in concrete examples, especially if we're dealing with matrices with ten or more rows and columns. For example, if we wanted discuss the minor corresponding to the  $(12, 3)$ -entry of a matrix, writing  $A_{123}$  will cause confusion, while  $A_{12,3}$  is easily understood. Where there is no risk of confusion, we frequently omit the comma to reduce clutter.

**Example 6.3.4 Computing minors and cofactors.**

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 0 & 8 \\ -3 & 5 & 7 & 2 \\ -1 & 9 & -4 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Find  $A_{1,3}$ ,  $A_{3,2}$ ,  $B_{2,1}$ ,  $B_{4,3}$  and their respective cofactors.

**Solution.** To compute the minor  $A_{1,3}$ , we remove the first row and third column of  $A$  then take the determinant.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ 4 & 5 & \cancel{6} \\ 7 & 8 & \cancel{9} \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$A_{1,3} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 32 - 35 = -3.$$

The corresponding cofactor,  $C_{1,3}$ , is

$$C_{1,3} = (-1)^{1+3} A_{1,3} = (-1)^4 (-3) = -3.$$

The minor  $A_{3,2}$  is found by removing the third row and second column of  $A$  then taking the determinant.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \cancel{2} & 3 \\ 4 & \cancel{5} & 6 \\ 7 & \cancel{8} & \cancel{9} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$$

$$A_{3,2} = \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 6 - 12 = -6.$$

The corresponding cofactor,  $C_{3,2}$ , is

$$C_{3,2} = (-1)^{3+2} A_{3,2} = (-1)^5 (-6) = 6.$$

The minor  $A_{3,2}$  is found by removing the third row and second column of  $A$  then taking the determinant.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \cancel{2} & 3 \\ 4 & \cancel{5} & 6 \\ 7 & \cancel{8} & \cancel{9} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$$

$$A_{3,2} = \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 6 - 12 = -6.$$

The corresponding cofactor,  $C_{3,2}$ , is

$$C_{3,2} = (-1)^{3+2} A_{3,2} = (-1)^5 (-6) = 6.$$

The minor  $B_{2,1}$  is found by removing the second row and first column of  $B$  then taking the determinant.

$$B = \begin{bmatrix} 1 & 2 & 0 & 8 \\ -3 & 5 & 7 & 2 \\ -1 & 9 & -4 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \cancel{1} & 2 & 0 & 8 \\ \cancel{-3} & \cancel{5} & \cancel{7} & \cancel{2} \\ \cancel{-1} & 9 & -4 & 6 \\ \cancel{1} & 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 8 \\ 9 & -4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$$



$$B_{2,1} = \begin{vmatrix} 2 & 0 & 8 \\ 9 & -4 & 6 \\ 1 & 1 & 1 \end{vmatrix} \stackrel{!}{=} ?$$

We're a bit stuck. We don't know how to find the determinant of this  $3 \times 3$  matrix. We'll come back to this later. The corresponding cofactor is

$$C_{2,1} = (-1)^{2+1} B_{2,1} = -B_{2,1},$$

whatever this number happens to be. The minor  $B_{4,3}$  is found by removing the fourth row and third column of  $B$  then taking the determinant.

$$B = \begin{bmatrix} 1 & 2 & 0 & 8 \\ -3 & 5 & 7 & 2 \\ -1 & 9 & -4 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & \emptyset & 8 \\ -3 & 5 & \cancel{7} & 2 \\ -1 & 9 & \cancel{-4} & 6 \\ \cancel{1} & \cancel{1} & \cancel{1} & \cancel{1} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 8 \\ -3 & 5 & 2 \\ -1 & 9 & 6 \end{bmatrix}$$

$$B_{4,3} = \begin{vmatrix} 1 & 2 & 8 \\ -3 & 5 & 2 \\ -1 & 9 & 6 \end{vmatrix} \stackrel{!}{=} ?$$

Again, we're stuck. We won't be able to fully compute  $C_{4,3}$ ; all we know so far is that

$$C_{4,3} = (-1)^{4+3} B_{4,3} = (-1) B_{4,3}.$$

Once we learn how to compute determinants for matrices larger than  $2 \times 2$  we can come back and finish this exercise.

In our previous example we ran into a bit of trouble. By our definition, in order to compute a minor of an  $n \times n$  matrix we needed to compute the determinant of a  $(n-1) \times (n-1)$  matrix. This was fine when we started with a  $3 \times 3$  matrix, but when we got up to a  $4 \times 4$  matrix (and larger) we run into trouble.

We are almost ready to define the determinant for any square matrix; we need one last definition.

#### Definition 6.3.5 Cofactor Expansion.

Let  $A = [a_{ij}]$  be an  $n \times n$  matrix. The **cofactor expansion** of  $A$  along the  $i$ th row is the sum

$$a_{i,1}C_{i,1} + a_{i,2}C_{i,2} + \cdots + a_{i,n}C_{i,n}.$$

The cofactor expansion of  $A$  down the  $j$ th column is the sum

$$a_{1,j}C_{1,j} + a_{2,j}C_{2,j} + \cdots + a_{n,j}C_{n,j}.$$

The notation of this definition might be a little intimidating, so let's look at an example.

The reader may find it helpful to review the determinant formula for the cross product in Section 2.4. Our method for computing the cross product follows exactly the same pattern as the cofactor expansion of a  $3 \times 3$  matrix along the first row.

**Example 6.3.6 Computing cofactor expansions.**

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Find the cofactor expansions along the second row and down the first column.

**Solution.** By the definition, the cofactor expansion along the second row is the sum

$$a_{2,1}C_{2,1} + a_{2,2}C_{2,2} + a_{2,3}C_{2,3}.$$

(Be sure to compare the above line to the definition of cofactor expansion, and see how the “ $i$ ” in the definition is replaced by “2” here.)

We’ll find each cofactor and then compute the sum.

$$\begin{aligned} C_{2,1} &= (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = (-1)(-6) = 6 && \left( \begin{array}{l} \text{we removed the second row} \\ \text{and first column of } A \\ \text{to compute the minor} \end{array} \right) \\ C_{2,2} &= (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = (1)(-12) = -12 && \left( \begin{array}{l} \text{we removed the second row} \\ \text{and second column of } A \\ \text{to compute the minor} \end{array} \right) \\ C_{2,3} &= (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = (-1)(-6) = 6 && \left( \begin{array}{l} \text{we removed the second row} \\ \text{and third column of } A \\ \text{to compute the minor} \end{array} \right). \end{aligned}$$

Thus the cofactor expansion along the second row is

$$\begin{aligned} a_{2,1}C_{2,1} + a_{2,2}C_{2,2} + a_{2,3}C_{2,3} &= 4(6) + 5(-12) + 6(6) \\ &= 24 - 60 + 36 \\ &= 0. \end{aligned}$$

At the moment, we don’t know what to do with this cofactor expansion; we’ve just successfully found it.

We move on to find the cofactor expansion down the first column. By the definition, this sum is

$$a_{1,1}C_{1,1} + a_{2,1}C_{2,1} + a_{3,1}C_{3,1}.$$

(Again, compare this to the above definition and see how we replaced the “ $j$ ” with “1.”)

We find each cofactor:

$$\begin{aligned} C_{1,1} &= (-1)^{1+1} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = (1)(-3) = -3 && \left( \begin{array}{l} \text{we removed the first row} \\ \text{and first column of } A \\ \text{to compute the minor} \end{array} \right) \\ C_{2,1} &= (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = (-1)(-6) = 6 && \text{(we computed this cofactor above)} \\ C_{3,1} &= (-1)^{3+1} \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = (1)(-3) = -3 && \left( \begin{array}{l} \text{we removed the third row} \\ \text{and first column of } A \\ \text{to compute the minor} \end{array} \right) \end{aligned}$$

The cofactor expansion down the first column is

$$\begin{aligned} a_{1,1}C_{1,1} + a_{2,1}C_{2,1} + a_{3,1}C_{3,1} &= 1(-3) + 4(6) + 7(-3) \\ &= -3 + 24 - 21 \\ &= 0. \end{aligned}$$

Is it a coincidence that both cofactor expansions were 0? We'll answer that in a while. Now that we've gotten the hang of minors and cofactors, we're ready to finally define the determinant.

#### Definition 6.3.7 The Determinant.

The *determinant* of an  $n \times n$  matrix  $A$ , denoted  $\det(A)$  or  $|A|$ , is a number given by the following:

- if  $A$  is a  $1 \times 1$  matrix  $A = [a]$ , then  $\det(A) = a$ .
- if  $A$  is a  $2 \times 2$  matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then  $\det(A) = ad - bc$ .

- if  $A$  is an  $n \times n$  matrix, where  $n \geq 2$ , then  $\det(A)$  is the number found by taking the cofactor expansion along the first row of  $A$ . That is,

$$\det(A) = a_{1,1}C_{1,1} + a_{1,2}C_{1,2} + \cdots + a_{1,n}C_{1,n}.$$

Notice that in order to compute the determinant of an  $n \times n$  matrix, we need to compute the determinants of  $n(n-1) \times (n-1)$  matrices. This can be a lot of work. We'll later learn how to shorten some of this. First, let's practice.

#### Example 6.3.8 Computing a $3 \times 3$ determinant.

Find the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

**Solution.** Notice that this is the matrix from Example 6.3.6. The cofactor expansion along the first row is

$$\det(A) = a_{1,1}C_{1,1} + a_{1,2}C_{1,2} + a_{1,3}C_{1,3}.$$

We'll compute each cofactor first then take the appropriate sum.

$$\begin{aligned} C_{1,1} &= (-1)^{1+1}A_{1,1} & C_{1,2} &= (-1)^{1+2}A_{1,2} & C_{1,3} &= (-1)^{1+3}A_{1,3} \\ &= 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} & &= (-1) \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} & &= 1 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= 45 - 48 & &= (-1)(36 - 42) & &= 32 - 35 \\ &= -3 & &= 6 & &= -3. \end{aligned}$$

Therefore the determinant of  $A$  is

$$\det(A) = 1(-3) + 2(6) + 3(-3) = 0.$$

**Example 6.3.9 Another  $3 \times 3$  determinant.**

Find the determinant of

$$A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & -1 \\ 3 & -1 & 1 \end{bmatrix}.$$

**Solution.** We'll compute each cofactor first then find the determinant.

$$\begin{aligned} C_{1,1} &= (-1)^{1+1}A_{1,1} & C_{1,2} &= (-1)^{1+2}A_{1,2} & C_{1,3} &= (-1)^{1+3}A_{1,3} \\ &= 1 \cdot \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} & &= (-1) \cdot \begin{vmatrix} 0 & -1 \\ 3 & 1 \end{vmatrix} & &= 1 \cdot \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} \\ &= 2 - 1 & &= (-1)(0 + 3) & &= 0 - 6 \\ &= 1 & &= -3 & &= -6. \end{aligned}$$

Thus the determinant is

$$\det(A) = 3(1) + 6(-3) + 7(-6) = -57.$$

**Example 6.3.10 Computing a  $4 \times 4$  determinant.**

Find the determinant of

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & 2 & 3 & 4 \\ 8 & 5 & -3 & 1 \\ 5 & 9 & -6 & 3 \end{bmatrix}.$$

**Solution.** This, quite frankly, will take quite a bit of work. In order to compute this determinant, we need to compute 4 minors, each of which requires finding the determinant of a  $3 \times 3$  matrix! Complaining won't get us any closer to the solution, (But it might make us feel a little better. Glance ahead: do you see how much work we have to do!?) so let's get started. We first compute the cofactors:

$$\begin{aligned} C_{1,1} &= (-1)^{1+1}A_{1,1} \\ &= 1 \cdot \begin{vmatrix} 2 & 3 & 4 \\ 5 & -3 & 1 \\ 9 & -6 & 3 \end{vmatrix} && \left( \begin{array}{c} \text{we must compute the determinant} \\ \text{of this } 3 \times 3 \text{ matrix} \end{array} \right) \\ &= 2 \cdot (-1)^{1+1} \begin{vmatrix} -3 & 1 \\ -6 & 3 \end{vmatrix} + 3 \cdot (-1)^{1+2} \begin{vmatrix} 5 & 1 \\ 9 & 3 \end{vmatrix} + 4 \cdot (-1)^{1+3} \begin{vmatrix} 5 & -3 \\ 9 & -6 \end{vmatrix} \\ &= 2(-3) + 3(-6) + 4(-3) \\ &= -36 \end{aligned}$$

$$C_{1,2} = (-1)^{1+2}A_{1,2}$$

$$\begin{aligned}
&= (-1) \cdot \begin{vmatrix} -1 & 3 & 4 \\ 8 & -3 & 1 \\ 5 & -6 & 3 \end{vmatrix} \quad \left( \begin{array}{c} \text{we must compute the determinant} \\ \text{of this } 3 \times 3 \text{ matrix} \end{array} \right) \\
&= (-1) \left[ \underbrace{(-1) \cdot (-1)^{1+1} \begin{vmatrix} -3 & 1 \\ -6 & 3 \end{vmatrix} + 3 \cdot (-1)^{1+2} \begin{vmatrix} 8 & 1 \\ 5 & 3 \end{vmatrix} + 4 \cdot (-1)^{1+3} \begin{vmatrix} 8 & -3 \\ 5 & -6 \end{vmatrix}}_{\text{the determinant of the } 3 \times 3 \text{ matrix}} \right] \\
&= (-1) [(-1)(-3) + 3(-19) + 4(-33)] \\
&= 186
\end{aligned}$$

$$\begin{aligned}
C_{1,3} &= (-1)^{1+3} A_{1,3} \\
&= 1 \cdot \begin{vmatrix} -1 & 2 & 4 \\ 8 & 5 & 1 \\ 5 & 9 & 3 \end{vmatrix} \quad \left( \begin{array}{c} \text{we must compute the determinant} \\ \text{of this } 3 \times 3 \text{ matrix} \end{array} \right) \\
&= (-1) \cdot (-1)^{1+1} \begin{vmatrix} 5 & 1 \\ 9 & 3 \end{vmatrix} + 2 \cdot (-1)^{1+2} \begin{vmatrix} 8 & 1 \\ 5 & 3 \end{vmatrix} + 4 \cdot (-1)^{1+3} \begin{vmatrix} 8 & 5 \\ 5 & 9 \end{vmatrix} \\
&= (-1)(6) + 2(-19) + 4(47) \\
&= 144
\end{aligned}$$

$$\begin{aligned}
C_{1,4} &= (-1)^{1+4} A_{1,4} \\
&= (-1) \cdot \begin{vmatrix} -1 & 2 & 3 \\ 8 & 5 & -3 \\ 5 & 9 & -6 \end{vmatrix} \quad \left( \begin{array}{c} \text{we must compute the determinant} \\ \text{of this } 3 \times 3 \text{ matrix} \end{array} \right) \\
&= (-1) \left[ \underbrace{(-1) \cdot (-1)^{1+1} \begin{vmatrix} 5 & -3 \\ 9 & -6 \end{vmatrix} + 2 \cdot (-1)^{1+2} \begin{vmatrix} 8 & -3 \\ 5 & -6 \end{vmatrix} + 3 \cdot (-1)^{1+3} \begin{vmatrix} 8 & 5 \\ 5 & 9 \end{vmatrix}}_{\text{the determinant of the } 3 \times 3 \text{ matrix}} \right] \\
&= (-1) [(-1)(-3) + 2(33) + 3(47)] \\
&= -210
\end{aligned}$$

We've computed our four cofactors. All that is left is to compute the cofactor expansion.

$$\det(A) = 1(-36) + 2(186) + 1(144) + 2(-210) = 60.$$

As a way of “visualizing” this, let's write out the cofactor expansion again but including the matrices in their place.

$$\begin{aligned}
\det(A) &= a_{1,1}C_{1,1} + a_{1,2}C_{1,2} + a_{1,3}C_{1,3} + a_{1,4}C_{1,4} \\
&= 1(-1)^2 \underbrace{\begin{vmatrix} 2 & 3 & 4 \\ 5 & -3 & 1 \\ 9 & -6 & 3 \end{vmatrix}}_{=-36} + 2(-1)^3 \underbrace{\begin{vmatrix} -1 & 3 & 4 \\ 8 & -3 & 1 \\ 5 & -6 & 3 \end{vmatrix}}_{=-186} \\
&\quad + \\
&\quad 1(-1)^4 \underbrace{\begin{vmatrix} -1 & 2 & 4 \\ 8 & 5 & 1 \\ 5 & 9 & 3 \end{vmatrix}}_{=144} + 2(-1)^5 \underbrace{\begin{vmatrix} -1 & 2 & 3 \\ 8 & 5 & -3 \\ 5 & 9 & -6 \end{vmatrix}}_{=-210}
\end{aligned}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 60.$$

That certainly took a while; it required more than 50 multiplications (we didn't count the additions). To compute the determinant of a  $5 \times 5$  matrix, we'll need to compute the determinants of five  $4 \times 4$  matrices, meaning that we'll need over 250 multiplications! Not only is this a lot of work, but there are just too many ways to make silly mistakes. (The author made three when the above example was originally typed.) There are some tricks to make this job easier, but regardless we see the need to employ technology. Even then, technology quickly bogs down. A  $25 \times 25$  matrix is considered "small" by today's standards, but it is essentially impossible for a computer to compute its determinant by only using cofactor expansion; it too needs to employ "tricks."

It is common for mathematicians, scientists and engineers to consider linear systems with thousands of equations and variables.

In the next section we will learn some of these tricks as we learn some of the properties of the determinant. Right now, let's review the essentials of what we have learned.

1. The determinant of a square matrix is a number that is determined by the matrix.
2. We find the determinant by computing the cofactor expansion along the first row.
3. To compute the determinant of an  $n \times n$  matrix, we need to compute  $n$  determinants of  $(n - 1) \times (n - 1)$  matrices.

## 6.3.1 Exercises

**Exercise Group.** Find the determinant of the  $2 \times 2$  matrix.

1.  $\begin{bmatrix} 10 & 7 \\ 8 & 9 \end{bmatrix}$

2.  $\begin{bmatrix} 6 & -1 \\ -7 & 8 \end{bmatrix}$

3.  $\begin{bmatrix} -1 & -7 \\ -5 & 9 \end{bmatrix}$

4.  $\begin{bmatrix} -10 & -1 \\ -4 & 7 \end{bmatrix}$

5.  $\begin{bmatrix} 8 & 10 \\ 2 & -3 \end{bmatrix}$

6.  $\begin{bmatrix} 10 & -10 \\ -10 & 0 \end{bmatrix}$

7.  $\begin{bmatrix} 1 & -3 \\ 7 & 7 \end{bmatrix}$

8.  $\begin{bmatrix} -4 & -5 \\ -1 & -4 \end{bmatrix}$

**Exercise Group.** A matrix  $A$  is given.

(a) Construct the submatrices used to compute the minors  $A_{1,1}$ ,  $A_{1,2}$  and  $A_{1,3}$ .

(b) Find the cofactors  $C_{1,1}$ ,  $C_{1,2}$ , and  $C_{1,3}$ .

9.  $\begin{bmatrix} -7 & -3 & 10 \\ 3 & 7 & 6 \\ 1 & 6 & 10 \end{bmatrix}$

10.  $\begin{bmatrix} -2 & -9 & 6 \\ -10 & -6 & 8 \\ 0 & -3 & -2 \end{bmatrix}$

11.  $\begin{bmatrix} -5 & -3 & 3 \\ -3 & 3 & 10 \\ -9 & 3 & 9 \end{bmatrix}$

12.  $\begin{bmatrix} -6 & -4 & 6 \\ -8 & 0 & 0 \\ -10 & 8 & -1 \end{bmatrix}$

**Exercise Group.** Find the determinant of the given matrix using cofactor expansion along the first row.

13.  $\begin{bmatrix} 3 & 2 & 3 \\ -6 & 1 & -10 \\ -8 & -9 & -9 \end{bmatrix}$

14.  $\begin{bmatrix} 8 & -9 & -2 \\ -9 & 9 & -7 \\ 5 & -1 & 9 \end{bmatrix}$

15.  $\begin{bmatrix} -4 & 3 & -4 \\ -4 & -5 & 3 \\ 3 & -4 & 5 \end{bmatrix}$

16.  $\begin{bmatrix} 1 & -2 & 1 \\ 5 & 5 & 4 \\ 4 & 0 & 0 \end{bmatrix}$

17.  $\begin{bmatrix} 1 & -4 & 1 \\ 0 & 3 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

18.  $\begin{bmatrix} 3 & -1 & 0 \\ -3 & 0 & -4 \\ 0 & -1 & -4 \end{bmatrix}$

19.  $\begin{bmatrix} -5 & 0 & -4 \\ 2 & 4 & -1 \\ -5 & 0 & -4 \end{bmatrix}$

20.  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

21.  $\begin{bmatrix} 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$

22.  $\begin{bmatrix} -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & -1 \end{bmatrix}$

23.  $\begin{bmatrix} -5 & 1 & 0 & 0 \\ -3 & -5 & 2 & 5 \\ -2 & 4 & -3 & 4 \\ 5 & 4 & -3 & 3 \end{bmatrix}$

24.  $\begin{bmatrix} 2 & -1 & 4 & 4 \\ 3 & -3 & 3 & 2 \\ 0 & 4 & -5 & 1 \\ -2 & -5 & -2 & -5 \end{bmatrix}$

25. Let  $A$  be a  $2 \times 2$  matrix;  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Show why  $\det(A) = ad - bc$  by computing the cofactor expansion of  $A$

along the first row.



## 6.4 Properties of the Determinant

In the previous section we learned how to compute the determinant. In this section we learn some of the properties of the determinant, and this will allow us to compute determinants more easily. In the next section we will see one application of determinants.

We start with a theorem that gives us more freedom when computing determinants.

### Theorem 6.4.1 Cofactor Expansion Along Any Row or Column.

*Let  $A$  be an  $n \times n$  matrix. The determinant of  $A$  can be computed using cofactor expansion along any row or column of  $A$ .*

We alluded to this fact way back after Example 6.3.6. We had just learned what cofactor expansion was and we practised along the second row and down the third column. Later, we found the determinant of this matrix by computing the cofactor expansion along the first row. In all three cases, we got the number 0. This wasn't a coincidence. The above theorem states that all three expansions were actually computing the determinant.

How does this help us? By giving us freedom to choose any row or column to use for the expansion, we can choose a row or column that looks “most appealing.” This usually means “it has lots of zeros.” We demonstrate this principle below.

**Note:** Theorem 6.4.1 is sometimes called the *Laplace Expansion Theorem*, after Pierre-Simon, Marquis de Laplace (1749-1827), a French mathematician and physicist whose importance and influence rivals that of Newton. The reader may be wondering why we have not included a proof of this result, which is one of the more important computational facts about the determinant. The answer is that in most textbooks we checked that included the proof, the complete details take up some four pages or so, and don't really add all that much to the understanding of what's going on.

### Example 6.4.2 Computing a $4 \times 4$ determinant.

Find the determinant of

$$A = \begin{bmatrix} 1 & 2 & 0 & 9 \\ 2 & -3 & 0 & 5 \\ 7 & 2 & 3 & 8 \\ -4 & 1 & 0 & 2 \end{bmatrix}.$$

**Solution.** Our first reaction may well be “Oh no! Not another  $4 \times 4$  determinant!” However, we can use cofactor expansion along any row or column that we choose. The third column looks great; it has lots of zeros in it. The cofactor expansion along this column is

$$\begin{aligned} \det(A) &= a_{1,3}C_{1,3} + a_{2,3}C_{2,3} + a_{3,3}C_{3,3} + a_{4,3}C_{4,3} \\ &= 0 \cdot C_{1,3} + 0 \cdot C_{2,3} + 3 \cdot C_{3,3} + 0 \cdot C_{4,3}. \end{aligned}$$

The wonderful thing here is that three of our cofactors are multiplied by 0. We won't bother computing them since they will not contribute to the determinant. Thus

$$\begin{aligned} \det(A) &= 3 \cdot C_{3,3} \\ &= 3 \cdot (-1)^{3+3} \cdot \begin{vmatrix} 1 & 2 & 9 \\ 2 & -3 & 5 \\ -4 & 1 & 2 \end{vmatrix} \\ &= 3 \cdot (-147) \quad \left( \begin{array}{l} \text{we computed the determinant of the} \\ 3 \times 3 \text{ matrix without showing our work;} \\ \text{it is } -147 \end{array} \right) \\ &= -447. \end{aligned}$$

Wow. That was a lot simpler than computing all that we did in Example 6.3.10. Of course, in that example, we didn't really have any shortcuts that we could have employed. Our next example involves a  $5 \times 5$  determinant. At first, this looks like trouble, until we realize that the matrix is *triangular*. As we'll see, this makes our job much easier.

**Example 6.4.3** Computing the determinant of a  $5 \times 5$  (triangular) matrix.

Find the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 10 & 11 & 12 \\ 0 & 0 & 0 & 13 & 14 \\ 0 & 0 & 0 & 0 & 15 \end{bmatrix}.$$

**Solution.** Since we can expand along any row or column, things are not as bad as they might at first seem. In fact, this problem is very easy. What row or column should we choose to find the determinant along? There are two obvious choices: the first column or the last row. Both have 4 zeros in them. We choose the first column. We omit most of the cofactor expansion, since most of it is just 0:

$$\det(A) = 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 6 & 7 & 8 & 9 \\ 0 & 10 & 11 & 12 \\ 0 & 0 & 13 & 14 \\ 0 & 0 & 0 & 15 \end{vmatrix}.$$

Similarly, this determinant is not bad to compute; we again choose to use cofactor expansion along the first column. Note: technically, this cofactor expansion is  $6 \cdot (-1)^{1+1} A_{1,1}$ ; we are going to drop the  $(-1)^{1+1}$  terms from here on out in this example (it will show up a lot...).

$$\det(A) = 1 \cdot 6 \cdot \begin{vmatrix} 10 & 11 & 12 \\ 0 & 13 & 14 \\ 0 & 0 & 15 \end{vmatrix}.$$

You can probably see a trend. We'll finish out the steps without explaining each one.

$$\begin{aligned} \det(A) &= 1 \cdot 6 \cdot 10 \cdot \begin{vmatrix} 13 & 14 \\ 0 & 15 \end{vmatrix} \\ &= 1 \cdot 6 \cdot 10 \cdot 13 \cdot 15 \\ &= 11700. \end{aligned}$$

We see that the final determinant is the product of the diagonal entries. This works for any triangular matrix (and since diagonal matrices are triangular, it works for diagonal matrices as well). This is an important enough idea that we'll put it into a box.

**Key Idea 6.4.4 The Determinant of Triangular Matrices.**

The determinant of a triangular matrix is the product of its diagonal elements.

It is now again time to start thinking like a mathematician. Remember, mathematicians see something new and often ask “How does this relate to things I already know?” So now we ask, “If we change a matrix in some way, how is its determinant changed?”

The standard way that we change matrices is through elementary row operations. If we perform an elementary row operation on a matrix, how will the determinant of the new matrix compare to the determinant of the original matrix?

Let’s experiment first and then we’ll officially state what happens.

**Example 6.4.5 Row operations and determinants.**

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Let  $B$  be formed from  $A$  by doing one of the following elementary row operations:

1.  $2R_1 + R_2 \rightarrow R_2$
2.  $5R_1 \rightarrow R_1$
3.  $R_1 \leftrightarrow R_2$

Find  $\det(A)$  as well as  $\det(B)$  for each of the row operations above.

**Solution.** It is straightforward to compute  $\det(A) = -2$ .

Let  $B$  be formed by performing the first row operation on  $A$ ; thus

$$B = \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix}.$$

It is clear that  $\det(B) = -2$ , the same as  $\det(A)$ .

Now let  $B$  be formed by performing the second elementary row operation on  $A$ ; that is,

$$B = \begin{bmatrix} 5 & 10 \\ 3 & 4 \end{bmatrix}.$$

We can see that  $\det(B) = -10$ , which is  $5 \cdot \det(A)$ .

Finally, let  $B$  be formed by the third row operation given; swap the two rows of  $A$ . We see that

$$B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

and that  $\det(B) = 2$ , which is  $(-1) \cdot \det(A)$ .

We’ve seen in the above example that there seems to be a relationship between the determinants of matrices “before and after” being changed by elementary row operations. Certainly, one example isn’t enough to base a theory on, and we have not proved anything yet. Regardless, the following theorem is true.

**Theorem 6.4.6 The Determinant and Elementary Row Operations.**

Let  $A$  be an  $n \times n$  matrix and let  $B$  be formed by performing one elementary row operation on  $A$ .

1. If  $B$  is formed from  $A$  by adding a scalar multiple of one row to another, then  $\det(B) = \det(A)$ .
2. If  $B$  is formed from  $A$  by multiplying one row of  $A$  by a scalar  $k$ , then  $\det(B) = k \cdot \det(A)$ .
3. If  $B$  is formed from  $A$  by interchanging two rows of  $A$ , then  $\det(B) = -\det(A)$ .

Let's put this theorem to use in a couple of examples.

**Example 6.4.7 Using row operations to compute a determinant.**

Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Compute  $\det(A)$ , then find the determinants of the following matrices by inspection using Theorem 6.4.6.

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 7 & 7 & 7 \end{bmatrix} \quad D = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

**Solution.** Computing  $\det(A)$  by cofactor expansion down the first column or along the second row seems like the best choice, utilizing the one zero in the matrix. We can quickly confirm that  $\det(A) = 1$ .

To compute  $\det(B)$ , notice that the rows of  $A$  were rearranged to form  $B$ . There are different ways to describe what happened; saying  $R_1 \leftrightarrow R_2$  was followed by  $R_1 \leftrightarrow R_3$  produces  $B$  from  $A$ . Since there were two row swaps,  $\det(B) = (-1)(-1)\det(A) = \det(A) = 1$ .

Notice that  $C$  is formed from  $A$  by multiplying the third row by 7. Thus  $\det(C) = 7 \cdot \det(A) = 7$ .

It takes a little thought, but we can form  $D$  from  $A$  by the operation  $-3R_2 + R_1 \rightarrow R_1$ . This type of elementary row operation does not change determinants, so  $\det(D) = \det(A)$ .

**Example 6.4.8 Effect of elementary row operations on the determinant.**

The matrix  $B$  was formed from  $A$  using the following elementary row operations, though not necessarily in this order. Find  $\det(A)$ .

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

$$2R_1 \rightarrow R_1$$

$$\frac{1}{3}R_3 \rightarrow R_3$$

$$R_1 \leftrightarrow R_2$$

$$6R_1 + R_2 \rightarrow R_2$$

**Solution.** It doesn't take too much work to compute  $\det B = 24$ . In looking at our list of elementary row operations, we see that only the first three have an effect on the determinant. Therefore

$$24 = \det(B) = 2 \cdot \frac{1}{3} \cdot (-1) \cdot \det(A)$$

and hence

$$\det(A) = -36.$$

In the previous example, we may have been tempted to “rebuild”  $A$  using the elementary row operations and then computing the determinant. This can be done, but in general it is a bad idea; it takes too much work and it is too easy to make a mistake.

Let's continue to think like mathematicians; mathematicians tend to remember “problems” they've encountered in the past, and when they learn something new, in the backs of their minds they try to apply their new knowledge to solve their old problem. (This is why mathematicians rarely smile: they are remembering their problems)

What “problem” did we recently uncover? We stated in the last chapter that even computers could not compute the determinant of large matrices with co-factor expansion. How then can we compute the determinant of large matrices?

We just learned two interesting and useful facts about matrix determinants. First, the determinant of a triangular matrix is easy to compute: just multiply the diagonal elements. Secondly, we know that given any square matrix, we can use elementary row operations to put the matrix in triangular form. We can then find the determinant of the new matrix (which is easy), and adjust that number by recalling what elementary operations we performed.

#### Example 6.4.9 Reducing a determinant to triangular form.

Find the determinant of  $A$  by first putting  $A$  into a triangular form, where

$$A = \begin{bmatrix} 2 & 4 & -2 \\ -1 & -2 & 5 \\ 3 & 2 & 1 \end{bmatrix}.$$

**Solution.** In putting  $A$  into a triangular form, we need not worry about getting leading 1s, but it does tend to make our life easier as we work out a problem by hand. So let's scale the first row by  $1/2$ :

$$\frac{1}{2}R_1 \rightarrow R_1 \quad \begin{bmatrix} 1 & 2 & -1 \\ -1 & -2 & 5 \\ 3 & 2 & 1 \end{bmatrix}.$$

Now let's get 0s below this leading 1:

$$\begin{array}{l} R_1 + R_2 \rightarrow R_2 \\ -3R_1 + R_3 \rightarrow R_3 \end{array} \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 4 \\ 0 & -4 & 4 \end{bmatrix}.$$

We can finish in one step; by interchanging rows 2 and 3 we'll have our matrix in triangular form.

$$R_2 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \end{bmatrix}.$$

Let's name this last matrix  $B$ . The determinant of  $B$  is easy to compute as it is triangular;  $\det(B) = -16$ . We can use this to find  $\det(A)$ .

Recall the steps we used to transform  $A$  into  $B$ . They are:

$$\begin{array}{l} \frac{1}{2}R_1 \rightarrow R_1 \\ R_1 + R_2 \rightarrow R_2 \\ -3R_1 + R_3 \rightarrow R_3 \\ R_2 \leftrightarrow R_3 \end{array}$$

The first operation multiplied a row of  $A$  by  $\frac{1}{2}$ . This means that the resulting matrix had a determinant that was  $\frac{1}{2}$  the determinant of  $A$ .

The next two operations did not affect the determinant at all. The last operation, the row swap, changed the sign. Combining these effects, we know that

$$-16 = \det(B) = (-1)\frac{1}{2}\det(A).$$

Solving for  $\det(A)$  we have that  $\det(A) = 32$ .

In practice, we don't need to keep track of operations where we add multiples of one row to another; they simply do not affect the determinant. Also, in practice, these steps are carried out by a computer, and computers don't care about leading 1s. Therefore, row scaling operations are rarely used. The only things to keep track of are row swaps, and even then all we care about are the number of row swaps. An odd number of row swaps means that the original determinant has the opposite sign of the triangular form matrix; an even number of row swaps means they have the same determinant.

If you find yourself needing to compute a determinant by hand (say, on an exam), it's a good idea to keep the following principles in mind:

1. Stick to row operations of the type  $R_i + kR_j \rightarrow R_i$  as much as possible: *they don't change the determinant.*
2. Getting all the way to triangular form isn't really necessary. Use row operations of the above type to create as many zeros as possible in one of the columns, and then expand along that column.

To see how these principles work in practice, let's repeat Example 6.4.9. This time we'll focus on creating zeros, but we won't worry about getting to triangular form. Since adding a multiple of one row to another doesn't change the

**Note:** If you want to get really fancy, since  $\det(A^T) = \det(A)$ , and since performing row operations on  $A^T$  is the same as performing *column* operations on  $A$ , you can also add a multiple of one column to another without changing the determinant!

determinant, we can compute  $\det(A)$  with a string of equalities, as follows:

$$\begin{aligned}
 \begin{vmatrix} 2 & 4 & -2 \\ -1 & 2 & 5 \\ 3 & 2 & 1 \end{vmatrix} &= \begin{vmatrix} 0 & 0 & 8 \\ -1 & 2 & 5 \\ 3 & 2 & 1 \end{vmatrix} && (\text{Add } 2R_2 \text{ to } R_1) \\
 &= 0 + 0 + 8(-1)^{1+3} \begin{vmatrix} -1 & -2 \\ 3 & 2 \end{vmatrix} && (\text{Expand along Row 1}) \\
 &= 8((-1)(2) - (-2)(3)) \\
 &= 8(4) = 32.
 \end{aligned}$$

Of course, in this case we got lucky and ended up with two zeros in the first row after one row operation. However, had this not been the case, we would have simply done one more row operation ( $R_3 + 3R_2 \rightarrow R_3$ ) to create a second zero in the first column, and then done a cofactor expansion along that column.

For larger determinants, we can follow the same routine: create zeros in one column, expand along that column to get a smaller determinant, and repeat.

Let's think some more like a mathematician. How does the determinant work with other matrix operations that we know? Specifically, how does the determinant interact with matrix addition, scalar multiplication, matrix multiplication, the transpose and the trace? We'll again do an example to get an idea of what is going on, then give a theorem to state what is true.

#### Example 6.4.10 Determinants and matrix operations.

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}.$$

Find the determinants of the matrices  $A$ ,  $B$ ,  $A - B$ ,  $3A$ ,  $AB$ ,  $A^T$ , and  $A^{-1}$ . Can you find any connections between these values?

**Solution.** We can quickly compute that  $\det(A) = -2$  and  $\det(B) = 7$ .

$$\begin{aligned}
 \det(A - B) &= \det \left( \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \right) \\
 &= \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} \\
 &= 1.
 \end{aligned}$$

It's tough to find a connection between  $\det(A - B)$ ,  $\det(A)$  and  $\det(B)$ .

$$\begin{aligned}
 \det(3A) &= \begin{vmatrix} 3 & 6 \\ 9 & 12 \end{vmatrix} \\
 &= -18.
 \end{aligned}$$

We can figure this one out; multiplying one row of  $A$  by 3 increases the determinant by a factor of 3; doing it again (and hence multiplying both rows by 3) increases the determinant again by a factor of 3. Therefore  $\det(3A) = 3 \cdot 3 \cdot \det(A)$ , or  $3^2 \cdot \det(A)$ .

$$\det(AB) = \det \left( \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \right)$$

$$\begin{aligned}
 &= \begin{vmatrix} 8 & 11 \\ 18 & 23 \end{vmatrix} \\
 &= -14.
 \end{aligned}$$

This one seems clear;  $\det(AB) = \det(A) \det(B)$ .

$$\begin{aligned}
 \det(A^T) &= \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \\
 &= -2.
 \end{aligned}$$

Obviously  $\det(A^T) = \det(A)$ ; is this always going to be the case? If we think about it, we can see that the cofactor expansion along the first row of  $A$  will give us the same result as the cofactor expansion along the first column of  $A^T$ .

$$\begin{aligned}
 \det(A^{-1}) &= \begin{vmatrix} -2 & 1 \\ 3/2 & -1/2 \end{vmatrix} \\
 &= 1 - 3/2 \\
 &= -1/2.
 \end{aligned}$$

It seems as though

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Seeing that expansion along the first row agrees with expansion along the first column can be a bit tricky to think out in your head. Try it with a  $3 \times 3$  matrix  $A$  and see how it works. All the  $2 \times 2$  submatrices that are created in  $A^T$  are the transpose of those found in  $A$ ; this doesn't matter since it is easy to see that the determinant isn't affected by the transpose in a  $2 \times 2$  matrix.

We now state a few theorems that confirm our conjectures from the previous example.

**Theorem 6.4.11 The determinant of a non-invertible matrix.**

*If an  $n \times n$  matrix  $A$  is not invertible, then  $\det(A) = 0$ .*

To see that Theorem 6.4.11 is true, note that if  $A$  is not invertible, then the reduced row echelon form  $R$  of  $A$  must have a row of zeros. Performing a cofactor expansion along this row, we immediately see that  $\det(R) = 0$ . Since  $R$  is obtained from  $A$  by a series of elementary row operations, we know from Theorem 6.4.6 that  $\det(A)$  is a multiple of  $\det(R)$ , and thus  $\det(A) = 0$ .

It follows from Theorem 6.4.11 (using the logical principle known as the *contrapositive*) that if  $\det(A) \neq 0$ , we're guaranteed that  $A$  is invertible.

At this point, we naturally should ask whether or not the converse to Theorem 6.4.11 is true as well: suppose we know  $\det(A) = 0$ . Does that imply that  $A$  is not invertible? (Or equivalently, if we know  $A$  is invertible, does this imply that  $\det(A) \neq 0$ ?) The answer is yes, but to see this, we first need a more general result.

**Theorem 6.4.12 Determinant Properties.**

*Let  $A$  and  $B$  be  $n \times n$  matrices and let  $k$  be a scalar. The following are true:*

1.  $\det(kA) = k^n \cdot \det(A)$



2.  $\det(A^T) = \det(A)$
3.  $\det(AB) = \det(A) \det(B)$

Proving that  $\det(AB) = \det(A) \det(B)$  is most easily done using elementary matrices. (See Section 4.6.) Recall that multiplying a matrix *on the left* by an elementary matrix is the same as doing the corresponding row operation: if  $A$  is any  $3 \times 3$  matrix, then  $EA$  can be obtained from  $A$  using the same row operation used to create  $E$ .

Theorem 6.4.6 then tells us that

$$\det(EB) = \det(E) \det(B)$$

for any matrix  $B$  and elementary matrix  $E$ . The rest boils down to two cases: either  $\det(A) = 0$ , in which case  $A$  is not invertible, so neither is  $AB$ , and thus  $\det(AB) = 0 = 0 \det(B)$ , or  $\det(A) \neq 0$ . In the latter case,  $A$  is invertible, and can be written as a product of elementary matrices. We can then prove that  $\det(AB) = \det(A) \det(B)$  by applying Theorem 6.4.6 repeatedly.

From Theorem 6.4.12, we see that  $\det(AB) = \det(A) \det(B)$  for any matrices  $A$  and  $B$ . What does this tell us in the case of an invertible matrix? Recall that if  $A$  is invertible, then we can determine the inverse matrix  $A^{-1}$  such that

$$AA^{-1} = I_n.$$

Now, the identity matrix is triangular, and all of its diagonal entries are equal to 1, so we immediately see that  $\det(I_n) = 1$ . Thus, taking the determinant of both sides of the above equation, we have

$$\det(AA^{-1}) = \det(A) \det(A^{-1}) = 1.$$

We have a product of two numbers equal to one, which tells us that neither of these numbers can be zero. (Otherwise, the product would be zero as well.) Thus, if  $A$  is invertible, it must be the case that  $\det(A) \neq 0$ , so a matrix  $A$  is invertible if and only if  $\det(A) \neq 0$ .

As an added bonus, we can rearrange the above equation to give us one more property of the determinant:

**Theorem 6.4.13 The determinant of an inverse.**

If  $A$  is an invertible matrix, then  $\det(A) \neq 0$ , and

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Combining Theorems Theorem 6.4.11 and Theorem 6.4.13 allows us to add on to our Invertible Matrix Theorem.

**Theorem 6.4.14 Invertible Matrix Theorem.**

Let  $A$  be an  $n \times n$  matrix. The following statements are equivalent.

- (a)  $A$  is invertible.
- (b) The equation  $A\vec{x} = \vec{0}$  has exactly one solution (namely,  $\vec{x} = \vec{0}$ ).
- (c) The reduced row echelon form of  $A$  is  $I$ .

- (d) The equation  $A\vec{x} = \vec{b}$  has exactly one solution for every  $n \times 1$  vector  $\vec{b}$ .
- (e) There exists a matrix  $C$  such that  $AC = I$ .
- (f) There exists a matrix  $B$  such that  $BA = I$ .
- (g)  $\det(A) \neq 0$ .

This new addition to the Invertible Matrix Theorem is very useful; we'll refer back to it in Chapter 7 when we discuss eigenvalues.

In the next section we'll see how the determinant can be used to solve systems of linear equations.

## 6.4.1 Exercises

**Exercise Group.** Find the determinant of the given matrix using cofactor expansion along any row or column you choose.

1. 
$$\begin{bmatrix} 1 & 2 & 3 \\ -5 & 0 & 3 \\ 4 & 0 & 6 \end{bmatrix}$$

3. 
$$\begin{bmatrix} -4 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -5 \end{bmatrix}$$

5. 
$$\begin{bmatrix} -2 & -3 & 5 \\ 5 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

7. 
$$\begin{bmatrix} -3 & 0 & -5 \\ -2 & -3 & 3 \\ -1 & 0 & 1 \end{bmatrix}$$

9. 
$$\begin{bmatrix} 5 & -5 & 0 & 1 \\ 2 & 4 & -1 & -1 \\ 5 & 0 & 0 & 4 \\ -1 & -2 & 0 & 5 \end{bmatrix}$$

11. 
$$\begin{bmatrix} -5 & -5 & 0 & -2 \\ 0 & 0 & 5 & 0 \\ 1 & 3 & 3 & 1 \\ -4 & -2 & -1 & -5 \end{bmatrix}$$

13. 
$$\begin{bmatrix} 4 & 0 & 5 & 1 & 0 \\ 1 & 0 & 3 & 1 & 5 \\ 2 & 2 & 0 & 2 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 4 & 4 & 2 & 5 & 3 \end{bmatrix}$$

2. 
$$\begin{bmatrix} -4 & 4 & -4 \\ 0 & 0 & -3 \\ -2 & -2 & -1 \end{bmatrix}$$

4. 
$$\begin{bmatrix} 0 & -3 & 1 \\ 0 & 0 & 5 \\ -4 & 1 & 0 \end{bmatrix}$$

6. 
$$\begin{bmatrix} -2 & -2 & 0 \\ 2 & -5 & -3 \\ -5 & 1 & 0 \end{bmatrix}$$

8. 
$$\begin{bmatrix} 0 & 4 & -4 \\ 3 & 1 & -3 \\ -3 & -4 & 0 \end{bmatrix}$$

10. 
$$\begin{bmatrix} -1 & 3 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 4 & -5 & -2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

12. 
$$\begin{bmatrix} -1 & 0 & -2 & 5 \\ 3 & -5 & 1 & -2 \\ -5 & -2 & -1 & -3 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

14. 
$$\begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 4 & 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 3 \\ 5 & 0 & 5 & 0 & 4 \end{bmatrix}$$

**Exercise Group.** A matrix  $M$  and  $\det(M)$  are given. Matrices  $A$ ,  $B$  and  $C$  are formed by performing operations on  $M$ . Determine the determinants of  $A$ ,  $B$  and  $C$  using Theorem 6.4.6 and Theorem 6.4.12, and indicate the operations used to form  $A$ ,  $B$  and  $C$ .

15.  $M = \begin{bmatrix} 9 & 7 & 8 \\ 1 & 3 & 7 \\ 6 & 3 & 3 \end{bmatrix}, \det(M) = 45.$

(a)  $A = \begin{bmatrix} 18 & 14 & 16 \\ 1 & 3 & 7 \\ 6 & 3 & 3 \end{bmatrix}$

(b)  $B = \begin{bmatrix} 9 & 7 & 8 \\ 1 & 3 & 7 \\ 96 & 73 & 83 \end{bmatrix}$

(c)  $C = \begin{bmatrix} 9 & 1 & 6 \\ 7 & 3 & 3 \\ 8 & 7 & 3 \end{bmatrix}$

16.  $M = \begin{bmatrix} 0 & 3 & 5 \\ 3 & 1 & 0 \\ -2 & -4 & -1 \end{bmatrix}, \det(M) = -41.$

(a)  $A = \begin{bmatrix} 0 & 3 & 5 \\ -2 & -4 & -1 \\ 3 & 1 & 0 \end{bmatrix}$

(b)  $B = \begin{bmatrix} 0 & 3 & 5 \\ 3 & 1 & 0 \\ 8 & 16 & 4 \end{bmatrix}$

(c)  $C = \begin{bmatrix} 3 & 4 & 5 \\ 3 & 1 & 0 \\ -2 & -4 & -1 \end{bmatrix}$

$$17. \quad M = \begin{bmatrix} 5 & 1 & 5 \\ 4 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \det(M) = -16.$$

$$(a) \quad A = \begin{bmatrix} 0 & 0 & 4 \\ 5 & 1 & 5 \\ 4 & 0 & 2 \end{bmatrix}$$

$$(b) \quad B = \begin{bmatrix} -5 & -1 & -5 \\ -4 & 0 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$(c) \quad C = \begin{bmatrix} 15 & 3 & 15 \\ 12 & 0 & 6 \\ 0 & 0 & 12 \end{bmatrix}$$

$$18. \quad M = \begin{bmatrix} 5 & 4 & 0 \\ 7 & 9 & 3 \\ 1 & 3 & 9 \end{bmatrix}, \det(M) = 120.$$

$$(a) \quad A = \begin{bmatrix} 1 & 3 & 9 \\ 7 & 9 & 3 \\ 5 & 4 & 0 \end{bmatrix}$$

$$(b) \quad B = \begin{bmatrix} 5 & 4 & 0 \\ 14 & 18 & 6 \\ 3 & 9 & 27 \end{bmatrix}$$

$$(c) \quad C = \begin{bmatrix} -5 & -4 & 0 \\ -7 & -9 & -3 \\ -1 & -3 & -9 \end{bmatrix}$$

**Exercise Group.** Matrices  $A$  and  $B$  are given. Verify part 3 of Theorem 6.4.12 by computing  $\det(A)$ ,  $\det(B)$  and  $\det(AB)$ .

$$19. \quad A = \begin{bmatrix} 2 & 0 & 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & -4 \\ 1 & 3 \end{bmatrix}.$$

$$20. \quad A = \begin{bmatrix} 3 & -1 & 4 & 1 \end{bmatrix}, B = \begin{bmatrix} -4 & -1 \\ -5 & 3 \end{bmatrix}.$$

$$21. \quad A = \begin{bmatrix} -4 & 4 & 5 & -2 \end{bmatrix}, B = \begin{bmatrix} -3 & -4 \\ 5 & -3 \end{bmatrix}.$$

$$22. \quad A = \begin{bmatrix} -3 & -1 & 2 & -3 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 4 & -4 \end{bmatrix}.$$

**Exercise Group.** Find the determinant of the given matrix.

$$23. \quad \begin{bmatrix} 3 & 2 & 3 \\ -6 & 1 & -10 \\ -8 & -9 & -9 \end{bmatrix}$$

$$24. \quad \begin{bmatrix} 8 & -9 & -2 \\ -9 & 9 & -7 \\ 5 & -1 & 9 \end{bmatrix}$$

$$25. \quad \begin{bmatrix} -4 & 3 & -4 \\ -4 & -5 & 3 \\ 3 & -4 & 5 \end{bmatrix}$$

$$26. \quad \begin{bmatrix} 1 & -2 & 1 \\ 5 & 5 & 4 \\ 4 & 0 & 0 \end{bmatrix}$$

$$27. \quad \begin{bmatrix} 1 & -4 & 1 \\ 0 & 3 & 0 \\ 1 & 2 & 2 \end{bmatrix}$$

$$28. \quad \begin{bmatrix} 3 & -1 & 0 \\ -3 & 0 & -4 \\ 0 & -1 & -4 \end{bmatrix}$$

$$29. \quad \begin{bmatrix} -5 & 0 & -4 \\ 2 & 4 & -1 \\ -5 & 0 & -4 \end{bmatrix}$$

$$30. \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

## 6.5 Applications of the Determinant

In the previous sections we have learned about the determinant, but we haven't given a really good reason *why* we would want to compute it. This section shows two applications of the determinant: solving systems of linear equations and computing the inverse of a matrix.

### 6.5.1 Cramer's Rule

#### Theorem 6.5.1 Cramer's Rule.

Let  $A$  be an  $n \times n$  matrix with  $\det(A) \neq 0$  and let  $\vec{b}$  be an  $n \times 1$  column vector. Then the linear system

$$A\vec{x} = \vec{b}$$

has solution

$$x_i = \frac{\det(A_i(\vec{b}))}{\det(A)},$$

where  $A_i(\vec{b})$  is the matrix formed by replacing the  $i$ th column of  $A$  with  $\vec{b}$ .

#### Example 6.5.2 Using Cramer's Rule.

Use Cramer's Rule to solve the linear system  $A\vec{x} = \vec{b}$  where

$$A = \begin{bmatrix} 1 & 5 & -3 \\ 1 & 4 & 2 \\ 2 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} -36 \\ -11 \\ 7 \end{bmatrix}.$$

**Solution.** We first compute the determinant of  $A$  to see if we can apply Cramer's Rule.

$$\det(A) = \begin{vmatrix} 1 & 5 & -3 \\ 1 & 4 & 2 \\ 2 & -1 & 0 \end{vmatrix} = 49.$$

Since  $\det(A) \neq 0$ , we can apply Cramer's Rule. Following Theorem 6.5.1, we compute  $\det(A_1(\vec{b}))$ ,  $\det(A_2(\vec{b}))$  and  $\det(A_3(\vec{b}))$ .

$$\det(A_1(\vec{b})) = \begin{vmatrix} \mathbf{-36} & 5 & -3 \\ \mathbf{-11} & 4 & 2 \\ \mathbf{7} & -1 & 0 \end{vmatrix} = 49.$$

(We used a bold font to show where replaced the first column of  $A$ .)

$$\det(A_2(\vec{b})) = \begin{vmatrix} 1 & \mathbf{-36} & -3 \\ 1 & \mathbf{-11} & 2 \\ 2 & \mathbf{7} & 0 \end{vmatrix} = -245$$

$$\det(A_3(\vec{b})) = \begin{vmatrix} 1 & 5 & \mathbf{-36} \\ 1 & 4 & \mathbf{-11} \\ 2 & -1 & \mathbf{7} \end{vmatrix} = 196.$$

Therefore we can compute  $\vec{x}$ :

$$x_1 = \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{49}{49} = 1$$

The closest we came to motivating the determinant is that if  $\det(A) = 0$ , then we know that  $A$  is not invertible. But it seems that there may be easier ways to check.

It is interesting to note that despite the presentation given here, determinants actually pre-date the modern usage of matrices by more than a century. Cramer's rule<sup>1</sup> was published by Cramer in 1750, and the term Matrix<sup>2</sup> was introduced by James Joseph Sylvester in 1850. (Even then, Sylvester's description of matrices was in terms of minors — that's right, determinants.) The interested reader is encouraged to read up on the history of the subject. (Wikipedia is not a bad place to start.)

$$x_2 = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{-245}{49} = -5$$

$$x_3 = \frac{\det(A_3(\vec{b}))}{\det(A)} = \frac{196}{49} = 4.$$

Therefore

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}.$$

### Example 6.5.3 Using Cramer's Rule.

Use Cramer's Rule to solve the linear system  $A\vec{x} = \vec{b}$  where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

**Solution.** The determinant of  $A$  is  $-2$ , so we can apply Cramer's Rule.

$$\det(A_1(\vec{b})) = \begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix} = -6$$

$$\det(A_2(\vec{b})) = \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = 4.$$

Therefore

$$x_1 = \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{-6}{-2} = 3$$

$$x_2 = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{4}{-2} = -2,$$

and

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

We learned in Section 6.4 that when considering a linear system  $A\vec{x} = \vec{b}$  where  $A$  is square, if  $\det(A) \neq 0$  then  $A$  is invertible and  $A\vec{x} = \vec{b}$  has exactly one solution. We also stated in Key Idea 4.5.2 that if  $\det(A) = 0$ , then  $A$  is not invertible and so therefore either  $A\vec{x} = \vec{b}$  has no solution or infinitely many solutions. Our method of figuring out which of these cases applied was to form the augmented matrix  $\begin{bmatrix} A & \vec{b} \end{bmatrix}$ , put it into reduced row echelon form, and then interpret the results.

Cramer's Rule specifies that  $\det(A) \neq 0$  (so we are guaranteed a solution). When  $\det(A) = 0$  we are not able to discern whether infinitely many solutions or no solution exists for a given vector  $\vec{b}$ . Cramer's Rule is only applicable to the case when exactly one solution exists.

We end this section with a practical consideration. We have mentioned before that finding determinants is a computationally intensive operation. To solve a linear system with 3 equations and 3 unknowns, we need to compute 4 determinants. Just think: with 10 equations and 10 unknowns, we'd need to compute 11 really hard determinants of  $10 \times 10$  matrices! That is a lot of work!

The upshot of this is that Cramer's Rule makes for a poor choice in solving numerical linear systems. It simply is not done in practice; it is hard to beat Gaussian elimination.

So why include it? *Because its truth is amazing.* The determinant is a very strange operation; it produces a number in a very odd way. It should seem incredible to the reader that by manipulating determinants in a particular way, we can solve linear systems.

### 6.5.2 The Adjugate Formula

Recall that Theorem 4.4.10 in Section 4.4 gave us a “shortcut” for computing the inverse of a  $2 \times 2$  matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ : as long as  $\det(A) \neq 0$ , we have

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

This result can be easily verified by checking that  $AA^{-1} = I_2$  as required. The reader may have wondered if there is a similar formula for  $A^{-1}$  for a general  $n \times n$  matrix  $A$ , and whether or not such a formula would still constitute a “shortcut”. The results here are mixed. Yes, there's a formula, and we will present it shortly. However, as with Cramer's rule, it is *not* a shortcut. The reasons are the same as those we just mentioned for Cramer's rule: as long as we're dealing with a matrix whose entries are numbers, computing the inverse using row operations is vastly more efficient.

We begin with a definition.

#### Definition 6.5.4 The adjugate of a matrix.

Let  $A$  be an  $n \times n$  matrix.

- The **matrix of cofactors** of  $A$  is the  $n \times n$  matrix

$$\text{cof}(A) = [C_{ij}]$$

whose  $(i, j)$ -entry is given by the  $(i, j)$ -cofactor of  $A$ .

- The **adjugate** of  $A$  is the  $n \times n$  matrix

$$\text{adj}(A) = (\text{cof}(A))^T = [C_{ij}]^T.$$

Thus to obtain the matrix of cofactors for  $A$ , we replace each entry of  $A$  by the corresponding cofactor. Taking the transpose of this matrix produces the adjugate of  $A$ .

Why do we care about the adjugate matrix? Consider the product  $A \cdot \text{adj}(A)$ :

$$A \cdot \text{adj}(A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

Notice that the indices for  $\text{adj}(A)$  are reversed, since we took the transpose of the cofactor matrix. What is the  $(i, j)$  entry of this product? Consider first

A version of Cramer's Rule is often taught in introductory differential equations courses, as it can be used to find solutions to certain linear differential equations. In this situation, the entries of the matrices are functions, not numbers, and hence computing determinants is easier than using Gaussian elimination. Again, though, as the matrices get large, other solution methods are resorted to.

the case where  $i = j$ . We find that the  $(i, i)$ -entry is

$$a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

Here we see exactly why we want to take the transpose in our definition of  $\text{adj}(A)$ : when we multiply matrices, we multiply rows times columns, and taking the transpose ensures that each column of  $\text{adj}(A)$  is the corresponding row of  $\text{cof}(A)$ .

But this is just the cofactor expansion of  $\det(A)$  along the  $i^{\text{th}}$  row! Thus, the  $(i, i)$  entry of  $A \cdot \text{adj}(A)$  is simply  $\det(A)$ . This tells us what the diagonal is. What about the off-diagonal entries?

When  $i \neq j$ , we have the  $(i, j)$ -entry

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}.$$

This is no longer a cofactor expansion for the determinant of  $A$ , since we're taking entries from one row of  $A$ , and cofactors from another. This is, however, a cofactor expansion for the determinant of the matrix  $B$  that we obtain if we replace Row  $j$  of  $A$  with another copy of Row  $i$ . (Take a moment to think about why this is true.) But this means that the matrix  $B$  has two identical rows, and using Theorem 6.4.6, we can see that we must have  $\det(B) = 0$ . This means that all of the off-diagonal entries of our product are zero! We have

$$A \cdot \text{adj}(A) = \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \det(A) \end{bmatrix} = \det(A)I_n.$$

Now, we know that  $A$  is invertible if and only if  $\det(A) \neq 0$ , and as long as  $\det(A) \neq 0$ , we can multiply both sides of the above equation by  $\frac{1}{\det(A)}$ . With a bit of rearranging, we find

$$A \cdot \left( \frac{1}{\det(A)} \text{adj}(A) \right) = I_n.$$

But we know that if we can find *any* matrix  $B$  such that  $AB = I_n$ , then  $B$  is necessarily the inverse of  $A$ . We have established the following theorem.

**Theorem 6.5.5 The adjugate formula for the inverse.**

Let  $A$  be an  $n \times n$  matrix. If  $\det(A) \neq 0$ , then  $A$  is invertible, and

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A).$$

Notice that Theorem 4.4.10 is a special case of Theorem 6.5.5. The cofactors of a  $2 \times 2$  matrix are simply numbers, and it's easy to check that the adjugate of  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is  $\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ .

Let us repeat our words of caution from the beginning of this discussion. Just because we have a formula for the inverse does not mean we need to use it! Consider the case of a  $5 \times 5$  matrix (remember that this is a relatively small matrix by practical standards). Would you want to use Theorem 6.5.5 to compute the inverse? What would this require? Well, we'd need to compute  $\det(A)$ , since that appears in the formula, so there's already a  $5 \times 5$  determinant to deal with. But don't forget what  $\text{adj}(A)$  is: a matrix of cofactors. In this case,  $\text{adj}(A)$  would consist of *twenty-five* different  $4 \times 4$  determinants that would all need to be computed. What do you think would be less work? Computing one  $5 \times 5$  determinant and 25  $4 \times 4$  determinants, or using row operations? Now consider doing this for  $10 \times 10$ , or  $100 \times 100$  matrices. Sometimes the first method is also the best!

Let's do one example to see that even for a  $3 \times 3$  matrix, there's a fair amount of work involved.



**Example 6.5.6 Using the adjugate formula.**

Use Theorem 6.5.5 to compute the inverse of the matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & -2 \\ 1 & 5 & -3 \end{bmatrix}.$$

**Solution.** We begin by computing  $\det(A)$ , to make sure that the inverse exists. Using the  $-1$  in the first row to create a zero in the  $(3, 2)$  spot below it, we have

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 & -1 & 3 \\ 4 & 0 & -2 \\ 1 & 5 & -3 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 3 \\ 4 & 0 & -2 \\ 11 & 0 & 12 \end{vmatrix} \\ &= (-1)(-1)^{1+2} \begin{vmatrix} 4 & -2 \\ 11 & 12 \end{vmatrix} = 1(4(12) - 11(-2)) = 70. \end{aligned}$$

Next, we set about computing all nine cofactors of  $A$ . We have

$$\begin{aligned} C_{11} &= (+1) \begin{vmatrix} 0 & -2 \\ 5 & -3 \end{vmatrix} = 10 \\ C_{12} &= (-1) \begin{vmatrix} 4 & -2 \\ 1 & -3 \end{vmatrix} = 10 \\ C_{13} &= (+1) \begin{vmatrix} 4 & 0 \\ 1 & 5 \end{vmatrix} = 20 \\ C_{21} &= (-1) \begin{vmatrix} -1 & 3 \\ 5 & -3 \end{vmatrix} = 12 \\ C_{22} &= (+1) \begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix} = -9 \\ C_{23} &= (-1) \begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} = -11 \\ C_{31} &= (+1) \begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} = 2 \\ C_{32} &= (-1) \begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix} = 16 \\ C_{33} &= (+1) \begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix} = 4. \end{aligned}$$

Thus, we obtain

$$\operatorname{adj}(A) = \begin{bmatrix} 10 & 10 & 20 \\ 12 & -9 & -11 \\ 2 & 16 & 4 \end{bmatrix}^T = \begin{bmatrix} 10 & 12 & 2 \\ 10 & -9 & 16 \\ 20 & -11 & 4 \end{bmatrix}.$$

If we haven't made any computational errors (and there's a good chance that we have!) then Theorem 6.5.5 tells us that

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{70} \begin{bmatrix} 10 & 12 & 2 \\ 10 & -9 & 16 \\ 20 & -11 & 4 \end{bmatrix} = \begin{bmatrix} 1/7 & 6/35 & 1/35 \\ 1/7 & -9/70 & 8/35 \\ 2/7 & -11/70 & 2/35 \end{bmatrix}.$$

**Caution:** The entries of  $\operatorname{adj}(A)$  are the cofactors  $C_{ij}$ , and *not* the products  $a_{ij}C_{ij}$  that appear in the cofactor expansion theorem.

The reader should verify that  $AA^{-1} = I$  to make sure that we haven't made any mistakes. (The author made two mistakes that were caught doing this verification!)

In the next chapter we'll see another use for the determinant. Meanwhile, try to develop a deeper appreciation of math: odd, complicated things that seem completely unrelated often are intricately tied together. Mathematicians see these connections and describe them as "beautiful."

## 6.5.3 Exercises

**Exercise Group.** Matrices  $A$  and  $\vec{b}$  are given.

(a) Give  $\det(A)$  and  $\det(A_i)$  for all  $i$ .

(b) Use Cramer's Rule to solve  $A\vec{x} = \vec{b}$ . If Cramer's Rule cannot be used to find the solution, then state whether or not a solution exists.

$$1. \quad A = \begin{bmatrix} 3 & 0 & -3 & 5 & 4 & 4 \\ 5 & 5 & -4 & & & \end{bmatrix}, \vec{b} = \begin{bmatrix} 24 \\ 0 \\ 31 \end{bmatrix}.$$

$$2. \quad A = \begin{bmatrix} 4 & -4 & 0 & 5 & 1 & -1 \\ 3 & -1 & 2 & & & \end{bmatrix}, \vec{b} = \begin{bmatrix} 16 \\ 22 \\ 8 \end{bmatrix}.$$

$$3. \quad A = \begin{bmatrix} 1 & 0 & -10 & 4 & -3 & -10 \\ -9 & 6 & -2 & & & \end{bmatrix}, \vec{b} = \begin{bmatrix} -40 \\ -94 \\ 132 \end{bmatrix}.$$

$$4. \quad A = \begin{bmatrix} -6 & -7 & -7 & 5 & 4 & 1 \\ 5 & 4 & 8 & & & \end{bmatrix}, \vec{b} = \begin{bmatrix} 58 \\ -35 \\ -49 \end{bmatrix}.$$

$$5. \quad A = [9 \quad 5 \quad -4 \quad -7], \vec{b} = \begin{bmatrix} -45 \\ 20 \end{bmatrix}.$$

$$6. \quad A = [0 \quad -6 \quad 9 \quad -10], \vec{b} = \begin{bmatrix} 6 \\ -17 \end{bmatrix}.$$

$$7. \quad A = [2 \quad 10 \quad -1 \quad 3], \vec{b} = \begin{bmatrix} 42 \\ 19 \end{bmatrix}.$$

$$8. \quad A = [7 \quad -7 \quad -7 \quad 9], \vec{b} = \begin{bmatrix} 28 \\ -26 \end{bmatrix}.$$

$$9. \quad A = [-8 \quad 16 \quad 10 \quad -20], \vec{b} = \begin{bmatrix} -48 \\ 60 \end{bmatrix}.$$

$$10. \quad A = [7 \quad 14 \quad -2 \quad -4], \vec{b} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}.$$

$$11. \quad A = \begin{bmatrix} 4 & 9 & 3 & -5 & -2 & -13 \\ -1 & 10 & -13 & & & \end{bmatrix}, \vec{b} = \begin{bmatrix} -28 \\ 35 \\ 7 \end{bmatrix}.$$

$$12. \quad A = \begin{bmatrix} 7 & -4 & 25 & -2 & 1 & -7 \\ 9 & -7 & 34 & & & \end{bmatrix}, \vec{b} = \begin{bmatrix} -1 \\ -3 \\ 5 \end{bmatrix}.$$

**Exercise Group.** Use Theorem 6.5.5 to compute the inverse of  $A$ , if it exists.

$$13. \quad A = \begin{bmatrix} 2 & -1 & 4 \\ 3 & -5 & 7 \\ 0 & 3 & -2 \end{bmatrix}$$

$$14. \quad A = \begin{bmatrix} 3 & 2 & -5 \\ 1 & 0 & -1 \\ 7 & 4 & 2 \end{bmatrix}$$

$$15. \quad A = \begin{bmatrix} 2 & -4 & 73 & 1 & 5 \\ 5 & -5 & 2 & & \end{bmatrix}$$

$$16. \quad A = \begin{bmatrix} 5 & 2 & 0 \\ 0 & -2 & 3 \\ 5 & -2 & 6 \end{bmatrix}$$

$$17. \quad A = \begin{bmatrix} 1 & -4 & 3 & 2 \\ 5 & 0 & -3 & 6 \\ 2 & -3 & 1 & 4 \\ 7 & 2 & -5 & 1 \end{bmatrix}$$

$$18. \quad A = \begin{bmatrix} 3 & 1 & 0 & -1 \\ 6 & 4 & 2 & 0 \\ -3 & -1 & -5 & 2 \\ 1 & 0 & -1 & 4 \end{bmatrix}$$



## Chapter 7

# Eigenvalues and Eigenvectors

We have often explored new ideas in Linear Algebra by making connections to our previous algebraic experience. Adding two numbers,  $x + y$ , led us to adding vectors  $\vec{x} + \vec{y}$  and adding matrices  $A + B$ . We explored multiplication, which then led us to solving the matrix equation  $A\vec{x} = \vec{b}$ , which was reminiscent of solving the algebra equation  $ax = b$ .

This chapter is motivated by another analogy. Consider: when we multiply an unknown number  $x$  by another number such as 5, what do we know about the result? Unless,  $x = 0$ , we know that in some sense  $5x$  will be “5 times bigger than  $x$ .” Applying this to vectors, we would readily agree that  $5\vec{x}$  gives a vector that is “5 times bigger than  $\vec{x}$ ”; we know from Part 8 of Theorem 2.2.16 that  $\|5\vec{x}\| = 5\|\vec{x}\|$ .

Within the linear algebra context, though, we have two types of multiplication: scalar and matrix multiplication. What happens to  $\vec{x}$  when we multiply it by a matrix  $A$ ? Our first response is likely along the lines of “You just get another vector. There is no definable relationship.” We might wonder if there is ever the case where a matrix–vector multiplication is very similar to a scalar–vector multiplication. That is, do we ever have the case where  $A\vec{x} = a\vec{x}$ , where  $a$  is some scalar? That is the motivating question of this chapter.

## 7.1 Eigenvalues and Eigenvectors

We start by considering the matrix  $A$  and vector  $\vec{x}$  as given below.

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Multiplying  $A\vec{x}$  gives:

$$\begin{aligned} A\vec{x} &= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ 5 \end{bmatrix} \\ &= 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} ! \end{aligned}$$

Wow! It looks like multiplying  $A\vec{x}$  is the same as  $5\vec{x}$ ! This makes us wonder lots of things: is this the only case in the world where something like this happens? (Probably not.) Is  $A$  somehow a special matrix, and  $A\vec{x} = 5\vec{x}$  for any

Eigenvalues and eigenvectors are frequently encountered in the context of “characteristic directions” for linear transformations. Before continuing with this section, the reader might find it useful to take a look at the list of linear transformations of the Cartesian plane in Subsection 5.1.3. For each of the transformations listed, see if you can predict which vectors will be transformed into scalar multiples of themselves: for which  $\vec{x} \in \mathbb{R}^2$  is  $T(\vec{x}) = k\vec{x}$ ? (We’ll answer this later in the section.) This question leads to the idea of *invariant subspaces* for a linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . These are subspaces  $V \subseteq \mathbb{R}^n$  such that  $T(\vec{x}) \in V$  for each  $\vec{x} \in V$ . Such subspaces are of great importance in many areas of Mathematics and Physics.

vector  $\vec{x}$  we pick? (Probably not.) Or maybe  $\vec{x}$  was a special vector, and no matter what  $2 \times 2$  matrix  $A$  we picked, we would have  $A\vec{x} = 5\vec{x}$ . (Again, probably not.)

A more likely explanation is this: given the matrix  $A$ , the number 5 and the vector  $\vec{x}$  formed a special pair that happened to work together in a nice way. It is then natural to wonder if other “special” pairs exist. For instance, could we find a vector  $\vec{x}$  where  $A\vec{x} = 3\vec{x}$ ?

This equation is hard to solve *at first*; we are not used to matrix equations where  $\vec{x}$  appears on both sides of “=.” Therefore we put off solving this for just a moment to state a definition and make a few comments.

### Definition 7.1.1 Eigenvalues and Eigenvectors.

Let  $A$  be an  $n \times n$  matrix,  $\vec{x}$  a nonzero  $n \times 1$  column vector and  $\lambda$  a scalar. If

$$A\vec{x} = \lambda\vec{x},$$

then  $\vec{x}$  is an *eigenvector* of  $A$  and  $\lambda$  is an *eigenvalue* of  $A$ .

The word “eigen” is German for “proper” or “characteristic.” Therefore, an *eigenvector* of  $A$  is a “characteristic vector of  $A$ .” This vector tells us something about  $A$ .

Why do we use the Greek letter (lambda)? It is pure tradition. Above, we used  $a$  to represent the unknown scalar, since we are used to that notation. We now switch to  $\lambda$  because that is how everyone else does it. (An example of mathematical peer pressure.) Don’t get hung up on this; it is just a number.

Note that our definition requires that  $A$  be a square matrix. If  $A$  isn’t square then  $A\vec{x}$  and  $\lambda\vec{x}$  will have different sizes, and so they cannot be equal. Also note that  $\vec{x}$  must be nonzero. Why? What if  $\vec{x} = \vec{0}$ ? Then *no matter what* is,  $A\vec{x} = \lambda\vec{x}$ . This would then imply that *every number* is an eigenvalue; if every number is an eigenvalue, then we wouldn’t need a definition for it. Therefore we specify that  $\vec{x} \neq \vec{0}$ .

Our last comment before trying to find eigenvalues and eigenvectors for given matrices deals with “why we care.” Did we stumble upon a mathematical curiosity, or does this somehow help us build better bridges, heal the sick, send astronauts into orbit, design optical equipment, and understand quantum mechanics? The answer, of course, is “Yes.” (Except for the “understand quantum mechanics” part. Nobody truly understands that stuff; they just *probably* understand it.) This is a wonderful topic in and of itself: we need no external application to appreciate its worth. At the same time, it has many, many applications to “the real world.”

Back to our math. Given a square matrix  $A$ , we want to find a nonzero vector  $\vec{x}$  and a scalar such that  $A\vec{x} = \lambda\vec{x}$ . We will solve this using the skills we developed in Chapter 4.

$$A\vec{x} = \lambda\vec{x} \quad (\text{original equation})$$

$$A\vec{x} - \lambda\vec{x} = \vec{0} \quad (\text{subtract } \lambda\vec{x} \text{ from both sides})$$

$$(A - \lambda I)\vec{x} = \vec{0} \quad (\text{factor out } \vec{x}).$$

Think about this last factorization. We are likely tempted to say

$$A\vec{x} - \lambda\vec{x} = (A - \lambda)\vec{x},$$

but this really doesn’t make sense. After all, what does “a matrix minus a number” mean? We need the identity matrix in order for this to be logical.

We frequently make claims about topics in this text having “many applications to the real world.” You don’t need to take our word for it on this. The reader is encouraged to look up the Wikipedia page<sup>1</sup> on eigenvalues and eigenvectors. (This is one of the better written and accessible Wikipedia pages on a mathematics topic we’ve encountered, by the way!) Scrolling down to the section on applications reveals an extensive list of applications, from Physics (analyzing inertial tensors in Mechanics, the Schrödinger equation in Quantum Mechanics, ...), Engineering, Chemistry and Geology, to image processing, data analysis, and epidemiology. Even Google’s famous “Page Rank” algorithm involves eigenvalues and eigenvectors. For information on this, search (on Google?) for the “\$25,000,000,000 Eigenvector”.

Let us now think about the equation  $(A - \lambda I)\vec{x} = \vec{0}$ . While it looks complicated, it really is just matrix equation of the type we solved in Section 3.6. We are just trying to solve  $B\vec{x} = \vec{0}$ , where  $B = (A - \lambda I)$ .

We know from our previous work that this type of equation always has a solution, namely,  $\vec{x} = \vec{0}$ . (Recall this is a *homogeneous* system of equations.) However, we want  $\vec{x}$  to be an eigenvector and, by the definition, eigenvectors cannot be  $\vec{0}$ .

This means that we want solutions to  $(A - \lambda I)\vec{x} = \vec{0}$  other than  $\vec{x} = \vec{0}$ . Recall that Theorem 4.4.12 says that if the matrix  $(A - \lambda I)$  is invertible, then the *only* solution to  $(A - \lambda I)\vec{x} = \vec{0}$  is  $\vec{x} = \vec{0}$ . Therefore, in order to have other solutions, we need  $(A - \lambda I)$  to not be invertible.

Finally, recall from Theorem 6.4.12 that noninvertible matrices all have a determinant of 0. Therefore, if we want to find eigenvalues and eigenvectors  $\vec{x}$ , we need  $\det(A - \lambda I) = 0$ .

Let's start our practice of this theory by finding such that  $\det(A - \lambda I) = 0$ ; that is, let's find the eigenvalues of a matrix.

### Example 7.1.2 Computing the eigenvalues of a matrix.

Find the eigenvalues of  $A$ , that is, find such that  $\det(A - \lambda I) = 0$ , where

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}.$$

**Solution.** (Note that this is the matrix we used at the beginning of this section.) First, we write out what  $A - \lambda I$  is:

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\ &= \begin{bmatrix} 1 - \lambda & 4 \\ 2 & 3 - \lambda \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1 - \lambda & 4 \\ 2 & 3 - \lambda \end{vmatrix} \\ &= (1 - \lambda)(3 - \lambda) - 8 \\ &= \lambda^2 - 4\lambda - 5. \end{aligned}$$

Since we want  $\det(A - \lambda I) = 0$ , we want  $\lambda^2 - 4\lambda - 5 = 0$ . This is a simple quadratic equation that is easy to factor:

$$\begin{aligned} \lambda^2 - 4\lambda - 5 &= 0 \\ (\lambda - 5)(\lambda + 1) &= 0 \\ \lambda &= -1, 5. \end{aligned}$$

According to our above work,  $\det(A - \lambda I) = 0$  when  $\lambda = -1, 5$ . Thus, the eigenvalues of  $A$  are  $-1$  and  $5$ .

Earlier, when looking at the same matrix as used in our example, we wondered if we could find a vector  $\vec{x}$  such that  $A\vec{x} = 3\vec{x}$ . According to this example,

the answer is “No.” With this matrix  $A$ , the only values of  $\lambda$  that work are  $-1$  and  $5$ .

Let’s restate the above in a different way: It is pointless to try to find  $\vec{x}$  where  $A\vec{x} = 3\vec{x}$ , for there is no such  $\vec{x}$ . There are only 2 equations of this form that have a solution, namely

$$A\vec{x} = -\vec{x} \quad \text{and} \quad A\vec{x} = 5\vec{x}.$$

As we introduced this section, we gave a vector  $\vec{x}$  such that  $A\vec{x} = 5\vec{x}$ . Is this the only one? Let’s find out while calling our work an example; this will amount to finding the eigenvectors of  $A$  that correspond to the eigenvector of  $5$ .

**Example 7.1.3** Computing an eigenvector corresponding to a given eigenvalue.

Find  $\vec{x}$  such that  $A\vec{x} = 5\vec{x}$ , where

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}.$$

**Solution.** Recall that our algebra from before showed that if

$$A\vec{x} = \lambda\vec{x} \quad \text{then} \quad (A - \lambda I)\vec{x} = \vec{0}.$$

Therefore, we need to solve the equation  $(A - \lambda I)\vec{x} = \vec{0}$  for  $\vec{x}$  when  $\lambda = 5$ :

$$\begin{aligned} A - 5I &= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}. \end{aligned}$$

To solve  $(A - 5I)\vec{x} = \vec{0}$ , we form the augmented matrix and put it into reduced row echelon form:

$$\left[ \begin{array}{cc|c} -4 & 4 & 0 \\ 2 & -2 & 0 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right].$$

Thus

$$\begin{aligned} x_1 &= t \\ x_2 &= t \text{ is free} \end{aligned}$$

and

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We have infinitely many solutions to the equation  $A\vec{x} = 5\vec{x}$ ; any nonzero scalar multiple of the vector  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  is a solution. We can do a few examples to confirm this:

$$\begin{aligned} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} &= \begin{bmatrix} 10 \\ 10 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 7 \end{bmatrix} &= \begin{bmatrix} 35 \\ 35 \end{bmatrix} = 5 \begin{bmatrix} 7 \\ 7 \end{bmatrix} \end{aligned}$$



$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \end{bmatrix} = \begin{bmatrix} -15 \\ -15 \end{bmatrix} = 5 \begin{bmatrix} -3 \\ -3 \end{bmatrix}.$$

Of course, this works in general. For any  $t$ , we have

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \left( t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = t \left( \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = t \left( 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = 5 \left( t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right).$$

Our method of finding the eigenvalues of a matrix  $A$  boils down to determining which values of  $\lambda$  give the matrix  $(A - \lambda I)$  a determinant of 0. In computing  $\det(A - \lambda I)$ , we get a polynomial in  $\lambda$  whose roots are the eigenvalues of  $A$ . This polynomial is important and so it gets its own name.

**Definition 7.1.4 Characteristic Polynomial.**

Let  $A$  be an  $n \times n$  matrix. The *characteristic polynomial* of  $A$  is the  $n$ th degree polynomial  $p(\lambda) = \det(A - \lambda I)$ .

Our definition just states *what* the characteristic polynomial is. We know from our work so far *why* we care: the roots of the characteristic polynomial of an  $n \times n$  matrix  $A$  are the eigenvalues of  $A$ .

In Examples 2 and 3, we found eigenvalues and eigenvectors, respectively, of a given matrix. That is, given a matrix  $A$ , we found values  $\lambda$  and vectors  $\vec{x}$  such that  $A\vec{x} = \lambda\vec{x}$ . The steps that follow outline the general procedure for finding eigenvalues and eigenvectors; we'll follow this up with some examples.

**Key Idea 7.1.5 Finding Eigenvalues and Eigenvectors.**

Let  $A$  be an  $n \times n$  matrix.

1. To find the eigenvalues of  $A$ , compute  $p(\lambda)$ , the characteristic polynomial of  $A$ , set it equal to 0, then solve for  $\lambda$ .
2. To find the eigenvectors of  $A$ , *for each eigenvalue* solve the homogeneous system  $(A - \lambda I)\vec{x} = \vec{0}$ .

**Example 7.1.6 Computing eigenvalues and eigenvectors.**

Find the eigenvalues of  $A$ , and for each eigenvalue, find an eigenvector where

$$A = \begin{bmatrix} -3 & 15 \\ 3 & 9 \end{bmatrix}.$$

**Solution.** To find the eigenvalues, we must compute  $\det(A - \lambda I)$  and set it equal to 0.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -3 - \lambda & 15 \\ 3 & 9 - \lambda \end{vmatrix} \\ &= (-3 - \lambda)(9 - \lambda) - 45 \\ &= \lambda^2 - 6\lambda - 27 - 45 \\ &= \lambda^2 - 6\lambda - 72 \\ &= (\lambda - 12)(\lambda + 6). \end{aligned}$$

Therefore,  $\det(A - \lambda I) = 0$  when  $\lambda = -6$  and  $12$ ; these are our eigenvalues. (We should note that  $p(\lambda) = \lambda^2 - 6\lambda - 72$  is our characteristic polynomial.)

It sometimes helps to give them names, so we'll say  $\lambda_1 = -6$  and  $\lambda_2 = 12$ . Now we find eigenvectors.

For  $\lambda_1 = -6$ , we need to solve the equation  $(A - (-6)I)\vec{x} = \vec{0}$ . To do this, we form the appropriate augmented matrix and put it into reduced row echelon form.

$$\begin{bmatrix} 3 & 15 & 0 \\ 3 & 15 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Our solution is

$$\begin{aligned} x_1 &= -5t \\ x_2 &= t \quad \text{is free;} \end{aligned}$$

in vector form, we have

$$\vec{x} = t \begin{bmatrix} -5 \\ 1 \end{bmatrix}.$$

We may pick any nonzero value for  $t$  to get an eigenvector; a simple option is  $x_2 = 1$ . Thus we have the eigenvector

$$\vec{x}_1 = \begin{bmatrix} -5 \\ 1 \end{bmatrix}.$$

(We used the notation  $\vec{x}_1$  to associate this eigenvector with the eigenvalue  $\lambda_1$ .)

We now repeat this process to find an eigenvector for  $\lambda_2 = 12$ . In solving  $(A - 12I)\vec{x} = \vec{0}$ , we find

$$\begin{bmatrix} -15 & 15 & 0 \\ 3 & -3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In vector form, we have

$$\vec{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Again, we may pick any nonzero value for  $t$ , and so we choose  $t = 1$ . Thus an eigenvector for  $\lambda_2$  is

$$\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

To summarize, we have:

$$\text{eigenvalue } \lambda_1 = -6 \text{ with eigenvector } \vec{x}_1 = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

and

$$\text{eigenvalue } \lambda_2 = 12 \text{ with eigenvector } \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We should take a moment and check our work: is it true that  $A\vec{x}_1 = \lambda_1\vec{x}_1$ ?

$$A\vec{x}_1 = \begin{bmatrix} -3 & 15 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 30 \\ -6 \end{bmatrix} = (-6) \begin{bmatrix} -5 \\ 1 \end{bmatrix} = \lambda_1\vec{x}_1.$$

Yes; it appears we have truly found an eigenvalue/eigenvector pair for the matrix  $A$ .

**Example 7.1.7 Computing eigenvalues and eigenvectors.**

Let  $A = \begin{bmatrix} -3 & 0 \\ 5 & 1 \end{bmatrix}$ . Find the eigenvalues of  $A$  and an eigenvector for each eigenvalue.

**Solution.** We first compute the characteristic polynomial, set it equal to 0, then solve for  $\lambda$ .

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -3 - \lambda & 0 \\ 5 & 1 - \lambda \end{vmatrix} \\ &= (-3 - \lambda)(1 - \lambda). \end{aligned}$$

From this, we see that  $\det(A - \lambda I) = 0$  when  $\lambda = -3, 1$ . We'll set  $\lambda_1 = -3$  and  $\lambda_2 = 1$ .

Finding an eigenvector for  $\lambda_1$ :

We solve  $(A - (-3)I)\vec{x} = \vec{0}$  for  $\vec{x}$  by row reducing the appropriate matrix:

$$\begin{bmatrix} 0 & 0 & 0 \\ 5 & 4 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 4/5 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Our solution, in vector form, is

$$\vec{x} = t \begin{bmatrix} -4/5 \\ 1 \end{bmatrix}.$$

Again, we can pick any nonzero value for  $t$ ; a nice choice would eliminate the fraction. Therefore we pick  $t = 5$ , and find

$$\vec{x}_1 = \begin{bmatrix} -4 \\ 5 \end{bmatrix}.$$

Finding an eigenvector for  $\lambda_2$ :

We solve  $(A - (1)I)\vec{x} = \vec{0}$  for  $\vec{x}$  by row reducing the appropriate matrix:

$$\begin{bmatrix} -4 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We've seen a matrix like this before, but we may need a bit of a refreshing. Our first row tells us that  $x_1 = 0$ , and we see that no rows/equations involve  $x_2$ . We conclude that  $x_2$  is free. Therefore, our solution, in vector form, is

$$\vec{x} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We pick  $t = 1$ , and find

$$\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

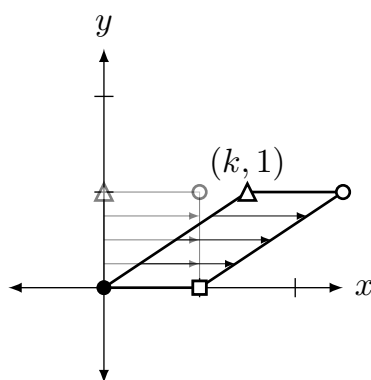
To summarize, we have:

$$\text{eigenvalue } \lambda_1 = -3 \text{ with eigenvector } \vec{x}_1 = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$$

and

$$\text{eigenvalue } \lambda_2 = 1 \text{ with eigenvector } \vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Notice that in both of our examples so far, we were able to completely factor the characteristic polynomial and obtain two distinct eigenvalues. For  $2 \times 2$  matrices, the characteristic polynomial will always be quadratic, and we know that finding roots of quadratic polynomials falls into three categories: those with two distinct roots, like in the examples above, those with one repeated root (for example,  $x^2 - 2x + 1 = (x - 1)^2$ ), and those with no real roots (for example,  $x^2 + 1$ ). In the case of a repeated root, we will have only one eigenvalue. Will we have only one eigenvector, or could there be two? (We'll have more to say about this later.) What if there are no real roots? Then there are no (real) eigenvalues, so presumably there are no eigenvectors, either. What if we allow for complex roots? Let's look at some examples.



**Figure 7.1.9** A horizontal shear by a factor of  $k$

### Example 7.1.8 A matrix with only one eigenvalue.

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}.$$

**Solution.** The transformation  $T(\vec{x}) = A\vec{x}$  defined by  $A$  is an example of a *horizontal shear*. (See Section 5.1.) Such a transformation leaves horizontal vectors unaffected, but vectors with a nonzero vertical component get pulled to the right: see Figure 7.1.9.

From the diagram we can probably guess that the horizontal vector  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  will be an eigenvector with eigenvalue 1, since it is left untouched by the shear transformation. Let's confirm this analytically.

We begin as usual by finding the characteristic polynomial. We have

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 4 \\ 0 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^2.$$

Here, we see that we have only one eigenvalue; namely,  $\lambda = 1$ . Let's look for a corresponding eigenvector. We have

$$A - \lambda = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The corresponding system  $(A - 1 \cdot I)\vec{x} = \vec{0}$  has augmented matrix with reduced row echelon form

$$\left[ \begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right],$$

which tells us that in our solution,  $x_1 = t$  is free, and  $x_2 = 0$ . Setting  $t = 1$ , we get the single eigenvector

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

as expected.

In each of our examples to this point, every eigenvalue corresponded to a single (independent) eigenvector. Is this always the case? We will not prove it in this textbook, but it turns out that in general, the power to which the factor

$(\lambda - x)$  appears in the characteristic polynomial (called the **multiplicity** of the eigenvalue) places an upper limit on the number of independent eigenvectors that can correspond to that eigenvalue.

In Example 7.1.7, we had  $\det(A - \lambda I) = (-3 - \lambda)^1(1 - \lambda)^1$ , so the two eigenvalues  $\lambda = -3$  and  $\lambda = 1$  each have multiplicity one, and therefore they each have one corresponding eigenvector. In Example 7.1.8, the eigenvalue  $\lambda = 3$  has multiplicity two, but we still had only one corresponding eigenvector. Can we ever have such an eigenvalue with two corresponding eigenvectors?

**Example 7.1.10 An eigenvalue of multiplicity two.**

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

**Solution.** Here, we notice that  $A$  is a scalar multiple of the identity. As a transformation of the Cartesian plane, the transformation  $T(\vec{x}) = A\vec{x}$  is a *dilation*: it expands the size of every vector in the plane by a factor of 4. Knowing that this is a transformation that stretches, but does not rotate, we might expect that every nonzero vector is an eigenvector of  $A$ ! Indeed, given  $\vec{x} \neq \vec{0}$ , we have

$$A\vec{x} = (4I)\vec{x} = 4(I\vec{x}) = 4\vec{x},$$

so  $\vec{x}$  is an eigenvector corresponding to the eigenvalue 4.

Of course, this is pretty much the end of the story here, but let's get some practice with our algorithm for finding eigenvalues and eigenvectors and confirm our results. We can immediately see that

$$\det(A - \lambda I) = (4 - \lambda)^2,$$

so that  $\lambda = 4$  is an eigenvalue of multiplicity 2.

What about the eigenvectors? Well, computing  $A - 4I$  is somewhat interesting: we get

$$A - 4I = \begin{bmatrix} 4 - 4 & 0 \\ 0 & 4 - 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

the zero matrix. Again, we see that literally any nonzero vector  $\vec{x} \in \mathbb{R}^2$  qualifies as an eigenvector. We know that we can find at most two independent vectors in  $\mathbb{R}^2$ , so a simple choice is to take the standard basis vectors  $\vec{e}_1$  and  $\vec{e}_2$ .

Notice that we could have proceeded as usual and attempted to solve the system  $(A - 4I)\vec{x} = \vec{0}$ . In this case we get a rather strange augmented matrix:

$$[A \quad \vec{0}] = [\mathbf{0} \quad \vec{0}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}!$$

It might seem like there's absolutely nothing to do here, but we can read off a solution. In this case neither row places any conditions on the variables  $x_1$  and  $x_2$ , so both are free:  $x_1 = s$  and  $x_2 = t$  are both parameters, and

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Setting  $s = 1$  and  $t = 0$  gives us the eigenvector  $\vec{e}_1$ , and setting  $s = 0$ ,  $t = 1$  gives us the eigenvector  $\vec{e}_2$ .

We mentioned above that another possibility is that the characteristic polynomial has no real zeros at all, in which case our matrix has no (real) eigenvalues. Let's see what we can say in such a situation.

**Example 7.1.11 A matrix with complex eigenvalues.**

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

**Solution.** Before we proceed, let's pause and think about this in the context of matrix transformations. If we define the transformation  $T(\vec{x}) = A\vec{x}$ , we have

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

Notice that  $T(\vec{x})$  is orthogonal to  $\vec{x}$ :

$$T(\vec{x}) \cdot \vec{x} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -x_2x_1 + x_1x_2 = 0.$$

This is because the transformation  $T$  represents a rotation through an angle of  $\frac{\pi}{2}$  (90 degrees). Indeed,  $A$  is a rotation matrix (see Section 5.1) of the form

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix},$$

where  $\theta = \frac{\pi}{2}$ .

Now, think about the eigenvalue equation  $A\vec{x} = \lambda\vec{x}$ . In this case, an eigenvector  $\vec{x}$  would be a vector in the plane such that rotating it by 90 degrees produces a parallel vector! Clearly, this is nonsense, and indeed, we find that

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1,$$

which has no real roots, so the matrix  $A$  has no eigenvalues, which makes sense from a geometric point of view.

However, this is not the end of the story, provided that we're willing to work with complex numbers. Over the *complex* numbers, we *do* have two eigenvalues:

$$\lambda^2 + 1 = (\lambda + i)(\lambda - i),$$

so  $\lambda = i$  and  $\lambda = -i$  are eigenvalues. What are the eigenvectors? We proceed as always, except that the arithmetic in the row operations is a bit trickier with complex numbers. For  $\lambda = i$ , we have the system  $(A - iI)\vec{x} = \vec{0}$ . We set up the augmented matrix below, and in this case, we'll proceed step-by-step to the reduced row echelon form.

We have

$$A - iI = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix},$$

so we get the augmented matrix

$$\left[ \begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[ \begin{array}{cc|c} 1 & -i & 0 \\ -i & -1 & 0 \end{array} \right] \xrightarrow{R_2 + iR_1 \rightarrow R_2} \left[ \begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array} \right].$$

Notice in the last step that  $-i + i(1) = 0$  gives the zero in the first column, and  $-1 + i(-i) = -1 + 1 = 0$  gives the zero in the second column. This tells us that  $x_2 = t$  is a free (complex!) parameter while  $x_1 - ix_2 = 0$ , so  $x_1 = ix_2 = it$ . Our vector solution is thus

$$\vec{x}_1 = \begin{bmatrix} it \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix},$$

and we can check that

$$A\vec{x}_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = \begin{bmatrix} i(i) \\ i(1) \end{bmatrix} = i \begin{bmatrix} i \\ 1 \end{bmatrix} = i\vec{x}_1,$$

as expected.

We can similarly set up and solve

$$[(A + iI) \quad \vec{0}] = \left[ \begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array} \right],$$

giving us  $x_2 = t$  as a free parameter, and  $x_1 = -ix_2 = -it$ , so

$$\vec{x}_2 = \begin{bmatrix} -it \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

In this context we're free to choose any complex value for  $t$ . Choosing  $t = i$  gives us the solution  $\vec{x}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ .

Our last few examples provided interesting departures from the earlier ones where we had two distinct eigenvalues; they also provided examples where we were able to analyze the situation geometrically, by considering the linear transformations defined by the matrix. The reader is encouraged to consider the other examples of transformations given in Section 5.1 and attempt a similar analysis.

So far, our examples have involved  $2 \times 2$  matrices. Let's do an example with a  $3 \times 3$  matrix. The only real additional complication here is that our characteristic polynomial will now be a *cubic* polynomial, so factoring it is going to take some more work.

#### Example 7.1.12 Eigenvalues and eigenvectors for a $3 \times 3$ matrix.

Find the eigenvalues of  $A$ , and for each eigenvalue, give one eigenvector, where

$$A = \begin{bmatrix} -7 & -2 & 10 \\ -3 & 2 & 3 \\ -6 & -2 & 9 \end{bmatrix}.$$

**Solution.** We first compute the characteristic polynomial, set it equal to 0, then solve for  $\lambda$ . A warning: this process is rather long. We'll use cofactor expansion along the first row; don't get bogged down with the arithmetic that comes from each step; just try to get the basic idea of what was done from step to step.

You may have learned the basics of factoring degree three polynomials in high school. As a reminder, possible roots can be found by factoring the constant term (in this case,  $-6$ ) of the polynomial. That is, the roots of this equation could be  $\pm 1, \pm 2, \pm 3$  and  $\pm 6$ . That's 12 things to check.

$$\begin{aligned}\det(A - \lambda I) &= \begin{vmatrix} -7 - \lambda & -2 & 10 \\ -3 & 2 - \lambda & 3 \\ -6 & -2 & 9 - \lambda \end{vmatrix} \\ &= (-7 - \lambda) \begin{vmatrix} 2 - \lambda & 3 \\ -2 & 9 - \lambda \end{vmatrix} - (-2) \begin{vmatrix} -3 & 3 \\ -6 & 9 - \lambda \end{vmatrix} + 10 \begin{vmatrix} -3 & 2 - \lambda \\ -6 & -2 \end{vmatrix} \\ &= (-7 - \lambda)(\lambda^2 - 11\lambda + 24) + 2(3\lambda - 9) + 10(-6\lambda + 18) \\ &= -\lambda^3 + 4\lambda^2 - \lambda - 6 \\ &= -(\lambda + 1)(\lambda - 2)(\lambda - 3).\end{aligned}$$

In the last step we factored the characteristic polynomial  $-\lambda^3 + 4\lambda^2 - \lambda - 6$ . Factoring polynomials of degree  $> 2$  is not trivial; we'll assume the reader has access to methods for doing this accurately.

One could also graph this polynomial to find the roots. Graphing will show us that  $\lambda = 3$  looks like a root, and a simple calculation will confirm that it is.

Our eigenvalues are  $\lambda_1 = -1$ ,  $\lambda_2 = 2$  and  $\lambda_3 = 3$ . We now find corresponding eigenvectors.

For  $\lambda_1 = -1$ :

We need to solve the equation  $(A - (-1)I)\vec{x} = \vec{0}$ . To do this, we form the appropriate augmented matrix and put it into reduced row echelon form.

$$\begin{bmatrix} -6 & -2 & 10 & 0 \\ -3 & 3 & 3 & 0 \\ -6 & -2 & 10 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1.5 & 0 \\ 0 & 1 & -.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Our solution, in vector form, is

$$\vec{x} = x_3 \begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix}.$$

We can pick any nonzero value for  $x_3$ ; a nice choice would get rid of the fractions. So we'll set  $x_3 = 2$  and choose  $\vec{x}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$  as our eigenvector.

For  $\lambda_2 = 2$ :

We need to solve the equation  $(A - 2I)\vec{x} = \vec{0}$ . To do this, we form the appropriate augmented matrix and put it into reduced row echelon form.

$$\begin{bmatrix} -9 & -2 & 10 & 0 \\ -3 & 0 & 3 & 0 \\ -6 & -2 & 7 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Our solution, in vector form, is

$$\vec{x} = x_3 \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}.$$



We can pick any nonzero value for  $x_3$ ; again, a nice choice would get rid of the fractions. So we'll set  $x_3 = 2$  and choose  $\vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$  as our eigenvector.

For  $\lambda_3 = 3$ :

We need to solve the equation  $(A - 3I)\vec{x} = \vec{0}$ . To do this, we form the appropriate augmented matrix and put it into reduced row echelon form.

$$\begin{bmatrix} -10 & -2 & 10 & 0 \\ -3 & -1 & 3 & 0 \\ -6 & -2 & 6 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Our solution, in vector form, is (note that  $x_2 = 0$ ):

$$\vec{x} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

We can pick any nonzero value for  $x_3$ ; an easy choice is  $x_3 = 1$ , so

$\vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$  as our eigenvector.

To summarize, we have the following eigenvalue/eigenvector pairs:

eigenvalue  $\lambda_1 = -1$  with eigenvector  $\vec{x}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

eigenvalue  $\lambda_2 = 2$  with eigenvector  $\vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

eigenvalue  $\lambda_3 = 3$  with eigenvector  $\vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ .

#### Example 7.1.13 Computing eigenvalues and eigenvectors.

Find the eigenvalues of  $A$ , and for each eigenvalue, give one eigenvector, where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 6 \\ 0 & 3 & 4 \end{bmatrix}.$$

**Solution.** We first compute the characteristic polynomial, set it equal to 0, then solve for  $\lambda$ . We'll use cofactor expansion down the first column (since it has lots of zeros).

$$\det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & -1 & 1 \\ 0 & 1 - \lambda & 6 \\ 0 & 3 & 4 - \lambda \end{vmatrix}$$

$$\begin{aligned}
&= (2 - \lambda) \begin{vmatrix} 1 - \lambda & 6 \\ 3 & 4 - \lambda \end{vmatrix} \\
&= (2 - \lambda)(\lambda^2 - 5\lambda - 14) \\
&= (2 - \lambda)(\lambda - 7)(\lambda + 2).
\end{aligned}$$

Notice that while the characteristic polynomial is cubic, we never actually saw a cubic; we never distributed the  $(2 - \lambda)$  across the quadratic. Instead, we realized that this was a factor of the cubic, and just factored the remaining quadratic. (This makes this example quite a bit simpler than the previous example.)

Our eigenvalues are  $\lambda_1 = -2$ ,  $\lambda_2 = 2$  and  $\lambda_3 = 7$ . We now find corresponding eigenvectors.

For  $\lambda_1 = -2$ :

We need to solve the equation  $(A - (-2)I)\vec{x} = \vec{0}$ . To do this, we form the appropriate augmented matrix and put it into reduced row echelon form.

$$\begin{bmatrix} 4 & -1 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 3 & 6 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3/4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Our solution, in vector form, is

$$\vec{x} = x_3 \begin{bmatrix} -3/4 \\ -2 \\ 1 \end{bmatrix}.$$

We can pick any nonzero value for  $x_3$ ; a nice choice would get rid of the fractions. So we'll set  $x_3 = 4$  and choose  $\vec{x}_1 = \begin{bmatrix} -3 \\ -8 \\ 4 \end{bmatrix}$  as our eigenvector.

For  $\lambda_2 = 2$ :

We need to solve the equation  $(A - 2I)\vec{x} = \vec{0}$ . To do this, we form the appropriate augmented matrix and put it into reduced row echelon form.

$$\begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & 3 & 2 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This looks funny, so we'll look remind ourselves how to solve this. The first two rows tell us that  $x_2 = 0$  and  $x_3 = 0$ , respectively. Notice that no row/equation uses  $x_1$ ; we conclude that it is free. Therefore, our solution in vector form is

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We can pick any nonzero value for  $x_1$ ; an easy choice is  $x_1 = 1$ , which gives  $\vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  as our eigenvector.

For  $\lambda_3 = 7$ :

We need to solve the equation  $(A - 7I)\vec{x} = \vec{0}$ . To do this, we form the appropriate augmented matrix and put it into reduced row echelon form.

$$\begin{bmatrix} -5 & -1 & 1 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Our solution, in vector form, is (note that  $x_1 = 0$ ):

$$\vec{x} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

We can pick any nonzero value for  $x_3$ ; an easy choice is  $x_3 = 1$ , so

$$\vec{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ is our eigenvector.}$$

To summarize, we have the following eigenvalue/eigenvector pairs:

$$\text{eigenvalue } \lambda_1 = -2 \text{ with eigenvector } \vec{x}_1 = \begin{bmatrix} -3 \\ -8 \\ 4 \end{bmatrix}$$

$$\text{eigenvalue } \lambda_2 = 2 \text{ with eigenvector } \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{eigenvalue } \lambda_3 = 7 \text{ with eigenvector } \vec{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

In this section we have learned about a new concept: given a matrix  $A$  we can find certain values and vectors  $\vec{x}$  where  $A\vec{x} = \lambda\vec{x}$ . In the next section we will continue to the pattern we have established in this text: after learning a new concept, we see how it interacts with other concepts we know about. That is, we'll look for connections between eigenvalues and eigenvectors and things like the inverse, determinants, the trace, the transpose, etc..

## 7.1.1 Exercises

**Exercise Group.** A matrix  $A$  and one of its eigenvectors are given. Find the eigenvalue of  $A$  for the given eigenvector.

1.  $A = \begin{bmatrix} 9 & 8 \\ -6 & -5 \end{bmatrix}, \vec{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

2.  $A = \begin{bmatrix} 19 & -6 \\ 48 & -15 \end{bmatrix}, \vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

3.  $A = \begin{bmatrix} -11 & -19 & 14 \\ -6 & -8 & 6 \\ -12 & -22 & 15 \end{bmatrix}, \vec{x} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$

4.  $A = \begin{bmatrix} -7 & 1 & 3 \\ 10 & 2 & -3 \\ -20 & -14 & 1 \end{bmatrix}, \vec{x} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$

5.  $A = \begin{bmatrix} -12 & -10 & 0 \\ 15 & 13 & 0 \\ 15 & 18 & -5 \end{bmatrix}, \vec{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

6.  $A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}, \vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

**Exercise Group.** A matrix  $A$  and one of its eigenvalues are given. Find an eigenvector of  $A$  for the given eigenvalue.

7.  $A = \begin{bmatrix} -16 & -28 & -19 \\ 42 & 69 & 46 \\ -42 & -72 & -49 \end{bmatrix}, \lambda = 5$

8.  $A = \begin{bmatrix} 7 & -5 & -10 \\ 6 & 2 & -6 \\ 2 & -5 & -5 \end{bmatrix}, \lambda = -3$

9.  $A = \begin{bmatrix} 4 & 5 & -3 \\ -7 & -8 & 3 \\ 1 & -5 & 8 \end{bmatrix}, \lambda = 2$

10.  $A = \begin{bmatrix} 16 & 6 \\ -18 & -5 \end{bmatrix}, \lambda = 4$

11.  $A = \begin{bmatrix} -2 & 6 \\ -9 & 13 \end{bmatrix}, \lambda = 7$

**Exercise Group.** Find the eigenvalues of the given matrix. For each eigenvalue, give an eigenvector.

12.  $\begin{bmatrix} 1 & -2 & -3 \\ 0 & 3 & 0 \\ 0 & -1 & -1 \end{bmatrix}$

13.  $\begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 7 \end{bmatrix}$

14.  $\begin{bmatrix} 2 & -12 \\ 2 & -8 \end{bmatrix}$

15.  $\begin{bmatrix} 5 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 5 & -2 \end{bmatrix}$

16.  $\begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix}$

17.  $\begin{bmatrix} -1 & 18 & 0 \\ 1 & 2 & 0 \\ 5 & -3 & -1 \end{bmatrix}$

18.  $\begin{bmatrix} 1 & 0 & 12 \\ 2 & -5 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

19.  $\begin{bmatrix} 3 & 12 \\ 1 & -1 \end{bmatrix}$

20.  $\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$

21.  $\begin{bmatrix} 1 & 0 & -18 \\ -4 & 3 & -1 \\ 1 & 0 & -8 \end{bmatrix}$

22.  $\begin{bmatrix} 3 & 5 & -5 \\ -2 & 3 & 2 \\ -2 & 5 & 0 \end{bmatrix}$

23.  $\begin{bmatrix} -1 & -4 \\ -3 & -2 \end{bmatrix}$

24.  $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$

25.  $\begin{bmatrix} 0 & 1 \\ 25 & 0 \end{bmatrix}$

26.  $\begin{bmatrix} 5 & 9 \\ -1 & -5 \end{bmatrix}$

27.  $\begin{bmatrix} -4 & 72 \\ -1 & 13 \end{bmatrix}$

**28.**  $\begin{bmatrix} 5 & -2 & 3 \\ 0 & 4 & 0 \\ 0 & -1 & 3 \end{bmatrix}$

## 7.2 Properties of Eigenvalues and Eigenvectors

In this section we'll explore how the eigenvalues and eigenvectors of a matrix relate to other properties of that matrix. This section is essentially a hodgepodge of interesting facts about eigenvalues; the goal here is not to memorize various facts about matrix algebra, but to again be amazed at the many connections between mathematical concepts.

We'll begin our investigations with an example that will give a foundation for other discoveries.

### Example 7.2.1 Eigenvalues of a triangular matrix.

Let  $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ . Find the eigenvalues of  $A$ .

**Solution.** To find the eigenvalues, we compute  $\det A - \lambda I$ :

$$\begin{aligned} \det A - \lambda I &= \begin{vmatrix} 1 - \lambda & 2 & 3 \\ 0 & 4 - \lambda & 5 \\ 0 & 0 & 6 - \lambda \end{vmatrix} \\ &= (1 - \lambda)(4 - \lambda)(6 - \lambda). \end{aligned}$$

Since our matrix is triangular, the determinant is easy to compute; it is just the product of the diagonal elements. Therefore, we found (and factored) our characteristic polynomial very easily, and we see that we have eigenvalues of  $\lambda = 1, 4$ , and  $6$ .

This examples demonstrates a wonderful fact for us: the eigenvalues of a triangular matrix are simply the entries on the diagonal. Finding the corresponding eigenvectors still takes some work, but finding the eigenvalues is easy.

With that fact in the backs of our minds, let us proceed to the next example where we will come across some more interesting facts about eigenvalues and eigenvectors.

### Example 7.2.2 Exploring properties of eigenvalues.

Let  $A = \begin{bmatrix} -3 & 15 \\ 3 & 9 \end{bmatrix}$  and let  $B = \begin{bmatrix} -7 & -2 & 10 \\ -3 & 2 & 3 \\ -6 & -2 & 9 \end{bmatrix}$  (as used in Examples Example 7.1.6 and Example 7.1.12, respectively). Find the following:

1. The eigenvalues and eigenvectors of  $A$  and  $B$
2. The eigenvalues and eigenvectors of  $A^{-1}$  and  $B^{-1}$
3. eigenvalues and eigenvectors of  $A^T$  and  $B^T$
4. The trace of  $A$  and  $B$
5. The determinant of  $A$  and  $B$

**Solution.** We'll answer each in turn.

1. We already know the answer to these for we did this work in previous examples. Therefore we just list the answers.

For  $A$ , we have eigenvalues  $\lambda = -6$  and  $12$ , with eigenvectors

$$\vec{x} = x_2 \begin{bmatrix} -5 \\ 1 \end{bmatrix} \text{ and } x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ respectively.}$$

For  $B$ , we have eigenvalues  $\lambda = -1, 2$ , and  $3$  with eigenvectors

$$\vec{x} = x_3 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, x_3 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \text{ and } x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \text{ respectively.}$$

2. We first compute the inverses of  $A$  and  $B$ . They are:

$$A^{-1} = \begin{bmatrix} -1/8 & 5/24 \\ 1/24 & 1/24 \end{bmatrix} \text{ and } B^{-1} = \begin{bmatrix} -4 & 1/3 & 13/3 \\ -3/2 & 1/2 & 3/2 \\ -3 & 1/3 & 10/3 \end{bmatrix}.$$

Finding the eigenvalues and eigenvectors of these matrices is not terribly hard, but it is not “easy,” either. Therefore, we omit showing the intermediate steps and go right to the conclusions.

For  $A^{-1}$ , we have eigenvalues  $\lambda = -1/6$  and  $1/12$ , with eigenvectors

$$\vec{x} = x_2 \begin{bmatrix} -5 \\ 1 \end{bmatrix} \text{ and } x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ respectively.}$$

For  $B^{-1}$ , we have eigenvalues  $\lambda = -1, 1/2$  and  $1/3$  with eigenvectors

$$\vec{x} = x_3 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, x_3 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \text{ and } x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \text{ respectively.}$$

3. Of course, computing the transpose of  $A$  and  $B$  is easy; computing their eigenvalues and eigenvectors takes more work. Again, we omit the intermediate steps.

For  $A^T$ , we have eigenvalues  $\lambda = -6$  and  $12$  with eigenvectors

$$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } x_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \text{ respectively.}$$

For  $B^T$ , we have eigenvalues  $\lambda = -1, 2$  and  $3$  with eigenvectors

$$\vec{x} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \text{ and } x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \text{ respectively.}$$

4. The trace of  $A$  is  $6$ ; the trace of  $B$  is  $4$ .

5. The determinant of  $A$  is  $-72$ ; the determinant of  $B$  is  $-6$ .

Now that we have completed the “grunt work,” let’s analyze the results of

the previous example. We are looking for any patterns or relationships that we can find.

**The eigenvalues and eigenvectors of  $A$  and  $A^{-1}$ .** In our example, we found that the eigenvalues of  $A$  are  $-6$  and  $12$ ; the eigenvalues of  $A^{-1}$  are  $-1/6$  and  $1/12$ . Also, the eigenvalues of  $B$  are  $-1$ ,  $2$  and  $3$ , whereas the eigenvalues of  $B^{-1}$  are  $-1$ ,  $1/2$  and  $1/3$ . There is an obvious relationship here; it seems that if  $\lambda$  is an eigenvalue of  $A$ , then  $1/\lambda$  will be an eigenvalue of  $A^{-1}$ . We can also note that the corresponding eigenvectors matched, too.

Why is this the case? Consider an invertible matrix  $A$  with eigenvalue  $\lambda \neq 0$  and eigenvector  $\vec{x}$ . Then, by definition, we know that  $A\vec{x} = \lambda\vec{x}$ . Now multiply both sides by  $A^{-1}$ :

$$\begin{aligned} A\vec{x} &= \lambda\vec{x} \\ A^{-1}A\vec{x} &= A^{-1}\lambda\vec{x} \\ \vec{x} &= \lambda A^{-1}\vec{x} \\ \frac{1}{\lambda}\vec{x} &= A^{-1}\vec{x}. \end{aligned}$$

We have just shown that  $A^{-1}\vec{x} = 1/\lambda\vec{x}$ ; this, by definition, shows that  $\vec{x}$  is an eigenvector of  $A^{-1}$  with eigenvalue  $1/\lambda$ . This explains the result we saw above. Of course, this all falls apart if  $\lambda = 0$ , but this is impossible for an invertible matrix: see Theorem 7.2.5 below.

**The eigenvalues and eigenvectors of  $A$  and  $A^T$ .** Our example showed that  $A$  and  $A^T$  had the same eigenvalues but different (but somehow similar) eigenvectors; it also showed that  $B$  and  $B^T$  had the same eigenvalues but unrelated eigenvectors. Why is this?

We can answer the eigenvalue question relatively easily; it follows from the properties of the determinant and the transpose. Recall the following two facts:

1.  $(A + B)^T = A^T + B^T$  (Theorem 6.1.9)
2.  $\det A = \det A^T$  (Theorem 6.4.12)

We find the eigenvalues of a matrix by computing the characteristic polynomial; that is, we find  $\det A - \lambda I$ . What is the characteristic polynomial of  $A^T$ ? Consider:

$$\begin{aligned} \det A^T - \lambda I &= \det A^T - \lambda I^T \quad \text{since } I = I^T \\ &= \det (A - \lambda I)^T \quad (\text{Theorem 6.1.9}) \\ &= \det A - \lambda I \quad (\text{Theorem 6.4.12}). \end{aligned}$$

So we see that the characteristic polynomial of  $A^T$  is the same as that for  $A$ . Therefore they have the same eigenvalues.

What about their respective eigenvectors? Is there any relationship? The simple answer is “No.”

**The eigenvalues and eigenvectors of  $A$  and the trace.** Note that the eigenvalues of  $A$  are  $-6$  and  $12$ , and the trace is  $6$ ; the eigenvalues of  $B$  are  $-1$ ,  $2$  and  $3$ , and the trace of  $B$  is  $4$ . Do we notice any relationship?

It seems that the sum of the eigenvalues is the trace! Why is this the case?

The answer to this is a bit out of the scope of this text; we can justify part of this fact, and another part we'll just state as being true without justification.



Suppose  $A$  is an  $n \times n$  matrix with no complex eigenvalues (that is, the characteristic polynomial can be completely factored over the real numbers). When this is the case, it turns out that we can find a square matrix  $P$  such that  $P^{-1}AP$  is an upper triangular matrix with the eigenvalues of  $A$  on the diagonal. (We unfortunately do not have the time to prove this, or to explain how the matrix  $P$  is determined.)

Now, recall from Theorem 6.2.3 that  $\text{tr}(AB) = \text{tr}(BA)$ . Since  $P^{-1}AP$  is upper-triangular, we know that  $\text{tr}(P^{-1}AP)$  is the sum of the eigenvalues; also, using our Theorem 6.2.3, we know that  $\text{tr}(P^{-1}AP) = \text{tr}(P^{-1}PA) = \text{tr}(A)$ . Thus the trace of  $A$  is the sum of the eigenvalues. It turns out that this result remains true when  $A$  has complex eigenvalues, and the proof is similar, except that the entries of the matrix  $P$  will be complex numbers.

**The eigenvalues and eigenvectors of  $A$  and the determinant.** Again, the eigenvalues of  $A$  are  $-6$  and  $12$ , and the determinant of  $A$  is  $-72$ . The eigenvalues of  $B$  are  $-1$ ,  $2$  and  $3$ ; the determinant of  $B$  is  $-6$ . It seems as though the product of the eigenvalues is the determinant.

This is indeed true; we defend this with our argument from above. We know that the determinant of a triangular matrix is the product of the diagonal elements. Therefore, given a matrix  $A$ , we can find  $P$  such that  $P^{-1}AP$  is upper triangular with the eigenvalues of  $A$  on the diagonal. Thus  $\det P^{-1}AP$  is the product of the eigenvalues. Using Theorem 6.4.12, we know that  $\det P^{-1}AP = \det P^{-1}PA = \det A$ . Thus the determinant of  $A$  is the product of the eigenvalues.

We summarize the results of our example with the following theorem.

**Theorem 7.2.3 Properties of Eigenvalues and Eigenvectors.**

Let  $A$  be an  $n \times n$  invertible matrix. The following are true:

1. If  $A$  is triangular, then the diagonal elements of  $A$  are the eigenvalues of  $A$ .
2. If  $\lambda$  is an eigenvalue of  $A$  with eigenvector  $\vec{x}$ , then  $\frac{1}{\lambda}$  is an eigenvalue of  $A^{-1}$  with eigenvector  $\vec{x}$ .
3. If  $\lambda$  is an eigenvalue of  $A$ , then  $\lambda$  is an eigenvalue of  $A^T$ .
4. The sum of the eigenvalues of  $A$  is equal to  $\text{tr}(A)$ , the trace of  $A$ .
5. The product of the eigenvalues of  $A$  is equal to  $\det A$ , the determinant of  $A$ .

There is one more concept concerning eigenvalues and eigenvectors that we will explore. We do so in the context of an example.

**Example 7.2.4 Eigenvalues of a non-invertible matrix.**

Find the eigenvalues and eigenvectors of the matrix  $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ .

**Solution.** To find the eigenvalues, we compute  $\det A - \lambda I$ :

$$\det A - \lambda I = \begin{vmatrix} 1 - \lambda & 2 \\ 1 & 2 - \lambda \end{vmatrix}$$

**Note:** the reader would be justified in wondering about the case where  $A$  is a real matrix with complex eigenvalues. Any such eigenvalues will arise from irreducible quadratic factors of the characteristic polynomial, and we know from Theorem 1.4.10 that the complex zeros of a quadratic with real coefficients occur in conjugate pairs. That is, if  $\lambda$  is a complex eigenvalue of  $A$ , so is the complex conjugate  $\bar{\lambda}$ . When we multiply the eigenvalues of  $A$  together, the product will contain  $\lambda \cdot \bar{\lambda} = |\lambda|^2$ , which is always a real number. Thus, it is always true that  $\det(A)$  is equal to the product of the eigenvalues of  $A$ , provided that we include any possible complex eigenvalues.

$$\begin{aligned}
 &= (1 - \lambda)(2 - \lambda) - 2 \\
 &= \lambda^2 - 3\lambda \\
 &= \lambda(\lambda - 3)
 \end{aligned}$$

Our eigenvalues are therefore  $\lambda = 0, 3$ .

For  $\lambda = 0$ , we find the eigenvectors:

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This shows that  $x_1 = -2x_2$ , and so our eigenvectors  $\vec{x}$  are

$$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

For  $\lambda = 3$ , we find the eigenvectors:

$$\begin{bmatrix} -2 & 2 & 0 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This shows that  $x_1 = x_2$ , and so our eigenvectors  $\vec{x}$  are

$$\vec{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

One interesting thing about the above example is that we see that 0 is an eigenvalue of  $A$ ; we have not officially encountered this before. Does this mean anything significant?

Think about what an eigenvalue of 0 means: there exists a nonzero vector  $\vec{x}$  where  $A\vec{x} = 0\vec{x} = \vec{0}$ . That is, we have a nontrivial solution to  $A\vec{x} = \vec{0}$ . We know this only happens when  $A$  is not invertible.

So if  $A$  is invertible, there is no nontrivial solution to  $A\vec{x} = \vec{0}$ , and hence 0 is not an eigenvalue of  $A$ . If  $A$  is not invertible, then there is a nontrivial solution to  $A\vec{x} = \vec{0}$ , and hence 0 is an eigenvalue of  $A$ . This leads us to our final addition to the Invertible Matrix Theorem.

#### Theorem 7.2.5 Invertible Matrix Theorem.

Let  $A$  be an  $n \times n$  matrix. The following statements are equivalent.

- (a)  $A$  is invertible.
- (b) The equation  $A\vec{x} = \vec{0}$  has exactly one solution (namely,  $\vec{x} = \vec{0}$ ).
- (c) The reduced row echelon form of  $A$  is  $I$ .
- (d) The equation  $A\vec{x} = \vec{b}$  has exactly one solution for every  $n \times 1$  vector  $\vec{b}$ .
- (e) There exists a matrix  $C$  such that  $AC = I$ .
- (f) There exists a matrix  $B$  such that  $BA = I$ .
- (g)  $\det(A) \neq 0$ .
- (h)  $A$  does not have an eigenvalue of 0.

This section is about the properties of eigenvalues and eigenvectors. Of course, we have not investigated all of the numerous properties of eigenvalues and eigenvectors; we have just surveyed some of the most common (and most important) concepts. One of the more important topics — which we will describe briefly in our final section — is the question of *diagonalization*. For some  $n \times n$  matrices  $A$ , it is possible to find an invertible matrix  $P$  such that  $P^{-1}AP$  is a **diagonal** matrix, and the diagonal entries of this matrix are precisely the eigenvalues of  $A$ . When is this possible? When is it not? To a large extent, this comes down to the question of *multiplicity*: when we have a repeated eigenvalue, how many independent eigenvectors are associated with it?

Finally, we have found the eigenvalues of matrices by finding the roots of the characteristic polynomial. We have limited our examples to quadratic and cubic polynomials; one would expect for larger sized matrices that a computer would be used to factor the characteristic polynomials. However, in general, this is *not* how the eigenvalues are found. Factoring high order polynomials is too unreliable, even with a computer — round off errors can cause unpredictable results. Also, to even compute the characteristic polynomial, one needs to compute the determinant, which is also expensive (as discussed in the previous chapter).

So how are eigenvalues found? There are *iterative* processes that can progressively transform a matrix  $A$  into another matrix that is *almost* an upper triangular matrix (the entries below the diagonal are almost zero) where the entries on the diagonal are the eigenvalues. The more iterations one performs, the better the approximation is.

These methods are so fast and reliable that some computer programs convert polynomial root finding problems into eigenvalue problems!

Most textbooks on Linear Algebra will provide direction on exploring the above topics and give further insight to what is going on. We have mentioned all the eigenvalue and eigenvector properties in this section for the same reasons we gave in the previous section. First, knowing these properties helps us solve numerous real world problems, and second, it is fascinating to see how rich and deep the theory of matrices is.

### 7.2.1 Exercises

**Exercise Group.** A matrix  $A$  is given.

- (a) Find the eigenvalues of  $A$ , and for each eigenvalue, find an eigenvector.
- (b) Do the same for  $A^T$ .
- (c) Do the same for  $A^{-1}$ .
- (d) Find  $\text{tr}(A)$ .
- (e) Find  $\det A$ .

Use Theorem 7.2.3 to verify your results.

1.  $\begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix}$

3.  $\begin{bmatrix} 5 & 30 \\ -1 & -6 \end{bmatrix}$

5.  $\begin{bmatrix} 5 & -9 & 0 \\ 1 & -5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

2.  $\begin{bmatrix} -2 & -14 \\ -1 & 3 \end{bmatrix}$

4.  $\begin{bmatrix} -4 & 72 \\ -1 & 13 \end{bmatrix}$

6.  $\begin{bmatrix} 0 & 25 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -3 \end{bmatrix}$

## 7.3 Eigenvalues and Diagonalization

Up to now we have concentrated primarily on finding eigenvalues and eigenvectors, and while we may have claimed that this is a particularly useful endeavour, we haven't offered much to back up this claim, aside from referring the reader to a Wikipedia page. While a complete treatment of the theory and applications of eigenvalues and eigenvectors goes well beyond the scope of this book, we can offer a brief taste of what's to come (should the reader choose to further their studies in linear algebra).

**Similar matrices.** We begin by defining what it means for two matrices to be **similar**. Shortly, we will see that similar matrices share a number of properties, and ultimately we will define diagonalizable matrices as those matrices that are similar to a diagonal matrix.

### Definition 7.3.1 Similar Matrices.

Let  $A$  and  $B$  be  $n \times n$  matrices. We say that  $A$  is **similar** to  $B$ , and write  $A \sim B$ , if there exists an invertible  $n \times n$  matrix  $P$  such that

$$A = P^{-1}BP.$$

Although our definition above is not symmetric (it defines what it means for  $A$  to be similar to  $B$ , but not vice-versa), there is in fact no ambiguity in using a statement such as “ $A$  and  $B$  are similar matrices,” as the next theorem demonstrates.

### Theorem 7.3.2 Properties of matrix similarity.

Let  $A$ ,  $B$ , and  $C$  be  $n \times n$  matrices. Then:

1.  $A \sim A$
2. If  $A \sim B$ , then  $B \sim A$
3. If  $A \sim B$  and  $B \sim C$ , then  $A \sim C$ .

Let's see why these results are true.

1. Setting  $P = I$ , the  $n \times n$  identity matrix, we have  $A = I^{-1}AI$ , and thus  $A \sim A$ .
2. Suppose  $A \sim B$ . Then we know that  $A = P^{-1}BP$  for some invertible matrix  $P$ . Setting  $Q = P^{-1}$ , (and noting  $Q^{-1} = P$ ) we have

$$B = (PP^{-1})B(PP^{-1}) = P(P^{-1}BP)P^{-1} = Q^{-1}AQ,$$

so  $B \sim A$ .

3. Suppose that  $A \sim B$  and  $B \sim C$ . Then there exist  $n \times n$  matrices  $P$  and  $Q$  such that  $A = P^{-1}BP$  and  $B = Q^{-1}CQ$ . But then

$$A = P^{-1}BP = P^{-1}(Q^{-1}CQ)P = (P^{-1}Q^{-1})C(QP) = (QP)^{-1}C(QP),$$

which shows that  $A \sim C$ .

The fact that the definition of matrix similarity satisfies the three properties in the theorem above tells us that matrix similarity is an example of what

is called an *equivalence relation*. This equivalence relation breaks the set of all  $n \times n$  matrices into sets of “equivalent” matrices. Similar matrices are so-called because, although they often have very different entries, they share many of the same properties. (At the end of this section, we’ll discuss the fact that from the point of view of linear transformations, similar matrices describe the same linear transformation, if we represent that linear transformation using different choices of basis.) The following theorem gives some of the properties similar matrices share.

**Theorem 7.3.3 Shared properties of similar matrices.**

Let  $A$  and  $B$  be  $n \times n$  matrices. If  $A \sim B$ , then:

1.  $\text{tr}(A) = \text{tr}(B)$
2.  $\det(A) = \det(B)$
3.  $A$  and  $B$  have the same eigenvalues.

Let us again check the details on each of the points in the above theorem to see why they’re true.

1. Recall that the trace satisfies  $\text{tr}(AB) = \text{tr}(BA)$  for any  $n \times n$  matrices  $A$  and  $B$ . Thus, if  $A = P^{-1}BP$ , we have

$$\text{tr}(A) = \text{tr}(P^{-1}BP) = \text{tr}(B(P^{-1}P)) = \text{tr}(BI) = \text{tr}(B).$$

2. Recall that the determinant satisfies  $\det(AB) = \det(A)\det(B)$  for any  $n \times n$  matrices  $A$  and  $B$ , and  $\det(A^{-1}) = \frac{1}{\det(A)}$  for any invertible matrix  $A$ . Thus, if  $A = P^{-1}BP$ , we have

$$\begin{aligned} \det(A) &= \det(P^{-1}BP) = \det(P^{-1})\det(B)\det(P) \\ &= \frac{1}{\det(P)}\det(B)\det(P) = \det(B). \end{aligned}$$

3. Suppose that  $\lambda$  is an eigenvalue of  $A$ , and that  $A \sim B$ . Since  $\lambda$  is an eigenvalue of  $A$ , we have

$$A\vec{v} = \lambda\vec{v}$$

for some nonzero vector  $\vec{v}$ . But if  $A \sim B$ , then  $A = P^{-1}BP$  for some invertible matrix  $P$ , so

$$P^{-1}BP\vec{v} = \lambda\vec{v}.$$

Multiplying both sides on the left by  $P$ , we have

$$B(P\vec{v}) = P(\lambda\vec{v}) = \lambda(P\vec{v}).$$

Since  $P$  is invertible and  $\vec{v} \neq \vec{0}$ , we know that  $\vec{w} = P\vec{v}$  is nonzero, and thus  $\lambda$  is an eigenvalue of  $B$  with corresponding eigenvector  $P\vec{v}$ .

**Multiplicity of an eigenvalue.** To clarify the presentation in this section, we will shift our notation somewhat from what we’ve been using so far in the textbook. Previously, we used the definition

$$p_A(\lambda) = \det(A - \lambda I)$$

for the characteristic polynomial of a matrix  $A$ . We will adjust this in two ways: first, we will use  $x$  as the variable in our polynomial; second, we will shift the sign of the characteristic polynomial and make the following definition:

**Definition 7.3.4 Characteristic Polynomial (revised definition).**

The **characteristic polynomial** of an  $n \times n$  matrix  $A$  is the degree- $n$  polynomial  $p_A(x)$  defined by

$$p_A(x) = \det(xI - A).$$

Notice that we are using  $xI - A$  rather than  $A - xI$ . This guarantees that the highest-degree term in  $p_A(x)$  is always  $x^n$  (with coefficient 1), whereas before this term was  $\pm x^n$ , depending on whether  $n$  is even or odd. We are using  $x$  as our variable to make it easier to talk about factors of the characteristic polynomial: with this change, the statements “ $\lambda$  is an eigenvalue of  $A$ ” and “ $(x - \lambda)$  is a factor of  $p_A(x)$ ” are equivalent.

In the previous sections, we saw examples of matrices whose characteristic polynomial had a repeated root. In such cases, it is sometimes possible (but not guaranteed) that there is more than one independent eigenvector associated with the repeated eigenvalue. Indeed, consider the matrices

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

All three matrices are upper-triangular, so they have the same characteristic polynomial, namely:

$$p(x) = (x - 2)^3.$$

Notice that the eigenvalue  $\lambda = 2$  is “repeated”, in the sense that it is a repeated root of the characteristic polynomial. This leads to the following definition:

**Definition 7.3.5 Algebraic Multiplicity of an Eigenvalue.**

We say that  $\lambda$  is an eigenvalue with **algebraic multiplicity**  $k$  if  $(x - \lambda)^k$  is a factor of  $p_A(x)$ , but  $(x - \lambda)^{k+1}$  is not.

In other words, the algebraic multiplicity of an eigenvalue  $\lambda$  is the (largest) power to which the corresponding factor  $(x - \lambda)$  appears in the characteristic polynomial. In our example above, all three of the matrices  $A$ ,  $B$ , and  $C$  have the eigenvalue 2 with algebraic multiplicity 3. However, consider the corresponding eigenvectors in each case:

For  $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ , we have  $A - 2I = 0$ , so every non-zero vector  $\vec{x}$  is an eigenvector. Indeed,  $A = 2I$ , so

$$A\vec{x} = (2I)\vec{x} = I(2\vec{x}) = 2\vec{x}$$

for every vector. In particular, we can choose the standard unit basis vectors

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

as our basic eigenvectors, and we see that there are three independent eigenvectors corresponding to the eigenvalue  $\lambda = 2$ .

For  $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ , we have  $B - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ , which is row-equivalent to  $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ . The general solution to the equation  $(B - 2I)\vec{x} = \vec{0}$  is thus given by

$$\vec{x} = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

since the reduced row-echelon form of  $B - 2I$  tells us that we must have  $z = 0$ , while  $x$  and  $y$  are free. Thus, in this case we have two independent basic eigenvectors, given by the standard unit basis vectors  $\vec{e}_1$  and  $\vec{e}_2$ .

Finally, for  $C = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$  we have  $C - 2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ , which is already in reduced-row echelon form. From this, we can see that the system  $(C - 2I)\vec{x} = \vec{0}$  has general solution

$$\vec{x} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t\vec{e}_1,$$

so we have only one independent eigenvector associated to  $\lambda = 2$ ; namely,  $\vec{e}_1$ .

The above shows us that when the algebraic multiplicity of an eigenvalue is greater than one, we might have multiple independent eigenvectors associated to that eigenvalue, but this is not guaranteed. To quantify this situation, we introduce further terminology. First, we make a definition: the **null space** of an  $n \times n$  matrix  $A$  is the set defined by

$$\text{null}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}.$$

In this language, we note that every eigenvector associated to an eigenvalue  $\lambda$  belongs to the null space of  $A - \lambda I$ . Indeed,  $\text{null}(A - \lambda I)$  is equal to the set of all eigenvectors associated to the eigenvalue  $\lambda$ , along with the zero vector. This leads to another definition:

#### Definition 7.3.6 Eigenspace.

Let  $A$  be an  $n \times n$  matrix, and let  $\lambda$  be any real number. The  $\lambda$ -**eigenspace** of  $A$ , denoted by  $E(A, \lambda)$ , is defined by

$$E(A, \lambda) = \text{null}(A - \lambda I).$$

A couple of remarks on this definition are in order. First, notice that  $\lambda$  is not assumed to be an eigenvalue in the definition above. However, for any real number  $\lambda$  which is not an eigenvalue, we have

$$E(A, \lambda) = \{\vec{0}\}.$$

The eigenvalues of  $A$  can thus be described as those real numbers for which  $E(A, \lambda) \neq \{\vec{0}\}$ . When  $\lambda$  is an eigenvalue, the *basic* eigenvectors associated



to  $\lambda$  (that is, the basic solutions to the system  $(A - \lambda I)\vec{x} = \vec{0}$ ) form a **basis** for  $E(A, \lambda)$ . This tells us that the dimension of each eigenspace is given by the number of independent basic eigenvectors associated to the corresponding eigenvalue, and gives us one more definition:

**Definition 7.3.7 Geometric Multiplicity of an Eigenvalue.**

Let  $A$  be an  $n \times n$  matrix. The **geometric multiplicity** of an eigenvalue  $\lambda$  of  $A$  is defined to be the dimension of the eigenspace  $E(A, \lambda)$ .

**Example 7.3.8 Computing eigenvalues and their multiplicities.**

Determine the eigenvalues of the matrix  $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ , along

with their algebraic and geometric multiplicities.

**Solution.** We first compute the eigenvalues of  $A$ . The characteristic polynomial of  $A$  is given by

$$\begin{aligned} c_A(x) &= \begin{vmatrix} x & -1 & -1 \\ -1 & x & -1 \\ -1 & -1 & x \end{vmatrix} \\ &= x \begin{vmatrix} x & -1 \\ -1 & x \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ -1 & x \end{vmatrix} - \begin{vmatrix} -1 & x \\ -1 & -1 \end{vmatrix} \\ &= x(x^2 - 1) - (x + 1) - (x + 1) \\ &= x(x - 1)(x + 1) - 2(x + 1) \\ &= (x^2 - x - 2)(x + 1) \\ &= (x - 2)(x + 1)^2. \end{aligned}$$

We see that the roots of  $c_A(x)$  are  $\lambda_1 = 2$ , which has algebraic multiplicity 1, and  $\lambda_2 = -1$ , which has algebraic multiplicity 2.

To determine the geometric multiplicities, we compute the corresponding eigenvectors. For  $\lambda = 2$ , we have

$$A - 2I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus,  $(A - 2I)\vec{x} = \vec{0}$  for  $\vec{x} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ , so  $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  is the (single) basic eigenvector associated to  $\lambda = 2$ . This gives us

$$E(A, 2) = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\},$$

so the geometric multiplicity of  $\lambda = 2$  is  $\dim E(A, 2) = 1$ .

For  $\lambda = -1$ , we find

$$A - (-1)I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This tells us that the general solution to  $(A + I)\vec{x} = \vec{0}$  is

$$\vec{x} = \begin{bmatrix} -s - t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

so

$$E(A, -1) = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\},$$

and the geometric multiplicity of  $\lambda = -1$  is  $\dim E(A, -1) = 2$ .

**Diagonalization.** One common application of matrix algebra is to a class of dynamical systems known as *linear recurrences*. A linear recurrence is a process that occurs in discrete steps, such that the state of the system at any step  $k$  depends linearly on the state of the system after the previous step. An example might be a population model for an ecosystem involving several species. For a simple two-species “predator-prey” model, we might have two populations  $P_1$  (of rabbits, perhaps?) and  $P_2$  (let’s say these are foxes) where the populations at a time  $t_k$  satisfy a relationship such as

$$\begin{aligned} P_1(t_k) &= aP_1(t_{k-1}) + bP_2(t_{k-1}) \\ P_2(t_k) &= cP_1(t_{k-1}) + dP_2(t_{k-1}). \end{aligned}$$

A reasonable model would probably have positive values for the coefficients  $a$  and  $d$  (absent competition, predators, etc. we expect the populations to increase), while  $b$  would be negative (more foxes mean fewer rabbits) and  $c$  would be positive (more rabbits provide more fox food). If we introduce a “population vector”  $\vec{P}(t) = \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix}$  to describe the situation at a time  $t$ , then the above

system can be written in the form  $\vec{P}(t_k) = A\vec{P}(t_{k-1})$ , where  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

Now, given an initial population state  $\vec{P}(t_0)$ , we might want to know what to expect after several generations (or seasons, or some other reasonable unit of time). Notice that we have

$$\begin{aligned} \vec{P}(t_1) &= A\vec{P}(t_0) \\ \vec{P}(t_2) &= A\vec{P}(t_1) = A(A\vec{P}(t_0)) = A^2\vec{P}(t_0) \\ \vec{P}(t_3) &= A\vec{P}(t_2) = A(A^2\vec{P}(t_0)) = A^3\vec{P}(t_0) \\ &\vdots \\ \vec{P}(t_k) &= A^k\vec{P}(t_0). \end{aligned}$$

Although most students in a first course in linear algebra would prefer to avoid complex eigenvalues (and trigonometric functions!), it turns out that these are a key feature of stable ecosystems for a model like this! Complex eigenvalues lead to sinusoidal terms, which correspond to oscillations in the two populations: the predator and prey populations will go up and down, in a cyclical fashion.

Thus, to model the time evolution of our system, it suffices to compute powers of our matrix  $A$ . If we want to study a system over long periods of time, we need to be able to compute large powers of our matrix, and for complex systems, being able to do so efficiently will become increasingly important. (For example, we may be interested in whether or not the system settles into an equilibrium state. This would be the case, for example, if we were able to show that  $A^k$  was approximately equal to the identity matrix for large values of  $k$ .)

One case where this is easily done is when the initial state  $\vec{P}_0 = \vec{P}(t_0)$  is an eigenvector for our matrix: if  $A\vec{P}_0 = \lambda\vec{P}_0$ , then we have

$$\begin{aligned} A^2\vec{P}_0 &= A(A\vec{P}_0) = A(\lambda\vec{P}_0) = \lambda(A\vec{P}_0) = \lambda(\lambda\vec{P}_0) = \lambda^2\vec{P}_0 \\ A^3\vec{P}_0 &= A(A^2\vec{P}_0) = A(\lambda^2\vec{P}_0) = \lambda^2(A\vec{P}_0) = \lambda^2(\lambda\vec{P}_0) = \lambda^3\vec{P}_0, \end{aligned}$$

and so on: in general,  $A^k\vec{P}_0 = \lambda^k\vec{P}_0$ . This is useful, since computing powers of a number is much simpler than computing powers of a matrix.

What if the initial state is not an eigenvector? It turns out that the same trick works as long as  $\vec{P}_0$  can be written as a *linear combination* of eigenvectors: suppose that

$$\vec{P}_0 = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_k\vec{x}_k,$$

where  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k$  are eigenvectors of  $A$  with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_k$ , respectively. (Note that at this point we've moved away from the original premise of a  $2 \times 2$  matrix  $A$  to consider a more general situation.) In this case, for any natural number  $n$ , we have

$$\begin{aligned} A^n\vec{P}_0 &= A^n(c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_k\vec{x}_k) \\ &= c_1(A^n\vec{x}_1) + c_2(A^n\vec{x}_2) + \cdots + c_k(A^n\vec{x}_k) \\ &= c_1(\lambda_1^n\vec{x}_1) + c_2(\lambda_2^n\vec{x}_2) + \cdots + c_k(\lambda_k^n\vec{x}_k), \end{aligned}$$

using our previous result. Again, we only need to be able to compute powers of the eigenvalues, and not of the original matrix.

With the above in mind, we will clearly be in an advantageous situation if we can guarantee that every vector in our space can be written as a linear combination of eigenvectors. We know from our earlier work that in order to guarantee this, we would need to be able to construct a *basis of eigenvectors*. We will see how such a basis can be used after we make a definition.

#### Definition 7.3.9 Diagonalizable Matrix.

We say that an  $n \times n$  matrix  $A$  is **diagonalizable** (can be diagonalized) if  $A \sim D$  for some diagonal matrix  $D$ . In other words,  $A$  is diagonalizable if there exists an invertible matrix  $P$  such that  $D = P^{-1}AP$  is a diagonal matrix.

#### Theorem 7.3.10 A matrix with a basis of eigenvectors is diagonalizable.

An  $n \times n$  matrix  $A$  is diagonalizable if and only if there exists a basis for  $\mathbb{R}^n$  consisting of eigenvectors of  $A$ .

We will now proceed with a proof of this theorem. We will see that in addition to the theoretical interest in providing a proof, the ideas in this proof will have practical use as well.

First, suppose that  $A$  can be diagonalized. Then there exists an invertible matrix  $P$  such that  $P^{-1}AP = D$ , where

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

and  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the (not necessarily distinct) eigenvalues of  $A$ . Letting  $\vec{e}_1, \dots, \vec{e}_n$  denote the standard basis for  $\mathbb{R}^n$  we have, for each  $j = 1, 2, \dots, n$ :

$$\begin{aligned} D\vec{e}_j &= \lambda_j \vec{e}_j \\ (P^{-1}AP)\vec{e}_j &= \lambda_j \vec{e}_j \\ (AP)\vec{e}_j &= P(\lambda_j \vec{e}_j) \\ A(P\vec{e}_j) &= \lambda_j (P\vec{e}_j), \end{aligned}$$

Suppose that  $c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_k\vec{x}_k = \vec{0}$  for some scalars  $c_1, c_2, \dots, c_k$ . Then we have

$$\begin{aligned} \vec{0} &= c_1 P\vec{e}_1 + \dots + c_n P\vec{e}_n \\ &= P(c_1\vec{e}_1 + \dots + c_n\vec{e}_n). \end{aligned}$$

Since  $P$  is invertible, we can conclude that  $c_1\vec{e}_1 + \dots + c_n\vec{e}_n = \vec{0}$ , and thus  $c_1 = c_2 = \dots = c_n = 0$ , which shows that the  $\vec{x}_i$  are independent.

so  $\vec{x}_j = P\vec{e}_j$  is an eigenvector for  $A$  with eigenvalue  $\lambda_j$ . Moreover, since  $P$  is invertible, we know that the vectors  $\vec{x}_1, \dots, \vec{x}_n$  are linearly independent, and thus form our desired basis of eigenvectors. (See the margin note for a proof of this fact.)

Conversely, suppose that we have a basis  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$  of eigenvectors for  $A$ , with corresponding eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ . Let  $P$  be the matrix whose columns are given by these vectors. Then  $P$  is invertible, since its columns are linearly independent. Let  $D$  be defined as above. Then

$$PD = [\vec{x}_1 \quad \vec{x}_2 \quad \dots \quad \vec{x}_n] \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} = [\lambda_1 \vec{x}_1 \quad \lambda_2 \vec{x}_2 \quad \dots \quad \lambda_n \vec{x}_n],$$

while

$$AP = A[\vec{x}_1 \quad \vec{x}_2 \quad \dots \quad \vec{x}_n] = [A\vec{x}_1 \quad A\vec{x}_2 \quad \dots \quad A\vec{x}_n] = [\lambda_1 \vec{x}_1 \quad \lambda_2 \vec{x}_2 \quad \dots \quad \lambda_n \vec{x}_n].$$

Thus,  $PD = AP$ , and since  $P$  is invertible, this gives us  $D = P^{-1}AP$ , as required.

As we discussed above, the diagonalizability of  $A$  makes it easy to compute powers of  $A$ . Indeed, suppose  $P^{-1}AP = D$  is diagonal, for some invertible matrix  $P$ . Then  $A = PDP^{-1}$ , and

$$\begin{aligned} A^n &= (PDP^{-1})^n = (PDP^{-1}) \dots (PDP^{-1}) \\ &= PD(P^{-1}P)D \dots (P^{-1}P)DP^{-1} = PD^nP^{-1}. \end{aligned}$$

This makes it easy to compute  $A^n$ , since it's easy to compute  $D^n$ :

$$\text{If } D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}, \text{ then } D^n = \begin{bmatrix} \lambda_1^n & 0 & \dots & 0 \\ 0 & \lambda_2^n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^n \end{bmatrix}.$$

The proof Theorem 7.3.10 is quite useful, since it tells us two things: first, we now know what it takes to diagonalize a matrix: for an  $n \times n$  matrix, we need to be able to find  $n$  independent eigenvectors. Second, the proof of the theorem tells us *how* to diagonalize: once we've found our eigenvectors, we construct the matrix  $P$  whose columns are given by the eigenvectors of  $A$ ; the matrix  $P^{-1}AP$  will then be a diagonal matrix with the eigenvalues of  $A$  on the main diagonal. This leaves us with the question: when can we be sure we have enough eigenvectors? To answer this, we begin with a theorem:

**Theorem 7.3.11 Linear independence of eigenvectors.**

*The eigenvectors  $\vec{x}_1, \dots, \vec{x}_k$  corresponding to distinct eigenvalues of  $A$  are linearly independent.*

The above theorem leads to the following corollary:

**Theorem 7.3.12 Distinct eigenvalues and diagonalization.**

*If an  $n \times n$  matrix  $A$  has  $n$  distinct eigenvalues, then  $A$  is diagonalizable.*

Let us first see how the above corollary follows from our theorem. We know that every eigenvalue has at least one associated eigenvector. If there are  $n$  distinct eigenvalues, then we have  $n$  associated eigenvectors which, by the theorem above, are linearly independent, and we know that any set of  $n$  linearly independent vectors in  $\mathbb{R}^n$  forms a basis for  $\mathbb{R}^n$ .

As for the proof of the theorem, we use an argument by contradiction: suppose, to the contrary, that the eigenvectors  $\vec{x}_1, \dots, \vec{x}_k$  are linearly *dependent*. Then there is some  $m$ ,  $2 \leq m \leq k - 1$ , such that the vectors  $\vec{x}_1, \dots, \vec{x}_m$  are linearly independent, but  $\vec{x}_{m+1}$  can be written as a linear combination of the vectors  $\vec{x}_1, \dots, \vec{x}_m$ ; that is,

$$\vec{x}_{m+1} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_m\vec{x}_m, \quad (7.3.1)$$

for some scalars  $c_1, \dots, c_m$ . Then, multiplying both sides of (7.3.1) on the left by  $A$ , we must have

$$A\vec{x}_{m+1} = A(c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_m\vec{x}_m) = c_1(A\vec{x}_1) + c_2(A\vec{x}_2) + \cdots + c_m(A\vec{x}_m),$$

but each vector  $\vec{x}_i$  is an eigenvector of  $A$  with eigenvalue  $\lambda_i$ .

Thus,

$$\lambda_{m+1}\vec{x}_{m+1} = c_1\lambda_1\vec{x}_1 + c_2\lambda_2\vec{x}_2 + \cdots + c_m\lambda_m\vec{x}_m.$$

On the other hand, if we multiply both sides of \eqref{eq-indepeigen} by the scalar  $\lambda_{m+1}$ , we have

$$\lambda_{m+1}\vec{x}_{m+1} = c_1\lambda_{m+1}\vec{x}_1 + c_2\lambda_{m+1}\vec{x}_2 + \cdots + c_m\lambda_{m+1}\vec{x}_m.$$

Subtracting these last two equations, we find:

$$\vec{0} = c_1(\lambda_{m+1} - \lambda_1)\vec{x}_1 + c_2(\lambda_{m+1} - \lambda_2)\vec{x}_2 + \cdots + c_m(\lambda_{m+1} - \lambda_m)\vec{x}_m.$$

Now, we are assuming that our eigenvalues are all *distinct*. Thus,  $\lambda_{m+1} - \lambda_i \neq 0$  for all  $i = 1, 2, \dots, m$ . On the other hand, the vectors  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_m$  are assumed to be linearly independent, so we must have

$$c_1(\lambda_{m+1} - \lambda_1) = 0, c_2(\lambda_{m+1} - \lambda_2) = 0, \dots, c_m(\lambda_{m+1} - \lambda_m) = 0.$$

Since none of the  $\lambda_{m+1} - \lambda_i$  terms vanish, it must be the case that  $c_1 = c_2 = \cdots = c_m = 0$ . But if that is true, then

$$\vec{x}_{m+1} = 0\vec{x}_1 + 0\vec{x}_2 + \cdots + 0\vec{x}_m = \vec{0},$$

which is impossible, since  $\vec{x}_{m+1}$  was assumed to be an eigenvector, and eigenvectors are nonzero. Thus, it must be the case that all of our eigenvectors are linearly independent.

The above tells us that any  $n \times n$  matrix  $A$  with  $n$  distinct eigenvalues is automatically diagonalizable. There are two other possibilities. One is that the characteristic polynomial of  $A$  cannot be completely factored over the real numbers. In this case, diagonalization is impossible, *unless* we are willing to consider complex eigenvalues. Recall that over the complex numbers, every polynomial can be completely factored. (This is the Fundamental Theorem of Algebra.) In this case, we may still be able to diagonalize, but the eigenvectors corresponding to the complex eigenvalues may well be complex themselves, so the matrix  $P$  used to diagonalize  $A$  could have complex entries.

The other possibility is that we have repeated eigenvalues. Here, it is possible that we encounter truly insurmountable obstacles. First, we have the following theorem, which we will state without proof. The interested reader can easily find the details in other linear algebra textbooks or online.

**Theorem 7.3.13 Algebraic versus geometric multiplicity.**

*Let  $A$  be an  $n \times n$  matrix, and for each real number  $\lambda$ , let  $m_\lambda(A)$  denote the algebraic multiplicity of  $\lambda$ . (If  $\lambda$  is not an eigenvalue of  $A$ , we set  $m_\lambda(A) = 0$ .) Then for all numbers  $\lambda$ , we have*

$$\dim \text{null}(A - \lambda I) \leq m_\lambda(A).$$

In other words, the geometric multiplicity of an eigenvalue is always less than or equal to the algebraic multiplicity. Note that the sum of the algebraic multiplicities is always equal to the degree of the characteristic polynomial, which, in turn, is equal to  $n$ , for an  $n \times n$  matrix  $A$ . On the other hand, the total number of independent eigenvectors for  $A$  is equal to the sum of the geometric multiplicities. (We know that eigenvectors for *different* eigenvalues are independent, and the geometric multiplicity tells us how many independent eigenvectors we'll have for a single eigenvalue.) As a result, we have the following:

**Theorem 7.3.14 Diagonalization and multiplicities.**

*An  $n \times n$  matrix  $A$  is diagonalizable if and only if  $\dim \text{null}(A - \lambda I) = m_\lambda(A)$  for each eigenvalue  $\lambda$  of  $A$ ; that is, if the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity.*

This also tells us exactly when we can expect a matrix to fail to be diagonalizable: this will be the case if  $A$  has an eigenvalue of algebraic multiplicity greater than 1 for which the geometric multiplicity is less than the algebraic multiplicity. (That is, if we don't get "enough" basic eigenvectors corresponding to a repeated eigenvalue.)

The reader may wonder if there is anything further that can be said in the case that a matrix  $A$  is not diagonalizable, and indeed, there is. As long as we are willing to work over the complex numbers (to avoid situations where the characteristic polynomial cannot be factored), one can prove that in the worst-case scenario, there exists an invertible matrix  $P$  such that the matrix  $P^{-1}AP$  is triangular. (If we can't get a diagonal matrix, upper or lower-triangular is the next best thing.) The standard form of such a matrix is called the *Jordan Canonical Form*; it is obtained using what are called *generalized eigenvectors*. The Jordan Canonical Form of a matrix is studied in more advanced courses in linear algebra.

One last important result that we do not have time to study concerns the

case of symmetric matrices. Recall that an  $n \times n$  matrix  $A$  is *symmetric* if  $A^T = A$ . An important result called the *Spectral Theorem* guarantees that every symmetric matrix  $A$  is diagonalizable. In fact, the spectral theorem goes one step further. Recall that two vectors  $\vec{u}$  and  $\vec{v}$  are *orthogonal* if  $\vec{u} \cdot \vec{v} = 0$ . We say that a set of vectors  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$  is **orthogonal** if  $\vec{x}_i \neq \vec{0}$  for each  $i$ , and  $\vec{x}_i \cdot \vec{x}_j = 0$  for all  $i \neq j$ . Orthogonality is a “stronger” condition than linear independence. Indeed, if the vectors  $\vec{x}_1, \dots, \vec{x}_k$  are orthogonal and

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_k\vec{x}_k = \vec{0}$$

for some scalars  $c_1, c_2, \dots, c_k$ , taking the dot product of both sides of the above equation with  $\vec{x}_1$  gives us

$$c_1(\vec{x}_1 \cdot \vec{x}_1) + c_2(0) + \dots + c_k(0) = 0,$$

and since  $\vec{x}_1 \cdot \vec{x}_1 = \|\vec{x}_1\|^2 \neq 0$ , it must be that  $c_1 = 0$ , and similarly, all the other scalars must be zero as well.

For a symmetric matrix, eigenvectors corresponding to distinct eigenvalues are not just independent, they’re orthogonal. Indeed, since  $\vec{x}_1 \cdot \vec{x}_2$  can be written as the matrix product  $\vec{x}_1^T \vec{x}_2$ , we have (using the fact that  $A^T = A$ ):

$$\lambda_1(\vec{x}_1 \cdot \vec{x}_2) = (\lambda_1 \vec{x}_1)^T \vec{x}_2 = (A\vec{x}_1)^T \vec{x}_2 = (\vec{x}_1^T A^T) \vec{x}_2 = \vec{x}_1^T (A\vec{x}_2) = \vec{x}_1^T (\lambda_2 \vec{x}_2) = \lambda_2(\vec{x}_1 \cdot \vec{x}_2),$$

Thus,  $(\lambda_1 - \lambda_2)(\vec{x}_1 \cdot \vec{x}_2) = 0$ , and since  $\lambda_1 \neq \lambda_2$ , we must have  $\vec{x}_1 \cdot \vec{x}_2 = 0$ .

It is possible to prove that for a symmetric matrix, one can find an **orthogonal basis** of eigenvectors. This observation has a number of important applications. (Try an online search for “orthogonal diagonalization” and you should be able to find several examples.)

**Matrix similarity and change of basis for transformations.** We end this section with the promised explanation of how similar matrices can be viewed as different representations of the same linear transformation. If we think of matrices in terms of the matrix transformations they define, then two matrix transformations obtained from similar matrices should be viewed as two descriptions of the same linear transformation using different “coordinate systems”. Let us explain what we mean here. Suppose

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

is a matrix transformation defined by  $T(\vec{x}) = A\vec{x}$  for some  $n \times n$  matrix  $A$ . Let us consider this to be a transformation defined in terms of the standard basis vectors  $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$  for  $\mathbb{R}^n$ .

We can then write

$$\vec{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T = x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_n\vec{e}_n$$

and think of  $T$  as a function of the variables  $x_1, x_2, \dots, x_n$ . To put this another way, when we speak of a matrix transformation with standard matrix  $A$ , that matrix  $A$  is defined with respect to the standard basis. In general, let  $\mathcal{A} = \{\vec{e}_1, \dots, \vec{e}_n\}$  denote our standard basis, and suppose

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$$

is another basis for  $\mathbb{R}^n$ . Recall that when we say that the set  $\mathcal{B}$  is a *basis*, we mean that  $\text{span } \mathcal{B} = \mathbb{R}^n$ , so that every vector in  $\mathbb{R}^n$  can be written as a linear combination of the vectors in  $\mathcal{B}$ , and  $\mathcal{B}$  is linearly independent, which means

that every vector in  $\mathbb{R}^n$  can be written *uniquely* as a linear combination of the vectors in  $\mathcal{B}$ .

Let  $\vec{v} = y_1\vec{b}_1 + y_2\vec{b}_2 + \cdots + y_n\vec{b}_n$  any vector in  $\mathbb{R}^n$ . Using the same linear transformation  $T$  as above, we have

$$T(\vec{v}) = T(y_1\vec{b}_1 + y_2\vec{b}_2 + \cdots + y_n\vec{b}_n) = y_1T(\vec{b}_1) + y_2T(\vec{b}_2) + \cdots + y_nT(\vec{b}_n).$$

Now, each of the vectors  $T(\vec{b}_i)$  can, in turn, be written in terms of the basis  $\mathcal{B}$ : we have

$$\begin{aligned} T(\vec{b}_1) &= b_{11}\vec{b}_1 + b_{21}\vec{b}_2 + \cdots + b_{n1}\vec{b}_n \\ T(\vec{b}_2) &= b_{12}\vec{b}_1 + b_{22}\vec{b}_2 + \cdots + b_{n2}\vec{b}_n \\ &\vdots \\ T(\vec{b}_n) &= b_{1n}\vec{b}_1 + b_{2n}\vec{b}_2 + \cdots + b_{nn}\vec{b}_n. \end{aligned}$$

The coefficients  $b_{ij}$  obtained above determine a new matrix  $B$ , which we define to be the matrix of  $T$  with respect to the basis  $\mathcal{B}$ . Notice that if we write  $\vec{y} = [y_1 \ y_2 \ \cdots \ y_n]^T$ , we have

$$B\vec{y} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_{11}y_1 + b_{12}y_2 + \cdots + b_{1n}y_n \\ b_{21}y_1 + b_{22}y_2 + \cdots + b_{2n}y_n \\ \vdots \\ b_{n1}y_1 + b_{n2}y_2 + \cdots + b_{nn}y_n \end{bmatrix}.$$

Let's introduce some notation: for any vector  $\vec{w}$  in  $\mathbb{R}^n$ , we will let  $[\vec{w}]_{\mathcal{B}}$  denote the  $n \times 1$  column vector defined as follows:

$$\text{If } \vec{w} = w_1\vec{b}_1 + w_2\vec{b}_2 + \cdots + w_n\vec{b}_n, \text{ then } [\vec{w}]_{\mathcal{B}} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}.$$

In this notation, we have  $\vec{y} = [\vec{v}]_{\mathcal{B}}$ , and

$$[T(\vec{v})]_{\mathcal{B}} = \begin{bmatrix} b_{11}y_1 + b_{12}y_2 + \cdots + b_{1n}y_n \\ b_{21}y_1 + b_{22}y_2 + \cdots + b_{2n}y_n \\ \vdots \\ b_{n1}y_1 + b_{n2}y_2 + \cdots + b_{nn}y_n \end{bmatrix},$$

since

$$\begin{aligned} &(b_{11}y_1 + \cdots + b_{1n}y_n)\vec{b}_1 + \cdots + (b_{n1}y_1 + \cdots + b_{nn}y_n)\vec{b}_n \\ &= y_1(b_{11}\vec{b}_1 + \cdots + b_{n1}\vec{b}_n) + \cdots + y_n(b_{1n}\vec{b}_1 + \cdots + b_{nn}\vec{b}_n) \\ &= y_1T(\vec{b}_1) + \cdots + y_nT(\vec{b}_n) \\ &= T(y_1\vec{b}_1 + \cdots + y_n\vec{b}_n) \\ &= T(\vec{v}). \end{aligned}$$

Let's tie everything together. First, let  $P$  be the  $n \times n$  matrix whose columns are the vectors in  $\mathcal{B}$ . Then we know that

$$\vec{b}_1 = P\vec{e}_1, \vec{b}_2 = P\vec{e}_2, \dots, \vec{b}_n = P\vec{e}_n,$$



and conversely,

$$\vec{e}_1 = P^{-1}\vec{b}_1, \vec{e}_2 = P^{-1}\vec{b}_2, \dots, \vec{e}_n = P^{-1}\vec{b}_n.$$

If  $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ , then we have

$$\vec{v} = v_1\vec{b}_1 + \dots + v_n\vec{b}_n = v_1P\vec{e}_1 + \dots + v_nP\vec{e}_n = P(v_1\vec{e}_1 + \dots + v_n\vec{e}_n) = P[\vec{v}]_{\mathcal{B}}.$$

Note that we can also write  $\vec{v} = [\vec{v}]_{\mathcal{A}}$ , where  $\mathcal{A}$  is the standard basis, since by default, all of our vectors are written in terms of the standard basis. Thus, we can conclude that

$$[\vec{v}]_{\mathcal{A}} = P[\vec{v}]_{\mathcal{B}} \quad (7.3.2)$$

defines the relationship between the column vector representations of our vector with respect to the two bases. Similarly, we can write

$$[T(\vec{v})]_{\mathcal{A}} = P[T(\vec{v})]_{\mathcal{B}} \quad (7.3.3)$$

for the column representations of the output of our linear transformation  $T$ .

However, we also know that our transformation was originally defined using the matrix  $A$  according to

$$[T(\vec{v})]_{\mathcal{A}} = A[\vec{v}]_{\mathcal{A}} \quad (7.3.4)$$

and that the matrix  $B$  above was defined such that

$$[T(\vec{v})]_{\mathcal{B}} = B[\vec{v}]_{\mathcal{B}}. \quad (7.3.5)$$

Equating the two expressions above for  $[T(\vec{v})]_{\mathcal{A}}$ , in Equations (7.3.3) and (7.3.4) we have

$$P[T(\vec{v})]_{\mathcal{B}} = A[\vec{v}]_{\mathcal{A}}.$$

Plugging in Equations (7.3.5) on the left and (7.3.2) on the right, we have

$$PB[\vec{v}]_{\mathcal{B}} = AP[\vec{v}]_{\mathcal{B}}.$$

Since this must be true for any vector, we conclude that  $PB = AP$ , and thus

$$B = P^{-1}AP,$$

so  $B$  is similar to  $A$ .

### 7.3.1 Exercises

**Exercise Group.** Compute the characteristic polynomial of the given matrix, and the eigenvalues of the matrix, along with their algebraic and geometric multiplicities. If possible, construct a matrix  $P$  such that  $P^{-1}AP$  is diagonal, and verify your result. If no such matrix  $P$  exists, explain why.

1. 
$$\begin{bmatrix} 7 & 0 & 0 \\ 10 & -3 & 0 \\ 10 & -8 & 5 \end{bmatrix}$$

2. 
$$\begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & 3 \\ 2 & -1 & 3 \end{bmatrix}$$

3. 
$$\begin{bmatrix} 9 & -7 \\ -6 & 20 \end{bmatrix}$$

4. 
$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

5. 
$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

6. 
$$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$$

7. 
$$\begin{bmatrix} 3 & 2 & -2 \\ 2 & 2 & -2 \\ 2 & 2 & -1 \end{bmatrix}$$

8. 
$$\begin{bmatrix} 4 & -2 \\ 1 & 3 \end{bmatrix}$$

# Appendix A

## Answers to Selected Exercises

### 1 • The Real and Complex Numbers

#### 1.2 • Real Number Arithmetic

##### 1.2 • Exercises

1.2.1.  $\frac{23}{9}$

1.2.5.  $\frac{9}{22}$

1.2.9.  $\frac{\sqrt{61}}{2}$

1.2.13. 6

1.2.17. undefined

1.2.21.  $\frac{3}{5}$

1.2.25. 5

1.2.29.  $-1$

1.2.33.  $\frac{25}{4}$

1.2.3.  $-\frac{3\sqrt[5]{3}}{8} = -\frac{3^{6/5}}{8}$

1.2.7.  $\frac{-4 + \sqrt{2}}{7}$

1.2.11.  $-\frac{24}{7}$

1.2.15. 18

1.2.19.  $\frac{15}{16}$

1.2.23.  $-1$

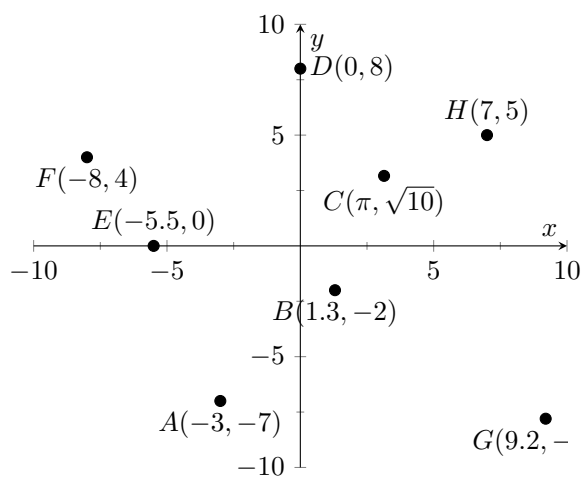
1.2.27.  $\sqrt{10}$

1.2.31.  $-\frac{1}{3}$

### 1.3 • The Cartesian Coordinate Plane

#### 1.3 • Exercises

1.3.1. The required points  $A(-3, -7)$ ,  $B(1.3, -2)$ ,  $C(\pi, \sqrt{10})$ ,  $D(0, 8)$ ,  $E(-5.5, 0)$ ,  $F(-8, 4)$ ,  $G(9.2, -7.8)$ , and  $H(7, 5)$  are plotted in the Cartesian Coordinate Plane below.



**1.3.3.**  $d = 4\sqrt{10}$ ,  $M = (1, -4)$

**1.3.5.**  $d = \frac{\sqrt{37}}{2}$ ,  $M = (\frac{5}{6}, \frac{7}{4})$

**1.3.7.**  $d = 3\sqrt{5}$ ,  $M = (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{3}}{2})$

**1.3.9.**  $d = \sqrt{x^2 + y^2}$ ,  $M = (\frac{x}{2}, \frac{y}{2})$

**1.3.11.**  $(0, 3)$

**1.3.13.**  $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}), (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

## 1.4 • Complex Numbers

### 1.4 • Exercises

**1.4.1.**

- $z + w = 2 + 7i$
- $zw = -12 + 8i$
- $z^2 = -5 + 12i$
- $\frac{1}{z} = \frac{2}{13} - \frac{3}{13}i$
- $\frac{z}{w} = \frac{3}{4} - \frac{1}{2}i$
- $\frac{w}{z} = \frac{12}{13} + \frac{8}{13}i$
- $\bar{z} = 2 - 3i$
- $z\bar{z} = 13$
- $(\bar{z})^2 = -5 - 12i$

**1.4.3.**

- $z + w = -1 + 3i$
- $zw = -2 - i$
- $z^2 = -1$
- $\frac{1}{z} = -i$
- $\frac{z}{w} = \frac{2}{5} - \frac{1}{5}i$
- $\frac{w}{z} = 2 + i$
- $\bar{z} = -i$
- $z\bar{z} = 1$
- $(\bar{z})^2 = -1$

**1.4.5.**

- $z + w = 5 + 2i$
- $zw = 41 + 11i$
- $z^2 = -16 - 30i$
- $\frac{1}{z} = \frac{3}{34} + \frac{5}{34}i$
- $\frac{z}{w} = -\frac{29}{53} - \frac{31}{53}i$
- $\frac{w}{z} = -\frac{29}{34} + \frac{31}{34}i$
- $\bar{z} = 3 + 5i$
- $z\bar{z} = 34$
- $(\bar{z})^2 = -16 + 30i$

**1.4.9.**

- $z + w = i\sqrt{3}$
- $zw = -1$
- $z^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
- $\frac{1}{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$
- $\frac{z}{w} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$
- $\frac{w}{z} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$
- $\bar{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$
- $z\bar{z} = 1$
- $(\bar{z})^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

**1.4.11.**  $7i$

**1.4.15.**  $-12$

**1.4.19.**  $i^5 = i^4 \cdot i = 1 \cdot i = i$

**1.4.23.**  $i^{15} = (i^4)^3 \cdot i^3 = 1 \cdot (-i) = -i$

**1.4.27.**  $x = \frac{2 \pm i\sqrt{14}}{3}$

**1.4.31.**  $y = \pm \frac{3i\sqrt{2}}{2}$

**1.4.35.**  $z = \pm 2, \pm 2i$

**1.4.7.**

- $z + w = 2\sqrt{2}$
- $zw = 4$
- $z^2 = -4i$
- $\frac{1}{z} = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4}i$
- $\frac{z}{w} = -i$
- $\frac{w}{z} = i$
- $\bar{z} = \sqrt{2} + i\sqrt{2}$
- $z\bar{z} = 4$
- $(\bar{z})^2 = 4i$

**1.4.13.**  $-10$

**1.4.17.**  $3$

**1.4.21.**  $i^7 = i^4 \cdot i^3 = 1 \cdot (-i) = -i$

**1.4.25.**  $i^{117} = (i^4)^{29} \cdot i = 1 \cdot i = i$

**1.4.29.**  $y = \pm 2, \pm i$

**1.4.33.**  $x = \frac{\sqrt{5} \pm i\sqrt{3}}{2}$

**2 • Vectors****2.1 • Introduction to Cartesian Coordinates in Space**

## 2.1 • Exercises

2.1.1. (a).  $\sqrt{6}$

(b).  $\sqrt{17}$

(c).  $\sqrt{11}$

(d). do

## 2.2 • An Introduction to Vectors

### 2.2 • Exercises

2.2.1.

(a)  $\langle 1, 6 \rangle$

(b)  $i + 6j$

2.2.5.

(a)  $\vec{u} + \vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}; \vec{u} - \vec{v} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}; 2\vec{u} - 3\vec{v} = \begin{bmatrix} -1 \\ -7 \end{bmatrix}.$

(c)  $\vec{x} = \begin{bmatrix} 1/2 \\ 2 \end{bmatrix}.$

2.2.11. (a).  $\sqrt{5}$

(b).  $\sqrt{13}$

(c).  $\sqrt{26}$

(d).  $\sqrt{10}$

2.2.17.  $\langle 0.6, 0.8 \rangle$

2.2.21.  $\left\langle \frac{-1}{2}, \frac{\sqrt{3}}{2} \right\rangle$

## 2.3 • The Dot Product

### 2.3 • Exercises

2.3.1.  $-22$

2.3.5. not defined

2.3.7. Answers will vary.

2.3.9.  $\cos^{-1}\left(\frac{3}{\sqrt{10}}\right)$

2.3.13. (a).  $\langle -7, 4 \rangle$

(b).  $\langle 4, 7 \rangle$

2.3.17.  $\left\langle \frac{-5}{10}, \frac{15}{10} \right\rangle$

2.3.21.  $\left\langle \frac{14}{14}, \frac{28}{14}, \frac{42}{14} \right\rangle$

2.3.23. (a).  $\left\langle \frac{-5}{10}, \frac{15}{10} \right\rangle$

(b).  $\left\langle \frac{15}{10}, \frac{5}{10} \right\rangle$

2.3.27. (a).  $\left\langle \frac{14}{14}, \frac{28}{14}, \frac{42}{14} \right\rangle$

(b).  $\left\langle \frac{0}{14}, \frac{42}{14}, \frac{-28}{14} \right\rangle$

2.3.29. 1.96lb

2.2.3.

(a)  $\langle 6, -1, 6 \rangle$

(b)  $6i - j + 6k$

2.2.13. (a).  $\sqrt{5}$

(b).  $3\sqrt{5}$

(c).  $2\sqrt{5}$

(d).  $4\sqrt{5}$

2.2.19.  $\left\langle \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

2.3.3. 3

2.3.11.  $\frac{\pi}{4}$

2.3.15. (a).  $\langle 1, 0, -1 \rangle$

(b).  $\langle 1, 1, 1 \rangle$

2.3.19.  $\left\langle \frac{-1}{2}, \frac{-1}{2} \right\rangle$

2.3.25. (a).  $\left\langle \frac{-1}{2}, \frac{-1}{2} \right\rangle$

(b).  $\left\langle \frac{-5}{2}, \frac{5}{2} \right\rangle$

2.3.31. 141.42ft-lb

2.3.33. 500ft-lb

2.3.35. 500ft-lb

## 2.4 • The Cross Product

### 2.4 • Exercises

2.4.1.  $\langle 12, -15, 3 \rangle$

2.4.5.  $\langle 0, -2, 0 \rangle$

2.4.9.  $-j$

2.4.11. Answers will vary.

2.4.13. 5

2.4.17.  $\sqrt{14}$

2.4.21.  $\frac{5\sqrt{2}}{2}$

2.4.25. 7

2.4.27. 2

2.4.29.

$\langle 0.408248, 0.408248, -0.816497 \rangle$  or  $\langle -0.408248, -0.408248, 0.816497 \rangle$

2.4.3.  $\langle -5, -31, 27 \rangle$

2.4.7.  $\vec{u} \times \vec{v} = \begin{bmatrix} 0 \\ 0 \\ ad - bc \end{bmatrix}$

2.4.15. 0

2.4.19. 3

2.4.23. 1

2.4.33. 87.5ft-lb

2.4.35.  $200/3 \approx 66.67\text{ft-lb}$ 

2.4.31.  $\langle 0, 1, 0 \rangle$  or  $\langle 0, -1, 0 \rangle$

## 2.5 • Lines

### 2.5 • Exercises

2.5.7. (a).  $(7, 2, -1) + t\langle 1, -1, 2 \rangle$

(b).  $x = 7 + t, y = 2 - t, z = -1 + 2t$

(c).  $x - 7 = 2 - y = \frac{z+1}{2}$

2.5.11. parallel

2.5.19.  $\sqrt{41}/3$

2.5.23.  $3/\sqrt{2}$

2.5.15. skew

2.5.21.  $5\sqrt{2}/2$

## 2.6 • Planes

### 2.6 • Exercises

2.6.1. Answers will vary.

2.6.15. (a).  $x - 5 + y - 7 + z - 3 = 0$

(b).  $x + y + z = 15$

2.6.25.  $\sqrt{5/7}$

2.6.3. Answers will vary.

2.6.17. (a).  $3(x + 4) + 8(y - 7) - 10(z - 2) = 0$

(b).  $3x + 8y - 10z = 24$

2.6.27.  $1/\sqrt{3}$

## 2.7 • Span and Linear Independence

### 2.7 • Exercises

$$2.7.1. \begin{bmatrix} -9 \\ -8 \\ 16 \\ 12 \end{bmatrix}$$

$$2.7.3. \begin{bmatrix} -22 \\ 2 \\ 12 \\ 39 \end{bmatrix}$$

$$2.7.5. \sqrt{71}$$

$$2.7.7. 15$$

**2.7.9.** True. Suppose  $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$  is linearly independent. Let  $T \subseteq S$  be a subset. By re-ordering the vectors we can assume  $T = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$  for some  $m \leq n$ . If  $T$  were linearly dependent, then there would exist scalars  $c_1, \dots, c_m$ , not all zero, such that  $c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{0}$ . But if this is the case, then we would have  $c_1\vec{v}_1 + \dots + c_m\vec{v}_m + 0\vec{v}_{m+1} + \dots + 0\vec{v}_n = \vec{0}$ , which is impossible if  $S$  is independent.

**2.7.11.** True, since we can add any scalar multiple of the zero vector to a linear combination without affecting the value of that linear combination.

**2.7.13.** False. The set  $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$  is linearly dependent, since  $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , but  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  does not belong to the span of the set  $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ .

## 3 • Systems of Linear Equations

### 3.1 • Introduction to Linear Equations

#### 3.1 • Exercises

**3.1.1.** Yes

**3.1.3.** Yes

**3.1.5.** No

**3.1.7.** Yes

**3.1.9.** Yes

$$3.1.11. x = 1, y = -2$$

$$3.1.13. x = -1, y = 0, z = 2$$

**3.1.15.** 35 blue, 40 green, 20 red, 5 yellow

**3.1.17.** 12 \$0.30 trinkets, 8 \$0.65 trinkets

### 3.2 • Using Matrices To Solve Systems of Linear Equations

#### 3.2 • Exercises

$$3.2.1. \left[ \begin{array}{ccc|c} 3 & 4 & 5 & 7 \\ -1 & 1 & -3 & 1 \\ 2 & -2 & 3 & 5 \end{array} \right]$$

$$3.2.3. \left[ \begin{array}{cccc|c} 1 & 3 & -4 & 5 & 17 \\ -1 & 0 & 4 & 8 & 1 \\ 2 & 3 & 4 & 5 & 6 \end{array} \right]$$

$$3.2.5. \begin{aligned} x_1 + 2x_2 &= 3 \\ -x_1 + 3x_2 &= 9 \end{aligned}$$

$$3.2.7. \begin{aligned} x_1 + x_2 - x_3 - x_4 &= 2 \\ 2x_1 + x_2 + 3x_3 + 5x_4 &= 7 \end{aligned}$$

$$3.2.9. \begin{aligned} x_1 &= 2 \\ x_2 &= -1 \\ x_3 &= 5 \\ x_4 &= 3 \end{aligned}$$

$$3.2.11. \left[ \begin{array}{ccc} -2 & 1 & -7 \\ 0 & 4 & -2 \\ 5 & 0 & 3 \end{array} \right]$$

$$3.2.13. \left[ \begin{array}{ccc} 2 & -1 & 7 \\ 2 & 3 & 5 \\ 5 & 0 & 3 \end{array} \right]$$



$$3.2.15. \begin{bmatrix} 2 & -1 & 7 \\ 0 & 2 & -1 \\ 5 & 0 & 3 \end{bmatrix}$$

$$3.2.17. 2R_2 \rightarrow R_2$$

$$3.2.21. -R_2 + R_3 \leftrightarrow R_3$$

$$3.2.23. x = -1, y = 3$$

$$3.2.27. x_1 = 1, x_2 = 5, x_3 = 7$$

$$3.2.19. 2R_3 + R_1 \rightarrow R_1$$

$$3.2.25. x = \frac{1}{2}, y = \frac{1}{3}$$

### 3.3 • Elementary Row Operations and Gaussian Elimination

#### 3.3 • Exercises

3.3.1.

(a) yes

(b) no

(c) no

(d) yes

$$3.3.5. \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$3.3.9. \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 7 \end{bmatrix}$$

$$3.3.13. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$3.3.17. \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$3.3.21. \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 4 \end{bmatrix}$$

3.3.3.

(a) no

(b) yes

(c) yes

(d) yes

$$3.3.7. \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

$$3.3.11. \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$3.3.15. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$3.3.19. \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

### 3.4 • Existence and Uniqueness of Solutions

#### 3.4 • Exercises

3.4.1.  $x_1 = 1 - 2x_2$ ;  $x_2$  is free. Possible solutions:  $x_1 = 1, x_2 = 0$  and  $x_1 = -1, x_2 = 1$ . Geometrically, both equations describe the same line, and every point on this line is a solution to the system.

3.4.5.  $x_1 = 1 - x_2 - x_4$ ;  $x_2$  is free;  $x_3 = 1 - 2x_4$ ;  $x_4$  is free. Possible solutions:  $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0$  and  $x_1 = -2, x_2 = 1, x_3 = -3, x_4 = 2$ . Since there are four variables, a geometric description is more difficult (but see if you can come up with one!).

3.4.9.  $x_1 = 1; x_2 = 2$ . Geometrically, the system represents two lines intersecting in a single point.

3.4.3. No solution; the system is inconsistent.

Geometrically, the system describes three planes, any two of which intersect along a line. However, there is no point common to all three.

3.4.7.  $x_1 = 3 - x_3 - 2x_4; x_2 = -3 - 5x_3 - 7x_4; x_3$  is free;  $x_4$  is free. Possible solutions:  $x_1 = 3, x_2 = -3, x_3 = 0, x_4 = 0$  and  $x_1 = 0, x_2 = -5, x_3 = -1, x_4 = 1$ . Since there are four variables, a geometric description is more difficult (but see if you can come up with one!).

3.4.11.  $x_1 = 0; x_2 = -1$ . Geometrically, the system represents two lines intersecting in a single point.

**3.4.13.**  $x_1 = \frac{1}{3} - \frac{4}{3}x_3$ ;  $x_2 = \frac{1}{3} - \frac{1}{3}x_3$ ;  $x_3$  is free.  
Possible solutions:  $x_1 = \frac{1}{3}$ ,  $x_2 = \frac{1}{3}$ ,  $x_3 = 0$  and  $x_1 = -1$ ,  $x_2 = 0$ ,  $x_3 = 1$ . Geometrically, the system represents three planes that all intersect along a common line.

**3.4.15.** Exactly 1 solution if  $k \neq 2$ ; infinitely many solutions if  $k = 2$ ; never no solution.

**3.4.17.** Never exactly 1 solution; infinitely many solutions if  $k = 2$ ; no solution if  $k \neq 2$ .

## 3.5 • Applications of Linear Systems

### 3.5 • Exercises

**3.5.1.**  $f(x) = 6x - 3$

**3.5.3.**  $f(x) = -x^2 + x + 5$

**3.5.5.**  $f(x) = -x^3 + x^2 - x + 1$

**3.5.7.**  $f(x) = 3x - 5$

**3.5.9.**  $f(x) = x^2 + 1$

**3.5.11.**

(a) Substitution yields the equations  $2 = ae^b$  and  $4 = ae^{2b}$ ; these are not linear equations.

(b)  $y = ae^{bx}$  implies that  $\ln y = \ln(ae^{bx}) = \ln a + \ln e^{bx} = \ln a + bx$ .

(c) Plugging in the points for  $x$  and  $y$  in the equation  $\ln y = \ln a + bx$ , we have equations

$$\begin{array}{rcl} \ln a & + & b = \ln 2 \\ \ln a & + & 2b = \ln 4 \end{array}.$$

To solve,

$$\begin{bmatrix} 1 & 1 & \ln 2 \\ 1 & 2 & \ln 4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \ln 2 \end{bmatrix}.$$

Therefore  $\ln a = 0$  and  $b = \ln 2$ .

(d) Since  $\ln a = 0$ , we know that  $a = e^0 = 1$ . Thus our exponential function is  $f(x) = e^{x \ln 2}$ .

**3.5.13.** The augmented matrix from this system is  $\begin{bmatrix} 1 & 1 & 1 & 1 & 8 \\ 6 & 1 & 2 & 3 & 29 \\ 0 & 1 & -1 & 0 & 2 \end{bmatrix}$ . From this we find the solution

$$\begin{aligned} t &= 4 - \frac{1}{3}f \\ x &= 3 - \frac{1}{3}f \\ w &= 1 - \frac{1}{3}f. \end{aligned}$$

The only time each of these variables are nonnegative integers is when  $f = 0$  or  $f = 3$ . If  $f = 0$ , then we have 4 touchdowns, 3 extra points and 1 two point conversions (no field goals). If  $f = 3$ , then we have 3 touchdowns, 2 extra points and no two point conversions (and 3 field goals).

**3.5.15.** Let  $x_1$ ,  $x_2$  and  $x_3$  represent the number of free throws, 2 point and 3 point shots taken. The augmented matrix from this system is  $\begin{bmatrix} 1 & 1 & 1 & 70 \\ 1 & 2 & 3 & 110 \end{bmatrix}$ . From this we find the solution

$$x_1 = 30 + x_3$$

$$x_2 = 40 - 2x_3.$$

In order for  $x_2$  to be nonnegative, we need  $x_3 \leq 20$ . Thus there are 21 different scenarios: the “first” is where 0 three point shots are taken ( $x_3 = 0$ ), 30 free throws and 40 two point shots; the “last” is where 20 three point shots are taken, 50 free throws, and no two point shots.

**3.5.17.** Let  $y = ax + b$ ; all linear functions through (2,5) come in the form  $y = (2.5 - \frac{1}{2}b)x + b$ . Examples:  $b = 1$  yields  $y = 2x + 1$ ;  $b = -1$  yields  $y = 3x - 1$ .

**3.5.19.** Let  $y = ax^2 + bx + c$ ; we find that  $a = 2 - \frac{1}{2}c$  and  $b = -1 + \frac{1}{2}c$ . Examples:  $c = 0$  yields  $y = 2x^2 - x$ ;  $c = -2$  yields  $y = 3x^2 - 2x - 2$ .

**3.5.21.** No.

**3.5.23.** Yes.  $\vec{x} = -2\vec{w}_1 + 2\vec{w}_2$ .

**3.5.25.** Yes.

**3.5.27.** No.

## 3.6 • Vector Solutions to Linear Systems

### 3.6 • Exercises

**3.6.1.** Multiply  $A\vec{u}$  and  $A\vec{v}$  to verify.

**3.6.3.** Multiply  $A\vec{u}$  and  $A\vec{v}$  to verify.

**3.6.5.** Multiply  $A\vec{u}$  and  $A\vec{v}$  to verify.

**3.6.7.** Multiply  $A\vec{u}$ ,  $A\vec{v}$  and  $A(\vec{u} + \vec{v})$  to verify.

**3.6.9.** Multiply  $A\vec{u}$ ,  $A\vec{v}$  and  $A(\vec{u} + \vec{v})$  to verify.

**3.6.11.**

$$(a) \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(b) \vec{x} = \begin{bmatrix} 2/5 \\ -13/5 \end{bmatrix}$$

**3.6.15.**

$$(a) \vec{x} = x_3 \begin{bmatrix} 5/4 \\ 1 \\ 1 \end{bmatrix}$$

$$(b) \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5/4 \\ 1 \\ 1 \end{bmatrix}$$

**3.6.19.**

$$(a) \vec{x} = x_3 \begin{bmatrix} 2 \\ 2/5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 2/5 \\ 0 \\ 1 \end{bmatrix}$$

$$(b) \vec{x} = \begin{bmatrix} -2 \\ 2/5 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 2/5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 2/5 \\ 0 \\ 1 \end{bmatrix}$$

**3.6.13.**

$$(a) \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(b) \vec{x} = \begin{bmatrix} -2 \\ -9/4 \end{bmatrix}$$

**3.6.17.**

$$(a) \vec{x} = x_3 \begin{bmatrix} 6 \\ -4 \\ 1 \end{bmatrix}$$

$$(b) \vec{x} = \begin{bmatrix} -12 \\ 8 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ -4 \\ 1 \end{bmatrix}$$

**3.6.21.**

$$(a) \vec{x} = x_2 \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 13/2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

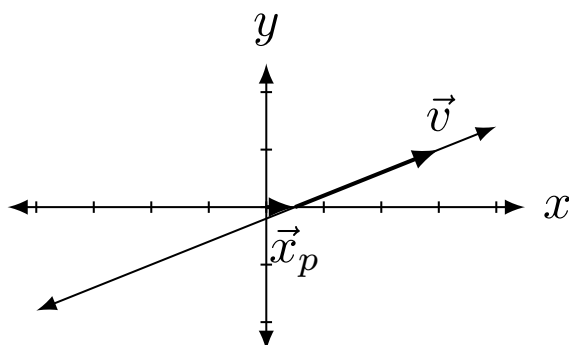
$$(b) \vec{x} = \begin{bmatrix} -5 \\ 0 \\ 3/2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 13/2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

3.6.23.

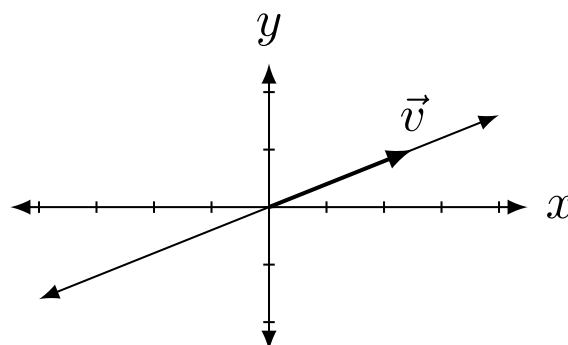
$$(a) \vec{x} = x_4 \begin{bmatrix} 1 \\ 13/9 \\ -1/3 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$(b) \vec{x} = \begin{bmatrix} 1 \\ 1/9 \\ 5/3 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 13/9 \\ -1/3 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$3.6.25. \vec{x} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2.5 \\ 1 \end{bmatrix} = \vec{x}_p + x_2 \vec{v}$$



$$3.6.27. \vec{x} = x_2 \begin{bmatrix} 2.5 \\ 1 \end{bmatrix} = x_2 \vec{v}$$



## 4 • Matrix Algebra

### 4.1 • Matrix Addition and Scalar Multiplication

#### 4.1 • Exercises

$$4.1.1. \begin{bmatrix} -2 & -1 \\ 12 & 13 \end{bmatrix}$$

$$4.1.3. \begin{bmatrix} 2 & -2 \\ 14 & 8 \end{bmatrix}$$

$$4.1.5. \begin{bmatrix} 9 & -7 \\ 11 & -6 \end{bmatrix}$$

$$4.1.7. \begin{bmatrix} -14 \\ 6 \end{bmatrix}$$

$$4.1.9. \begin{bmatrix} -15 \\ -25 \end{bmatrix}$$

$$4.1.11. X = \begin{bmatrix} -5 & 91 & -14 \end{bmatrix}$$

$$4.1.13. X = \begin{bmatrix} -5 & -29/2 & -19/2 \end{bmatrix}$$

$$4.1.15. a = 2, b = 1$$

$$4.1.17. a = 5/2 + 3/2b$$

4.1.19. No solution.

4.1.21. No solution.

### 4.2 • Matrix Multiplication

#### 4.2 • Exercises

$$4.2.1. -22$$

$$4.2.3. 0$$

$$4.2.5. 5$$

$$4.2.7. 15$$

$$4.2.9. -2$$

$$4.2.11. \text{Not possible.}$$

$$\mathbf{4.2.13.} \quad AB = \begin{bmatrix} 8 & 3 \\ 10 & -9 \end{bmatrix}, BA = \begin{bmatrix} -3 & 24 \\ 4 & 2 \end{bmatrix}$$

**4.2.17.**  $AB$  is not possible,

$$BA = \begin{bmatrix} 27 & -33 & 39 \\ -27 & -3 & -15 \end{bmatrix}$$

$$\mathbf{4.2.21.} \quad AB = \begin{bmatrix} -56 & 2 & -36 \\ 20 & 19 & -30 \\ -50 & -13 & 0 \end{bmatrix},$$

$$BA = \begin{bmatrix} -46 & 40 \\ 72 & 9 \end{bmatrix}$$

$$\mathbf{4.2.25.} \quad AB = \begin{bmatrix} 0 & 0 & 4 \\ 6 & 4 & -2 \\ 2 & -4 & -6 \end{bmatrix},$$

$$BA = \begin{bmatrix} 2 & -2 & 6 \\ 2 & 2 & 4 \\ 4 & 0 & -6 \end{bmatrix}$$

$$\mathbf{4.2.29.} \quad DA = \begin{bmatrix} 2 & 2 & 2 \\ -6 & -6 & -6 \\ -15 & -15 & -15 \end{bmatrix},$$

$$AD = \begin{bmatrix} 2 & -3 & 5 \\ 4 & -6 & 10 \\ -6 & 9 & -15 \end{bmatrix}$$

$$\mathbf{4.2.33.} \quad DA = \begin{bmatrix} d_1a & d_1b & d_1c \\ d_2d & d_2e & d_2f \\ d_3g & d_3h & d_3i \end{bmatrix},$$

$$AD = \begin{bmatrix} d_1a & d_2b & d_3c \\ d_1d & d_2e & d_3f \\ d_1g & d_2h & d_3i \end{bmatrix}$$

$$\mathbf{4.2.35.} \quad A\vec{x} = \begin{bmatrix} -6 \\ 11 \end{bmatrix}$$

$$\mathbf{4.2.39.} \quad A\vec{x} = \begin{bmatrix} 2x_1 - x_2 \\ 4x_1 + 3x_2 \end{bmatrix}$$

$$\mathbf{4.2.41.} \quad A^2 = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}; A^3 = \begin{bmatrix} 8 & 0 \\ 0 & 27 \end{bmatrix}$$

$$\mathbf{4.2.43.} \quad A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**4.2.45.**

$$(a) \quad \begin{bmatrix} 0 & -2 \\ -5 & -1 \end{bmatrix}$$

$$\mathbf{4.2.15.} \quad AB = \begin{bmatrix} -1 & -2 & 12 \\ 10 & 4 & 32 \end{bmatrix}, BA \text{ is not possible.}$$

$$\mathbf{4.2.19.} \quad AB = \begin{bmatrix} -32 & 34 & -24 \\ -32 & 38 & -8 \\ -16 & 21 & 4 \end{bmatrix},$$

$$BA = \begin{bmatrix} 22 & -14 \\ -4 & -12 \end{bmatrix}$$

$$\mathbf{4.2.23.} \quad AB = \begin{bmatrix} -15 & -22 & -21 & -1 \\ 16 & -53 & -59 & -31 \end{bmatrix}, BA \text{ is not possible.}$$

$$\mathbf{4.2.27.} \quad AB = \begin{bmatrix} 21 & -17 & -5 \\ 19 & 5 & 19 \\ 5 & 9 & 4 \end{bmatrix},$$

$$BA = \begin{bmatrix} 19 & 5 & 23 \\ 5 & -7 & -1 \\ -14 & 6 & 18 \end{bmatrix}$$

$$\mathbf{4.2.31.} \quad DA = \begin{bmatrix} 4 & -6 \\ 4 & -6 \end{bmatrix}, AD = \begin{bmatrix} 4 & 8 \\ -3 & -6 \end{bmatrix}$$

$$\mathbf{4.2.37.} \quad A\vec{x} = \begin{bmatrix} -5 \\ 5 \\ 21 \end{bmatrix}$$

(b)  $\begin{bmatrix} 10 & 2 \\ 5 & 11 \end{bmatrix}$

(c)  $[-11 \quad -15 \ 37 \ 32]$

(d) No.

(e)  $(A+B)(A+B) = AA + AB + BA + BB = A^2 + AB + BA + B^2.$

### 4.3 • Solving Matrix Equations $AX = B$

#### 4.3 • Exercises

4.3.1.  $X = \begin{bmatrix} 0 & -2 & -8 & 6 \end{bmatrix}$

4.3.5.  $X = \begin{bmatrix} -5 & 2 & -3 & -4 & -3 & -2 \end{bmatrix}$

4.3.9.  $X = \begin{bmatrix} 3 & -3 & 3 & 2 & -2 & -3 \\ -3 & -1 & -2 & & & \end{bmatrix}$

4.3.3.  $X = \begin{bmatrix} -1 & 2 & -4 & -6 & -2 & 3 \end{bmatrix}$

4.3.7.  $X = \begin{bmatrix} 1 & -9 & -4 & -5 \end{bmatrix}$

4.3.11.  $X = \begin{bmatrix} -1/2 & -1/2 & 0 & -1/2 & -1 & 1/2 \\ -1/2 & -3/4 & & 3/4 & & \end{bmatrix}$

### 4.4 • The Matrix Inverse

#### 4.4 • Exercises

4.4.5.  $A^{-1}$  does not exist.

4.4.21.  $A^{-1}$  does not exist.

4.4.33.  $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

### 4.5 • Properties of the Matrix Inverse

#### 4.5 • Exercises

4.5.1.  $(AB)^{-1} = \begin{bmatrix} -2 & 3 \\ 1 & -1.4 \end{bmatrix}$

4.5.3.  $(AB)^{-1} = \begin{bmatrix} 29/5 & -18/5 \\ -11/5 & 7/5 \end{bmatrix}$

4.5.5.  $A^{-1} = \begin{bmatrix} -2 & -5 \\ -1 & -3 \end{bmatrix}, (A^{-1})^{-1} = \begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix}$

4.5.7.  $A^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}, (A^{-1})^{-1} = \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}$

4.5.9. Solutions will vary.

4.5.11. Likely some entries that should be 0 will not be exactly 0, but rather very small values.

### 4.6 • Elementary Matrices

#### 4.6 • Exercises

4.6.1.  $R_1 - 3R_2 \rightarrow R_1$

4.6.3.  $R_2 - 4R_1 \rightarrow R_2$

4.6.5.  $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

4.6.7.  $\begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$

4.6.9.  $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} R_1 + 3R_2 \rightarrow R_1$

4.6.11.  $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, R_2 + 4R_1 \rightarrow R_2$

4.6.13. Answers may vary. One possibility:

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

4.6.15. Answers may vary. One possibility:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

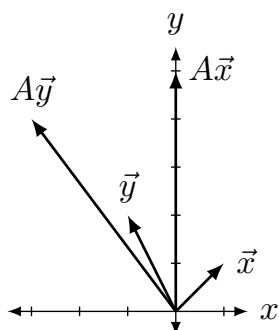
$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

## 5 • Matrix Transformations

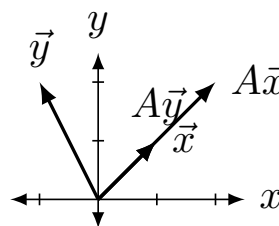
### 5.1 • Matrix Transformations

#### 5.1 • Exercises

5.1.1.



5.1.3.



5.1.5.  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

5.1.7.  $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

5.1.9.  $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

5.1.11.  $A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$

5.1.13.  $A = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$

5.1.15. Yes, these are the same; the transformation matrix in each is  $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ .

5.1.17. Yes, these are the same. Each produces the transformation matrix  $\begin{bmatrix} 1/2 & 0 \\ 0 & 3 \end{bmatrix}$ .

## 5.2 • Properties of Linear Transformations

### 5.2 • Exercises

5.2.1. Yes

5.2.3. No; cannot add a constant.

5.2.5. Yes.

5.2.7.  $[T] = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$

$$5.2.9. [T] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$5.2.11. [T] = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

## 5.3 • Subspaces of $\mathbb{R}^n$

### 5.3 • Exercises

5.3.1. Not a subspace. The vector  $\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$  belongs to  $S$ , but  $2\vec{v}$  does not.

5.3.3. Subspace. If  $y = 2x$ , we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

so  $U$  is equal to the span of the vector  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ , and therefore a subspace.

## 5.4 • Null Space and Column Space

### 5.4 • Exercises

$$5.4.1. \text{null}(A) = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ has dimension 1; } \text{col}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -4 \end{bmatrix} \right\} \text{ has dimension 3;}$$

$4 = 1 + 3.$

$$5.4.3. \text{null}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ has dimension 1; } \text{col}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ -8 \end{bmatrix} \right\} \text{ has dimension 2; } 3 = 1 + 2.$$

$$5.4.5. \text{null}(A) = \text{span} \left\{ \begin{bmatrix} -6 \\ -4 \\ -1 \\ 3 \end{bmatrix} \right\} \text{ has dimension 1; } \text{col}(A) = \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix} \right\} \text{ has dimension 3;}$$

$4 = 1 + 3.$

$$5.4.7. \text{null}(A) = \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -8 \\ 5 \end{bmatrix} \right\} \text{ has dimension 2; } \text{col}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \right\} \text{ has dimension 2;}$$

$4 = 2 + 2.$

## 6 • Operations on Matrices

### 6.1 • The Matrix Transpose

#### 6.1 • Exercises

$$6.1.1. A \text{ is skew symmetric. } \begin{bmatrix} 0 & -6 & 1 \\ 6 & 0 & 4 \\ -1 & -4 & 0 \end{bmatrix}$$

$$6.1.3. \begin{bmatrix} -9 & 6 & -8 \\ 4 & -3 & 1 \\ 10 & -7 & -1 \end{bmatrix}$$

$$6.1.5. A \text{ is diagonal, as is } A^T. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$6.1.7. A \text{ is symmetric. } \begin{bmatrix} 6 & -4 & -5 \\ -4 & 0 & 2 \\ -5 & 2 & -2 \end{bmatrix}$$



$$6.1.9. \begin{bmatrix} -7 \\ -8 \\ 2 \\ -3 \end{bmatrix}$$

$$6.1.13. \begin{bmatrix} 4 & -9 \\ -7 & 6 \\ -4 & 3 \\ -9 & -9 \end{bmatrix}$$

6.1.17.  $A$  is upper triangular;  $A^T$  is lower triangular.

$$\begin{bmatrix} -3 & 0 & 0 \\ -4 & -3 & 0 \\ -5 & 5 & -3 \end{bmatrix}$$

$$6.1.21. \begin{bmatrix} -5 & 3 & -10 \\ -9 & 1 & -8 \end{bmatrix}$$

$$6.1.11. A \text{ is symmetric. } \begin{bmatrix} 13 & -3 \\ -3 & 1 \end{bmatrix}$$

$$6.1.15. \begin{bmatrix} 2 & 5 & 7 \\ -5 & 5 & -4 \\ -3 & -6 & -10 \end{bmatrix}$$

$$6.1.19. A \text{ is symmetric. } \begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & 3 \\ -2 & 3 & 6 \end{bmatrix}$$

$$6.1.23. A \text{ is skew symmetric. } \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -4 \\ -2 & 4 & 0 \end{bmatrix}$$

## 6.2 • The Matrix Trace

### 6.2 • Exercises

6.2.1. Not defined; the matrix must be square.

6.2.5.  $-9$

6.2.9.  $1$

6.2.13.  $0$

6.2.17.

$$(a) \operatorname{tr}(A) = -1; \operatorname{tr}(B) = ; \operatorname{tr}(A + B) = 5$$

$$(b) \operatorname{tr}(AB) = 201 = \operatorname{tr}(BA)$$

6.2.3.  $3$

6.2.7.  $-5$

6.2.11. Not defined; the matrix must be square.

6.2.15.  $n$

6.2.19.

$$(a) \operatorname{tr}(A) = -5; \operatorname{tr}(B) = -4; \operatorname{tr}(A + B) = -9$$

$$(b) \operatorname{tr}(AB) = 23 = \operatorname{tr}(BA)$$

## 6.3 • The Determinant

### 6.3 • Exercises

6.3.1.  $34$

6.3.5.  $-44$

6.3.9.

(a) The submatrices are  $\begin{bmatrix} 7 & 6 & 10 \\ 3 & 6 & 10 \end{bmatrix}$ , and  $\begin{bmatrix} 3 & 7 & 6 \end{bmatrix}$ , respectively.

$$(b) C_{1,2} = 34, C_{1,2} = -24, C_{1,3} = 11$$

6.3.13.  $-59$

6.3.17.  $3$

6.3.21.  $0$

6.3.25. Hint:  $C_{1,1} = d$ .

6.3.3.  $-44$

6.3.7.  $28$

6.3.11.

(a) The submatrices are  $\begin{bmatrix} 3 & 10 \\ 3 & 9 \end{bmatrix}$ ,  $\begin{bmatrix} -3 & 10 & -9 & 9 \end{bmatrix}$ , and  $\begin{bmatrix} -3 & 3 & -9 & 3 \end{bmatrix}$ , respectively.

$$(b) C_{1,2} = -3, C_{1,2} = -63, C_{1,3} = 18$$

6.3.15.  $15$

6.3.19.  $0$

6.3.23.  $-113$

## 6.4 • Properties of the Determinant

### 6.4 • Exercises

6.4.1.  $84$

6.4.3.  $0$

6.4.5. 10  
 6.4.9. 175  
 6.4.13. 34

6.4.15.

- (a)  $\det(A) = 90; 2R_1 \rightarrow R_1$   
 (b)  $\det(B) = 45; 10R_1 + R_3 \rightarrow R_3$   
 (c)  $\det(C) = 45; C = A^T$

6.4.19.  $\det(A) = 4, \det(B) = 4, \det(AB) = 16$

6.4.23.  $-59$   
 6.4.27. 3

6.4.7. 24  
 6.4.11.  $-200$

6.4.17.

- (a)  $\det(A) = -16; R_1 \leftrightarrow R_2$  then  $R_1 \leftrightarrow R_3$   
 (b)  $\det(B) = -16; -R_1 \rightarrow R_1$  and  $-R_2 \rightarrow R_2$   
 (c)  $\det(C) = -432; C = 3 * M$

6.4.21.  $\det(A) = -12, \det(B) = 29,$   
 $\det(AB) = -348$

6.4.25. 15  
 6.4.29. 0

## 6.5 • Applications of the Determinant

### 6.5 • Exercises

6.5.1.

- (a)  $\det(A) = -123, \det(A_1) = -492,$   
 $\det(A_2) = 123, \det(A_3) = 492$

(b)  $\vec{x} = \begin{bmatrix} 4 \\ -1 \\ -4 \end{bmatrix}$

6.5.5.

- (a)  $\det(A) = -43, \det(A_1) = 215, \det(A_2) = 0$

(b)  $\vec{x} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$

6.5.9.

- (a)  $\det(A) = 0, \det(A_1) = 0, \det(A_2) = 0$   
 (b) Infinitely many solutions exist.

6.5.13.  $A^{-1} = \frac{1}{8} \begin{bmatrix} -11 & 10 & 13 \\ 6 & -4 & -2 \\ 9 & -6 & -7 \end{bmatrix}$

6.5.17.  $A^{-1} = \frac{1}{10} \begin{bmatrix} 9 & 5 & -12 & 0 \\ 15 & 13 & -26 & -4 \\ 19 & 13 & -28 & -4 \\ 2 & 4 & -4 & -2 \end{bmatrix}$

6.5.3.

- (a)  $\det(A) = 96, \det(A_1) = -960,$   
 $\det(A_2) = 768, \det(A_3) = 288$

(b)  $\vec{x} = \begin{bmatrix} -10 \\ 8 \\ 3 \end{bmatrix}$

6.5.7.

- (a)  $\det(A) = 16, \det(A_1) = -64, \det(A_2) = 80$

(b)  $\vec{x} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$

6.5.11.

- (a)  $\det(A) = 0, \det(A_1) = 0, \det(A_2) = 0,$   
 $\det(A_3) = 0$   
 (b) Infinitely many solutions exist.

6.5.15.  $A$  is not invertible.

## 7 • Eigenvalues and Eigenvectors

### 7.1 • Eigenvalues and Eigenvectors

#### 7.1 • Exercises

7.1.1.  $\lambda = 3$   
 7.1.5.  $\lambda = -2$

7.1.3.  $\lambda = -5$

$$7.1.7. \vec{x} = \begin{bmatrix} 3 \\ -7 \\ 7 \end{bmatrix}$$

$$7.1.11. \vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$7.1.13. \lambda_1 = 2 \text{ with } \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \lambda_2 = 3 \text{ with}$$

$$\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; \lambda_3 = 7 \text{ with } \vec{x}_3 = \begin{bmatrix} -1 \\ 15 \\ 10 \end{bmatrix}$$

$$7.1.17. \lambda_1 = -4 \text{ with } \vec{x}_1 = \begin{bmatrix} -6 \\ 1 \\ 11 \end{bmatrix}; \lambda_2 = -1 \text{ with}$$

$$\vec{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \lambda_3 = 5 \text{ with } \vec{x}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$7.1.21. \lambda_1 = -5 \text{ with } \vec{x}_1 = \begin{bmatrix} 24 \\ 13 \\ 8 \end{bmatrix}; \lambda_2 = -2 \text{ with}$$

$$\vec{x}_2 = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}; \lambda_3 = 3 \text{ with } \vec{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$7.1.25. \lambda_1 = -5 \text{ with } \vec{x}_1 = \begin{bmatrix} -1 \\ 5 \end{bmatrix}; \lambda_2 = 5 \text{ with}$$

$$\vec{x}_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$7.1.9. \vec{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$7.1.15. \lambda_1 = -2 \text{ with } \vec{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \lambda_2 = 1 \text{ with}$$

$$\vec{x}_2 = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}; \lambda_3 = 5 \text{ with } \vec{x}_3 = \begin{bmatrix} 28 \\ 7 \\ 1 \end{bmatrix}$$

$$7.1.19. \lambda_1 = -3 \text{ with } \vec{x}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}; \lambda_2 = 5 \text{ with}$$

$$\vec{x}_2 = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

$$7.1.23. \lambda_1 = -5 \text{ with } \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \lambda_2 = 2 \text{ with}$$

$$\vec{x}_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$7.1.27. \lambda_1 = 4 \text{ with } \vec{x}_1 = \begin{bmatrix} 9 \\ 1 \end{bmatrix}; \lambda_2 = 5 \text{ with } \vec{x}_2 = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

## 7.2 • Properties of Eigenvalues and Eigenvectors

### 7.2 • Exercises

7.2.1.

(a)  $\lambda_1 = 1$  with  $\vec{x}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ ;  $\lambda_2 = 4$  with  $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b)  $\lambda_1 = 1$  with  $\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ;  $\lambda_2 = 4$  with

$$\vec{x}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

(c)  $\lambda_1 = 1/4$  with  $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ;  $\lambda_2 = 1$  with

$$\vec{x}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

(d) 5

(e) 4

7.2.3.

(a)  $\lambda_1 = -1$  with  $\vec{x}_1 = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$ ;  $\lambda_2 = 0$  with

$$\vec{x}_2 = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

(b)  $\lambda_1 = -1$  with  $\vec{x}_1 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ ;  $\lambda_2 = 0$  with  $\vec{x}_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

(c)  $A$  is not invertible.

(d)  $-1$

(e) 0

## 7.2.5.

$$(a) \lambda_1 = -4 \text{ with } \vec{x}_1 = \begin{bmatrix} -7 \\ -7 \\ 6 \end{bmatrix}; \lambda_2 = 3 \text{ with}$$

$$\vec{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \lambda_3 = 4 \text{ with } \vec{x}_3 = \begin{bmatrix} 9 \\ 1 \\ 22 \end{bmatrix}$$

$$(b) \lambda_1 = -4 \text{ with } \vec{x}_1 = \begin{bmatrix} -1 \\ 9 \\ 0 \end{bmatrix}; \lambda_2 = 3 \text{ with}$$

$$\vec{x}_2 = \begin{bmatrix} -20 \\ 26 \\ 7 \end{bmatrix}; \lambda_3 = 4 \text{ with } \vec{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$(c) \lambda_1 = -1/4 \text{ with } \vec{x}_1 = \begin{bmatrix} -7 \\ -7 \\ 6 \end{bmatrix}; \lambda_2 = 1/3 \text{ with}$$

$$\vec{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \lambda_3 = 1/4 \text{ with } \vec{x}_3 = \begin{bmatrix} 9 \\ 1 \\ 22 \end{bmatrix}$$

$$(d) 3$$

$$(e) -48$$

## 7.3 • Eigenvalues and Diagonalization

## 7.3 • Exercises

$$7.3.1. c_A(x) = (x - 7)(x - 5)(x + 3)$$

Eigenvalue  $\lambda_1 = 7$  has algebraic and geometric multiplicity 1.

Eigenvalue  $\lambda_2 = 5$  has algebraic and geometric multiplicity 1.

Eigenvalue  $\lambda_3 = -3$  has algebraic and geometric multiplicity 1.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$7.3.5. c_A(x) = (x + 1)^2(x - 2)$$

Eigenvalue  $\lambda_1 = -1$  has algebraic and geometric multiplicity 2.

Eigenvalue  $\lambda_2 = 2$  has algebraic and geometric multiplicity 1.

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$7.3.3. c_A(x) = (x - 6)(x - 23)$$

Eigenvalue  $\lambda_1 = 6$  has algebraic and geometric multiplicity 1.

Eigenvalue  $\lambda_2 = 23$  has algebraic and geometric multiplicity 1.

$$P = \begin{bmatrix} 7 & -1 \\ 3 & 2 \end{bmatrix}.$$

$$7.3.7. c_A(x) = (x - 1)^2(x - 2)$$

Eigenvalue  $\lambda_1 = 1$  has algebraic multiplicity 2 and geometric multiplicity 1.

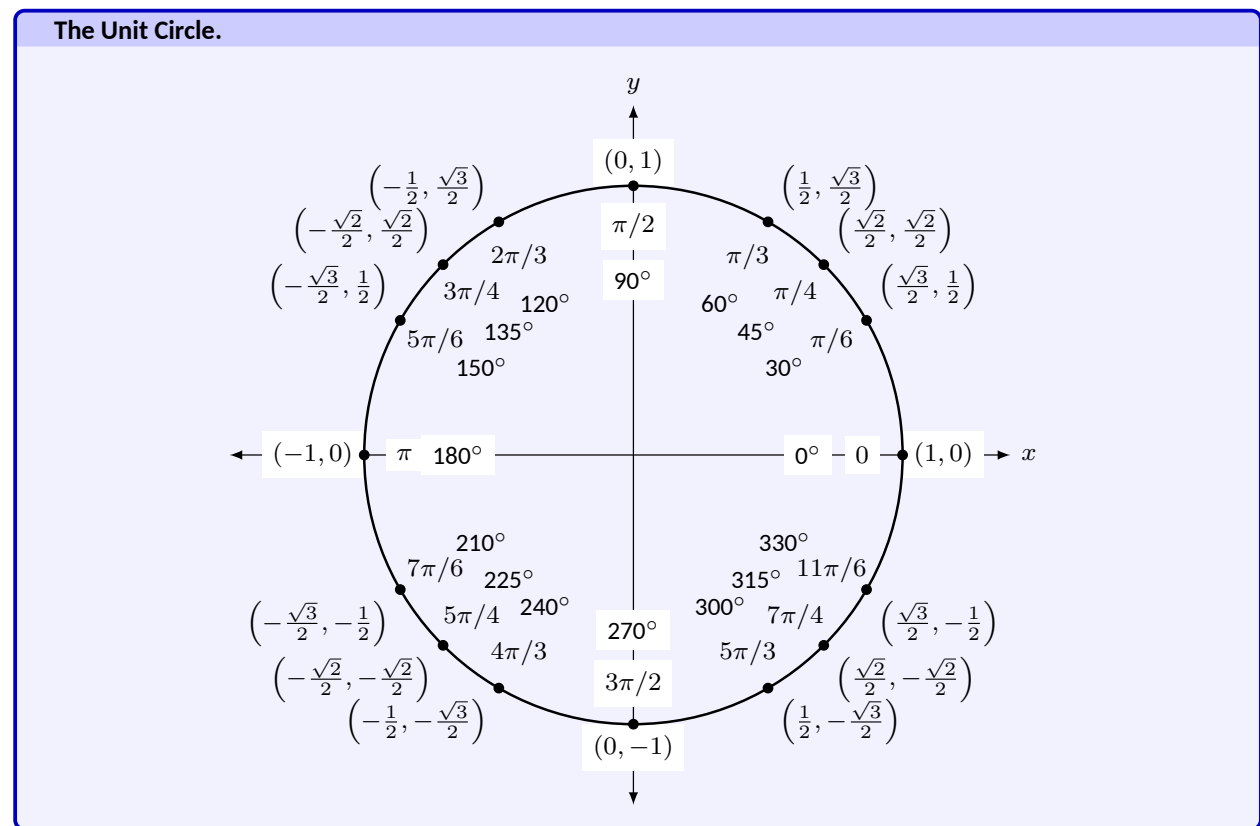
Eigenvalue  $\lambda_2 = 2$  has algebraic and geometric multiplicity 1.

Since the geometric multiplicity of  $\lambda_1$  is less than its algebraic multiplicity, no such  $P$  exists; the matrix cannot be diagonalized.

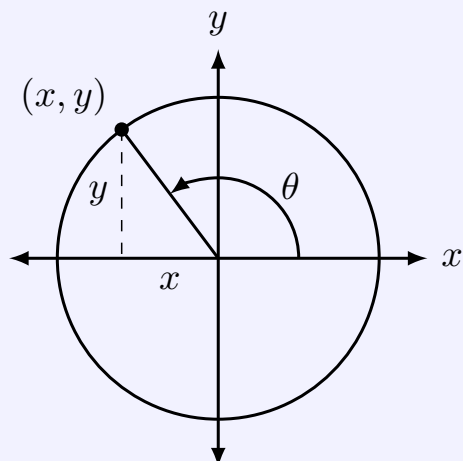
## Appendix B

# Quick Reference

### B.1 Trigonometry Reference



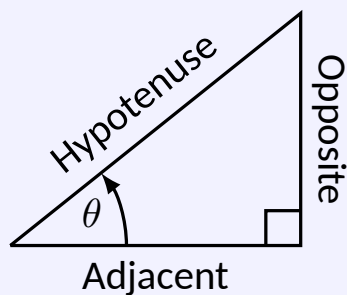
#### B.1.1 Definitions of the Trigonometric Functions

**Unit Circle Definition.**

$$\sin \theta = y \quad \cos \theta = x$$

$$\csc \theta = \frac{1}{y} \quad \sec \theta = \frac{1}{x}$$

$$\tan \theta = \frac{y}{x} \quad \cot \theta = \frac{x}{y}$$

**Right Triangle Definition.**

$$\sin \theta = \frac{O}{H} \quad \csc \theta = \frac{H}{O}$$

$$\cos \theta = \frac{A}{H} \quad \sec \theta = \frac{H}{A}$$

$$\tan \theta = \frac{O}{A} \quad \cot \theta = \frac{A}{O}$$

**B.1.2 Common Trigonometric Identities**

$$1. \sin^2 x + \cos^2 x = 1$$

$$2. \tan^2 x + 1 = \sec^2 x$$

$$3. 1 + \cot^2 x = \csc^2 x$$

**List B.1.1 Pythagorean Identities**

$$1. \sin 2x = 2 \sin x \cos x$$

2.

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= 2 \cos^2 x - 1$$

$$= 1 - 2 \sin^2 x$$

$$3. \tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

**List B.1.2 Double Angle Formulas**

1.  $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
2.  $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
3.  $\tan\left(\frac{\pi}{2} - x\right) = \cot x$
4.  $\csc\left(\frac{\pi}{2} - x\right) = \sec x$
5.  $\sec\left(\frac{\pi}{2} - x\right) = \csc x$
6.  $\cot\left(\frac{\pi}{2} - x\right) = \tan x$

**List B.1.3 Cofunction Identities**

1.  $\sin^2 x = \frac{1 - \cos 2x}{2}$
2.  $\cos^2 x = \frac{1 + \cos 2x}{2}$
3.  $\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$

**List B.1.5 Power-Reducing Formulas**

1.  $\sin(-x) = -\sin x$
2.  $\cos(-x) = \cos x$
3.  $\tan(-x) = -\tan x$
4.  $\csc(-x) = -\csc x$
5.  $\sec(-x) = \sec x$
6.  $\cot(-x) = -\cot x$

**List B.1.4 Even/Odd Identities**

1.  $\sin x + \sin y = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$
2.  $\sin x - \sin y = 2 \sin\left(\frac{x-y}{2}\right) \cos\left(\frac{x+y}{2}\right)$
3.  $\cos x + \cos y = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$  =
4.  $\cos x - \cos y = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$  =

**List B.1.6 Sum to Product Formulas****List B.1.7 Product to Sum Formulas**

1.  $\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$
2.  $\cos x \cos y = \frac{1}{2}(\cos(x-y) + \cos(x+y))$
3.  $\sin x \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$

**List B.1.8 Angle Sum/Difference Formulas**

1.  $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
2.  $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
3.  $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

## B.2 Areas and Volumes

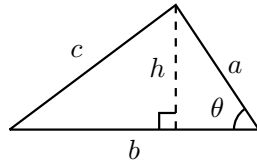
### Triangles

$$h = a \sin \theta$$

$$\text{Area} = \frac{1}{2}bh$$

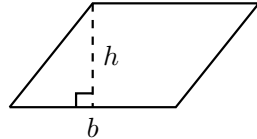
Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$



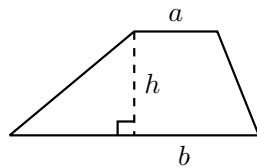
### Parallelograms

$$\text{Area} = bh$$



### Trapezoids

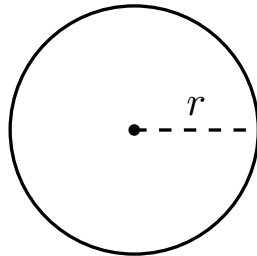
$$\text{Area} = \frac{1}{2}(a + b)h$$



### Circles

$$\text{Area} = \pi r^2$$

$$\text{Circumference} = 2\pi r$$

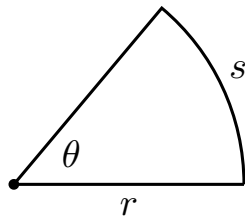


### Sectors of Circles

$\theta$  in radians

$$\text{Area} = \frac{1}{2}\theta r^2$$

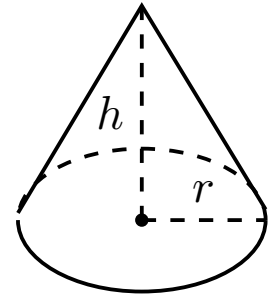
$$s = r\theta$$



### Right Circular Cone

$$\text{Volume} = \frac{1}{3}\pi r^2 h$$

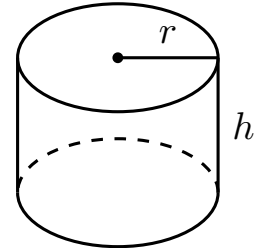
$$\text{Surface Area} = \pi r \sqrt{r^2 + h^2} + \pi r^2$$



### Right Circular Cylinder

$$\text{Volume} = \pi r^2 h$$

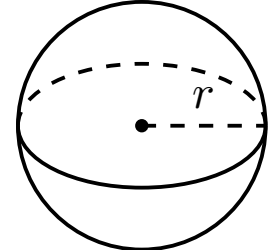
$$\text{Surface Area} = 2\pi r h + 2\pi r^2$$



### Sphere

$$\text{Volume} = \frac{4}{3}\pi r^3$$

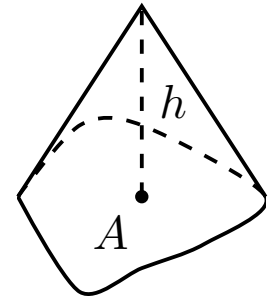
$$\text{Surface Area} = 4\pi r^2$$



### General Cone

$$\text{Area of Base} = A$$

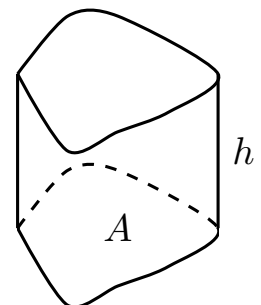
$$\text{Volume} = \frac{1}{3}Ah$$



### General Right Cylinder

$$\text{Area of Base} = A$$

$$\text{Volume} = Ah$$





## B.3 Algebra

### Factors and Zeros of Polynomials.

Let  $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$  be a polynomial. If  $p(a) = 0$ , then  $a$  is a *zero* of the polynomial and a solution of the equation  $p(x) = 0$ . Furthermore,  $(x - a)$  is a *factor* of the polynomial.

### Fundamental Theorem of Algebra.

An  $n$ th degree polynomial has  $n$  (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

### Quadratic Formula.

If  $p(x) = ax^2 + bx + c$ , and  $0 \leq b^2 - 4ac$ , then the real zeros of  $p$  are  $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

### Special Factors.

$$x^2 - a^2 = (x - a)(x + a)$$

$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$

$$x^4 - a^4 = (x^2 - a^2)(x^2 + a^2)$$

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n$$

$$(x - y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \cdots \pm nxy^{n-1} \mp y^n$$

### Binomial Theorem.

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x - y)^2 = x^2 - 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$

### Rational Zero Theorem.

If  $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$  has integer coefficients, then every *rational zero* of  $p$  is of the form  $x = r/s$ , where  $r$  is a factor of  $a_0$  and  $s$  is a factor of  $a_n$ .

### Factoring by Grouping.

$$acx^3 + adx^2 + bcx + bd = ax^2(cx + d) + b(cx + d) = (ax^2 + b)(cx + d)$$

**Arithmetic Operations.**

$$ab + ac = a(b + c)$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right) \left(\frac{d}{c}\right) = \frac{ad}{bc}$$

$$a \left(\frac{b}{c}\right) = \frac{ab}{c}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc}$$

$$\frac{a - b}{c - d} = \frac{b - a}{d - c}$$

$$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

$$\frac{ab + ac}{a} = b + c$$

**Exponents and Radicals.**

$$a^0 = 1, a \neq 0$$

$$\frac{a^x}{a^y} = a^{x-y}$$

$$a^{-x} = \frac{1}{a^x}$$

$$(ab)^x = a^x b^x$$

$$\sqrt[n]{a} = a^{1/n}$$

$$\sqrt[n]{ab} = \sqrt[n]{a} \sqrt[n]{b}$$

$$a^x a^y = a^{x+y}$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

$$(a^x)^y = a^{xy}$$

$$\sqrt{a} = a^{1/2}$$

$$\sqrt[n]{a^m} = a^{m/n}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

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