Math 2570 Calculus III

Fall 2018 Edition, University of Lethbridge
An adaptation of the APEX Calculus textbook, edited by Sean Fitzpatrick

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Preface

This a custom textbook that covers the entire curriculum for the course Math 2570 (Calculus III) at the University of Lethbridge at minimal cost to the student. It is also an *Open Education Resource*. As a student, you are free to keep as many copies as you want, for as long as you want. You can print it, in whole or in part, or share it with a friend. As an instructor, I am free to modify the content as I see fit, whether this means editing to fit our curriculum, or simply correcting typos.

Most of this textbook is adapted from the *APEX Calculus* textbook project, which originated in the Department of Applied Mathematics at the Virginia Military Institute. (See apexcalculus.com.) On the following page you'll find the original preface from their text, which explains their project in more detail. They have produced calculus textbook that is **free** in two regards: it's free to download from their website, and the authors have made all the files needed to produce the textbook freely available, allowing others (such as myself) to edit the text to suit the needs of various courses (such as Math 2570).

What's even better is that the textbook is of remarkably high production quality: unlike many free texts, it is polished and professionally produced, with graphics on almost every page, and a large collection of exercises (with selected answers!).

I hope that you find this textbook useful. If you find any errors, or if you have any suggestions as to how the material could be better arranged or presented, please let me know. (The great thing about an open source textbook is that it can be edited at any time!) In particular, if you find a particular topic that you think needs further explanation, or more examples, or more exercises, please let us know. My hope is that this text will be improved every time it is used for this course.

Sean Fitzpatrick
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May, 2018

Preface to APEX Calculus

A Note on Using this Text

Thank you for reading this short preface. Allow us to share a few key points about the text so that you may better understand what you will find beyond this page.

This text is Part I of a three–text series on Calculus. The first part covers material taught in many "Calc 1" courses: limits, derivatives, and the basics of integration, found in Chapters 1 through 6.1. The second text covers material often taught in "Calc 2:" integration and its applications, along with an introduction to sequences, series and Taylor Polynomials, found in Chapters 5 through 8. The third text covers topics common in "Calc 3" or "multivariable calc:" parametric equations, polar coordinates, vector–valued functions, and functions of more than one variable, found in Chapters 9 through 13. All three are available separately for free at `www.apexcalculus.com`. These three texts are intended to work together and make one cohesive text, *APEX Calculus*, which can also be downloaded from the website.

Printing the entire text as one volume makes for a large, heavy, cumbersome book. One can certainly only print the pages they currently need, but some prefer to have a nice, bound copy of the text. Therefore this text has been split into these three manageable parts, each of which can be purchased for under $15 at Amazon.com.

A result of this splitting is that sometimes a concept is said to be explored in a "later section," though that section does not actually appear in this particular text. Also, the index makes reference to topics and page numbers that do not appear in this text. This is done intentionally to show the reader what topics are available for study. Downloading the .pdf of *APEX Calculus* will ensure that you have all the content.

For Students: How to Read this Text

Mathematics textbooks have a reputation for being hard to read. High–level mathematical writing often seeks to say much with few words, and this style often seeps into texts of lower–level topics. This book was written with the goal of being easier to read than many other calculus textbooks, without becoming too verbose.

Each chapter and section starts with an introduction of the coming material, hopefully setting the stage for "why you should care," and ends with a look ahead to see how the just–learned material helps address future problems.

Please read the text; it is written to explain the concepts of Calculus. There are numerous examples to demonstrate the meaning of definitions, the truth of theorems, and the application of mathematical techniques. When you encounter a sentence you don't understand, read it again. If it still doesn't make sense, read on anyway, as sometimes confusing sentences are explained by later sentences.

You don't have to read every equation. The examples generally show "all" the steps needed to solve a problem. Sometimes reading through each step is helpful; sometimes it is confusing. When the steps are illustrating a new technique, one probably should follow each step closely to learn the new technique. When the steps are showing the mathematics needed to find a number to be used later, one can usually skip ahead and see how that number is being used, instead of getting bogged down in reading how the number was found.

Most proofs have been omitted. In mathematics, *proving* something is always true is extremely important, and entails much more than testing to see if it works twice. However, students often are confused by the details of a proof, or become concerned that they should have been able to construct this proof on their own. To alleviate this potential problem, we do not include the proofs to most theorems in the text. The interested reader is highly encouraged to find proofs online or from their instructor. In most cases, one is very capable of understanding what a theorem *means* and *how to apply it* without knowing fully *why* it is true.

Interactive, 3D Graphics

New to Version 3.0 is the addition of interactive, 3D graphics in the .pdf version. Nearly all graphs of objects in space can be rotated, shifted, and zoomed in/out so the reader can better understand the object illustrated.

As of this writing, the only pdf viewers that support these 3D graphics are Adobe Reader & Acrobat (and only the versions for PC/Mac/Unix/Linux computers, not tablets or smartphones). To activate the interactive mode, click on the image. Once activated, one can click/drag to rotate the object and use the scroll wheel on a mouse to zoom in/out. (A great way to investigate an image is to first zoom in on the page of the pdf viewer so the graphic itself takes up much of the screen, then zoom inside the graphic itself.) A CTRL-click/drag pans the object left/right or up/down. By right-clicking on the graph one can access a menu of other options, such as changing the lighting scheme or perspective. One can also revert the graph back to its default view. If you wish to deactivate the interactivity, one can right-click and choose the "Disable Content" option.

Thanks

There are many people who deserve recognition for the important role they have played in the development of this text. First, I thank Michelle for her support and encouragement, even as this "project from work" occupied my time and attention at home. Many thanks to Troy Siemers, whose most important contributions extend far beyond the sections he wrote or the 227 figures he coded in Asymptote for 3D interaction. He provided incredible support, advice and encouragement for which I am very grateful. My thanks to Brian Heinold and Dimplekumar Chalishajar for their contributions and to Jennifer Bowen for reading through so much material and providing great feedback early on. Thanks to Troy, Lee Dewald, Dan Joseph, Meagan Herald, Bill Lowe, John David, Vonda Walsh, Geoff Cox, Jessica Libertini and other faculty of VMI who have given me numerous suggestions and corrections based on their experience with teaching from the text. (Special thanks to Troy, Lee & Dan for their patience in teaching Calc III while I was still writing the Calc III material.) Thanks to Randy Cone for encouraging his tutors of VMI's Open Math Lab to read through the text and check the solutions, and thanks to the tutors for spending their time doing so. A very special thanks to Kristi Brown and Paul Janiczek who took this opportunity far above & beyond what I expected, meticulously checking every solution and carefully reading every example. Their comments have been extraordinarily helpful. I am also thankful for the support provided by Wane Schneiter, who as my Dean provided me with extra time to work on this project. I am blessed to have so many people give of their time to make this book better.

APEX – Affordable Print and Electronic teXts

APEX is a consortium of authors who collaborate to produce high–quality, low–cost textbooks. The current textbook–writing paradigm is facing a potential revolution as desktop publishing and electronic formats increase in popularity. However, writing a good textbook is no easy task, as the time requirements alone are substantial. It takes countless hours of work to produce text, write examples and exercises, edit and publish. Through collaboration, however, the cost to any individual can be lessened, allowing us to create texts that we freely distribute electronically and sell in printed form for an incredibly low cost. Having said that, nothing is entirely free; someone always bears some cost. This text "cost" the authors of this book their time, and that was not enough. *APEX Calculus* would not exist had not the Virginia Military Institute, through a generous Jackson–Hope grant, given the lead author significant time away from teaching so he could focus on this text.

We encourage others to adapt this work to fit their own needs. One might add sections that are "missing" or remove sections that your students won't need. The source files can be found at `github.com/APEXCalculus`.

You can learn more at `www.vmi.edu/APEX`.

Version 4.0

Key changes from Version 3.0 to 4.0:

- Numerous typographical and "small" mathematical corrections (again, thanks to all my close readers!).
- "Large" mathematical corrections and adjustments. There were a number of places in Version 3.0 where a definition/theorem was not correct as stated. See `www.apexcalculus.com` for more information.
- More useful numbering of Examples, Theorems, etc. "Definition 11.4.2" refers to the second definition of Chapter 11, Section 4.
- The addition of Section 13.7: Triple Integration with Cylindrical and Spherical Coordinates
- The addition of Chapter 14: Vector Analysis.

10: Sequences and Series

This chapter introduces **sequences** and **series**, important mathematical constructions that are useful when solving a large variety of mathematical problems. The content of this chapter is considerably different from the content of the chapters before it. While the material we learn here definitely falls under the scope of "calculus," we will make very little use of derivatives or integrals. Limits are extremely important, though, especially limits that involve infinity.

Notation: We use $\mathbb{N}$ to describe the set of natural numbers, that is, the integers 1, 2, 3, ...

One of the problems addressed by this chapter is this: suppose we know information about a function and its derivatives at a point, such as $f(1) = 3$, $f'(1) = 1$, $f''(1) = -2$, $f'''(1) = 7$, and so on. What can I say about $f(x)$ itself? Is there any reasonable approximation of the value of $f(2)$? The topic of Taylor Series addresses this problem, and allows us to make excellent approximations of functions when limited knowledge of the function is available.

Factorial: The expression 4! refers to the number $4 \cdot 3 \cdot 2 \cdot 1 = 24$.

In general, $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$, where n is a natural number.

We define $0! = 1$. While this does not immediately make sense, it makes many mathematical formulas work properly.

10.1 Sequences

We commonly refer to a set of events that occur one after the other as a *sequence* of events. In mathematics, we use the word *sequence* to refer to an ordered set of numbers, i.e., a set of numbers that "occur one after the other."

For instance, the numbers 2, 4, 6, 8, ..., form a sequence. The order is important; the first number is 2, the second is 4, etc. It seems natural to seek a formula that describes a given sequence, and often this can be done. For instance, the sequence above could be described by the function $a(n) = 2n$, for the values of $n = 1, 2, \ldots$ To find the 10^{th} term in the sequence, we would compute $a(10)$. This leads us to the following, formal definition of a sequence.

Definition 10.1.1 Sequence

A **sequence** is a function $a(n)$ whose domain is $\mathbb{N}$. The **range** of a sequence is the set of all distinct values of $a(n)$.

The **terms** of a sequence are the values $a(1)$, $a(2)$, ..., which are usually denoted with subscripts as a_1, a_2, ...

A sequence $a(n)$ is often denoted as $\{a_n\}$.

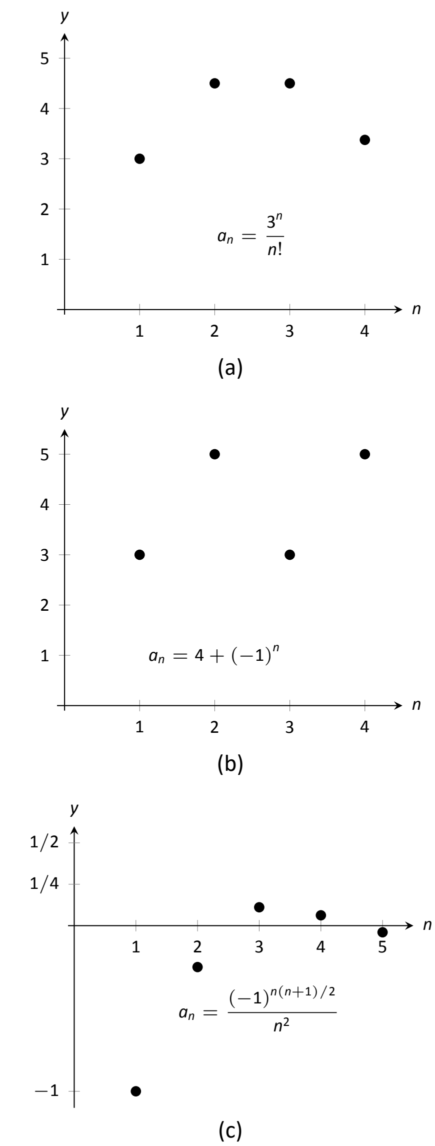


Figure 10.1.1: Plotting sequences in Example 10.1.1.

Example 10.1.1 Listing terms of a sequence

List the first four terms of the following sequences.

1. $\{a_n\} = \left\{\dfrac{3^n}{n!}\right\}$ 2. $\{a_n\} = \{4+(-1)^n\}$ 3. $\{a_n\} = \left\{\dfrac{(-1)^{n(n+1)/2}}{n^2}\right\}$

SOLUTION

1. $a_1 = \dfrac{3^1}{1!} = 3;\quad a_2 = \dfrac{3^2}{2!} = \dfrac{9}{2};\quad a_3 = \dfrac{3^3}{3!} = \dfrac{9}{2};\quad a_4 = \dfrac{3^4}{4!} = \dfrac{27}{8}$

   We can plot the terms of a sequence with a scatter plot. The "x"-axis is used for the values of n, and the values of the terms are plotted on the y-axis. To visualize this sequence, see Figure 10.1.1(a).

2. $a_1 = 4 + (-1)^1 = 3;\quad a_2 = 4 + (-1)^2 = 5;$

   $a_3 = 4+(-1)^3 = 3;\quad a_4 = 4+(-1)^4 = 5.$ Note that the range of this sequence is finite, consisting of only the values 3 and 5. This sequence is plotted in Figure 10.1.1(b).

3. $a_1 = \dfrac{(-1)^{1(2)/2}}{1^2} = -1;\quad a_2 = \dfrac{(-1)^{2(3)/2}}{2^2} = -\dfrac{1}{4}$

   $a_3 = \dfrac{(-1)^{3(4)/2}}{3^2} = \dfrac{1}{9}\quad a_4 = \dfrac{(-1)^{4(5)/2}}{4^2} = \dfrac{1}{16};$

   $a_5 = \dfrac{(-1)^{5(6)/2}}{5^2} = -\dfrac{1}{25}.$

   We gave one extra term to begin to show the pattern of signs is "$-, -, +, +, -, -, \ldots$" due to the fact that the exponent of -1 is a special quadratic. This sequence is plotted in Figure 10.1.1(c).

Example 10.1.2 Determining a formula for a sequence

Find the n^{th} term of the following sequences, i.e., find a function that describes each of the given sequences.

1. 2, 5, 8, 11, 14, . . .
2. 2, −5, 10, −17, 26, −37, . . .
3. 1, 1, 2, 6, 24, 120, 720, . . .
4. $\dfrac{5}{2}, \dfrac{5}{2}, \dfrac{15}{8}, \dfrac{5}{4}, \dfrac{25}{32}, \ldots$

SOLUTION We should first note that there is never exactly one function that describes a finite set of numbers as a sequence. There are many sequences that start with 2, then 5, as our first example does. We are looking for a simple formula that describes the terms given, knowing there is possibly more than one answer.

1. Note how each term is 3 more than the previous one. This implies a linear function would be appropriate: $a(n) = a_n = 3n + b$ for some appropriate value of b. As we want $a_1 = 2$, we set $b = -1$. Thus $a_n = 3n - 1$.

2. First notice how the sign changes from term to term. This is most commonly accomplished by multiplying the terms by either $(-1)^n$ or $(-1)^{n+1}$. Using $(-1)^n$ multiplies the odd terms by (-1); using $(-1)^{n+1}$ multiplies

the even terms by (-1). As this sequence has negative even terms, we will multiply by $(-1)^{n+1}$.

After this, we might feel a bit stuck as to how to proceed. At this point, we are just looking for a pattern of some sort: what do the numbers 2, 5, 10, 17, etc., have in common? There are many correct answers, but the one that we'll use here is that each is one more than a perfect square. That is, $2 = 1^2 + 1$, $5 = 2^2 + 1$, $10 = 3^2 + 1$, etc. Thus our formula is $a_n = (-1)^{n+1}(n^2 + 1)$.

3. One who is familiar with the factorial function will readily recognize these numbers. They are 0!, 1!, 2!, 3!, etc. Since our sequences start with $n = 1$, we cannot write $a_n = n!$, for this misses the 0! term. Instead, we shift by 1, and write $a_n = (n-1)!$.

4. This one may appear difficult, especially as the first two terms are the same, but a little "sleuthing" will help. Notice how the terms in the numerator are always multiples of 5, and the terms in the denominator are always powers of 2. Does something as simple as $a_n = \frac{5n}{2^n}$ work?

   When $n = 1$, we see that we indeed get $5/2$ as desired. When $n = 2$, we get $10/4 = 5/2$. Further checking shows that this formula indeed matches the other terms of the sequence.

A common mathematical endeavour is to create a new mathematical object (for instance, a sequence) and then apply previously known mathematics to the new object. We do so here. The fundamental concept of calculus is the limit, so we will investigate what it means to find the limit of a sequence.

Definition 10.1.2 Limit of a Sequence, Convergent, Divergent

Let $\{a_n\}$ be a sequence and let L be a real number. Given any $\varepsilon > 0$, if an m can be found such that $|a_n - L| < \varepsilon$ for all $n > m$, then we say the **limit of $\{a_n\}$, as n approaches infinity, is** L, denoted

$$\lim_{n\to\infty} a_n = L.$$

If $\lim\limits_{n\to\infty} a_n$ exists, we say the sequence **converges**; otherwise, the sequence **diverges**.

This definition states, informally, that if the limit of a sequence is L, then if you go far enough out along the sequence, all subsequent terms will be *really close* to L. Of course, the terms "far enough" and "really close" are subjective terms, but hopefully the intent is clear.

This definition is reminiscent of the ε–δ proofs of Chapter 1. In that chapter we developed other tools to evaluate limits apart from the formal definition; we do so here as well.

Theorem 10.1.1 Limit of a Sequence

Let $\{a_n\}$ be a sequence and let $f(x)$ be a function whose domain contains the positive real numbers where $f(n) = a_n$ for all n in $\mathbb{N}$.

If $\lim\limits_{x\to\infty} f(x) = L$, then $\lim\limits_{n\to\infty} a_n = L$.

Theorem 10.1.1 allows us, in certain cases, to apply the tools developed in Chapter 1 to limits of sequences. Note two things *not* stated by the theorem:

1. If $\lim\limits_{x\to\infty} f(x)$ does not exist, we cannot conclude that $\lim\limits_{n\to\infty} a_n$ does not exist. It may, or may not, exist. For instance, we can define a sequence $\{a_n\} = \{\cos(2\pi n)\}$. Let $f(x) = \cos(2\pi x)$. Since the cosine function oscillates over the real numbers, the limit $\lim\limits_{x\to\infty} f(x)$ does not exist.

   However, for every positive integer n, $\cos(2\pi n) = 1$, so $\lim\limits_{n\to\infty} a_n = 1$.

2. If we cannot find a function $f(x)$ whose domain contains the positive real numbers where $f(n) = a_n$ for all n in $\mathbb{N}$, we cannot conclude $\lim\limits_{n\to\infty} a_n$ does not exist. It may, or may not, exist.

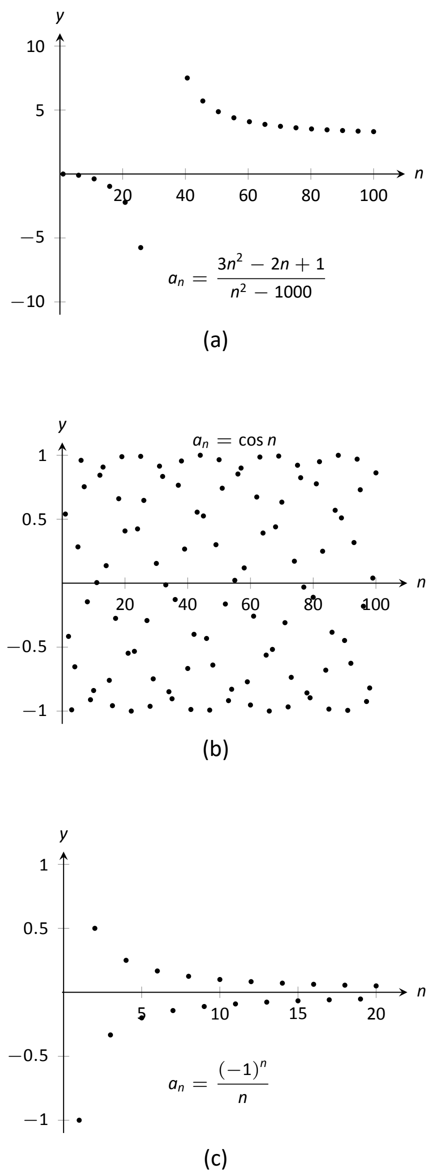


Figure 10.1.2: Scatter plots of the sequences in Example 10.1.3.

Example 10.1.3 Determining convergence/divergence of a sequence

Determine the convergence or divergence of the following sequences.

1. $\{a_n\} = \left\{\dfrac{3n^2-2n+1}{n^2-1000}\right\}$ 2. $\{a_n\} = \{\cos n\}$ 3. $\{a_n\} = \left\{\dfrac{(-1)^n}{n}\right\}$

SOLUTION

1. Using Theorem 1.5.1, we can state that $\lim\limits_{x\to\infty} \dfrac{3x^2-2x+1}{x^2-1000} = 3$. (We could have also directly applied l'Hospital's Rule.) Thus the sequence $\{a_n\}$ converges, and its limit is 3. A scatter plot of every 5 values of a_n is given in Figure 10.1.2 (a). The values of a_n vary widely near $n = 30$, ranging from about -73 to 125, but as n grows, the values approach 3.

2. The limit $\lim\limits_{x\to\infty} \cos x$ does not exist, as $\cos x$ oscillates (and takes on every value in $[-1, 1]$ infinitely many times). Thus we cannot apply Theorem 10.1.1.

   The fact that the cosine function oscillates strongly hints that $\cos n$, when n is restricted to $\mathbb{N}$, will also oscillate. Figure 10.1.2 (b), where the sequence is plotted, implies that this is true. Because only discrete values of cosine are plotted, it does not bear strong resemblance to the familiar cosine wave. The proof of the following statement is beyond the scope of this text, but it is true: there are infinitely many integers n that are arbitrarily (i.e., *very*) close to an even multiple of π, so that $\cos n \approx 1$. Similarly, there are infinitely many integers m that are arbitrarily close to an odd multiple of π, so that $\cos m \approx -1$. As the sequence takes on values near 1 and -1 infinitely many times, we conclude that $\lim\limits_{n\to\infty} a_n$ does not exist.

3. We cannot actually apply Theorem 10.1.1 here, as the function $f(x) = (-1)^x/x$ is not well defined. (What does $(-1)^{\sqrt{2}}$ mean? In actuality, there is an answer, but it involves *complex analysis*, beyond the scope of this text.)

   Instead, we invoke the definition of the limit of a sequence. By looking at the plot in Figure 10.1.2 (c), we would like to conclude that the sequence converges to $L = 0$. Let $\varepsilon > 0$ be given. We can find a natural number m

such that $1/m < \varepsilon$. Let $n > m$, and consider $|a_n - L|$:

$$\begin{aligned} |a_n - L| &= \left|\frac{(-1)^n}{n} - 0\right| \\ &= \frac{1}{n} \\ &< \frac{1}{m} \quad \text{(since } n > m\text{)} \\ &< \varepsilon. \end{aligned}$$

We have shown that by picking m large enough, we can ensure that a_n is arbitrarily close to our limit, $L = 0$, hence by the definition of the limit of a sequence, we can say $\lim\limits_{n\to\infty} a_n = 0$.

In the previous example we used the definition of the limit of a sequence to determine the convergence of a sequence as we could not apply Theorem 10.1.1. In general, we like to avoid invoking the definition of a limit, and the following theorem gives us tool that we could use in that example instead.

Theorem 10.1.2 Absolute Value Theorem

Let $\{a_n\}$ be a sequence. If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$

Example 10.1.4 Determining the convergence/divergence of a sequence

Determine the convergence or divergence of the following sequences.

1. $\{a_n\} = \left\{\dfrac{(-1)^n}{n}\right\}$ 2. $\{a_n\} = \left\{\dfrac{(-1)^n(n+1)}{n}\right\}$

SOLUTION

1. This appeared in Example 10.1.3. We want to apply Theorem 10.1.2, so consider the limit of $\{|a_n|\}$:

$$\begin{aligned} \lim_{n\to\infty} |a_n| &= \lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| \\ &= \lim_{n\to\infty} \frac{1}{n} \\ &= 0. \end{aligned}$$

Since this limit is 0, we can apply Theorem 10.1.2 and state that $\lim\limits_{n\to\infty} a_n = 0$.

2. Because of the alternating nature of this sequence (i.e., every other term is multiplied by -1), we cannot simply look at the limit $\lim\limits_{x\to\infty} \dfrac{(-1)^x(x+1)}{x}$. We can try to apply the techniques of Theorem 10.1.2:

$$\begin{aligned} \lim_{n\to\infty} |a_n| &= \lim_{n\to\infty} \left|\frac{(-1)^n(n+1)}{n}\right| \\ &= \lim_{n\to\infty} \frac{n+1}{n} \\ &= 1. \end{aligned}$$

We have concluded that when we ignore the alternating sign, the sequence approaches 1. This means we cannot apply Theorem 10.1.2; it states the the limit must be 0 in order to conclude anything.

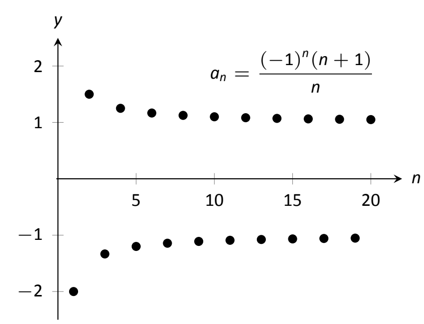


Figure 10.1.3: A plot of a sequence in Example 10.1.4, part 2.

Since we know that the signs of the terms alternate *and* we know that the limit of $|a_n|$ is 1, we know that as n approaches infinity, the terms will alternate between values close to 1 and -1, meaning the sequence diverges. A plot of this sequence is given in Figure 10.1.3.

We continue our study of the limits of sequences by considering some of the properties of these limits.

Theorem 10.1.3 Properties of the Limits of Sequences

Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim\limits_{n\to\infty} a_n = L$, $\lim\limits_{n\to\infty} b_n = K$, and let c be a real number.

1. $\lim\limits_{n\to\infty} (a_n \pm b_n) = L \pm K$
2. $\lim\limits_{n\to\infty} (a_n \cdot b_n) = L \cdot K$
3. $\lim\limits_{n\to\infty} (a_n / b_n) = L/K$, $K \neq 0$
4. $\lim\limits_{n\to\infty} c \cdot a_n = c \cdot L$

Example 10.1.5 Applying properties of limits of sequences

Let the following sequences, and their limits, be given:

- $\{a_n\} = \left\{\dfrac{n+1}{n^2}\right\}$, and $\lim\limits_{n\to\infty} a_n = 0$;
- $\{b_n\} = \left\{\left(1+\dfrac{1}{n}\right)^n\right\}$, and $\lim\limits_{n\to\infty} b_n = e$; and
- $\{c_n\} = \{n \cdot \sin(5/n)\}$, and $\lim\limits_{n\to\infty} c_n = 5$.

Evaluate the following limits.

1. $\lim\limits_{n\to\infty} (a_n + b_n)$ 2. $\lim\limits_{n\to\infty} (b_n \cdot c_n)$ 3. $\lim\limits_{n\to\infty} (1000 \cdot a_n)$

SOLUTION We will use Theorem 10.1.3 to answer each of these.

1. Since $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} b_n = e$, we conclude that $\lim\limits_{n\to\infty} (a_n + b_n) = 0 + e = e$. So even though we are adding something to each term of the sequence b_n, we are adding something so small that the final limit is the same as before.

2. Since $\lim\limits_{n\to\infty} b_n = e$ and $\lim\limits_{n\to\infty} c_n = 5$, we conclude that $\lim\limits_{n\to\infty} (b_n \cdot c_n) = e \cdot 5 = 5e$.

3. Since $\lim\limits_{n\to\infty} a_n = 0$, we have $\lim\limits_{n\to\infty} 1000a_n = 1000 \cdot 0 = 0$. It does not matter that we multiply each term by 1000; the sequence still approaches 0. (It just takes longer to get close to 0.)

There is more to learn about sequences than just their limits. We will also study their range and the relationships terms have with the terms that follow. We start with some definitions describing properties of the range.

Definition 10.1.3 Bounded and Unbounded Sequences

A sequence $\{a_n\}$ is said to be **bounded** if there exist real numbers m and M such that $m < a_n < M$ for all n in $\mathbb{N}$.

A sequence $\{a_n\}$ is said to be **unbounded** if it is not bounded.

A sequence $\{a_n\}$ is said to be **bounded above** if there exists an M such that $a_n < M$ for all n in $\mathbb{N}$; it is **bounded below** if there exists an m such that $m < a_n$ for all n in $\mathbb{N}$.

It follows from this definition that an unbounded sequence may be bounded above or bounded below; a sequence that is both bounded above and below is simply a bounded sequence.

Example 10.1.6 Determining boundedness of sequences

Determine the boundedness of the following sequences.

1. $\{a_n\} = \left\{\dfrac{1}{n}\right\}$ 2. $\{a_n\} = \{2^n\}$

Solution

1. The terms of this sequence are always positive but are decreasing, so we have $0 < a_n < 2$ for all n. Thus this sequence is bounded. Figure 10.1.4(a) illustrates this.

2. The terms of this sequence obviously grow without bound. However, it is also true that these terms are all positive, meaning $0 < a_n$. Thus we can say the sequence is unbounded, but also bounded below. Figure 10.1.4(b) illustrates this.

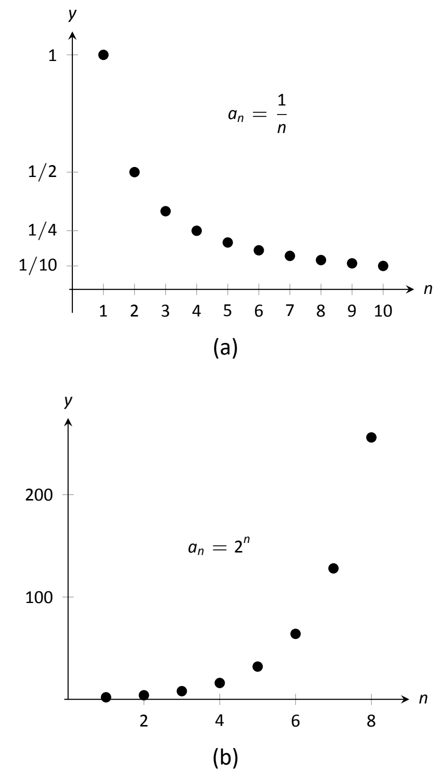


Figure 10.1.4: A plot of $\{a_n\} = \{1/n\}$ and $\{a_n\} = \{2^n\}$ from Example 10.1.6.

The previous example produces some interesting concepts. First, we can recognize that the sequence $\{1/n\}$ converges to 0. This says, informally, that "most" of the terms of the sequence are "really close" to 0. This implies that the sequence is bounded, using the following logic. First, "most" terms are near 0, so we could find some sort of bound on these terms (using Definition 10.1.2, the bound is ε). That leaves a "few" terms that are not near 0 (i.e., a *finite* number of terms). A finite list of numbers is always bounded.

This logic implies that if a sequence converges, it must be bounded. This is indeed true, as stated by the following theorem.

Theorem 10.1.4 Convergent Sequences are Bounded

Let $\{a_n\}$ be a convergent sequence. Then $\{a_n\}$ is bounded.

Note: Keep in mind what Theorem 10.1.4 does *not* say. It does not say that bounded sequences must converge, nor does it say that if a sequence does not converge, it is not bounded.

In Example 10.1.5 we saw the sequence $\{b_n\} = \left\{(1 + 1/n)^n\right\}$, where it was stated that $\lim\limits_{n\to\infty} b_n = e$. (Note that this is simply restating part of Theorem 1.3.5.) Even though it may be difficult to intuitively grasp the behaviour of this sequence, we know immediately that it is bounded.

Another interesting concept to come out of Example 10.1.6 again involves the sequence $\{1/n\}$. We stated, without proof, that the terms of the sequence were decreasing. That is, that $a_{n+1} < a_n$ for all n. (This is easy to show. Clearly $n < n + 1$. Taking reciprocals flips the inequality: $1/n > 1/(n + 1)$. This is the

Note: It is sometimes useful to call a monotonically increasing sequence *strictly increasing* if $a_n < a_{n+1}$ for all n; i.e, we remove the possibility that subsequent terms are equal.
A similar statement holds for *strictly decreasing.*

same as $a_n > a_{n+1}$.) Sequences that either steadily increase or decrease are important, so we give this property a name.

Definition 10.1.4 Monotonic Sequences

1. A sequence $\{a_n\}$ is **monotonically increasing** if $a_n \leq a_{n+1}$ for all n, i.e.,
$$a_1 \leq a_2 \leq a_3 \leq \cdots a_n \leq a_{n+1} \cdots$$
2. A sequence $\{a_n\}$ is **monotonically decreasing** if $a_n \geq a_{n+1}$ for all n, i.e.,
$$a_1 \geq a_2 \geq a_3 \geq \cdots a_n \geq a_{n+1} \cdots$$
3. A sequence is **monotonic** if it is monotonically increasing or monotonically decreasing.

Example 10.1.7 Determining monotonicity

Determine the monotonicity of the following sequences.

1. $\{a_n\} = \left\{\dfrac{n+1}{n}\right\}$
2. $\{a_n\} = \left\{\dfrac{n^2+1}{n+1}\right\}$
3. $\{a_n\} = \left\{\dfrac{n^2-9}{n^2-10n+26}\right\}$
4. $\{a_n\} = \left\{\dfrac{n^2}{n!}\right\}$

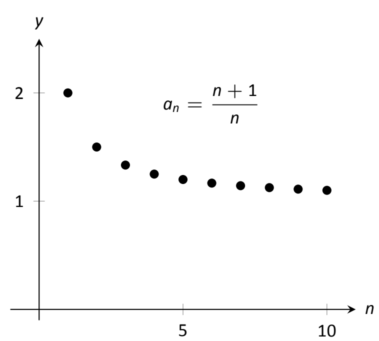


Figure 10.1.5: A plot of $\{a_n\} = \{(n+1)/n\}$ in Example 10.1.7.

SOLUTION In each of the following, we will examine $a_{n+1}-a_n$. If $a_{n+1}-a_n > 0$, we conclude that $a_n < a_{n+1}$ and hence the sequence is increasing. If $a_{n+1}-a_n < 0$, we conclude that $a_n > a_{n+1}$ and the sequence is decreasing. Of course, a sequence need not be monotonic and perhaps neither of the above will apply.

We also give a scatter plot of each sequence. These are useful as they suggest a pattern of monotonicity, but analytic work should be done to confirm a graphical trend.

1.
$$\begin{aligned} a_{n+1}-a_n &= \frac{n+2}{n+1} - \frac{n+1}{n} \\ &= \frac{(n+2)(n)-(n+1)^2}{(n+1)n} \\ &= \frac{-1}{n(n+1)} \\ &< 0 \quad \text{for all } n. \end{aligned}$$

Since $a_{n+1}-a_n < 0$ for all n, we conclude that the sequence is decreasing.

2.
$$\begin{aligned} a_{n+1}-a_n &= \frac{(n+1)^2+1}{n+2} - \frac{n^2+1}{n+1} \\ &= \frac{\big((n+1)^2+1\big)(n+1)-(n^2+1)(n+2)}{(n+1)(n+2)} \\ &= \frac{n^2+4n+1}{(n+1)(n+2)} \\ &> 0 \quad \text{for all } n. \end{aligned}$$

Since $a_{n+1} - a_n > 0$ for all n, we conclude the sequence is increasing; see Figure 10.1.6(a).

3. We can clearly see in Figure 10.1.6(b), where the sequence is plotted, that it is not monotonic. However, it does seem that after the first 4 terms it is decreasing. To understand why, perform the same analysis as done before:

$$\begin{aligned} a_{n+1} - a_n &= \frac{(n+1)^2 - 9}{(n+1)^2 - 10(n+1) + 26} - \frac{n^2 - 9}{n^2 - 10n + 26} \\ &= \frac{n^2 + 2n - 8}{n^2 - 8n + 17} - \frac{n^2 - 9}{n^2 - 10n + 26} \\ &= \frac{(n^2 + 2n - 8)(n^2 - 10n + 26) - (n^2 - 9)(n^2 - 8n + 17)}{(n^2 - 8n + 17)(n^2 - 10n + 26)} \\ &= \frac{-10n^2 + 60n - 55}{(n^2 - 8n + 17)(n^2 - 10n + 26)}. \end{aligned}$$

We want to know when this is greater than, or less than, 0. The denominator is always positive, therefore we are only concerned with the numerator. For small values of n, the numerator is positive. As n grows large, the numerator is dominated by $-10n^2$, meaning the entire fraction will be negative; i.e., for large enough n, $a_{n+1} - a_n < 0$. Using the quadratic formula we can determine that the numerator is negative for $n \geq 5$.

In short, the sequence is simply not monotonic, though it is useful to note that for $n \geq 5$, the sequence is monotonically decreasing.

4. Again, the plot in Figure 10.1.6(c) shows that the sequence is not monotonic, but it suggests that it is monotonically decreasing after the first term. We perform the usual analysis to confirm this.

$$\begin{aligned} a_{n+1} - a_n &= \frac{(n+1)^2}{(n+1)!} - \frac{n^2}{n!} \\ &= \frac{(n+1)^2 - n^2(n+1)}{(n+1)!} \\ &= \frac{-n^3 + 2n + 1}{(n+1)!} \end{aligned}$$

When $n = 1$, the above expression is > 0; for $n \geq 2$, the above expression is < 0. Thus this sequence is not monotonic, but it is monotonically decreasing after the first term.

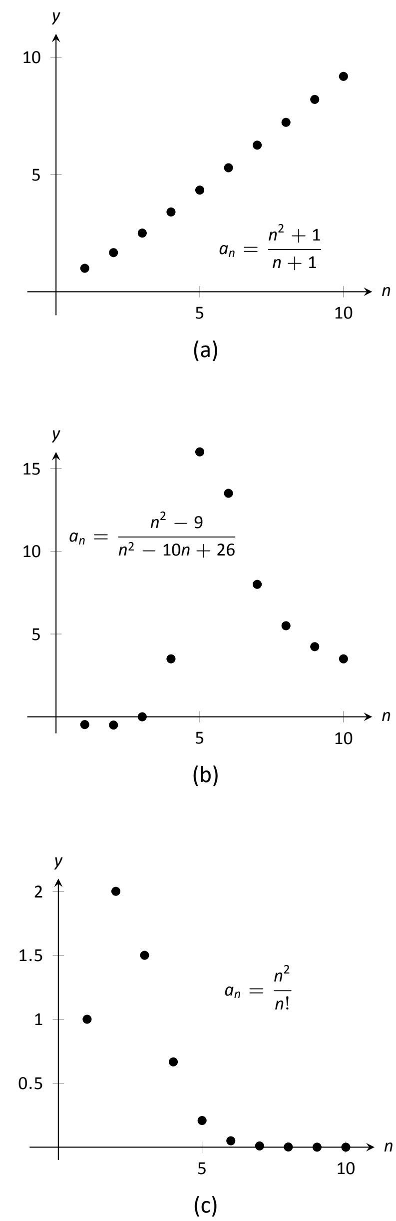


Figure 10.1.6: Plots of sequences in Example 10.1.7.

Knowing that a sequence is monotonic can be useful. Consider, for example, a sequence that is monotonically decreasing and is bounded below. We know the sequence is always getting smaller, but that there is a bound to how small it can become. This is enough to prove that the sequence will converge, as stated in the following theorem.

Theorem 10.1.5 Bounded Monotonic Sequences are Convergent

1. Let $\{a_n\}$ be a monotonically increasing sequence that is bounded above. Then $\{a_n\}$ converges.
2. Let $\{a_n\}$ be a monotonically decreasing sequence that is bounded below. Then $\{a_n\}$ converges.

Consider once again the sequence $\{a_n\} = \{1/n\}$. It is easy to show it is monotonically decreasing and that it is always positive (i.e., bounded below by 0). Therefore we can conclude by Theorem 10.1.5 that the sequence converges. We already knew this by other means, but in the following section this theorem will become very useful.

We can replace Theorem 10.1.5 with the statement "Let $\{a_n\}$ be a bounded, monotonic sequence. Then $\{a_n\}$ converges; i.e., $\lim\limits_{n\to\infty} a_n$ exists." We leave it to the reader in the exercises to show the theorem and the above statement are equivalent.

Sequences are a great source of mathematical inquiry. The On-Line Encyclopedia of Integer Sequences (`http://oeis.org`) contains thousands of sequences and their formulae. (As of this writing, there are 297,573 sequences in the database.) Perusing this database quickly demonstrates that a single sequence can represent several different "real life" phenomena.

Interesting as this is, our interest actually lies elsewhere. We are more interested in the *sum* of a sequence. That is, given a sequence $\{a_n\}$, we are very interested in $a_1 + a_2 + a_3 + \cdots$. Of course, one might immediately counter with "Doesn't this just add up to 'infinity'?" Many times, yes, but there are many important cases where the answer is no. This is the topic of *series*, which we begin to investigate in the next section.

Exercises 10.1

Terms and Concepts

1. Use your own words to define a *sequence.*

2. The domain of a sequence is the ________ numbers.

3. Use your own words to describe the *range* of a sequence.

4. Describe what it means for a sequence to be *bounded*.

Problems

In Exercises 5 – 8, give the first five terms of the given sequence.

5. $\{a_n\} = \left\{\dfrac{4^n}{(n+1)!}\right\}$

6. $\{b_n\} = \left\{\left(-\dfrac{3}{2}\right)^n\right\}$

7. $\{c_n\} = \left\{-\dfrac{n^{n+1}}{n+2}\right\}$

8. $\{d_n\} = \left\{\dfrac{1}{\sqrt{5}}\left(\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right)\right\}$

In Exercises 9 – 12, determine the n^{th} term of the given sequence.

9. 4, 7, 10, 13, 16, ...

10. $3,\ -\dfrac{3}{2},\ \dfrac{3}{4},\ -\dfrac{3}{8},\ \ldots$

11. $10,\ 20,\ 40,\ 80,\ 160,\ \ldots$

12. $1, 1,\ \dfrac{1}{2},\ \dfrac{1}{6},\ \dfrac{1}{24},\ \dfrac{1}{120},\ \ldots$

In Exercises 13 – 16, use the following information to determine the limit of the given sequences.

- $\{a_n\} = \left\{\dfrac{2^n - 20}{2^n}\right\}$; $\quad \lim_{n\to\infty} a_n = 1$
- $\{b_n\} = \left\{\left(1+\dfrac{2}{n}\right)^n\right\}$; $\quad \lim_{n\to\infty} b_n = e^2$
- $\{c_n\} = \{\sin(3/n)\}$; $\quad \lim_{n\to\infty} c_n = 0$

13. $\{a_n\} = \left\{\dfrac{2^n - 20}{7\cdot 2^n}\right\}$

14. $\{a_n\} = \{3b_n - a_n\}$

15. $\{a_n\} = \left\{\sin(3/n)\left(1+\dfrac{2}{n}\right)^n\right\}$

16. $\{a_n\} = \left\{\left(1+\dfrac{2}{n}\right)^{2n}\right\}$

In Exercises 17 – 28, determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.

17. $\{a_n\} = \left\{(-1)^n \dfrac{n}{n+1}\right\}$

18. $\{a_n\} = \left\{\dfrac{4n^2 - n + 5}{3n^2 + 1}\right\}$

19. $\{a_n\} = \left\{\dfrac{4^n}{5^n}\right\}$

20. $\{a_n\} = \left\{\dfrac{n-1}{n} - \dfrac{n}{n-1}\right\}, n \geq 2$

21. $\{a_n\} = \{\ln(n)\}$

22. $\{a_n\} = \left\{\dfrac{3n}{\sqrt{n^2+1}}\right\}$

23. $\{a_n\} = \left\{\left(1+\dfrac{1}{n}\right)^n\right\}$

24. $\{a_n\} = \left\{5 - \dfrac{1}{n}\right\}$

25. $\{a_n\} = \left\{\dfrac{(-1)^{n+1}}{n}\right\}$

26. $\{a_n\} = \left\{\dfrac{1.1^n}{n}\right\}$

27. $\{a_n\} = \left\{\dfrac{2n}{n+1}\right\}$

28. $\{a_n\} = \left\{(-1)^n \dfrac{n^2}{2^n - 1}\right\}$

In Exercises 29 – 34, determine whether the sequence is bounded, bounded above, bounded below, or none of the above.

29. $\{a_n\} = \{\sin n\}$

30. $\{a_n\} = \{\tan n\}$

31. $\{a_n\} = \left\{(-1)^n \dfrac{3n-1}{n}\right\}$

32. $\{a_n\} = \left\{\dfrac{3n^2 - 1}{n}\right\}$

33. $\{a_n\} = \{n\cos n\}$

34. $\{a_n\} = \{2^n - n!\}$

In Exercises 35 – 38, determine whether the sequence is monotonically increasing or decreasing. If it is not, determine if there is an m such that it is monotonic for all $n \geq m$.

35. $\{a_n\} = \left\{\dfrac{n}{n+2}\right\}$

36. $\{a_n\} = \left\{\dfrac{n^2 - 6n + 9}{n}\right\}$

37. $\{a_n\} = \left\{(-1)^n \dfrac{1}{n^3}\right\}$

38. $\{a_n\} = \left\{\dfrac{n^2}{2^n}\right\}$

Exercises 39 – 42 explore further the theory of sequences.

39. Prove Theorem 10.1.2; that is, use the definition of the limit of a sequence to show that if $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

40. Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = K$.

    (a) Show that if $a_n < b_n$ for all n, then $L \leq K$.

    (b) Give an example where $L = K$.

41. Prove the Squeeze Theorem for sequences: Let $\{a_n\}$ and $\{b_n\}$ be such that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = L$, and let $\{c_n\}$ be such that $a_n \leq c_n \leq b_n$ for all n. Then $\lim\limits_{n\to\infty} c_n = L$

42. Prove the statement "Let $\{a_n\}$ be a bounded, monotonic sequence. Then $\{a_n\}$ converges; i.e., $\lim\limits_{n\to\infty} a_n$ exists." is equivalent to Theorem 10.1.5. That is,

    (a) Show that if Theorem 10.1.5 is true, then above statement is true, and

    (b) Show that if the above statement is true, then Theorem 10.1.5 is true.

10.2 Infinite Series

Given the sequence $\{a_n\} = \{1/2^n\} = 1/2,\ 1/4,\ 1/8,\ \ldots$, consider the following sums:

$$\begin{array}{ccccc} a_1 & = & 1/2 & = & 1/2 \\ a_1 + a_2 & = & 1/2 + 1/4 & = & 3/4 \\ a_1 + a_2 + a_3 & = & 1/2 + 1/4 + 1/8 & = & 7/8 \\ a_1 + a_2 + a_3 + a_4 & = & 1/2 + 1/4 + 1/8 + 1/16 & = & 15/16 \end{array}$$

In general, we can show that

$$a_1 + a_2 + a_3 + \cdots + a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}.$$

Let S_n be the sum of the first n terms of the sequence $\{1/2^n\}$. From the above, we see that $S_1 = 1/2$, $S_2 = 3/4$, etc. Our formula at the end shows that $S_n = 1 - 1/2^n$.

Now consider the following limit: $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \big(1 - 1/2^n\big) = 1$. This limit can be interpreted as saying something amazing: *the sum of* all *the terms of the sequence* $\{1/2^n\}$ *is 1.*

This example illustrates some interesting concepts that we explore in this section. We begin this exploration with some definitions.

Definition 10.2.1 Infinite Series, n^{th} Partial Sums, Convergence, Divergence

Let $\{a_n\}$ be a sequence.

1. The sum $\displaystyle\sum_{n=1}^{\infty} a_n$ is an **infinite series** (or, simply **series**).
2. Let $S_n = \displaystyle\sum_{i=1}^{n} a_i$; the sequence $\{S_n\}$ is the sequence of n^{th} **partial sums** of $\{a_n\}$.
3. If the sequence $\{S_n\}$ converges to L, we say the series $\displaystyle\sum_{n=1}^{\infty} a_n$ **converges** to L, and we write $\displaystyle\sum_{n=1}^{\infty} a_n = L$.
4. If the sequence $\{S_n\}$ diverges, the series $\displaystyle\sum_{n=1}^{\infty} a_n$ **diverges**.

Using our new terminology, we can state that the series $\displaystyle\sum_{n=1}^{\infty} 1/2^n$ converges, and $\displaystyle\sum_{n=1}^{\infty} 1/2^n = 1$.

We will explore a variety of series in this section. We start with two series that diverge, showing how we might discern divergence.

Example 10.2.1 Showing series diverge

1. Let $\{a_n\} = \{n^2\}$. Show $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

2. Let $\{b_n\} = \{(-1)^{n+1}\}$. Show $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

SOLUTION

1. Consider S_n, the n^{th} partial sum.

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &= 1^2 + 2^2 + 3^2 \cdots + n^2. \end{aligned}$$

By Theorem 5.3.1, this is

$$= \frac{n(n+1)(2n+1)}{6}.$$

Since $\displaystyle\lim_{n\to\infty} S_n = \infty$, we conclude that the series $\displaystyle\sum_{n=1}^{\infty} n^2$ diverges. It is instructive to write $\displaystyle\sum_{n=1}^{\infty} n^2 = \infty$ for this tells us *how* the series diverges: it grows without bound.

A scatter plot of the sequences $\{a_n\}$ and $\{S_n\}$ is given in Figure 10.2.1(a). The terms of $\{a_n\}$ are growing, so the terms of the partial sums $\{S_n\}$ are growing even faster, illustrating that the series diverges.

2. The sequence $\{b_n\}$ starts with 1, -1, 1, -1, ... Consider some of the partial sums S_n of $\{b_n\}$:

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 0 \\ S_3 &= 1 \\ S_4 &= 0 \end{aligned}$$

This pattern repeats; we find that $S_n = \begin{cases} 1 & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$. As $\{S_n\}$ oscillates, repeating 1, 0, 1, 0, ..., we conclude that $\displaystyle\lim_{n\to\infty} S_n$ does not exist, hence $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}$ diverges.

A scatter plot of the sequence $\{b_n\}$ and the partial sums $\{S_n\}$ is given in Figure 10.2.1(b). When n is odd, $b_n = S_n$ so the marks for b_n are drawn oversized to show they coincide.

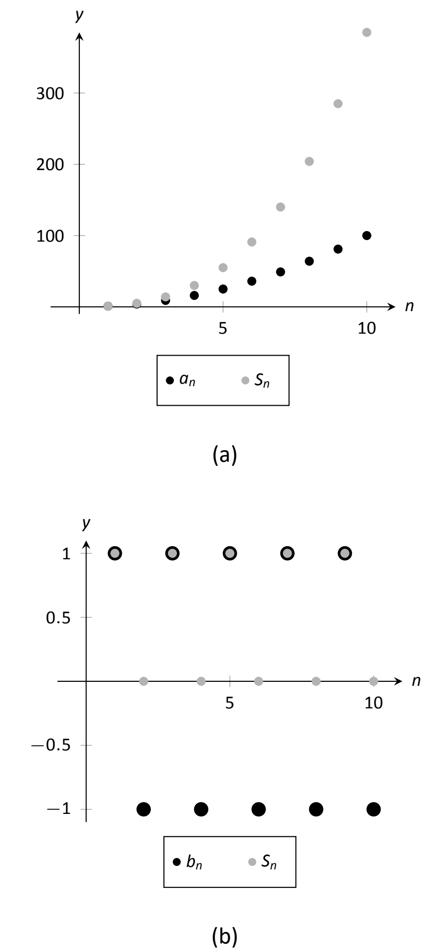


Figure 10.2.1: Scatter plots relating to Example 10.2.1.

While it is important to recognize when a series diverges, we are generally more interested in the series that converge. In this section we will demonstrate a few general techniques for determining convergence; later sections will delve deeper into this topic.

Geometric Series

One important type of series is a *geometric series*.

Definition 10.2.2 Geometric Series

A **geometric series** is a series of the form

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots + r^n + \cdots$$

Note that the index starts at $n = 0$, not $n = 1$.

We started this section with a geometric series, although we dropped the first term of 1. One reason geometric series are important is that they have nice convergence properties.

Theorem 10.2.1 Geometric Series Test

Consider the geometric series $\displaystyle\sum_{n=0}^{\infty} r^n$.

1. The n^{th} partial sum is: $S_n = \dfrac{1 - r^{n+1}}{1 - r}$.
2. The series converges if, and only if, $|r| < 1$. When $|r| < 1$,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

According to Theorem 10.2.1, the series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^2 = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

converges as $r = 1/2$, and $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-1/2} = 2$. This concurs with our introductory example; while there we got a sum of 1, we skipped the first term of 1.

Example 10.2.2 Exploring geometric series

Check the convergence of the following series. If the series converges, find its sum.

1. $\displaystyle\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n$
2. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n$
3. $\displaystyle\sum_{n=0}^{\infty} 3^n$

Solution

1. Since $r = 3/4 < 1$, this series converges. By Theorem 10.2.1, we have that

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{1-3/4} = 4.$$

However, note the subscript of the summation in the given series: we are to start with $n = 2$. Therefore we subtract off the first two terms, giving:

$$\sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^n = 4 - 1 - \frac{3}{4} = \frac{9}{4}.$$

This is illustrated in Figure 10.2.2.

2. Since $|r| = 1/2 < 1$, this series converges, and by Theorem 10.2.1,

$$\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n = \frac{1}{1-(-1/2)} = \frac{2}{3}.$$

The partial sums of this series are plotted in Figure 10.2.3(a). Note how the partial sums are not purely increasing as some of the terms of the sequence $\{(-1/2)^n\}$ are negative.

3. Since $r > 1$, the series diverges. (This makes "common sense"; we expect the sum

$$1 + 3 + 9 + 27 + 81 + 243 + \cdots$$

to diverge.) This is illustrated in Figure 10.2.3(b).

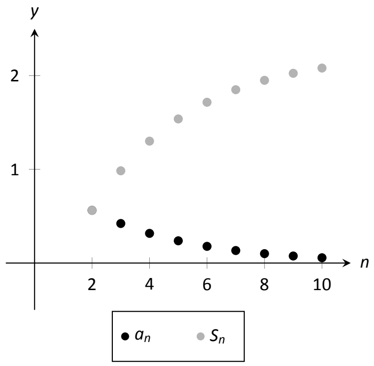


Figure 10.2.2: Scatter plots relating to the series in Example 10.2.2.

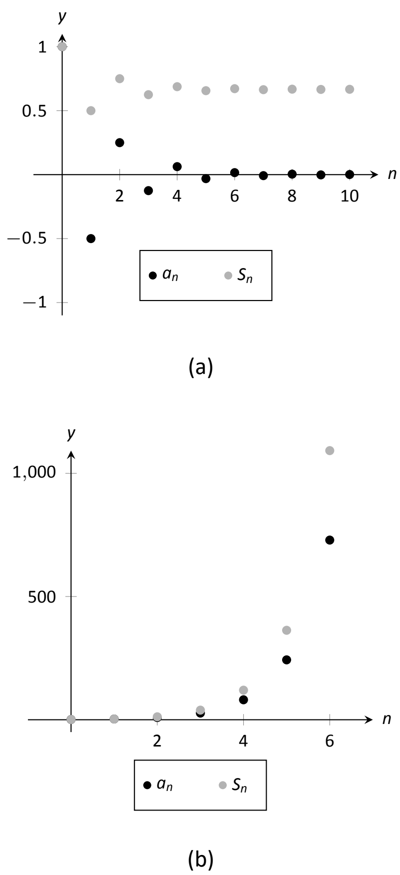


Figure 10.2.3: Scatter plots relating to the series in Example 10.2.2.

p–Series

Another important type of series is the *p-series*.

Definition 10.2.3 $\quad$ ***p*–Series, General *p*–Series**

1. A ***p*–series** is a series of the form

$$\sum_{n=1}^{\infty}\frac{1}{n^p}, \qquad \text{where } p > 0.$$

2. A **general *p*–series** is a series of the form

$$\sum_{n=1}^{\infty}\frac{1}{(an+b)^p}, \qquad \text{where } p > 0 \text{ and } a, b \text{ are real numbers.}$$

Like geometric series, one of the nice things about p–series is that they have easy to determine convergence properties.

Theorem 10.2.2 $\quad$ ***p*–Series Test**

A general p–series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(an+b)^p}$ will converge if, and only if, $p > 1$.

Note: Theorem 10.2.2 assumes that $an + b \neq 0$ for all n. If $an + b = 0$ for some n, then of course the series does not converge regardless of p as not all of the terms of the sequence are defined.

Example 10.2.3 Determining convergence of series

Determine the convergence of the following series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
5. $\displaystyle\sum_{n=11}^{\infty} \frac{1}{(\frac{1}{2}n-5)^3}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

Solution

1. This is a p–series with $p=1$. By Theorem 10.2.2, this series diverges.

   This series is a famous series, called the *Harmonic Series*, so named because of its relationship to *harmonics* in the study of music and sound.

2. This is a p–series with $p=2$. By Theorem 10.2.2, it converges. Note that the theorem does not give a formula by which we can determine *what* the series converges to; we just know it converges. A famous, unexpected result is that this series converges to $\pi^2/6$.

3. This is a p–series with $p=1/2$; the theorem states that it diverges.

4. This is not a p–series; the definition does not allow for alternating signs. Therefore we cannot apply Theorem 10.2.2. (Another famous result states that this series, the *Alternating Harmonic Series*, converges to $\ln 2$.)

5. This is a general p–series with $p=3$, therefore it converges.

6. This is not a p–series, but a geometric series with $r=1/2$. It converges.

Later sections will provide tests by which we can determine whether or not a given series converges. This, in general, is much easier than determining *what* a given series converges to. There are many cases, though, where the sum can be determined.

Example 10.2.4 Telescoping series

Evaluate the sum $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$.

Solution It will help to write down some of the first few partial sums of this series.

$$S_1 = \frac{1}{1}-\frac{1}{2} \qquad = 1-\frac{1}{2}$$
$$S_2 = \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right) \qquad = 1-\frac{1}{3}$$
$$S_3 = \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right) \qquad = 1-\frac{1}{4}$$
$$S_4 = \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right) \qquad = 1-\frac{1}{5}$$

Note how most of the terms in each partial sum are cancelled out! In general, we see that $S_n = 1-\dfrac{1}{n+1}$. The sequence $\{S_n\}$ converges, as $\displaystyle\lim_{n\to\infty} S_n =$

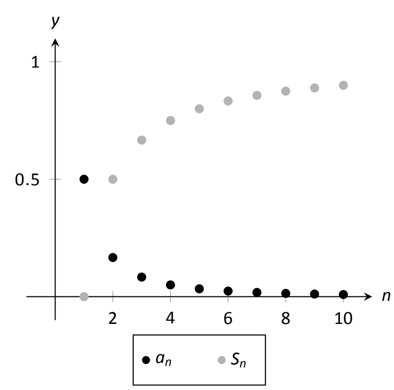


Figure 10.2.4: Scatter plots relating to the series of Example 10.2.4.

$\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1$, and so we conclude that $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=1$. Partial sums of the series are plotted in Figure 10.2.4.

The series in Example 10.2.4 is an example of a **telescoping series**. Informally, a telescoping series is one in which most terms cancel with preceding or following terms, reducing the number of terms in each partial sum. The partial sum S_n did not contain n terms, but rather just two: 1 and $1/(n+1)$.

When possible, seek a way to write an explicit formula for the n^{th} partial sum S_n. This makes evaluating the limit $\lim_{n\to\infty} S_n$ much more approachable. We do so in the next example.

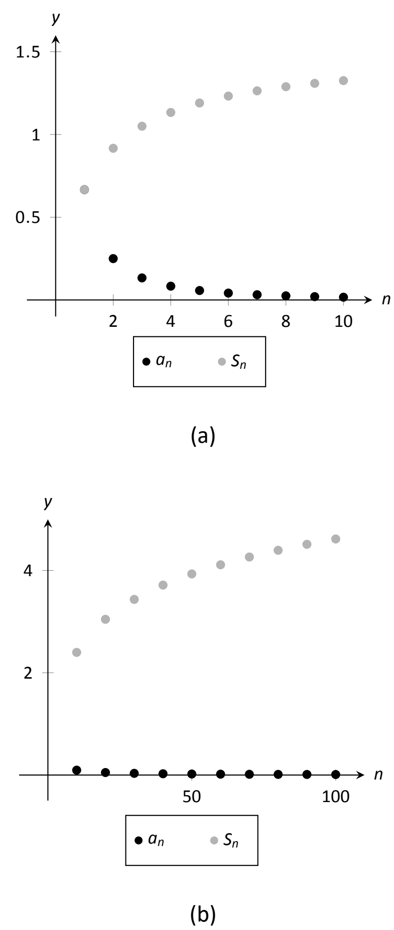


Figure 10.2.5: Scatter plots relating to the series in Example 10.2.5.

Example 10.2.5 Evaluating series

Evaluate each of the following infinite series.

1. $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n^2+2n}$ 2. $\displaystyle\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right)$

Solution

1. We can decompose the fraction $2/(n^2+2n)$ as

$$\frac{2}{n^2+2n}=\frac{1}{n}-\frac{1}{n+2}.$$

(See Section 6.5, Partial Fraction Decomposition, to recall how this is done, if necessary.)

Expressing the terms of $\{S_n\}$ is now more instructive:

$$\begin{aligned}
S_1 &= 1-\frac{1}{3} && = 1-\frac{1}{3}\\
S_2 &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right) && = 1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}\\
S_3 &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right) && = 1+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}\\
S_4 &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right) && = 1+\frac{1}{2}-\frac{1}{5}-\frac{1}{6}\\
S_5 &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right) && = 1+\frac{1}{2}-\frac{1}{6}-\frac{1}{7}
\end{aligned}$$

We again have a telescoping series. In each partial sum, most of the terms cancel and we obtain the formula $S_n=1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}$. Taking limits allows us to determine the convergence of the series:

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{3}{2},\quad\text{so}\quad\sum_{n=1}^{\infty}\frac{1}{n^2+2n}=\frac{3}{2}.$$

This is illustrated in Figure 10.2.5(a).

2. We begin by writing the first few partial sums of the series:

$$\begin{aligned}
S_1 &= \ln(2)\\
S_2 &= \ln(2) + \ln\left(\frac{3}{2}\right)\\
S_3 &= \ln(2) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right)\\
S_4 &= \ln(2) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \ln\left(\frac{5}{4}\right)
\end{aligned}$$

At first, this does not seem helpful, but recall the logarithmic identity: $\ln x + \ln y = \ln(xy)$. Applying this to S_4 gives:

$$S_4 = \ln(2) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \ln\left(\frac{5}{4}\right) = \ln\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\right) = \ln(5).$$

We can conclude that $\{S_n\} = \big\{\ln(n+1)\big\}$. This sequence does not converge, as $\lim\limits_{n\to\infty} S_n = \infty$. Therefore $\sum\limits_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \infty$; the series diverges. Note in Figure 10.2.5(b) how the sequence of partial sums grows slowly; after 100 terms, it is not yet over 5. Graphically we may be fooled into thinking the series converges, but our analysis above shows that it does not.

We are learning about a new mathematical object, the series. As done before, we apply "old" mathematics to this new topic.

Theorem 10.2.3 Properties of Infinite Series

Let $\sum\limits_{n=1}^{\infty} a_n = L$, $\sum\limits_{n=1}^{\infty} b_n = K$, and let c be a constant.

1. Constant Multiple Rule: $\sum\limits_{n=1}^{\infty} c\cdot a_n = c\cdot\sum\limits_{n=1}^{\infty} a_n = c\cdot L$.
2. Sum/Difference Rule: $\sum\limits_{n=1}^{\infty} \big(a_n \pm b_n\big) = \sum\limits_{n=1}^{\infty} a_n \pm \sum\limits_{n=1}^{\infty} b_n = L \pm K$.

Before using this theorem, we provide a few "famous" series.

Key Idea 10.2.1 Important Series

1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = e.$ (Note that the index starts with $n = 0$.)

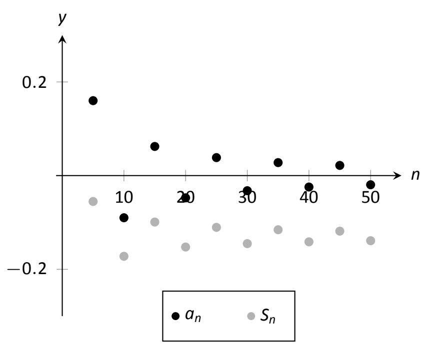
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$

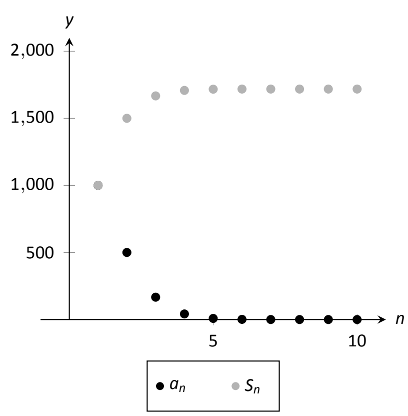
4. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}}{2n+1} = \frac{\pi}{4}.$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (This is called the *Harmonic Series*.)

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$ (This is called the *Alternating Harmonic Series*.)



(a)



(b)

Figure 10.2.6: Scatter plots relating to the series in Example 10.2.6.

Example 10.2.6 Evaluating series

Evaluate the given series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n^2-n)}{n^3}$ 2. $\displaystyle\sum_{n=1}^{\infty} \frac{1000}{n!}$ 3. $\displaystyle\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\frac{1}{49}+\cdots$

Solution

1. We start by using algebra to break the series apart:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n^2-n)}{n^3} &= \sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}n^2}{n^3} - \frac{(-1)^{n+1}n}{n^3}\right)\\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\\ &= \ln(2) - \frac{\pi^2}{12} \approx -0.1293.\end{aligned}$$

This is illustrated in Figure 10.2.6(a).

2. This looks very similar to the series that involves e in Key Idea 10.2.1. Note, however, that the series given in this example starts with $n = 1$ and not $n = 0$. The first term of the series in the Key Idea is $1/0! = 1$, so we will subtract this from our result below:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1000}{n!} &= 1000 \cdot \sum_{n=1}^{\infty} \frac{1}{n!}\\ &= 1000 \cdot (e-1) \approx 1718.28.\end{aligned}$$

This is illustrated in Figure 10.2.6(b). The graph shows how this particular series converges very rapidly.

3. The denominators in each term are perfect squares; we are adding $\displaystyle\sum_{n=4}^{\infty}\frac{1}{n^2}$ (note we start with $n = 4$, not $n = 1$). This series will converge. Using the formula from Key Idea 10.2.1, we have the following:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2} &= \sum_{n=1}^{3}\frac{1}{n^2}+\sum_{n=4}^{\infty}\frac{1}{n^2}\\ \sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{3}\frac{1}{n^2} &= \sum_{n=4}^{\infty}\frac{1}{n^2}\\ \frac{\pi^2}{6}-\left(\frac{1}{1}+\frac{1}{4}+\frac{1}{9}\right) &= \sum_{n=4}^{\infty}\frac{1}{n^2}\\ \frac{\pi^2}{6}-\frac{49}{36} &= \sum_{n=4}^{\infty}\frac{1}{n^2}\\ 0.2838 &\approx \sum_{n=4}^{\infty}\frac{1}{n^2}\end{aligned}$$

It may take a while before one is comfortable with this statement, whose truth lies at the heart of the study of infinite series: *it is possible that the sum of an infinite list of nonzero numbers is finite.* We have seen this repeatedly in this section, yet it still may "take some getting used to."

As one contemplates the behaviour of series, a few facts become clear.

1. In order to add an infinite list of nonzero numbers and get a finite result, "most" of those numbers must be "very near" 0.

2. If a series diverges, it means that the sum of an infinite list of numbers is not finite (it may approach $\pm\infty$ or it may oscillate), and:

   (a) The series will still diverge if the first term is removed.

   (b) The series will still diverge if the first 10 terms are removed.

   (c) The series will still diverge if the first $1,000,000$ terms are removed.

   (d) The series will still diverge if any finite number of terms from anywhere in the series are removed.

These concepts are very important and lie at the heart of the next two theorems.

Theorem 10.2.4 $\quad$ **n^{th}–Term Test for Divergence**

Consider the series $\displaystyle\sum_{n=1}^{\infty}a_n$. If $\displaystyle\lim_{n\to\infty}a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty}a_n$ diverges.

Important! This theorem *does not state* that if $\displaystyle\lim_{n\to\infty}a_n = 0$ then $\displaystyle\sum_{n=1}^{\infty}a_n$ converges. The standard example of this is the Harmonic Series, as given in Key Idea 10.2.1. The Harmonic Sequence, $\{1/n\}$, converges to 0; the Harmonic Series, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$, diverges.

Looking back, we can apply this theorem to the series in Example 10.2.1. In that example, the n^{th} terms of both sequences do not converge to 0, therefore we can quickly conclude that each series diverges.

One can rewrite Theorem 10.2.4 to state "If a series converges, then the underlying sequence converges to 0." While it is important to understand the truth of this statement, in practice it is rarely used. It is generally far easier to prove the convergence of a sequence than the convergence of a series.

Theorem 10.2.5 Infinite Nature of Series

The convergence or divergence of an infinite series remains unchanged by the addition or subtraction of any finite number of terms. That is:

1. A divergent series will remain divergent with the addition or subtraction of any finite number of terms.
2. A convergent series will remain convergent with the addition or subtraction of any finite number of terms. (Of course, the *sum* will likely change.)

Consider once more the Harmonic Series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges; that is, the sequence of partial sums $\{S_n\}$ grows (very, very slowly) without bound. One might think that by removing the "large" terms of the sequence that perhaps the series will converge. This is simply not the case. For instance, the sum of the first 10 million terms of the Harmonic Series is about 16.7. Removing the first 10 million terms from the Harmonic Series changes the n^{th} partial sums, effectively subtracting 16.7 from the sum. However, a sequence that is growing without bound will still grow without bound when 16.7 is subtracted from it.

The equations below illustrate this. The first line shows the infinite sum of the Harmonic Series split into the sum of the first 10 million terms plus the sum of "everything else." The next equation shows us subtracting these first 10 million terms from both sides. The final equation employs a bit of "psuedo–math": subtracting 16.7 from "infinity" still leaves one with "infinity."

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n} &= \sum_{n=1}^{10,000,000}\frac{1}{n} + \sum_{n=10,000,001}^{\infty}\frac{1}{n}\\
\sum_{n=1}^{\infty}\frac{1}{n} - \sum_{n=1}^{10,000,000}\frac{1}{n} &= \sum_{n=10,000,001}^{\infty}\frac{1}{n}\\
\infty - 16.7 &= \infty.
\end{aligned}$$

This section introduced us to series and defined a few special types of series whose convergence properties are well known: we know when a p-series or a geometric series converges or diverges. Most series that we encounter are not one of these types, but we are still interested in knowing whether or not they converge. The next three sections introduce tests that help us determine whether or not a given series converges.

Exercises 10.2

Terms and Concepts

1. Use your own words to describe how sequences and series are related.

2. Use your own words to define a *partial sum*.

3. Given a series $\sum_{n=1}^{\infty} a_n$, describe the two sequences related to the series that are important.

4. Use your own words to explain what a geometric series is.

5. T/F: If $\{a_n\}$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

6. T/F: If $\{a_n\}$ converges to 0, then $\sum_{n=0}^{\infty} a_n$ converges.

Problems

In Exercises 7 – 14, a series $\sum_{n=1}^{\infty} a_n$ is given.

(a) Give the first 5 partial sums of the series.

(b) Give a graph of the first 5 terms of a_n and S_n on the same axes.

7. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

8. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

9. $\sum_{n=1}^{\infty} \cos(\pi n)$

10. $\sum_{n=1}^{\infty} n$

11. $\sum_{n=1}^{\infty} \frac{1}{n!}$

12. $\sum_{n=1}^{\infty} \frac{1}{3^n}$

13. $\sum_{n=1}^{\infty} \left(-\frac{9}{10}\right)^n$

14. $\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$

In Exercises 15 – 20, use Theorem 10.2.4 to show the given series diverges.

15. $\sum_{n=1}^{\infty} \frac{3n^2}{n(n+2)}$

16. $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

17. $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

18. $\sum_{n=1}^{\infty} \frac{5^n - n^5}{5^n + n^5}$

19. $\sum_{n=1}^{\infty} \frac{2^n + 1}{2^{n+1}}$

20. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$

In Exercises 21 – 30, state whether the given series converges or diverges.

21. $\sum_{n=1}^{\infty} \frac{1}{n^5}$

22. $\sum_{n=0}^{\infty} \frac{1}{5^n}$

23. $\sum_{n=0}^{\infty} \frac{6^n}{5^n}$

24. $\sum_{n=1}^{\infty} n^{-4}$

25. $\sum_{n=1}^{\infty} \sqrt{n}$

26. $\sum_{n=1}^{\infty} \frac{10}{n!}$

27. T/F: If $\{a_n\}$ converges to 0, then $\sum_{n=0}^{\infty} a_n$ converges.

28. $\sum_{n=1}^{\infty} \frac{2}{(2n+8)^2}$

29. $\sum_{n=1}^{\infty} \frac{1}{2n}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$

In Exercises 31 – 46, a series is given.

(a) Find a formula for S_n, the n^{th} partial sum of the series.

(b) Determine whether the series converges or diverges. If it converges, state what it converges to.

31. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n}$

32. $\displaystyle\sum_{n=1}^{\infty} 2$

33. $1^3 + 2^3 + 3^3 + 4^3 + \cdots$

34. $\displaystyle\sum_{n=1}^{\infty} (-1)^n n$

35. $\displaystyle\sum_{n=0}^{\infty} \frac{5}{2^n}$

36. $\displaystyle\sum_{n=1}^{\infty} e^{-n}$

37. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} + \cdots$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+2)}$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$

41. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

43. $\frac{1}{1\cdot 4} + \frac{1}{2\cdot 5} + \frac{1}{3\cdot 6} + \frac{1}{4\cdot 7} + \cdots$

44. $2 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{9}\right) + \left(\frac{1}{8} + \frac{1}{27}\right) + \cdots$

45. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$

46. $\displaystyle\sum_{n=0}^{\infty} \left(\sin 1\right)^n$

47. Break the Harmonic Series into the sum of the odd and even terms:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{2n-1} + \sum_{n=1}^{\infty} \frac{1}{2n}.$$

The goal is to show that each of the series on the right diverge.

(a) Show why $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1} > \sum_{n=1}^{\infty} \frac{1}{2n}$.

(Compare each n^{th} partial sum.)

(b) Show why $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1} < 1 + \sum_{n=1}^{\infty} \frac{1}{2n}$

(c) Explain why (a) and (b) demonstrate that the series of odd terms is convergent, if, and only if, the series of even terms is also convergent. (That is, show both converge or both diverge.)

(d) Explain why knowing the Harmonic Series is divergent determines that the even and odd series are also divergent.

48. Show the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(2n-1)(2n+1)}$ diverges.

10.3 Integral and Comparison Tests

Knowing whether or not a series converges is very important, especially when we discuss Power Series in Section 10.6. Theorems 10.2.1 and 10.2.2 give criteria for when Geometric and p-series converge, and Theorem 10.2.4 gives a quick test to determine if a series diverges. There are many important series whose convergence cannot be determined by these theorems, though, so we introduce a set of tests that allow us to handle a broad range of series. We start with the Integral Test.

Integral Test

We stated in Section 10.1 that a sequence $\{a_n\}$ is a function $a(n)$ whose domain is $\mathbb{N}$, the set of natural numbers. If we can extend $a(n)$ to $\mathbb{R}$, the real numbers, and it is both positive and decreasing on $[1, \infty)$, then the convergence of $\displaystyle\sum_{n=1}^{\infty} a_n$ is the same as $\displaystyle\int_1^{\infty} a(x)\,dx$.

Note: Theorem 10.3.1 does not state that the integral and the summation have the same value.

> **Theorem 10.3.1 Integral Test**
>
> Let a sequence $\{a_n\}$ be defined by $a_n = a(n)$, where $a(n)$ is continuous, positive and decreasing on $[1, \infty)$. Then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, if, and only if, $\displaystyle\int_1^{\infty} a(x)\,dx$ converges.

We can demonstrate the truth of the Integral Test with two simple graphs. In Figure 10.3.1(a), the height of each rectangle is $a(n) = a_n$ for $n = 1, 2, \ldots$, and clearly the rectangles enclose more area than the area under $y = a(x)$. Therefore we can conclude that

$$\int_1^{\infty} a(x)\,dx < \sum_{n=1}^{\infty} a_n. \tag{10.1}$$

In Figure 10.3.1(b), we draw rectangles under $y = a(x)$ with the Right-Hand rule, starting with $n = 2$. This time, the area of the rectangles is less than the area under $y = a(x)$, so $\displaystyle\sum_{n=2}^{\infty} a_n < \int_1^{\infty} a(x)\,dx$. Note how this summation starts with $n = 2$; adding a_1 to both sides lets us rewrite the summation starting with $n = 1$:

$$\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\,dx. \tag{10.2}$$

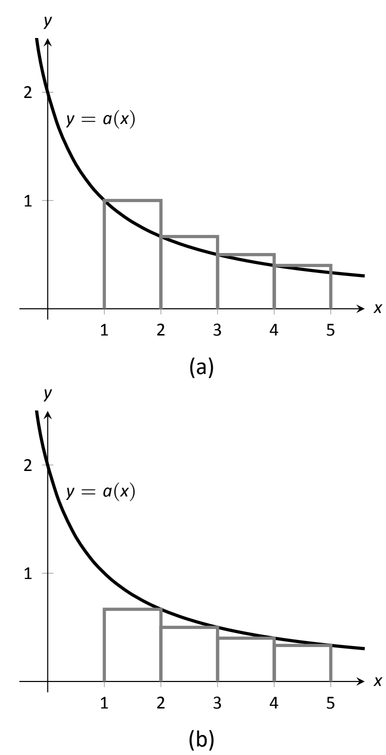


Figure 10.3.1: Illustrating the truth of the Integral Test.

Combining Equations (10.1) and (10.2), we have

$$\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\,dx < a_1 + \sum_{n=1}^{\infty} a_n. \tag{10.3}$$

From Equation (10.3) we can make the following two statements:

1. If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, so does $\displaystyle\int_1^{\infty} a(x)\,dx$ (because $\displaystyle\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\,dx$)

2. If $\sum_{n=1}^{\infty} a_n$ converges, so does $\int_1^{\infty} a(x)\,dx$ (because $\int_1^{\infty} a(x)\,dx < \sum_{n=1}^{\infty} a_n$.)

Therefore the series and integral either both converge or both diverge. Theorem 10.2.5 allows us to extend this theorem to series where $a(n)$ is positive and decreasing on $[b, \infty)$ for some $b > 1$.

Example 10.3.1 Using the Integral Test

Determine the convergence of $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$. (The terms of the sequence $\{a_n\} = \{\ln n/n^2\}$ and the n[th] partial sums are given in Figure 10.3.2.)

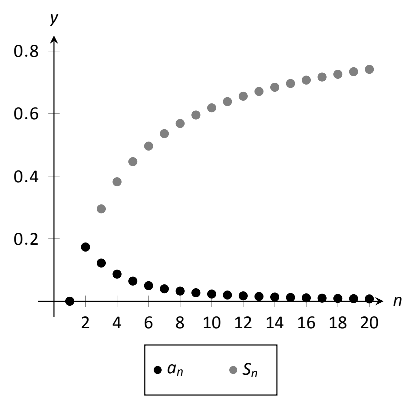


Figure 10.3.2: Plotting the sequence and series in Example 10.3.1.

Solution Figure 10.3.2 implies that $a(n) = (\ln n)/n^2$ is positive and decreasing on $[2, \infty)$. We can determine this analytically, too. We know $a(n)$ is positive as both $\ln n$ and n^2 are positive on $[2, \infty)$. To determine that $a(n)$ is decreasing, consider $a'(n) = (1 - 2\ln n)/n^3$, which is negative for $n \geq 2$. Since $a'(n)$ is negative, $a(n)$ is decreasing.

Applying the Integral Test, we test the convergence of $\int_1^{\infty} \frac{\ln x}{x^2}\,dx$. Integrating this improper integral requires the use of Integration by Parts, with $u = \ln x$ and $dv = 1/x^2\,dx$.

$$\begin{aligned}
\int_1^{\infty} \frac{\ln x}{x^2}\,dx &= \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx \\
&= \lim_{b\to\infty} -\frac{1}{x}\ln x\Big|_1^b + \int_1^b \frac{1}{x^2}\,dx
\end{aligned}$$

$$\begin{aligned}
&= \lim_{b\to\infty} -\frac{1}{x}\ln x - \frac{1}{x}\Big|_1^b \\
&= \lim_{b\to\infty} 1 - \frac{1}{b} - \frac{\ln b}{b}. \qquad \text{Apply L'Hospital's Rule:} \\
&= 1.
\end{aligned}$$

Since $\int_1^{\infty} \frac{\ln x}{x^2}\,dx$ converges, so does $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$.

Theorem 10.2.2 was given without justification, stating that the general p-series $\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$ converges if, and only if, $p > 1$. In the following example, we prove this to be true by applying the Integral Test.

Example 10.3.2 Using the Integral Test to establish Theorem 10.2.2.

Use the Integral Test to prove that $\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$ converges if, and only if, $p > 1$.

Solution Consider the integral $\displaystyle\int_1^\infty \frac{1}{(ax+b)^p}\,dx$; assuming $p \neq 1$,

$$\begin{aligned}\int_1^\infty \frac{1}{(ax+b)^p}\,dx &= \lim_{c\to\infty}\int_1^c \frac{1}{(ax+b)^p}\,dx\\ &= \lim_{c\to\infty}\frac{1}{a(1-p)}(ax+b)^{1-p}\Big|_1^c\\ &= \lim_{c\to\infty}\frac{1}{a(1-p)}\big((ac+b)^{1-p}-(a+b)^{1-p}\big).\end{aligned}$$

This limit converges if, and only if, $p > 1$. It is easy to show that the integral also diverges in the case of $p = 1$. (This result is similar to the work preceding Key Idea 6.8.1.)

Therefore $\displaystyle\sum_{n=1}^\infty \frac{1}{(an+b)^p}$ converges if, and only if, $p > 1$.

Note: A sequence $\{a_n\}$ is a **positive sequence** if $a_n > 0$ for all n.

Because of Theorem 10.2.5, any theorem that relies on a positive sequence still holds true when $a_n > 0$ for all but a finite number of values of n.

We consider two more convergence tests in this section, both *comparison* tests. That is, we determine the convergence of one series by comparing it to another series with known convergence.

Direct Comparison Test

Theorem 10.3.2 Direct Comparison Test

Let $\{a_n\}$ and $\{b_n\}$ be positive sequences where $a_n \leq b_n$ for all $n \geq N$, for some $N \geq 1$.

1. If $\displaystyle\sum_{n=1}^\infty b_n$ converges, then $\displaystyle\sum_{n=1}^\infty a_n$ converges.
2. If $\displaystyle\sum_{n=1}^\infty a_n$ diverges, then $\displaystyle\sum_{n=1}^\infty b_n$ diverges.

Example 10.3.3 Applying the Direct Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^\infty \frac{1}{3^n+n^2}$.

Solution This series is neither a geometric or p-series, but seems related. We predict it will converge, so we look for a series with larger terms that converges. (Note too that the Integral Test seems difficult to apply here.)

Since $3^n < 3^n + n^2$, $\dfrac{1}{3^n} > \dfrac{1}{3^n+n^2}$ for all $n \geq 1$. The series $\displaystyle\sum_{n=1}^\infty \frac{1}{3^n}$ is a convergent geometric series; by Theorem 10.3.2, $\displaystyle\sum_{n=1}^\infty \frac{1}{3^n+n^2}$ converges.

Example 10.3.4 Applying the Direct Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^\infty \frac{1}{n-\ln n}$.

Solution We know the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and it seems that the given series is closely related to it, hence we predict it will diverge.

Since $n \geq n - \ln n$ for all $n \geq 1$, $\frac{1}{n} \leq \frac{1}{n - \ln n}$ for all $n \geq 1$.

The Harmonic Series diverges, so we conclude that $\sum_{n=1}^{\infty} \frac{1}{n - \ln n}$ diverges as well.

The concept of direct comparison is powerful and often relatively easy to apply. Practice helps one develop the necessary intuition to quickly pick a proper series with which to compare. However, it is easy to construct a series for which it is difficult to apply the Direct Comparison Test.

Consider $\sum_{n=1}^{\infty} \frac{1}{n + \ln n}$. It is very similar to the divergent series given in Example 10.3.4. We suspect that it also diverges, as $\frac{1}{n} \approx \frac{1}{n + \ln n}$ for large n. However, the inequality that we naturally want to use "goes the wrong way": since $n \leq n + \ln n$ for all $n \geq 1$, $\frac{1}{n} \geq \frac{1}{n + \ln n}$ for all $n \geq 1$. The given series has terms *less than* the terms of a divergent series, and we cannot conclude anything from this.

Fortunately, we can apply another test to the given series to determine its convergence.

Limit Comparison Test

Theorem 10.3.3 Limit Comparison Test

Let $\{a_n\}$ and $\{b_n\}$ be positive sequences.

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where L is a positive real number, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then if $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then if $\sum_{n=1}^{\infty} b_n$ diverges, then so does $\sum_{n=1}^{\infty} a_n$.

Theorem 10.3.3 is most useful when the convergence of the series from $\{b_n\}$ is known and we are trying to determine the convergence of the series from $\{a_n\}$.

We use the Limit Comparison Test in the next example to examine the series $\sum_{n=1}^{\infty} \frac{1}{n + \ln n}$ which motivated this new test.

Example 10.3.5 Applying the Limit Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + \ln n}$ using the Limit Comparison Test.

SOLUTION We compare the terms of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + \ln n}$ to the terms of the Harmonic Sequence $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{1/(n+\ln n)}{1/n} &= \lim_{n\to\infty} \frac{n}{n+\ln n} \\ &= 1 \quad \text{(after applying L'Hôpital's Rule).}\end{aligned}$$

Since the Harmonic Series diverges, we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + \ln n}$ diverges as well.

Example 10.3.6 Applying the Limit Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - n^2}$

SOLUTION This series is similar to the one in Example 10.3.3, but now we are considering "$3^n - n^2$" instead of "$3^n + n^2$." This difference makes applying the Direct Comparison Test difficult.

Instead, we use the Limit Comparison Test and compare with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{1/(3^n - n^2)}{1/3^n} &= \lim_{n\to\infty} \frac{3^n}{3^n - n^2} \\ &= 1 \quad \text{(after applying L'Hospital's Rule twice).}\end{aligned}$$

We know $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent geometric series, hence $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - n^2}$ converges as well.

As mentioned before, practice helps one develop the intuition to quickly choose a series with which to compare. A general rule of thumb is to pick a series based on the dominant term in the expression of $\{a_n\}$. It is also helpful to note that factorials dominate exponentials, which dominate algebraic functions (e.g., polynomials), which dominate logarithms. In the previous example, the dominant term of $\dfrac{1}{3^n - n^2}$ was 3^n, so we compared the series to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$. It is hard to apply the Limit Comparison Test to series containing factorials, though, as we have not learned how to apply L'Hospital's Rule to $n!$.

Example 10.3.7 Applying the Limit Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^2 - n + 1}$.

SOLUTION We naïvely attempt to apply the rule of thumb given above and note that the dominant term in the expression of the series is $1/n^2$. Knowing that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we attempt to apply the Limit Comparison Test:

$$\begin{aligned}\lim_{n\to\infty} \frac{(\sqrt{n}+3)/(n^2-n+1)}{1/n^2} &= \lim_{n\to\infty} \frac{n^2(\sqrt{n}+3)}{n^2-n+1}\\ &= \infty \quad \text{(Apply L'Hôpital's Rule).}\end{aligned}$$

Theorem 10.3.3 part (3) only applies when $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges; in our case, it converges. Ultimately, our test has not revealed anything about the convergence of our series.

The problem is that we chose a poor series with which to compare. Since the numerator and denominator of the terms of the series are both algebraic functions, we should have compared our series to the dominant term of the numerator divided by the dominant term of the denominator.

The dominant term of the numerator is $n^{1/2}$ and the dominant term of the denominator is n^2. Thus we should compare the terms of the given series to $n^{1/2}/n^2 = 1/n^{3/2}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{(\sqrt{n}+3)/(n^2-n+1)}{1/n^{3/2}} &= \lim_{n\to\infty} \frac{n^{3/2}(\sqrt{n}+3)}{n^2-n+1}\\ &= 1 \quad \text{(Apply L'Hôpital's Rule).}\end{aligned}$$

Since the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^2-n+1}$ converges as well.

We mentioned earlier that the Integral Test did not work well with series containing factorial terms. The next section introduces the Ratio Test, which does handle such series well. We also introduce the Root Test, which is good for series where each term is raised to a power.

Exercises 10.3

Terms and Concepts

1. In order to apply the Integral Test to a sequence $\{a_n\}$, the function $a(n) = a_n$ must be ______, ______ and ______.

2. T/F: The Integral Test can be used to determine the sum of a convergent series.

3. What test(s) in this section do not work well with factorials?

4. Suppose $\displaystyle\sum_{n=0}^{\infty} a_n$ is convergent, and there are sequences $\{b_n\}$ and $\{c_n\}$ such that $0 \leq b_n \leq a_n \leq c_n$ for all n. What can be said about the series $\displaystyle\sum_{n=0}^{\infty} b_n$ and $\displaystyle\sum_{n=0}^{\infty} c_n$?

Problems

In Exercises 5 – 12, use the Integral Test to determine the convergence of the given series.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

8. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

10. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

In Exercises 13 – 22, use the Direct Comparison Test to determine the convergence of the given series; state what series is used for comparison.

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n-5}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n+n^2-n}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!+n}$

17. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$

18. $\displaystyle\sum_{n=5}^{\infty} \frac{1}{\sqrt{n}-2}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^3-5}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{5^n+10}$

21. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2-1}$

22. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$

In Exercises 23 – 32, use the Limit Comparison Test to determine the convergence of the given series; state what series is used for comparison.

23. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2-3n+5}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n-n^2}$

25. $\displaystyle\sum_{n=4}^{\infty} \frac{\ln n}{n-3}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{n-10}{n^2+10n+10}$

29. $\displaystyle\sum_{n=1}^{\infty} \sin\left(1/n\right)$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{n^3-5}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^2+17}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+100}$

In Exercises 33 – 40, determine the convergence of the given series. State the test used; more than one test may be appropriate.

33. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+5)^3}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n!}$

37. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n+n}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{10n+5}$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(1/n)}{\sqrt{n}}$

41. Given that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, state which of the following series converges, may converge, or does not converge.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} a_n a_{n+1}$

   (c) $\displaystyle\sum_{n=1}^{\infty} (a_n)^2$

   (d) $\displaystyle\sum_{n=1}^{\infty} n a_n$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$

10.4 Ratio and Root Tests

The n^{th}–Term Test of Theorem 10.2.4 states that in order for a series $\sum_{n=1}^{\infty} a_n$ to converge, $\lim_{n\to\infty} a_n = 0$. That is, the terms of $\{a_n\}$ must get very small. Not only must the terms approach 0, they must approach 0 "fast enough": while $\lim_{n\to\infty} 1/n = 0$, the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges as the terms of $\{1/n\}$ do not approach 0 "fast enough."

The comparison tests of the previous section determine convergence by comparing terms of a series to terms of another series whose convergence is known. This section introduces the Ratio and Root Tests, which determine convergence by analyzing the terms of a series to see if they approach 0 "fast enough."

Ratio Test

Theorem 10.4.1 Ratio Test

Let $\{a_n\}$ be a positive sequence where $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$.

1. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $L > 1$ or $L = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, the Ratio Test is inconclusive.

Note: Theorem 10.2.5 allows us to apply the Ratio Test to series where $\{a_n\}$ is positive for all but a finite number of terms.

The principle of the Ratio Test is this: if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$, then for large n, each term of $\{a_n\}$ is significantly smaller than its previous term which is enough to ensure convergence.

Example 10.4.1 Applying the Ratio Test

Use the Ratio Test to determine the convergence of the following series:

1. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ 2. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ 3. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Solution

1. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$:

$$\begin{aligned} \lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} &= \lim_{n\to\infty} \frac{2^{n+1}n!}{2^n(n+1)!} \\ &= \lim_{n\to\infty} \frac{2}{n+1} \\ &= 0. \end{aligned}$$

Since the limit is $0 < 1$, by the Ratio Test $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{3^{n+1}/(n+1)^3}{3^n/n^3} &= \lim_{n\to\infty} \frac{3^{n+1}n^3}{3^n(n+1)^3}\\ &= \lim_{n\to\infty} \frac{3n^3}{(n+1)^3}\\ &= 3.\end{aligned}$$

Since the limit is $3 > 1$, by the Ratio Test $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ diverges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{1/\big((n+1)^2+1\big)}{1/(n^2+1)} &= \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1}\\ &= 1.\end{aligned}$$

Since the limit is 1, the Ratio Test is inconclusive. We can easily show this series converges using the Direct or Limit Comparison Tests, with each comparing to the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

The Ratio Test is not effective when the terms of a series *only* contain algebraic functions (e.g., polynomials). It is most effective when the terms contain some factorials or exponentials. The previous example also reinforces our developing intuition: factorials dominate exponentials, which dominate algebraic functions, which dominate logarithmic functions. In Part 1 of the example, the factorial in the denominator dominated the exponential in the numerator, causing the series to converge. In Part 2, the exponential in the numerator dominated the algebraic function in the denominator, causing the series to diverge.

While we have used factorials in previous sections, we have not explored them closely and one is likely to not yet have a strong intuitive sense for how they behave. The following example gives more practice with factorials.

Example 10.4.2 Applying the Ratio Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$.

Solution Before we begin, be sure to note the difference between $(2n)!$ and $2n!$. When $n = 4$, the former is $8! = 8 \cdot 7 \cdot \ldots \cdot 2 \cdot 1 = 40,320$, whereas the latter is $2(4 \cdot 3 \cdot 2 \cdot 1) = 48$.

Applying the Ratio Test:

$$\lim_{n\to\infty} \frac{(n+1)!(n+1)!/\big(2(n+1)\big)!}{n!n!/(2n)!} = \lim_{n\to\infty} \frac{(n+1)!(n+1)!(2n)!}{n!n!(2n+2)!}$$

Noting that $(2n+2)! = (2n+2)\cdot(2n+1)\cdot(2n)!$, we have

$$\begin{aligned}&= \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)}\\ &= 1/4.\end{aligned}$$

Since the limit is $1/4 < 1$, by the Ratio Test we conclude $\displaystyle\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$ converges.

Root Test

The final test we introduce is the Root Test, which works particularly well on series where each term is raised to a power, and does not work well with terms containing factorials.

Note: Theorem 10.2.5 allows us to apply the Root Test to series where $\{a_n\}$ is positive for all but a finite number of terms.

Theorem 10.4.2 Root Test

Let $\{a_n\}$ be a positive sequence, and let $\displaystyle\lim_{n\to\infty} (a_n)^{1/n} = L$.

1. If $L < 1$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.
2. If $L > 1$ or $L = \infty$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, the Root Test is inconclusive.

Example 10.4.3 Applying the Root Test

Determine the convergence of the following series using the Root Test:

1. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n+1}{5n-2}\right)^n$ 2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{(\ln n)^n}$ 3. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$.

Solution

1. $\displaystyle\lim_{n\to\infty} \left(\left(\frac{3n+1}{5n-2}\right)^n\right)^{1/n} = \lim_{n\to\infty} \frac{3n+1}{5n-2} = \frac{3}{5}.$

   Since the limit is less than 1, we conclude the series converges. Note: it is difficult to apply the Ratio Test to this series.

2. $\displaystyle\lim_{n\to\infty} \left(\frac{n^4}{(\ln n)^n}\right)^{1/n} = \lim_{n\to\infty} \frac{\left(n^{1/n}\right)^4}{\ln n}.$

   As n grows, the numerator approaches 1 (apply L'Hospital's Rule) and the denominator grows to infinity. Thus

   $$\lim_{n\to\infty} \frac{\left(n^{1/n}\right)^4}{\ln n} = 0.$$

   Since the limit is less than 1, we conclude the series converges.

3. $\displaystyle\lim_{n\to\infty} \left(\frac{2^n}{n^2}\right)^{1/n} = \lim_{n\to\infty} \frac{2}{\left(n^{1/n}\right)^2} = 2.$

   Since this is greater than 1, we conclude the series diverges.

Each of the tests we have encountered so far has required that we analyze series from *positive* sequences. The next section relaxes this restriction by considering *alternating series*, where the underlying sequence has terms that alternate between being positive and negative.

Exercises 10.4

Terms and Concepts

1. The Ratio Test is not effective when the terms of a sequence only contain ______ functions.

2. The Ratio Test is most effective when the terms of a sequence contains ______ and/or ______ functions.

3. What three convergence tests do not work well with terms containing factorials?

4. The Root Test works particularly well on series where each term is ______ to a ______.

Problems

In Exercises 5 – 14, determine the convergence of the given series using the Ratio Test. If the Ratio Test is inconclusive, state so and determine convergence with another test.

5. $\displaystyle\sum_{n=0}^{\infty} \frac{2n}{n!}$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{5^n - 3n}{4^n}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{n!10^n}{(2n)!}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n + n^4}{7^n + n^2}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^3 + 7}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{10 \cdot 5^n}{7^n - 3}$

12. $\displaystyle\sum_{n=1}^{\infty} n \cdot \left(\frac{3}{5}\right)^n$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots 2n}{3 \cdot 6 \cdot 9 \cdot 12 \cdots 3n}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{5 \cdot 10 \cdot 15 \cdots (5n)}$

In Exercises 15 – 24, determine the convergence of the given series using the Root Test. If the Root Test is inconclusive, state so and determine convergence with another test.

15. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+5}{3n+11}\right)^n$

16. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{.9n^2 - n - 3}{n^2 + n + 3}\right)^n$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n^2}{3^n}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^2 2^{n+1}}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+7}}{7^n}$

21. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2 - n}{n^2 + n}\right)^n$

22. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\left(\ln n\right)^n}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{\left(\ln n\right)^n}$

In Exercises 25 – 34, determine the convergence of the given series. State the test used; more than one test may be appropriate.

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 4n - 2}{n^3 + 4n^2 - 3n + 7}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n^4 4^n}{n!}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n + n}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^n}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2 + 4n + 1}}$

30. $\sum_{n=1}^{\infty} \frac{n!n!n!}{(3n)!}$

31. $\sum_{n=1}^{\infty} \frac{1}{\ln n}$

32. $\sum_{n=1}^{\infty} \left(\frac{n+2}{n+1}\right)^{n}$

33. $\sum_{n=1}^{\infty} \frac{n^3}{\left(\ln n\right)^{n}}$

34. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$

10.5 Alternating Series and Absolute Convergence

All of the series convergence tests we have used require that the underlying sequence $\{a_n\}$ be a positive sequence. (We can relax this with Theorem 10.2.5 and state that there must be an $N > 0$ such that $a_n > 0$ for all $n > N$; that is, $\{a_n\}$ is positive for all but a finite number of values of n.)

In this section we explore series whose summation includes negative terms. We start with a very specific form of series, where the terms of the summation alternate between being positive and negative.

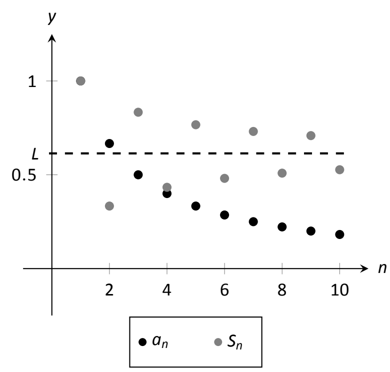


Figure 10.5.1: Illustrating convergence with the Alternating Series Test.

Definition 10.5.1 Alternating Series

Let $\{a_n\}$ be a positive sequence. An **alternating series** is a series of either the form

$$\sum_{n=1}^{\infty}(-1)^n a_n \qquad \text{or} \qquad \sum_{n=1}^{\infty}(-1)^{n+1} a_n.$$

Recall the terms of Harmonic Series come from the Harmonic Sequence $\{a_n\} = \{1/n\}$. An important alternating series is the **Alternating Harmonic Series**:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Geometric Series can also be alternating series when $r < 0$. For instance, if $r = -1/2$, the geometric series is

$$\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots$$

Theorem 10.2.1 states that geometric series converge when $|r| < 1$ and gives the sum: $\displaystyle\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$. When $r = -1/2$ as above, we find

$$\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n = \frac{1}{1-(-1/2)} = \frac{1}{3/2} = \frac{2}{3}.$$

A powerful convergence theorem exists for other alternating series that meet a few conditions.

Theorem 10.5.1 Alternating Series Test

Let $\{a_n\}$ be a positive, decreasing sequence where $\lim\limits_{n\to\infty} a_n = 0$. Then

$$\sum_{n=1}^{\infty}(-1)^n a_n \qquad \text{and} \qquad \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

converge.

The basic idea behind Theorem 10.5.1 is illustrated in Figure 10.5.1. A positive, decreasing sequence $\{a_n\}$ is shown along with the partial sums

$$S_n = \sum_{i=1}^{n}(-1)^{i+1}a_i = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1}a_n.$$

Because $\{a_n\}$ is decreasing, the amount by which S_n bounces up/down decreases. Moreover, the odd terms of S_n form a decreasing, bounded sequence, while the even terms of S_n form an increasing, bounded sequence. Since bounded, monotonic sequences converge (see Theorem 10.1.5) and the terms of $\{a_n\}$ approach 0, one can show the odd and even terms of S_n converge to the same common limit L, the sum of the series.

Example 10.5.1 Applying the Alternating Series Test

Determine if the Alternating Series Test applies to each of the following series.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ 2. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{\ln n}{n}$ 3. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{|\sin n|}{n^2}$

Solution

1. This is the Alternating Harmonic Series as seen previously. The underlying sequence is $\{a_n\} = \{1/n\}$, which is positive, decreasing, and approaches 0 as $n \to \infty$. Therefore we can apply the Alternating Series Test and conclude this series converges.

   While the test does not state what the series converges to, we will see later that $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \ln 2$.

2. The underlying sequence is $\{a_n\} = \{\ln n/n\}$. This is positive and approaches 0 as $n \to \infty$ (use L'Hospital's Rule). However, the sequence is not decreasing for all n. It is straightforward to compute $a_1 = 0$, $a_2 \approx 0.347$, $a_3 \approx 0.366$, and $a_4 \approx 0.347$: the sequence is increasing for at least the first 3 terms.

   We do not immediately conclude that we cannot apply the Alternating Series Test. Rather, consider the long–term behaviour of $\{a_n\}$. Treating $a_n = a(n)$ as a continuous function of n defined on $[1, \infty)$, we can take its derivative:
$$a'(n) = \frac{1 - \ln n}{n^2}.$$
   The derivative is negative for all $n \geq 3$ (actually, for all $n > e$), meaning $a(n) = a_n$ is decreasing on $[3, \infty)$. We can apply the Alternating Series Test to the series when we start with $n = 3$ and conclude that $\displaystyle\sum_{n=3}^{\infty}(-1)^n\frac{\ln n}{n}$ converges; adding the terms with $n = 1$ and $n = 2$ do not change the convergence (i.e., we apply Theorem 10.2.5).

   The important lesson here is that as before, if a series fails to meet the criteria of the Alternating Series Test on only a finite number of terms, we can still apply the test.

3. The underlying sequence is $\{a_n\} = |\sin n|/n$. This sequence is positive and approaches 0 as $n \to \infty$. However, it is not a decreasing sequence; the value of $|\sin n|$ oscillates between 0 and 1 as $n \to \infty$. We cannot remove a finite number of terms to make $\{a_n\}$ decreasing, therefore we cannot apply the Alternating Series Test.

   Keep in mind that this does not mean we conclude the series diverges; in fact, it does converge. We are just unable to conclude this based on Theorem 10.5.1.

Key Idea 10.2.1 gives the sum of some important series. Two of these are

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.64493 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \approx 0.82247.$$

These two series converge to their sums at different rates. To be accurate to two places after the decimal, we need 202 terms of the first series though only 13 of the second. To get 3 places of accuracy, we need 1069 terms of the first series though only 33 of the second. Why is it that the second series converges so much faster than the first?

While there are many factors involved when studying rates of convergence, the alternating structure of an alternating series gives us a powerful tool when approximating the sum of a convergent series.

Theorem 10.5.2 The Alternating Series Approximation Theorem

Let $\{a_n\}$ be a sequence that satisfies the hypotheses of the Alternating Series Test, and let S_n and L be the n^{th} partial sums and sum, respectively, of either $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$ or $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}a_n$. Then

1. $|S_n - L| < a_{n+1}$, and
2. L is between S_n and S_{n+1}.

Part 1 of Theorem 10.5.2 states that the n^{th} partial sum of a convergent alternating series will be within a_{n+1} of its total sum. Consider the alternating series we looked at before the statement of the theorem, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$. Since $a_{14} = 1/14^2 \approx 0.0051$, we know that S_{13} is within 0.0051 of the total sum.

Moreover, Part 2 of the theorem states that since $S_{13} \approx 0.8252$ and $S_{14} \approx 0.8201$, we know the sum L lies between 0.8201 and 0.8252. One use of this is the knowledge that S_{14} is accurate to two places after the decimal.

Some alternating series converge slowly. In Example 10.5.1 we determined the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$ converged. With $n = 1001$, we find $\ln n/n \approx 0.0069$, meaning that $S_{1000} \approx 0.1633$ is accurate to one, maybe two, places after the decimal. Since $S_{1001} \approx 0.1564$, we know the sum L is $0.1564 \leq L \leq 0.1633$.

Example 10.5.2 Approximating the sum of convergent alternating series

Approximate the sum of the following series, accurate to within 0.001.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^3}$ 2. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$.

Solution

1. Using Theorem 10.5.2, we want to find n where $1/n^3 < 0.001$:

$$\begin{aligned}\frac{1}{n^3} &\leq 0.001 = \frac{1}{1000}\\ n^3 &\geq 1000\\ n &\geq \sqrt[3]{1000}\\ n &\geq 10.\end{aligned}$$

   Let L be the sum of this series. By Part 1 of the theorem, $|S_9 - L| < a_{10} = 1/1000$. We can compute $S_9 = 0.902116$, which our theorem states is within 0.001 of the total sum.

   We can use Part 2 of the theorem to obtain an even more accurate result. As we know the 10$^{\text{th}}$ term of the series is $-1/1000$, we can easily compute $S_{10} = 0.901116$. Part 2 of the theorem states that L is between S_9 and S_{10}, so $0.901116 < L < 0.902116$.

2. We want to find n where $\ln(n)/n < 0.001$. We start by solving $\ln(n)/n = 0.001$ for n. This cannot be solved algebraically, so we will use Newton's Method to approximate a solution.

   Let $f(x) = \ln(x)/x - 0.001$; we want to know where $f(x) = 0$. We make a guess that x must be "large," so our initial guess will be $x_1 = 1000$. Recall how Newton's Method works: given an approximate solution x_n, our next approximation x_{n+1} is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

   We find $f'(x) = \big(1 - \ln(x)\big)/x^2$. This gives

$$\begin{aligned}x_2 &= 1000 - \frac{\ln(1000)/1000 - 0.001}{\big(1 - \ln(1000)\big)/1000^2}\\ &= 2000.\end{aligned}$$

   Using a computer, we find that Newton's Method seems to converge to a solution $x = 9118.01$ after 8 iterations. Taking the next integer higher, we have $n = 9119$, where $\ln(9119)/9119 = 0.000999903 < 0.001$.

   Again using a computer, we find $S_{9118} = -0.160369$. Part 1 of the theorem states that this is within 0.001 of the actual sum L. Already knowing the 9,119$^{\text{th}}$ term, we can compute $S_{9119} = -0.159369$, meaning $-0.159369 < L < -0.160369$.

Notice how the first series converged quite quickly, where we needed only 10 terms to reach the desired accuracy, whereas the second series took over 9,000 terms.

One of the famous results of mathematics is that the Harmonic Series, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, yet the Alternating Harmonic Series, $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$, converges. The notion that alternating the signs of the terms in a series can make a series converge leads us to the following definitions.

Note: In Definition 10.5.2, $\sum_{n=1}^{\infty} a_n$ is not necessarily an alternating series; it just may have some negative terms.

Definition 10.5.2 Absolute and Conditional Convergence

1. A series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if $\sum_{n=1}^{\infty} |a_n|$ converges.
2. A series $\sum_{n=1}^{\infty} a_n$ **converges conditionally** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Thus we say the Alternating Harmonic Series converges conditionally.

Example 10.5.3 Determining absolute and conditional convergence.
Determine if the following series converge absolutely, conditionally, or diverge.

1. $\sum_{n=1}^{\infty} (-1)^n \frac{n+3}{n^2+2n+5}$ 2. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+2n+5}{2^n}$ 3. $\sum_{n=3}^{\infty} (-1)^n \frac{3n-3}{5n-10}$

Solution

1. We can show the series
$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{n+3}{n^2+2n+5} \right| = \sum_{n=1}^{\infty} \frac{n+3}{n^2+2n+5}$$
diverges using the Limit Comparison Test, comparing with $1/n$.

   The series $\sum_{n=1}^{\infty} (-1)^n \frac{n+3}{n^2+2n+5}$ converges using the Alternating Series Test; we conclude it converges conditionally.

2. We can show the series
$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{n^2+2n+5}{2^n} \right| = \sum_{n=1}^{\infty} \frac{n^2+2n+5}{2^n}$$
converges using the Ratio Test.

   Therefore we conclude $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+2n+5}{2^n}$ converges absolutely.

3. The series
$$\sum_{n=3}^{\infty} \left| (-1)^n \frac{3n-3}{5n-10} \right| = \sum_{n=3}^{\infty} \frac{3n-3}{5n-10}$$
diverges using the n^{th} Term Test, so it does not converge absolutely.

   The series $\sum_{n=3}^{\infty} (-1)^n \frac{3n-3}{5n-10}$ fails the conditions of the Alternating Series Test as $(3n-3)/(5n-10)$ does not approach 0 as $n \to \infty$. We can state further that this series diverges; as $n \to \infty$, the series effectively adds and subtracts $3/5$ over and over. This causes the sequence of partial sums to oscillate and not converge.

   Therefore the series $\sum_{n=1}^{\infty} (-1)^n \frac{3n-3}{5n-10}$ diverges.

Knowing that a series converges absolutely allows us to make two important statements, given in the following theorem. The first is that absolute convergence is "stronger" than regular convergence. That is, just because $\sum_{n=1}^{\infty} a_n$ converges, we cannot conclude that $\sum_{n=1}^{\infty}|a_n|$ will converge, but knowing a series converges absolutely tells us that $\sum_{n=1}^{\infty} a_n$ will converge.

One reason this is important is that our convergence tests all require that the underlying sequence of terms be positive. By taking the absolute value of the terms of a series where not all terms are positive, we are often able to apply an appropriate test and determine absolute convergence. This, in turn, determines that the series we are given also converges.

The second statement relates to **rearrangements** of series. When dealing with a finite set of numbers, the sum of the numbers does not depend on the order which they are added. (So $1+2+3=3+1+2$.) One may be surprised to find out that when dealing with an infinite set of numbers, the same statement does not always hold true: some infinite lists of numbers may be rearranged in different orders to achieve different sums. The theorem states that the terms of an absolutely convergent series can be rearranged in any way without affecting the sum.

Theorem 10.5.3 Absolute Convergence Theorem

Let $\sum_{n=1}^{\infty} a_n$ be a series that converges absolutely.

1. $\sum_{n=1}^{\infty} a_n$ converges.
2. Let $\{b_n\}$ be any rearrangement of the sequence $\{a_n\}$. Then

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

In Example 10.5.3, we determined the series in part 2 converges absolutely. Theorem 10.5.3 tells us the series converges (which we could also determine using the Alternating Series Test).

The theorem states that rearranging the terms of an absolutely convergent series does not affect its sum. This implies that perhaps the sum of a conditionally convergent series can change based on the arrangement of terms. Indeed, it can. The Riemann Rearrangement Theorem (named after Bernhard Riemann) states that any conditionally convergent series can have its terms rearranged so that the sum is any desired value, including ∞!

As an example, consider the Alternating Harmonic Series once more. We have stated that

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}\cdots = \ln 2,$$

(see Key Idea 10.2.1 or Example 10.5.1).

Consider the rearrangement where every positive term is followed by two negative terms:

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}\cdots$$

(Convince yourself that these are exactly the same numbers as appear in the Alternating Harmonic Series, just in a different order.) Now group some terms and simplify:

$$\begin{aligned}\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\cdots &= \\ \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\cdots &= \\ \frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots\right) &= \frac{1}{2}\ln 2.\end{aligned}$$

By rearranging the terms of the series, we have arrived at a different sum! (One could *try* to argue that the Alternating Harmonic Series does not actually converge to $\ln 2$, because rearranging the terms of the series *shouldn't* change the sum. However, the Alternating Series Test proves this series converges to L, for some number L, and if the rearrangement does not change the sum, then $L = L/2$, implying $L = 0$. But the Alternating Series Approximation Theorem quickly shows that $L > 0$. The only conclusion is that the rearrangement *did* change the sum.) This is an incredible result.

We end here our study of tests to determine convergence. The end of this text contains a table summarizing the tests that one may find useful.

While series are worthy of study in and of themselves, our ultimate goal within calculus is the study of Power Series, which we will consider in the next section. We will use power series to create functions where the output is the result of an infinite summation.

Exercises 10.5

Terms and Concepts

1. Why is $\sum_{n=1}^{\infty} \sin n$ not an alternating series?

2. A series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges when $\{a_n\}$ is ______, ______ and $\lim_{n\to\infty} a_n =$______.

3. Give an example of a series where $\sum_{n=0}^{\infty} a_n$ converges but $\sum_{n=0}^{\infty} |a_n|$ does not.

4. The sum of a ______ convergent series can be changed by rearranging the order of its terms.

Problems

In Exercises 5 – 20, an alternating series $\sum_{n=i}^{\infty} a_n$ is given.

(a) Determine if the series converges or diverges.

(b) Determine if $\sum_{n=0}^{\infty} |a_n|$ converges or diverges.

(c) If $\sum_{n=0}^{\infty} a_n$ converges, determine if the convergence is conditional or absolute.

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

6. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n!}}$

7. $\sum_{n=0}^{\infty} (-1)^n \frac{n+5}{3n-5}$

8. $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2}$

9. $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{3n+5}{n^2-3n+1}$

10. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n + 1}$

11. $\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln n}$

12. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1+3+5+\cdots+(2n-1)}$

13. $\sum_{n=1}^{\infty} \cos(\pi n)$

14. $\sum_{n=2}^{\infty} \frac{\sin\left((n+1/2)\pi\right)}{n \ln n}$

15. $\sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n$

16. $\sum_{n=0}^{\infty} (-e)^{-n}$

17. $\sum_{n=0}^{\infty} \frac{(-1)^n n^2}{n!}$

18. $\sum_{n=0}^{\infty} (-1)^n 2^{-n^2}$

19. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

20. $\sum_{n=1}^{\infty} \frac{(-1000)^n}{n!}$

Let S_n be the n^{th} partial sum of a series. In Exercises 21 – 24, a convergent alternating series is given and a value of n. Compute S_n and S_{n+1} and use these values to find bounds on the sum of the series.

21. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}, \quad n=5$

22. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}, \quad n=4$

23. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}, \quad n=6$

24. $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n, \quad n=9$

In Exercises 25 – 28, a convergent alternating series is given along with its sum and a value of ε. Use Theorem 10.5.2 to find n such that the n^{th} partial sum of the series is within ε of the sum of the series.

25. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}, \quad \varepsilon = 0.001$

26. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}, \quad \varepsilon = 0.0001$

27. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}, \quad \varepsilon = 0.001$

28. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} = \cos 1, \quad \varepsilon = 10^{-8}$

10.6 Power Series

So far, our study of series has examined the question of "Is the sum of these infinite terms finite?," i.e., "Does the series converge?" We now approach series from a different perspective: as a function. Given a value of x, we evaluate $f(x)$ by finding the sum of a particular series that depends on x (assuming the series converges). We start this new approach to series with a definition.

Definition 10.6.1 Power Series

Let $\{a_n\}$ be a sequence, let x be a variable, and let c be a real number.

1. The **power series in** x is the series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

2. The **power series in** x **centred at** c is the series

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$$

Example 10.6.1 Examples of power series

Write out the first five terms of the following power series:

1. $\displaystyle\sum_{n=0}^{\infty} x^n$ 2. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{(x+1)^n}{n}$ 3. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1}\frac{(x-\pi)^{2n}}{(2n)!}$.

Solution

1. One of the conventions we adopt is that $x^0 = 1$ regardless of the value of x. Therefore
$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots$$
This is a geometric series in x.

2. This series is centred at $c = -1$. Note how this series starts with $n = 1$. We could rewrite this series starting at $n = 0$ with the understanding that $a_0 = 0$, and hence the first term is 0.
$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{(x+1)^n}{n} = (x+1) - \frac{(x+1)^2}{2} + \frac{(x+1)^3}{3} - \frac{(x+1)^4}{4} + \frac{(x+1)^5}{5} \dots$$

3. This series is centred at $c = \pi$. Recall that $0! = 1$.
$$\sum_{n=0}^{\infty} (-1)^{n+1}\frac{(x-\pi)^{2n}}{(2n)!} = -1 + \frac{(x-\pi)^2}{2} - \frac{(x-\pi)^4}{24} + \frac{(x-\pi)^6}{6!} - \frac{(x-\pi)^8}{8!} \dots$$

We introduced power series as a type of function, where a value of x is given and the sum of a series is returned. Of course, not every series converges. For instance, in part 1 of Example 10.6.1, we recognized the series $\displaystyle\sum_{n=0}^{\infty} x^n$ as a geometric series in x. Theorem 10.2.1 states that this series converges only when $|x| < 1$.

This raises the question: "For what values of x will a given power series converge?," which leads us to a theorem and definition.

Theorem 10.6.1 Convergence of Power Series

Let a power series $\displaystyle\sum_{n=0}^{\infty} a_n(x-c)^n$ be given. Then one of the following is true:

1. The series converges only at $x = c$.
2. There is an $R > 0$ such that the series converges for all x in $(c - R, c + R)$ and diverges for all $x < c - R$ and $x > c + R$.
3. The series converges for all x.

The value of R is important when understanding a power series, hence it is given a name in the following definition. Also, note that part 2 of Theorem 10.6.1 makes a statement about the interval $(c - R, c + R)$, but the not the endpoints of that interval. A series may/may not converge at these endpoints.

Definition 10.6.2 Radius and Interval of Convergence

1. The number R given in Theorem 10.6.1 is the **radius of convergence** of a given series. When a series converges for only $x = c$, we say the radius of convergence is 0, i.e., $R = 0$. When a series converges for all x, we say the series has an infinite radius of convergence, i.e., $R = \infty$.
2. The **interval of convergence** is the set of all values of x for which the series converges.

To find the values of x for which a given series converges, we will use the convergence tests we studied previously (especially the Ratio Test). However, the tests all required that the terms of a series be positive. The following theorem gives us a work–around to this problem.

Theorem 10.6.2 The Radius of Convergence of a Series and Absolute Convergence

The series $\displaystyle\sum_{n=0}^{\infty} a_n(x-c)^n$ and $\displaystyle\sum_{n=0}^{\infty} \left|a_n(x-c)^n\right|$ have the same radius of convergence R.

Theorem 10.6.2 allows us to find the radius of convergence R of a series by applying the Ratio Test (or any applicable test) to the absolute value of the terms of the series. We practice this in the following example.

Example 10.6.2 Determining the radius and interval of convergence.

Find the radius and interval of convergence for each of the following series:

1. $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$ 2. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ 3. $\displaystyle\sum_{n=0}^{\infty}2^n(x-3)^n$ 4. $\displaystyle\sum_{n=0}^{\infty}n!x^n$

Solution

1. We apply the Ratio Test to the series $\displaystyle\sum_{n=0}^{\infty}\left|\frac{x^n}{n!}\right|$:

$$\begin{aligned}\lim_{n\to\infty}\frac{\left|x^{n+1}/(n+1)!\right|}{\left|x^n/n!\right|} &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\cdot\frac{n!}{(n+1)!}\right|\\ &= \lim_{n\to\infty}\left|\frac{x}{n+1}\right|\\ &= 0 \text{ for all } x.\end{aligned}$$

The Ratio Test shows us that regardless of the choice of x, the series converges. Therefore the radius of convergence is $R=\infty$, and the interval of convergence is $(-\infty,\infty)$.

2. We apply the Ratio Test to the series $\displaystyle\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{x^n}{n}\right| = \sum_{n=1}^{\infty}\left|\frac{x^n}{n}\right|$:

$$\begin{aligned}\lim_{n\to\infty}\frac{\left|x^{n+1}/(n+1)\right|}{\left|x^n/n\right|} &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\cdot\frac{n}{n+1}\right|\\ &= \lim_{n\to\infty}|x|\frac{n}{n+1}\\ &= |x|.\end{aligned}$$

The Ratio Test states a series converges if the limit of $|a_{n+1}/a_n| = L < 1$. We found the limit above to be $|x|$; therefore, the power series converges when $|x|<1$, or when x is in $(-1,1)$. Thus the radius of convergence is $R=1$.

To determine the interval of convergence, we need to check the endpoints of $(-1,1)$. When $x=-1$, we have the opposite of the Harmonic Series:

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^n}{n} &= \sum_{n=1}^{\infty}\frac{-1}{n}\\ &= -\infty.\end{aligned}$$

The series diverges when $x=-1$.

When $x=1$, we have the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1)^n}{n}$, which is the Alternating Harmonic Series, which converges. Therefore the interval of convergence is $(-1,1]$.

3. We apply the Ratio Test to the series $\displaystyle\sum_{n=0}^{\infty}\left|2^n(x-3)^n\right|$:

$$\begin{aligned}\lim_{n\to\infty}\frac{\left|2^{n+1}(x-3)^{n+1}\right|}{\left|2^n(x-3)^n\right|} &= \lim_{n\to\infty}\left|\frac{2^{n+1}}{2^n}\cdot\frac{(x-3)^{n+1}}{(x-3)^n}\right|\\ &= \lim_{n\to\infty}\left|2(x-3)\right|.\end{aligned}$$

According to the Ratio Test, the series converges when $\big|2(x-3)\big| < 1 \implies \big|x-3\big| < 1/2$. The series is centred at 3, and x must be within $1/2$ of 3 in order for the series to converge. Therefore the radius of convergence is $R = 1/2$, and we know that the series converges absolutely for all x in $(3 - 1/2, 3 + 1/2) = (2.5, 3.5)$.

We check for convergence at the endpoints to find the interval of convergence. When $x = 2.5$, we have:

$$\begin{aligned}\sum_{n=0}^{\infty} 2^n(2.5-3)^n &= \sum_{n=0}^{\infty} 2^n(-1/2)^n \\ &= \sum_{n=0}^{\infty}(-1)^n,\end{aligned}$$

which diverges. A similar process shows that the series also diverges at $x = 3.5$. Therefore the interval of convergence is $(2.5, 3.5)$.

4. We apply the Ratio Test to $\displaystyle\sum_{n=0}^{\infty}\big|n!x^n\big|$:

$$\begin{aligned}\lim_{n\to\infty}\frac{\big|(n+1)!x^{n+1}\big|}{\big|n!x^n\big|} &= \lim_{n\to\infty}\big|(n+1)x\big| \\ &= \infty \quad \text{for all } x, \text{ except } x = 0.\end{aligned}$$

The Ratio Test shows that the series diverges for all x except $x = 0$. Therefore the radius of convergence is $R = 0$.

We can use a power series to define a function:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

where the domain of f is a subset of the interval of convergence of the power series. One can apply calculus techniques to such functions; in particular, we can find derivatives and antiderivatives.

Theorem 10.6.3 Derivatives and Indefinite Integrals of Power Series Functions

Let $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n(x-c)^n$ be a function defined by a power series, with radius of convergence R.

1. $f(x)$ is continuous and differentiable on $(c-R, c+R)$.
2. $f'(x) = \displaystyle\sum_{n=1}^{\infty} a_n \cdot n \cdot (x-c)^{n-1}$, with radius of convergence R.
3. $\displaystyle\int f(x)\,dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$, with radius of convergence R.

A few notes about Theorem 10.6.3:

1. The theorem states that differentiation and integration do not change the radius of convergence. It does not state anything about the *interval* of convergence. They are not always the same.

2. Notice how the summation for $f'(x)$ starts with $n = 1$. This is because the constant term a_0 of $f(x)$ goes to 0.

3. Differentiation and integration are simply calculated term–by–term using the Power Rules.

Example 10.6.3 Derivatives and indefinite integrals of power series

Let $f(x) = \displaystyle\sum_{n=0}^{\infty} x^n$. Find $f'(x)$ and $F(x) = \displaystyle\int f(x)\,dx$, along with their respective intervals of convergence.

Solution We find the derivative and indefinite integral of $f(x)$, following Theorem 10.6.3.

1. $f'(x) = \displaystyle\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$.

   In Example 10.6.1, we recognized that $\displaystyle\sum_{n=0}^{\infty} x^n$ is a geometric series in x. We know that such a geometric series converges when $|x| < 1$; that is, the interval of convergence is $(-1, 1)$.

   To determine the interval of convergence of $f'(x)$, we consider the endpoints of $(-1, 1)$:

   $$f'(-1) = 1 - 2 + 3 - 4 + \cdots, \quad \text{which diverges.}$$

   $$f'(1) = 1 + 2 + 3 + 4 + \cdots, \quad \text{which diverges.}$$

   Therefore, the interval of convergence of $f'(x)$ is $(-1, 1)$.

2. $F(x) = \displaystyle\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

   To find the interval of convergence of $F(x)$, we again consider the endpoints of $(-1, 1)$:

   $$F(-1) = C - 1 + 1/2 - 1/3 + 1/4 + \cdots$$

   The value of C is irrelevant; notice that the rest of the series is an Alternating Series that whose terms converge to 0. By the Alternating Series Test, this series converges. (In fact, we can recognize that the terms of the series after C are the opposite of the Alternating Harmonic Series. We can thus say that $F(-1) = C - \ln 2$.)

   $$F(1) = C + 1 + 1/2 + 1/3 + 1/4 + \cdots$$

   Notice that this summation is $C +$ the Harmonic Series, which diverges. Since F converges for $x = -1$ and diverges for $x = 1$, the interval of convergence of $F(x)$ is $[-1, 1)$.

The previous example showed how to take the derivative and indefinite integral of a power series without motivation for why we care about such operations. We may care for the sheer mathematical enjoyment "that we can", which is motivation enough for many. However, we would be remiss to not recognize that we can learn a great deal from taking derivatives and indefinite integrals.

Recall that $f(x) = \sum_{n=0}^{\infty} x^n$ in Example 10.6.3 is a geometric series. According to Theorem 10.2.1, this series converges to $1/(1-x)$ when $|x| < 1$. Thus we can say

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{on} \quad (-1, 1).$$

Integrating the power series, (as done in Example 10.6.3,) we find

$$F(x) = C_1 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \tag{10.4}$$

while integrating the function $f(x) = 1/(1-x)$ gives

$$F(x) = -\ln|1-x| + C_2. \tag{10.5}$$

Equating Equations (10.4) and (10.5), we have

$$F(x) = C_1 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln|1-x| + C_2.$$

Letting $x = 0$, we have $F(0) = C_1 = C_2$. This implies that we can drop the constants and conclude

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln|1-x|.$$

We established in Example 10.6.3 that the series on the left converges at $x = -1$; substituting $x = -1$ on both sides of the above equality gives

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots = -\ln 2.$$

On the left we have the opposite of the Alternating Harmonic Series; on the right, we have $-\ln 2$. We conclude that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

Important: We stated in Key Idea 10.2.1 (in Section 10.2) that the Alternating Harmonic Series converges to $\ln 2$, and referred to this fact again in Example 10.5.1 of Section 10.5. However, we never gave an argument for why this was the case. The work above finally shows how we conclude that the Alternating Harmonic Series converges to $\ln 2$.

We use this type of analysis in the next example.

Example 10.6.4 Analyzing power series functions

Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Find $f'(x)$ and $\int f(x)\,dx$, and use these to analyze the behaviour of $f(x)$.

Solution We start by making two notes: first, in Example 10.6.2, we found the interval of convergence of this power series is $(-\infty, \infty)$. Second, we will find it useful later to have a few terms of the series written out:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \tag{10.6}$$

We now find the derivative:

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} n\frac{x^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = 1 + x + \frac{x^2}{2!} + \cdots . \end{aligned}$$

Since the series starts at $n = 1$ and each term refers to $(n-1)$, we can re-index the series starting with $n = 0$:

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= f(x). \end{aligned}$$

We found the derivative of $f(x)$ is $f(x)$. The only functions for which this is true are of the form $y = ce^x$ for some constant c. As $f(0) = 1$ (see Equation (10.6)), c must be 1. Therefore we conclude that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

for all x.

We can also find $\int f(x)\,dx$:

$$\begin{aligned} \int f(x)\,dx &= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!(n+1)} \\ &= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \end{aligned}$$

We write out a few terms of this last series:

$$C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} = C + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

The integral of $f(x)$ differs from $f(x)$ only by a constant, again indicating that $f(x) = e^x$.

Example 10.6.4 and the work following Example 10.6.3 established relationships between a power series function and "regular" functions that we have dealt with in the past. In general, given a power series function, it is difficult (if not impossible) to express the function in terms of elementary functions. We chose examples where things worked out nicely.

In this section's last example, we show how to solve a simple differential equation with a power series.

Example 10.6.5 Solving a differential equation with a power series.
Give the first 4 terms of the power series solution to $y' = 2y$, where $y(0) = 1$.

SOLUTION The differential equation $y' = 2y$ describes a function $y = f(x)$ where the derivative of y is twice y and $y(0) = 1$. This is a rather simple differential equation; with a bit of thought one should realize that if $y = Ce^{2x}$, then $y' = 2Ce^{2x}$, and hence $y' = 2y$. By letting $C = 1$ we satisfy the initial condition of $y(0) = 1$.

Let's ignore the fact that we already know the solution and find a power series function that satisfies the equation. The solution we seek will have the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

for unknown coefficients a_n. We can find $f'(x)$ using Theorem 10.6.3:

$$f'(x) = \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \cdots .$$

Since $f'(x) = 2f(x)$, we have

$$\begin{aligned} a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \cdots &= 2\big(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\big) \\ &= 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + \cdots \end{aligned}$$

The coefficients of like powers of x must be equal, so we find that

$$a_1 = 2a_0, \quad 2a_2 = 2a_1, \quad 3a_3 = 2a_2, \quad 4a_4 = 2a_3, \quad \text{etc.}$$

The initial condition $y(0) = f(0) = 1$ indicates that $a_0 = 1$; with this, we can find the values of the other coefficients:

$$\begin{aligned} a_0 = 1 \text{ and } a_1 = 2a_0 &\Rightarrow a_1 = 2; \\ a_1 = 2 \text{ and } 2a_2 = 2a_1 &\Rightarrow a_2 = 4/2 = 2; \\ a_2 = 2 \text{ and } 3a_3 = 2a_2 &\Rightarrow a_3 = 8/(2\cdot 3) = 4/3; \\ a_3 = 4/3 \text{ and } 4a_4 = 2a_3 &\Rightarrow a_4 = 16/(2\cdot 3\cdot 4) = 2/3. \end{aligned}$$

Thus the first 5 terms of the power series solution to the differential equation $y' = 2y$ is

$$f(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots$$

In Section 10.8, as we study Taylor Series, we will learn how to recognize this series as describing $y = e^{2x}$.

Our last example illustrates that it can be difficult to recognize an elementary function by its power series expansion. It is far easier to start with a known function, expressed in terms of elementary functions, and represent it as a power series function. One may wonder why we would bother doing so, as the latter function probably seems more complicated. In the next two sections, we show both *how* to do this and *why* such a process can be beneficial.

Exercises 10.6

Terms and Concepts

1. We adopt the convention that $x^0 =$______, regardless of the value of x.

2. What is the difference between the radius of convergence and the interval of convergence?

3. If the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ is 5, what is the radius of convergence of $\displaystyle\sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$?

4. If the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ is 5, what is the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} (-1)^n a_n x^n$?

Problems

In Exercises 5 – 8, write out the sum of the first 5 terms of the given power series.

5. $\displaystyle\sum_{n=0}^{\infty} 2^n x^n$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^n$

8. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

In Exercises 9 – 24, a power series is given.

(a) Find the radius of convergence.

(b) Find the interval of convergence.

9. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} x^n$

10. $\displaystyle\sum_{n=0}^{\infty} n x^n$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{n}$

12. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+4)^n}{n!}$

13. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

14. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x-5)^n}{10^n}$

15. $\displaystyle\sum_{n=0}^{\infty} 5^n (x-1)^n$

16. $\displaystyle\sum_{n=0}^{\infty} (-2)^n x^n$

17. $\displaystyle\sum_{n=0}^{\infty} \sqrt{n} x^n$

18. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{3^n} x^n$

19. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!} (x-5)^n$

20. $\displaystyle\sum_{n=0}^{\infty} (-1)^n n! (x-10)^n$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{n^3}$

23. $\displaystyle\sum_{n=0}^{\infty} n! \left(\frac{x}{10}\right)^n$

24. $\displaystyle\sum_{n=0}^{\infty} n^2 \left(\frac{x+4}{4}\right)^n$

In Exercises 25 – 30, a function $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ is given.

(a) Give a power series for $f'(x)$ and its interval of convergence.

(b) Give a power series for $\int f(x)\, dx$ and its interval of convergence.

25. $\displaystyle\sum_{n=0}^{\infty} n x^n$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

27. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$

28. $\displaystyle\sum_{n=0}^{\infty}(-3x)^n$

29. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

30. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$

In Exercises 31 – 36, give the first 5 terms of the series that is a solution to the given differential equation.

31. $y' = 3y, \quad y(0) = 1$

32. $y' = 5y, \quad y(0) = 5$

33. $y' = y^2, \quad y(0) = 1$

34. $y' = y + 1, \quad y(0) = 1$

35. $y'' = -y, \quad y(0) = 0, y'(0) = 1$

36. $y'' = 2y, \quad y(0) = 1, y'(0) = 1$

10.7 Taylor Polynomials

Consider a function $y = f(x)$ and a point $\big(c, f(c)\big)$. The derivative, $f'(c)$, gives the instantaneous rate of change of f at $x = c$. Of all lines that pass through the point $\big(c, f(c)\big)$, the line that best approximates f at this point is the tangent line; that is, the line whose slope (rate of change) is $f'(c)$.

In Figure 10.7.1, we see a function $y = f(x)$ graphed. The table below the graph shows that $f(0) = 2$ and $f'(0) = 1$; therefore, the tangent line to f at $x = 0$ is $p_1(x) = 1(x-0)+2 = x+2$. The tangent line is also given in the figure. Note that "near" $x = 0$, $p_1(x) \approx f(x)$; that is, the tangent line approximates f well.

One shortcoming of this approximation is that the tangent line only matches the slope of f; it does not, for instance, match the concavity of f. We can find a polynomial, $p_2(x)$, that does match the concavity without much difficulty, though. The table in Figure 10.7.1 gives the following information:

$$f(0) = 2 \qquad f'(0) = 1 \qquad f''(0) = 2.$$

Therefore, we want our polynomial $p_2(x)$ to have these same properties. That is, we need

$$p_2(0) = 2 \qquad p_2'(0) = 1 \qquad p_2''(0) = 2.$$

Let's start with a general quadratic function

$$p(x) = a_0 + a_1x + a_2x^2$$

We find the following:

$$\begin{aligned} p_2(x) &= a_0 + a_1x + a_2x^2 & p_2(0) &= a_0 \\ p_2{}'(x) &= a_1 + 2a_2x & p_2{}'(0) &= a_1 \\ p_2{}''(x) &= 2a_2 & p_2{}''(0) &= 2a_2. \end{aligned}$$

To get the desired properties above, we must have

$$a_0 = f(0) = 2,\ a_1 = f'(0) = 1,\ 2a_2 = f''(0) = 2,$$

so $a_0 = 2, a_1 = 1$, and $a_2 = 2/2 = 1$, giving us the polynomial

$$p_2(x) = 2 + x + x^2.$$

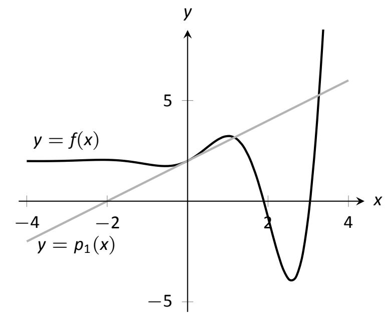
We can repeat this approximation process by creating polynomials of higher degree that match more of the derivatives of f at $x = 0$. In general, a polynomial of degree n can be created to match the first n derivatives of f. Figure 10.7.2 also shows $p_4(x) = -x^4/2 - x^3/6 + x^2 + x + 2$, whose first four derivatives at 0 match those of f.

How do we ensure that the derivatives of our polynomial match those of f? We simply begin with a polynomial of the desired degree, compute its derivatives, and compare them to those of f! Recall that each term in a polynomial consists of a power of x, and a coefficient, like so: a_nx^n. Our goal is to determine the value for each coefficient a_n so that the derivatives of our polynomial match those of our function f. If we take k derivatives of the term a_nx^n, with $k \leq n$, we obtain

$$\frac{d^k}{dx^k}(a_nx^n) = n(n-1)\cdots(n-k+1)a_nx^{n-k}.$$

For $k < n$, the expression above vanishes when we set $x = 0$. However, for $n = k$, we obtain the constant value

$$\frac{d^k}{dx^k}(a_kx^k) = k\cdot(k-1)\cdots 2\cdot 1a_k. \qquad (10.7)$$



$f(0) = 2$	$f'''(0) = -1$
$f'(0) = 1$	$f^{(4)}(0) = -12$
$f''(0) = 2$	$f^{(5)}(0) = -19$

Figure 10.7.1: Plotting $y = f(x)$ and a table of derivatives of f evaluated at 0.

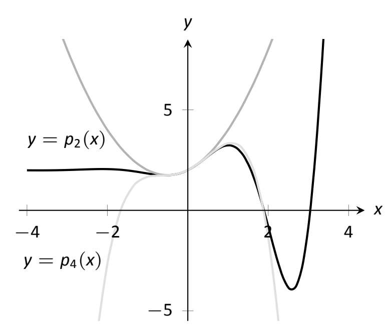


Figure 10.7.2: Plotting f, p_2 and p_4.

Consider a polynomial

$$p_n(x) = a_0 + a_1x + \cdots + a_kx^k + \cdots + a_nx^n$$

of degree n. If we take k derivatives, all of the terms involving powers of x less than k disappear, and when we set $x = 0$, all of the terms involving powers of x larger than k disappear, leaving us with the single constant given in (10.7).

The notation $k!$ is read as "k factorial". By convention, we also define $0! = 1$, mostly because it makes our formulas look a lot nicer.

Recalling the notation $k! = 1 \cdot 2 \cdot 3 \cdots k$ for the product of the first k integers, we have shown that

$$p_n^{(k)}(0) = k!a_k.$$

If we want the derivatives of p_n to agree with some unknown function f when $x = 0$, then we must have

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

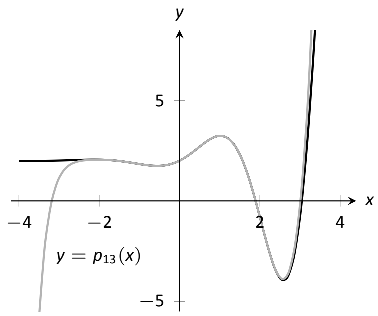


Figure 10.7.3: Plotting f and p_{13}.

As we use more and more derivatives, our polynomial approximation to f gets better and better. In this example, the interval on which the approximation is "good" gets bigger and bigger. Figure 10.7.3 shows $p_{13}(x)$; we can visually affirm that this polynomial approximates f very well on $[-2, 3]$. (The polynomial $p_{13}(x)$ is not particularly "nice". It is

$$\frac{16901x^{13}}{6227020800}+\frac{13x^{12}}{1209600}-\frac{1321x^{11}}{39916800}-\frac{779x^{10}}{1814400}-\frac{359x^9}{362880}+\frac{x^8}{240}+\frac{139x^7}{5040}+\frac{11x^6}{360}-\frac{19x^5}{120}-\frac{x^4}{2}-\frac{x^3}{6}+x^2+x+2.)$$

Historical note: Colin Maclaurin was a Scottish mathematician, born in 1698. He lived until 1746, and made a number of contributions to the development of mathematics and physics. His election as professor of mathematics at the University of Aberdeen at the age of 19 made him the world's youngest professor, a record he held until 2008! He was also a staunch foe of the Jacobite Rebellion, and was instrumental in the defence of Edinburgh against the army of Bonnie Prince Charlie. (For more details, see Wikipedia.)

The polynomials we have created are examples of *Taylor polynomials*, named after the British mathematician Brook Taylor who made important discoveries about such functions. In the discussion above, we concentrated on evaluating the derivatives of f at 0; however, there is nothing special about this point. Just as we can consider the linear approximation of a function near any point, so too can we determine a polynomial approximation about any value c in the domain of f. The only catch is that our polynomial will then be given in terms of powers of $x - c$, rather than powers of x, as we see in the following definition.

Definition 10.7.1 Taylor Polynomial, Maclaurin Polynomial

Let f be a function whose first n derivatives exist at $x = c$.

1. The **Taylor polynomial of degree n of f at** $x = c$ is

$$p_n(x) = f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n.$$

2. A special case of the Taylor polynomial is the Maclaurin polynomial, where $c = 0$. That is, the **Maclaurin polynomial of degree n of f** is

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

We will practice creating Taylor and Maclaurin polynomials in the following examples.

Example 10.7.1 Finding and using Maclaurin polynomials

1. Find the n^{th} Maclaurin polynomial for $f(x) = e^x$.
2. Use $p_5(x)$ to approximate the value of e.

Solution

1. We start with creating a table of the derivatives of e^x evaluated at $x = 0$. In this particular case, this is relatively simple, as shown in Figure 10.7.4. By the definition of the Maclaurin series, we have

$$\begin{aligned} p_n(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\ &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots + \frac{1}{n!}x^n. \end{aligned}$$

2. Using our answer from part 1, we have

$$p_5 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5.$$

To approximate the value of e, note that $e = e^1 = f(1) \approx p_5(1)$. It is very straightforward to evaluate $p_5(1)$:

$$p_5(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{163}{60} \approx 2.71667.$$

A plot of $f(x) = e^x$ and $p_5(x)$ is given in Figure 10.7.5.

$$\begin{array}{lcl} f(x) = e^x & \Rightarrow & f(0) = 1 \\ f'(x) = e^x & \Rightarrow & f'(0) = 1 \\ f''(x) = e^x & \Rightarrow & f''(0) = 1 \\ \vdots & & \vdots \\ f^{(n)}(x) = e^x & \Rightarrow & f^{(n)}(0) = 1 \end{array}$$

Figure 10.7.4: The derivatives of $f(x) = e^x$ evaluated at $x = 0$.

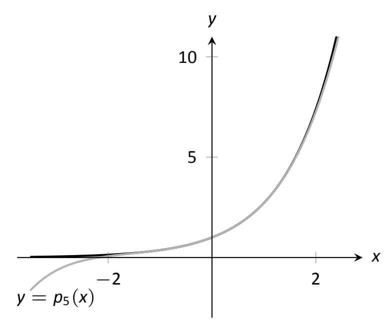


Figure 10.7.5: A plot of $f(x) = e^x$ and its 5^{th} degree Maclaurin polynomial $p_5(x)$.

Example 10.7.2 Finding and using Taylor polynomials

1. Find the n^{th} Taylor polynomial of $y = \ln x$ at $x = 1$.
2. Use $p_6(x)$ to approximate the value of $\ln 1.5$.
3. Use $p_6(x)$ to approximate the value of $\ln 2$.

Solution

1. We begin by creating a table of derivatives of $\ln x$ evaluated at $x = 1$. While this is not as straightforward as it was in the previous example, a pattern does emerge, as shown in Figure 10.7.6.

   Using Definition 10.7.1, we have

$$\begin{aligned} p_n(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &= 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots + \frac{(-1)^{n+1}}{n}(x-1)^n. \end{aligned}$$

Note how the coefficients of the $(x-1)$ terms turn out to be "nice."

$$\begin{array}{lcl} f(x) = \ln x & \Rightarrow & f(1) = 0 \\ f'(x) = 1/x & \Rightarrow & f'(1) = 1 \\ f''(x) = -1/x^2 & \Rightarrow & f''(1) = -1 \\ f'''(x) = 2/x^3 & \Rightarrow & f'''(1) = 2 \\ f^{(4)}(x) = -6/x^4 & \Rightarrow & f^{(4)}(1) = -6 \\ \vdots & & \vdots \\ f^{(n)}(x) = & \Rightarrow & f^{(n)}(1) = \\ \dfrac{(-1)^{n+1}(n-1)!}{x^n} & & (-1)^{n+1}(n-1)! \end{array}$$

Figure 10.7.6: Derivatives of $\ln x$ evaluated at $x = 1$.

2. We can compute $p_6(x)$ using our work above:

$$p_6(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6.$$

Since $p_6(x)$ approximates $\ln x$ well near $x = 1$, we approximate $\ln 1.5 \approx p_6(1.5)$:

$$\begin{aligned} p_6(1.5) &= (1.5-1) - \frac{1}{2}(1.5-1)^2 + \frac{1}{3}(1.5-1)^3 - \frac{1}{4}(1.5-1)^4 + \cdots \\ &\cdots + \frac{1}{5}(1.5-1)^5 - \frac{1}{6}(1.5-1)^6 \\ &= \frac{259}{640} \\ &\approx 0.404688. \end{aligned}$$

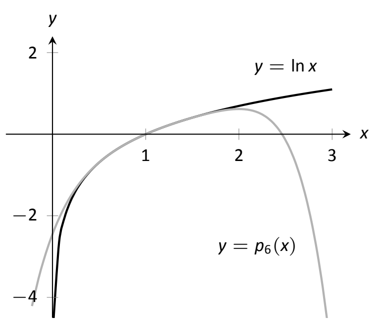


Figure 10.7.7: A plot of $y = \ln x$ and its 6$^{\text{th}}$ degree Taylor polynomial at $x = 1$.

This is a good approximation as a calculator shows that $\ln 1.5 \approx 0.4055$. Figure 10.7.7 plots $y = \ln x$ with $y = p_6(x)$. We can see that $\ln 1.5 \approx p_6(1.5)$.

3. We approximate $\ln 2$ with $p_6(2)$:

$$\begin{aligned} p_6(2) &= (2-1) - \frac{1}{2}(2-1)^2 + \frac{1}{3}(2-1)^3 - \frac{1}{4}(2-1)^4 + \cdots \\ &\cdots + \frac{1}{5}(2-1)^5 - \frac{1}{6}(2-1)^6 \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \\ &= \frac{37}{60} \\ &\approx 0.616667. \end{aligned}$$

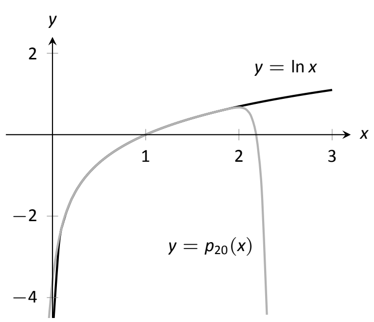


Figure 10.7.8: A plot of $y = \ln x$ and its 20$^{\text{th}}$ degree Taylor polynomial at $x = 1$.

This approximation is not terribly impressive: a hand held calculator shows that $\ln 2 \approx 0.693147$. The graph in Figure 10.7.7 shows that $p_6(x)$ provides less accurate approximations of $\ln x$ as x gets close to 0 or 2.

Surprisingly enough, even the 20$^{\text{th}}$ degree Taylor polynomial fails to approximate $\ln x$ for $x > 2$, as shown in Figure 10.7.8. We'll soon discuss why this is.

Taylor polynomials are used to approximate functions $f(x)$ in mainly two situations:

Note: Even though Taylor polynomials *could* be used in calculators and computers to calculate values of trigonometric functions, in practice they generally aren't. Other more efficient and accurate methods have been developed, such as the CORDIC algorithm.

1. When $f(x)$ is known, but perhaps "hard" to compute directly. For instance, we can define $y = \cos x$ as either the ratio of sides of a right triangle ("adjacent over hypotenuse") or with the unit circle. However, neither of these provides a convenient way of computing $\cos 2$. A Taylor polynomial of sufficiently high degree can provide a reasonable method of computing such values using only operations usually hard–wired into a computer ($+$, $-$, $\times$ and $\div$).

2. When $f(x)$ is not known, but information about its derivatives is known. This occurs more often than one might think, especially in the study of differential equations.

In both situations, a critical piece of information to have is "How good is my approximation?" If we use a Taylor polynomial to compute $\cos 2$, how do we know how accurate the approximation is?

Although much of the content presented in Calculus concerns the search for exact answers to problems such as integration and differentiation, many practical applications of calculus involve attempts to find *approximations*; for example, using Newton's Method to approximate the zeros of a function or numerical integration to approximate the value of an integral that cannot be solved exactly. Whenever an approximation is used, one naturally wishes to know how good the approximation is. In other words, we look for a bound on the error introduced by working with an approximation. The following theorem gives bounds on the error introduced in using a Taylor (and hence Maclaurin) polynomial to approximate a function.

Theorem 10.7.1 Taylor's Theorem

1. Let f be a function whose $n+1^{\text{th}}$ derivative exists on an interval I and let c be in I. Then, for each x in I, there exists z_x between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x),$$

where $R_n(x) = \dfrac{f^{(n+1)}(z_x)}{(n+1)!}(x-c)^{(n+1)}$.

2. $|R_n(x)| \leq \dfrac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|(x-c)^{(n+1)}\right|$

The first part of Taylor's Theorem states that $f(x) = p_n(x) + R_n(x)$, where $p_n(x)$ is the n^{th} order Taylor polynomial and $R_n(x)$ is the remainder, or error, in the Taylor approximation. The second part gives bounds on how big that error can be. If the $(n+1)^{\text{th}}$ derivative is large on I, the error may be large; if x is far from c, the error may also be large. However, the $(n+1)!$ term in the denominator tends to ensure that the error gets smaller as n increases.

The following example computes error estimates for the approximations of $\ln 1.5$ and $\ln 2$ made in Example 10.7.2.

Note: one way of quantifying the extent to which one function approximates another is using the *order* to which they agree. We say that two functions f and g **agree to order** n **at** c if n is the *largest* integer for which

$$\lim_{x\to c}\frac{f(x)-g(x)}{(x-c)^n} = 0.$$

Taylor's Theorem tells us that a function and its degree n Taylor polynomial agree to order n. Roughly speaking, this means that their difference goes to zero faster than the n^{th} power of $x-c$ as x approaches c.

Example 10.7.3 Finding error bounds of a Taylor polynomial

Use Theorem 10.7.1 to find error bounds when approximating $\ln 1.5$ and $\ln 2$ with $p_6(x)$, the Taylor polynomial of degree 6 of $f(x) = \ln x$ at $x = 1$, as calculated in Example 10.7.2.

SOLUTION

1. We start with the approximation of $\ln 1.5$ with $p_6(1.5)$. The theorem references an open interval I that contains both x and c. The smaller the interval we use the better; it will give us a more accurate (and smaller!) approximation of the error. We let $I = (0.9, 1.6)$, as this interval contains both $c = 1$ and $x = 1.5$.

   The theorem references $\max\left|f^{(n+1)}(z)\right|$. In our situation, this is asking "How big can the 7^{th} derivative of $y = \ln x$ be on the interval $(0.9, 1.6)$?" The seventh derivative is $y = -6!/x^7$. The largest value it attains on I is about 1506. Thus we can bound the error as:

$$\begin{aligned}\left|R_6(1.5)\right| &\leq \frac{\max\left|f^{(7)}(z)\right|}{7!}\left|(1.5-1)^7\right| \\ &\leq \frac{1506}{5040}\cdot\frac{1}{2^7} \\ &\approx 0.0023.\end{aligned}$$

We computed $p_6(1.5) = 0.404688$; using a calculator, we find $\ln 1.5 \approx 0.405465$, so the actual error is about 0.000778, which is less than our bound of 0.0023. This affirms Taylor's Theorem; the theorem states that our approximation would be within about 2 thousandths of the actual value, whereas the approximation was actually closer.

2. We again find an interval I that contains both $c = 1$ and $x = 2$; we choose $I = (0.9, 2.1)$. The maximum value of the seventh derivative of f on this interval is again about 1506 (as the largest values come near $x = 0.9$). Thus

$$\begin{aligned}\left|R_6(2)\right| &\leq \frac{\max\left|f^{(7)}(z)\right|}{7!}\left|(2-1)^7\right| \\ &\leq \frac{1506}{5040}\cdot 1^7 \\ &\approx 0.30.\end{aligned}$$

This bound is not as nearly as good as before. Using the degree 6 Taylor polynomial at $x = 1$ will bring us within 0.3 of the correct answer. As $p_6(2) \approx 0.61667$, our error estimate guarantees that the actual value of $\ln 2$ is somewhere between 0.31667 and 0.91667. These bounds are not particularly useful.

In reality, our approximation was only off by about 0.07. However, we are approximating ostensibly because we do not know the real answer. In order to be assured that we have a good approximation, we would have to resort to using a polynomial of higher degree.

We practice again. This time, we use Taylor's theorem to find n that guarantees our approximation is within a certain amount.

$f(x) = \cos x$	$\Rightarrow$	$f(0) = 1$
$f'(x) = -\sin x$	$\Rightarrow$	$f'(0) = 0$
$f''(x) = -\cos x$	$\Rightarrow$	$f''(0) = -1$
$f'''(x) = \sin x$	$\Rightarrow$	$f'''(0) = 0$
$f^{(4)}(x) = \cos x$	$\Rightarrow$	$f^{(4)}(0) = 1$
$f^{(5)}(x) = -\sin x$	$\Rightarrow$	$f^{(5)}(0) = 0$
$f^{(6)}(x) = -\cos x$	$\Rightarrow$	$f^{(6)}(0) = -1$
$f^{(7)}(x) = \sin x$	$\Rightarrow$	$f^{(7)}(0) = 0$
$f^{(8)}(x) = \cos x$	$\Rightarrow$	$f^{(8)}(0) = 1$
$f^{(9)}(x) = -\sin x$	$\Rightarrow$	$f^{(9)}(0) = 0$

Figure 10.7.9: A table of the derivatives of $f(x) = \cos x$ evaluated at $x = 0$.

Example 10.7.4 Finding sufficiently accurate Taylor polynomials

Find n such that the n^{th} Taylor polynomial of $f(x) = \cos x$ at $x = 0$ approximates $\cos 2$ to within 0.001 of the actual answer. What is $p_n(2)$?

Solution Following Taylor's theorem, we need bounds on the size of the derivatives of $f(x) = \cos x$. In the case of this trigonometric function, this is easy. All derivatives of cosine are $\pm\sin x$ or $\pm\cos x$. In all cases, these functions are never greater than 1 in absolute value. We want the error to be less than 0.001. To find the appropriate n, consider the following inequalities:

$$\begin{aligned}\frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|(2-0)^{(n+1)}\right| &\leq 0.001 \\ \frac{1}{(n+1)!}\cdot 2^{(n+1)} &\leq 0.001\end{aligned}$$

We find an n that satisfies this last inequality with trial–and–error. When $n = 8$, we have $\dfrac{2^{8+1}}{(8+1)!} \approx 0.0014$; when $n = 9$, we have $\dfrac{2^{9+1}}{(9+1)!} \approx 0.000282 <$

0.001. Thus we want to approximate $\cos 2$ with $p_9(2)$.

We now set out to compute $p_9(x)$. We again need a table of the derivatives of $f(x) = \cos x$ evaluated at $x = 0$. A table of these values is given in Figure 10.7.9. Notice how the derivatives, evaluated at $x = 0$, follow a certain pattern. All the odd powers of x in the Taylor polynomial will disappear as their coefficient is 0. While our error bounds state that we need $p_9(x)$, our work shows that this will be the same as $p_8(x)$.

Since we are forming our polynomial at $x = 0$, we are creating a Maclaurin polynomial, and:

$$\begin{aligned} p_8(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(8)}(0)}{8!}x^8 \\ &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 \end{aligned}$$

We finally approximate $\cos 2$:

$$\cos 2 \approx p_8(2) = -\frac{131}{315} \approx -0.41587.$$

Our error bound guarantee that this approximation is within 0.001 of the correct answer. Technology shows us that our approximation is actually within about 0.0003 of the correct answer.

Figure 10.7.10 shows a graph of $y = p_8(x)$ and $y = \cos x$. Note how well the two functions agree on about $(-\pi, \pi)$.

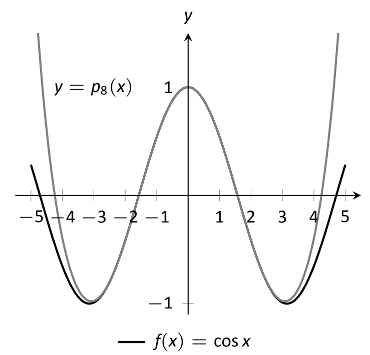


Figure 10.7.10: A graph of $f(x) = \cos x$ and its degree 8 Maclaurin polynomial.

Example 10.7.5 Finding and using Taylor polynomials

1. Find the degree 4 Taylor polynomial, $p_4(x)$, for $f(x) = \sqrt{x}$ at $x = 4$.
2. Use $p_4(x)$ to approximate $\sqrt{3}$.
3. Find bounds on the error when approximating $\sqrt{3}$ with $p_4(3)$.

Solution

1. We begin by evaluating the derivatives of f at $x = 4$. This is done in Figure 10.7.11. These values allow us to form the Taylor polynomial $p_4(x)$:

$$p_4(x) = 2 + \frac{1}{4}(x-4) + \frac{-1/32}{2!}(x-4)^2 + \frac{3/256}{3!}(x-4)^3 + \frac{-15/2048}{4!}(x-4)^4.$$

2. As $p_4(x) \approx \sqrt{x}$ near $x = 4$, we approximate $\sqrt{3}$ with $p_4(3) = 1.73212$.

3. To find a bound on the error, we need an open interval that contains $x = 3$ and $x = 4$. We set $I = (2.9, 4.1)$. The largest value the fifth derivative of $f(x) = \sqrt{x}$ takes on this interval is near $x = 2.9$, at about 0.0273. Thus

$$\left|R_4(3)\right| \le \frac{0.0273}{5!}\left|(3-4)^5\right| \approx 0.00023.$$

This shows our approximation is accurate to at least the first 2 places after the decimal. (It turns out that our approximation is actually accurate to 4 places after the decimal.) A graph of $f(x) = \sqrt{x}$ and $p_4(x)$ is given in Figure 10.7.12. Note how the two functions are nearly indistinguishable on $(2, 7)$.

$$\begin{aligned} f(x) &= \sqrt{x} & &\Rightarrow & f(4) &= 2 \\ f'(x) &= \frac{1}{2\sqrt{x}} & &\Rightarrow & f'(4) &= \frac{1}{4} \\ f''(x) &= \frac{-1}{4x^{3/2}} & &\Rightarrow & f''(4) &= \frac{-1}{32} \\ f'''(x) &= \frac{3}{8x^{5/2}} & &\Rightarrow & f'''(4) &= \frac{3}{256} \\ f^{(4)}(x) &= \frac{-15}{16x^{7/2}} & &\Rightarrow & f^{(4)}(4) &= \frac{-15}{2048} \end{aligned}$$

Figure 10.7.11: A table of the derivatives of $f(x) = \sqrt{x}$ evaluated at $x = 4$.

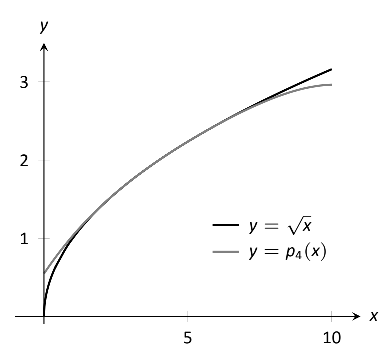


Figure 10.7.12: A graph of $f(x) = \sqrt{x}$ and its degree 4 Taylor polynomial at $x = 4$.

Our final example gives a brief introduction to using Taylor polynomials to solve differential equations.

Example 10.7.6 Approximating an unknown function

A function $y = f(x)$ is unknown save for the following two facts.

1. $y(0) = f(0) = 1$, and
2. $y' = y^2$

(This second fact says that amazingly, the derivative of the function is actually the function squared!)

Find the degree 3 Maclaurin polynomial $p_3(x)$ of $y = f(x)$.

SOLUTION One might initially think that not enough information is given to find $p_3(x)$. However, note how the second fact above actually lets us know what $y'(0)$ is:

$$y' = y^2 \Rightarrow y'(0) = y^2(0).$$

Since $y(0) = 1$, we conclude that $y'(0) = 1$.

Now we find information about y''. Starting with $y' = y^2$, take derivatives of both sides, *with respect to x*. That means we must use implicit differentiation.

$$\begin{aligned} y' &= y^2 \\ \frac{d}{dx}\left(y'\right) &= \frac{d}{dx}\left(y^2\right) \\ y'' &= 2y \cdot y'. \end{aligned}$$

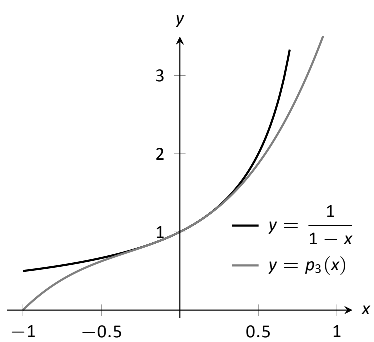


Figure 10.7.13: A graph of $y = -1/(x-1)$ and $y = p_3(x)$ from Example 10.7.6.

Now evaluate both sides at $x = 0$:

$$\begin{aligned} y''(0) &= 2y(0) \cdot y'(0) \\ y''(0) &= 2 \end{aligned}$$

We repeat this once more to find $y'''(0)$. We again use implicit differentiation; this time the Product Rule is also required.

$$\begin{aligned} \frac{d}{dx}\left(y''\right) &= \frac{d}{dx}\left(2yy'\right) \\ y''' &= 2y' \cdot y' + 2y \cdot y''. \end{aligned}$$

Now evaluate both sides at $x = 0$:

$$\begin{aligned} y'''(0) &= 2y'(0)^2 + 2y(0)y''(0) \\ y'''(0) &= 2 + 4 = 6 \end{aligned}$$

In summary, we have:

$$y(0) = 1 \qquad y'(0) = 1 \qquad y''(0) = 2 \qquad y'''(0) = 6.$$

We can now form $p_3(x)$:

$$\begin{aligned} p_3(x) &= 1 + x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 \\ &= 1 + x + x^2 + x^3. \end{aligned}$$

It turns out that the differential equation we started with, $y' = y^2$, where $y(0) = 1$, can be solved without too much difficulty: $y = \dfrac{1}{1-x}$. Figure 10.7.13 shows this function plotted with $p_3(x)$. Note how similar they are near $x = 0$.

It is beyond the scope of this text to pursue error analysis when using Taylor polynomials to approximate solutions to differential equations. This topic is often broached in introductory Differential Equations courses and usually covered in depth in Numerical Analysis courses. Such an analysis is very important; one needs to know how good their approximation is. We explored this example simply to demonstrate the usefulness of Taylor polynomials.

Most of this chapter has been devoted to the study of infinite series. This section has taken a step back from this study, focusing instead on finite summation of terms. In the next section, we explore **Taylor Series**, where we represent a function with an infinite series.

Exercises 10.7

Terms and Concepts

1. What is the difference between a Taylor polynomial and a Maclaurin polynomial?

2. T/F: In general, $p_n(x)$ approximates $f(x)$ better and better as n gets larger.

3. For some function $f(x)$, the Maclaurin polynomial of degree 4 is $p_4(x) = 6 + 3x - 4x^2 + 5x^3 - 7x^4$. What is $p_2(x)$?

4. For some function $f(x)$, the Maclaurin polynomial of degree 4 is $p_4(x) = 6 + 3x - 4x^2 + 5x^3 - 7x^4$. What is $f'''(0)$?

Problems

In Exercises 5 – 12, find the Maclaurin polynomial of degree n for the given function.

5. $f(x) = e^{-x}, \quad n = 3$

6. $f(x) = \sin x, \quad n = 8$

7. $f(x) = x \cdot e^x, \quad n = 5$

8. $f(x) = \tan x, \quad n = 6$

9. $f(x) = e^{2x}, \quad n = 4$

10. $f(x) = \dfrac{1}{1-x}, \quad n = 4$

11. $f(x) = \dfrac{1}{1+x}, \quad n = 4$

12. $f(x) = \dfrac{1}{1+x}, \quad n = 7$

In Exercises 13 – 20, find the Taylor polynomial of degree n, at $x = c$, for the given function.

13. $f(x) = \sqrt{x}, \quad n = 4, \quad c = 1$

14. $f(x) = \ln(x+1), \quad n = 4, \quad c = 1$

15. $f(x) = \cos x, \quad n = 6, \quad c = \pi/4$

16. $f(x) = \sin x, \quad n = 5, \quad c = \pi/6$

17. $f(x) = \dfrac{1}{x}, \quad n = 5, \quad c = 2$

18. $f(x) = \dfrac{1}{x^2}, \quad n = 8, \quad c = 1$

19. $f(x) = \dfrac{1}{x^2+1}, \quad n = 3, \quad c = -1$

20. $f(x) = x^2 \cos x, \quad n = 2, \quad c = \pi$

In Exercises 21 – 24, approximate the function value with the indicated Taylor polynomial and give approximate bounds on the error.

21. Approximate $\sin 0.1$ with the Maclaurin polynomial of degree 3.

22. Approximate $\cos 1$ with the Maclaurin polynomial of degree 4.

23. Approximate $\sqrt{10}$ with the Taylor polynomial of degree 2 centered at $x = 9$.

24. Approximate $\ln 1.5$ with the Taylor polynomial of degree 3 centered at $x = 1$.

Exercises 25 – 28 ask for an n to be found such that $p_n(x)$ approximates $f(x)$ within a certain bound of accuracy.

25. Find n such that the Maclaurin polynomial of degree n of $f(x) = e^x$ approximates e within 0.0001 of the actual value.

26. Find n such that the Taylor polynomial of degree n of $f(x) = \sqrt{x}$, centered at $x = 4$, approximates $\sqrt{3}$ within 0.0001 of the actual value.

27. Find n such that the Maclaurin polynomial of degree n of $f(x) = \cos x$ approximates $\cos \pi/3$ within 0.0001 of the actual value.

28. Find n such that the Maclaurin polynomial of degree n of $f(x) = \sin x$ approximates $\cos \pi$ within 0.0001 of the actual value.

In Exercises 29 – 34, find the n^{th} term of the indicated Taylor polynomial.

29. Find a formula for the n^{th} term of the Maclaurin polynomial for $f(x) = e^x$.

30. Find a formula for the n^{th} term of the Maclaurin polynomial for $f(x) = \cos x$.

31. Find a formula for the n^{th} term of the Maclaurin polynomial for $f(x) = \sin x$.

32. Find a formula for the n^{th} term of the Maclaurin polynomial for $f(x) = \dfrac{1}{1-x}$.

33. Find a formula for the n^{th} term of the Maclaurin polynomial for $f(x) = \dfrac{1}{1+x}$.

34. Find a formula for the n^{th} term of the Taylor polynomial for $f(x) = \ln x$ centred at $x = 1$.

In Exercises 35 – 37, approximate the solution to the given differential equation with a degree 4 Maclaurin polynomial.

35. $y' = y, \qquad y(0) = 1$

36. $y' = 5y, \qquad y(0) = 3$

37. $y' = \dfrac{2}{y}, \qquad y(0) = 1$

10.8 Taylor Series

In Section 10.6, we showed how certain functions can be represented by a power series function. In Section 10.7, we showed how we can approximate functions with polynomials, given that enough derivative information is available. In this section we combine these concepts: if a function $f(x)$ is infinitely differentiable, we show how to represent it with a power series function.

Definition 10.8.1 Taylor and Maclaurin Series

Let $f(x)$ have derivatives of all orders at $x = c$.

1. The **Taylor Series of** $f(x)$**, centred at** c is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

2. Setting $c = 0$ gives the **Maclaurin Series of** $f(x)$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

If $p_n(x)$ is the n^{th} degree Taylor polynomial for $f(x)$ centred at $x = c$, we saw how $f(x)$ is *approximately equal* to $p_n(x)$ near $x = c$. We also saw how increasing the degree of the polynomial generally reduced the error.

We are now considering *series*, where we sum an infinite set of terms. Our ultimate hope is to see the error vanish and claim a function is *equal* to its Taylor series.

When creating the Taylor polynomial of degree n for a function $f(x)$ at $x = c$, we needed to evaluate f, and the first n derivatives of f, at $x = c$. When creating the Taylor series of f, it helps to find a pattern that describes the n^{th} derivative of f at $x = c$. We demonstrate this in the next two examples.

$$\begin{array}{lcl}
f(x) = \cos x & \Rightarrow & f(0) = 1 \\
f'(x) = -\sin x & \Rightarrow & f'(0) = 0 \\
f''(x) = -\cos x & \Rightarrow & f''(0) = -1 \\
f'''(x) = \sin x & \Rightarrow & f'''(0) = 0 \\
f^{(4)}(x) = \cos x & \Rightarrow & f^{(4)}(0) = 1 \\
f^{(5)}(x) = -\sin x & \Rightarrow & f^{(5)}(0) = 0 \\
f^{(6)}(x) = -\cos x & \Rightarrow & f^{(6)}(0) = -1 \\
f^{(7)}(x) = \sin x & \Rightarrow & f^{(7)}(0) = 0 \\
f^{(8)}(x) = \cos x & \Rightarrow & f^{(8)}(0) = 1 \\
f^{(9)}(x) = -\sin x & \Rightarrow & f^{(9)}(0) = 0
\end{array}$$

Figure 10.8.1: A table of the derivatives of $f(x) = \cos x$ evaluated at $x = 0$.

Example 10.8.1 The Maclaurin series of $f(x) = \cos x$

Find the Maclaurin series of $f(x) = \cos x$.

Solution In Example 10.7.4 we found the 8^{th} degree Maclaurin polynomial of $\cos x$. In doing so, we created the table shown in Figure 10.8.1. Notice how $f^{(n)}(0) = 0$ when n is odd, $f^{(n)}(0) = 1$ when n is divisible by 4, and $f^{(n)}(0) = -1$ when n is even but not divisible by 4. Thus the Maclaurin series of $\cos x$ is

$$1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

We can go further and write this as a summation. Since we only need the terms where the power of x is even, we write the power series in terms of x^{2n}:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Example 10.8.2 The Taylor series of $f(x) = \ln x$ **at** $x = 1$

Find the Taylor series of $f(x) = \ln x$ centred at $x = 1$.

Solution Figure 10.8.2 shows the n^{th} derivative of $\ln x$ evaluated at $x = 1$ for $n = 0, \ldots, 5$, along with an expression for the n^{th} term:

$$f^{(n)}(1) = (-1)^{n+1}(n-1)! \quad \text{for } n \geq 1.$$

Remember that this is what distinguishes Taylor series from Taylor polynomials; we are very interested in finding a pattern for the n^{th} term, not just finding a finite set of coefficients for a polynomial. Since $f(1) = \ln 1 = 0$, we skip the first term and start the summation with $n = 1$, giving the Taylor series for $\ln x$, centred at $x = 1$, as

$$\sum_{n=1}^{\infty}(-1)^{n+1}(n-1)!\frac{1}{n!}(x-1)^n = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}.$$

$$\begin{aligned}
f(x) &= \ln x & &\Rightarrow & f(1) &= 0\\
f'(x) &= 1/x & &\Rightarrow & f'(1) &= 1\\
f''(x) &= -1/x^2 & &\Rightarrow & f''(1) &= -1\\
f'''(x) &= 2/x^3 & &\Rightarrow & f'''(1) &= 2\\
f^{(4)}(x) &= -6/x^4 & &\Rightarrow & f^{(4)}(1) &= -6\\
f^{(5)}(x) &= 24/x^5 & &\Rightarrow & f^{(5)}(1) &= 24\\
&\vdots & & & &\vdots\\
f^{(n)}(x) &= \frac{(-1)^{n+1}(n-1)!}{x^n} & &\Rightarrow & f^{(n)}(1) &= (-1)^{n+1}(n-1)!
\end{aligned}$$

Figure 10.8.2: Derivatives of $\ln x$ evaluated at $x = 1$.

It is important to note that Definition 10.8.1 defines a Taylor series given a function $f(x)$; however, we *cannot* yet state that $f(x)$ *is equal* to its Taylor series. We will find that "most of the time" they are equal, but we need to consider the conditions that allow us to conclude this.

Theorem 10.7.1 states that the error between a function $f(x)$ and its n^{th}–degree Taylor polynomial $p_n(x)$ is $R_n(x)$, where

$$|R_n(x)| \leq \frac{\max|f^{(n+1)}(z)|}{(n+1)!}\left|(x-c)^{(n+1)}\right|.$$

If $R_n(x)$ goes to 0 for each x in an interval I as n approaches infinity, we conclude that the function is equal to its Taylor series expansion.

Theorem 10.8.1 Function and Taylor Series Equality

Let $f(x)$ have derivatives of all orders at $x = c$, let $R_n(x)$ be as stated in Theorem 10.7.1, and let I be an interval on which the Taylor series of $f(x)$ converges. If $\lim\limits_{n\to\infty} R_n(x) = 0$ for all x in I, then

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n \text{ on } I.$$

We demonstrate the use of this theorem in an example.

Example 10.8.3 Establishing equality of a function and its Taylor series

Show that $f(x) = \cos x$ is equal to its Maclaurin series, as found in Example 10.8.1, for all x.

Solution Given a value x, the magnitude of the error term $R_n(x)$ is bounded by

$$|R_n(x)| \leq \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|x^{n+1}\right|.$$

Since all derivatives of $\cos x$ are $\pm\sin x$ or $\pm\cos x$, whose magnitudes are bounded by 1, we can state

$$|R_n(x)| \leq \frac{1}{(n+1)!}\left|x^{n+1}\right|$$

which implies

$$-\frac{|x^{n+1}|}{(n+1)!} \leq R_n(x) \leq \frac{|x^{n+1}|}{(n+1)!}. \qquad (10.8)$$

For any x, $\lim_{n\to\infty} \dfrac{x^{n+1}}{(n+1)!} = 0$. Applying the Squeeze Theorem to Equation (10.8), we conclude that $\lim_{n\to\infty} R_n(x) = 0$ for all x, and hence

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{for all } x.$$

It is natural to assume that a function is equal to its Taylor series on the series' interval of convergence, but this is not always the case. In order to properly establish equality, one must use Theorem 10.8.1. This is a bit disappointing, as we developed beautiful techniques for determining the interval of convergence of a power series, and proving that $R_n(x) \to 0$ can be difficult. For instance, it is not a simple task to show that $\ln x$ equals its Taylor series on $(0, 2]$ as found in Example 10.8.2; in the Exercises, the reader is only asked to show equality on $(1, 2)$, which is simpler.

There is good news. A function $f(x)$ that is equal to its Taylor series, centred at any point the domain of $f(x)$, is said to be an **analytic function**, and most, if not all, functions that we encounter within this course are analytic functions. Generally speaking, any function that one creates with elementary functions (polynomials, exponentials, trigonometric functions, etc.) that is not piecewise defined is probably analytic. For most functions, we assume the function is equal to its Taylor series on the series' interval of convergence and only use Theorem 10.8.1 when we suspect something may not work as expected.

We develop the Taylor series for one more important function, then give a table of the Taylor series for a number of common functions.

Example 10.8.4 The Binomial Series

Find the Maclaurin series of $f(x) = (1+x)^k$, $k \neq 0$.

SOLUTION When k is a positive integer, the Maclaurin series is finite. For instance, when $k = 4$, we have

$$f(x) = (1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4.$$

The coefficients of x when k is a positive integer are known as the *binomial coefficients*, giving the series we are developing its name.

When $k = 1/2$, we have $f(x) = \sqrt{1+x}$. Knowing a series representation of this function would give a useful way of approximating $\sqrt{1.3}$, for instance.

To develop the Maclaurin series for $f(x) = (1+x)^k$ for any value of $k \neq 0$, we consider the derivatives of f evaluated at $x = 0$:

$$\begin{aligned}
f(x) &= (1+x)^k & f(0) &= 1\\
f'(x) &= k(1+x)^{k-1} & f'(0) &= k\\
f''(x) &= k(k-1)(1+x)^{k-2} & f''(0) &= k(k-1)\\
f'''(x) &= k(k-1)(k-2)(1+x)^{k-3} & f'''(0) &= k(k-1)(k-2)\\
&\vdots & &\vdots\\
f^{(n)}(x) &= k(k-1)\cdots\big(k-(n-1)\big)(1+x)^{k-n} & f^{(n)}(0) &= k(k-1)\cdots\big(k-(n-1)\big)
\end{aligned}$$

Thus the Maclaurin series for $f(x) = (1+x)^k$ is

$$1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \ldots + \frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!}x^n + \ldots$$

It is important to determine the interval of convergence of this series. With

$$a_n = \frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!}x^n,$$

we apply the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty}\left|\frac{k(k-1)\cdots(k-n)}{(n+1)!}x^{n+1}\right| \bigg/ \left|\frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!}x^n\right| \\
&= \lim_{n\to\infty}\left|\frac{k-n}{n}x\right| \\
&= |x|.
\end{aligned}$$

The series converges absolutely when the limit of the Ratio Test is less than 1; therefore, we have absolute convergence when $|x| < 1$.

While outside the scope of this text, the interval of convergence depends on the value of k. When $k > 0$, the interval of convergence is $[-1, 1]$. When $-1 < k < 0$, the interval of convergence is $[-1, 1)$. If $k \leq -1$, the interval of convergence is $(-1, 1)$.

We learned that Taylor polynomials offer a way of approximating a "difficult to compute" function with a polynomial. Taylor series offer a way of exactly representing a function with a series. One probably can see the use of a good approximation; is there any use of representing a function exactly as a series?

While we should not overlook the mathematical beauty of Taylor series (which is reason enough to study them), there are practical uses as well. They provide a valuable tool for solving a variety of problems, including problems relating to integration and differential equations.

In Key Idea 10.8.1 (on the following page) we give a table of the Taylor series of a number of common functions. We then give a theorem about the "algebra of power series," that is, how we can combine power series to create power series of new functions. This allows us to find the Taylor series of functions like $f(x) = e^x \cos x$ by knowing the Taylor series of e^x and $\cos x$.

Before we investigate combining functions, consider the Taylor series for the arctangent function (see Key Idea 10.8.1). Knowing that $\tan^{-1}(1) = \pi/4$, we can use this series to approximate the value of π:

$$\begin{aligned}
\frac{\pi}{4} &= \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \\
\pi &= 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)
\end{aligned}$$

Unfortunately, this particular expansion of π converges very slowly. The first 100 terms approximate π as 3.13159, which is not particularly good.

Key Idea 10.8.1 Important Taylor Series Expansions

Function and Series	First Few Terms	Interval of Convergence
$e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	$(-\infty, \infty)$
$\sin x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$	$(-\infty, \infty)$
$\cos x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$	$(-\infty, \infty)$
$\ln x = \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$	$(x-1) - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3} - \cdots$	$(0, 2]$
$\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + \cdots$	$(-1, 1)$
$(1+x)^k = \displaystyle\sum_{n=0}^{\infty} \frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!} x^n$	$1 + kx + \dfrac{k(k-1)}{2!} x^2 + \cdots$	$(-1, 1)$[a]
$\tan^{-1} x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$	$[-1, 1]$

[a] Convergence at $x = \pm 1$ depends on the value of k.

Theorem 10.8.2 Algebra of Power Series

Let $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \displaystyle\sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and let $h(x)$ be continuous.

1. $f(x) \pm g(x) = \displaystyle\sum_{n=0}^{\infty} (a_n \pm b_n) x^n \quad$ for $|x| < R$.
2. $f(x)g(x) = \left(\displaystyle\sum_{n=0}^{\infty} a_n x^n\right)\left(\displaystyle\sum_{n=0}^{\infty} b_n x^n\right) = \displaystyle\sum_{n=0}^{\infty} \big(a_0 b_n + a_1 b_{n-1} + \ldots a_n b_0\big) x^n$ for $|x| < R$.
3. $f\big(h(x)\big) = \displaystyle\sum_{n=0}^{\infty} a_n \big(h(x)\big)^n \quad$ for $|h(x)| < R$.

Example 10.8.5 Combining Taylor series

Write out the first 3 terms of the Taylor Series for $f(x) = e^x \cos x$ using Key Idea 10.8.1 and Theorem 10.8.2.

Solution Key Idea 10.8.1 informs us that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{and} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots .$$

Applying Theorem 10.8.2, we find that

$$e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right).$$

Distribute the right hand expression across the left:

$$= 1\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{x^2}{2!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$$
$$+ \frac{x^3}{3!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{x^4}{4!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \cdots$$

Distribute again and collect like terms.

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \frac{x^7}{630} + \cdots$$

While this process is a bit tedious, it is much faster than evaluating all the necessary derivatives of $e^x \cos x$ and computing the Taylor series directly.

Because the series for e^x and $\cos x$ both converge on $(-\infty, \infty)$, so does the series expansion for $e^x \cos x$.

Note: In Example 10.8.6, one could create a series for $\ln(\sqrt{x})$ by simply recognizing that $\ln(\sqrt{x}) = \ln(x^{1/2}) = 1/2 \ln x$, and hence multiplying the Taylor series for $\ln x$ by $1/2$. This example was chosen to demonstrate other aspects of series, such as the fact that the interval of convergence changes.

Example 10.8.6 Creating new Taylor series

Use Theorem 10.8.2 to create series for $y = \sin(x^2)$ and $y = \ln(\sqrt{x})$.

Solution Given that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

we simply substitute x^2 for x in the series, giving

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} \cdots .$$

Since the Taylor series for $\sin x$ has an infinite radius of convergence, so does the Taylor series for $\sin(x^2)$.

The Taylor expansion for $\ln x$ given in Key Idea 10.8.1 is centred at $x = 1$, so we will center the series for $\ln(\sqrt{x})$ at $x = 1$ as well. With

$$\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots,$$

we substitute $\sqrt{x}$ for x to obtain

$$\ln(\sqrt{x}) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\sqrt{x}-1)^n}{n} = (\sqrt{x}-1) - \frac{(\sqrt{x}-1)^2}{2} + \frac{(\sqrt{x}-1)^3}{3} - \cdots .$$

While this is not strictly a power series, it is a series that allows us to study the function $\ln(\sqrt{x})$. Since the interval of convergence of $\ln x$ is $(0, 2]$, and the range of $\sqrt{x}$ on $(0, 4]$ is $(0, 2]$, the interval of convergence of this series expansion of $\ln(\sqrt{x})$ is $(0, 4]$.

Example 10.8.7 Using Taylor series to evaluate definite integrals

Use the Taylor series of e^{-x^2} to evaluate $\displaystyle\int_0^1 e^{-x^2}\,dx$.

Solution We learned, when studying Numerical Integration, that e^{-x^2} does not have an antiderivative expressible in terms of elementary functions. This means any definite integral of this function must have its value approximated, and not computed exactly.

We can quickly write out the Taylor series for e^{-x^2} using the Taylor series of e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and so

$$\begin{aligned} e^{-x^2} &= \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots . \end{aligned}$$

We use Theorem 10.6.3 to integrate:

$$\int e^{-x^2}\,dx = C + x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots$$

This *is* the antiderivative of e^{-x^2}; while we can write it out as a series, we cannot write it out in terms of elementary functions. We can evaluate the definite integral $\displaystyle\int_0^1 e^{-x^2}\,dx$ using this antiderivative; substituting 1 and 0 for x and subtracting gives

$$\int_0^1 e^{-x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} \cdots .$$

Summing the 5 terms shown above give the approximation of 0.74749. Since this is an alternating series, we can use the Alternating Series Approximation Theorem, (Theorem 10.5.2), to determine how accurate this approximation is. The next term of the series is $1/(11\cdot 5!) \approx 0.00075758$. Thus we know our approximation is within 0.00075758 of the actual value of the integral. This is arguably much less work than using Simpson's Rule to approximate the value of the integral.

Example 10.8.8 Using Taylor series to solve differential equations

Solve the differential equation $y' = 2y$ in terms of a power series, and use the theory of Taylor series to recognize the solution in terms of an elementary function.

Solution We found the first 5 terms of the power series solution to this differential equation in Example 10.6.5 in Section 10.6. These are:

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = \frac{4}{2} = 2, \quad a_3 = \frac{8}{2\cdot 3} = \frac{4}{3}, \quad a_4 = \frac{16}{2\cdot 3\cdot 4} = \frac{2}{3}.$$

We include the "unsimplified" expressions for the coefficients found in Example 10.6.5 as we are looking for a pattern. It can be shown that $a_n = 2^n/n!$. Thus the solution, written as a power series, is

$$y = \sum_{n=0}^{\infty} \frac{2^n}{n!}x^n = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}.$$

Using Key Idea 10.8.1 and Theorem 10.8.2, we recognize $f(x) = e^{2x}$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \Rightarrow \qquad e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}.$$

Finding a pattern in the coefficients that match the series expansion of a known function, such as those shown in Key Idea 10.8.1, can be difficult. What if the coefficients in the previous example were given in their reduced form; how could we still recover the function $y = e^{2x}$?

Suppose that all we know is that

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = 2, \quad a_3 = \frac{4}{3}, \quad a_4 = \frac{2}{3}.$$

Definition 10.8.1 states that each term of the Taylor expansion of a function includes an $n!$. This allows us to say that

$$a_2 = 2 = \frac{b_2}{2!}, \quad a_3 = \frac{4}{3} = \frac{b_3}{3!}, \quad \text{and} \quad a_4 = \frac{2}{3} = \frac{b_4}{4!}$$

for some values b_2, b_3 and b_4. Solving for these values, we see that $b_2 = 4$, $b_3 = 8$ and $b_4 = 16$. That is, we are recovering the pattern we had previously seen, allowing us to write

$$\begin{aligned} f(x) = \sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} \frac{b_n}{n!}x^n \\ &= 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4 + \cdots \end{aligned}$$

From here it is easier to recognize that the series is describing an exponential function.

There are simpler, more direct ways of solving the differential equation $y' = 2y$. We applied power series techniques to this equation to demonstrate its utility, and went on to show how *sometimes* we are able to recover the solution in terms of elementary functions using the theory of Taylor series. Most differential equations faced in real scientific and engineering situations are much more complicated than this one, but power series can offer a valuable tool in finding, or at least approximating, the solution.

This chapter introduced sequences, which are ordered lists of numbers, followed by series, wherein we add up the terms of a sequence. We quickly saw that such sums do not always add up to "infinity," but rather converge. We studied tests for convergence, then ended the chapter with a formal way of defining functions based on series. Such "series–defined functions" are a valuable tool in solving a number of different problems throughout science and engineering.

Coming in the next chapters are new ways of defining curves in the plane apart from using functions of the form $y = f(x)$. Curves created by these new methods can be beautiful, useful, and important.

Exercises 10.8

Terms and Concepts

1. What is the difference between a Taylor polynomial and a Taylor series?

2. What theorem must we use to show that a function is equal to its Taylor series?

Problems

Key Idea 10.8.1 gives the n^{th} term of the Taylor series of common functions. In Exercises 3 – 6, verify the formula given in the Key Idea by finding the first few terms of the Taylor series of the given function and identifying a pattern.

3. $f(x) = e^x; \quad c = 0$

4. $f(x) = \sin x; \quad c = 0$

5. $f(x) = 1/(1-x); \quad c = 0$

6. $f(x) = \tan^{-1} x; \quad c = 0$

In Exercises 7 – 12, find a formula for the n^{th} term of the Taylor series of $f(x)$, centered at c, by finding the coefficients of the first few powers of x and looking for a pattern. (The formulas for several of these are found in Key Idea 10.8.1; show work verifying these formula.)

7. $f(x) = \cos x; \quad c = \pi/2$

8. $f(x) = 1/x; \quad c = 1$

9. $f(x) = e^{-x}; \quad c = 0$

10. $f(x) = \ln(1+x); \quad c = 0$

11. $f(x) = x/(x+1); \quad c = 1$

12. $f(x) = \sin x; \quad c = \pi/4$

In Exercises 13 – 16, show that the Taylor series for $f(x)$, as given in Key Idea 10.8.1, is equal to $f(x)$ by applying Theorem 10.8.1; that is, show $\lim\limits_{n\to\infty} R_n(x) = 0$.

13. $f(x) = e^x$

14. $f(x) = \sin x$

15. $f(x) = \ln x$ (show equality only on $(1,2)$)

16. $f(x) = 1/(1-x)$ (show equality only on $(-1,0)$)

In Exercises 17 – 20, use the Taylor series given in Key Idea 10.8.1 to verify the given identity.

17. $\cos(-x) = \cos x$

18. $\sin(-x) = -\sin x$

19. $\frac{d}{dx}\big(\sin x\big) = \cos x$

20. $\frac{d}{dx}\big(\cos x\big) = -\sin x$

In Exercises 21 – 24, write out the first 5 terms of the Binomial series with the given k-value.

21. $k = 1/2$

22. $k = -1/2$

23. $k = 1/3$

24. $k = 4$

In Exercises 25 – 30, use the Taylor series given in Key Idea 10.8.1 to create the Taylor series of the given functions.

25. $f(x) = \cos\left(x^2\right)$

26. $f(x) = e^{-x}$

27. $f(x) = \sin\left(2x+3\right)$

28. $f(x) = \tan^{-1}\left(x/2\right)$

29. $f(x) = e^x \sin x$ (only find the first 4 terms)

30. $f(x) = (1+x)^{1/2}\cos x$ (only find the first 4 terms)

In Exercises 31 – 32, approximate the value of the given definite integral by using the first 4 nonzero terms of the integrand's Taylor series.

31. $\displaystyle\int_0^{\sqrt{\pi}} \sin\left(x^2\right)\,dx$

32. $\displaystyle\int_0^{\pi^2/4} \cos\left(\sqrt{x}\right)\,dx$

11: VECTORS

This chapter introduces a new mathematical object, the **vector**. Defined in Section 11.2, we will see that vectors provide a powerful language for describing quantities that have magnitude and direction aspects. A simple example of such a quantity is force: when applying a force, one is generally interested in how much force is applied (i.e., the magnitude of the force) and the direction in which the force was applied. Vectors will play an important role in many of the subsequent chapters in this text.

This chapter begins with moving our mathematics out of the plane and into "space." That is, we begin to think mathematically not only in two dimensions, but in three. With this foundation, we can explore vectors both in the plane and in space.

11.1 Introduction to Cartesian Coordinates in Space

Up to this point in this text we have considered mathematics in a 2–dimensional world. We have plotted graphs on the *x*-*y* plane using rectangular and polar coordinates and found the area of regions in the plane. We have considered properties of *solid* objects, such as volume and surface area, but only by first defining a curve in the plane and then rotating it out of the plane.

While there is wonderful mathematics to explore in "2D," we live in a "3D" world and eventually we will want to apply mathematics involving this third dimension. In this section we introduce Cartesian coordinates in space and explore basic surfaces. This will lay a foundation for much of what we do in the remainder of the text.

Each point P in space can be represented with an ordered triple, $P = (a, b, c)$, where a, b and c represent the relative position of P along the x-, y- and z-axes, respectively. Each axis is perpendicular to the other two.

Visualizing points in space on paper can be problematic, as we are trying to represent a 3-dimensional concept on a 2–dimensional medium. We cannot draw three lines representing the three axes in which each line is perpendicular to the other two. Despite this issue, standard conventions exist for plotting shapes in space that we will discuss that are more than adequate.

One convention is that the axes must conform to the **right hand rule**. This rule states that when the index finger of the right hand is extended in the direction of the positive x-axis, and the middle finger (bent "inward" so it is perpendicular to the palm) points along the positive y-axis, then the extended thumb will point in the direction of the positive z-axis. (It may take some thought to verify this, but this system is inherently different from the one created by using the "left hand rule.")

As long as the coordinate axes are positioned so that they follow this rule, it does not matter how the axes are drawn on paper. There are two popular methods that we briefly discuss.

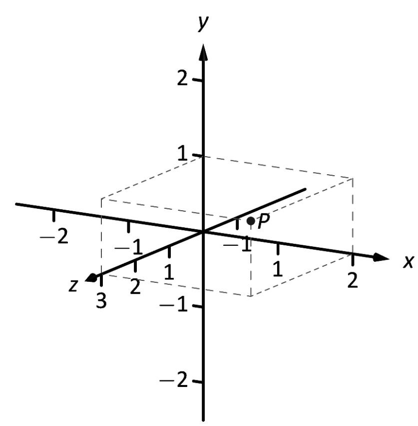


Figure 11.1.1: Plotting the point $P = (2, 1, 3)$ in space.

In Figure 11.1.1 we see the point $P = (2, 1, 3)$ plotted on a set of axes. The basic convention here is that the *x*-*y* plane is drawn in its standard way, with the *z*-axis down to the left. The perspective is that the paper represents the *x*-*y* plane and the positive *z* axis is coming up, off the page. This method is preferred by many engineers. Because it can be hard to tell where a single point lies in relation to all the axes, dashed lines have been added to let one see how far along each axis the point lies.

One can also consider the x-y plane as being a horizontal plane in, say, a room, where the positive z-axis is pointing up. When one steps back and looks at this room, one might draw the axes as shown in Figure 11.1.2. The same point P is drawn, again with dashed lines. This point of view is preferred by most mathematicians, and is the convention adopted by this text.

Just as the x- and y-axes divide the plane into four *quadrants*, the x-, y-, and z-coordinate planes divide space into eight *octants*. The octant in which x, y, and z are positive is called the **first octant**. We do not name the other seven octants in this text.

Measuring Distances

It is of critical importance to know how to measure distances between points in space. The formula for doing so is based on measuring distance in the plane, and is known (in both contexts) as the Euclidean measure of distance.

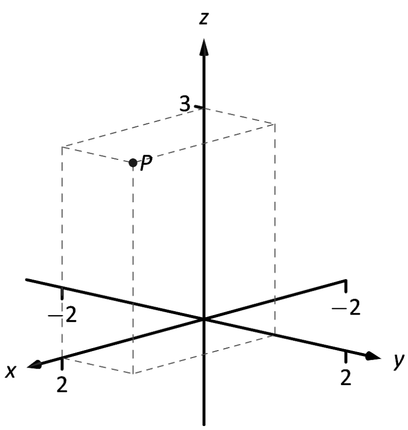


Figure 11.1.2: Plotting the point $P = (2, 1, 3)$ in space with a perspective used in this text.

Definition 11.1.1 Distance In Space

Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ be points in space. The distance D between P and Q is

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

We refer to the line segment that connects points P and Q in space as $\overline{PQ}$, and refer to the length of this segment as $\|\overline{PQ}\|$. The above distance formula allows us to compute the length of this segment.

Example 11.1.1 Length of a line segment

Let $P = (1, 4, -1)$ and let $Q = (2, 1, 1)$. Draw the line segment $\overline{PQ}$ and find its length.

Solution The points P and Q are plotted in Figure 11.1.3; no special consideration need be made to draw the line segment connecting these two points; simply connect them with a straight line. One *cannot* actually measure this line on the page and deduce anything meaningful; its true length must be measured analytically. Applying Definition 11.1.1, we have

$$\|\overline{PQ}\| = \sqrt{(2-1)^2 + (1-4)^2 + (1-(-1))^2} = \sqrt{14} \approx 3.74.$$

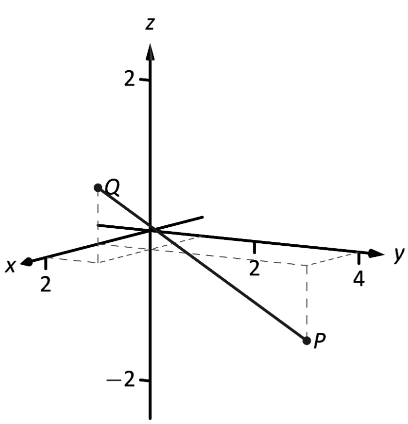


Figure 11.1.3: Plotting points P and Q in Example 11.1.1.

Spheres

Just as a circle is the set of all points in the *plane* equidistant from a given point (its center), a sphere is the set of all points in *space* that are equidistant from a given point. Definition 11.1.1 allows us to write an equation of the sphere.

We start with a point $C = (a, b, c)$ which is to be the center of a sphere with radius r. If a point $P = (x, y, z)$ lies on the sphere, then P is r units from C; that is,

$$\|\overline{PC}\| = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r.$$

Squaring both sides, we get the standard equation of a sphere in space with center at $C = (a, b, c)$ with radius r, as given in the following Key Idea.

Key Idea 11.1.1 Standard Equation of a Sphere in Space

The standard equation of the sphere with radius r, centred at $C = (a, b, c)$, is
$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

Example 11.1.2 Equation of a sphere

Find the center and radius of the sphere defined by $x^2+2x+y^2-4y+z^2-6z = 2$.

Solution To determine the center and radius, we must put the equation in standard form. This requires us to complete the square (three times).
$$\begin{aligned} x^2+2x+y^2-4y+z^2-6z &= 2 \\ (x^2+2x+1)+(y^2-4y+4)+(z^2-6z+9)-14 &= 2 \\ (x+1)^2+(y-2)^2+(z-3)^2 &= 16 \end{aligned}$$
The sphere is centred at $(-1, 2, 3)$ and has a radius of 4.

The equation of a sphere is an example of an implicit function defining a surface in space. In the case of a sphere, the variables x, y and z are all used. We now consider situations where surfaces are defined where one or two of these variables are absent.

Introduction to Planes in Space

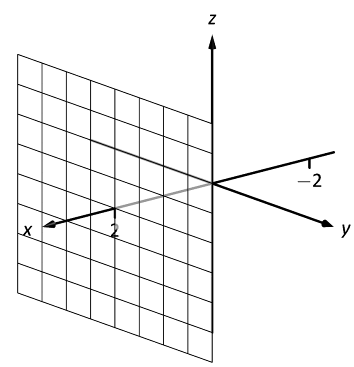
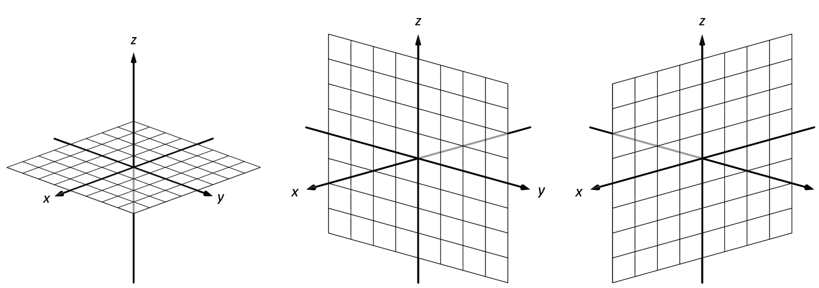


Figure 11.1.5: The plane $x = 2$.

The coordinate axes naturally define three planes (shown in Figure 11.1.4), the **coordinate planes**: the x-y plane, the y-z plane and the x-z plane. The x-y plane is characterized as the set of all points in space where the z-value is 0. This, in fact, gives us an equation that describes this plane: $z = 0$. Likewise, the x-z plane is all points where the y-value is 0, characterized by $y = 0$.



the x-y plane　the y-z plane　the x-z plane

Figure 11.1.4: The coordinate planes.

The equation $x = 2$ describes all points in space where the x-value is 2. This is a plane, parallel to the y-z coordinate plane, shown in Figure 11.1.5.

Example 11.1.3 Regions defined by planes

Sketch the region defined by the inequalities $-1 \leq y \leq 2$.

Solution The region is all points between the planes $y = -1$ and $y = 2$. These planes are sketched in Figure 11.1.6, which are parallel to the x-z plane. Thus the region extends infinitely in the x and z directions, and is bounded by planes in the y direction.

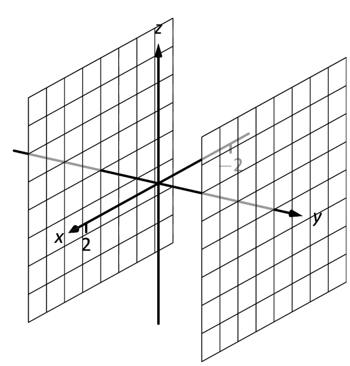
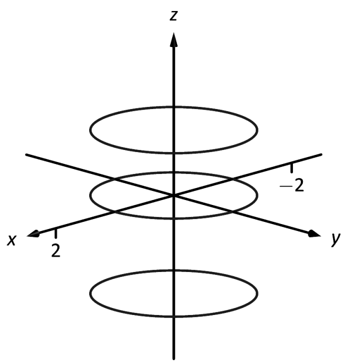


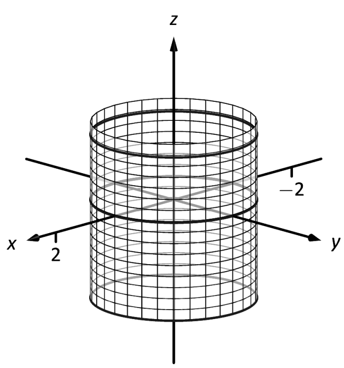
Figure 11.1.6: Sketching the boundaries of a region in Example 11.1.3.

Cylinders

The equation $x = 1$ obviously lacks the y and z variables, meaning it defines points where the y and z coordinates can take on any value. Now consider the equation $x^2 + y^2 = 1$ *in space*. In *the plane*, this equation describes a circle of radius 1, centred at the origin. In space, the z coordinate is not specified, meaning it can take on any value. In Figure 11.1.8 (a), we show part of the graph of the equation $x^2 + y^2 = 1$ by sketching 3 circles: the bottom one has a constant z-value of -1.5, the middle one has a z-value of 0 and the top circle has a z-value of 1. By plotting *all* possible z-values, we get the surface shown in Figure 11.1.8(b). This surface looks like a "tube," or a "cylinder"; mathematicians call this surface a **cylinder** for an entirely different reason.



(a)



(b)

Figure 11.1.8: Sketching $x^2 + y^2 = 1$.

Definition 11.1.2 Cylinder

Let C be a curve in a plane and let L be a line not parallel to C. A **cylinder** is the set of all lines parallel to L that pass through C. The curve C is the **directrix** of the cylinder, and the lines are the **rulings**.

In this text, we consider curves C that lie in planes parallel to one of the coordinate planes, and lines L that are perpendicular to these planes, forming **right cylinders**. Thus the directrix can be defined using equations involving 2 variables, and the rulings will be parallel to the axis of the 3rd variable.

In the example preceding the definition, the curve $x^2 + y^2 = 1$ in the x-y plane is the directrix and the rulings are lines parallel to the z-axis. (Any circle shown in Figure 11.1.8 can be considered a directrix; we simply choose the one where $z = 0$.) Sample rulings can also be viewed in part (b) of the figure. More examples will help us understand this definition.

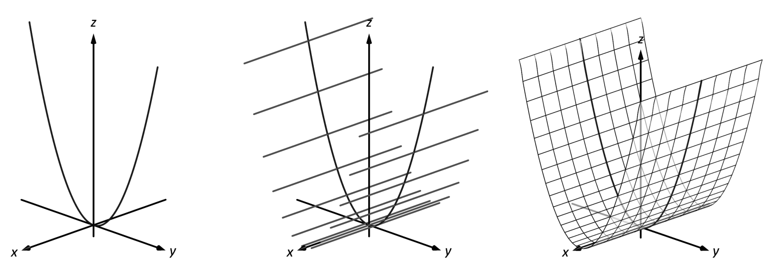
Example 11.1.4 Graphing cylinders

Graph the following cylinders.

1. $z = y^2$ 2. $x = \sin z$

Solution

1. We can view the equation $z = y^2$ as a parabola in the y-z plane, as illustrated in Figure 11.1.7(a). As x does not appear in the equation, the rulings are lines through this parabola parallel to the x-axis, shown in (b). These rulings give an idea as to what the surface looks like, drawn in (c).



(a) (b) (c)

Figure 11.1.7: Sketching the cylinder defined by $z = y^2$.

2. We can view the equation $x = \sin z$ as a sine curve that exists in the x-z plane, as shown in Figure 11.1.9 (a). The rules are parallel to the y axis as the variable y does not appear in the equation $x = \sin z$; some of these are shown in part (b). The surface is shown in part (c) of the figure.

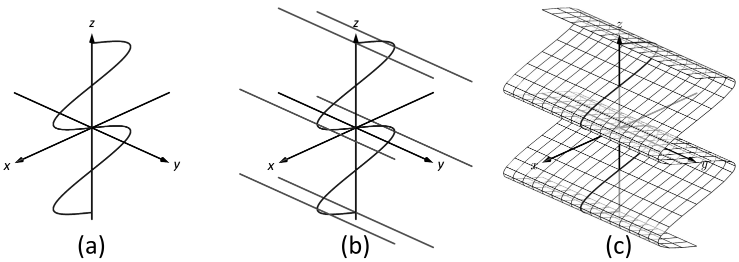


Figure 11.1.9: Sketching the cylinder defined by $x = \sin z$.

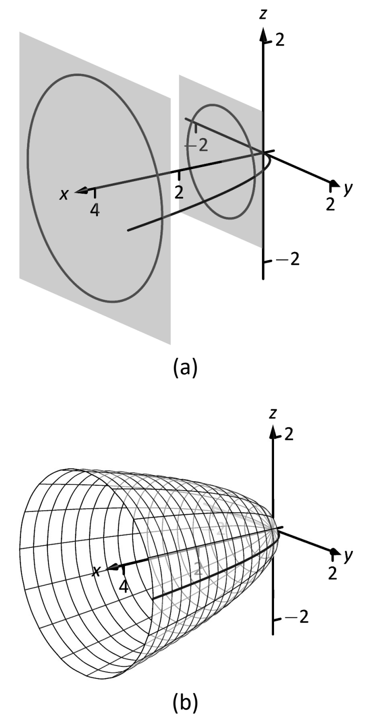


Figure 11.1.10: Introducing surfaces of revolution.

Surfaces of Revolution

One of the applications of integration we learned previously was to find the volume of solids of revolution – solids formed by revolving a curve about a horizontal or vertical axis. We now consider how to find the equation of the surface of such a solid.

Consider the surface formed by revolving $y = \sqrt{x}$ about the x-axis. Cross–sections of this surface parallel to the y-z plane are circles, as shown in Figure 11.1.10(a). Each circle has equation of the form $y^2 + z^2 = r^2$ for some radius r. The radius is a function of x; in fact, it is $r(x) = \sqrt{x}$. Thus the equation of the surface shown in Figure 11.1.10b is $y^2 + z^2 = (\sqrt{x})^2$.

We generalize the above principles to give the equations of surfaces formed by revolving curves about the coordinate axes.

Key Idea 11.1.2 Surfaces of Revolution, Part 1

Let r be a radius function.

1. The equation of the surface formed by revolving $y = r(x)$ or $z = r(x)$ about the x-axis is $y^2 + z^2 = r(x)^2$.
2. The equation of the surface formed by revolving $x = r(y)$ or $z = r(y)$ about the y-axis is $x^2 + z^2 = r(y)^2$.
3. The equation of the surface formed by revolving $x = r(z)$ or $y = r(z)$ about the z-axis is $x^2 + y^2 = r(z)^2$.

Example 11.1.5 Finding equation of a surface of revolution

Let $y = \sin z$ on $[0, \pi]$. Find the equation of the surface of revolution formed by revolving $y = \sin z$ about the z-axis.

Solution Using Key Idea 11.1.2, we find the surface has equation $x^2 + y^2 = \sin^2 z$. The curve is sketched in Figure 11.1.11(a) and the surface is drawn in Figure 11.1.11(b).

Note how the surface (and hence the resulting equation) is the same if we began with the curve $x = \sin z$, which is also drawn in Figure 11.1.11(a).

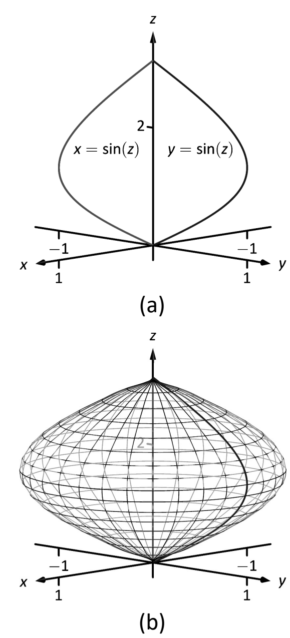


Figure 11.1.11: Revolving $y = \sin z$ about the z-axis in Example 11.1.5.

This particular method of creating surfaces of revolution is limited. For instance, in Example 7.3.4 of Section 7.3 we found the volume of the solid formed by revolving $y = \sin x$ about the y-axis. Our current method of forming surfaces can only rotate $y = \sin x$ about the x-axis. Trying to rewrite $y = \sin x$ as a function of y is not trivial, as simply writing $x = \sin^{-1} y$ only gives part of the region we desire.

What we desire is a way of writing the surface of revolution formed by rotating $y = f(x)$ about the y-axis. We start by first recognizing this surface is the same as revolving $z = f(x)$ about the z-axis. This will give us a more natural way of viewing the surface.

A value of x is a measurement of distance from the z-axis. At the distance r, we plot a z-height of $f(r)$. When rotating $f(x)$ about the z-axis, we want all points a distance of r from the z-axis in the x-y plane to have a z-height of $f(r)$. All such points satisfy the equation $r^2 = x^2 + y^2$; hence $r = \sqrt{x^2 + y^2}$. Replacing r with $\sqrt{x^2 + y^2}$ in $f(r)$ gives $z = f(\sqrt{x^2 + y^2})$. This is the equation of the surface.

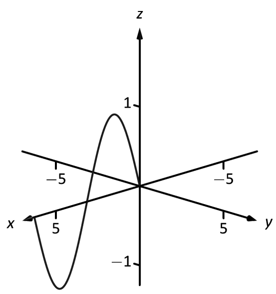
Key Idea 11.1.3 Surfaces of Revolution, Part 2

Let $z = f(x)$, $x \geq 0$, be a curve in the x-z plane. The surface formed by revolving this curve about the z-axis has equation $z = f\big(\sqrt{x^2 + y^2}\big)$.

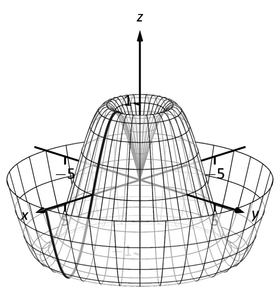
Example 11.1.6 Finding equation of surface of revolution

Find the equation of the surface found by revolving $z = \sin x$ about the z-axis.

Solution Using Key Idea 11.1.3, the surface has equation $z = \sin\left(\sqrt{x^2 + y^2}\right)$. The curve and surface are graphed in Figure 11.1.12.



(a)



(b)

Figure 11.1.12: Revolving $z = \sin x$ about the z-axis in Example 11.1.6.

Quadric Surfaces

Spheres, planes and cylinders are important surfaces to understand. We now consider one last type of surface, a **quadric surface**. The definition may look intimidating, but we will show how to analyze these surfaces in an illuminating way.

> **Definition 11.1.3 Quadric Surface**
>
> A **quadric surface** is the graph of the general second–degree equation in three variables:
>
> $$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

When the coefficients D, E or F are not zero, the basic shapes of the quadric surfaces are rotated in space. We will focus on quadric surfaces where these coefficients are 0; we will not consider rotations. There are six basic quadric surfaces: the elliptic paraboloid, elliptic cone, ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, and the hyperbolic paraboloid.

We study each shape by considering **traces**, that is, intersections of each surface with a plane parallel to a coordinate plane. For instance, consider the elliptic paraboloid $z = x^2/4 + y^2$, shown in Figure 11.1.13. If we intersect this shape with the plane $z = d$ (i.e., replace z with d), we have the equation:

$$d = \frac{x^2}{4} + y^2.$$

Divide both sides by d:

$$1 = \frac{x^2}{4d} + \frac{y^2}{d}.$$

This describes an ellipse – so cross sections parallel to the x-y coordinate plane are ellipses. This ellipse is drawn in the figure.

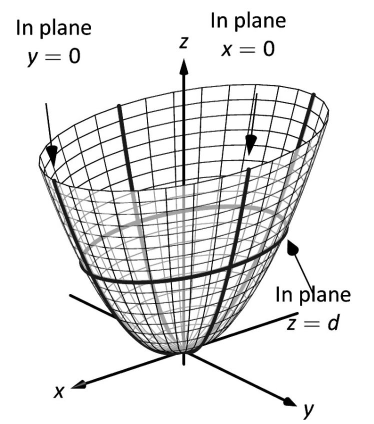


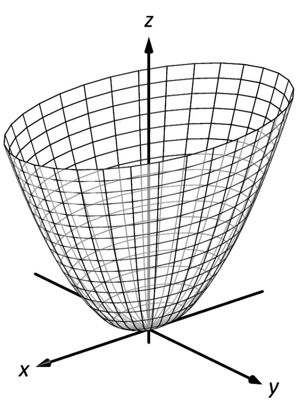
Figure 11.1.13: The elliptic paraboloid $z = x^2/4 + y^2$.

Now consider cross sections parallel to the x-z plane. For instance, letting $y = 0$ gives the equation $z = x^2/4$, clearly a parabola. Intersecting with the plane $x = 0$ gives a cross section defined by $z = y^2$, another parabola. These parabolas are also sketched in the figure.

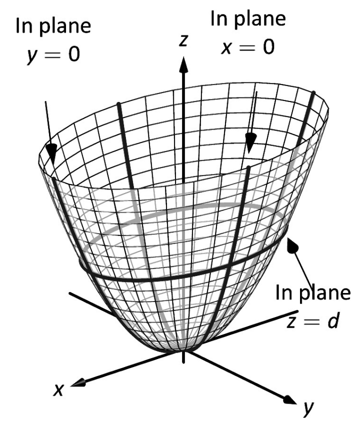
Thus we see where the elliptic paraboloid gets its name: some cross sections are ellipses, and others are parabolas.

Such an analysis can be made with each of the quadric surfaces. We give a sample equation of each, provide a sketch with representative traces, and describe these traces.

Elliptic Paraboloid, $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$



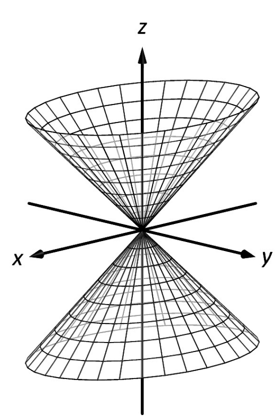
Plane	Trace
$x = d$	Parabola
$y = d$	Parabola
$z = d$	Ellipse



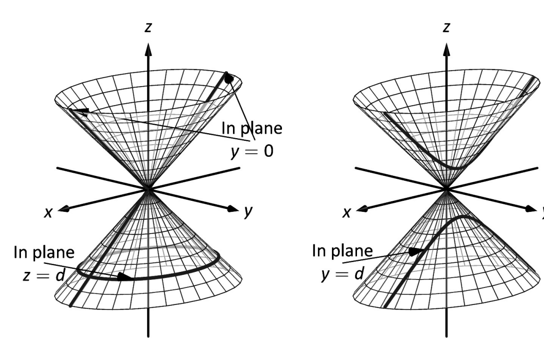
One variable in the equation of the elliptic paraboloid will be raised to the first power; above, this is the z variable. The paraboloid will "open" in the direction of this variable's axis. Thus $x = y^2/a^2 + z^2/b^2$ is an elliptic paraboloid that opens along the x-axis.

Multiplying the right hand side by (-1) defines an elliptic paraboloid that "opens" in the opposite direction.

Elliptic Cone, $z^2 = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

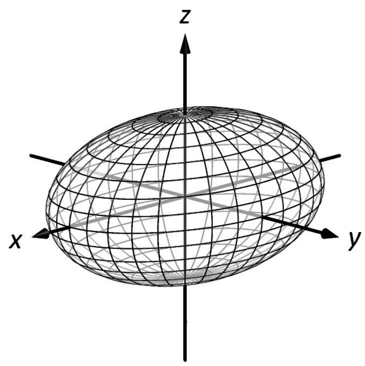


Plane	Trace
$x = 0$	Crossed Lines
$y = 0$	Crossed Lines
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse

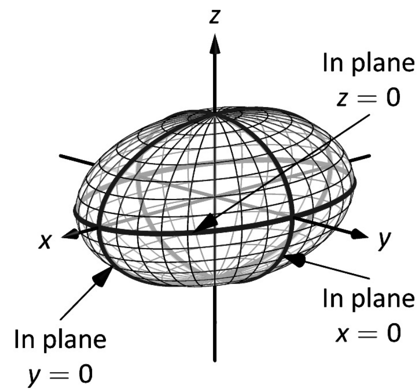


One can rewrite the equation as $z^2 - x^2/a^2 - y^2/b^2 = 0$. The one variable with a positive coefficient corresponds to the axis that the cones "open" along.

Ellipsoid, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

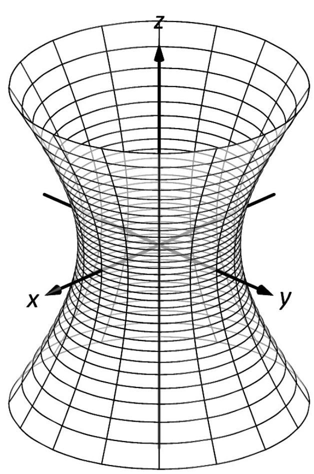


Plane	Trace
$x=d$	Ellipse
$y=d$	Ellipse
$z=d$	Ellipse

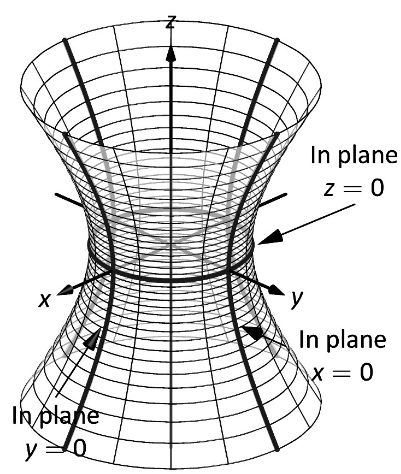


If $a=b=c\neq 0$, the ellipsoid is a sphere with radius a; compare to Key Idea 11.1.1.

Hyperboloid of One Sheet, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$

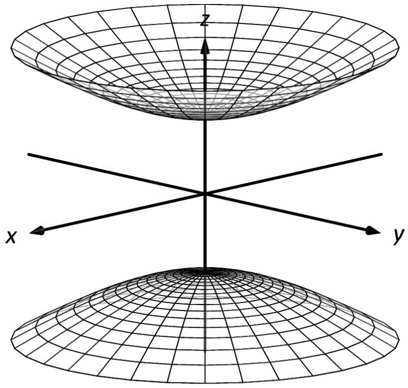


Plane	Trace
$x=d$	Hyperbola
$y=d$	Hyperbola
$z=d$	Ellipse

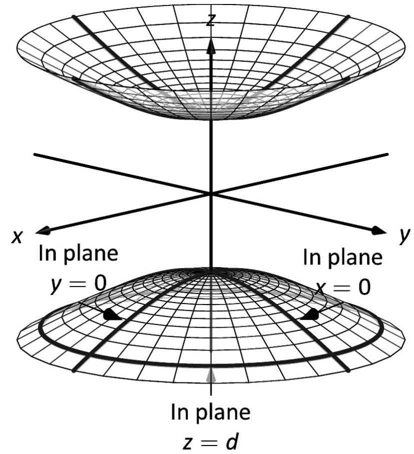


The one variable with a negative coefficient corresponds to the axis that the hyperboloid “opens” along.

Hyperboloid of Two Sheets, $\dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

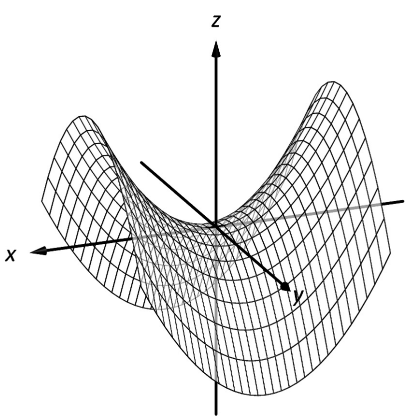


Plane	Trace
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse

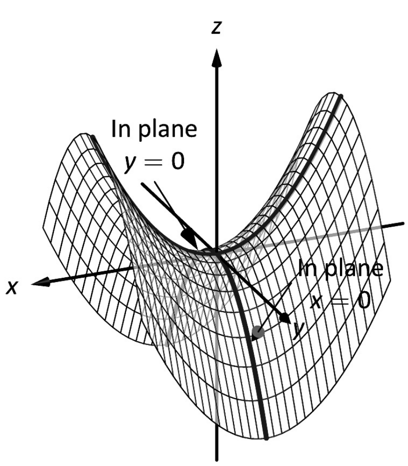


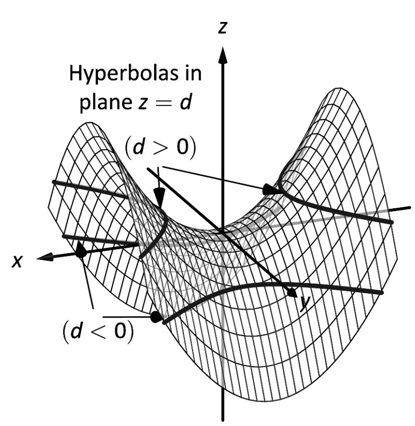
The one variable with a positive coefficient corresponds to the axis that the hyperboloid “opens” along. In the case illustrated, when $|d| < |c|$, there is no trace.

Hyperbolic Paraboloid, $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$



Plane	Trace
$x = d$	Parabola
$y = d$	Parabola
$z = d$	Hyperbola





The parabolic traces will open along the axis of the one variable that is raised to the first power.

Example 11.1.7 Sketching quadric surfaces

Sketch the quadric surface defined by the given equation.

1. $y = \dfrac{x^2}{4} + \dfrac{z^2}{16}$ 2. $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1.$ 3. $z = y^2 - x^2.$

Solution

1. $y = \dfrac{x^2}{4} + \dfrac{z^2}{16}$:

   We first identify the quadric by pattern–matching with the equations given previously. Only two surfaces have equations where one variable is raised to the first power, the elliptic paraboloid and the hyperbolic paraboloid. In the latter case, the other variables have different signs, so we conclude that this describes a hyperbolic paraboloid. As the variable with the first power is y, we note the paraboloid opens along the y-axis.

   To make a decent sketch by hand, we need only draw a few traces. In this case, the traces $x = 0$ and $z = 0$ form parabolas that outline the shape.

   $x = 0$: The trace is the parabola $y = z^2/16$

   $z = 0$: The trace is the parabola $y = x^2/4$.

   Graphing each trace in the respective plane creates a sketch as shown in Figure 11.1.14(a). This is enough to give an idea of what the paraboloid looks like. The surface is filled in in (b).

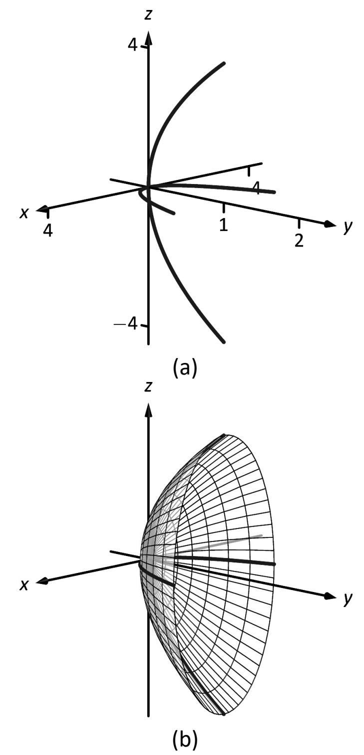


Figure 11.1.14: Sketching an elliptic paraboloid.

2. $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$:

   This is an ellipsoid. We can get a good idea of its shape by drawing the traces in the coordinate planes.

   $x = 0$: The trace is the ellipse $\dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$. The major axis is along the y–axis with length 6 (as $b = 3$, the length of the axis is 6); the minor axis is along the z-axis with length 4.

   $y = 0$: The trace is the ellipse $x^2 + \dfrac{z^2}{4} = 1$. The major axis is along the z-axis, and the minor axis has length 2 along the x-axis.

   $z = 0$: The trace is the ellipse $x^2 + \dfrac{y^2}{9} = 1$, with major axis along the y-axis.

   Graphing each trace in the respective plane creates a sketch as shown in Figure 11.1.15(a). Filling in the surface gives Figure 11.1.15(b).

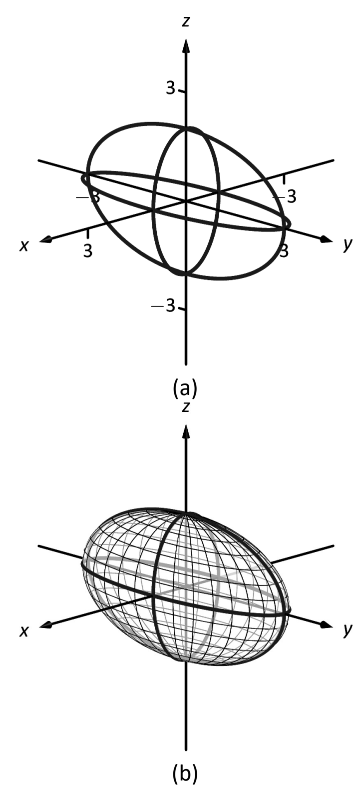


Figure 11.1.15: Sketching an ellipsoid.

3. $z = y^2 - x^2$:

   This defines a hyperbolic paraboloid, very similar to the one shown in the gallery of quadric sections. Consider the traces in the $y - z$ and $x - z$ planes:

   $x = 0$: The trace is $z = y^2$, a parabola opening up in the $y - z$ plane.

   $y = 0$: The trace is $z = -x^2$, a parabola opening down in the $x - z$ plane.

   Sketching these two parabolas gives a sketch like that in Figure 11.1.16(a), and filling in the surface gives a sketch like (b).

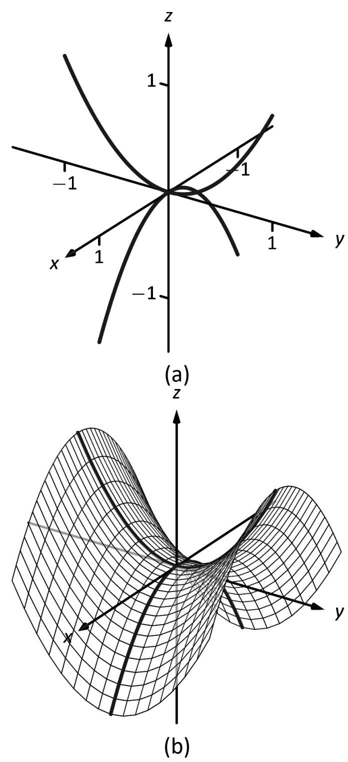


Figure 11.1.16: Sketching a hyperbolic paraboloid.

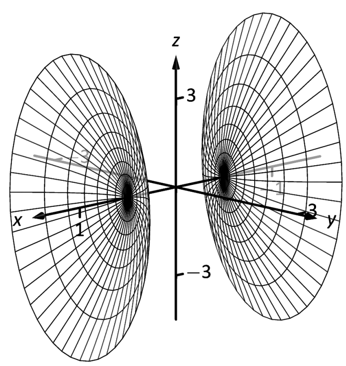


Figure 11.1.17: A possible equation of this quadric surface is found in Example 11.1.8.

Example 11.1.8 Identifying quadric surfaces

Consider the quadric surface shown in Figure 11.1.17. Which of the following equations best fits this surface?

(a) $x^2 - y^2 - \dfrac{z^2}{9} = 0$ (c) $z^2 - x^2 - y^2 = 1$

(b) $x^2 - y^2 - z^2 = 1$ (d) $4x^2 - y^2 - \dfrac{z^2}{9} = 1$

Solution The image clearly displays a hyperboloid of two sheets. The gallery informs us that the equation will have a form similar to $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

We can immediately eliminate option (a), as the constant in that equation is not 1.

The hyperboloid "opens" along the x-axis, meaning x must be the only variable with a positive coefficient, eliminating (c).

The hyperboloid is wider in the z-direction than in the y-direction, so we need an equation where $c > b$. This eliminates (b), leaving us with (d). We should verify that the equation given in (d), $4x^2 - y^2 - \frac{z^2}{9} = 1$, fits.

We already established that this equation describes a hyperboloid of two sheets that opens in the x-direction and is wider in the z-direction than in the y. Now note the coefficient of the x-term. Rewriting $4x^2$ in standard form, we have: $4x^2 = \dfrac{x^2}{(1/2)^2}$. Thus when $y = 0$ and $z = 0$, x must be $1/2$; i.e., each hyperboloid "starts" at $x = 1/2$. This matches our figure.

We conclude that $4x^2 - y^2 - \dfrac{z^2}{9} = 1$ best fits the graph.

This section has introduced points in space and shown how equations can describe surfaces. The next sections explore *vectors*, an important mathematical object that we'll use to explore curves in space.

Exercises 11.1

Terms and Concepts

1. Axes drawn in space must conform to the ________ ________ rule.

2. In the plane, the equation $x = 2$ defines a ________; in space, $x = 2$ defines a ________.

3. In the plane, the equation $y = x^2$ defines a ________; in space, $y = x^2$ defines a ________.

4. Which quadric surface looks like a Pringles® chip?

5. Consider the hyperbola $x^2 - y^2 = 1$ in the plane. If this hyperbola is rotated about the x-axis, what quadric surface is formed?

6. Consider the hyperbola $x^2 - y^2 = 1$ in the plane. If this hyperbola is rotated about the y-axis, what quadric surface is formed?

Problems

7. The points $A = (1, 4, 2)$, $B = (2, 6, 3)$ and $C = (4, 3, 1)$ form a triangle in space. Find the distances between each pair of points and determine if the triangle is a right triangle.

8. The points $A = (1, 1, 3)$, $B = (3, 2, 7)$, $C = (2, 0, 8)$ and $D = (0, -1, 4)$ form a quadrilateral $ABCD$ in space. Is this a parallelogram?

9. Find the center and radius of the sphere defined by $x^2 - 8x + y^2 + 2y + z^2 + 8 = 0$.

10. Find the center and radius of the sphere defined by $x^2 + y^2 + z^2 + 4x - 2y - 4z + 4 = 0$.

In Exercises 11 – 14, describe the region in space defined by the inequalities.

11. $x^2 + y^2 + z^2 < 1$

12. $0 \leq x \leq 3$

13. $x \geq 0,\ y \geq 0,\ z \geq 0$

14. $y \geq 3$

In Exercises 15 – 18, sketch the cylinder in space.

15. $z = x^3$

16. $y = \cos z$

17. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

18. $y = \dfrac{1}{x}$

In Exercises 19 – 22, give the equation of the surface of revolution described.

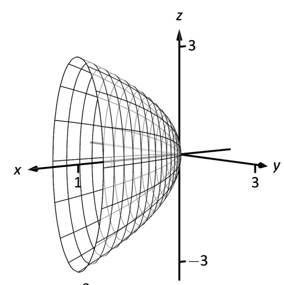
19. Revolve $z = \dfrac{1}{1 + y^2}$ about the y-axis.

20. Revolve $y = x^2$ about the x-axis.

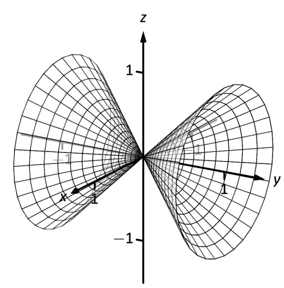
21. Revolve $z = x^2$ about the z-axis.

22. Revolve $z = 1/x$ about the z-axis.

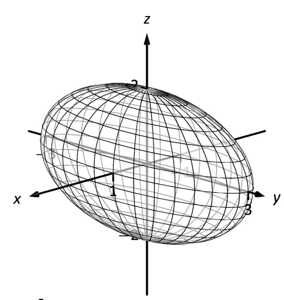
In Exercises 23 – 26, a quadric surface is sketched. Determine which of the given equations best fits the graph.

23. 

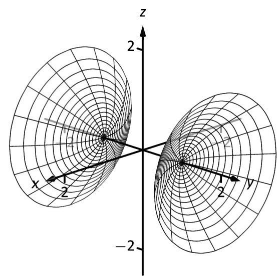
(a) $x = y^2 + \dfrac{z^2}{9}$ (b) $x = y^2 + \dfrac{z^2}{3}$

24. 

(a) $x^2 - y^2 - z^2 = 0$ (b) $x^2 - y^2 + z^2 = 0$

25. 

(a) $x^2 + \dfrac{y^2}{3} + \dfrac{z^2}{2} = 1$ (b) $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$

26. 

(a) $y^2 - x^2 - z^2 = 1$

(b) $y^2 + x^2 - z^2 = 1$

In Exercises 27 – 32, sketch the quadric surface.

27. $z - y^2 + x^2 = 0$

28. $z^2 = x^2 + \frac{y^2}{4}$

29. $x = -y^2 - z^2$

30. $16x^2 - 16y^2 - 16z^2 = 1$

31. $\frac{x^2}{9} - y^2 + \frac{z^2}{25} = 1$

32. $4x^2 + 2y^2 + z^2 = 4$

11.2 An Introduction to Vectors

Many quantities we think about daily can be described by a single number: temperature, speed, cost, weight and height. There are also many other concepts we encounter daily that cannot be described with just one number. For instance, a weather forecaster often describes wind with its speed and its direction ("... with winds from the southeast gusting up to 30 mph ..."). When applying a force, we are concerned with both the magnitude and direction of that force. In both of these examples, *direction* is important. Because of this, we study *vectors*, mathematical objects that convey both magnitude and direction information.

One "bare–bones" definition of a vector is based on what we wrote above: "a vector is a mathematical object with magnitude and direction parameters." This definition leaves much to be desired, as it gives no indication as to how such an object is to be used. Several other definitions exist; we choose here a definition rooted in a geometric visualization of vectors. It is very simplistic but readily permits further investigation.

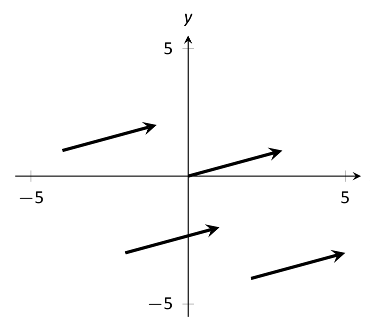


Figure 11.2.1: Drawing the same vector with different initial points.

Definition 11.2.1 Vector

A **vector** is a directed line segment.

Given points P and Q (either in the plane or in space), we denote with $\overrightarrow{PQ}$ the vector *from P to Q*. The point P is said to be the **initial point** of the vector, and the point Q is the **terminal point**.

The **magnitude**, **length** or **norm** of $\overrightarrow{PQ}$ is the length of the line segment $\overline{PQ}$: $\parallel \overrightarrow{PQ} \parallel = \parallel \overline{PQ} \parallel$.

Two vectors are **equal** if they have the same magnitude and direction.

Figure 11.2.1 shows multiple instances of the same vector. Each directed line segment has the same direction and length (magnitude), hence each is the same vector.

We use $\mathbb{R}^2$ (pronounced "r two") to represent all the vectors in the plane, and use $\mathbb{R}^3$ (pronounced "r three") to represent all the vectors in space.

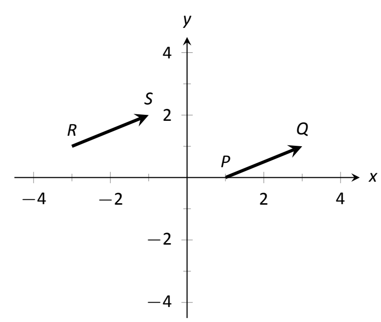


Figure 11.2.2: Illustrating how equal vectors have the same displacement.

Consider the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ as shown in Figure 11.2.2. The vectors look to be equal; that is, they seem to have the same length and direction. Indeed, they are. Both vectors move 2 units to the right and 1 unit up from the initial point to reach the terminal point. One can analyze this movement to measure the magnitude of the vector, and the movement itself gives direction information (one could also measure the slope of the line passing through P and Q or R and S). Since they have the same length and direction, these two vectors are equal.

This demonstrates that inherently all we care about is *displacement*; that is, how far in the x, y and possibly z directions the terminal point is from the initial point. Both the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ in Figure 11.2.2 have an x-displacement of 2 and a y-displacement of 1. This suggests a standard way of describing vectors in the plane. A vector whose x-displacement is a and whose y-displacement is b will have terminal point (a, b) when the initial point is the origin, $(0, 0)$. This leads us to a definition of a standard and concise way of referring to vectors.

Definition 11.2.2 Component Form of a Vector

1. The **component form** of a vector $\vec{v}$ in $\mathbb{R}^2$, whose terminal point is (a,b) when its initial point is $(0,0)$, is $\langle a,b\rangle$.
2. The **component form** of a vector $\vec{v}$ in $\mathbb{R}^3$, whose terminal point is (a,b,c) when its initial point is $(0,0,0)$, is $\langle a,b,c\rangle$.

The numbers a, b (and c, respectively) are the **components** of $\vec{v}$.

It follows from the definition that the component form of the vector $\overrightarrow{PQ}$, where $P=(x_1,y_1)$ and $Q=(x_2,y_2)$ is

$$\overrightarrow{PQ}=\langle x_2-x_1,y_2-y_1\rangle\,;$$

in space, where $P=(x_1,y_1,z_1)$ and $Q=(x_2,y_2,z_2)$, the component form of $\overrightarrow{PQ}$ is

$$\overrightarrow{PQ}=\langle x_2-x_1,y_2-y_1,z_2-z_1\rangle\,.$$

We practice using this notation in the following example.

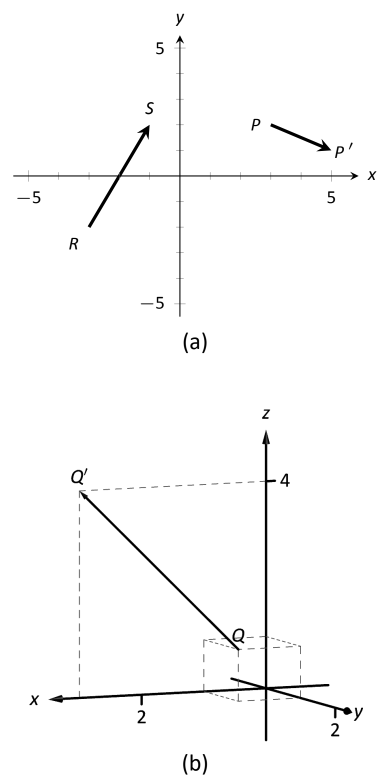


Figure 11.2.3: Graphing vectors in Example 11.2.1.

Example 11.2.1 Using component form notation for vectors

1. Sketch the vector $\vec{v}=\langle 2,-1\rangle$ starting at $P=(3,2)$ and find its magnitude.
2. Find the component form of the vector $\vec{w}$ whose initial point is $R=(-3,-2)$ and whose terminal point is $S=(-1,2)$.
3. Sketch the vector $\vec{u}=\langle 2,-1,3\rangle$ starting at the point $Q=(1,1,1)$ and find its magnitude.

Solution

1. Using P as the initial point, we move 2 units in the positive x-direction and -1 units in the positive y-direction to arrive at the terminal point $P'=(5,1)$, as drawn in Figure 11.2.3(a).

   The magnitude of $\vec{v}$ is determined directly from the component form:

   $$\|\,\vec{v}\,\|=\sqrt{2^2+(-1)^2}=\sqrt{5}.$$

2. Using the note following Definition 11.2.2, we have

   $$\overrightarrow{RS}=\langle -1-(-3),2-(-2)\rangle=\langle 2,4\rangle\,.$$

   One can readily see from Figure 11.2.3(a) that the x- and y-displacement of $\overrightarrow{RS}$ is 2 and 4, respectively, as the component form suggests.

3. Using Q as the initial point, we move 2 units in the positive x-direction, -1 unit in the positive y-direction, and 3 units in the positive z-direction to arrive at the terminal point $Q'=(3,0,4)$, illustrated in Figure 11.2.3(b).

   The magnitude of $\vec{u}$ is:

   $$\|\,\vec{u}\,\|=\sqrt{2^2+(-1)^2+3^2}=\sqrt{14}.$$

Now that we have defined vectors, and have created a nice notation by which to describe them, we start considering how vectors interact with each other. That is, we define an *algebra* on vectors.

Definition 11.2.3 Vector Algebra

1. Let $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$ be vectors in $\mathbb{R}^2$, and let c be a scalar.

   (a) The addition, or sum, of the vectors $\vec{u}$ and $\vec{v}$ is the vector
   $$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle .$$

   (b) The scalar product of c and $\vec{v}$ is the vector
   $$c\vec{v} = c\,\langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle .$$

2. Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be vectors in $\mathbb{R}^3$, and let c be a scalar.

   (a) The addition, or sum, of the vectors $\vec{u}$ and $\vec{v}$ is the vector
   $$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle .$$

   (b) The scalar product of c and $\vec{v}$ is the vector
   $$c\vec{v} = c\,\langle v_1, v_2, v_3 \rangle = \langle cv_1, cv_2, cv_3 \rangle .$$

In short, we say addition and scalar multiplication are computed "component–wise."

Example 11.2.2 Adding vectors

Sketch the vectors $\vec{u} = \langle 1, 3 \rangle$, $\vec{v} = \langle 2, 1 \rangle$ and $\vec{u} + \vec{v}$ all with initial point at the origin.

Solution We first compute $\vec{u} + \vec{v}$.

$$\begin{aligned}\vec{u} + \vec{v} &= \langle 1, 3 \rangle + \langle 2, 1 \rangle \\ &= \langle 3, 4 \rangle .\end{aligned}$$

These are all sketched in Figure 11.2.4.

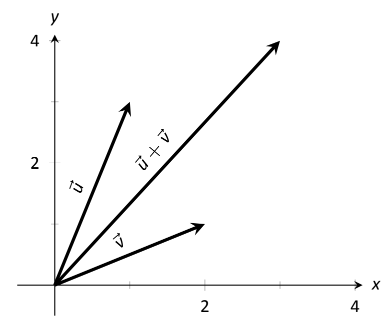


Figure 11.2.4: Graphing the sum of vectors in Example 11.2.2.

As vectors convey magnitude and direction information, the sum of vectors also convey length and magnitude information. Adding $\vec{u} + \vec{v}$ suggests the following idea:

"Starting at an initial point, go out $\vec{u}$, then go out $\vec{v}$."

This idea is sketched in Figure 11.2.5, where the initial point of $\vec{v}$ is the terminal point of $\vec{u}$. This is known as the "Head to Tail Rule" of adding vectors. Vector addition is very important. For instance, if the vectors $\vec{u}$ and $\vec{v}$ represent forces acting on a body, the sum $\vec{u} + \vec{v}$ gives the resulting force. Because of various physical applications of vector addition, the sum $\vec{u} + \vec{v}$ is often referred to as the **resultant vector**, or just the "resultant."

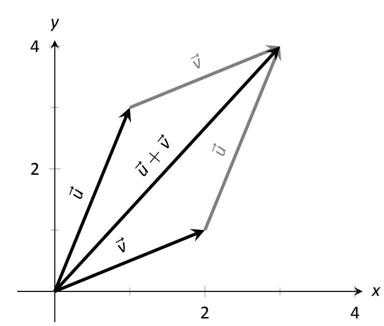


Figure 11.2.5: Illustrating how to add vectors using the Head to Tail Rule and Parallelogram Law.

Analytically, it is easy to see that $\vec{u} + \vec{v} = \vec{v} + \vec{u}$. Figure 11.2.5 also gives a graphical representation of this, using gray vectors. Note that the vectors $\vec{u}$

and $\vec{v}$, when arranged as in the figure, form a parallelogram. Because of this, the Head to Tail Rule is also known as the Parallelogram Law: the vector $\vec{u}+\vec{v}$ is defined by forming the parallelogram defined by the vectors $\vec{u}$ and $\vec{v}$; the initial point of $\vec{u}+\vec{v}$ is the common initial point of parallelogram, and the terminal point of the sum is the common terminal point of the parallelogram.

While not illustrated here, the Head to Tail Rule and Parallelogram Law hold for vectors in $\mathbb{R}^3$ as well.

It follows from the properties of the real numbers and Definition 11.2.3 that

$$\vec{u}-\vec{v}=\vec{u}+(-1)\vec{v}.$$

The Parallelogram Law gives us a good way to visualize this subtraction. We demonstrate this in the following example.

Example 11.2.3 Vector Subtraction

Let $\vec{u}=\langle 3,1\rangle$ and $\vec{v}=\langle 1,2\rangle$. Compute and sketch $\vec{u}-\vec{v}$.

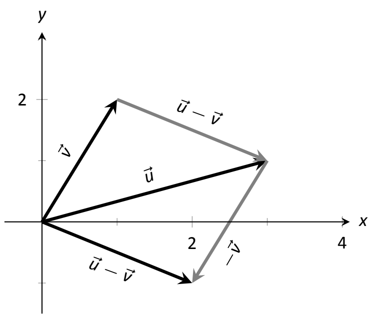


Figure 11.2.6: Illustrating how to subtract vectors graphically.

Solution The computation of $\vec{u}-\vec{v}$ is straightforward, and we show all steps below. Usually the formal step of multiplying by (-1) is omitted and we "just subtract."

$$\begin{aligned}\vec{u}-\vec{v}&=\vec{u}+(-1)\vec{v}\\&=\langle 3,1\rangle+\langle -1,-2\rangle\\&=\langle 2,-1\rangle .\end{aligned}$$

Figure 11.2.6 illustrates, using the Head to Tail Rule, how the subtraction can be viewed as the sum $\vec{u}+(-\vec{v})$. The figure also illustrates how $\vec{u}-\vec{v}$ can be obtained by looking only at the terminal points of $\vec{u}$ and $\vec{v}$ (when their initial points are the same).

Example 11.2.4 Scaling vectors

1. Sketch the vectors $\vec{v}=\langle 2,1\rangle$ and $2\vec{v}$ with initial point at the origin.
2. Compute the magnitudes of $\vec{v}$ and $2\vec{v}$.

Solution

1. We compute $2\vec{v}$:

$$\begin{aligned}2\vec{v}&=2\langle 2,1\rangle\\&=\langle 4,2\rangle .\end{aligned}$$

Both $\vec{v}$ and $2\vec{v}$ are sketched in Figure 11.2.7. Make note that $2\vec{v}$ does not start at the terminal point of $\vec{v}$; rather, its initial point is also the origin.

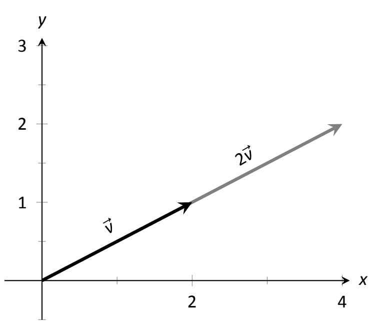


Figure 11.2.7: Graphing vectors $\vec{v}$ and $2\vec{v}$ in Example 11.2.4.

2. The figure suggests that $2\vec{v}$ is twice as long as $\vec{v}$. We compute their magnitudes to confirm this.

$$\begin{aligned}\parallel\vec{v}\parallel&=\sqrt{2^2+1^2}\\&=\sqrt{5}.\\\parallel 2\vec{v}\parallel&=\sqrt{4^2+2^2}\\&=\sqrt{20}\\&=\sqrt{4\cdot 5}=2\sqrt{5}.\end{aligned}$$

As we suspected, $2\vec{v}$ is twice as long as $\vec{v}$.

In Example 11.2.4 above, we saw that $\| 2\vec{v} \| = 2\| \vec{v} \|$, which makes sense geometrically: $2\vec{v} = \vec{v} + \vec{v}$, and adding a vector to itself should produce a vector twice as long with the same direction. The following theorem tells us that this is true in general.

The **zero vector** is the vector whose initial point is also its terminal point. It is denoted by $\vec{0}$. Its component form, in $\mathbb{R}^2$, is $\langle 0, 0\rangle$; in $\mathbb{R}^3$, it is $\langle 0, 0, 0\rangle$. Usually the context makes is clear whether $\vec{0}$ is referring to a vector in the plane or in space.

Our examples have illustrated key principles in vector algebra: how to add and subtract vectors and how to multiply vectors by a scalar. The following theorem states formally the properties of these operations.

Theorem 11.2.1 Properties of Vector Operations

The following are true for all scalars c and d, and for all vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, where $\vec{u}$, $\vec{v}$ and $\vec{w}$ are all in $\mathbb{R}^2$ or where $\vec{u}$, $\vec{v}$ and $\vec{w}$ are all in $\mathbb{R}^3$:

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ — Commutative Property
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ — Associative Property
3. $\vec{v} + \vec{0} = \vec{v}$ — Additive Identity
4. $(cd)\vec{v} = c(d\vec{v})$
5. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ — Distributive Property
6. $(c + d)\vec{v} = c\vec{v} + d\vec{v}$ — Distributive Property
7. $0\vec{v} = \vec{0}$
8. $\| c\vec{v} \| = |c| \cdot \| \vec{v} \|$
9. $\| \vec{u} \| = 0$ if, and only if, $\vec{u} = \vec{0}$.

To prove Property 8, let $\vec{v} = \langle a, b\rangle$ be any vector in $\mathbb{R}^2$ (the proof for $\mathbb{R}^3$ is similar), and let c be any scalar. Then

$$\begin{aligned} \| c\vec{v} \| &= \| c\, \langle a, b\rangle \| \\ &= \| \langle ca, cb\rangle \| \\ &= \sqrt{(ca)^2 + (cb)^2} \\ &= \sqrt{c^2a^2 + c^2b^2} \\ &= \sqrt{c^2(a^2 + b^2)} \\ &= \sqrt{c^2}\sqrt{a^2 + b^2} \\ &= |c|\| \vec{v} \|, \end{aligned}$$

as required.
(Recall that $\sqrt{c^2} = |c|$, the absolute value of c, since c might be negative, but the square root is always positive.)

As stated before, each nonzero vector $\vec{v}$ conveys magnitude and direction information. We have a method of extracting the magnitude, which we write as $\| \vec{v} \|$. *Unit vectors* are a way of extracting just the direction information from a vector.

Definition 11.2.4 Unit Vector

A **unit vector** is a vector $\vec{v}$ with a magnitude of 1; that is,

$$\| \vec{v} \| = 1.$$

Consider this scenario: you are given a vector $\vec{v}$ and are told to create a vector of length 10 in the direction of $\vec{v}$. How does one do that? If we knew that $\vec{u}$ was the unit vector in the direction of $\vec{v}$, the answer would be easy: $10\vec{u}$. So how do we find $\vec{u}$?

Property 8 of Theorem 11.2.1 holds the key. If we divide $\vec{v}$ by its magnitude, it becomes a vector of length 1. Consider:

$$\left\| \frac{1}{\| \vec{v} \|}\vec{v} \right\| = \frac{1}{\| \vec{v} \|}\| \vec{v} \| \qquad \text{(we can pull out } \frac{1}{\| \vec{v} \|} \text{ as it is a positive scalar)}$$
$$= 1.$$

So the vector of length 10 in the direction of $\vec{v}$ is $10\left(\frac{1}{\| \vec{v} \|}\vec{v}\right) = \frac{10}{\| \vec{v} \|}\vec{v}$.

An example will make this more clear.

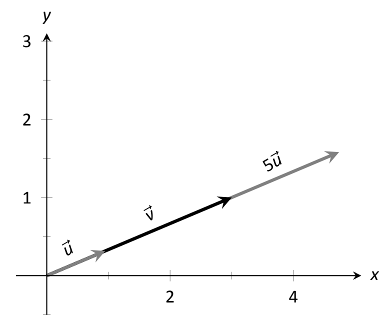


Figure 11.2.8: Graphing vectors in Example 11.2.5. All vectors shown have their initial point at the origin.

Example 11.2.5 Using Unit Vectors

Let $\vec{v} = \langle 3, 1\rangle$ and let $\vec{w} = \langle 1, 2, 2\rangle$.

1. Find the unit vector in the direction of $\vec{v}$.
2. Find the unit vector in the direction of $\vec{w}$.
3. Find the vector in the direction of $\vec{v}$ with magnitude 5.

Solution

1. We find $\| \vec{v} \| = \sqrt{10}$. So the unit vector $\vec{u}$ in the direction of $\vec{v}$ is
$$\vec{u} = \frac{1}{\sqrt{10}}\vec{v} = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle.$$
2. We find $\| \vec{w} \| = 3$, so the unit vector $\vec{z}$ in the direction of $\vec{w}$ is
$$\vec{u} = \frac{1}{3}\vec{w} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle.$$
3. To create a vector with magnitude 5 in the direction of $\vec{v}$, we multiply the unit vector $\vec{u}$ by 5. Thus $5\vec{u} = \langle 15/\sqrt{10}, 5/\sqrt{10}\rangle$ is the vector we seek. This is sketched in Figure 11.2.8.

Note: $\vec{0}$ is directionless; because $\| \vec{0} \| = 0$, there is no unit vector in the "direction" of $\vec{0}$.
Some texts define two vectors as being parallel if one is a scalar multiple of the other. By this definition, $\vec{0}$ is parallel to all vectors as $\vec{0} = 0\vec{v}$ for all $\vec{v}$.

We define what it means for two vectors to be perpendicular in Definition 11.3.2, which is written to exclude $\vec{0}$. It could be written to include $\vec{0}$; by such a definition, $\vec{0}$ is perpendicular to all vectors. While counter-intuitive, it is mathematically sound to allow $\vec{0}$ to be both parallel and perpendicular to all vectors.

We prefer the given definition of parallel as it is grounded in the fact that unit vectors provide direction information. One may adopt the convention that $\vec{0}$ is parallel to all vectors if they desire. (See also the marginal note on page 567.)

The basic formation of the unit vector $\vec{u}$ in the direction of a vector $\vec{v}$ leads to a interesting equation. It is:

$$\vec{v} = \| \vec{v} \| \frac{1}{\| \vec{v} \|}\vec{v}.$$

We rewrite the equation with parentheses to make a point:

$$\vec{v} = \underbrace{\| \vec{v} \|}_{\text{magnitude}} \cdot \underbrace{\left(\frac{1}{\| \vec{v} \|}\vec{v}\right)}_{\text{direction}}.$$

This equation illustrates the fact that a nonzero vector has both magnitude and direction, where we view a unit vector as supplying *only* direction information. Identifying unit vectors with direction allows us to define **parallel vectors**.

Definition 11.2.5 Parallel Vectors

1. Unit vectors $\vec{u}_1$ and $\vec{u}_2$ are **parallel** if $\vec{u}_1 = \pm\vec{u}_2$.
2. Nonzero vectors $\vec{v}_1$ and $\vec{v}_2$ are **parallel** if their respective unit vectors are parallel.

It is equivalent to say that vectors $\vec{v}_1$ and $\vec{v}_2$ are parallel if there is a scalar $c \neq 0$ such that $\vec{v}_1 = c\vec{v}_2$ (see marginal note).

If one graphed all unit vectors in $\mathbb{R}^2$ with the initial point at the origin, then the terminal points would all lie on the unit circle. Based on what we know from trigonometry, we can then say that the component form of all unit vectors in $\mathbb{R}^2$ is $\langle \cos\theta, \sin\theta\rangle$ for some angle θ.

A similar construction in $\mathbb{R}^3$ shows that the terminal points all lie on the unit sphere. These vectors also have a particular component form, but its derivation is not as straightforward as the one for unit vectors in $\mathbb{R}^2$. Important concepts about unit vectors are given in the following Key Idea.

Key Idea 11.2.1 Unit Vectors

1. The unit vector in the direction of $\vec{v}$ is
$$\vec{u} = \frac{1}{\parallel \vec{v} \parallel}\vec{v}.$$
2. A vector $\vec{u}$ in $\mathbb{R}^2$ is a unit vector if, and only if, its component form is $\langle \cos\theta, \sin\theta\rangle$ for some angle θ.
3. A vector $\vec{u}$ in $\mathbb{R}^3$ is a unit vector if, and only if, its component form is $\langle \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\rangle$ for some angles θ and φ.

These formulas can come in handy in a variety of situations, especially the formula for unit vectors in the plane.

Example 11.2.6 Finding Component Forces

Consider a weight of 50 lb hanging from two chains, as shown in Figure 11.2.9. One chain makes an angle of 30° with the vertical, and the other an angle of 45°. Find the force applied to each chain.

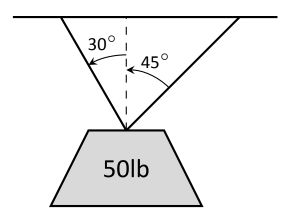


Figure 11.2.9: A diagram of a weight hanging from 2 chains in Example 11.2.6.

SOLUTION Knowing that gravity is pulling the 50 lb weight straight down, we can create a vector $\vec{F}$ to represent this force.

$$\vec{F} = 50\langle 0, -1\rangle = \langle 0, -50\rangle.$$

We can view each chain as "pulling" the weight up, preventing it from falling. We can represent the force from each chain with a vector. Let $\vec{F}_1$ represent the force from the chain making an angle of 30° with the vertical, and let $\vec{F}_2$ represent the force form the other chain. Convert all angles to be measured from the horizontal (as shown in Figure 11.2.10), and apply Key Idea 11.2.1. As we do not yet know the magnitudes of these vectors, (that is the problem at hand), we use m_1 and m_2 to represent them.

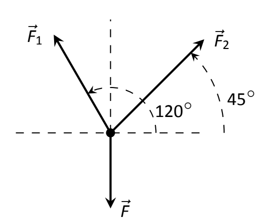


Figure 11.2.10: A diagram of the force vectors from Example 11.2.6.

$$\vec{F}_1 = m_1\langle \cos 120°, \sin 120°\rangle$$
$$\vec{F}_2 = m_2\langle \cos 45°, \sin 45°\rangle$$

As the weight is not moving, we know the sum of the forces is $\vec{0}$. This gives:

$$\vec{F} + \vec{F}_1 + \vec{F}_2 = \vec{0}$$
$$\langle 0, -50\rangle + m_1\langle \cos 120°, \sin 120°\rangle + m_2\langle \cos 45°, \sin 45°\rangle = \vec{0}$$

The sum of the entries in the first component is 0, and the sum of the entries in the second component is also 0. This leads us to the following two equations:

$$\begin{aligned} m_1 \cos 120^\circ + m_2 \cos 45^\circ &= 0 \\ m_1 \sin 120^\circ + m_2 \sin 45^\circ &= 50 \end{aligned}$$

This is a simple 2-equation, 2-unknown system of linear equations. We leave it to the reader to verify that the solution is

$$m_1 = 50(\sqrt{3}-1) \approx 36.6; \qquad m_2 = \frac{50\sqrt{2}}{1+\sqrt{3}} \approx 25.88.$$

It might seem odd that the sum of the forces applied to the chains is more than 50 lb. We leave it to a physics class to discuss the full details, but offer this short explanation. Our equations were established so that the *vertical* components of each force sums to 50 lb, thus supporting the weight. Since the chains are at an angle, they also pull against each other, creating an "additional" horizontal force while holding the weight in place.

Unit vectors were very important in the previous calculation; they allowed us to define a vector in the proper direction but with an unknown magnitude. Our computations were then computed component–wise. Because such calculations are often necessary, the *standard unit vectors* can be useful.

Definition 11.2.6 Standard Unit Vectors

1. In $\mathbb{R}^2$, the standard unit vectors are
$$\vec{i} = \langle 1, 0 \rangle \quad \text{and} \quad \vec{j} = \langle 0, 1 \rangle .$$
2. In $\mathbb{R}^3$, the standard unit vectors are
$$\vec{i} = \langle 1, 0, 0 \rangle \quad \text{and} \quad \vec{j} = \langle 0, 1, 0 \rangle \quad \text{and} \quad \vec{k} = \langle 0, 0, 1 \rangle .$$

Example 11.2.7 Using standard unit vectors

1. Rewrite $\vec{v} = \langle 2, -3 \rangle$ using the standard unit vectors.
2. Rewrite $\vec{w} = 4\vec{i} - 5\vec{j} + 2\vec{k}$ in component form.

Solution

1.
$$\begin{aligned} \vec{v} &= \langle 2, -3 \rangle \\ &= \langle 2, 0 \rangle + \langle 0, -3 \rangle \\ &= 2\langle 1, 0 \rangle - 3\langle 0, 1 \rangle \\ &= 2\vec{i} - 3\vec{j} \end{aligned}$$

2.
$$\begin{aligned} \vec{w} &= 4\vec{i} - 5\vec{j} + 2\vec{k} \\ &= \langle 4, 0, 0 \rangle + \langle 0, -5, 0 \rangle + \langle 0, 0, 2 \rangle \\ &= \langle 4, -5, 2 \rangle \end{aligned}$$

These two examples demonstrate that converting between component form and the standard unit vectors is rather straightforward. Many mathematicians prefer component form, and it is the preferred notation in this text. Many engineers prefer using the standard unit vectors, and many engineering texts use that notation.

Example 11.2.8 Finding Component Force

A weight of 25 lb is suspended from a chain of length 2 ft while a wind pushes the weight to the right with constant force of 5 lb as shown in Figure 11.2.11. What angle will the chain make with the vertical as a result of the wind's pushing? How much higher will the weight be?

Solution The force of the wind is represented by the vector $\vec{F}_w = 5\vec{i}$. The force of gravity on the weight is represented by $\vec{F}_g = -25\vec{j}$. The direction and magnitude of the vector representing the force on the chain are both unknown. We represent this force with

$$\vec{F}_c = m\,\langle \cos\varphi, \sin\varphi\rangle = m\cos\varphi\,\vec{i} + m\sin\varphi\,\vec{j}$$

for some magnitude m and some angle with the horizontal φ. (Note: θ is the angle the chain makes with the *vertical*; φ is the angle with the *horizontal*.)

As the weight is at equilibrium, the sum of the forces is $\vec{0}$:

$$\begin{aligned} \vec{F}_c + \vec{F}_w + \vec{F}_g &= \vec{0} \\ m\cos\varphi\,\vec{i} + m\sin\varphi\,\vec{j} + 5\vec{i} - 25\vec{j} &= \vec{0} \end{aligned}$$

Thus the sum of the $\vec{i}$ and $\vec{j}$ components are 0, leading us to the following system of equations:

$$\begin{aligned} 5 + m\cos\varphi &= 0 \\ -25 + m\sin\varphi &= 0 \end{aligned} \tag{11.1}$$

This is enough to determine $\vec{F}_c$ already, as we know $m\cos\varphi = -5$ and $m\sin\varphi = 25$. Thus $F_c = \langle -5, 25\rangle$. We can use this to find the magnitude m:

$$m = \sqrt{(-5)^2 + 25^2} = 5\sqrt{26} \approx 25.5 \text{ lb}.$$

We can then use either equality from Equation (11.1) to solve for φ. We choose the first equality as using arccosine will return an angle in the 2$^{\text{nd}}$ quadrant:

$$5 + 5\sqrt{26}\cos\varphi = 0 \quad\Rightarrow\quad \varphi = \cos^{-1}\left(\frac{-5}{5\sqrt{26}}\right) \approx 1.7682 \approx 101.31^\circ.$$

Subtracting 90° from this angle gives us an angle of 11.31° with the vertical.

We can now use trigonometry to find out how high the weight is lifted. The diagram shows that a right triangle is formed with the 2 ft chain as the hypotenuse with an interior angle of 11.31°. The length of the adjacent side (in the diagram, the dashed vertical line) is $2\cos 11.31^\circ \approx 1.96$ ft. Thus the weight is lifted by about 0.04 ft, almost 1/2 in.

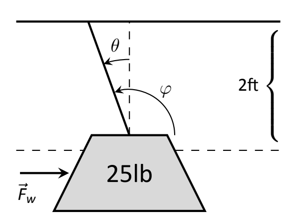


Figure 11.2.11: A figure of a weight being pushed by the wind in Example 11.2.8.

The algebra we have applied to vectors is already demonstrating itself to be very useful. There are two more fundamental operations we can perform with vectors, the *dot product* and the *cross product*. The next two sections explore each in turn.

Exercises 11.2

Terms and Concepts

1. Name two different things that cannot be described with just one number, but rather need 2 or more numbers to fully describe them.

2. What is the difference between $(1, 2)$ and $\langle 1, 2\rangle$?

3. What is a unit vector?

4. Unit vectors can be thought of as conveying what type of information?

5. What does it mean for two vectors to be parallel?

6. What effect does multiplying a vector by -2 have?

Problems

In Exercises 7 – 10, points P and Q are given. Write the vector $\overrightarrow{PQ}$ in component form and using the standard unit vectors.

7. $P = (2, -1), \quad Q = (3, 5)$

8. $P = (3, 2), \quad Q = (7, -2)$

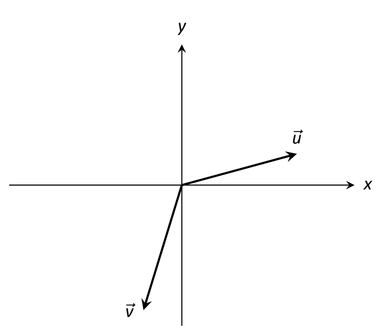
9. $P = (0, 3, -1), \quad Q = (6, 2, 5)$

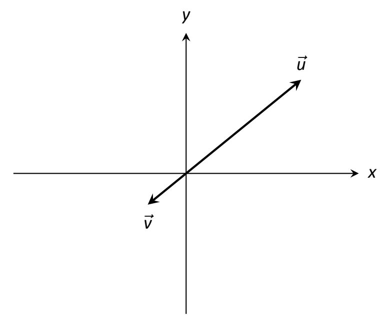
10. $P = (2, 1, 2), \quad Q = (4, 3, 2)$

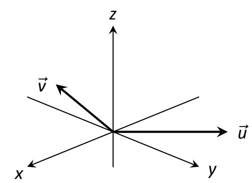
11. Let $\vec{u} = \langle 1, -2\rangle$ and $\vec{v} = \langle 1, 1\rangle$.
    (a) Find $\vec{u}+\vec{v}$, $\vec{u}-\vec{v}$, $2\vec{u}-3\vec{v}$.
    (b) Sketch the above vectors on the same axes, along with $\vec{u}$ and $\vec{v}$.
    (c) Find $\vec{x}$ where $\vec{u}+\vec{x} = 2\vec{v}-\vec{x}$.

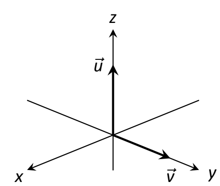
12. Let $\vec{u} = \langle 1, 1, -1\rangle$ and $\vec{v} = \langle 2, 1, 2\rangle$.
    (a) Find $\vec{u}+\vec{v}$, $\vec{u}-\vec{v}$, $\pi\vec{u}-\sqrt{2}\vec{v}$.
    (b) Sketch the above vectors on the same axes, along with $\vec{u}$ and $\vec{v}$.
    (c) Find $\vec{x}$ where $\vec{u}+\vec{x} = \vec{v}+2\vec{x}$.

In Exercises 13 – 16, sketch $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ on the same axes.

13.


14.


15.


16.


In Exercises 17 – 20, find $\| \vec{u} \|$, $\| \vec{v} \|$, $\| \vec{u}+\vec{v} \|$ and $\| \vec{u}-\vec{v} \|$.

17. $\vec{u} = \langle 2, 1\rangle, \quad \vec{v} = \langle 3, -2\rangle$

18. $\vec{u} = \langle -3, 2, 2\rangle, \quad \vec{v} = \langle 1, -1, 1\rangle$

19. $\vec{u} = \langle 1, 2\rangle, \quad \vec{v} = \langle -3, -6\rangle$

20. $\vec{u} = \langle 2, -3, 6\rangle, \quad \vec{v} = \langle 10, -15, 30\rangle$

21. Under what conditions is $\| \vec{u} \| + \| \vec{v} \| = \| \vec{u}+\vec{v} \|$?

In Exercises 22 – 25, find the unit vector $\vec{u}$ in the direction of $\vec{v}$.

22. $\vec{v} = \langle 3, 7\rangle$

23. $\vec{v} = \langle 6, 8\rangle$

24. $\vec{v} = \langle 1, -2, 2\rangle$

25. $\vec{v} = \langle 2, -2, 2\rangle$

26. Find the unit vector in the first quadrant of $\mathbb{R}^2$ that makes a 50° angle with the x-axis.

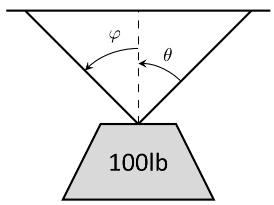
27. Find the unit vector in the second quadrant of $\mathbb{R}^2$ that makes a 30° angle with the y-axis.

28. Verify, from Key Idea 11.2.1, that

$$\vec{u} = \langle \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta \rangle$$

is a unit vector for all angles θ and φ.

A weight of 100lb is suspended from two chains, making angles with the vertical of θ and φ as shown in the figure below.



In Exercises 29 – 32, angles θ and φ are given. Find the magnitude of the force applied to each chain.

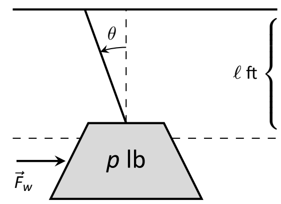
29. $\theta = 30^\circ$, $\varphi = 30^\circ$

30. $\theta = 60^\circ$, $\varphi = 60^\circ$

31. $\theta = 20^\circ$, $\varphi = 15^\circ$

32. $\theta = 0^\circ$, $\varphi = 0^\circ$

A weight of $p$lb is suspended from a chain of length ℓ while a constant force of $\vec{F}_w$ pushes the weight to the right, making an angle of θ with the vertical, as shown in the figure below.



In Exercises 33 – 36, a force $\vec{F}_w$ and length ℓ are given. Find the angle θ and the height the weight is lifted as it moves to the right.

33. $\vec{F}_w = 1\text{lb}$, $\ell = 1\text{ft}$, $p = 1\text{lb}$

34. $\vec{F}_w = 1\text{lb}$, $\ell = 1\text{ft}$, $p = 10\text{lb}$

35. $\vec{F}_w = 1\text{lb}$, $\ell = 10\text{ft}$, $p = 1\text{lb}$

36. $\vec{F}_w = 10\text{lb}$, $\ell = 10\text{ft}$, $p = 1\text{lb}$

11.3 The Dot Product

The previous section introduced vectors and described how to add them together and how to multiply them by scalars. This section introduces *a* multiplication on vectors called the **dot product**.

Definition 11.3.1 Dot Product

1. Let $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$ in $\mathbb{R}^2$. The **dot product** of $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \cdot \vec{v}$, is
$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2.$$
2. Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$. The **dot product** of $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \cdot \vec{v}$, is
$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Note how this product of vectors returns a *scalar*, not another vector. We practice evaluating a dot product in the following example, then we will discuss why this product is useful.

Example 11.3.1 Evaluating dot products

1. Let $\vec{u} = \langle 1, 2 \rangle$, $\vec{v} = \langle 3, -1 \rangle$ in $\mathbb{R}^2$. Find $\vec{u} \cdot \vec{v}$.
2. Let $\vec{x} = \langle 2, -2, 5 \rangle$ and $\vec{y} = \langle -1, 0, 3 \rangle$ in $\mathbb{R}^3$. Find $\vec{x} \cdot \vec{y}$.

Solution

1. Using Definition 11.3.1, we have
$$\vec{u} \cdot \vec{v} = 1(3) + 2(-1) = 1.$$
2. Using the definition, we have
$$\vec{x} \cdot \vec{y} = 2(-1) - 2(0) + 5(3) = 13.$$

The dot product, as shown by the preceding example, is very simple to evaluate. It is only the sum of products. While the definition gives no hint as to why we would care about this operation, there is an amazing connection between the dot product and angles formed by the vectors. Before stating this connection, we give a theorem stating some of the properties of the dot product.

Theorem 11.3.1 Properties of the Dot Product

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let c be a scalar.

1. $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$ Commutative Property
2. $\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$ Distributive Property
3. $c(\vec{u}\cdot\vec{v})=(c\vec{u})\cdot\vec{v}=\vec{u}\cdot(c\vec{v})$
4. $\vec{0}\cdot\vec{v}=0$
5. $\vec{v}\cdot\vec{v}=\|\,\vec{v}\,\|^2$

Note: proving Theorem 11.3.1 is straightforward and left to the reader. The reader is reminded, however, that proofs must be *general*: choosing particular numbers for the vectors $\vec{u}$, $\vec{v}$, etc. only shows that the properties hold for those particular numbers. Instead, one should write $\vec{u}=\langle u_1,u_2,u_3\rangle$, $\vec{v}=\langle v_1,v_2,v_3\rangle$, etc. and then proceed using the rules of algebra for real numbers. For example, $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$ since

$$\begin{aligned}\vec{u}\cdot\vec{v}&=u_1v_1+u_2v_2+u_3v_3\\&=v_1u_1+v_2u_2+v_3u_3\\&=\vec{v}\cdot\vec{u},\end{aligned}$$

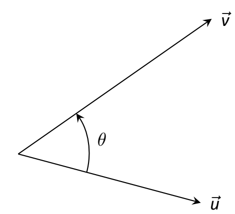
and this argument is valid no matter what values are substituted for the components of the two vectors.

The last statement of the theorem makes a handy connection between the magnitude of a vector and the dot product with itself. Our definition and theorem give properties of the dot product, but we are still likely wondering "What does the dot product *mean*?" It is helpful to understand that the dot product of a vector with itself is connected to its magnitude.

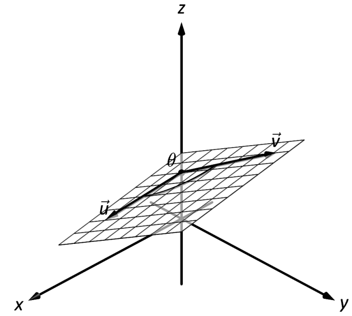
The next theorem extends this understanding by connecting the dot product to magnitudes and angles. Given vectors $\vec{u}$ and $\vec{v}$ in the plane, an angle θ is clearly formed when $\vec{u}$ and $\vec{v}$ are drawn with the same initial point as illustrated in Figure 11.3.1(a). (We always take θ to be the angle in $[0,\pi]$ as two angles are actually created.)

The same is also true of 2 vectors in space: given $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ with the same initial point, there is a plane that contains both $\vec{u}$ and $\vec{v}$. (When $\vec{u}$ and $\vec{v}$ are co-linear, there are infinitely many planes that contain both vectors.) In that plane, we can again find an angle θ between them (and again, $0\leq\theta\leq\pi$). This is illustrated in Figure 11.3.1(b).

The following theorem connects this angle θ to the dot product of $\vec{u}$ and $\vec{v}$.



(a)



(b)

Figure 11.3.1: Illustrating the angle formed by two vectors with the same initial point.

Theorem 11.3.2 The Dot Product and Angles

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then

$$\vec{u}\cdot\vec{v}=\|\,\vec{u}\,\|\,\|\,\vec{v}\,\|\cos\theta,$$

where θ, $0\leq\theta\leq\pi$, is the angle between $\vec{u}$ and $\vec{v}$.

The proof of Theorem 11.3.2 is an application of the Law of Cosines, using the properties in Theorem 11.3.1. Referring to Figure 11.3.2, if we let $a=\|\,\vec{u}\,\|$, $b=\|\,\vec{v}\,\|$, and $c=\|\,\vec{u}-\vec{v}\,\|$, then the Law of Cosines tells us that

$$c^2=a^2+b^2-2ab\cos(\theta).$$

Thus, we have

$$\begin{aligned}\|\,\vec{u}-\vec{v}\,\|^2&=\|\,\vec{u}\,\|^2+\|\,\vec{v}\,\|^2-2\|\,\vec{u}\,\|\|\,\vec{v}\,\|\cos\theta\\(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})&=\vec{u}\cdot\vec{u}+\vec{v}\cdot\vec{v}-2\|\,\vec{u}\,\|\|\,\vec{v}\,\|\cos\theta\\\vec{u}\cdot\vec{u}-\vec{u}\cdot\vec{v}-\vec{v}\cdot\vec{u}+\vec{v}\cdot\vec{v}&=\vec{u}\cdot\vec{u}+\vec{v}\cdot\vec{v}-2\|\,\vec{u}\,\|\|\,\vec{v}\,\|\cos\theta\\-2\vec{u}\cdot\vec{v}&=-2\|\,\vec{u}\,\|\|\,\vec{v}\,\|\cos\theta\\\vec{u}\cdot\vec{v}&=\|\,\vec{u}\,\|\|\,\vec{v}\,\|\cos\theta,\end{aligned}$$

as required.

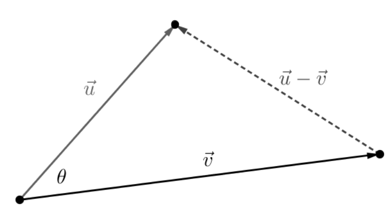


Figure 11.3.2: Proving Theorem 11.3.2

Using Theorem 11.3.1, we can rewrite this theorem as

$$\frac{\vec{u}}{\parallel \vec{u} \parallel} \cdot \frac{\vec{v}}{\parallel \vec{v} \parallel} = \cos\theta.$$

Note how on the left hand side of the equation, we are computing the dot product of two unit vectors. Recalling that unit vectors essentially only provide direction information, we can informally restate Theorem 11.3.2 as saying "The dot product of two directions gives the cosine of the angle between them."

When θ is an acute angle (i.e., $0 \leq \theta < \pi/2$), $\cos\theta$ is positive; when $\theta = \pi/2$, $\cos\theta = 0$; when θ is an obtuse angle ($\pi/2 < \theta \leq \pi$), $\cos\theta$ is negative. Thus the sign of the dot product gives a general indication of the angle between the vectors, illustrated in Figure 11.3.3.

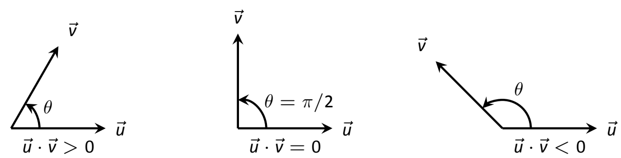


Figure 11.3.3: Illustrating the relationship between the angle between vectors and the sign of their dot product.

We *can* use Theorem 11.3.2 to compute the dot product, but generally this theorem is used to find the angle between known vectors (since the dot product is generally easy to compute). To this end, we rewrite the theorem's equation as

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\parallel \vec{u} \parallel \parallel \vec{v} \parallel} \quad \Leftrightarrow \quad \theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{\parallel \vec{u} \parallel \parallel \vec{v} \parallel}\right).$$

We practice using this theorem in the following example.

Example 11.3.2 Using the dot product to find angles

Let $\vec{u} = \langle 3, 1\rangle$, $\vec{v} = \langle -2, 6\rangle$ and $\vec{w} = \langle -4, 3\rangle$, as shown in Figure 11.3.4. Find the angles α, β and θ.

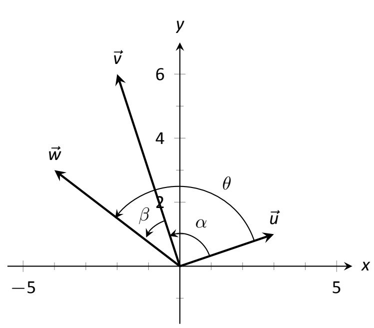


Figure 11.3.4: Vectors used in Example 11.3.2.

Solution We start by computing the magnitude of each vector.

$$\parallel \vec{u} \parallel = \sqrt{10}; \quad \parallel \vec{v} \parallel = 2\sqrt{10}; \quad \parallel \vec{w} \parallel = 5.$$

We now apply Theorem 11.3.2 to find the angles.

$$\begin{aligned} \alpha &= \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{(\sqrt{10})(2\sqrt{10})}\right) \\ &= \cos^{-1}(0) = \frac{\pi}{2} = 90^\circ. \end{aligned}$$

$$\begin{aligned}\beta &= \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{(2\sqrt{10})(5)}\right)\\ &= \cos^{-1}\left(\frac{26}{10\sqrt{10}}\right)\\ &\approx 0.6055 \approx 34.7^\circ.\end{aligned}$$

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{w}}{(\sqrt{10})(5)}\right)\\ &= \cos^{-1}\left(\frac{-9}{5\sqrt{10}}\right)\\ &\approx 2.1763 \approx 124.7^\circ\end{aligned}$$

We see from our computation that $\alpha+\beta=\theta$, as indicated by Figure 11.3.4. While we knew this should be the case, it is nice to see that this non-intuitive formula indeed returns the results we expected. We do a similar example next in the context of vectors in space.

Example 11.3.3 Using the dot product to find angles

Let $\vec{u}=\langle 1,1,1\rangle$, $\vec{v}=\langle -1,3,-2\rangle$ and $\vec{w}=\langle -5,1,4\rangle$, as illustrated in Figure 11.3.5. Find the angle between each pair of vectors.

Solution

1. Between $\vec{u}$ and $\vec{v}$:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)\\ &= \cos^{-1}\left(\frac{0}{\sqrt{3}\sqrt{14}}\right)\\ &= \frac{\pi}{2}.\end{aligned}$$

2. Between $\vec{u}$ and $\vec{w}$:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{w}}{\|\vec{u}\|\|\vec{w}\|}\right)\\ &= \cos^{-1}\left(\frac{0}{\sqrt{3}\sqrt{42}}\right)\\ &= \frac{\pi}{2}.\end{aligned}$$

3. Between $\vec{v}$ and $\vec{w}$:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right)\\ &= \cos^{-1}\left(\frac{0}{\sqrt{14}\sqrt{42}}\right)\\ &= \frac{\pi}{2}.\end{aligned}$$

While our work shows that each angle is $\pi/2$, i.e., 90°, none of these angles looks to be a right angle in Figure 11.3.5. Such is the case when drawing three–dimensional objects on the page.

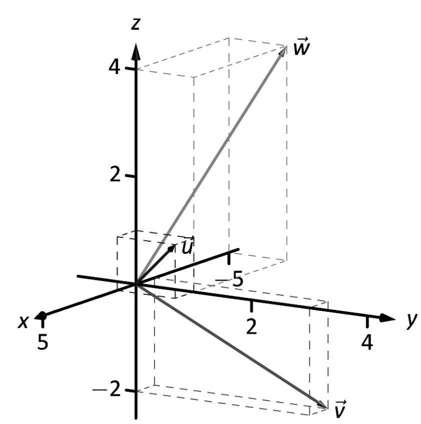


Figure 11.3.5: Vectors used in Example 11.3.3.

All three angles between these vectors was $\pi/2$, or 90°. We know from geometry and everyday life that 90° angles are "nice" for a variety of reasons, so it should seem significant that these angles are all $\pi/2$. Notice the common feature in each calculation (and also the calculation of α in Example 11.3.2): the dot products of each pair of angles was 0. We use this as a basis for a definition of the term **orthogonal**, which is essentially synonymous to *perpendicular*.

Definition 11.3.2 Orthogonal

Vectors $\vec{u}$ and $\vec{v}$ are **orthogonal** if their dot product is 0.

Example 11.3.4 Finding orthogonal vectors

Let $\vec{u} = \langle 3, 5\rangle$ and $\vec{v} = \langle 1, 2, 3\rangle$.

1. Find two vectors in $\mathbb{R}^2$ that are orthogonal to $\vec{u}$.
2. Find two non–parallel vectors in $\mathbb{R}^3$ that are orthogonal to $\vec{v}$.

Note: The term *perpendicular* originally referred to lines. As mathematics progressed, the concept of "being at right angles to" was applied to other objects, such as vectors and planes, and the term *orthogonal* was introduced. It is especially used when discussing objects that are hard, or impossible, to visualize: two vectors in 5-dimensional space are orthogonal if their dot product is 0. It is not wrong to say they are *perpendicular*, but common convention gives preference to the word *orthogonal*.
Note also that Definition 11.3.2 makes sense if either $\vec{u}$ or $\vec{v}$ is the zero vector, but this is not the case for the conventional understanding of the word perpendicular.

Solution

1. Recall that a line perpendicular to a line with slope m has slope $-1/m$, the "opposite reciprocal slope." We can think of the slope of $\vec{u}$ as 5/3, its "rise over run." A vector orthogonal to $\vec{u}$ will have slope $-3/5$. There are many such choices, though all parallel:

$$\langle -5, 3\rangle \quad \text{or} \quad \langle 5, -3\rangle \quad \text{or} \quad \langle -10, 6\rangle \quad \text{or} \quad \langle 15, -9\rangle\,, \text{etc.}$$

2. There are infinite directions in space orthogonal to any given direction, so there are an infinite number of non–parallel vectors orthogonal to $\vec{v}$. Since there are so many, we have great leeway in finding some.

   One way is to arbitrarily pick values for the first two components, leaving the third unknown. For instance, let $\vec{v}_1 = \langle 2, 7, z\rangle$. If $\vec{v}_1$ is to be orthogonal to $\vec{v}$, then $\vec{v}_1 \cdot \vec{v} = 0$, so

$$2 + 14 + 3z = 0 \quad \Rightarrow z = \frac{-16}{3}.$$

   So $\vec{v}_1 = \langle 2, 7, -16/3\rangle$ is orthogonal to $\vec{v}$. We can apply a similar technique by leaving the first or second component unknown.

   Another method of finding a vector orthogonal to $\vec{v}$ mirrors what we did in part 1. Let $\vec{v}_2 = \langle -2, 1, 0\rangle$. Here we switched the first two components of $\vec{v}$, changing the sign of one of them (similar to the "opposite reciprocal" concept before). Letting the third component be 0 effectively ignores the third component of $\vec{v}$, and it is easy to see that

$$\vec{v}_2 \cdot \vec{v} = \langle -2, 1, 0\rangle \cdot \langle 1, 2, 3\rangle = 0.$$

   Clearly $\vec{v}_1$ and $\vec{v}_2$ are not parallel.

An important construction is illustrated in Figure 11.3.6, where vectors $\vec{u}$ and $\vec{v}$ are sketched. In part (a), a dotted line is drawn from the tip of $\vec{u}$ to the line containing $\vec{v}$, where the dotted line is orthogonal to $\vec{v}$. In part (b), the dotted line is replaced with the vector $\vec{z}$ and $\vec{w}$ is formed, parallel to $\vec{v}$. It is clear by the diagram that $\vec{u} = \vec{w} + \vec{z}$. What is important about this construction is this: $\vec{u}$ is *decomposed* as the sum of two vectors, one of which is parallel to $\vec{v}$ and one that is perpendicular to $\vec{v}$. It is hard to overstate the importance of this construction (as we'll see in upcoming examples).

The vectors $\vec{w}$, $\vec{z}$ and $\vec{u}$ as shown in Figure 11.3.6 (b) form a right triangle, where the angle between $\vec{v}$ and $\vec{u}$ is labelled θ. We can find $\vec{w}$ in terms of $\vec{v}$ and $\vec{u}$.

Using trigonometry, we can state that

$$\parallel \vec{w} \parallel = \parallel \vec{u} \parallel \cos\theta. \tag{11.2}$$

We also know that $\vec{w}$ is parallel to to $\vec{v}$; that is, the direction of $\vec{w}$ is the direction of $\vec{v}$, described by the unit vector $\frac{1}{\parallel \vec{v} \parallel}\vec{v}$. The vector $\vec{w}$ is the vector in the direction $\frac{1}{\parallel \vec{v} \parallel}\vec{v}$ with magnitude $\parallel \vec{u} \parallel \cos\theta$:

$$\vec{w} = \Big(\parallel \vec{u} \parallel \cos\theta\Big)\frac{1}{\parallel \vec{v} \parallel}\vec{v}.$$

Replace $\cos\theta$ using Theorem 11.3.2:

$$\begin{aligned} &= \left(\parallel \vec{u} \parallel \frac{\vec{u}\cdot\vec{v}}{\parallel \vec{u} \parallel\parallel \vec{v} \parallel}\right)\frac{1}{\parallel \vec{v} \parallel}\vec{v} \\ &= \frac{\vec{u}\cdot\vec{v}}{\parallel \vec{v} \parallel^2}\vec{v}. \end{aligned}$$

Now apply Theorem 11.3.1.

$$= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}.$$

Since this construction is so important, it is given a special name.

Definition 11.3.3 Orthogonal Projection

Let nonzero vectors $\vec{u}$ and $\vec{v}$ be given. The **orthogonal projection of $\vec{u}$ onto $\vec{v}$**, denoted $\text{proj}_{\,\vec{v}}\,\vec{u}$, is

$$\text{proj}_{\,\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}.$$

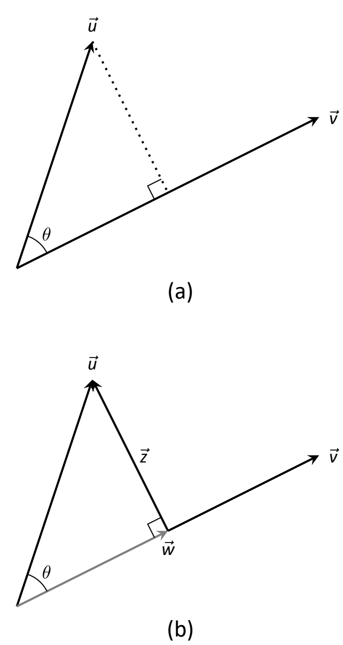


Figure 11.3.6: Developing the construction of the *orthogonal projection*.

Example 11.3.5 Computing the orthogonal projection

1. Let $\vec{u} = \langle -2, 1\rangle$ and $\vec{v} = \langle 3, 1\rangle$. Find $\text{proj}_{\,\vec{v}}\,\vec{u}$, and sketch all three vectors with initial points at the origin.

2. Let $\vec{w} = \langle 2, 1, 3\rangle$ and $\vec{x} = \langle 1, 1, 1\rangle$. Find $\text{proj}_{\,\vec{x}}\,\vec{w}$, and sketch all three vectors with initial points at the origin.

Solution

1. Applying Definition 11.3.3, we have

$$\begin{aligned}\operatorname{proj}_{\vec{v}}\vec{u} &= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}\\ &= \frac{-5}{10}\langle 3,1\rangle\\ &= \left\langle -\frac{3}{2}, -\frac{1}{2}\right\rangle.\end{aligned}$$

   Vectors $\vec{u}$, $\vec{v}$ and $\operatorname{proj}_{\vec{v}}\vec{u}$ are sketched in Figure 11.3.7(a). Note how the projection is parallel to $\vec{v}$; that is, it lies on the same line through the origin as $\vec{v}$, although it points in the opposite direction. That is because the angle between $\vec{u}$ and $\vec{v}$ is obtuse (i.e., greater than 90°).

2. Apply the definition:

$$\begin{aligned}\operatorname{proj}_{\vec{x}}\vec{w} &= \frac{\vec{w}\cdot\vec{x}}{\vec{x}\cdot\vec{x}}\vec{x}\\ &= \frac{6}{3}\langle 1,1,1\rangle\\ &= \langle 2,2,2\rangle.\end{aligned}$$

   These vectors are sketched in Figure 11.3.7(b), and again in part (c) from a different perspective. Because of the nature of graphing these vectors, the sketch in part (b) makes it difficult to recognize that the drawn projection has the geometric properties it should. The graph shown in part (c) illustrates these properties better.

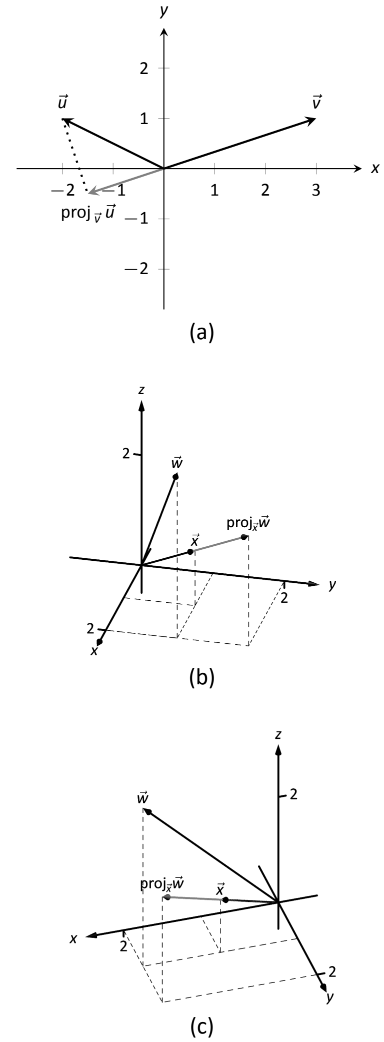


Figure 11.3.7: Graphing the vectors used in Example 11.3.5.

We can use the properties of the dot product found in Theorem 11.3.1 to rearrange the formula found in Definition 11.3.3:

$$\begin{aligned}\operatorname{proj}_{\vec{v}}\vec{u} &= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}\\ &= \frac{\vec{u}\cdot\vec{v}}{\|\,\vec{v}\,\|^2}\vec{v}\\ &= \left(\vec{u}\cdot\frac{\vec{v}}{\|\,\vec{v}\,\|}\right)\frac{\vec{v}}{\|\,\vec{v}\,\|}.\end{aligned}$$

The above formula shows that the orthogonal projection of $\vec{u}$ onto $\vec{v}$ is only concerned with the *direction* of $\vec{v}$, as both instances of $\vec{v}$ in the formula come in the form $\vec{v}/\|\,\vec{v}\,\|$, the unit vector in the direction of $\vec{v}$.

A special case of orthogonal projection occurs when $\vec{v}$ is a unit vector. In this situation, the formula for the orthogonal projection of a vector $\vec{u}$ onto $\vec{v}$ reduces to just $\operatorname{proj}_{\vec{v}}\vec{u} = (\vec{u}\cdot\vec{v})\vec{v}$, as $\vec{v}\cdot\vec{v} = 1$.

This gives us a new understanding of the dot product. When $\vec{v}$ is a unit vector, essentially providing only direction information, the dot product of $\vec{u}$ and $\vec{v}$ gives "how much of $\vec{u}$ is in the direction of $\vec{v}$." This use of the dot product will be very useful in future sections.

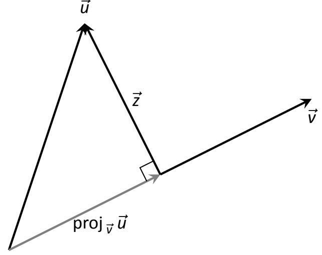


Figure 11.3.8: Illustrating the orthogonal projection.

Now consider Figure 11.3.8 where the concept of the orthogonal projection is again illustrated. It is clear that

$$\vec{u} = \operatorname{proj}_{\vec{v}}\vec{u} + \vec{z}. \tag{11.3}$$

As we know what $\vec{u}$ and $\text{proj}_{\vec{v}}\,\vec{u}$ are, we can solve for $\vec{z}$ and state that

$$\vec{z} = \vec{u} - \text{proj}_{\vec{v}}\,\vec{u}.$$

This leads us to rewrite Equation (11.3) in a seemingly silly way:

$$\vec{u} = \text{proj}_{\vec{v}}\,\vec{u} + (\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}).$$

This is not nonsense, as pointed out in the following Key Idea. (Notation note: the expression "$\parallel \vec{y}$" means "is parallel to $\vec{y}$." We can use this notation to state "$\vec{x} \parallel \vec{y}$" which means "$\vec{x}$ is parallel to $\vec{y}$." The expression "$\perp \vec{y}$" means "is orthogonal to $\vec{y}$," and is used similarly.)

Key Idea 11.3.1 Orthogonal Decomposition of Vectors

Let nonzero vectors $\vec{u}$ and $\vec{v}$ be given. Then $\vec{u}$ can be written as the sum of two vectors, one of which is parallel to $\vec{v}$, and one of which is orthogonal to $\vec{v}$:

$$\vec{u} = \underbrace{\text{proj}_{\vec{v}}\,\vec{u}}_{\parallel \vec{v}} + (\underbrace{\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}}_{\perp \vec{v}}).$$

Note: The argument leading to Definition 11.3.3 is not quite a proof, since it depended on choices made in forming the diagram in Figure 11.3.6. However, we can easily verify that the result in Key Idea 11.3.1 is always valid: since

$$\begin{aligned}\vec{v}\cdot(\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}) &= \vec{v}\cdot\vec{u} - \vec{v}\cdot\left(\frac{\vec{u}\cdot\vec{v}}{\|\,\vec{v}\,\|^2}\vec{v}\right)\\ &= \vec{v}\cdot\vec{u} - \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}(\vec{v}\cdot\vec{v})\\ &= \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v} = 0\end{aligned}$$

for **any** vectors $\vec{u}$ and $\vec{v} \neq \vec{0}$, we are guaranteed that the vector $u - \text{proj}_{\vec{v}}\,\vec{u}$ will always be orthogonal to $\vec{v}$.

We illustrate the use of this equality in the following example.

Example 11.3.6 Orthogonal decomposition of vectors

1. Let $\vec{u} = \langle -2, 1\rangle$ and $\vec{v} = \langle 3, 1\rangle$ as in Example 11.3.5. Decompose $\vec{u}$ as the sum of a vector parallel to $\vec{v}$ and a vector orthogonal to $\vec{v}$.

2. Let $\vec{w} = \langle 2, 1, 3\rangle$ and $\vec{x} = \langle 1, 1, 1\rangle$ as in Example 11.3.5. Decompose $\vec{w}$ as the sum of a vector parallel to $\vec{x}$ and a vector orthogonal to $\vec{x}$.

Solution

1. In Example 11.3.5, we found that $\text{proj}_{\vec{v}}\,\vec{u} = \langle -1.5, -0.5\rangle$. Let

$$\vec{z} = \vec{u} - \text{proj}_{\vec{v}}\,\vec{u} = \langle -2, 1\rangle - \langle -1.5, -0.5\rangle = \langle -0.5, 1.5\rangle\,.$$

   Is $\vec{z}$ orthogonal to $\vec{v}$? (I.e, is $\vec{z} \perp \vec{v}$?) We check for orthogonality with the dot product:

$$\vec{z}\cdot\vec{v} = \langle -0.5, 1.5\rangle \cdot \langle 3, 1\rangle = 0.$$

   Since the dot product is 0, we know $\vec{z} \perp \vec{v}$. Thus:

$$\begin{aligned}\vec{u} &= \text{proj}_{\vec{v}}\,\vec{u} \;+\; (\vec{u} - \text{proj}_{\vec{v}}\,\vec{u})\\ \langle -2, 1\rangle &= \underbrace{\langle -1.5, -0.5\rangle}_{\parallel \vec{v}} \;+\; \underbrace{\langle -0.5, 1.5\rangle}_{\perp \vec{v}}.\end{aligned}$$

2. We found in Example 11.3.5 that $\text{proj}_{\vec{x}}\,\vec{w} = \langle 2, 2, 2\rangle$. Applying the Key Idea, we have:

$$\vec{z} = \vec{w} - \text{proj}_{\vec{x}}\,\vec{w} = \langle 2, 1, 3\rangle - \langle 2, 2, 2\rangle = \langle 0, -1, 1\rangle\,.$$

   We check to see if $\vec{z} \perp \vec{x}$:

$$\vec{z}\cdot\vec{x} = \langle 0, -1, 1\rangle \cdot \langle 1, 1, 1\rangle = 0.$$

Since the dot product is 0, we know the two vectors are orthogonal. We now write $\vec{w}$ as the sum of two vectors, one parallel and one orthogonal to $\vec{x}$:

$$\begin{aligned}\vec{w} &= \text{proj}_{\vec{x}}\,\vec{w} \;+\; (\vec{w} - \text{proj}_{\vec{x}}\,\vec{w})\\ \langle 2,1,3\rangle &= \underbrace{\langle 2,2,2\rangle}_{\parallel\,\vec{x}} \;+\; \underbrace{\langle 0,-1,1\rangle}_{\perp\,\vec{x}}\end{aligned}$$

We give an example of where this decomposition is useful.

Example 11.3.7 Orthogonally decomposing a force vector

Consider Figure 11.3.9(a), showing a box weighing 50 lb on a ramp that rises 5 ft over a span of 20 ft. Find the components of force, and their magnitudes, acting on the box (as sketched in part (b) of the figure):

1. in the direction of the ramp, and
2. orthogonal to the ramp.

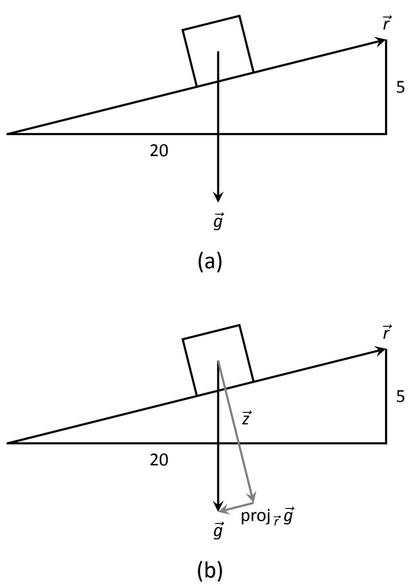


Figure 11.3.9: Sketching the ramp and box in Example 11.3.7. Note: *The vectors are not drawn to scale.*

Solution As the ramp rises 5 ft over a horizontal distance of 20 ft, we can represent the direction of the ramp with the vector $\vec{r} = \langle 20, 5\rangle$. Gravity pulls down with a force of 50 lb, which we represent with $\vec{g} = \langle 0, -50\rangle$.

1. To find the force of gravity in the direction of the ramp, we compute $\text{proj}_{\vec{r}}\,\vec{g}$:

$$\begin{aligned}\text{proj}_{\vec{r}}\,\vec{g} &= \frac{\vec{g}\cdot\vec{r}}{\vec{r}\cdot\vec{r}}\vec{r}\\ &= \frac{-250}{425}\langle 20,5\rangle\\ &= \left\langle -\frac{200}{17}, -\frac{50}{17}\right\rangle \approx \langle -11.76, -2.94\rangle .\end{aligned}$$

The magnitude of $\text{proj}_{\vec{r}}\,\vec{g}$ is $\|\,\text{proj}_{\vec{r}}\,\vec{g}\,\| = 50/\sqrt{17} \approx 12.13$ lb. Though the box weighs 50 lb, a force of about 12 lb is enough to keep the box from sliding down the ramp.

2. To find the component $\vec{z}$ of gravity orthogonal to the ramp, we use Key Idea 11.3.1.

$$\begin{aligned}\vec{z} &= \vec{g} - \text{proj}_{\vec{r}}\,\vec{g}\\ &= \left\langle \frac{200}{17}, -\frac{800}{17}\right\rangle \approx \langle 11.76, -47.06\rangle .\end{aligned}$$

The magnitude of this force is $\|\,\vec{z}\,\| \approx 48.51$ lb. In physics and engineering, knowing this force is important when computing things like static frictional force. (For instance, we could easily compute if the static frictional force alone was enough to keep the box from sliding down the ramp.)

Application to Work

In physics, the application of a force F to move an object in a straight line a distance d produces *work*; the amount of work W is $W = Fd$, (where F is in the direction of travel). The orthogonal projection allows us to compute work when the force is not in the direction of travel.

Consider Figure 11.3.10, where a force $\vec{F}$ is being applied to an object moving in the direction of $\vec{d}$. (The distance the object travels is the magnitude of $\vec{d}$.) The work done is the amount of force in the direction of $\vec{d}$, $\| \operatorname{proj}_{\vec{d}} \vec{F} \|$, times $\| \vec{d} \|$:

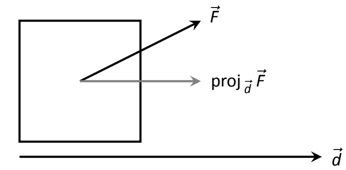


Figure 11.3.10: Finding work when the force and direction of travel are given as vectors.

$$\begin{aligned}
\| \operatorname{proj}_{\vec{d}} \vec{F} \| \cdot \| \vec{d} \| &= \left\| \frac{\vec{F} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \vec{d} \right\| \cdot \| \vec{d} \| \\
&= \left| \frac{\vec{F} \cdot \vec{d}}{\| \vec{d} \|^2} \right| \cdot \| \vec{d} \| \cdot \| \vec{d} \| \\
&= \frac{\left| \vec{F} \cdot \vec{d} \right|}{\| \vec{d} \|^2} \| \vec{d} \|^2 \\
&= \left| \vec{F} \cdot \vec{d} \right|.
\end{aligned}$$

The expression $\vec{F} \cdot \vec{d}$ will be positive if the angle between $\vec{F}$ and $\vec{d}$ is acute; when the angle is obtuse (hence $\vec{F} \cdot \vec{d}$ is negative), the force is causing motion in the opposite direction of $\vec{d}$, resulting in "negative work." We want to capture this sign, so we drop the absolute value and find that $W = \vec{F} \cdot \vec{d}$.

Definition 11.3.4 Work

Let $\vec{F}$ be a constant force that moves an object in a straight line from point P to point Q. Let $\vec{d} = \overrightarrow{PQ}$. The **work** W done by $\vec{F}$ along $\vec{d}$ is $W = \vec{F} \cdot \vec{d}$.

Example 11.3.8 Computing work

A man slides a box along a ramp that rises 3 ft over a distance of 15 ft by applying 50 lb of force as shown in Figure 11.3.11. Compute the work done.

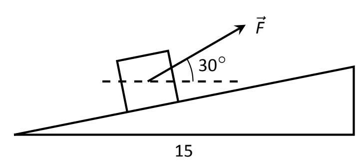


Figure 11.3.11: Computing work when sliding a box up a ramp in Example 11.3.8.

Solution The figure indicates that the force applied makes a 30° angle with the horizontal, so $\vec{F} = 50 \langle \cos 30^\circ, \sin 30^\circ \rangle \approx \langle 43.3, 25 \rangle$. The ramp is represented by $\vec{d} = \langle 15, 3 \rangle$. The work done is simply

$$\vec{F} \cdot \vec{d} = 50 \langle \cos 30^\circ, \sin 30^\circ \rangle \cdot \langle 15, 3 \rangle \approx 724.5 \text{ ft–lb}.$$

Note how we did not actually compute the distance the object travelled, nor the magnitude of the force in the direction of travel; this is all inherently computed by the dot product!

The dot product is a powerful way of evaluating computations that depend on angles without actually using angles. The next section explores another "product" on vectors, the *cross product.* Once again, angles play an important role, though in a much different way.

Exercises 11.3

Terms and Concepts

1. The dot product of two vectors is a ________, not a vector.

2. How are the concepts of the dot product and vector magnitude related?

3. How can one quickly tell if the angle between two vectors is acute or obtuse?

4. Give a synonym for "orthogonal."

Problems

In Exercises 5 – 10, find the dot product of the given vectors.

5. $\vec{u} = \langle 2, -4 \rangle$, $\vec{v} = \langle 3, 7 \rangle$

6. $\vec{u} = \langle 5, 3 \rangle$, $\vec{v} = \langle 6, 1 \rangle$

7. $\vec{u} = \langle 1, -1, 2 \rangle$, $\vec{v} = \langle 2, 5, 3 \rangle$

8. $\vec{u} = \langle 3, 5, -1 \rangle$, $\vec{v} = \langle 4, -1, 7 \rangle$

9. $\vec{u} = \langle 1, 1 \rangle$, $\vec{v} = \langle 1, 2, 3 \rangle$

10. $\vec{u} = \langle 1, 2, 3 \rangle$, $\vec{v} = \langle 0, 0, 0 \rangle$

11. Create your own vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$ and show that $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.

12. Create your own vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ and scalar c and show that $c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$.

In Exercises 13 – 16, find the measure of the angle between the two vectors in both radians and degrees.

13. $\vec{u} = \langle 1, 1 \rangle$, $\vec{v} = \langle 1, 2 \rangle$

14. $\vec{u} = \langle -2, 1 \rangle$, $\vec{v} = \langle 3, 5 \rangle$

15. $\vec{u} = \langle 8, 1, -4 \rangle$, $\vec{v} = \langle 2, 2, 0 \rangle$

16. $\vec{u} = \langle 1, 7, 2 \rangle$, $\vec{v} = \langle 4, -2, 5 \rangle$

In Exercises 17 – 20, a vector $\vec{v}$ is given. Give two vectors that are orthogonal to $\vec{v}$.

17. $\vec{v} = \langle 4, 7 \rangle$

18. $\vec{v} = \langle -3, 5 \rangle$

19. $\vec{v} = \langle 1, 1, 1 \rangle$

20. $\vec{v} = \langle 1, -2, 3 \rangle$

In Exercises 21 – 26, vectors $\vec{u}$ and $\vec{v}$ are given. Find $\text{proj}_{\vec{v}}\,\vec{u}$, the orthogonal projection of $\vec{u}$ onto $\vec{v}$, and sketch all three vectors with the same initial point.

21. $\vec{u} = \langle 1, 2 \rangle$, $\vec{v} = \langle -1, 3 \rangle$

22. $\vec{u} = \langle 5, 5 \rangle$, $\vec{v} = \langle 1, 3 \rangle$

23. $\vec{u} = \langle -3, 2 \rangle$, $\vec{v} = \langle 1, 1 \rangle$

24. $\vec{u} = \langle -3, 2 \rangle$, $\vec{v} = \langle 2, 3 \rangle$

25. $\vec{u} = \langle 1, 5, 1 \rangle$, $\vec{v} = \langle 1, 2, 3 \rangle$

26. $\vec{u} = \langle 3, -1, 2 \rangle$, $\vec{v} = \langle 2, 2, 1 \rangle$

In Exercises 27 – 32, vectors $\vec{u}$ and $\vec{v}$ are given. Write $\vec{u}$ as the sum of two vectors, one of which is parallel to $\vec{v}$ and one of which is perpendicular to $\vec{v}$. Note: these are the same pairs of vectors as found in Exercises 21 – 26.

27. $\vec{u} = \langle 1, 2 \rangle$, $\vec{v} = \langle -1, 3 \rangle$

28. $\vec{u} = \langle 5, 5 \rangle$, $\vec{v} = \langle 1, 3 \rangle$

29. $\vec{u} = \langle -3, 2 \rangle$, $\vec{v} = \langle 1, 1 \rangle$

30. $\vec{u} = \langle -3, 2 \rangle$, $\vec{v} = \langle 2, 3 \rangle$

31. $\vec{u} = \langle 1, 5, 1 \rangle$, $\vec{v} = \langle 1, 2, 3 \rangle$

32. $\vec{u} = \langle 3, -1, 2 \rangle$, $\vec{v} = \langle 2, 2, 1 \rangle$

33. A 10lb box sits on a ramp that rises 4ft over a distance of 20ft. How much force is required to keep the box from sliding down the ramp?

34. A 10lb box sits on a 15ft ramp that makes a 30° angle with the horizontal. How much force is required to keep the box from sliding down the ramp?

35. How much work is performed in moving a box horizontally 10ft with a force of 20lb applied at an angle of 45° to the horizontal?

36. How much work is performed in moving a box horizontally 10ft with a force of 20lb applied at an angle of 10° to the horizontal?

37. How much work is performed in moving a box up the length of a ramp that rises 2ft over a distance of 10ft, with a force of 50lb applied horizontally?

38. How much work is performed in moving a box up the length of a ramp that rises 2ft over a distance of 10ft, with a force of 50lb applied at an angle of 45° to the horizontal?

39. How much work is performed in moving a box up the length of a 10ft ramp that makes a 5° angle with the horizontal, with 50lb of force applied in the direction of the ramp?

11.4 The Cross Product

"Orthogonality" is immensely important. A quick scan of your current environment will undoubtedly reveal numerous surfaces and edges that are perpendicular to each other (including the edges of this page). The dot product provides a quick test for orthogonality: vectors $\vec{u}$ and $\vec{v}$ are perpendicular if, and only if, $\vec{u} \cdot \vec{v} = 0$.

Given two non–parallel, nonzero vectors $\vec{u}$ and $\vec{v}$ in space, it is very useful to find a vector $\vec{w}$ that is perpendicular to both $\vec{u}$ and $\vec{v}$. There is a operation, called the **cross product**, that creates such a vector. This section defines the cross product, then explores its properties and applications.

Definition 11.4.1 Cross Product

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be vectors in $\mathbb{R}^3$. The **cross product of $\vec{u}$ and $\vec{v}$**, denoted $\vec{u} \times \vec{v}$, is the vector

$$\vec{u} \times \vec{v} = \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1 \rangle .$$

The definition of the cross product may look strange (and complicated) at first, but it's more or less forced by the requirement that it be orthogonal to both $\vec{u}$ and $\vec{v}$. To begin to see why, suppose $\vec{w} = \langle a, b, c \rangle$ is an arbitrary vector such that $\vec{w} \cdot \vec{u} = 0$ and $\vec{w} \cdot \vec{v} = 0$. This gives us the pair of equations

$$\begin{aligned} u_1a + u_2b + u_3c &= 0 \\ v_1a + v_2b + v_3c &= 0. \end{aligned}$$

This is a *system of linear equations* in the variables a, b, and c. Using Gaussian elimination (recalling your linear algebra), it's easy to show that (up to a scalar multiple) the solution is given by Definition 11.4.1.

This definition can be a bit cumbersome to remember. After an example we will give a convenient method for computing the cross product. For now, careful examination of the products and differences given in the definition should reveal a pattern that is not too difficult to remember. (For instance, in the first component only 2 and 3 appear as subscripts; in the second component, only 1 and 3 appear as subscripts. Further study reveals the order in which they appear.)

Let's practice using this definition by computing a cross product.

Example 11.4.1 Computing a cross product

Let $\vec{u} = \langle 2, -1, 4 \rangle$ and $\vec{v} = \langle 3, 2, 5 \rangle$. Find $\vec{u} \times \vec{v}$, and verify that it is orthogonal to both $\vec{u}$ and $\vec{v}$.

SOLUTION Using Definition 11.4.1, we have

$$\begin{aligned} \vec{u} \times \vec{v} &= \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle \\ &= \langle (-1)5 - (4)2, (4)3 - (2)5, (2)2 - (-1)3 \rangle = \langle -13, 2, 7 \rangle . \end{aligned}$$

(We encourage the reader to compute this product on their own, then verify their result.)

We test whether or not $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$ using the dot product:

$$\begin{aligned} \big(\vec{u} \times \vec{v}\big) \cdot \vec{u} &= \langle -13, 2, 7 \rangle \cdot \langle 2, -1, 4 \rangle = 0, \\ \big(\vec{u} \times \vec{v}\big) \cdot \vec{v} &= \langle -13, 2, 7 \rangle \cdot \langle 3, 2, 5 \rangle = 0. \end{aligned}$$

Since both dot products are zero, $\vec{u} \times \vec{v}$ is indeed orthogonal to both $\vec{u}$ and $\vec{v}$.

We now introduce a method for computing the cross-product that is easier to remember, which you may recall from your first course in linear algebra.

Consider a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of four real numbers a, b, c, and d. A 2×2 determinant takes any such matrix and assigns the number $ad - bc$. This is commonly denoted as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Most people find it easiest to remember this in terms of the two *diagonals* of the array: we take the product of the two numbers on the *main diagonal* (top-left to bottom-right), and subtract the product of the two numbers on the other diagonal:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$bc \qquad\qquad ad$$

For example, we have $\begin{vmatrix} 4 & -2 \\ 6 & 3 \end{vmatrix} = 4(3) - (-2)(6) = 24$. Once we get comfortable with 2×2 determinants, we can write the cross product in terms of them, as follows:

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k} \\ &= (u_2v_3 - u_3v_2)\vec{i} - (u_3v_1 - u_1v_3)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}, \end{aligned} \tag{11.4}$$

as before. Now, this might not seem like much of an improvement over the previous formula, so we take things one step further. First, we form a 3×3 array as shown below.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

The first row comprises the standard unit vectors $\vec{i}, \vec{j}$, and $\vec{k}$. The second and third rows are the vectors $\vec{u}$ and $\vec{v}$, respectively. Next, we *expand* our 3×3 array as a vector, where the coefficient of each standard unit vector is given by the 2×2 determinant that's left over when we delete the row and column containing that unit vector.

For example, if we use $\vec{u}$ and $\vec{v}$ from Example 11.4.1, we obtain the array

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 3 & 2 & 5 \end{vmatrix}.$$

The expansion process used to obtain the coefficients of $\vec{i}, \vec{j}\vec{k}$ looks like the following:

$$\begin{vmatrix} \boxed{\vec{i}} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 3 & 2 & 5 \end{vmatrix} \longrightarrow \begin{vmatrix} -1 & 4 \\ 2 & 5 \end{vmatrix} \vec{i} = -13\vec{i}$$

Now repeat the first two columns after the original three:

$$\begin{vmatrix} \vec{i} & \boxed{\vec{j}} & \vec{k} \\ 2 & -1 & 4 \\ 3 & 2 & 5 \end{vmatrix} \longrightarrow \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} \vec{j} = -2\vec{j}$$

This gives three full "upper left to lower right" diagonals, and three full "upper right to lower left" diagonals, as shown. Compute the products along each diagonal, then add the products on the right and subtract the products on the left:

$$\begin{vmatrix} \vec{i} & \vec{j} & \boxed{\vec{k}} \\ 2 & -1 & 4 \\ 3 & 2 & 5 \end{vmatrix} \longrightarrow \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} \vec{k} = 7\vec{k}$$

$$\vec{u}\times\vec{v} = \big(-5\vec{i}+12\vec{j}+4\vec{k}\big) - \big(-3\vec{k}+8\vec{i}+10\vec{j}\big) = -13\vec{i}+2\vec{j}+7\vec{k} = \langle -13,2,7\rangle\,.$$

There is one more important detail to note: notice in Equation (11.4) that there is a **minus sign** in front of the coefficient of the unit vector $\vec{j}$. We need to make sure that the signs in front of each 2×2 determinant follow this $+$, $-$, $+$ pattern when we expand our array as a vector. For the vectors $\vec{u}$ and $\vec{v}$ in Example 11.4.1, we end up with the following:

$$\begin{aligned}\vec{u}\times\vec{v} &= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & -1 & 4\\ 3 & 2 & 5\end{vmatrix} = \begin{vmatrix}-1 & 4\\ 2 & 5\end{vmatrix}\vec{i} - \begin{vmatrix}2 & 4\\ 3 & 5\end{vmatrix}\vec{j} + \begin{vmatrix}2 & -1\\ 3 & 2\end{vmatrix}\vec{k}\\ &= -13\vec{i} - (-2)\vec{j} + 7\vec{k} = \langle -13,2,7\rangle\,,\end{aligned}$$

as before. The method will become more clear with a bit of practice.

Note: If the minus sign in front of the $\vec{j}$ coefficient seems out of place to you, it might help to imagine wrapping our 3×3 array around a cylinder (like the label on a tin can). If we read from left to right, *beginning in the $\vec{j}$ column*, then we should place the $\vec{k}$ column first, followed by the $\vec{i}$ column. For the vectors $\vec{u}$ and $\vec{v}$ in Example 11.4.1, this would result in the coefficient $\begin{vmatrix}4 & 2\\ 5 & 2\end{vmatrix} = 2$ for the $\vec{j}$ component, which has the correct sign. However, since our habit is to read starting from the far left, we tend to write the $\vec{i}$ column first, and then introduce the minus sign to compensate.

Example 11.4.2 Computing a cross product

Let $\vec{u} = \langle 1,3,6\rangle$ and $\vec{v} = \langle -1,2,1\rangle$. Compute both $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$.

Solution To compute $\vec{u}\times\vec{v}$, we form our 3×3 array as prescribed above, and expand it into a vector:

$$\begin{aligned}\vec{u}\times\vec{v} &= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 3 & 6\\ -1 & 2 & 1\end{vmatrix} = \begin{vmatrix}3 & 6\\ 2 & 1\end{vmatrix}\vec{i} - \begin{vmatrix}1 & 6\\ -1 & 1\end{vmatrix}\vec{j} + \begin{vmatrix}1 & 3\\ -1 & 2\end{vmatrix}\vec{k}\\ &= (3(1)-6(2))\vec{i} - (1(1)-6(-1))\vec{j} + (1(2)-3(-1))\vec{k}\\ &= -9\vec{i} - 7\vec{j} + 5\vec{k} = \langle -9,-7,5\rangle\,.\end{aligned}$$

To compute $\vec{v}\times\vec{u}$, we switch the second and third rows of the above matrix, then expand as before:

$$\begin{aligned}\vec{v}\times\vec{u} &= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & 2 & 1\\ 1 & 3 & 6\end{vmatrix} = \begin{vmatrix}2 & 1\\ 3 & 6\end{vmatrix}\vec{i} - \begin{vmatrix}-1 & 1\\ 1 & 6\end{vmatrix}\vec{j} + \begin{vmatrix}-1 & 2\\ 1 & 3\end{vmatrix}\vec{k}\\ &= (2(6)-1(3))\vec{i} - ((-1)(6)-1(1))\vec{j} + ((-1)(3)-2(1))\vec{k}\\ &= 9\vec{i} + 7\vec{j} - 5\vec{k} = \langle 9,7,-5\rangle = -\vec{u}\times\vec{v}.\end{aligned}$$

Note how with the rows being switched, the products that once appeared on the right now appear on the left, and vice–versa, so that the result is the opposite of $\vec{u}\times\vec{v}$. We leave it to the reader to verify that each of these vectors is orthogonal to $\vec{u}$ and $\vec{v}$.

Properties of the Cross Product

It is not coincidence that $\vec{v}\times\vec{u} = -(\vec{u}\times\vec{v})$ in the preceding example; one can show using Definition 11.4.1 that this will always be the case. The following theorem states several useful properties of the cross product, each of which can be verified by referring to the definition.

Theorem 11.4.1 Properties of the Cross Product

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^3$ and let c be a scalar. The following identities hold:

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ Anticommutative Property
2. (a) $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$ Distributive Properties
   (b) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
3. $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$
4. (a) $(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$ Orthogonality Properties
   (b) $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$
5. $\vec{u} \times \vec{u} = \vec{0}$
6. $\vec{u} \times \vec{0} = \vec{0}$
7. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$ Triple Scalar Product

We introduced the cross product as a way to find a vector orthogonal to two given vectors, but we did not give a proof that the construction given in Definition 11.4.1 satisfies this property. Theorem 11.4.1 asserts this property holds; we leave it as a problem in the Exercise section to verify this.

The algebraic properties of the cross product in Theorem 11.4.1 also give us an additional method for computing the cross product in terms of the unit vectors $\vec{i}, \vec{j}, \vec{k}$. We know from Property 5 that

$$\vec{i} \times \vec{i} = \vec{0}, \vec{j} \times \vec{j} = \vec{0}, \vec{k} \times \vec{k} = \vec{0},$$

and it's easy to check that

$$\vec{i} \times \vec{j} = \vec{k}, \vec{j} \times \vec{k} = \vec{i}, \vec{k} \times \vec{i} = \vec{j},$$

and then Property 1 guarantees that

$$\vec{j} \times \vec{i} = -\vec{k}, \vec{k} \times \vec{j} = -\vec{i}, \vec{i} \times \vec{k} = -\vec{j}.$$

Using Properties 2 and 3, we can then compute, for example,

$$\begin{aligned}\langle 2, 0, 3\rangle \times \langle -1, 4, 2\rangle &= (2\vec{i} + 3\vec{k}) \times (-\vec{i} + 4\vec{j} + 2\vec{k}) \\ &= -2(\vec{i} \times \vec{i}) + 8(\vec{i} \times \vec{j}) + 4(\vec{i} \times \vec{k}) \\ &\quad - 3(\vec{k} \times \vec{i}) + 12(\vec{k} \times \vec{j}) + 6(\vec{k} \times \vec{k}) \\ &= \vec{0} + 8\vec{k} - 4\vec{j} - 3\vec{j} - 12\vec{i} + \vec{0} = \langle -12, -7, 8\rangle .\end{aligned}$$

Property 5 from the theorem is also left to the reader to prove in the Exercise section, but it reveals something more interesting than "the cross product of a vector with itself is $\vec{0}$." Let $\vec{u}$ and $\vec{v}$ be parallel vectors; that is, let there be a scalar c such that $\vec{v} = c\vec{u}$. Consider their cross product:

$$\begin{aligned}\vec{u} \times \vec{v} &= \vec{u} \times (c\vec{u}) \\ &= c(\vec{u} \times \vec{u}) && \text{(by Property 3 of Theorem 11.4.1)} \\ &= \vec{0}. && \text{(by Property 5 of Theorem 11.4.1)}\end{aligned}$$

We have just shown that the cross product of parallel vectors is $\vec{0}$. This hints at something deeper. Theorem 11.3.2 related the angle between two vectors and their dot product; there is a similar relationship relating the cross product of two vectors and the angle between them, given by the following theorem.

Theorem 11.4.2 The Cross Product and Angles

Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^3$. Then

$$\parallel \vec{u} \times \vec{v} \parallel = \parallel \vec{u} \parallel \parallel \vec{v} \parallel \sin\theta,$$

where θ, $0 \leq \theta \leq \pi$, is the angle between $\vec{u}$ and $\vec{v}$.

Note that this theorem makes a statement about the *magnitude* of the cross product. When the angle between $\vec{u}$ and $\vec{v}$ is 0 or π (i.e., the vectors are parallel), the magnitude of the cross product is 0. The only vector with a magnitude of 0 is $\vec{0}$ (see Property 9 of Theorem 11.2.1), hence the cross product of parallel vectors is $\vec{0}$.

We provide some anecdotal evidence of the truth of this theorem in the following example.

Note: We could rewrite Definition 11.3.2 and Theorem 11.4.2 to include $\vec{0}$, then define that $\vec{u}$ and $\vec{v}$ are parallel if $\vec{u} \times \vec{v} = \vec{0}$. Since $\vec{0} \cdot \vec{v} = 0$ and $\vec{0} \times \vec{v} = \vec{0}$, this would mean that $\vec{0}$ is both parallel *and* orthogonal to all vectors. Apparent paradoxes such as this are not uncommon in mathematics and can be very useful. (See also the marginal note on page 546.)

Example 11.4.3 The cross product and angles

Let $\vec{u} = \langle 1, 3, 6 \rangle$ and $\vec{v} = \langle -1, 2, 1 \rangle$ as in Example 11.4.2. Verify Theorem 11.4.2 by finding θ, the angle between $\vec{u}$ and $\vec{v}$, and the magnitude of $\vec{u} \times \vec{v}$.

SOLUTION We use Theorem 11.3.2 to find the angle between $\vec{u}$ and $\vec{v}$.

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{\parallel \vec{u} \parallel \parallel \vec{v} \parallel}\right) \\ &= \cos^{-1}\left(\frac{11}{\sqrt{46}\sqrt{6}}\right) \\ &\approx 0.8471 = 48.54^\circ. \end{aligned}$$

Our work in Example 11.4.2 showed that $\vec{u} \times \vec{v} = \langle -9, -7, 5 \rangle$, hence $\parallel \vec{u} \times \vec{v} \parallel = \sqrt{155}$. Is $\parallel \vec{u} \times \vec{v} \parallel = \parallel \vec{u} \parallel \parallel \vec{v} \parallel \sin\theta$? Using numerical approximations, we find:

$$\begin{aligned} \parallel \vec{u} \times \vec{v} \parallel &= \sqrt{155} & \qquad \parallel \vec{u} \parallel \parallel \vec{v} \parallel \sin\theta &= \sqrt{46}\sqrt{6}\sin 0.8471 \\ &\approx 12.45. & &\approx 12.45. \end{aligned}$$

Numerically, they seem equal. Using a right triangle, one can show that

$$\sin\left(\cos^{-1}\left(\frac{11}{\sqrt{46}\sqrt{6}}\right)\right) = \frac{\sqrt{155}}{\sqrt{46}\sqrt{6}},$$

which allows us to verify the theorem exactly.

To see that Theorem 11.4.2 holds in general, let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be two arbitrary three-dimensional vectors. Since the angle between $\vec{u}$ and $\vec{v}$ is defined to lie between 0 and π, we know that $\sin\theta \geq 0$, so that both sides of the equation $\parallel \vec{u} \times \vec{v} \parallel = \parallel \vec{u} \parallel \parallel \vec{v} \parallel \sin\theta$ are positive. Thus, we can show that both sides are equal if we can show that their squares are equal. We have

$$\begin{aligned}
(\| \vec{u} \| \| \vec{v} \| \sin\theta)^2 &= \| \vec{u} \|^2 \| \vec{v} \|^2 \sin^2\theta \\
&= \| \vec{u} \|^2 \| \vec{v} \|^2 (1-\cos^2\theta) \qquad \text{since } \sin^2\theta + \cos^2\theta = 1 \\
&= \| \vec{u} \|^2 \| \vec{v} \|^2 - (\| \vec{u} \| \| \vec{v} \| \cos\theta)^2 \\
&= \| \vec{u} \|^2 \| \vec{v} \|^2 - (\vec{u}\cdot\vec{v})^2 \qquad \text{by Theorem 11.3.2} \\
&= (u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2) - (u_1v_1+u_2v_2+u_3v_3)^2 \\
&= u_2^2v_3^2 - 2u_2u_3v_2v_3 + u_3^2v_2^2 + u_1v_3^2 - 2u_1u_3v_1v_3 \\
&\qquad + u_3^2v_1^2 + u_1^2v_2^2 - 2u_1u_2v_1v_2 + u_2^2v_1^2 \\
&= (u_2v_3-u_3v_2)^2 + (u_3v_1-u_1v_3)^2 + (u_1v_2-u_2v_2)^2 \\
&= \| \vec{u}\times\vec{v} \|^2,
\end{aligned}$$

as required.

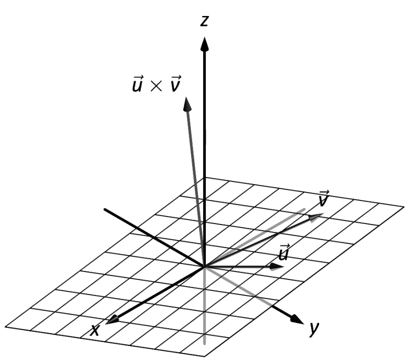


Figure 11.4.1: Illustrating the Right Hand Rule of the cross product.

Right Hand Rule

The anticommutative property of the cross product demonstrates that $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$ differ only by a sign – these vectors have the same magnitude but point in the opposite direction. When seeking a vector perpendicular to $\vec{u}$ and $\vec{v}$, we essentially have two directions to choose from, one in the direction of $\vec{u}\times\vec{v}$ and one in the direction of $\vec{v}\times\vec{u}$. Does it matter which we choose? How can we tell which one we will get without graphing, etc.?

Another wonderful property of the cross product, as defined, is that it follows the **right hand rule.** Given $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ with the same initial point, point the index finger of your right hand in the direction of $\vec{u}$ and let your middle finger point in the direction of $\vec{v}$ (much as we did when establishing the right hand rule for the 3-dimensional coordinate system). Your thumb will naturally extend in the direction of $\vec{u}\times\vec{v}$. One can "practice" this using Figure 11.4.1. If you switch, and point the index finder in the direction of $\vec{v}$ and the middle finger in the direction of $\vec{u}$, your thumb will now point in the opposite direction, allowing you to "visualize" the anticommutative property of the cross product.

Applications of the Cross Product

There are a number of ways in which the cross product is useful in mathematics, physics and other areas of science beyond "just" finding a vector perpendicular to two others. We highlight a few here.

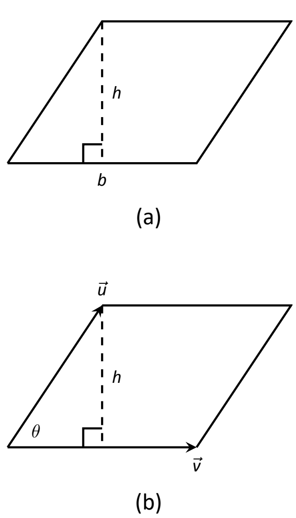


Figure 11.4.2: Using the cross product to find the area of a parallelogram.

Area of a Parallelogram

It is a standard geometry fact that the area of a parallelogram is $A = bh$, where b is the length of the base and h is the height of the parallelogram, as illustrated in Figure 11.4.2(a). As shown when defining the Parallelogram Law of vector addition, two vectors $\vec{u}$ and $\vec{v}$ define a parallelogram when drawn from the same initial point, as illustrated in Figure 11.4.2(b). Trigonometry tells us that $h = \| \vec{u} \| \sin\theta$, hence the area of the parallelogram is

$$A = \| \vec{u} \| \, \| \vec{v} \| \sin\theta = \| \vec{u}\times\vec{v} \|, \tag{11.5}$$

where the second equality comes from Theorem 11.4.2. We illustrate using Equation (11.5) in the following example.

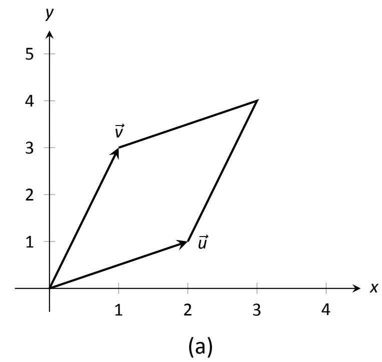
Example 11.4.4 Finding the area of a parallelogram

1. Find the area of the parallelogram defined by the vectors $\vec{u} = \langle 2, 1\rangle$ and $\vec{v} = \langle 1, 3\rangle$.
2. Verify that the points $A = (1, 1, 1)$, $B = (2, 3, 2)$, $C = (4, 5, 3)$ and $D = (3, 3, 2)$ are the vertices of a parallelogram. Find the area of the parallelogram.

Solution

1. Figure 11.4.3(a) sketches the parallelogram defined by the vectors $\vec{u}$ and $\vec{v}$. We have a slight problem in that our vectors exist in $\mathbb{R}^2$, not $\mathbb{R}^3$, and the cross product is only defined on vectors in $\mathbb{R}^3$. We skirt this issue by viewing $\vec{u}$ and $\vec{v}$ as vectors in the $x-y$ plane of $\mathbb{R}^3$, and rewrite them as $\vec{u} = \langle 2, 1, 0\rangle$ and $\vec{v} = \langle 1, 3, 0\rangle$. We can now compute the cross product. It is easy to show that $\vec{u}\times\vec{v} = \langle 0, 0, 5\rangle$; therefore the area of the parallelogram is $A = \parallel \vec{u} \times \vec{v} \parallel = 5$.
2. To show that the quadrilateral $ABCD$ is a parallelogram (shown in Figure 11.4.3(b)), we need to show that the opposite sides are parallel. We can quickly show that $\overrightarrow{AB} = \overrightarrow{DC} = \langle 1, 2, 1\rangle$ and $\overrightarrow{BC} = \overrightarrow{AD} = \langle 2, 2, 1\rangle$. We find the area by computing the magnitude of the cross product of $\overrightarrow{AB}$ and $\overrightarrow{BC}$:

$$\overrightarrow{AB} \times \overrightarrow{BC} = \langle 0, 1, -2\rangle \quad \Rightarrow \quad \parallel \overrightarrow{AB} \times \overrightarrow{BC} \parallel = \sqrt{5} \approx 2.236.$$



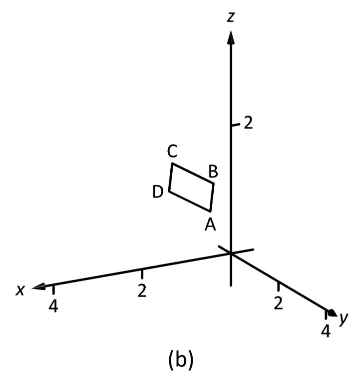


Figure 11.4.3: Sketching the parallelograms in Example 11.4.4.

This application is perhaps more useful in finding the area of a triangle (in short, triangles are used more often than parallelograms). We illustrate this in the following example.

Example 11.4.5 Area of a triangle

Find the area of the triangle with vertices $A = (1, 2)$, $B = (2, 3)$ and $C = (3, 1)$, as pictured in Figure 11.4.4.

Solution We found the area of this triangle in Example 7.1.4 to be 1.5 using integration. There we discussed the fact that finding the area of a triangle can be inconvenient using the "$\frac{1}{2}bh$" formula as one has to compute the height, which generally involves finding angles, etc. Using a cross product is much more direct.

We can choose any two sides of the triangle to use to form vectors; we choose $\overrightarrow{AB} = \langle 1, 1\rangle$ and $\overrightarrow{AC} = \langle 2, -1\rangle$. As in the previous example, we will rewrite these vectors with a third component of 0 so that we can apply the cross product. The area of the triangle is

$$\frac{1}{2}\parallel \overrightarrow{AB} \times \overrightarrow{AC} \parallel = \frac{1}{2}\parallel \langle 1, 1, 0\rangle \times \langle 2, -1, 0\rangle \parallel = \frac{1}{2}\parallel \langle 0, 0, -3\rangle \parallel = \frac{3}{2}.$$

We arrive at the same answer as before with less work.

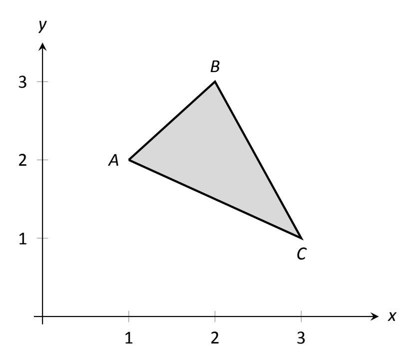


Figure 11.4.4: Finding the area of a triangle in Example 11.4.5.

Volume of a Parallelepiped

The three dimensional analogue to the parallelogram is the **parallelepiped**. Each face is parallel to the face opposite face, as illustrated in Figure 11.4.5. The volume of any three-dimensional solid whose cross-sectional area is a constant is given by $V = B\cdot h$, where B is the area of the base (the constant cross-sectional area), and h is the height. To determine a formula for the volume, we refer to Figure 11.4.6. By crossing $\vec{v}$ and $\vec{w}$, one gets a vector whose magnitude is the area of the base, and whose direction is perpendicular to the parallelogram forming the base of the solid. We can then see that the height of the parallelepiped is equal to the length of the projection of the vector $\vec{u}$ onto $\vec{v}\times\vec{w}$. Our volume is therefore:

Note: The word “parallelepiped” is pronounced “parallel–uh–pipe–ed.”

$$\begin{aligned} V &= B\cdot h \\ &= \parallel \vec{v}\times\vec{w} \parallel \cdot \parallel \text{proj}_{\vec{v}\times\vec{w}}\,\vec{u} \parallel \\ &= \parallel \vec{v}\times\vec{w} \parallel \cdot \parallel \left(\frac{\vec{u}\cdot(\vec{v}\times\vec{w})}{\parallel \vec{v}\times\vec{w}\parallel^2}\right)(\vec{v}\times\vec{w}) \parallel \\ &= \parallel \vec{v}\times\vec{w} \parallel \frac{|\vec{u}\cdot(\vec{v}\times\vec{w})|}{\parallel \vec{v}\times\vec{w}\parallel^2} \parallel \vec{v}\times\vec{w} \parallel \\ &= |\vec{u}\cdot(\vec{v}\times\vec{w})|. \end{aligned}$$

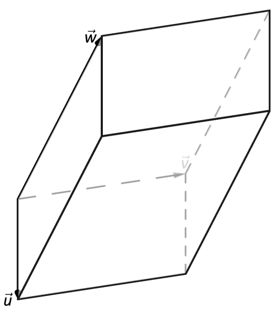


Figure 11.4.5: A parallelepiped is the three dimensional analogue to the parallelogram.

Thus the volume of a parallelepiped defined by vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ is

$$V = |\vec{u}\cdot(\vec{v}\times\vec{w})|. \tag{11.6}$$

Note how this is the Scalar Triple Product, first seen in Theorem 11.4.1. Applying the identities given in the theorem shows that we can apply the Scalar Triple Product in any “order” we choose to find the volume. That is,

$$V = |\vec{u}\cdot(\vec{v}\times\vec{w})| = |\vec{u}\cdot(\vec{w}\times\vec{v})| = |(\vec{u}\times\vec{v})\cdot\vec{w}|, \quad \text{etc.}$$

Example 11.4.6 Finding the volume of parallelepiped

Find the volume of the parallepiped defined by the vectors $\vec{u}=\langle 1,1,0\rangle$, $\vec{v}=\langle -1,1,0\rangle$ and $\vec{w}=\langle 0,1,1\rangle$.

Solution We apply Equation (11.6). We first find $\vec{v}\times\vec{w}=\langle 1,1,-1\rangle$. Then

$$|\vec{u}\cdot(\vec{v}\times\vec{w})| = |\langle 1,1,0\rangle\cdot\langle 1,1,-1\rangle| = 2.$$

So the volume of the parallelepiped is 2 cubic units.

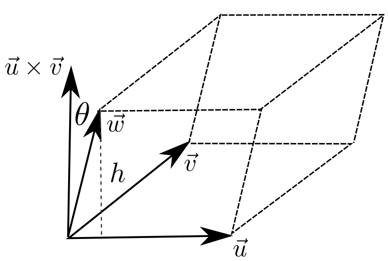


Figure 11.4.6: Determining the volume of a parallelepiped

Let’s take another look at how Equation (11.6) is computed in terms of our formulas for the dot and cross products. With $\vec{u}=\langle u_1,u_2,u_3\rangle$, $\vec{v}=\langle v_1,v_2,v_3\rangle$, and $\vec{w}=\langle w_1,w_2,w_3\rangle$, we have

$$\begin{aligned} \vec{u}\cdot(\vec{v}\times\vec{w}) &= \langle u_1,u_2,u_3\rangle \cdot \left\langle \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right\rangle \\ &= u_1\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}. \end{aligned}$$

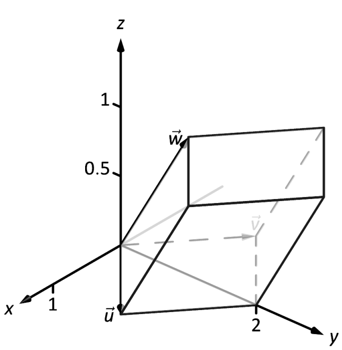


Figure 11.4.7: A parallelepiped in Example 11.4.6.

Compare this with our determinant formula for computing the cross product,

$$\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\vec{k}.$$

If we replace the unit vectors $\vec{i},\vec{j}$, *veck* in the above equation with the components of $\vec{u}$, we arrive at our first instance of a 3×3 **determinant**, along with a method for computing such an object:

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = \vec{u}\cdot(\vec{v}\times\vec{w}).$$

While this application of the Scalar Triple Product is interesting, it is not used all that often: parallelepipeds are not a common shape in physics and engineering. (It is, however, essential to understanding the change of variables formula for multiple integrals in Calculus.) The last application of the cross product is very applicable in engineering.

Torque

Torque is a measure of the turning force applied to an object. A classic scenario involving torque is the application of a wrench to a bolt. When a force is applied to the wrench, the bolt turns. When we represent the force and wrench with vectors $\vec{F}$ and $\vec{\ell}$, we see that the bolt moves (because of the threads) in a direction orthogonal to $\vec{F}$ and $\vec{\ell}$. Torque is usually represented by the Greek letter τ, or tau, and has units of N·m, a Newton–metre, or ft·lb, a foot–pound.

While a full understanding of torque is beyond the purposes of this book, when a force $\vec{F}$ is applied to a lever arm $\vec{\ell}$, the resulting torque is

$$\vec{\tau} = \vec{\ell} \times \vec{F}. \tag{11.7}$$

Example 11.4.7 Computing torque

A lever of length 2 ft makes an angle with the horizontal of 45°. Find the resulting torque when a force of 10 lb is applied to the end of the level where:

1. the force is perpendicular to the lever, and
2. the force makes an angle of 60° with the lever, as shown in Figure 11.4.8.

SOLUTION

1. We start by determining vectors for the force and lever arm. Since the lever arm makes a 45° angle with the horizontal and is 2 ft long, we can state that $\vec{\ell} = 2\langle\cos 45^\circ, \sin 45^\circ\rangle = \left\langle\sqrt{2}, \sqrt{2}\right\rangle$.

   Since the force vector is perpendicular to the lever arm (as seen in the left hand side of Figure 11.4.8), we can conclude it is making an angle of -45° with the horizontal. As it has a magnitude of 10 lb, we can state $\vec{F} = 10\langle\cos(-45^\circ), \sin(-45^\circ)\rangle = \left\langle 5\sqrt{2}, -5\sqrt{2}\right\rangle$.

   Using Equation (11.7) to find the torque requires a cross product. We again let the third component of each vector be 0 and compute the cross product:

$$\begin{aligned}\vec{\tau} &= \vec{\ell} \times \vec{F}\\ &= \left\langle\sqrt{2}, \sqrt{2}, 0\right\rangle \times \left\langle 5\sqrt{2}, -5\sqrt{2}, 0\right\rangle\\ &= \langle 0, 0, -20\rangle\end{aligned}$$

   This clearly has a magnitude of 20 ft-lb.

   We can view the force and lever arm vectors as lying "on the page"; our computation of $\vec{\tau}$ shows that the torque goes "into the page." This follows the Right Hand Rule of the cross product, and it also matches well with the example of the wrench turning the bolt. Turning a bolt clockwise moves it in.

2. Our lever arm can still be represented by $\vec{\ell} = \left\langle\sqrt{2}, \sqrt{2}\right\rangle$. As our force vector makes a 60° angle with $\vec{\ell}$, we can see (referencing the right hand

side of the figure) that $\vec{F}$ makes a -15° angle with the horizontal. Thus

$$\vec{F} = 10\,\langle \cos -15^\circ, \sin -15^\circ \rangle = \left\langle \frac{5(1+\sqrt{3})}{\sqrt{2}}, -\frac{5(1+\sqrt{3})}{\sqrt{2}} \right\rangle$$
$$\approx \langle 9.659, -2.588 \rangle\,.$$

We again make the third component 0 and take the cross product to find the torque:

$$\begin{aligned}\vec{\tau} &= \vec{\ell} \times \vec{F}\\ &= \left\langle \sqrt{2}, \sqrt{2}, 0 \right\rangle \times \left\langle \frac{5(1+\sqrt{3})}{\sqrt{2}}, -\frac{5(1+\sqrt{3})}{\sqrt{2}}, 0 \right\rangle\\ &= \left\langle 0, 0, -10\sqrt{3} \right\rangle\\ &\approx \langle 0, 0, -17.321 \rangle\,.\end{aligned}$$

As one might expect, when the force and lever arm vectors *are* orthogonal, the magnitude of force is greater than when the vectors *are not* orthogonal.

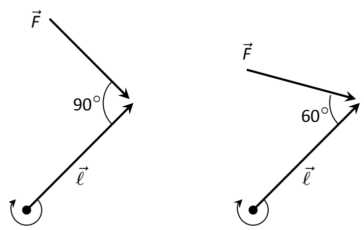


Figure 11.4.8: Showing a force being applied to a lever in Example 11.4.7.

While the cross product has a variety of applications (as noted in this chapter), its fundamental use is finding a vector perpendicular to two others. Knowing a vector is orthogonal to two others is of incredible importance, as it allows us to find the equations of lines and planes in a variety of contexts. The importance of the cross product, in some sense, relies on the importance of lines and planes, which see widespread use throughout engineering, physics and mathematics. We study lines and planes in the next two sections.

Exercises 11.4

Terms and Concepts

1. The cross product of two vectors is a ________, not a scalar.

2. One can visualize the direction of $\vec{u} \times \vec{v}$ using the ________ ________ ________.

3. Give a synonym for "orthogonal."

4. T/F: A fundamental principle of the cross product is that $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$.

5. ________ is a measure of the turning force applied to an object.

6. T/F: If $\vec{u}$ and $\vec{v}$ are parallel, then $\vec{u} \times \vec{v} = \vec{0}$.

Problems

In Exercises 7 – 16, vectors $\vec{u}$ and $\vec{v}$ are given. Compute $\vec{u} \times \vec{v}$ and show this is orthogonal to both $\vec{u}$ and $\vec{v}$.

7. $\vec{u} = \langle 3, 2, -2\rangle, \quad \vec{v} = \langle 0, 1, 5\rangle$

8. $\vec{u} = \langle 5, -4, 3\rangle, \quad \vec{v} = \langle 2, -5, 1\rangle$

9. $\vec{u} = \langle 4, -5, -5\rangle, \quad \vec{v} = \langle 3, 3, 4\rangle$

10. $\vec{u} = \langle -4, 7, -10\rangle, \quad \vec{v} = \langle 4, 4, 1\rangle$

11. $\vec{u} = \langle 1, 0, 1\rangle, \quad \vec{v} = \langle 5, 0, 7\rangle$

12. $\vec{u} = \langle 1, 5, -4\rangle, \quad \vec{v} = \langle -2, -10, 8\rangle$

13. $\vec{u} = \langle a, b, 0\rangle, \quad \vec{v} = \langle c, d, 0\rangle$

14. $\vec{u} = \vec{i}, \quad \vec{v} = \vec{j}$

15. $\vec{u} = \vec{i}, \quad \vec{v} = \vec{k}$

16. $\vec{u} = \vec{j}, \quad \vec{v} = \vec{k}$

17. Pick any vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ and show that $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.

18. Pick any vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ and show that $\vec{u}\cdot(\vec{v}\times\vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$.

In Exercises 19 – 22, the magnitudes of vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ are given, along with the angle θ between them. Use this information to find the magnitude of $\vec{u} \times \vec{v}$.

19. $\|\,\vec{u}\,\| = 2, \quad \|\,\vec{v}\,\| = 5, \quad \theta = 30^\circ$

20. $\|\,\vec{u}\,\| = 3, \quad \|\,\vec{v}\,\| = 7, \quad \theta = \pi/2$

21. $\|\,\vec{u}\,\| = 3, \quad \|\,\vec{v}\,\| = 4, \quad \theta = \pi$

22. $\|\,\vec{u}\,\| = 2, \quad \|\,\vec{v}\,\| = 5, \quad \theta = 5\pi/6$

In Exercises 23 – 26, find the area of the parallelogram defined by the given vectors.

23. $\vec{u} = \langle 1, 1, 2\rangle, \quad \vec{v} = \langle 2, 0, 3\rangle$

24. $\vec{u} = \langle -2, 1, 5\rangle, \quad \vec{v} = \langle -1, 3, 1\rangle$

25. $\vec{u} = \langle 1, 2\rangle, \quad \vec{v} = \langle 2, 1\rangle$

26. $\vec{u} = \langle 2, 0\rangle, \quad \vec{v} = \langle 0, 3\rangle$

In Exercises 27 – 30, find the area of the triangle with the given vertices.

27. Vertices: $(0, 0, 0)$, $(1, 3, -1)$ and $(2, 1, 1)$.

28. Vertices: $(5, 2, -1)$, $(3, 6, 2)$ and $(1, 0, 4)$.

29. Vertices: $(1, 1)$, $(1, 3)$ and $(2, 2)$.

30. Vertices: $(3, 1)$, $(1, 2)$ and $(4, 3)$.

In Exercises 31 – 32, find the area of the quadrilateral with the given vertices. (Hint: break the quadrilateral into 2 triangles.)

31. Vertices: $(0, 0)$, $(1, 2)$, $(3, 0)$ and $(4, 3)$.

32. Vertices: $(0, 0, 0)$, $(2, 1, 1)$, $(-1, 2, -8)$ and $(1, -1, 5)$.

In Exercises 33 – 34, find the volume of the parallelepiped defined by the given vectors.

33. $\vec{u} = \langle 1, 1, 1\rangle, \quad \vec{v} = \langle 1, 2, 3\rangle, \quad \vec{w} = \langle 1, 0, 1\rangle$

34. $\vec{u} = \langle -1, 2, 1\rangle, \quad \vec{v} = \langle 2, 2, 1\rangle, \quad \vec{w} = \langle 3, 1, 3\rangle$

In Exercises 35 – 38, find a unit vector orthogonal to both $\vec{u}$ and $\vec{v}$.

35. $\vec{u} = \langle 1, 1, 1\rangle, \quad \vec{v} = \langle 2, 0, 1\rangle$

36. $\vec{u} = \langle 1, -2, 1\rangle, \quad \vec{v} = \langle 3, 2, 1\rangle$

37. $\vec{u} = \langle 5, 0, 2\rangle, \quad \vec{v} = \langle -3, 0, 7\rangle$

38. $\vec{u} = \langle 1, -2, 1\rangle, \quad \vec{v} = \langle -2, 4, -2\rangle$

39. A bicycle rider applies 150lb of force, straight down, onto a pedal that extends 7in horizontally from the crankshaft. Find the magnitude of the torque applied to the crankshaft.

40. A bicycle rider applies 150lb of force, straight down, onto a pedal that extends 7in from the crankshaft, making a 30° angle with the horizontal. Find the magnitude of the torque applied to the crankshaft.

41. To turn a stubborn bolt, 80lb of force is applied to a 10in wrench. What is the maximum amount of torque that can be applied to the bolt?

42. To turn a stubborn bolt, 80lb of force is applied to a 10in wrench in a confined space, where the direction of applied force makes a 10° angle with the wrench. How much torque is subsequently applied to the wrench?

43. Show, using the definition of the Cross Product, that $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$; that is, that $\vec{u}$ is orthogonal to the cross product of $\vec{u}$ and $\vec{v}$.

44. Show, using the definition of the Cross Product, that $\vec{u} \times \vec{u} = \vec{0}$.

11.5 Lines

To find the equation of a line in the x-y plane, we need two pieces of information: a point and the slope. The slope conveys *direction* information. As vertical lines have an undefined slope, the following statement is more accurate:

> To define a line, one needs a point on the line and the direction of the line.

This holds true for lines in space.

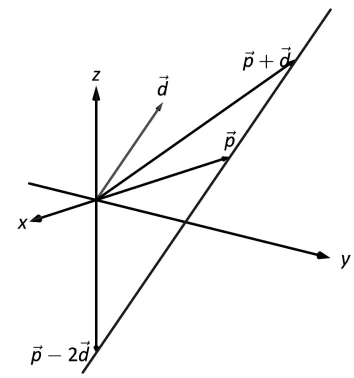


Figure 11.5.2: Defining a line in space.

Let P be a point in space, let $\vec{p}$ be the vector with initial point at the origin and terminal point at P (i.e., $\vec{p}$ "points" to P), and let $\vec{d}$ be a vector. Consider the points on the line through P in the direction of $\vec{d}$.

Clearly one point on the line is P; we can say that the *vector* $\vec{p}$ lies at this point on the line. To find another point on the line, we can start at $\vec{p}$ and move in a direction parallel to $\vec{d}$. For instance, starting at $\vec{p}$ and travelling one length of $\vec{d}$ places one at another point on the line. Consider Figure 11.5.2 where certain points along the line are indicated.

The figure illustrates how every point on the line can be obtained by starting with $\vec{p}$ and moving a certain distance in the direction of $\vec{d}$. That is, we can define the line as a function of t:

$$\vec{\ell}(t) = \vec{p} + t\,\vec{d}. \tag{11.8}$$

In many ways, this is *not* a new concept. Compare Equation (11.8) to the familiar "$y = mx + b$" equation of a line:

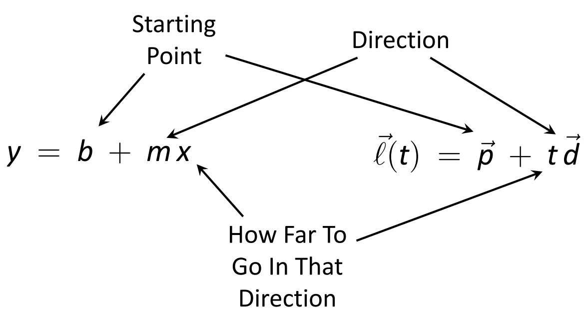


Figure 11.5.1: Understanding the vector equation of a line.

The equations exhibit the same structure: they give a starting point, define a direction, and state how far in that direction to travel.

Equation (11.8) is an example of a **vector–valued function**; the input of the function is a real number and the output is a vector. We will cover vector–valued functions extensively in the next chapter.

There are other ways to represent a line. Let $\vec{p} = \langle x_0, y_0, z_0\rangle$ and let $\vec{d} = \langle a, b, c\rangle$. Then the equation of the line through $\vec{p}$ in the direction of $\vec{d}$ is:

$$\begin{aligned}\vec{\ell}(t) &= \vec{p} + t\vec{d}\\ &= \langle x_0, y_0, z_0\rangle + t\,\langle a, b, c\rangle\\ &= \langle x_0 + at, y_0 + bt, z_0 + ct\rangle\,.\end{aligned}$$

The last line states that the x values of the line are given by $x = x_0 + at$, the y values are given by $y = y_0 + bt$, and the z values are given by $z = z_0 + ct$. These three equations, taken together, are the **parametric equations of the line** through $\vec{p}$ in the direction of $\vec{d}$.

Finally, each of the equations for x, y and z above contain the variable t. We can solve for t in each equation:

$$\begin{aligned} x = x_0 + at \quad &\Rightarrow \quad t = \frac{x - x_0}{a}, \\ y = y_0 + bt \quad &\Rightarrow \quad t = \frac{y - y_0}{b}, \\ z = z_0 + ct \quad &\Rightarrow \quad t = \frac{z - z_0}{c}, \end{aligned}$$

assuming $a, b, c \neq 0$. Since t is equal to each expression on the right, we can set these equal to each other, forming the **symmetric equations of the line** through $\vec{p}$ in the direction of $\vec{d}$:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Each representation has its own advantages, depending on the context. We summarize these three forms in the following definition, then give examples of their use.

Definition 11.5.1 Equations of Lines in Space

Consider the line in space that passes through $\vec{p} = \langle x_0, y_0, z_0 \rangle$ in the direction of $\vec{d} = \langle a, b, c \rangle$.

1. The **vector equation** of the line is

$$\vec{\ell}(t) = \vec{p} + t\vec{d}.$$

2. The **parametric equations** of the line are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

3. The **symmetric equations** of the line are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Example 11.5.1 Finding the equation of a line

Give all three equations, as given in Definition 11.5.1, of the line through $P = (2, 3, 1)$ in the direction of $\vec{d} = \langle -1, 1, 2 \rangle$. Does the point $Q = (-1, 6, 6)$ lie on this line?

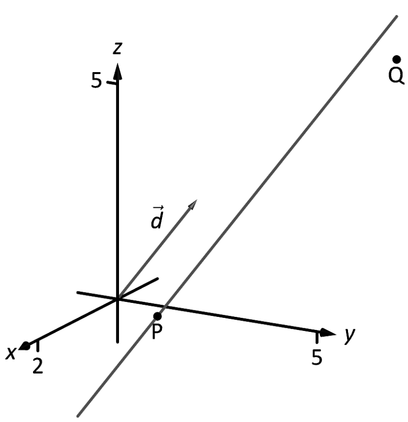


Figure 11.5.3: Graphing a line in Example 11.5.1.

Solution We identify the point $P = (2, 3, 1)$ with the vector $\vec{p} = \langle 2, 3, 1 \rangle$. Following the definition, we have

- the vector equation of the line is $\vec{\ell}(t) = \langle 2, 3, 1 \rangle + t \langle -1, 1, 2 \rangle$;
- the parametric equations of the line are

$$x = 2 - t, \quad y = 3 + t, \quad z = 1 + 2t; \text{ and}$$

- the symmetric equations of the line are

$$\frac{x - 2}{-1} = \frac{y - 3}{1} = \frac{z - 1}{2}.$$

The first two equations of the line are useful when a t value is given: one can immediately find the corresponding point on the line. These forms are good when calculating with a computer; most software programs easily handle equations in these formats. (For instance, the graphics program that made Figure 11.5.3 can be given the input "`(2-t,3+t,1+2*t)`" for $-1 \leq t \leq 3$.).

Does the point $Q = (-1, 6, 6)$ lie on the line? The graph in Figure 11.5.3 makes it clear that it does not. We can answer this question without the graph using any of the three equation forms. Of the three, the symmetric equations are probably best suited for this task. Simply plug in the values of x, y and z and see if equality is maintained:

$$\frac{-1-2}{-1} \stackrel{?}{=} \frac{6-3}{1} \stackrel{?}{=} \frac{6-1}{2} \quad \Rightarrow \quad 3 = 3 \neq 2.5.$$

We see that Q does not lie on the line as it did not satisfy the symmetric equations.

Example 11.5.2 Finding the equation of a line through two points

Find the parametric equations of the line through the points $P = (2, -1, 2)$ and $Q = (1, 3, -1)$.

Solution Recall the statement made at the beginning of this section: to find the equation of a line, we need a point and a direction. We have *two* points; either one will suffice. The direction of the line can be found by the vector with initial point P and terminal point Q: $\vec{PQ} = \langle -1, 4, -3\rangle$.

The parametric equations of the line ℓ through P in the direction of $\vec{PQ}$ are:

$$\ell: \quad x = 2 - t \quad y = -1 + 4t \quad z = 2 - 3t.$$

A graph of the points and line are given in Figure 11.5.4. Note how in the given parametrization of the line, $t = 0$ corresponds to the point P, and $t = 1$ corresponds to the point Q. This relates to the understanding of the vector equation of a line described in Figure 11.5.1. The parametric equations "start" at the point P, and t determines how far in the direction of $\vec{PQ}$ to travel. When $t = 0$, we travel 0 lengths of $\vec{PQ}$; when $t = 1$, we travel one length of $\vec{PQ}$, resulting in the point Q.

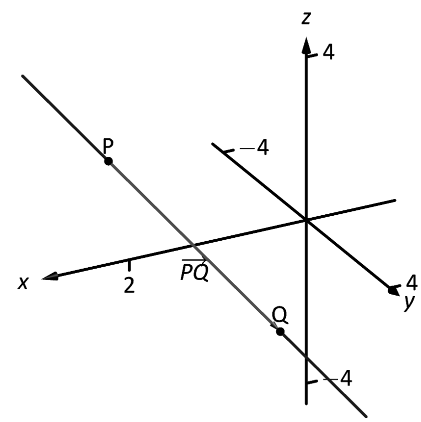


Figure 11.5.4: A graph of the line in Example 11.5.2.

Parallel, Intersecting and Skew Lines

In the plane, two *distinct* lines can either be parallel or they will intersect at exactly one point. In space, given equations of two lines, it can sometimes be difficult to tell whether the lines are distinct or not (i.e., the same line can be represented in different ways). Given lines $\vec{\ell_1}(t) = \vec{p}_1 + t\vec{d}_1$ and $\vec{\ell_2}(t) = \vec{p}_2 + t\vec{d}_2$, we have four possibilities: $\vec{\ell_1}$ and $\vec{\ell_2}$ are

the same line	they share all points;
intersecting lines	share only 1 point;
parallel lines	$\vec{d}_1 \parallel \vec{d}_2$, no points in common; or
skew lines	$\vec{d}_1 \nparallel \vec{d}_2$, no points in common.

The next two examples investigate these possibilities.

Example 11.5.3 Comparing lines

Consider lines ℓ_1 and ℓ_2, given in parametric equation form:

$$\ell_1: \begin{array}{rcl} x &=& 1+3t \\ y &=& 2-t \\ z &=& t \end{array} \qquad \ell_2: \begin{array}{rcl} x &=& -2+4s \\ y &=& 3+s \\ z &=& 5+2s. \end{array}$$

Determine whether ℓ_1 and ℓ_2 are the same line, intersect, are parallel, or skew.

Solution We start by looking at the directions of each line. Line ℓ_1 has the direction given by $\vec{d_1} = \langle 3, -1, 1\rangle$ and line ℓ_2 has the direction given by $\vec{d_2} = \langle 4, 1, 2\rangle$. It should be clear that $\vec{d_1}$ and $\vec{d_2}$ are not parallel, hence ℓ_1 and ℓ_2 are not the same line, nor are they parallel. Figure 11.5.5 verifies this fact (where the points and directions indicated by the equations of each line are identified).

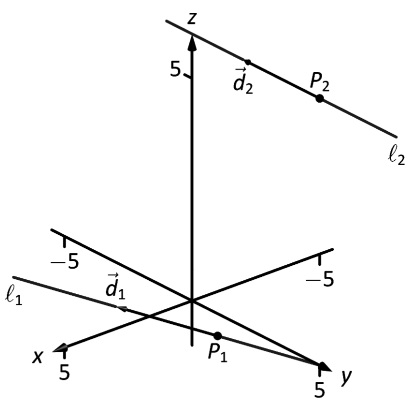


Figure 11.5.5: Sketching the lines from Example 11.5.3.

We next check to see if they intersect (if they do not, they are skew lines). To find if they intersect, we look for t and s values such that the respective x, y and z values are the same. That is, we want s and t such that:

$$\begin{array}{rcccl} 1+3t &=& x &=& -2+4s \\ 2-t &=& y &=& 3+s \\ t &=& z &=& 5+2s. \end{array}$$

This is a relatively simple system of linear equations. Since the last equation is already solved for t, substitute that value of t into the equation above it:

$$2-(5+2s) = 3+s \quad \Rightarrow \quad s=-2,\ t=1.$$

A key to remember is that we have *three* equations; we need to check if $s = -2,\ t = 1$ satisfies the first equation as well:

$$1+3(1) \neq -2+4(-2).$$

It does not. Therefore, we conclude that the lines ℓ_1 and ℓ_2 are skew.

Recall from linear algebra that a system of equations with no solution, such as the one in Example 11.5.3, is *inconsistent*. Although it is possible to find values that work for any two of the three equations, there is no set of values for s and t that work for all three equations simultaneously.

Example 11.5.4 Comparing lines

Consider the lines ℓ_1 and ℓ_2 given by the vector equations

$$\vec{\ell_1}(s) = \langle 2, -1, 4\rangle + s\langle 0, 4, -8\rangle$$
$$\vec{\ell_2}(t) = \langle -3, 4, -6\rangle + t\langle 2, -1, 2\rangle.$$

Determine if the lines are parallel, skew, or intersecting.

Solution We can immediately see that the lines cannot be parallel, since the x-component of the direction vector for ℓ_1 is zero, but this is not the case for the direction vector of ℓ_2. (There is no scalar c such that $0 \cdot c = 2$.) To determine if the lines intersect, we proceed as in the previous example. We must have

$$\begin{array}{rcccl} 2 &=& x &=& -3+2t \\ -1+4s &=& y &=& 4-t \\ 4-8s &=& z &=& -6+2t. \end{array}$$

The first equation immediately gives us $2t = 5$, so $t = \frac{5}{2}$. Plugging this into the second equation gives us

$$4s = 4 - \frac{5}{2} + 1 = \frac{5}{2} \quad \Rightarrow \quad s = \frac{5}{8}.$$

We now need to check to see if these values satisfy the third equation as well: we have

$$4 - 8s = 4 - 5 = -1,$$

and

$$-6 + 2t = -6 + 5 = -1,$$

so the values $s = \frac{5}{8}$, $t = \frac{5}{2}$ work for all three equations, and since

$$\vec{\ell}_1\left(\frac{5}{8}\right) = \langle 2, -1, 4\rangle + \frac{5}{8}\langle 0, 4, -8\rangle = \langle 2, \frac{3}{2}, -1\rangle \qquad \text{and}$$
$$\vec{\ell}_2\left(\frac{5}{2}\right) = \langle -3, 4, -6\rangle + \frac{5}{2}\langle 2, -1, 2\rangle = \langle 2, \frac{3}{2}, -1\rangle,$$

our point of intersection is $(2, \frac{3}{2}, -1)$.

Example 11.5.5 Comparing lines

Consider lines ℓ_1 and ℓ_2, given in parametric equation form:

$$\ell_1 : \begin{array}{rcl} x &=& -0.7 + 1.6t \\ y &=& 4.2 + 2.72t \\ z &=& 2.3 - 3.36t \end{array} \qquad \ell_2 : \begin{array}{rcl} x &=& 2.8 - 2.9s \\ y &=& 10.15 - 4.93s \\ z &=& -5.05 + 6.09s. \end{array}$$

Determine whether ℓ_1 and ℓ_2 are the same line, intersect, are parallel, or skew.

SOLUTION It is obviously very difficult to simply look at these equations and discern anything. This is done intentionally. In the "real world," most equations that are used do not have nice, integer coefficients. Rather, there are lots of digits after the decimal and the equations can look "messy."

We again start by deciding whether or not each line has the same direction. The direction of ℓ_1 is given by $\vec{d}_1 = \langle 1.6, 2.72, -3.36\rangle$ and the direction of ℓ_2 is given by $\vec{d}_2 = \langle -2.9, -4.93, 6.09\rangle$. When it is not clear through observation whether two vectors are parallel or not, the standard way of determining this is by comparing their respective unit vectors. Using a calculator, we find:

$$\vec{u}_1 = \frac{\vec{d}_1}{\parallel \vec{d}_1 \parallel} = \langle 0.3471, 0.5901, -0.7289\rangle$$
$$\vec{u}_2 = \frac{\vec{d}_2}{\parallel \vec{d}_2 \parallel} = \langle -0.3471, -0.5901, 0.7289\rangle .$$

The two vectors seem to be parallel (at least, their components are equal to 4 decimal places). In most situations, it would suffice to conclude that the lines are at least parallel, if not the same. One way to be sure is to rewrite $\vec{d}_1$ and $\vec{d}_2$ in terms of fractions, not decimals. We have

$$\vec{d}_1 = \left\langle \frac{16}{10}, \frac{272}{100}, -\frac{336}{100}\right\rangle \qquad \vec{d}_2 = \left\langle -\frac{29}{10}, -\frac{493}{100}, \frac{609}{100}\right\rangle .$$

One can then find the magnitudes of each vector in terms of fractions, then compute the unit vectors likewise. After a lot of manual arithmetic (or after briefly using a computer algebra system), one finds that

$$\vec{u}_1 = \left\langle \sqrt{\frac{10}{83}}, \frac{17}{\sqrt{830}}, -\frac{21}{\sqrt{830}}\right\rangle \qquad \vec{u}_2 = \left\langle -\sqrt{\frac{10}{83}}, -\frac{17}{\sqrt{830}}, \frac{21}{\sqrt{830}}\right\rangle .$$

We can now say without equivocation that these lines are parallel.

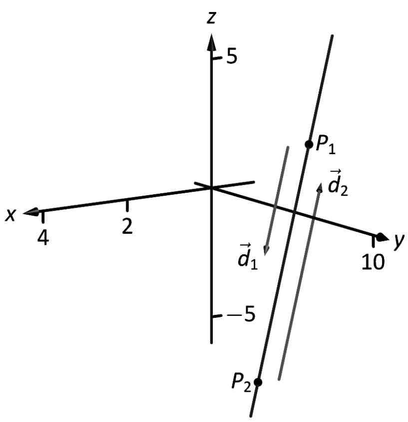


Figure 11.5.6: Graphing the lines in Example 11.5.5.

Are they the same line? The parametric equations for a line describe one point that lies on the line, so we know that the point $P_1 = (-0.7, 4.2, 2.3)$ lies on ℓ_1. To determine if this point also lies on ℓ_2, plug in the x, y and z values of P_1 into the symmetric equations for ℓ_2:

$$\frac{(-0.7)-2.8}{-2.9} \stackrel{?}{=} \frac{(4.2)-10.15}{-4.93} \stackrel{?}{=} \frac{(2.3)-(-5.05)}{6.09} \quad \Rightarrow \quad 1.2069 = 1.2069 = 1.2069.$$

The point P_1 lies on both lines, so we conclude they are the same line, just parametrized differently. Figure 11.5.6 graphs this line along with the points and vectors described by the parametric equations. Note how $\vec{d}_1$ and $\vec{d}_2$ are parallel, though point in opposite directions (as indicated by their unit vectors above).

Distances

Given a point Q and a line $\vec{\ell}(t) = \vec{p} + t\vec{d}$ in space, it is often useful to know the distance from the point to the line. (Here we use the standard definition of "distance," i.e., the length of the shortest line segment from the point to the line.) Identifying $\vec{p}$ with the point P, Figure 11.5.7 will help establish a general method of computing this distance h.

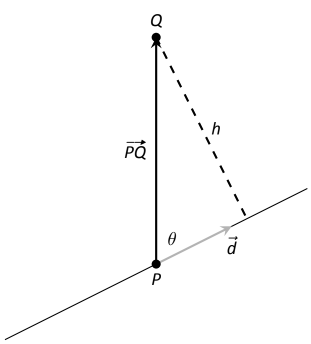


Figure 11.5.7: Establishing the distance from a point to a line.

From trigonometry, we know $h = \parallel \overrightarrow{PQ} \parallel \sin\theta$. We have a similar identity involving the cross product: $\parallel \overrightarrow{PQ} \times \vec{d} \parallel = \parallel \overrightarrow{PQ} \parallel \parallel \vec{d} \parallel \sin\theta$. Divide both sides of this latter equation by $\parallel \vec{d} \parallel$ to obtain h:

$$h = \frac{\parallel \overrightarrow{PQ} \times \vec{d} \parallel}{\parallel \vec{d} \parallel}. \tag{11.9}$$

We put Equation (11.9) to use in the following example.

Example 11.5.6 Finding the distance from a point to a line

Find the distance from the point $Q = (1, 1, 3)$ to the line $\vec{\ell}(t) = \langle 1, -1, 1\rangle + t\langle 2, 3, 1\rangle$.

Solution The equation of the line line gives us the point $P = (1, -1, 1)$ that lies on the line, hence $\overrightarrow{PQ} = \langle 0, 2, 2\rangle$. The equation also gives $\vec{d} = \langle 2, 3, 1\rangle$. Using Equation (11.9), we have the distance as

$$\begin{aligned} h &= \frac{\parallel \overrightarrow{PQ} \times \vec{d} \parallel}{\parallel \vec{d} \parallel} \\ &= \frac{\parallel \langle -4, 4, -4\rangle \parallel}{\sqrt{14}} \\ &= \frac{4\sqrt{3}}{\sqrt{14}}. \end{aligned}$$

It is also useful to determine the distance between lines, which we define as the length of the shortest line segment that connects the two lines (an argument from geometry shows that this line segments is perpendicular to both lines). Let lines $\vec{\ell}_1(t) = \vec{p}_1 + t\vec{d}_1$ and $\vec{\ell}_2(t) = \vec{p}_2 + t\vec{d}_2$ be given, as shown in Figure 11.5.8. To find the direction orthogonal to both $\vec{d}_1$ and $\vec{d}_2$, we take the cross product: $\vec{c} = \vec{d}_1 \times \vec{d}_2$. The magnitude of the orthogonal projection of $\overrightarrow{P_1P_2}$ onto $\vec{c}$ is the distance h we seek:

$$\begin{aligned} h &= \left\| \text{proj}_{\vec{c}}\, \overrightarrow{P_1P_2} \right\| \\ &= \left\| \frac{\overrightarrow{P_1P_2} \cdot \vec{c}}{\vec{c} \cdot \vec{c}} \vec{c} \right\| \\ &= \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{\| \vec{c} \|^2} \| \vec{c} \| \\ &= \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{\| \vec{c} \|}. \end{aligned}$$

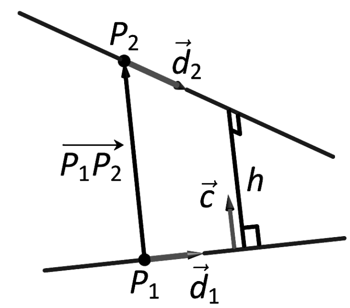


Figure 11.5.8: Establishing the distance between lines.

A problem in the Exercise section is to show that this distance is 0 when the lines intersect. Note the use of the Triple Scalar Product: $\overrightarrow{P_1P_2} \cdot c = \overrightarrow{P_1P_2} \cdot (\vec{d}_1 \times \vec{d}_2)$.

The following Key Idea restates these two distance formulas.

Key Idea 11.5.1 Distances to Lines

1. Let P be a point on a line ℓ that is parallel to $\vec{d}$. The distance h from a point Q to the line ℓ is:
$$h = \frac{\| \overrightarrow{PQ} \times \vec{d} \|}{\| \vec{d} \|}.$$

2. Let P_1 be a point on line ℓ_1 that is parallel to $\vec{d}_1$, and let P_2 be a point on line ℓ_2 parallel to $\vec{d}_2$, and let $\vec{c} = \vec{d}_1 \times \vec{d}_2$, where lines ℓ_1 and ℓ_2 are not parallel. The distance h between the two lines is:
$$h = \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{\| \vec{c} \|}.$$

Example 11.5.7 Finding the distance between lines

Find the distance between the lines

$$\ell_1: \begin{array}{rcl} x &=& 1+3t \\ y &=& 2-t \\ z &=& t \end{array} \qquad \ell_2: \begin{array}{rcl} x &=& -2+4s \\ y &=& 3+s \\ z &=& 5+2s. \end{array}$$

Solution These are the sames lines as given in Example 11.5.3, where we showed them to be skew. The equations allow us to identify the following points and vectors:

$$P_1 = (1,2,0) \quad P_2 = (-2,3,5) \quad \Rightarrow \quad \overrightarrow{P_1P_2} = \langle -3,1,5 \rangle.$$
$$\vec{d}_1 = \langle 3,-1,1 \rangle \quad \vec{d}_2 = \langle 4,1,2 \rangle \quad \Rightarrow \quad \vec{c} = \vec{d}_1 \times \vec{d}_2 = \langle -3,-2,7 \rangle.$$

Using Key Idea 11.5.1 we have that the distance h between the two lines is

$$h = \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{\| \vec{c} \|} = \frac{42}{\sqrt{62}} \approx 5.334.$$

While Key Idea 11.5.1 gives us a convenient formula for computing the distance, you are probably better off making sure you understand the argument used to obtain the formula, even though the formula is more efficient. For one thing, a formula is easily forgotten. For another, understanding the method will allow you to adapt it to similar situations still to come, such as computing the distance between skew lines, or from a point to a plane. The general method for these types of problems can be outlined as follows.

Key Idea 11.5.2 Steps for solving shortest distance problems

Suppose you are asked to find the distance between two objects, or to determine an object (such as a point) that is closest to a given object (a line or plane). Your solution to the problem should always include the following steps:

1. Make a list of all the information provided in the problem.
2. Make a note of what quantities you're asked to determine.
3. **Draw a diagram**. Label all relevant points and vectors, including those you know, and those you want to find.
4. Using your diagram as a reference, compute any unknown points or vectors.

Note: We can't overemphasize the fact that the diagram referred to in Key Idea 11.5.2 **does not have to be accurate** with respect to the coordinates and directions involved. It simply has to be capable of representing the information in the problem. Note that in Figure 11.5.9 in Example 11.5.8 we've drawn a line, some points, and some vectors that represent the problem, without reference to a coordinate system. The goal is to provide enough detail to allow us to set up the problem.

We put the method in Key Idea 11.5.2 to use in the following example. Note that in this example we're asked not just for the distance from a point to a line, but also for the point on the line that is *closest* to the given point, so simply using Equation (11.9) is not enough.

Example 11.5.8 Finding the closest point on a line

Find the distance from the point $Q=(1,3,-2)$ to the line $\vec{\ell}$ that passes through the point $P=(2,0,-1)$ in the direction of $\vec{d}=\langle 1,-1,0\rangle$, and find the point R on $\vec{\ell}$ that is closest to Q.

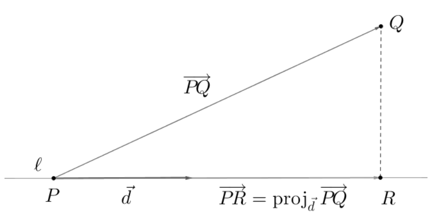


Figure 11.5.9: Setting up the solution in Example 11.5.8

Solution We're given a point P on the line, along with a direction vector $\vec{d}$, and a point Q not on the line. We seek the point R on the line that is closest to Q, as well as the distance from Q to R. We begin by diagramming the information in Figure 11.5.9. From the given points P and Q we can immediately construct the vector

$$\overrightarrow{PQ}=\langle 1-2,3-0,-2-(-1)\rangle=\langle -1,3,-1\rangle\,.$$

Rather than use Formula (11.9) to find the distance, we begin instead by finding the point R on the line that is closest to Q. From our diagram, we can see that the vector $\overrightarrow{PR}$ from P to R is equal to the projection of $\overrightarrow{PQ}$ onto the distance vector $\vec{d}$:

$$\overrightarrow{PR}=\mathrm{proj}_{\vec{d}}\,\overrightarrow{PQ}=\left(\frac{\langle -1,3,-1\rangle\cdot\langle 1,-1,0\rangle}{\langle 1,-1,0\rangle\cdot\langle 1,-1,0\rangle}\right)\langle 1,-1,0\rangle=\langle -2,2,0\rangle\,.$$

Now, we need to pause and take care that we don't make a very common mistake: the vector $\overrightarrow{PR}$ does **not** give the coordinates of the point R. Instead, $\overrightarrow{PR}$ tells us how to get *from* the point P *to* the point R. Letting O denote the origin, we can write $\overrightarrow{OP}$ and $\overrightarrow{OR}$ for the position vectors of P and R, respectively. Since $\overrightarrow{PR}=\overrightarrow{OR}-\overrightarrow{OP}$ using the "tip minus tail" rule for computing the vector between two points, we have

$$\overrightarrow{OR}=\overrightarrow{OP}+\overrightarrow{PR}=\langle 2,0,-1\rangle+\langle -2,2,0\rangle=\langle 0,2,-1\rangle\,.$$

Thus, we have $R = (0, 2, -1)$ as the point on the line closest to the point Q. We can now find the distance from Q to the line using the distance formula:

$$D = \sqrt{(1-0)^2 + (3-2)^2 + (-2-(-1))^2} = \sqrt{3}.$$

(You should verify that this agrees with the distance given by Formula (11.9).) An alternative way of computing the distance is to make use of the orthogonal decomposition in Key Idea 11.3.1. By definition of the distance from a point to a line, we know that the vector $\overrightarrow{RQ}$ must be orthogonal to the line, and thus to the direction vector $\vec{d}$. Using Key Idea 11.3.1, we have that

$$\overrightarrow{RQ} = \overrightarrow{PQ} - \overrightarrow{PR} = \langle -1, 3, -1\rangle - \langle -2, 2, 0\rangle = \langle -1, 1, 1\rangle,$$

and the shortest distance is given by $\| \overrightarrow{RQ} \| = \sqrt{3}$, as before.

In the case of skew lines, the key observation is that if we take the vector between **any** pair of points, one on each line, and project it onto the vector $\vec{c} = \vec{d}_1 \times \vec{d}_2$, the length of the resulting vector is the distance we seek.

Somewhat more challenging is the problem of finding the points on each line that actually *realize* this shortest distance.

Example 11.5.9 Finding the closest points on skew lines

Find the points R_1 on $\vec{\ell}_1$ and R_2 on $\vec{\ell}_2$, where $\vec{\ell}_1$ and $\vec{\ell}_2$ are the lines from Example 11.5.7, such that the distance from R_1 to R_2 is a minimum.

SOLUTION Since R_1 is a point on $\vec{\ell}_1$, we know that

$$R_1 = (1+3t, 2-t, t), \quad \text{for some real number } t, \tag{11.10}$$

and similarly,

$$R_2 = (-2+4s, 3+s, 5+2s), \quad \text{for some real number } s. \tag{11.11}$$

The vector $\overrightarrow{R_1R_2}$ is therefore given by

$$\overrightarrow{R_1R_2} = \langle -3+4s-3t, 1+s+t, 5+2s-t\rangle,$$

for some pair of real numbers s and t. We know that the line segment $\overline{R_1R_2}$ must be perpendicular to both $\vec{\ell}_1$ and $\vec{\ell}_2$ in order to minimize the distance, so the vector $\overrightarrow{R_1R_2}$ must be orthogonal to both $\vec{d}_1$ and $\vec{d}_2$. Thus,

$$\begin{aligned} 0 = \vec{d}_1 \cdot \overrightarrow{R_1R_2} &= 3(-3+4s-3t) - 1(1+s+t) + 1(5+2s-t) \\ &= 13s - 11t - 5, \text{ and} \\ 0 = \vec{d}_2 \cdot \overrightarrow{R_1R_2} &= 4(-3+4s-3t) + 1(1+s+t) + 2(5+2s-t) \\ &= 21s - 13t - 1. \end{aligned}$$

We end up having to solve a *system* of two linear equations in the two variables, s and t, given by

$$\begin{array}{rcrcl} 13s & - & 11t & = & 5, \\ 21s & - & 13t & = & 1. \end{array}$$

You probably had to solve such systems in high school. One option is to solve graphically, by plotting the lines given by each equation, and seeing where they intersect. However, this method has little hope of providing an accurate answer. Instead, we try a little algebra. Multiplying the first equation by 21 and the second by 13 gives us the equations $273s - 231t = 105$ and $273s - 169t = 13$, respectively. Subtracting the second equation from the first, we have $-62t = 92$, so $t = -\frac{92}{62} = -\frac{46}{31}$. Plugging this value back into any of the previous equations gives us $s = -\frac{351}{403} = -\frac{27}{31}$. (We didn't promise that the numbers would work out nicely!) Plugging these values back into equations (11.10) and (11.11), we find

$$R_1 = \left(-\frac{107}{31}, \frac{108}{31}, -\frac{46}{31}\right) \quad \text{and} \quad R_2 = \left(-\frac{170}{31}, \frac{66}{31}, \frac{101}{31}\right).$$

Our vector $\overrightarrow{R_1R_2}$ is then given by

$$\overrightarrow{R_1R_2} = \left\langle -\frac{63}{31}, -\frac{42}{31}, \frac{147}{31} \right\rangle = \frac{1}{31}\left\langle -63, -42, 147 \right\rangle,$$

and the distance between the two lines is given by

$$\| \overrightarrow{R_1R_2} \| = \frac{1}{31}\sqrt{63^2 + 42^2 + 147^2} = \frac{42}{\sqrt{62}},$$

as before.

You might be thinking, "Are those two values really the same?" A calculator can verify this of course, by computing the decimal approximations for the results in Examples 11.5.7 and 11.5.9. Alternatively, you can verify that

$$63^2 + 42^2 + 147^2 = 27,342 = 2(21)^2(31),$$

so

$$\begin{aligned} \frac{\sqrt{63^2 + 42^2 + 147^2}}{31} &= \sqrt{2(21^2)(31)31} \\ &= \frac{21\sqrt{2}\sqrt{31}}{31} \\ &= \frac{21\sqrt{2}}{\sqrt{31}} \\ &= \frac{21(2)}{\sqrt{31}\sqrt{2}} \\ &= \frac{42}{\sqrt{62}}. \end{aligned}$$

Example 11.5.9 required us to solve a system of two linear equations in two unknowns s and t. Although this involved some messy fractions, the algebra involved was fairly straightforward. In many real life problems it is necessary to be able to solve systems involving hundreds or even thousands of equations and variables. We will begin our study of how to systematically solve such systems in the next chapter.

One of the key points to understand from this section is this: to describe a line, we need a point and a direction. Whenever a problem is posed concerning a line, one needs to take whatever information is offered and glean point and direction information. Many questions can be asked (and *are* asked in the Exercise section) whose answer immediately follows from this understanding.

Lines are one of two fundamental objects of study in space. The other fundamental object is the *plane*, which we study in detail in the next section. Many complex three dimensional objects are studied by approximating their surfaces with lines and planes.

Exercises 11.5

Terms and Concepts

1. To find an equation of a line, what two pieces of information are needed?

2. Two distinct lines in the plane can intersect or be ________.

3. Two distinct lines in space can intersect, be ________ or be ________.

4. Use your own words to describe what it means for two lines in space to be skew.

Problems

In Exercises 5 – 14, write the vector, parametric and symmetric equations of the lines described.

5. Passes through $P = (2, -4, 1)$, parallel to $\vec{d} = \langle 9, 2, 5\rangle$.

6. Passes through $P = (6, 1, 7)$, parallel to $\vec{d} = \langle -3, 2, 5\rangle$.

7. Passes through $P = (2, 1, 5)$ and $Q = (7, -2, 4)$.

8. Passes through $P = (1, -2, 3)$ and $Q = (5, 5, 5)$.

9. Passes through $P = (0, 1, 2)$ and orthogonal to both $\vec{d}_1 = \langle 2, -1, 7\rangle$ and $\vec{d}_2 = \langle 7, 1, 3\rangle$.

10. Passes through $P = (5, 1, 9)$ and orthogonal to both $\vec{d}_1 = \langle 1, 0, 1\rangle$ and $\vec{d}_2 = \langle 2, 0, 3\rangle$.

11. Passes through the point of intersection of $\vec{\ell}_1(t)$ and $\vec{\ell}_2(t)$ and orthogonal to both lines, where $\vec{\ell}_1(t) = \langle 2, 1, 1\rangle + t\,\langle 5, 1, -2\rangle$ and $\vec{\ell}_2(t) = \langle -2, -1, 2\rangle + t\,\langle 3, 1, -1\rangle$.

12. Passes through the point of intersection of $\ell_1(t)$ and $\ell_2(t)$ and orthogonal to both lines, where
$$\ell_1 = \begin{cases} x = t \\ y = -2 + 2t \\ z = 1 + t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 2 + t \\ y = 2 - t \\ z = 3 + 2t \end{cases}.$$

13. Passes through $P = (1, 1)$, parallel to $\vec{d} = \langle 2, 3\rangle$.

14. Passes through $P = (-2, 5)$, parallel to $\vec{d} = \langle 0, 1\rangle$.

In Exercises 15 – 22, determine if the described lines are the same line, parallel lines, intersecting or skew lines. If intersecting, give the point of intersection.

15. $\vec{\ell}_1(t) = \langle 1, 2, 1\rangle + t\,\langle 2, -1, 1\rangle$,
$\vec{\ell}_2(t) = \langle 3, 3, 3\rangle + t\,\langle -4, 2, -2\rangle$.

16. $\vec{\ell}_1(t) = \langle 2, 1, 1\rangle + t\,\langle 5, 1, 3\rangle$,
$\vec{\ell}_2(t) = \langle 14, 5, 9\rangle + t\,\langle 1, 1, 1\rangle$.

17. $\vec{\ell}_1(t) = \langle 3, 4, 1\rangle + t\,\langle 2, -3, 4\rangle$,
$\vec{\ell}_2(t) = \langle -3, 3, -3\rangle + t\,\langle 3, -2, 4\rangle$.

18. $\vec{\ell}_1(t) = \langle 1, 1, 1\rangle + t\,\langle 3, 1, 3\rangle$,
$\vec{\ell}_2(t) = \langle 7, 3, 7\rangle + t\,\langle 6, 2, 6\rangle$.

19. $$\ell_1 = \begin{cases} x = 1 + 2t \\ y = 3 - 2t \\ z = t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 3 - t \\ y = 3 + 5t \\ z = 2 + 7t \end{cases}$$

20. $$\ell_1 = \begin{cases} x = 1.1 + 0.6t \\ y = 3.77 + 0.9t \\ z = -2.3 + 1.5t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 3.11 + 3.4t \\ y = 2 + 5.1t \\ z = 2.5 + 8.5t \end{cases}$$

21. $$\ell_1 = \begin{cases} x = 0.2 + 0.6t \\ y = 1.33 - 0.45t \\ z = -4.2 + 1.05t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 0.86 + 9.2t \\ y = 0.835 - 6.9t \\ z = -3.045 + 16.1t \end{cases}$$

22. $$\ell_1 = \begin{cases} x = 0.1 + 1.1t \\ y = 2.9 - 1.5t \\ z = 3.2 + 1.6t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 4 - 2.1t \\ y = 1.8 + 7.2t \\ z = 3.1 + 1.1t \end{cases}$$

In Exercises 23 – 26, find the distance from the point to the line.

23. $Q = (1, 1, 1)$, $\quad \vec{\ell}(t) = \langle 2, 1, 3\rangle + t\,\langle 2, 1, -2\rangle$

24. $Q = (2, 5, 6)$, $\quad \vec{\ell}(t) = \langle -1, 1, 1\rangle + t\,\langle 1, 0, 1\rangle$

25. $Q = (0, 3)$, $\quad \vec{\ell}(t) = \langle 2, 0\rangle + t\,\langle 1, 1\rangle$

26. $Q = (1, 1)$, $\quad \vec{\ell}(t) = \langle 4, 5\rangle + t\,\langle -4, 3\rangle$

In Exercises 27 – 28, find the distance between the two lines.

27. $\vec{\ell}_1(t) = \langle 1, 2, 1\rangle + t\,\langle 2, -1, 1\rangle$,
$\vec{\ell}_2(t) = \langle 3, 3, 3\rangle + t\,\langle 4, 2, -2\rangle$.

28. $\vec{\ell}_1(t) = \langle 0, 0, 1\rangle + t\,\langle 1, 0, 0\rangle$,
$\vec{\ell}_2(t) = \langle 0, 0, 3\rangle + t\,\langle 0, 1, 0\rangle$.

Exercises 29 – 31 explore special cases of the distance formulas found in Key Idea 11.5.1.

29. Let Q be a point on the line $\vec{\ell}(t)$. Show why the distance formula correctly gives the distance from the point to the line as 0.

30. Let lines $\vec{\ell}_1(t)$ and $\vec{\ell}_2(t)$ be intersecting lines. Show why the distance formula correctly gives the distance between these lines as 0.

31. Let lines $\vec{\ell}_1(t)$ and $\vec{\ell}_2(t)$ be parallel.

    (a) Show why the distance formula for distance between lines cannot be used as stated to find the distance between the lines.

    (b) Show why letting $\vec{c} = (\overrightarrow{P_1P_2} \times \vec{d}_2) \times \vec{d}_2$ allows one to use the formula.

    (c) Show how one can use the formula for the distance between a point and a line to find the distance between parallel lines.

11.6 Planes

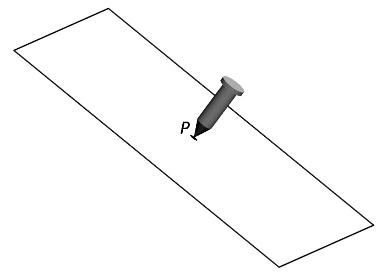


Figure 11.6.1: Illustrating defining a plane with a sheet of cardboard and a nail.

Any flat surface, such as a wall, table top or stiff piece of cardboard can be thought of as representing part of a plane. Consider a piece of cardboard with a point P marked on it. One can take a nail and stick it into the cardboard at P such that the nail is perpendicular to the cardboard; see Figure 11.6.1.

This nail provides a "handle" for the cardboard. Moving the cardboard around moves P to different locations in space. Tilting the nail (but keeping P fixed) tilts the cardboard. Both moving and tilting the cardboard defines a different plane in space. In fact, we can define a plane by: 1) the location of P in space, and 2) the direction of the nail.

The previous section showed that one can define a line given a point on the line and the direction of the line (usually given by a vector). One can make a similar statement about planes: we can define a plane in space given a point on the plane and the direction the plane "faces" (using the description above, the direction of the nail). Once again, the direction information will be supplied by a vector, called a **normal vector**, that is orthogonal to the plane.

What exactly does "orthogonal to the plane" mean? Choose any two points P and Q in the plane, and consider the vector $\overrightarrow{PQ}$. We say a vector $\vec{n}$ is orthogonal to the plane if $\vec{n}$ is perpendicular to $\overrightarrow{PQ}$ for all choices of P and Q; that is, if $\vec{n}\cdot\overrightarrow{PQ} = 0$ for all P and Q.

This gives us way of writing an equation describing the plane. Let $P = (x_0, y_0, z_0)$ be a point in the plane and let $\vec{n} = \langle a, b, c\rangle$ be a normal vector to the plane. A point $Q = (x, y, z)$ lies in the plane defined by P and $\vec{n}$ if, and only if, $\overrightarrow{PQ}$ is orthogonal to $\vec{n}$. Knowing $\overrightarrow{PQ} = \langle x - x_0, y - y_0, z - z_0\rangle$, consider:

$$\begin{aligned}\overrightarrow{PQ}\cdot\vec{n} &= 0\\ \langle x - x_0, y - y_0, z - z_0\rangle\cdot\langle a, b, c\rangle &= 0\\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \qquad (11.12)\end{aligned}$$

Equation (11.12) defines an *implicit* function describing the plane. More algebra produces:

$$ax + by + cz = ax_0 + by_0 + cz_0.$$

The right hand side is just a number, so we replace it with d:

$$ax + by + cz = d. \qquad (11.13)$$

As long as $c \neq 0$, we can solve for z:

$$z = \frac{1}{c}(d - ax - by). \qquad (11.14)$$

Equation (11.14) is especially useful as many computer programs can graph functions in this form. Equations (11.12) and (11.13) have specific names, given next.

Definition 11.6.1 Equations of a Plane in Standard and General Forms

The plane passing through the point $P = (x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c\rangle$ can be described by an equation with **standard form**

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0;$$

the equation's **general form** is

$$ax + by + cz = d.$$

A key to remember throughout this section is this: to find the equation of a plane, we need a point and a normal vector. We will give several examples of finding the equation of a plane, and in each one different types of information are given. In each case, we need to use the given information to find a point on the plane and a normal vector.

Example 11.6.1 Finding the equation of a plane.

Write the equation of the plane that passes through the points $P = (1, 1, 0)$, $Q = (1, 2, -1)$ and $R = (0, 1, 2)$ in standard form.

Solution We need a vector $\vec{n}$ that is orthogonal to the plane. Since P, Q and R are in the plane, so are the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$; $\overrightarrow{PQ} \times \overrightarrow{PR}$ is orthogonal to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and hence the plane itself.

It is straightforward to compute $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 2, 1, 1\rangle$. We can use any point we wish in the plane (any of P, Q or R will do) and we arbitrarily choose P. Following Definition 11.6.1, the equation of the plane in standard form is

$$2(x - 1) + (y - 1) + z = 0.$$

The plane is sketched in Figure 11.6.2.

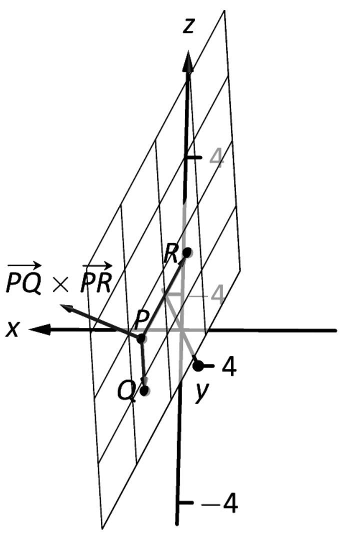


Figure 11.6.2: Sketching the plane in Example 11.6.1.

We have just demonstrated the fact that any three non-collinear points define a plane. (This is why a three-legged stool does not "rock;" it's three feet always lie in a plane. A four-legged stool will rock unless all four feet lie in the same plane.)

Example 11.6.2 Finding the equation of a plane.

Verify that lines ℓ_1 and ℓ_2, whose parametric equations are given below, intersect, then give the equation of the plane that contains these two lines in general form.

$$\ell_1: \begin{array}{rcl} x &=& -5 + 2s \\ y &=& 1 + s \\ z &=& -4 + 2s \end{array} \qquad \ell_2: \begin{array}{rcl} x &=& 2 + 3t \\ y &=& 1 - 2t \\ z &=& 1 + t \end{array}$$

Solution The lines clearly are not parallel. If they do not intersect, they are skew, meaning there is not a plane that contains them both. If they do intersect, there is such a plane.

To find their point of intersection, we set the x, y and z equations equal to each other and solve for s and t:

$$\begin{array}{rcl} -5 + 2s &=& 2 + 3t \\ 1 + s &=& 1 - 2t \\ -4 + 2s &=& 1 + t \end{array} \quad \Rightarrow \quad s = 2, \quad t = -1.$$

When $s = 2$ and $t = -1$, the lines intersect at the point $P = (-1, 3, 0)$.

Let $\vec{d}_1 = \langle 2, 1, 2\rangle$ and $\vec{d}_2 = \langle 3, -2, 1\rangle$ be the directions of lines ℓ_1 and ℓ_2, respectively. A normal vector to the plane containing these the two lines will also be orthogonal to $\vec{d}_1$ and $\vec{d}_2$. Thus we find a normal vector $\vec{n}$ by computing $\vec{n} = \vec{d}_1 \times \vec{d}_2 = \langle 5, 4 - 7\rangle$.

We can pick any point in the plane with which to write our equation; each line gives us infinite choices of points. We choose P, the point of intersection. We follow Definition 11.6.1 to write the plane's equation in general form:

$$\begin{aligned} 5(x+1) + 4(y-3) - 7z &= 0\\ 5x + 5 + 4y - 12 - 7z &= 0\\ 5x + 4y - 7z &= 7. \end{aligned}$$

The plane's equation in general form is $5x + 4y - 7z = 7$; it is sketched in Figure 11.6.3.

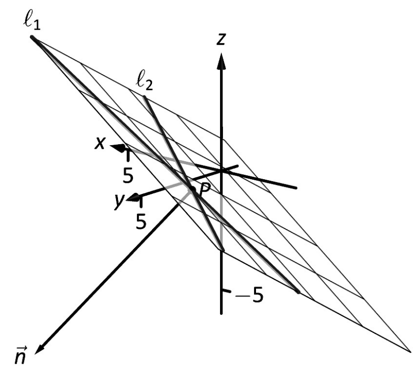


Figure 11.6.3: Sketching the plane in Example 11.6.2.

Example 11.6.3 Finding the equation of a plane

Give the equation, in standard form, of the plane that passes through the point $P = (-1, 0, 1)$ and is orthogonal to the line with vector equation $\vec{\ell}(t) = \langle -1, 0, 1\rangle + t\langle 1, 2, 2\rangle$.

Solution As the plane is to be orthogonal to the line, the plane must be orthogonal to the direction of the line given by $\vec{d} = \langle 1, 2, 2\rangle$. We use this as our normal vector. Thus the plane's equation, in standard form, is

$$(x+1) + 2y + 2(z-1) = 0.$$

The line and plane are sketched in Figure 11.6.4.

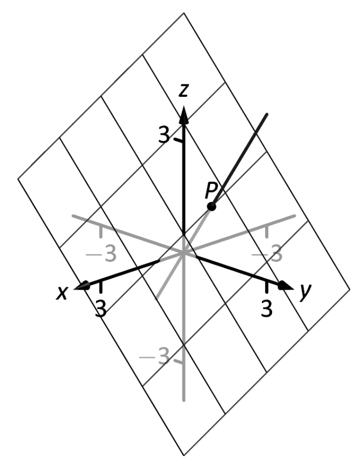


Figure 11.6.4: The line and plane in Example 11.6.3.

Example 11.6.4 Finding the intersection of two planes

Give the parametric equations of the line that is the intersection of the planes p_1 and p_2, where:

$$\begin{aligned} p_1 &: x - (y-2) + (z-1) = 0\\ p_2 &: -2(x-2) + (y+1) + (z-3) = 0 \end{aligned}$$

Solution To find an equation of a line, we need a point on the line and the direction of the line.

We can find a point on the line by solving each equation of the planes for z:

$$\begin{aligned} p_1 &: z = -x + y - 1\\ p_2 &: z = 2x - y - 2 \end{aligned}$$

We can now set these two equations equal to each other (i.e., we are finding values of x and y where the planes have the same z value):

$$\begin{aligned} -x + y - 1 &= 2x - y - 2\\ 2y &= 3x - 1\\ y &= \frac{1}{2}(3x - 1) \end{aligned}$$

We can choose any value for x; we choose $x = 1$. This determines that $y = 1$. We can now use the equations of either plane to find z: when $x = 1$ and $y = 1$, $z = -1$ on both planes. We have found a point P on the line: $P = (1, 1, -1)$.

We now need the direction of the line. Since the line lies in each plane, its direction is orthogonal to a normal vector for each plane. Considering the equations for p_1 and p_2, we can quickly determine their normal vectors. For p_1, $\vec{n}_1 = \langle 1, -1, 1\rangle$ and for p_2, $\vec{n}_2 = \langle -2, 1, 1\rangle$. A direction orthogonal to both of these directions is their cross product: $\vec{d} = \vec{n}_1 \times \vec{n}_2 = \langle -2, -3, -1\rangle$.

The parametric equations of the line through $P = (1, 1, -1)$ in the direction of $d = \langle -2, -3, -1\rangle$ is:

$$\ell: \quad x = -2t + 1 \quad y = -3t + 1 \quad z = -t - 1.$$

The planes and line are graphed in Figure 11.6.5.

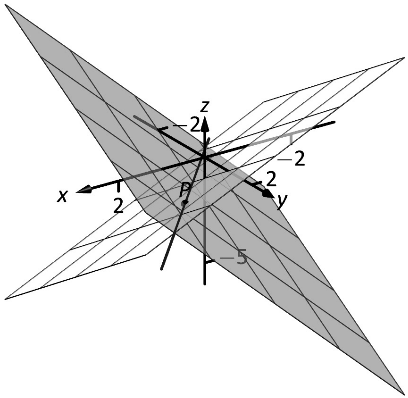


Figure 11.6.5: Graphing the planes and their line of intersection in Example 11.6.4.

Example 11.6.5 Finding the intersection of a plane and a line

Find the point of intersection, if any, of the line $\ell(t) = \langle 3, -3, -1\rangle + t\,\langle -1, 2, 1\rangle$ and the plane with equation in general form $2x + y + z = 4$.

Solution The equation of the plane shows that the vector $\vec{n} = \langle 2, 1, 1\rangle$ is a normal vector to the plane, and the equation of the line shows that the line moves parallel to $\vec{d} = \langle -1, 2, 1\rangle$. Since these are not orthogonal, we know there is a point of intersection. (If there were orthogonal, it would mean that the plane and line were parallel to each other, either never intersecting or the line was in the plane itself.)

To find the point of intersection, we need to find a t value such that $\ell(t)$ satisfies the equation of the plane. Rewriting the equation of the line with parametric equations will help:

$$\ell(t) = \begin{cases} x = 3 - t \\ y = -3 + 2t \\ z = -1 + t \end{cases}.$$

Replacing x, y and z in the equation of the plane with the expressions containing t found in the equation of the line allows us to determine a t value that indicates the point of intersection:

$$\begin{aligned} 2x + y + z &= 4 \\ 2(3 - t) + (-3 + 2t) + (-1 + t) &= 4 \\ t &= 2. \end{aligned}$$

When $t = 2$, the point on the line satisfies the equation of the plane; that point is $\ell(2) = \langle 1, 1, 1\rangle$. Thus the point $(1, 1, 1)$ is the point of intersection between the plane and the line, illustrated in Figure 11.6.6.

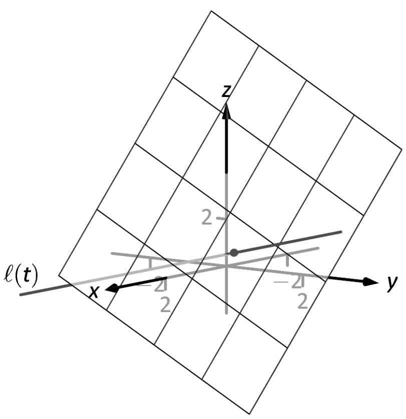


Figure 11.6.6: Illustrating the intersection of a line and a plane in Example 11.6.5.

Distances

Just as it was useful to find distances between points and lines in the previous section, it is also often necessary to find the distance from a point to a plane.

Consider Figure 11.6.7, where a plane with normal vector $\vec{n}$ is sketched containing a point P and a point Q, not on the plane, is given. We measure the distance from Q to the plane by measuring the length of the projection of $\overrightarrow{PQ}$ onto $\vec{n}$. That is, we want:

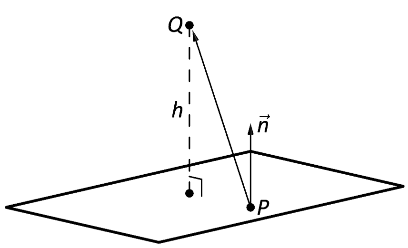


Figure 11.6.7: Illustrating finding the distance from a point to a plane.

$$\|\,\text{proj}_{\vec{n}}\,\overrightarrow{PQ}\,\| = \left\| \frac{\vec{n}\cdot\overrightarrow{PQ}}{\|\,\vec{n}\,\|^2}\vec{n} \right\| = \frac{|\vec{n}\cdot\overrightarrow{PQ}|}{\|\,\vec{n}\,\|} \tag{11.15}$$

Equation (11.15) is important as it does more than just give the distance between a point and a plane. We will see how it allows us to find several other distances as well: the distance between parallel planes and the distance from a line and a plane. Because Equation (11.15) is important, we restate it as a Key Idea.

Key Idea 11.6.1 Distance from a Point to a Plane

Let a plane with normal vector $\vec{n}$ be given, and let Q be a point. The distance h from Q to the plane is

$$h = \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{\| \vec{n} \|},$$

where P is any point in the plane.

Example 11.6.6 Distance between a point and a plane

Find the distance between the point $Q = (2, 1, 4)$ and the plane with equation $2x - 5y + 6z = 9$.

Solution Using the equation of the plane, we find the normal vector $\vec{n} = \langle 2, -5, 6 \rangle$. To find a point on the plane, we can let x and y be anything we choose, then let z be whatever satisfies the equation. Letting x and y be 0 seems simple; this makes $z = 1.5$. Thus we let $P = \langle 0, 0, 1.5 \rangle$, and $\overrightarrow{PQ} = \langle 2, 1, 2.5 \rangle$.

The distance h from Q to the plane is given by Key Idea 11.6.1:

$$\begin{aligned} h &= \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{\| \vec{n} \|} \\ &= \frac{|\langle 2, -5, -6 \rangle \cdot \langle 2, 1, 2.5 \rangle|}{\| \langle 2, -5, -6 \rangle \|} \\ &= \frac{|-16|}{\sqrt{65}} \\ &\approx 1.98. \end{aligned}$$

As we mentioned in 11.5, it is usually better to understand a method for calculating distances like that in 11.6.6, rather than memorizing a formula. Let us repeat the example using a more systematic approach.

Example 11.6.7 Distance between a point and a plane

Find the distance bewteen the point $Q = (2, 1, 4)$ and the plane with equation $2x - 5y + 6z = 9$.

Solution Referring to Figure 11.6.7, we need to determine the normal vector $\vec{n}$ and a point P on the plane. Using the equation of the plane, we find the normal vector $\vec{n} = \langle 2, -5, 6 \rangle$. To find a point on the plane, we can let x and y be anything we choose, then let z be whatever satisfies the equation. Letting x and y be 0 seems simple; this makes $z = \frac{3}{2}$. Thus we let $P = \left\langle 0, 0, \frac{3}{2} \right\rangle$, and $\overrightarrow{PQ} = \left\langle 2, 1, \frac{5}{2} \right\rangle$.

We can now compute the projection of $\overrightarrow{PQ}$ onto $\vec{n}$. We have:

$$\begin{aligned}\text{proj}_{\vec{n}}\,\overrightarrow{PQ} &= \left(\frac{\overrightarrow{PQ}\cdot\vec{n}}{\parallel\vec{n}\parallel^2}\right)\vec{n}\\ &= \left(\frac{\left\langle 2,1,\frac{5}{2}\right\rangle\cdot\langle 2,-5,6\rangle}{(2^2+5^2+6^2)}\right)\langle 2,-5,6\rangle\\ &= \frac{14}{65}\langle 2,-5,6\rangle.\end{aligned}$$

The desired distance is then given by

$$\parallel\text{proj}_{\vec{n}}\,\overrightarrow{PQ}\parallel = \frac{14}{65}\parallel\langle 2,-5,6\rangle\parallel = \frac{14}{\sqrt{65}}.$$

Although it was not requested in Example 11.6.6, note that we can also find the point R on the plane that is closest to Q. The desired point must be such that $\overrightarrow{RQ} = \text{proj}_{\vec{n}}\,\overrightarrow{PQ}$. Since we know the point Q and the vector $\overrightarrow{RQ}$, we can find the point R: since $\overrightarrow{RQ} = \overrightarrow{OQ} - \overrightarrow{OR}$, we find that

$$\begin{aligned}\overrightarrow{OR} &= \overrightarrow{OQ} - \overrightarrow{RQ}\\ &= \langle 2,1,4\rangle - \frac{14}{65}\langle 2,-5,6\rangle\\ &= \frac{1}{65}\langle 102,135,176\rangle.\end{aligned}$$

The desired point R thus has coordinates $\left(\frac{102}{65},\frac{135}{65},\frac{176}{65}\right)$. To make sure that we haven't made any mistakes, let's make sure that this point is indeed on the plane. We have

$$2\left(\frac{102}{65}\right) - 5\left(\frac{135}{65}\right) + 6\left(\frac{176}{65}\right) = \frac{1}{65}(204-675+1056) = \frac{585}{65} = 9,$$

as expected.

We can use Key Idea 11.6.1 to find other distances. Given two parallel planes, we can find the distance between these planes by letting P be a point on one plane and Q a point on the other. If ℓ is a line parallel to a plane, we can use the Key Idea to find the distance between them as well: again, let P be a point in the plane and let Q be any point on the line. (One can also use Key Idea 11.5.1.) The Exercise section contains problems of these types.

These past two sections have not explored lines and planes in space as an exercise of mathematical curiosity. However, there are many, many applications of these fundamental concepts. Complex shapes can be modelled (or, *approximated*) using planes. For instance, part of the exterior of an aircraft may have a complex, yet smooth, shape, and engineers will want to know how air flows across this piece as well as how heat might build up due to air friction. Many equations that help determine air flow and heat dissipation are difficult to apply to arbitrary surfaces, but simple to apply to planes. By approximating a surface with millions of small planes one can more readily model the needed behaviour.

Exercises 11.6

Terms and Concepts

1. In order to find the equation of a plane, what two pieces of information must one have?

2. What is the relationship between a plane and one of its normal vectors?

Problems

In Exercises 3 – 6, give any two points in the given plane.

3. $2x - 4y + 7z = 2$

4. $3(x + 2) + 5(y - 9) - 4z = 0$

5. $x = 2$

6. $4(y + 2) - (z - 6) = 0$

In Exercises 7 – 20, give the equation of the described plane in standard and general forms.

7. Passes through $(2, 3, 4)$ and has normal vector $\vec{n} = \langle 3, -1, 7\rangle$.

8. Passes through $(1, 3, 5)$ and has normal vector $\vec{n} = \langle 0, 2, 4\rangle$.

9. Passes through the points $(1, 2, 3)$, $(3, -1, 4)$ and $(1, 0, 1)$.

10. Passes through the points $(5, 3, 8)$, $(6, 4, 9)$ and $(3, 3, 3)$.

11. Contains the intersecting lines $\vec{\ell_1}(t) = \langle 2, 1, 2\rangle + t\langle 1, 2, 3\rangle$ and $\vec{\ell_2}(t) = \langle 2, 1, 2\rangle + t\langle 2, 5, 4\rangle$.

12. Contains the intersecting lines $\vec{\ell_1}(t) = \langle 5, 0, 3\rangle + t\langle -1, 1, 1\rangle$ and $\vec{\ell_2}(t) = \langle 1, 4, 7\rangle + t\langle 3, 0, -3\rangle$.

13. Contains the parallel lines $\vec{\ell_1}(t) = \langle 1, 1, 1\rangle + t\langle 1, 2, 3\rangle$ and $\vec{\ell_2}(t) = \langle 1, 1, 2\rangle + t\langle 1, 2, 3\rangle$.

14. Contains the parallel lines $\vec{\ell_1}(t) = \langle 1, 1, 1\rangle + t\langle 4, 1, 3\rangle$ and $\vec{\ell_2}(t) = \langle 4, 4, 4\rangle + t\langle 4, 1, 3\rangle$.

15. Contains the point $(2, -6, 1)$ and the line
$$\ell(t) = \begin{cases} x = 2 + 5t \\ y = 2 + 2t \\ z = -1 + 2t \end{cases}$$

16. Contains the point $(5, 7, 3)$ and the line
$$\ell(t) = \begin{cases} x = t \\ y = t \\ z = t \end{cases}$$

17. Contains the point $(5, 7, 3)$ and is orthogonal to the line $\vec{\ell}(t) = \langle 4, 5, 6\rangle + t\langle 1, 1, 1\rangle$.

18. Contains the point $(4, 1, 1)$ and is orthogonal to the line
$$\ell(t) = \begin{cases} x = 4 + 4t \\ y = 1 + 1t \\ z = 1 + 1t \end{cases}$$

19. Contains the point $(-4, 7, 2)$ and is parallel to the plane $3(x - 2) + 8(y + 1) - 10z = 0$.

20. Contains the point $(1, 2, 3)$ and is parallel to the plane $x = 5$.

In Exercises 21 – 22, give the equation of the line that is the intersection of the given planes.

21. $p1: \ 3(x - 2) + (y - 1) + 4z = 0$, and
$p2: \ 2(x - 1) - 2(y + 3) + 6(z - 1) = 0$.

22. $p1: \ 5(x - 5) + 2(y + 2) + 4(z - 1) = 0$, and
$p2: \ 3x - 4(y - 1) + 2(z - 1) = 0$.

In Exercises 23 – 26, find the point of intersection between the line and the plane.

23. line: $\langle 5, 1, -1\rangle + t\langle 2, 2, 1\rangle$,
plane: $5x - y - z = -3$

24. line: $\langle 4, 1, 0\rangle + t\langle 1, 0, -1\rangle$,
plane: $3x + y - 2z = 8$

25. line: $\langle 1, 2, 3\rangle + t\langle 3, 5, -1\rangle$,
plane: $3x - 2y - z = 4$

26. line: $\langle 1, 2, 3\rangle + t\langle 3, 5, -1\rangle$,
plane: $3x - 2y - z = -4$

In Exercises 27 – 30, find the given distances.

27. The distance from the point $(1, 2, 3)$ to the plane $3(x - 1) + (y - 2) + 5(z - 2) = 0$.

28. The distance from the point $(2, 6, 2)$ to the plane $2(x - 1) - y + 4(z + 1) = 0$.

29. The distance between the parallel planes
$x + y + z = 0$ and
$(x - 2) + (y - 3) + (z + 4) = 0$

30. The distance between the parallel planes
$2(x-1)+2(y+1)+(z-2)=0$ and
$2(x-3)+2(y-1)+(z-3)=0$

31. Show why if the point Q lies in a plane, then the distance formula correctly gives the distance from the point to the plane as 0.

32. How is Exercise 30 in Section 11.5 easier to answer once we have an understanding of planes?

12: Vector Valued Functions

In the previous chapter, we learned about vectors and were introduced to the power of vectors within mathematics. In this chapter, we'll build on this foundation to define functions whose input is a real number and whose output is a vector. We'll see how to graph these functions and apply calculus techniques to analyze their behaviour. Most importantly, we'll see *why* we are interested in doing this: we'll see beautiful applications to the study of moving objects.

12.1 Vector–Valued Functions

We are very familiar with **real valued functions**, that is, functions whose output is a real number. This section introduces **vector–valued functions** – functions whose output is a vector.

> **Definition 12.1.1 Vector–Valued Functions**
>
> A **vector–valued function** is a function of the form
>
> $$\vec{r}(t) = \langle\, f(t), g(t)\,\rangle \quad \text{or} \quad \vec{r}(t) = \langle\, f(t), g(t), h(t)\,\rangle\,,$$
>
> where f, g and h are real valued functions.
>
> The **domain** of $\vec{r}$ is the set of all values of t for which $\vec{r}(t)$ is defined. The **range** of $\vec{r}$ is the set of all possible output vectors $\vec{r}(t)$.

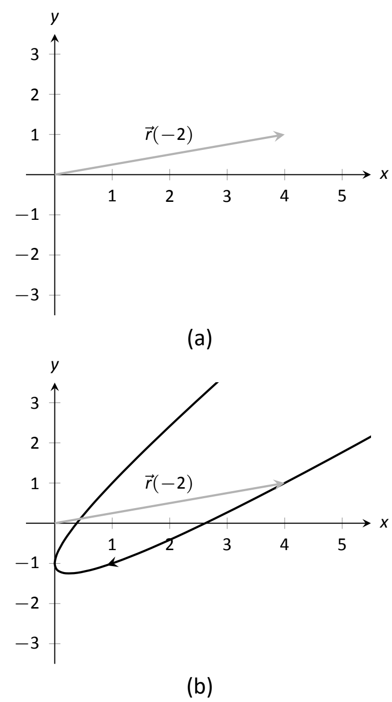


Figure 12.1.1: Sketching the graph of a vector–valued function.

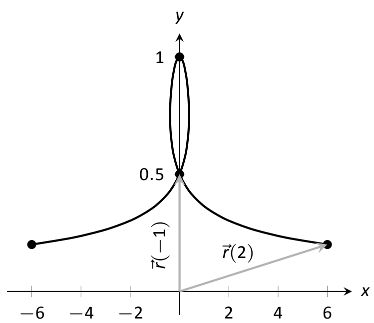
Evaluating and Graphing Vector–Valued Functions

Evaluating a vector–valued function at a specific value of t is straightforward; simply evaluate each component function at that value of t. For instance, if $\vec{r}(t) = \left\langle t^2, t^2+t-1\right\rangle$, then $\vec{r}(-2) = \langle 4, 1\rangle$. We can sketch this vector, as is done in Figure 12.1.1(a). Plotting lots of vectors is cumbersome, though, so generally we do not sketch the whole vector but just the terminal point. The **graph** of a vector–valued function is the set of all terminal points of $\vec{r}(t)$, where the initial point of each vector is always the origin. In Figure 12.1.1(b) we sketch the graph of $\vec{r}$; we can indicate individual points on the graph with their respective vector, as shown.

Vector–valued functions are closely related to parametric equations of graphs. While in both methods we plot points $\big(x(t), y(t)\big)$ or $\big(x(t), y(t), z(t)\big)$ to produce a graph, in the context of vector–valued functions each such point represents a vector. The implications of this will be more fully realized in the next section as we apply calculus ideas to these functions.

t	$t^3 - t$	$\dfrac{1}{t^2+1}$
-2	-6	1/5
-1	0	1/2
0	0	1
1	0	1/2
2	6	1/5

(a)



(b)

Figure 12.1.2: Sketching the vector–valued function of Example 12.1.1.

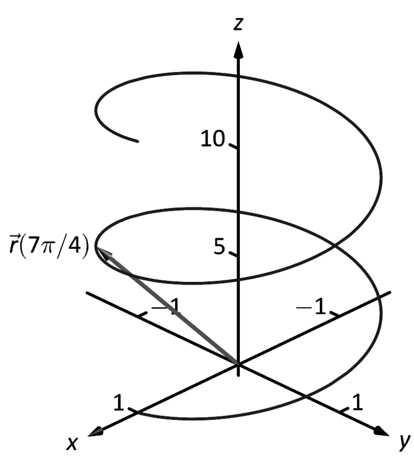


Figure 12.1.3: The graph of $\vec r(t)$ in Example 12.1.2.

Example 12.1.1 Graphing vector–valued functions

Graph $\vec r(t) = \left\langle t^3 - t, \dfrac{1}{t^2+1}\right\rangle$, for $-2 \le t \le 2$. Sketch $\vec r(-1)$ and $\vec r(2)$.

Solution We start by making a table of t, x and y values as shown in Figure 12.1.2(a). Plotting these points gives an indication of what the graph looks like. In Figure 12.1.2(b), we indicate these points and sketch the full graph. We also highlight $\vec r(-1)$ and $\vec r(2)$ on the graph.

Example 12.1.2 Graphing vector–valued functions.

Graph $\vec r(t) = \langle \cos t, \sin t, t\rangle$ for $0 \le t \le 4\pi$.

Solution We can again plot points, but careful consideration of this function is very revealing. Momentarily ignoring the third component, we see the x and y components trace out a circle of radius 1 centred at the origin. Noticing that the z component is t, we see that as the graph winds around the z-axis, it is also increasing at a constant rate in the positive z direction, forming a spiral. This is graphed in Figure 12.1.3. In the graph $\vec r(7\pi/4) \approx (0.707, -0.707, 5.498)$ is highlighted to help us understand the graph.

Algebra of Vector–Valued Functions

Definition 12.1.2 Operations on Vector–Valued Functions

Let $\vec r_1(t) = \langle f_1(t), g_1(t)\rangle$ and $\vec r_2(t) = \langle f_2(t), g_2(t)\rangle$ be vector–valued functions in $\mathbb{R}^2$ and let c be a scalar. Then:

1. $\vec r_1(t) \pm \vec r_2(t) = \langle\, f_1(t) \pm f_2(t), g_1(t) \pm g_2(t)\,\rangle$.
2. $c\vec r_1(t) = \langle\, cf_1(t), cg_1(t)\,\rangle$.

A similar definition holds for vector–valued functions in $\mathbb{R}^3$.

This definition states that we add, subtract and scale vector-valued functions component–wise. Combining vector–valued functions in this way can be very useful (as well as create interesting graphs).

Example 12.1.3 Adding and scaling vector–valued functions.

Let $\vec r_1(t) = \langle\, 0.2t, 0.3t\,\rangle$, $\vec r_2(t) = \langle\, \cos t, \sin t\,\rangle$ and $\vec r(t) = \vec r_1(t) + \vec r_2(t)$. Graph $\vec r_1(t)$, $\vec r_2(t)$, $\vec r(t)$ and $5\vec r(t)$ on $-10 \le t \le 10$.

Solution We can graph $\vec r_1$ and $\vec r_2$ easily by plotting points (or just using technology). Let's think about each for a moment to better understand how vector–valued functions work.

We can rewrite $\vec r_1(t) = \langle\, 0.2t, 0.3t\,\rangle$ as $\vec r_1(t) = t\,\langle 0.2, 0.3\rangle$. That is, the function $\vec r_1$ scales the vector $\langle 0.2, 0.3\rangle$ by t. This scaling of a vector produces a line in the direction of $\langle 0.2, 0.3\rangle$.

We are familiar with $\vec r_2(t) = \langle\, \cos t, \sin t\,\rangle$; it traces out a circle, centred at the origin, of radius 1. Figure 12.1.5(a) graphs $\vec r_1(t)$ and $\vec r_2(t)$.

Adding $\vec r_1(t)$ to $\vec r_2(t)$ produces $\vec r(t) = \langle\, \cos t + 0.2t, \sin t + 0.3t\,\rangle$, graphed in Figure 12.1.5(b). The linear movement of the line combines with the circle to create loops that move in the direction of $\langle 0.2, 0.3\rangle$. (We encourage the reader to experiment by changing $\vec r_1(t)$ to $\langle 2t, 3t\rangle$, etc., and observe the effects on the loops.)

Multiplying $\vec{r}(t)$ by 5 scales the function by 5, producing $5\vec{r}(t) = \langle 5\cos t + 1, 5\sin t + 1.5\rangle$, which is graphed in Figure 12.1.5(c) along with $\vec{r}(t)$. The new function is "5 times bigger" than $\vec{r}(t)$. Note how the graph of $5\vec{r}(t)$ in (c) looks identical to the graph of $\vec{r}(t)$ in (b). This is due to the fact that the x and y bounds of the plot in (c) are exactly 5 times larger than the bounds in (b).

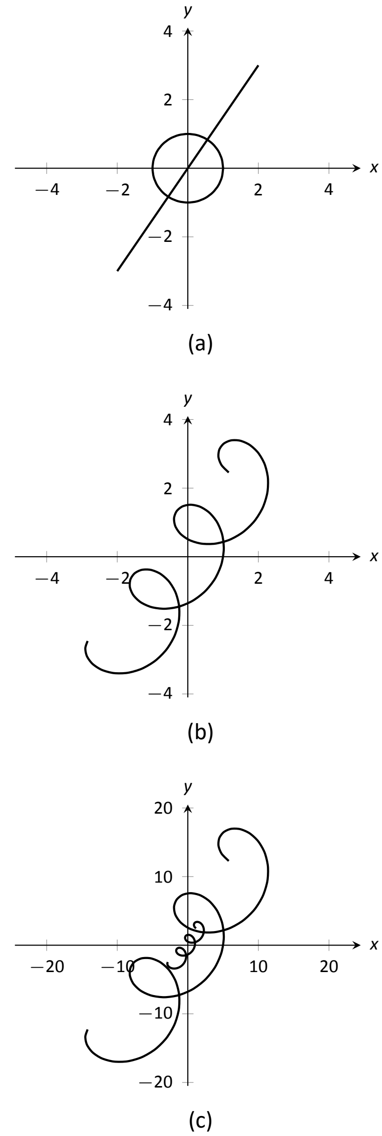


Figure 12.1.5: Graphing the functions in Example 12.1.3.

Example 12.1.4 Adding and scaling vector–valued functions.

A **cycloid** is a graph traced by a point p on a rolling circle, as shown in Figure 12.1.4. Find an equation describing the cycloid, where the circle has radius 1.

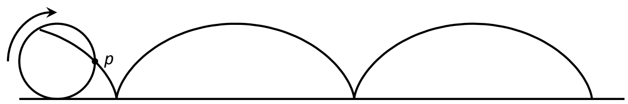


Figure 12.1.4: Tracing a cycloid.

SOLUTION This problem is not very difficult if we approach it in a clever way. We start by letting $\vec{p}(t)$ describe the position of the point p on the circle, where the circle is centred at the origin and only rotates clockwise (i.e., it does not roll). This is relatively simple given our previous experiences with parametric equations; $\vec{p}(t) = \langle \cos t, -\sin t\rangle$.

We now want the circle to roll. We represent this by letting $\vec{c}(t)$ represent the location of the center of the circle. It should be clear that the y component of $\vec{c}(t)$ should be 1; the center of the circle is always going to be 1 if it rolls on a horizontal surface.

The x component of $\vec{c}(t)$ is a linear function of t: $f(t) = mt$ for some scalar m. When $t = 0$, $f(t) = 0$ (the circle starts centred on the y-axis). When $t = 2\pi$, the circle has made one complete revolution, travelling a distance equal to its circumference, which is also 2π. This gives us a point on our line $f(t) = mt$, the point $(2\pi, 2\pi)$. It should be clear that $m = 1$ and $f(t) = t$. So $\vec{c}(t) = \langle t, 1\rangle$.

We now combine $\vec{p}$ and $\vec{c}$ together to form the equation of the cycloid: $\vec{r}(t) = \vec{p}(t) + \vec{c}(t) = \langle \cos t + t, -\sin t + 1\rangle$, which is graphed in Figure 12.1.6.

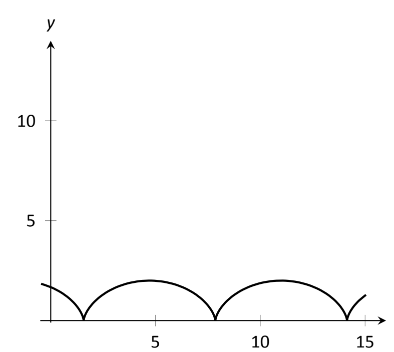


Figure 12.1.6: The cycloid in Example 12.1.4.

Displacement

A vector–valued function $\vec{r}(t)$ is often used to describe the position of a moving object at time t. At $t = t_0$, the object is at $\vec{r}(t_0)$; at $t = t_1$, the object is at $\vec{r}(t_1)$. Knowing the locations $\vec{r}(t_0)$ and $\vec{r}(t_1)$ give no indication of the path taken between them, but often we only care about the difference of the locations, $\vec{r}(t_1) - \vec{r}(t_0)$, the **displacement**.

Definition 12.1.3 Displacement

Let $\vec{r}(t)$ be a vector–valued function and let $t_0 < t_1$ be values in the domain. The **displacement** $\vec{d}$ of $\vec{r}$, from $t = t_0$ to $t = t_1$, is

$$\vec{d} = \vec{r}(t_1) - \vec{r}(t_0).$$

When the displacement vector is drawn with initial point at $\vec{r}(t_0)$, its terminal point is $\vec{r}(t_1)$. We think of it as the vector which points from a starting position to an ending position.

Example 12.1.5 Finding and graphing displacement vectors
Let $\vec{r}(t) = \left\langle \cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t) \right\rangle$. Graph $\vec{r}(t)$ on $-1 \le t \le 1$, and find the displacement of $\vec{r}(t)$ on this interval.

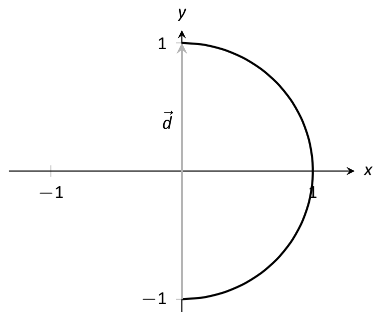


Figure 12.1.7: Graphing the displacement of a position function in Example 12.1.5.

Solution The function $\vec{r}(t)$ traces out the unit circle, though at a different rate than the "usual" $\langle \cos t, \sin t \rangle$ parametrization. At $t_0 = -1$, we have $\vec{r}(t_0) = \langle 0, -1 \rangle$; at $t_1 = 1$, we have $\vec{r}(t_1) = \langle 0, 1 \rangle$. The displacement of $\vec{r}(t)$ on $[-1, 1]$ is thus $\vec{d} = \langle 0, 1 \rangle - \langle 0, -1 \rangle = \langle 0, 2 \rangle$.

A graph of $\vec{r}(t)$ on $[-1, 1]$ is given in Figure 12.1.7, along with the displacement vector $\vec{d}$ on this interval.

Measuring displacement makes us contemplate related, yet very different, concepts. Considering the semi–circular path the object in Example 12.1.5 took, we can quickly verify that the object ended up a distance of 2 units from its initial location. That is, we can compute $\| \vec{d} \| = 2$. However, measuring *distance from the starting point* is different from measuring *distance travelled*. Being a semi–circle, we can measure the distance travelled by this object as $\pi \approx 3.14$ units. Knowing *distance from the starting point* allows us to compute **average rate of change.**

Definition 12.1.4 Average Rate of Change

Let $\vec{r}(t)$ be a vector–valued function, where each of its component functions is continuous on its domain, and let $t_0 < t_1$. The **average rate of change** of $\vec{r}(t)$ on $[t_0, t_1]$ is

$$\text{average rate of change} = \frac{\vec{r}(t_1) - \vec{r}(t_0)}{t_1 - t_0}.$$

Example 12.1.6 Average rate of change
Let $\vec{r}(t) = \left\langle \cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t) \right\rangle$ as in Example 12.1.5. Find the average rate of change of $\vec{r}(t)$ on $[-1, 1]$ and on $[-1, 5]$.

Solution We computed in Example 12.1.5 that the displacement of $\vec{r}(t)$ on $[-1, 1]$ was $\vec{d} = \langle 0, 2 \rangle$. Thus the average rate of change of $\vec{r}(t)$ on $[-1, 1]$ is:

$$\frac{\vec{r}(1) - \vec{r}(-1)}{1 - (-1)} = \frac{\langle 0, 2 \rangle}{2} = \langle 0, 1 \rangle .$$

We interpret this as follows: the object followed a semi–circular path, meaning it moved towards the right then moved back to the left, while climbing slowly, then quickly, then slowly again. *On average*, however, it progressed straight up at a constant rate of $\langle 0, 1 \rangle$ per unit of time.

We can quickly see that the displacement on $[-1, 5]$ is the same as on $[-1, 1]$, so $\vec{d} = \langle 0, 2 \rangle$. The average rate of change is different, though:

$$\frac{\vec{r}(5) - \vec{r}(-1)}{5 - (-1)} = \frac{\langle 0, 2 \rangle}{6} = \langle 0, 1/3 \rangle .$$

As it took "3 times as long" to arrive at the same place, this average rate of change on $[-1, 5]$ is $1/3$ the average rate of change on $[-1, 1]$.

We considered average rates of change in Sections 1.1 and 2.1 as we studied limits and derivatives. The same is true here; in the following section we apply calculus concepts to vector–valued functions as we find limits, derivatives, and integrals. Understanding the average rate of change will give us an understanding of the derivative; displacement gives us one application of integration.

Exercises 12.1

Terms and Concepts

1. Vector–valued functions are closely related to ________ ________ of graphs.

2. When sketching vector–valued functions, technically one isn't graphing points, but rather ________.

3. It can be useful to think of ________ as a vector that points from a starting position to an ending position.

4. In the context of vector–valued functions, average rate of change is ________ divided by time.

Problems

In Exercises 5 – 12, sketch the vector–valued function on the given interval.

5. $\vec{r}(t) = \langle t^2, t^2 - 1\rangle$, for $-2 \leq t \leq 2$.

6. $\vec{r}(t) = \langle t^2, t^3\rangle$, for $-2 \leq t \leq 2$.

7. $\vec{r}(t) = \langle 1/t, 1/t^2\rangle$, for $-2 \leq t \leq 2$.

8. $\vec{r}(t) = \langle \frac{1}{10}t^2, \sin t\rangle$, for $-2\pi \leq t \leq 2\pi$.

9. $\vec{r}(t) = \langle \frac{1}{10}t^2, \sin t\rangle$, for $-2\pi \leq t \leq 2\pi$.

10. $\vec{r}(t) = \langle 3\sin(\pi t), 2\cos(\pi t)\rangle$, on $[0, 2]$.

11. $\vec{r}(t) = \langle 3\cos t, 2\sin(2t)\rangle$, on $[0, 2\pi]$.

12. $\vec{r}(t) = \langle 2\sec t, \tan t\rangle$, on $[-\pi, \pi]$.

In Exercises 13 – 16, sketch the vector–valued function on the given interval in $\mathbb{R}^3$. Technology may be useful in creating the sketch.

13. $\vec{r}(t) = \langle 2\cos t, t, 2\sin t\rangle$, on $[0, 2\pi]$.

14. $\vec{r}(t) = \langle 3\cos t, \sin t, t/\pi\rangle$ on $[0, 2\pi]$.

15. $\vec{r}(t) = \langle \cos t, \sin t, \sin t\rangle$ on $[0, 2\pi]$.

16. $\vec{r}(t) = \langle \cos t, \sin t, \sin(2t)\rangle$ on $[0, 2\pi]$.

In Exercises 17 – 20, find $\| \vec{r}(t) \|$.

17. $\vec{r}(t) = \langle t, t^2\rangle$.

18. $\vec{r}(t) = \langle 5\cos t, 3\sin t\rangle$.

19. $\vec{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$.

20. $\vec{r}(t) = \langle \cos t, t, t^2\rangle$.

In Exercises 21 – 30, create a vector–valued function whose graph matches the given description.

21. A circle of radius 2, centered at $(1, 2)$, traced counter–clockwise once on $[0, 2\pi]$.

22. A circle of radius 3, centered at $(5, 5)$, traced clockwise once on $[0, 2\pi]$.

23. An ellipse, centered at $(0, 0)$ with vertical major axis of length 10 and minor axis of length 3, traced once counter–clockwise on $[0, 2\pi]$.

24. An ellipse, centered at $(3, -2)$ with horizontal major axis of length 6 and minor axis of length 4, traced once clockwise on $[0, 2\pi]$.

25. A line through $(2, 3)$ with a slope of 5.

26. A line through $(1, 5)$ with a slope of $-1/2$.

27. The line through points $(1, 2, 3)$ and $(4, 5, 6)$, where $\vec{r}(0) = \langle 1, 2, 3\rangle$ and $\vec{r}(1) = \langle 4, 5, 6\rangle$.

28. The line through points $(1, 2)$ and $(4, 4)$, where $\vec{r}(0) = \langle 1, 2\rangle$ and $\vec{r}(1) = \langle 4, 4\rangle$.

29. A vertically oriented helix with radius of 2 that starts at $(2, 0, 0)$ and ends at $(2, 0, 4\pi)$ after 1 revolution on $[0, 2\pi]$.

30. A vertically oriented helix with radius of 3 that starts at $(3, 0, 0)$ and ends at $(3, 0, 3)$ after 2 revolutions on $[0, 1]$.

In Exercises 31 – 34, find the average rate of change of $\vec{r}(t)$ on the given interval.

31. $\vec{r}(t) = \langle t, t^2\rangle$ on $[-2, 2]$.

32. $\vec{r}(t) = \langle t, t + \sin t\rangle$ on $[0, 2\pi]$.

33. $\vec{r}(t) = \langle 3\cos t, 2\sin t, t\rangle$ on $[0, 2\pi]$.

34. $\vec{r}(t) = \langle t, t^2, t^3\rangle$ on $[-1, 3]$.

12.2 Calculus and Vector–Valued Functions

The previous section introduced us to a new mathematical object, the vector–valued function. We now apply calculus concepts to these functions. We start with the limit, then work our way through derivatives to integrals.

Limits of Vector–Valued Functions

The initial definition of the limit of a vector–valued function is a bit intimidating, as was the definition of the limit in Definition 1.2.1. The theorem following the definition shows that in practice, taking limits of vector–valued functions is no more difficult than taking limits of real–valued functions.

Definition 12.2.1 Limits of Vector–Valued Functions

Let I be an open interval containing c, and let $\vec r(t)$ be a vector–valued function defined on I, except possibly at c. The **limit of $\vec r(t)$, as t approaches c, is $\vec L$**, expressed as

$$\lim_{t\to c}\vec r(t)=\vec L,$$

means that given any $\varepsilon>0$, there exists a $\delta>0$ such that for all $t\neq c$, if $|t-c|<\delta$, we have $\|\,\vec r(t)-\vec L\,\|<\varepsilon$.

Note: we can define one-sided limits in a manner very similar to Definition 12.2.1.

Note how the measurement of distance between real numbers is the absolute value of their difference; the measure of distance between vectors is the vector norm, or magnitude, of their difference.

Theorem 12.2.1 states that we can compute limits of vector–valued functions component–wise.

Theorem 12.2.1 Limits of Vector–Valued Functions

1. Let $\vec r(t)=\langle f(t),g(t)\,\rangle$ be a vector–valued function in $\mathbb{R}^2$ defined on an open interval I containing c, except possibly at c. Then

$$\lim_{t\to c}\vec r(t)=\left\langle \lim_{t\to c}f(t)\,,\ \lim_{t\to c}g(t)\right\rangle.$$

2. Let $\vec r(t)=\langle f(t),g(t),h(t)\,\rangle$ be a vector–valued function in $\mathbb{R}^3$ defined on an open interval I containing c, except possibly at c. Then

$$\lim_{t\to c}\vec r(t)=\left\langle \lim_{t\to c}f(t)\,,\ \lim_{t\to c}g(t)\,,\ \lim_{t\to c}h(t)\right\rangle$$

Example 12.2.1 Finding limits of vector–valued functions

Let $\vec r(t)=\left\langle \dfrac{\sin t}{t},\ t^2-3t+3,\ \cos t\right\rangle$. Find $\displaystyle\lim_{t\to 0}\vec r(t)$.

Solution We apply the theorem and compute limits component–wise.

$$\begin{aligned}\lim_{t\to 0}\vec r(t)&=\left\langle \lim_{t\to 0}\frac{\sin t}{t}\,,\ \lim_{t\to 0}t^2-3t+3\,,\ \lim_{t\to 0}\cos t\right\rangle\\&=\langle 1,3,1\rangle\,.\end{aligned}$$

Continuity

Definition 12.2.2 Continuity of Vector–Valued Functions

Let $\vec{r}(t)$ be a vector–valued function defined on an open interval I containing c.

1. $\vec{r}(t)$ is **continuous at** c if $\lim_{t\to c} \vec{r}(t) = \vec{r}(c)$.
2. If $\vec{r}(t)$ is continuous at all c in I, then $\vec{r}(t)$ is **continuous on** I.

We again have a theorem that lets us evaluate continuity component–wise.

Note: Using one-sided limits, we can also define continuity on closed intervals as done before.

Theorem 12.2.2 Continuity of Vector–Valued Functions

Let $\vec{r}(t)$ be a vector–valued function defined on an open interval I containing c. Then $\vec{r}(t)$ is continuous at c if, and only if, each of its component functions is continuous at c.

Example 12.2.2 Evaluating continuity of vector–valued functions

Let $\vec{r}(t) = \left\langle \dfrac{\sin t}{t},\ t^2 - 3t + 3,\ \cos t \right\rangle$. Determine whether $\vec{r}$ is continuous at $t = 0$ and $t = 1$.

Solution While the second and third components of $\vec{r}(t)$ are defined at $t = 0$, the first component, $(\sin t)/t$, is not. Since the first component is not even defined at $t = 0$, $\vec{r}(t)$ is not defined at $t = 0$, and hence it is not continuous at $t = 0$.

At $t = 1$ each of the component functions is continuous. Therefore $\vec{r}(t)$ is continuous at $t = 1$.

Derivatives

Consider a vector–valued function $\vec{r}$ defined on an open interval I containing t_0 and t_1. We can compute the displacement of $\vec{r}$ on $[t_0, t_1]$, as shown in Figure 12.2.1(a). Recall that dividing the displacement vector by $t_1 - t_0$ gives the average rate of change on $[t_0, t_1]$, as shown in (b).

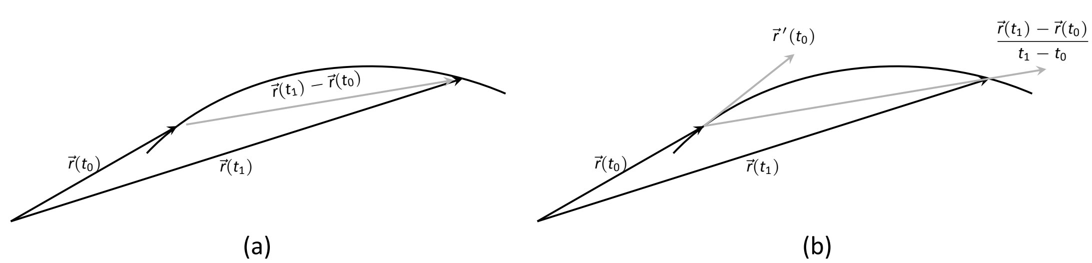


Figure 12.2.1: Illustrating displacement, leading to an understanding of the derivative of vector–valued functions.

The **derivative** of a vector–valued function is a measure of the *instantaneous* rate of change, measured by taking the limit as the length of $[t_0, t_1]$ goes to 0. Instead of thinking of an interval as $[t_0, t_1]$, we think of it as $[c, c+h]$ for some value of h (hence the interval has length h). The *average* rate of change is

$$\frac{\vec{r}(c+h)-\vec{r}(c)}{h}$$

for any value of $h \neq 0$. We take the limit as $h \to 0$ to measure the instantaneous rate of change; this is the derivative of $\vec{r}$.

Definition 12.2.3 Derivative of a Vector–Valued Function

Let $\vec{r}(t)$ be continuous on an open interval I containing c.

1. The **derivative of $\vec{r}$ at $t = c$** is

$$\vec{r}\,'(c) = \lim_{h\to 0}\frac{\vec{r}(c+h)-\vec{r}(c)}{h}.$$

2. The **derivative of $\vec{r}$** is

$$\vec{r}\,'(t) = \lim_{h\to 0}\frac{\vec{r}(t+h)-\vec{r}(t)}{h}.$$

Alternate notations for the derivative of $\vec{r}$ include:

$$\vec{r}\,'(t) = \frac{d}{dt}\big(\vec{r}(t)\big) = \frac{d\vec{r}}{dt}.$$

If a vector–valued function has a derivative for all c in an open interval I, we say that $\vec{r}(t)$ is **differentiable** on I.

Once again we might view this definition as intimidating, but recall that we can evaluate limits component–wise. The following theorem verifies that this means we can compute derivatives component–wise as well, making the task not too difficult.

Theorem 12.2.3 Derivatives of Vector–Valued Functions

1. Let $\vec{r}(t) = \langle f(t), g(t)\rangle$. Then

$$\vec{r}\,'(t) = \langle f'(t), g'(t)\rangle.$$

2. Let $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$. Then

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t)\rangle.$$

Note: again, using one-sided limits, we can define differentiability on closed intervals. We'll make use of this a few times in this chapter.

Example 12.2.3 Derivatives of vector–valued functions

Let $\vec{r}(t) = \langle t^2, t\rangle$.

1. Sketch $\vec{r}(t)$ and $\vec{r}\,'(t)$ on the same axes.
2. Compute $\vec{r}\,'(1)$ and sketch this vector with its initial point at the origin and at $\vec{r}(1)$.

SOLUTION

1. Theorem 12.2.3 allows us to compute derivatives component–wise, so

$$\vec{r}\,'(t) = \langle 2t, 1\rangle.$$

$\vec{r}(t)$ and $\vec{r}\,'(t)$ are graphed together in Figure 12.2.2(a). Note how plotting the two of these together, in this way, is not very illuminating. When

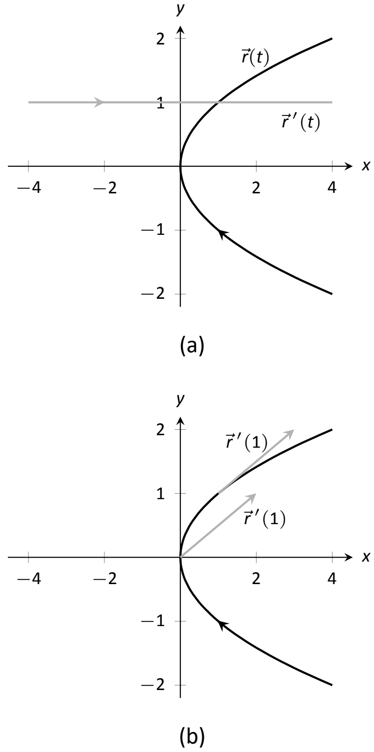


Figure 12.2.2: Graphing the derivative of a vector–valued function in Example 12.2.3.

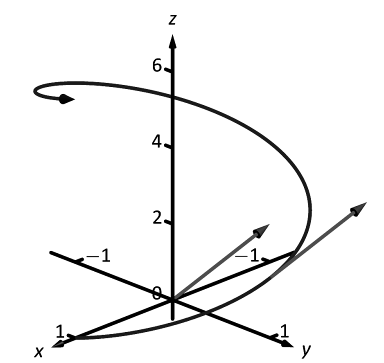


Figure 12.2.3: Viewing a vector–valued function and its derivative at one point.

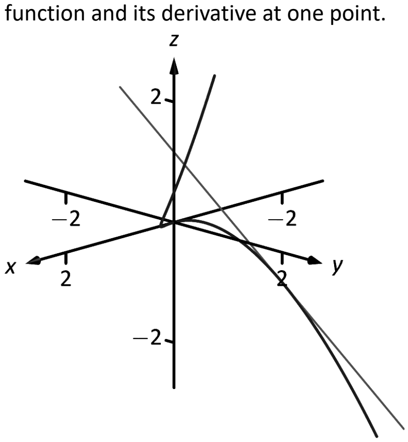


Figure 12.2.4: Graphing a curve in space with its tangent line.

dealing with real–valued functions, plotting $f(x)$ with $f'(x)$ gave us useful information as we were able to compare f and f' at the same x-values. When dealing with vector–valued functions, it is hard to tell which points on the graph of $\vec r\,'$ correspond to which points on the graph of $\vec r$.

2. We easily compute $\vec r\,'(1) = \langle 2, 1\rangle$, which is drawn in Figure 12.2.2 with its initial point at the origin, as well as at $\vec r(1) = \langle 1, 1\rangle$. These are sketched in Figure 12.2.2(b).

Example 12.2.4 Derivatives of vector–valued functions
Let $\vec r(t) = \langle \cos t, \sin t, t\rangle$. Compute $\vec r\,'(t)$ and $\vec r\,'(\pi/2)$. Sketch $\vec r\,'(\pi/2)$ with its initial point at the origin and at $\vec r(\pi/2)$.

Solution We compute $\vec r\,'$ as $\vec r\,'(t) = \langle -\sin t, \cos t, 1\rangle$. At $t = \pi/2$, we have $\vec r\,'(\pi/2) = \langle -1, 0, 1\rangle$. Figure 12.2.3 shows a graph of $\vec r(t)$, with $\vec r\,'(\pi/2)$ plotted with its initial point at the origin and at $\vec r(\pi/2)$.

In Examples 12.2.3 and 12.2.4, sketching a particular derivative with its initial point at the origin did not seem to reveal anything significant. However, when we sketched the vector with its initial point on the corresponding point on the graph, we did see something significant: the vector appeared to be *tangent* to the graph. We have not yet defined what "tangent" means in terms of curves in space; in fact, we use the derivative to define this term.

Definition 12.2.4 Tangent Vector, Tangent Line

Let $\vec r(t)$ be a differentiable vector–valued function on an open interval I containing c, where $\vec r\,'(c) \neq \vec 0$.

1. A vector $\vec v$ is **tangent to the graph of** $\vec r(t)$ **at** $t = c$ if $\vec v$ is parallel to $\vec r\,'(c)$.

2. The **tangent line** to the graph of $\vec r(t)$ at $t = c$ is the line through $\vec r(c)$ with direction parallel to $\vec r\,'(c)$. An equation of the tangent line is
$$\vec\ell(t) = \vec r(c) + t\,\vec r\,'(c).$$

Example 12.2.5 Finding tangent lines to curves in space
Let $\vec r(t) = \langle t, t^2, t^3\rangle$ on $[-1.5, 1.5]$. Find the vector equation of the line tangent to the graph of $\vec r$ at $t = -1$.

Solution To find the equation of a line, we need a point on the line and the line's direction. The point is given by $\vec r(-1) = \langle -1, 1, -1\rangle$. (To be clear, $\langle -1, 1, -1\rangle$ is a *vector*, not a point, but we use the point "pointed to" by this vector.)

The direction comes from $\vec r\,'(-1)$. We compute, component–wise, $\vec r\,'(t) = \langle 1, 2t, 3t^2\rangle$. Thus $\vec r\,'(-1) = \langle 1, -2, 3\rangle$.

The vector equation of the line is $\ell(t) = \langle -1, 1, -1\rangle + t\,\langle 1, -2, 3\rangle$. This line and $\vec r(t)$ are sketched in Figure 12.2.4.

Example 12.2.6 Finding tangent lines to curves
Find the equations of the lines tangent to $\vec r(t) = \langle t^3, t^2\rangle$ at $t = -1$ and $t = 0$.

Solution We find that $\vec{r}\,'(t) = \left\langle 3t^2, 2t\right\rangle$. At $t=-1$, we have

$$\vec{r}(-1) = \langle -1, 1\rangle \quad \text{and} \quad \vec{r}\,'(-1) = \langle 3, -2\rangle\,,$$

so the equation of the line tangent to the graph of $\vec{r}(t)$ at $t=-1$ is

$$\ell(t) = \langle -1, 1\rangle + t\,\langle 3, -2\rangle\,.$$

This line is graphed with $\vec{r}(t)$ in Figure 12.2.5.

At $t=0$, we have $\vec{r}\,'(0) = \langle 0,0\rangle = \vec{0}$! This implies that the tangent line "has no direction." We cannot apply Definition 12.2.4, hence cannot find the equation of the tangent line.

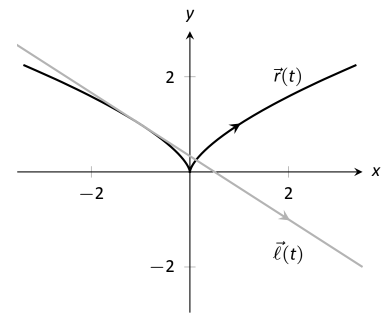


Figure 12.2.5: Graphing $\vec{r}(t)$ and its tangent line in Example 12.2.6.

We were unable to compute the equation of the tangent line to $\vec{r}(t) = \left\langle t^3, t^2\right\rangle$ at $t=0$ because $\vec{r}\,'(0) = \vec{0}$. The graph in Figure 12.2.5 shows that there is a cusp at this point. This leads us to another definition of **smooth**, previously defined by Definition 9.2.2 in Section 9.2.

Definition 12.2.5 Smooth Vector–Valued Functions

Let $\vec{r}(t)$ be a differentiable vector–valued function on an open interval I where $\vec{r}\,'(t)$ is continuous on I. $\vec{r}(t)$ is **smooth** on I if $\vec{r}\,'(t) \neq \vec{0}$ on I.

Having established derivatives of vector–valued functions, we now explore the relationships between the derivative and other vector operations. The following theorem states how the derivative interacts with vector addition and the various vector products.

Theorem 12.2.4 Properties of Derivatives of Vector–Valued Functions

Let $\vec{r}$ and $\vec{s}$ be differentiable vector–valued functions, let f be a differentiable real–valued function, and let c be a real number.

1. $\dfrac{d}{dt}\Big(\vec{r}(t) \pm \vec{s}(t)\Big) = \vec{r}\,'(t) \pm \vec{s}\,'(t)$
2. $\dfrac{d}{dt}\Big(c\vec{r}(t)\Big) = c\vec{r}\,'(t)$
3. $\dfrac{d}{dt}\Big(f(t)\vec{r}(t)\Big) = f'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$ **Product Rule**
4. $\dfrac{d}{dt}\Big(\vec{r}(t)\cdot\vec{s}(t)\Big) = \vec{r}\,'(t)\cdot\vec{s}(t) + \vec{r}(t)\cdot\vec{s}\,'(t)$ **Product Rule**
5. $\dfrac{d}{dt}\Big(\vec{r}(t)\times\vec{s}(t)\Big) = \vec{r}\,'(t)\times\vec{s}(t) + \vec{r}(t)\times\vec{s}\,'(t)$ **Product Rule**
6. $\dfrac{d}{dt}\Big(\vec{r}\big(f(t)\big)\Big) = \vec{r}\,'\big(f(t)\big)f'(t)$ **Chain Rule**

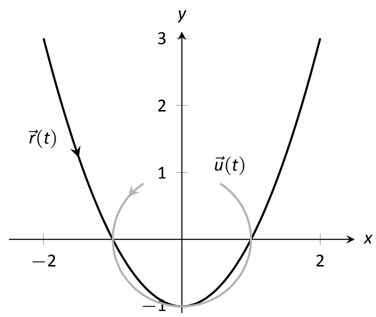


Figure 12.2.6: Graphing $\vec{r}(t)$ and $\vec{u}(t)$ in Example 12.2.7.

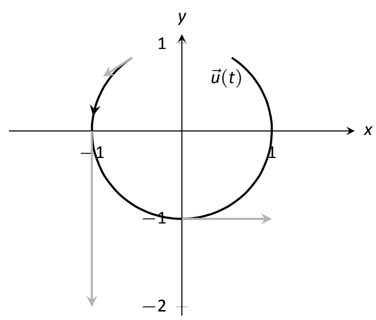


Figure 12.2.7: Graphing some of the derivatives of $\vec{u}(t)$ in Example 12.2.7.

Example 12.2.7 Using derivative properties of vector–valued functions

Let $\vec{r}(t) = \langle t, t^2 - 1\rangle$ and let $\vec{u}(t)$ be the unit vector that points in the direction of $\vec{r}(t)$.

1. Graph $\vec{r}(t)$ and $\vec{u}(t)$ on the same axes, on $[-2, 2]$.
2. Find $\vec{u}\,'(t)$ and sketch $\vec{u}\,'(-2)$, $\vec{u}\,'(-1)$ and $\vec{u}\,'(0)$. Sketch each with initial point the corresponding point on the graph of $\vec{u}$.

SOLUTION

1. To form the unit vector that points in the direction of $\vec{r}$, we need to divide $\vec{r}(t)$ by its magnitude.

$$\parallel \vec{r}(t) \parallel = \sqrt{t^2 + (t^2-1)^2} \quad \Rightarrow \quad \vec{u}(t) = \frac{1}{\sqrt{t^2+(t^2-1)^2}} \left\langle t, t^2-1\right\rangle.$$

   $\vec{r}(t)$ and $\vec{u}(t)$ are graphed in Figure 12.2.6. Note how the graph of $\vec{u}(t)$ forms part of a circle; this must be the case, as the length of $\vec{u}(t)$ is 1 for all t.

2. To compute $\vec{u}\,'(t)$, we use Theorem 12.2.4, writing

$$\vec{u}(t) = f(t)\vec{r}(t), \quad \text{where} \quad f(t) = \frac{1}{\sqrt{t^2+(t^2-1)^2}} = \left(t^2+(t^2-1)^2\right)^{-1/2}.$$

   (We *could* write

$$\vec{u}(t) = \left\langle \frac{t}{\sqrt{t^2+(t^2-1)^2}}, \frac{t^2-1}{\sqrt{t^2+(t^2-1)^2}}\right\rangle$$

   and then take the derivative. It is a matter of preference; this latter method requires two applications of the Quotient Rule where our method uses the Product and Chain Rules.)

   We find $f'(t)$ using the Chain Rule:

$$\begin{aligned} f'(t) &= -\frac{1}{2}\left(t^2+(t^2-1)^2\right)^{-3/2}\left(2t+2(t^2-1)(2t)\right) \\ &= -\frac{2t(2t^2-1)}{2\left(\sqrt{t^2+(t^2-1)^2}\right)^3} \end{aligned}$$

   We now find $\vec{u}\,'(t)$ using part 3 of Theorem 12.2.4:

$$\begin{aligned} \vec{u}\,'(t) &= f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t) \\ &= -\frac{2t(2t^2-1)}{2\left(\sqrt{t^2+(t^2-1)^2}\right)^3}\left\langle t, t^2-1\right\rangle + \frac{1}{\sqrt{t^2+(t^2-1)^2}}\langle 1, 2t\rangle. \end{aligned}$$

   This is admittedly very "messy;" such is usually the case when we deal with unit vectors. We can use this formula to compute $\vec{u}\,'(-2)$, $\vec{u}\,'(-1)$ and $\vec{u}\,'(0)$:

$$\begin{aligned} \vec{u}\,'(-2) &= \left\langle -\frac{15}{13\sqrt{13}}, -\frac{10}{13\sqrt{13}}\right\rangle \approx \langle -0.320, -0.213\rangle \\ \vec{u}\,'(-1) &= \langle 0, -2\rangle \\ \vec{u}\,'(0) &= \langle 1, 0\rangle \end{aligned}$$

Each of these is sketched in Figure 12.2.7. Note how the length of the vector gives an indication of how quickly the circle is being traced at that point. When $t=-2$, the circle is being drawn relatively slow; when $t=-1$, the circle is being traced much more quickly.

It is a basic geometric fact that a line tangent to a circle at a point P is perpendicular to the line passing through the center of the circle and P. This is illustrated in Figure 12.2.7; each tangent vector is perpendicular to the line that passes through its initial point and the center of the circle. Since the center of the circle is the origin, we can state this another way: $\vec{u}\,'(t)$ is orthogonal to $\vec{u}(t)$.

Recall that the dot product serves as a test for orthogonality: if $\vec{u}\cdot\vec{v}=0$, then $\vec{u}$ is orthogonal to $\vec{v}$. Thus in the above example, $\vec{u}(t)\cdot\vec{u}\,'(t)=0$.

This is true of any vector–valued function that has a constant length, that is, that traces out part of a circle. It has important implications later on, so we state it as a theorem (and leave its formal proof as an Exercise.)

Theorem 12.2.5 Vector–Valued Functions of Constant Length

Let $\vec{r}(t)$ be a vector–valued function of constant length that is differentiable on an open interval I. That is, $\|\,\vec{r}(t)\,\|=c$ for all t in I (equivalently, $\vec{r}(t)\cdot\vec{r}(t)=c^2$ for all t in I). Then $\vec{r}(t)\cdot\vec{r}\,'(t)=0$ for all t in I.

Integration

Before formally defining integrals of vector–valued functions, consider the following equation that our calculus experience tells us *should* be true:

$$\int_a^b \vec{r}\,'(t)\,dt=\vec{r}(b)-\vec{r}(a).$$

That is, the integral of a rate of change function should give total change. In the context of vector–valued functions, this total change is displacement. The above equation *is* true; we now develop the theory to show why.

We can define antiderivatives and the indefinite integral of vector–valued functions in the same manner we defined indefinite integrals in Definition 5.1.1. However, we cannot define the definite integral of a vector–valued function as we did in Definition 5.2.1. That definition was based on the signed area between a function $y=f(x)$ and the x-axis. An area–based definition will not be useful in the context of vector–valued functions. Instead, we define the definite integral of a vector–valued function in a manner similar to that of Theorem 5.3.2, utilizing Riemann sums.

Definition 12.2.6 Antiderivatives, Indefinite and Definite Integrals of Vector–Valued Functions

Let $\vec{r}(t)$ be a continuous vector–valued function on $[a,b]$. An **antiderivative** of $\vec{r}(t)$ is a function $\vec{R}(t)$ such that $\vec{R}'(t)=\vec{r}(t)$.

The set of all antiderivatives of $\vec{r}(t)$ is the **indefinite integral of** $\vec{r}(t)$, denoted by

$$\int \vec{r}(t)\ dt.$$

The definite integral of $\vec{r}(t)$ on $[a,b]$ is

$$\int_a^b \vec{r}(t)\ dt = \lim_{||\Delta t||\to 0}\sum_{i=1}^{n}\vec{r}(c_i)\Delta t_i,$$

where Δt_i is the length of the i^{th} subinterval of a partition of $[a,b]$, $||\Delta t||$ is the length of the largest subinterval in the partition, and c_i is any value in the i^{th} subinterval of the partition.

It is probably difficult to infer meaning from the definition of the definite integral. The important thing to realize from the definition is that it is built upon limits, which we can evaluate component–wise.

The following theorem simplifies the computation of definite integrals; the rest of this section and the following section will give meaning and application to these integrals.

Theorem 12.2.6 Indefinite and Definite Integrals of Vector–Valued Functions

Let $\vec{r}(t)=\langle f(t),g(t)\rangle$ be a vector–valued function in $\mathbb{R}^2$ that are continuous on $[a,b]$.

1. $\displaystyle\int \vec{r}(t)\ dt = \left\langle \int f(t)\ dt, \int g(t)\ dt\right\rangle$
2. $\displaystyle\int_a^b \vec{r}(t)\ dt = \left\langle \int_a^b f(t)\ dt, \int_a^b g(t)\ dt\right\rangle$

A similar statement holds for vector–valued functions in $\mathbb{R}^3$.

Example 12.2.8 Evaluating a definite integral of a vector–valued function

Let $\vec{r}(t)=\left\langle e^{2t},\sin t\right\rangle$. Evaluate $\displaystyle\int_0^1 \vec{r}(t)\ dt$.

Solution We follow Theorem 12.2.6.

$$\begin{aligned}\int_0^1 \vec{r}(t)\,dt &= \int_0^1 \left\langle e^{2t}, \sin t\right\rangle\,dt\\ &= \left\langle \int_0^1 e^{2t}\,dt\,, \int_0^1 \sin t\,dt\right\rangle\\ &= \left\langle \frac{1}{2}e^{2t}\Big|_0^1\,, -\cos t\Big|_0^1\right\rangle\\ &= \left\langle \frac{1}{2}(e^2-1)\,, -\cos(1)+1\right\rangle\\ &\approx \langle 3.19, 0.460\rangle\,.\end{aligned}$$

Example 12.2.9 Solving an initial value problem

Let $\vec{r}\,''(t) = \langle 2, \cos t, 12t\rangle$. Find $\vec{r}(t)$, where $\vec{r}(0) = \langle -7, -1, 2\rangle$ and $\vec{r}\,'(0) = \langle 5, 3, 0\rangle$.

Solution Knowing $\vec{r}\,''(t) = \langle 2, \cos t, 12t\rangle$, we find $\vec{r}\,'(t)$ by evaluating the indefinite integral.

$$\begin{aligned}\int \vec{r}\,''(t)\,dt &= \left\langle \int 2\,dt\,, \int \cos t\,dt\,, \int 12t\,dt\right\rangle\\ &= \left\langle 2t + C_1, \sin t + C_2, 6t^2 + C_3\right\rangle\\ &= \left\langle 2t, \sin t, 6t^2\right\rangle + \langle C_1, C_2, C_3\rangle\\ &= \left\langle 2t, \sin t, 6t^2\right\rangle + \vec{C}.\end{aligned}$$

Note how each indefinite integral creates its own constant which we collect as one constant vector $\vec{C}$. Knowing $\vec{r}\,'(0) = \langle 5, 3, 0\rangle$ allows us to solve for $\vec{C}$:

$$\begin{aligned}\vec{r}\,'(t) &= \left\langle 2t, \sin t, 6t^2\right\rangle + \vec{C}\\ \vec{r}\,'(0) &= \langle 0, 0, 0\rangle + \vec{C}\\ \langle 5, 3, 0\rangle &= \vec{C}.\end{aligned}$$

So $\vec{r}\,'(t) = \left\langle 2t, \sin t, 6t^2\right\rangle + \langle 5, 3, 0\rangle = \left\langle 2t+5, \sin t + 3, 6t^2\right\rangle$. To find $\vec{r}(t)$, we integrate once more.

$$\begin{aligned}\int \vec{r}\,'(t)\,dt &= \left\langle \int 2t+5\,dt, \int \sin t + 3\,dt, \int 6t^2\,dt\right\rangle\\ &= \left\langle t^2 + 5t, -\cos t + 3t, 2t^3\right\rangle + \vec{C}.\end{aligned}$$

With $\vec{r}(0) = \langle -7, -1, 2\rangle$, we solve for $\vec{C}$:

$$\begin{aligned}\vec{r}(t) &= \left\langle t^2 + 5t, -\cos t + 3t, 2t^3\right\rangle + \vec{C}\\ \vec{r}(0) &= \langle 0, -1, 0\rangle + \vec{C}\\ \langle -7, -1, 2\rangle &= \langle 0, -1, 0\rangle + \vec{C}\\ \langle -7, 0, 2\rangle &= \vec{C}.\end{aligned}$$

So $\vec{r}(t) = \left\langle t^2 + 5t, -\cos t + 3t, 2t^3\right\rangle + \langle -7, 0, 2\rangle = \left\langle t^2 + 5t - 7, -\cos t + 3t, 2t^3 + 2\right\rangle$.

What does the integration of a vector–valued function *mean*? There are many applications, but none as direct as "the area under the curve" that we used in understanding the integral of a real–valued function.

A key understanding for us comes from considering the integral of a derivative:

$$\int_a^b \vec{r}\,'(t)\,dt = \vec{r}(t)\Big|_a^b = \vec{r}(b) - \vec{r}(a).$$

Integrating a *rate of change* function gives *displacement*.

Noting that vector–valued functions are closely related to parametric equations, we can describe the arc length of the graph of a vector–valued function as an integral. Given parametric equations $x = f(t)$, $y = g(t)$, the arc length on $[a, b]$ of the graph is

$$\text{Arc Length} = \int_a^b \sqrt{f\,'(t)^2 + g\,'(t)^2}\,dt,$$

as stated in Theorem 9.3.1 in Section 9.3. If $\vec{r}(t) = \langle f(t), g(t)\rangle$, note that $\sqrt{f\,'(t)^2 + g\,'(t)^2} = \parallel \vec{r}\,'(t) \parallel$. Therefore we can express the arc length of the graph of a vector–valued function as an integral of the magnitude of its derivative.

Theorem 12.2.7 Arc Length of a Vector–Valued Function

Let $\vec{r}(t)$ be a vector–valued function where $\vec{r}\,'(t)$ is continuous on $[a, b]$. The arc length L of the graph of $\vec{r}(t)$ is

$$L = \int_a^b \parallel \vec{r}\,'(t) \parallel\,dt.$$

Note that we are actually integrating a scalar–function here, not a vector–valued function.

The next section takes what we have established thus far and applies it to objects in motion. We will let $\vec{r}(t)$ describe the path of an object in the plane or in space and will discover the information provided by $\vec{r}\,'(t)$ and $\vec{r}\,''(t)$.

Exercises 12.2

Terms and Concepts

1. Limits, derivatives and integrals of vector–valued functions are all evaluated ________–wise.

2. The definite integral of a rate of change function gives ________.

3. Why is it generally not useful to graph both $\vec{r}(t)$ and $\vec{r}\,'(t)$ on the same axes?

4. Theorem 12.2.4 contains three product rules. What are the three different types of products used in these rules?

Problems

In Exercises 5 – 8, evaluate the given limit.

5. $\displaystyle\lim_{t\to 5} \left\langle 2t+1, 3t^2-1, \sin t\right\rangle$

6. $\displaystyle\lim_{t\to 3} \left\langle e^t, \frac{t^2-9}{t+3}\right\rangle$

7. $\displaystyle\lim_{t\to 0} \left\langle \frac{t}{\sin t}, (1+t)^{\frac{1}{t}}\right\rangle$

8. $\displaystyle\lim_{h\to 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h}$, where $\vec{r}(t) = \left\langle t^2, t, 1\right\rangle$.

In Exercises 9 – 10, identify the interval(s) on which $\vec{r}(t)$ is continuous.

9. $\vec{r}(t) = \left\langle t^2, 1/t\right\rangle$

10. $\vec{r}(t) = \left\langle \cos t, e^t, \ln t\right\rangle$

In Exercises 11 – 16, find the derivative of the given function.

11. $\vec{r}(t) = \left\langle \cos t, e^t, \ln t\right\rangle$

12. $\vec{r}(t) = \left\langle \frac{1}{t}, \frac{2t-1}{3t+1}, \tan t\right\rangle$

13. $\vec{r}(t) = (t^2)\left\langle \sin t, 2t+5\right\rangle$

14. $r(t) = \left\langle t^2+1, t-1\right\rangle \cdot \left\langle \sin t, 2t+5\right\rangle$

15. $\vec{r}(t) = \left\langle t^2+1, t-1, 1\right\rangle \times \left\langle \sin t, 2t+5, 1\right\rangle$

16. $\vec{r}(t) = \left\langle \cosh t, \sinh t\right\rangle$

In Exercises 17 – 20, find $\vec{r}\,'(t)$. Sketch $\vec{r}(t)$ and $\vec{r}\,'(1)$, with the initial point of $\vec{r}\,'(1)$ at $\vec{r}(1)$.

17. $\vec{r}(t) = \left\langle t^2+t, t^2-t\right\rangle$

18. $\vec{r}(t) = \left\langle t^2-2t+2, t^3-3t^2+2t\right\rangle$

19. $\vec{r}(t) = \left\langle t^2+1, t^3-t\right\rangle$

20. $\vec{r}(t) = \left\langle t^2-4t+5, t^3-6t^2+11t-6\right\rangle$

In Exercises 21 – 24, give the equation of the line tangent to the graph of $\vec{r}(t)$ at the given t value.

21. $\vec{r}(t) = \left\langle t^2+t, t^2-t\right\rangle$ at $t=1$.

22. $\vec{r}(t) = \left\langle 3\cos t, \sin t\right\rangle$ at $t=\pi/4$.

23. $\vec{r}(t) = \left\langle 3\cos t, 3\sin t, t\right\rangle$ at $t=\pi$.

24. $\vec{r}(t) = \left\langle e^t, \tan t, t\right\rangle$ at $t=0$.

In Exercises 25 – 28, find the value(s) of t for which $\vec{r}(t)$ is not smooth.

25. $\vec{r}(t) = \left\langle \cos t, \sin t - t\right\rangle$

26. $\vec{r}(t) = \left\langle t^2-2t+1, t^3+t^2-5t+3\right\rangle$

27. $\vec{r}(t) = \left\langle \cos t - \sin t, \sin t - \cos t, \cos(4t)\right\rangle$

28. $\vec{r}(t) = \left\langle t^3-3t+2, -\cos(\pi t), \sin^2(\pi t)\right\rangle$

Exercises 29 – 32 ask you to verify parts of Theorem 12.2.4. In each let $f(t) = t^3$, $\vec{r}(t) = \left\langle t^2, t-1, 1\right\rangle$ and $\vec{s}(t) = \left\langle \sin t, e^t, t\right\rangle$. Compute the various derivatives as indicated.

29. Simplify $f(t)\vec{r}(t)$, then find its derivative; show this is the same as $f\,'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$.

30. Simplify $\vec{r}(t)\cdot\vec{s}(t)$, then find its derivative; show this is the same as $\vec{r}\,'(t)\cdot\vec{s}(t) + \vec{r}(t)\cdot\vec{s}\,'(t)$.

31. Simplify $\vec{r}(t)\times\vec{s}(t)$, then find its derivative; show this is the same as $\vec{r}\,'(t)\times\vec{s}(t) + \vec{r}(t)\times\vec{s}\,'(t)$.

32. Simplify $\vec{r}\big(f(t)\big)$, then find its derivative; show this is the same as $\vec{r}\,'\big(f(t)\big)f\,'(t)$.

In Exercises 33 – 36, evaluate the given definite or indefinite integral.

33. $\displaystyle\int \left\langle t^3, \cos t, te^t\right\rangle\,dt$

34. $\displaystyle\int \left\langle \frac{1}{1+t^2}, \sec^2 t\right\rangle\,dt$

35. $\displaystyle\int_0^{\pi} \left\langle -\sin t, \cos t\right\rangle\,dt$

36. $\displaystyle\int_{-2}^{2} \left\langle 2t+1, 2t-1\right\rangle\,dt$

In Exercises 37 – 40, solve the given initial value problems.

37. Find $\vec{r}(t)$, given that $\vec{r}\,'(t) = \langle t, \sin t\rangle$ and $\vec{r}(0) = \langle 2, 2\rangle$.

38. Find $\vec{r}(t)$, given that $\vec{r}\,'(t) = \langle 1/(t+1), \tan t\rangle$ and $\vec{r}(0) = \langle 1, 2\rangle$.

39. Find $\vec{r}(t)$, given that $\vec{r}\,''(t) = \left\langle t^2, t, 1\right\rangle$, $\vec{r}\,'(0) = \langle 1, 2, 3\rangle$ and $\vec{r}(0) = \langle 4, 5, 6\rangle$.

40. Find $\vec{r}(t)$, given that $\vec{r}\,''(t) = \left\langle \cos t, \sin t, e^t\right\rangle$, $\vec{r}\,'(0) = \langle 0, 0, 0\rangle$ and $\vec{r}(0) = \langle 0, 0, 0\rangle$.

In Exercises 41 – 44 , find the arc length of $\vec{r}(t)$ on the indicated interval.

41. $\vec{r}(t) = \langle 2\cos t, 2\sin t, 3t\rangle$ on $[0, 2\pi]$.

42. $\vec{r}(t) = \langle 5\cos t, 3\sin t, 4\sin t\rangle$ on $[0, 2\pi]$.

43. $\vec{r}(t) = \left\langle t^3, t^2, t^3\right\rangle$ on $[0, 1]$.

44. $\vec{r}(t) = \left\langle e^{-t}\cos t, e^{-t}\sin t\right\rangle$ on $[0, 1]$.

45. Prove Theorem 12.2.5; that is, show if $\vec{r}(t)$ has constant length and is differentiable, then $\vec{r}(t)\cdot\vec{r}\,'(t) = 0$. (Hint: use the Product Rule to compute $\frac{d}{dt}\big(\vec{r}(t)\cdot\vec{r}(t)\big)$.)

12.3 The Calculus of Motion

A common use of vector–valued functions is to describe the motion of an object in the plane or in space. A **position function** $\vec{r}(t)$ gives the position of an object at **time** t. This section explores how derivatives and integrals are used to study the motion described by such a function.

Definition 12.3.1 Velocity, Speed and Acceleration

Let $\vec{r}(t)$ be a position function in $\mathbb{R}^2$ or $\mathbb{R}^3$.

1. **Velocity**, denoted $\vec{v}(t)$, is the instantaneous rate of position change; that is, $\vec{v}(t) = \vec{r}\,'(t)$.
2. **Speed** is the magnitude of velocity, $\| \vec{v}(t) \|$.
3. **Acceleration**, denoted $\vec{a}(t)$, is the instantaneous rate of velocity change; that is, $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$.

Example 12.3.1 Finding velocity and acceleration

An object is moving with position function $\vec{r}(t) = \left\langle t^2 - t, t^2 + t\right\rangle$, $-3 \leq t \leq 3$, where distances are measured in feet and time is measured in seconds.

1. Find $\vec{v}(t)$ and $\vec{a}(t)$.
2. Sketch $\vec{r}(t)$; plot $\vec{v}(-1)$, $\vec{a}(-1)$, $\vec{v}(1)$ and $\vec{a}(1)$, each with their initial point at their corresponding point on the graph of $\vec{r}(t)$.
3. When is the object's speed minimized?

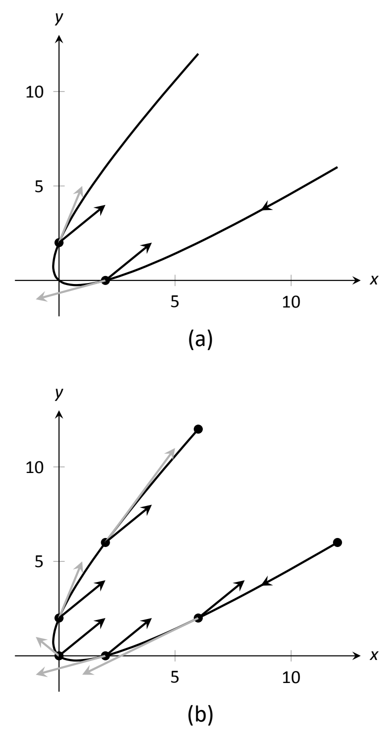


Figure 12.3.1: Graphing the position, velocity and acceleration of an object in Example 12.3.1.

Solution

1. Taking derivatives, we find

$$\vec{v}(t) = \vec{r}\,'(t) = \langle 2t - 1, 2t + 1\rangle \quad \text{and} \quad \vec{a}(t) = \vec{r}\,''(t) = \langle 2, 2\rangle\,.$$

Note that acceleration is constant.

2. $\vec{v}(-1) = \langle -3, -1\rangle$, $\vec{a}(-1) = \langle 2, 2\rangle$; $\vec{v}(1) = \langle 1, 3\rangle$, $\vec{a}(1) = \langle 2, 2\rangle$. These are plotted with $\vec{r}(t)$ in Figure 12.3.1(a).

We can think of acceleration as "pulling" the velocity vector in a certain direction. At $t = -1$, the velocity vector points down and to the left; at $t = 1$, the velocity vector has been pulled in the $\langle 2, 2\rangle$ direction and is now pointing up and to the right. In Figure 12.3.1(b) we plot more velocity/acceleration vectors, making more clear the effect acceleration has on velocity.

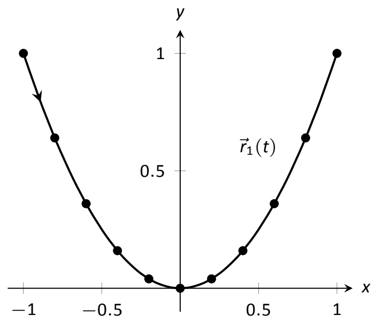
Since $\vec{a}(t)$ is constant in this example, as t grows large $\vec{v}(t)$ becomes almost parallel to $\vec{a}(t)$. For instance, when $t = 10$, $\vec{v}(10) = \langle 19, 21\rangle$, which is nearly parallel to $\langle 2, 2\rangle$.

3. The object's speed is given by

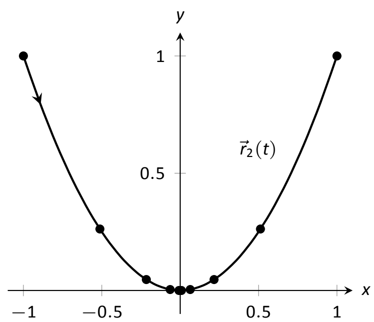
$$\| \vec{v}(t) \| = \sqrt{(2t-1)^2 + (2t+1)^2} = \sqrt{8t^2 + 2}.$$

To find the minimal speed, we could apply calculus techniques (such as set the derivative equal to 0 and solve for t, etc.) but we can find it by inspection. Inside the square root we have a quadratic which is minimized when $t = 0$. Thus the speed is minimized at $t = 0$, with a speed of $\sqrt{2}$ ft/s.

The graph in Figure 12.3.1(b) also implies speed is minimized here. The filled dots on the graph are located at integer values of t between -3 and 3. Dots that are far apart imply the object travelled a far distance in 1 second, indicating high speed; dots that are close together imply the object did not travel far in 1 second, indicating a low speed. The dots are closest together near $t = 0$, implying the speed is minimized near that value.



(a)



(b)

Figure 12.3.3: Comparing the positions of Objects 1 and 2 in Example 12.3.2.

Example 12.3.2 Analyzing Motion

Two objects follow an identical path at different rates on $[-1, 1]$. The position function for Object 1 is $\vec{r}_1(t) = \langle t, t^2\rangle$; the position function for Object 2 is $\vec{r}_2(t) = \langle t^3, t^6\rangle$, where distances are measured in feet and time is measured in seconds. Compare the velocity, speed and acceleration of the two objects on the path.

SOLUTION We begin by computing the velocity and acceleration function for each object:

$$\vec{v}_1(t) = \langle 1, 2t\rangle \qquad \vec{v}_2(t) = \langle 3t^2, 6t^5\rangle$$
$$\vec{a}_1(t) = \langle 0, 2\rangle \qquad \vec{a}_2(t) = \langle 6t, 30t^4\rangle$$

We immediately see that Object 1 has constant acceleration, whereas Object 2 does not.

At $t = -1$, we have $\vec{v}_1(-1) = \langle 1, -2\rangle$ and $\vec{v}_2(-1) = \langle 3, -6\rangle$; the velocity of Object 2 is three times that of Object 1 and so it follows that the speed of Object 2 is three times that of Object 1 ($3\sqrt{5}$ ft/s compared to $\sqrt{5}$ ft/s.)

At $t = 0$, the velocity of Object 1 is $\vec{v}(1) = \langle 1, 0\rangle$ and the velocity of Object 2 is $\vec{0}$! This tells us that Object 2 comes to a complete stop at $t = 0$.

In Figure 12.3.2, we see the velocity and acceleration vectors for Object 1 plotted for $t = -1, -1/2, 0, 1/2$ and $t = 1$. Note again how the constant acceleration vector seems to "pull" the velocity vector from pointing down, right to up, right. We could plot the analogous picture for Object 2, but the velocity and acceleration vectors are rather large ($\vec{a}_2(-1) = \langle -6, 30\rangle$!)

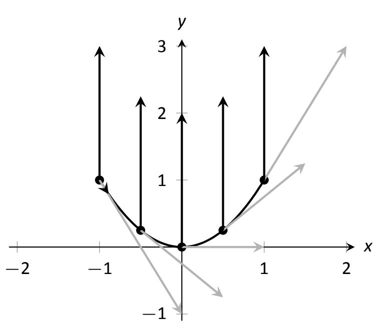


Figure 12.3.2: Plotting velocity and acceleration vectors for Object 1 in Example 12.3.2.

Instead, we simply plot the locations of Object 1 and 2 on intervals of $1/5^{\text{th}}$ of a second, shown in Figure 12.3.3(a) and (b). Note how the x-values of Object 1 increase at a steady rate. This is because the x-component of $\vec{a}(t)$ is 0; there is no acceleration in the x-component. The dots are not evenly spaced; the object is moving faster near $t = -1$ and $t = 1$ than near $t = 0$.

In part (b) of the Figure, we see the points plotted for Object 2. Note the large change in position from $t = -1$ to $t = -0.8$; the object starts moving very quickly. However, it slows considerably at it approaches the origin, and comes to a complete stop at $t = 0$. While it looks like there are 3 points near the origin, there are in reality 5 points there.

Since the objects begin and end at the same location, they have the same displacement. Since they begin and end at the same time, with the same displacement, they have the same average rate of change (i.e, they have the same average velocity). Since they follow the same path, they have the same distance travelled. Even though these three measurements are the same, the objects obviously travel the path in very different ways.

Example 12.3.3 Analyzing the motion of a whirling ball on a string
A young boy whirls a ball, attached to a string, above his head in a counter-clockwise circle. The ball follows a circular path and makes 2 revolutions per second. The string has length 2 ft.

1. Find the position function $\vec{r}(t)$ that describes this situation.
2. Find the acceleration of the ball and give a physical interpretation of it.
3. A tree stands 10 ft in front of the boy. At what t-values should the boy release the string so that the ball hits the tree?

SOLUTION

1. The ball whirls in a circle. Since the string is 2 ft long, the radius of the circle is 2. The position function $\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle$ describes a circle with radius 2, centred at the origin, but makes a full revolution every 2π seconds, not two revolutions per second. We modify the period of the trigonometric functions to be 1/2 by multiplying t by 4π. The final position function is thus
$$\vec{r}(t) = \langle 2\cos(4\pi t), 2\sin(4\pi t)\rangle\,.$$
(Plot this for $0 \leq t \leq 1/2$ to verify that one revolution is made in 1/2 a second.)

2. To find $\vec{a}(t)$, we take the derivative of $\vec{r}(t)$ twice.
$$\begin{aligned}\vec{v}(t) = \vec{r}\,'(t) &= \langle -8\pi\sin(4\pi t), 8\pi\cos(4\pi t)\rangle\\ \vec{a}(t) = \vec{r}\,''(t) &= \left\langle -32\pi^2\cos(4\pi t), -32\pi^2\sin(4\pi t)\right\rangle\\ &= -32\pi^2\left\langle\cos(4\pi t), \sin(4\pi t)\right\rangle\,.\end{aligned}$$
Note how $\vec{a}(t)$ is parallel to $\vec{r}(t)$, but has a different magnitude and points in the opposite direction. Why is this?

   Recall the classic physics equation, "Force = mass × acceleration." A force acting on a mass induces acceleration (i.e., the mass moves); acceleration acting on a mass induces a force (gravity gives our mass a *weight*). Thus force and acceleration are closely related. A moving ball "wants" to travel in a straight line. Why does the ball in our example move in a circle? It is attached to the boy's hand by a string. The string applies a force to the ball, affecting it's motion: the string *accelerates* the ball. This is not acceleration in the sense of "it travels faster;" rather, this acceleration is changing the velocity of the ball. In what direction is this force/acceleration being applied? In the direction of the string, towards the boy's hand.

   The magnitude of the acceleration is related to the speed at which the ball is travelling. A ball whirling quickly is rapidly changing direction/velocity. When velocity is changing rapidly, the acceleration must be "large."

3. When the boy releases the string, the string no longer applies a force to the ball, meaning acceleration is $\vec{0}$ and the ball can now move in a straight line in the direction of $\vec{v}(t)$.

   Let $t = t_0$ be the time when the boy lets go of the string. The ball will be at $\vec{r}(t_0)$, travelling in the direction of $\vec{v}(t_0)$. We want to find t_0 so that this line contains the point $(0, 10)$ (since the tree is 10 ft directly in front of the boy).

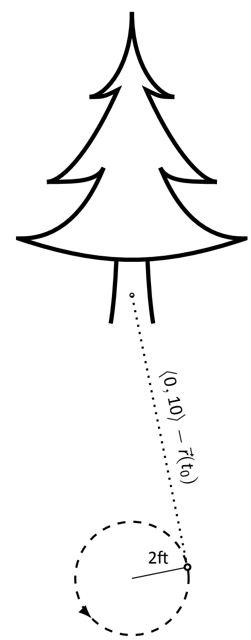


Figure 12.3.4: Modelling the flight of a ball in Example 12.3.3.

There are many ways to find this time value. We choose one that is relatively simple computationally. As shown in Figure 12.3.4, the vector from the release point to the tree is $\langle 0, 10\rangle - \vec{r}(t_0)$. This line segment is tangent to the circle, which means it is also perpendicular to $\vec{r}(t_0)$ itself, so their dot product is 0.

$$\begin{aligned}
\vec{r}(t_0)\cdot\big(\langle 0,10\rangle - \vec{r}(t_0)\big) &= 0\\
\langle 2\cos(4\pi t_0), 2\sin(4\pi t_0)\rangle\cdot\langle -2\cos(4\pi t_0), 10-2\sin(4\pi t_0)\rangle &= 0\\
-4\cos^2(4\pi t_0)+20\sin(4\pi t_0)-4\sin^2(4\pi t_0) &= 0\\
20\sin(4\pi t_0)-4 &= 0\\
\sin(4\pi t_0) &= 1/5\\
4\pi t_0 &= \sin^{-1}(1/5)\\
4\pi t_0 &\approx 0.2+2\pi n,
\end{aligned}$$

where n is an integer. Solving for t_0 we have:

$$t_0 \approx 0.016 + n/2$$

This is a wonderful formula. Every 1/2 second after $t = 0.016$s the boy can release the string (since the ball makes 2 revolutions per second, he has two chances each second to release the ball).

Example 12.3.4 Analyzing motion in space

An object moves in a spiral with position function $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$, where distances are measured in metres and time is in minutes. Describe the object's speed and acceleration at time t.

Solution With $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$, we have:

$$\begin{aligned}
\vec{v}(t) &= \langle -\sin t, \cos t, 1\rangle \quad \text{and}\\
\vec{a}(t) &= \langle -\cos t, -\sin t, 0\rangle .
\end{aligned}$$

The speed of the object is $\parallel \vec{v}(t) \parallel = \sqrt{(-\sin t)^2+\cos^2 t+1} = \sqrt{2}$ m/min; it moves at a constant speed. Note that the object does not accelerate in the z-direction, but rather moves up at a constant rate of 1 m/min.

The objects in Examples 12.3.3 and 12.3.4 travelled at a constant speed. That is, $\parallel \vec{v}(t) \parallel = c$ for some constant c. Recall Theorem 12.2.5, which states that if a vector–valued function $\vec{r}(t)$ has constant length, then $\vec{r}(t)$ is perpendicular to its derivative: $\vec{r}(t)\cdot\vec{r}\,'(t) = 0$. In these examples, the velocity function has constant length, therefore we can conclude that the velocity is perpendicular to the acceleration: $\vec{v}(t)\cdot\vec{a}(t) = 0$. A quick check verifies this.

There is an intuitive understanding of this. If acceleration is parallel to velocity, then it is only affecting the object's speed; it does not change the direction of travel. (For example, consider a dropped stone. Acceleration and velocity are parallel – straight down – and the direction of velocity never changes, though speed does increase.) If acceleration is not perpendicular to velocity, then there is some acceleration in the direction of travel, influencing the speed. If speed is constant, then acceleration must be orthogonal to velocity, as it then only affects direction, and not speed.

> **Key Idea 12.3.1 Objects With Constant Speed**
>
> If an object moves with constant speed, then its velocity and acceleration vectors are orthogonal. That is, $\vec{v}(t)\cdot\vec{a}(t)=0$.

Projectile Motion

An important application of vector–valued position functions is *projectile motion*: the motion of objects under only the influence of gravity. We will measure time in seconds, and distances will either be in metres or feet. We will show that we can completely describe the path of such an object knowing its initial position and initial velocity (i.e., where it *is* and where it *is going.*)

Suppose an object has initial position $\vec{r}(0)=\langle x_0,y_0\rangle$ and initial velocity $\vec{v}(0)=\langle v_x,v_y\rangle$. It is customary to rewrite $\vec{v}(0)$ in terms of its speed v_0 and direction $\vec{u}$, where $\vec{u}$ is a unit vector. Recall all unit vectors in $\mathbb{R}^2$ can be written as $\langle\cos\theta,\sin\theta\rangle$, where θ is an angle measure counter–clockwise from the x-axis. (We refer to θ as the **angle of elevation.**) Thus $\vec{v}(0)=v_0\left\langle\cos\theta,\sin\theta\right\rangle$.

Since the acceleration of the object is known, namely $\vec{a}(t)=\langle 0,-g\rangle$, where g is the gravitational constant, we can find $\vec{r}(t)$ knowing our two initial conditions. We first find $\vec{v}(t)$:

$$\begin{aligned}\vec{v}(t)&=\int\vec{a}(t)\,dt\\ \vec{v}(t)&=\int\langle 0,-g\rangle\,dt\\ \vec{v}(t)&=\langle 0,-gt\rangle+\vec{C}.\end{aligned}$$

Note: This text uses $g=32$ ft/s^2 when using Imperial units, and $g=9.8$ m/s^2 when using SI units.

Knowing $\vec{v}(0)=v_0\left\langle\cos\theta,\sin\theta\right\rangle$, we have $\vec{C}=v_0\left\langle\cos\theta,\sin\theta\right\rangle$ and so

$$\vec{v}(t)=\Big\langle v_0\cos\theta,-gt+v_0\sin\theta\Big\rangle.$$

We integrate once more to find $\vec{r}(t)$:

$$\begin{aligned}\vec{r}(t)&=\int\vec{v}(t)\,dt\\ \vec{r}(t)&=\int\Big\langle v_0\cos\theta,-gt+v_0\sin\theta\Big\rangle\,dt\\ \vec{r}(t)&=\left\langle\big(v_0\cos\theta\big)t,-\frac{1}{2}gt^2+\big(v_0\sin\theta\big)t\right\rangle+\vec{C}.\end{aligned}$$

Knowing $\vec{r}(0)=\langle x_0,y_0\rangle$, we conclude $\vec{C}=\langle x_0,y_0\rangle$ and

$$\vec{r}(t)=\left\langle\big(v_0\cos\theta\big)t+x_0\,,-\frac{1}{2}gt^2+\big(v_0\sin\theta\big)t+y_0\,\right\rangle.$$

Key Idea 12.3.2 Projectile Motion

The position function of a projectile propelled from an initial position of $\vec{r}_0 = \langle x_0, y_0 \rangle$, with initial speed v_0, with angle of elevation θ and neglecting all accelerations but gravity is

$$\vec{r}(t) = \left\langle \big(v_0 \cos\theta\big)t + x_0\ ,\ -\frac{1}{2}gt^2 + \big(v_0 \sin\theta\big)t + y_0 \right\rangle.$$

Letting $\vec{v}_0 = v_0 \langle \cos\theta, \sin\theta \rangle$, $\vec{r}(t)$ can be written as

$$\vec{r}(t) = \left\langle 0, -\frac{1}{2}gt^2 \right\rangle + \vec{v}_0 t + \vec{r}_0.$$

We demonstrate how to use this position function in the next two examples.

Example 12.3.5 Projectile Motion

Sydney shoots her Red Ryder® bb gun across level ground from an elevation of 4 ft, where the barrel of the gun makes a 5° angle with the horizontal. Find how far the bb travels before landing, assuming the bb is fired at the advertised rate of 350 ft/s and ignoring air resistance.

Solution A direct application of Key Idea 12.3.2 gives

$$\begin{aligned}\vec{r}(t) &= \big\langle (350\cos 5^\circ)t, -16t^2 + (350\sin 5^\circ)t + 4 \big\rangle \\ &\approx \big\langle 346.67t, -16t^2 + 30.50t + 4 \big\rangle,\end{aligned}$$

where we set her initial position to be $\langle 0, 4 \rangle$. We need to find *when* the bb lands, then we can find *where*. We accomplish this by setting the y-component equal to 0 and solving for t:

$$\begin{aligned}-16t^2 + 30.50t + 4 &= 0 \\ t &= \frac{-30.50 \pm \sqrt{30.50^2 - 4(-16)(4)}}{-32} \\ t &\approx 2.03 \text{ s}.\end{aligned}$$

(We discarded a negative solution that resulted from our quadratic equation.)

We have found that the bb lands 2.03 s after firing; with $t = 2.03$, we find the x-component of our position function is $346.67(2.03) = 703.74$ ft. The bb lands about 704 feet away.

Example 12.3.6 Projectile Motion

Alex holds his sister's bb gun at a height of 3 ft and wants to shoot a target that is 6 ft above the ground, 25 ft away. At what angle should he hold the gun to hit his target? (We still assume the muzzle velocity is 350 ft/s.)

Solution The position function for the path of Alex's bb is

$$\vec{r}(t) = \big\langle (350\cos\theta)t, -16t^2 + (350\sin\theta)t + 3 \big\rangle.$$

We need to find θ so that $\vec{r}(t) = \langle 25, 6 \rangle$ for some value of t. That is, we want to find θ and t such that

$$(350\cos\theta)t = 25 \quad \text{and} \quad -16t^2 + (350\sin\theta)t + 3 = 6.$$

This is not trivial (though not "hard"). We start by solving each equation for $\cos\theta$ and $\sin\theta$, respectively.

$$\cos\theta = \frac{25}{350t} \quad \text{and} \quad \sin\theta = \frac{3+16t^2}{350t}.$$

Using the Pythagorean Identity $\cos^2\theta + \sin^2\theta = 1$, we have

$$\left(\frac{25}{350t}\right)^2 + \left(\frac{3+16t^2}{350t}\right)^2 = 1$$

Multiply both sides by $(350t)^2$:

$$\begin{aligned} 25^2 + (3+16t^2)^2 &= 350^2t^2 \\ 256t^4 - 122,404t^2 + 634 &= 0. \end{aligned}$$

This is a quadratic *in* t^2. That is, we can apply the quadratic formula to find t^2, then solve for t itself.

$$\begin{aligned} t^2 &= \frac{122,404 \pm \sqrt{122,404^2 - 4(256)(634)}}{512} \\ t^2 &= 0.0052,\ 478.135 \\ t &= \pm 0.072,\ \pm 21.866 \end{aligned}$$

Clearly the negative t values do not fit our context, so we have $t = 0.072$ and $t = 21.866$. Using $\cos\theta = 25/(350t)$, we can solve for θ:

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{25}{350 \cdot 0.072}\right) \quad \text{and} \quad \cos^{-1}\left(\frac{25}{350 \cdot 21.866}\right) \\ \theta &= 7.03^\circ \quad \text{and} \quad 89.8^\circ. \end{aligned}$$

Alex has two choices of angle. He can hold the rifle at an angle of about 7° with the horizontal and hit his target 0.07 s after firing, or he can hold his rifle almost straight up, with an angle of 89.8°, where he'll hit his target about 22 s later. The first option is clearly the option he should choose.

Distance Travelled

Consider a driver who sets her cruise–control to 60 mph, and travels at this speed for an hour. We can ask:

1. How far did the driver travel?
2. How far from her starting position is the driver?

The first is easy to answer: she travelled 60 miles. The second is impossible to answer with the given information. We do not know if she travelled in a straight line, on an oval racetrack, or along a slowly–winding highway.

This highlights an important fact: to compute distance travelled, we need only to know the speed, given by $\parallel \vec{v}(t) \parallel$.

Theorem 12.3.1 Distance Travelled

Let $\vec{v}(t)$ be a velocity function for a moving object. The distance travelled by the object on $[a,b]$ is:

$$\text{distance travelled} = \int_a^b \parallel \vec{v}(t) \parallel \, dt.$$

Note that this is just a restatement of Theorem 12.2.7: arc length is the same as distance travelled, just viewed in a different context.

Example 12.3.7 Distance Travelled, Displacement, and Average Speed

A particle moves in space with position function $\vec{r}(t) = \langle t, t^2, \sin(\pi t) \rangle$ on $[-2, 2]$, where t is measured in seconds and distances are in metres. Find:

1. The distance travelled by the particle on $[-2, 2]$.
2. The displacement of the particle on $[-2, 2]$.
3. The particle's average speed.

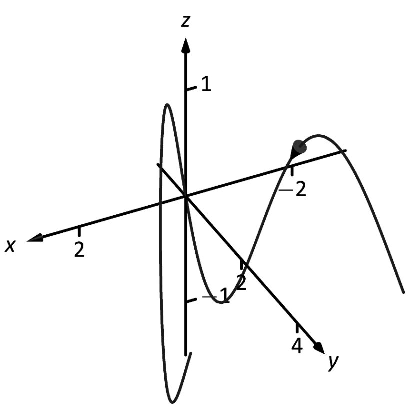


Figure 12.3.5: The path of the particle in Example 12.3.7.

Solution

1. We use Theorem 12.3.1 to establish the integral:

$$\begin{aligned}\text{distance travelled} &= \int_{-2}^{2} \| \vec{v}(t) \| \, dt \\ &= \int_{-2}^{2} \sqrt{1 + (2t)^2 + \pi^2 \cos^2(\pi t)} \, dt.\end{aligned}$$

This cannot be solved in terms of elementary functions so we turn to numerical integration, finding the distance to be 12.88 m.

2. The displacement is the vector

$$\vec{r}(2) - \vec{r}(-2) = \langle 2, 4, 0 \rangle - \langle -2, 4, 0 \rangle = \langle 4, 0, 0 \rangle .$$

That is, the particle ends with an x-value increased by 4 and with y- and z-values the same (see Figure 12.3.5).

3. We found above that the particle travelled 12.88 m over 4 seconds. We can compute average speed by dividing: 12.88/4 = 3.22 m/s.

We should also consider Definition 5.4.1 of Section 5.4, which says that the average value of a function f on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$. In our context, the average value of the speed is

$$\text{average speed} = \frac{1}{2-(-2)} \int_{-2}^{2} \| \vec{v}(t) \| \, dt \approx \frac{1}{4} 12.88 = 3.22 \text{ m/s}.$$

Note how the physical context of a particle travelling gives meaning to a more abstract concept learned earlier.

In Definition 5.4.1 of Chapter 5 we defined the average value of a function $f(x)$ on $[a, b]$ to be

$$\frac{1}{b-a} \int_a^b f(x) \, dx.$$

Note how in Example 12.3.7 we computed the average speed as

$$\frac{\text{distance travelled}}{\text{travel time}} = \frac{1}{2-(-2)} \int_{-2}^{2} \| \vec{v}(t) \| \, dt;$$

that is, we just found the average value of $\| \vec{v}(t) \|$ on $[-2, 2]$.

Likewise, given position function $\vec{r}(t)$, the average velocity on $[a, b]$ is

$$\frac{\text{displacement}}{\text{travel time}} = \frac{1}{b-a} \int_a^b \vec{r}\,'(t) \, dt = \frac{\vec{r}(b) - \vec{r}(a)}{b-a};$$

that is, it is the average value of $\vec{r}\,'(t)$, or $\vec{v}(t)$, on $[a, b]$.

Key Idea 12.3.3 Average Speed, Average Velocity

Let $\vec{r}(t)$ be a differentiable position function on $[a, b]$.

The **average speed** is:

$$\frac{\text{distance travelled}}{\text{travel time}} = \frac{\int_a^b \parallel \vec{v}(t) \parallel \, dt}{b-a} = \frac{1}{b-a}\int_a^b \parallel \vec{v}(t) \parallel \, dt.$$

The **average velocity** is:

$$\frac{\text{displacement}}{\text{travel time}} = \frac{\int_a^b \vec{r}\,'(t)\, dt}{b-a} = \frac{1}{b-a}\int_a^b \vec{r}\,'(t)\, dt.$$

The next two sections investigate more properties of the graphs of vector–valued functions and we'll apply these new ideas to what we just learned about motion.

Exercises 12.3

Terms and Concepts

1. How is *velocity* different from *speed*?

2. What is the difference between *displacement* and *distance travelled*?

3. What is the difference between *average velocity* and *average speed*?

4. *Distance travelled* is the same as ________ ________, just viewed in a different context.

5. Describe a scenario where an object's average speed is a large number, but the magnitude of the average velocity is not a large number.

6. Explain why it is not possible to have an average velocity with a large magnitude but a small average speed.

Problems

In Exercises 7 – 10 , a position function $\vec{r}(t)$ is given. Find $\vec{v}(t)$ and $\vec{a}(t)$.

7. $\vec{r}(t) = \langle 2t+1, 5t-2, 7\rangle$

8. $\vec{r}(t) = \left\langle 3t^2-2t+1, -t^2+t+14\right\rangle$

9. $\vec{r}(t) = \langle \cos t, \sin t\rangle$

10. $\vec{r}(t) = \langle t/10, -\cos t, \sin t\rangle$

In Exercises 11 – 14 , a position function $\vec{r}(t)$ is given. Sketch $\vec{r}(t)$ on the indicated interval. Find $\vec{v}(t)$ and $\vec{a}(t)$, then add $\vec{v}(t_0)$ and $\vec{a}(t_0)$ to your sketch, with their initial points at $\vec{r}(t_0)$, for the given value of t_0.

11. $\vec{r}(t) = \langle t, \sin t\rangle$ on $[0, \pi/2]$; $t_0 = \pi/4$

12. $\vec{r}(t) = \left\langle t^2, \sin t^2\right\rangle$ on $[0, \pi/2]$; $t_0 = \sqrt{\pi/4}$

13. $\vec{r}(t) = \left\langle t^2+t, -t^2+2t\right\rangle$ on $[-2, 2]$; $t_0 = 1$

14. $\vec{r}(t) = \left\langle \dfrac{2t+3}{t^2+1}, t^2\right\rangle$ on $[-1, 1]$; $t_0 = 0$

In Exercises 15 – 24 , a position function $\vec{r}(t)$ of an object is given. Find the speed of the object in terms of t, and find where the speed is minimized/maximized on the indicated interval.

15. $\vec{r}(t) = \left\langle t^2, t\right\rangle$ on $[-1, 1]$

16. $\vec{r}(t) = \left\langle t^2, t^2-t^3\right\rangle$ on $[-1, 1]$

17. $\vec{r}(t) = \langle 5\cos t, 5\sin t\rangle$ on $[0, 2\pi]$

18. $\vec{r}(t) = \langle 2\cos t, 5\sin t\rangle$ on $[0, 2\pi]$

19. $\vec{r}(t) = \langle \sec t, \tan t\rangle$ on $[0, \pi/4]$

20. $\vec{r}(t) = \langle t+\cos t, 1-\sin t\rangle$ on $[0, 2\pi]$

21. $\vec{r}(t) = \langle 12t, 5\cos t, 5\sin t\rangle$ on $[0, 4\pi]$

22. $\vec{r}(t) = \left\langle t^2-t, t^2+t, t\right\rangle$ on $[0, 1]$

23. $\vec{r}(t) = \left\langle t, t^2, \sqrt{1-t^2}\right\rangle$ on $[-1, 1]$

24. **Projectile Motion:** $\vec{r}(t) = \left\langle (v_0\cos\theta)t, -\dfrac{1}{2}gt^2 + (v_0\sin\theta)t\right\rangle$ on $\left[0, \dfrac{2v_0\sin\theta}{g}\right]$

In Exercises 25 – 28 , position functions $\vec{r}_1(t)$ and $\vec{r}_2(s)$ for two objects are given that follow the same path on the respective intervals.

(a) Show that the positions are the same at the indicated t_0 and s_0 values; i.e., show $\vec{r}_1(t_0) = \vec{r}_2(s_0)$.

(b) Find the velocity, speed and acceleration of the two objects at t_0 and s_0, respectively.

25. $\vec{r}_1(t) = \left\langle t, t^2\right\rangle$ on $[0, 1]$; $t_0 = 1$
$\vec{r}_2(s) = \left\langle s^2, s^4\right\rangle$ on $[0, 1]$; $s_0 = 1$

26. $\vec{r}_1(t) = \langle 3\cos t, 3\sin t\rangle$ on $[0, 2\pi]$; $t_0 = \pi/2$
$\vec{r}_2(s) = \langle 3\cos(4s), 3\sin(4s)\rangle$ on $[0, \pi/2]$; $s_0 = \pi/8$

27. $\vec{r}_1(t) = \langle 3t, 2t\rangle$ on $[0, 2]$; $t_0 = 2$
$\vec{r}_2(s) = \langle 6s-6, 4s-4\rangle$ on $[1, 2]$; $s_0 = 2$

28. $\vec{r}_1(t) = \left\langle t, \sqrt{t}\right\rangle$ on $[0, 1]$; $t_0 = 1$
$\vec{r}_2(s) = \left\langle \sin t, \sqrt{\sin t}\right\rangle$ on $[0, \pi/2]$; $s_0 = \pi/2$

In Exercises 29 – 32 , find the position function of an object given its acceleration and initial velocity and position.

29. $\vec{a}(t) = \langle 2, 3\rangle$; $\vec{v}(0) = \langle 1, 2\rangle$, $\vec{r}(0) = \langle 5, -2\rangle$

30. $\vec{a}(t) = \langle 2, 3\rangle$; $\vec{v}(1) = \langle 1, 2\rangle$, $\vec{r}(1) = \langle 5, -2\rangle$

31. $\vec{a}(t) = \langle \cos t, -\sin t\rangle$; $\vec{v}(0) = \langle 0, 1\rangle$, $\vec{r}(0) = \langle 0, 0\rangle$

32. $\vec{a}(t) = \langle 0, -32\rangle$; $\vec{v}(0) = \langle 10, 50\rangle$, $\vec{r}(0) = \langle 0, 0\rangle$

In Exercises 33 – 36 , find the displacement, distance travelled, average velocity and average speed of the described object on the given interval.

33. An object with position function $\vec{r}(t) = \langle 2\cos t, 2\sin t, 3t\rangle$, where distances are measured in feet and time is in seconds, on $[0, 2\pi]$.

34. An object with position function $\vec{r}(t) = \langle 5\cos t, -5\sin t\rangle$, where distances are measured in feet and time is in seconds, on $[0, \pi]$.

35. An object with velocity function $\vec{v}(t) = \langle \cos t, \sin t\rangle$, where distances are measured in feet and time is in seconds, on $[0, 2\pi]$.

36. An object with velocity function $\vec{v}(t) = \langle 1, 2, -1\rangle$, where distances are measured in feet and time is in seconds, on $[0, 10]$.

Exercises 37 – 42 ask you to solve a variety of problems based on the principles of projectile motion.

37. A boy whirls a ball, attached to a 3ft string, above his head in a counter–clockwise circle. The ball makes 2 revolutions per second.

    At what t-values should the boy release the string so that the ball heads directly for a tree standing 10ft in front of him?

38. David faces Goliath with only a stone in a 3ft sling, which he whirls above his head at 4 revolutions per second. They stand 20ft apart.

    (a) At what t-values must David release the stone in his sling in order to hit Goliath?

    (b) What is the speed at which the stone is travelling when released?

    (c) Assume David releases the stone from a height of 6ft and Goliath's forehead is 9ft above the ground. What angle of elevation must David apply to the stone to hit Goliath's head?

39. A hunter aims at a deer which is 40 yards away. Her crossbow is at a height of 5ft, and she aims for a spot on the deer 4ft above the ground. The crossbow fires her arrows at 300ft/s.

    (a) At what angle of elevation should she hold the crossbow to hit her target?

    (b) If the deer is moving perpendicularly to her line of sight at a rate of 20mph, by approximately how much should she lead the deer in order to hit it in the desired location?

40. A baseball player hits a ball at 100mph, with an initial height of 3ft and an angle of elevation of 20°, at Boston's Fenway Park. The ball flies towards the famed "Green Monster," a wall 37ft high located 310ft from home plate.

    (a) Show that as hit, the ball hits the wall.

    (b) Show that if the angle of elevation is 21°, the ball clears the Green Monster.

41. A Cessna flies at 1000ft at 150mph and drops a box of supplies to the professor (and his wife) on an island. Ignoring wind resistance, how far horizontally will the supplies travel before they land?

42. A football quarterback throws a pass from a height of 6ft, intending to hit his receiver 20yds away at a height of 5ft.

    (a) If the ball is thrown at a rate of 50mph, what angle of elevation is needed to hit his intended target?

    (b) If the ball is thrown at with an angle of elevation of 8°, what initial ball speed is needed to hit his target?

12.4 Unit Tangent and Normal Vectors

Unit Tangent Vector

Given a smooth vector–valued function $\vec{r}(t)$, we defined in Definition 12.2.4 that any vector parallel to $\vec{r}\,'(t_0)$ is *tangent* to the graph of $\vec{r}(t)$ at $t = t_0$. It is often useful to consider just the *direction* of $\vec{r}\,'(t)$ and not its magnitude. Therefore we are interested in the unit vector in the direction of $\vec{r}\,'(t)$. This leads to a definition.

Definition 12.4.1 Unit Tangent Vector

Let $\vec{r}(t)$ be a smooth function on an open interval I. The unit tangent vector $\vec{T}(t)$ is

$$\vec{T}(t) = \frac{1}{\|\,\vec{r}\,'(t)\,\|}\vec{r}\,'(t).$$

Example 12.4.1 Computing the unit tangent vector

Let $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$. Find $\vec{T}(t)$ and compute $\vec{T}(0)$ and $\vec{T}(1)$.

Solution We apply Definition 12.4.1 to find $\vec{T}(t)$.

$$\begin{aligned}\vec{T}(t) &= \frac{1}{\|\,\vec{r}\,'(t)\,\|}\vec{r}\,'(t)\\ &= \frac{1}{\sqrt{\big(-3\sin t\big)^2 + \big(3\cos t\big)^2 + 4^2}}\langle -3\sin t, 3\cos t, 4\rangle\\ &= \left\langle -\frac{3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5}\right\rangle.\end{aligned}$$

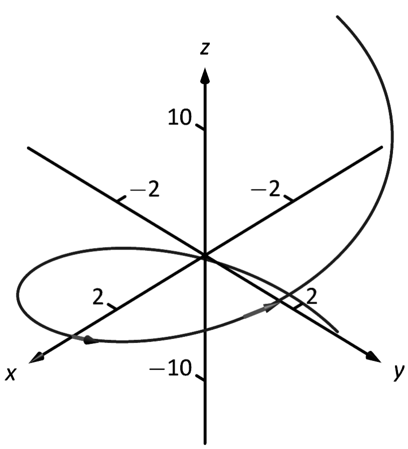


Figure 12.4.1: Plotting unit tangent vectors in Example 12.4.1.

We can now easily compute $\vec{T}(0)$ and $\vec{T}(1)$:

$$\vec{T}(0) = \left\langle 0, \frac{3}{5}, \frac{4}{5}\right\rangle; \quad \vec{T}(1) = \left\langle -\frac{3}{5}\sin 1, \frac{3}{5}\cos 1, \frac{4}{5}\right\rangle \approx \langle -0.505, 0.324, 0.8\rangle.$$

These are plotted in Figure 12.4.1 with their initial points at $\vec{r}(0)$ and $\vec{r}(1)$, respectively. (They look rather "short" since they are only length 1.)

The unit tangent vector $\vec{T}(t)$ always has a magnitude of 1, though it is sometimes easy to doubt that is true. We can help solidify this thought in our minds by computing $\|\,\vec{T}(1)\,\|$:

$$\|\,\vec{T}(1)\,\| \approx \sqrt{(-0.505)^2 + 0.324^2 + 0.8^2} = 1.000001.$$

We have rounded in our computation of $\vec{T}(1)$, so we don't get 1 exactly. We leave it to the reader to use the exact representation of $\vec{T}(1)$ to verify it has length 1.

In many ways, the previous example was "too nice." It turned out that $\vec{r}\,'(t)$ was always of length 5. In the next example the length of $\vec{r}\,'(t)$ is variable, leaving us with a formula that is not as clean.

Example 12.4.2 Computing the unit tangent vector

Let $\vec{r}(t) = \left\langle t^2 - t, t^2 + t \right\rangle$. Find $\vec{T}(t)$ and compute $\vec{T}(0)$ and $\vec{T}(1)$.

SOLUTION We find $\vec{r}\,'(t) = \langle 2t-1, 2t+1 \rangle$, and

$$\parallel \vec{r}\,'(t) \parallel = \sqrt{(2t-1)^2 + (2t+1)^2} = \sqrt{8t^2+2}.$$

Therefore

$$\vec{T}(t) = \frac{1}{\sqrt{8t^2+2}} \langle 2t-1, 2t+1 \rangle = \left\langle \frac{2t-1}{\sqrt{8t^2+2}}, \frac{2t+1}{\sqrt{8t^2+2}} \right\rangle.$$

When $t = 0$, we have $\vec{T}(0) = \left\langle -1/\sqrt{2}, 1/\sqrt{2} \right\rangle$; when $t = 1$, we have $\vec{T}(1) = \left\langle 1/\sqrt{10}, 3/\sqrt{10} \right\rangle$. We leave it to the reader to verify each of these is a unit vector. They are plotted in Figure 12.4.2

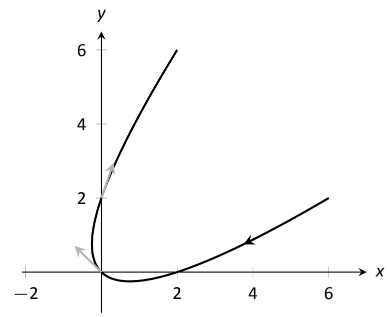


Figure 12.4.2: Plotting unit tangent vectors in Example 12.4.2.

Unit Normal Vector

Just as knowing the direction tangent to a path is important, knowing a direction orthogonal to a path is important. When dealing with real-valued functions, we defined the normal line at a point to the be the line through the point that was perpendicular to the tangent line at that point. We can do a similar thing with vector–valued functions. Given $\vec{r}(t)$ in $\mathbb{R}^2$, we have 2 directions perpendicular to the tangent vector, as shown in Figure 12.4.3. It is good to wonder "Is one of these two directions preferable over the other?"

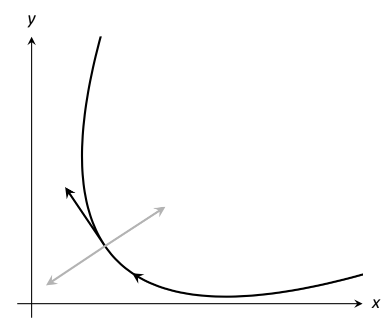


Figure 12.4.3: Given a direction in the plane, there are always two directions orthogonal to it.

Given $\vec{r}(t)$ in $\mathbb{R}^3$, there are infinitely many vectors orthogonal to the tangent vector at a given point. Again, we might wonder "Is one of these infinite choices preferable over the others? Is one of these the 'right' choice?"

The answer in both $\mathbb{R}^2$ and $\mathbb{R}^3$ is "Yes, there is one vector that is not only preferable, it is the 'right' one to choose." Recall Theorem 12.2.5, which states that if $\vec{r}(t)$ has constant length, then $\vec{r}(t)$ is orthogonal to $\vec{r}\,'(t)$ for all t. We know $\vec{T}(t)$, the unit tangent vector, has constant length. Therefore $\vec{T}(t)$ is orthogonal to $\vec{T}\,'(t)$.

We'll see that $\vec{T}\,'(t)$ is more than just a convenient choice of vector that is orthogonal to $\vec{r}\,'(t)$; rather, it is the "right" choice. Since all we care about is the direction, we define this newly found vector to be a unit vector.

Note: $\vec{T}(t)$ is a unit vector, by definition. This *does not* imply that $\vec{T}\,'(t)$ is also a unit vector.

Definition 12.4.2 Unit Normal Vector

Let $\vec{r}(t)$ be a vector–valued function where the unit tangent vector, $\vec{T}(t)$, is smooth on an open interval I. The **unit normal vector** $\vec{N}(t)$ is

$$\vec{N}(t) = \frac{1}{\parallel \vec{T}\,'(t) \parallel} \vec{T}\,'(t).$$

Example 12.4.3 Computing the unit normal vector

Let $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$ as in Example 12.4.1. Sketch both $\vec{T}(\pi/2)$ and $\vec{N}(\pi/2)$ with initial points at $\vec{r}(\pi/2)$.

SOLUTION In Example 12.4.1, we found $\vec{T}(t) = \Big\langle (-3/5)\sin t, (3/5)\cos t, 4/5\Big\rangle$. Therefore

$$\vec{T}\,'(t) = \left\langle -\frac{3}{5}\cos t, -\frac{3}{5}\sin t, 0\right\rangle \quad \text{and} \quad \parallel \vec{T}\,'(t) \parallel = \frac{3}{5}.$$

Thus

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{3/5} = \langle -\cos t, -\sin t, 0\rangle\,.$$

We compute $\vec{T}(\pi/2) = \langle -3/5, 0, 4/5\rangle$ and $\vec{N}(\pi/2) = \langle 0, -1, 0\rangle$. These are sketched in Figure 12.4.4.

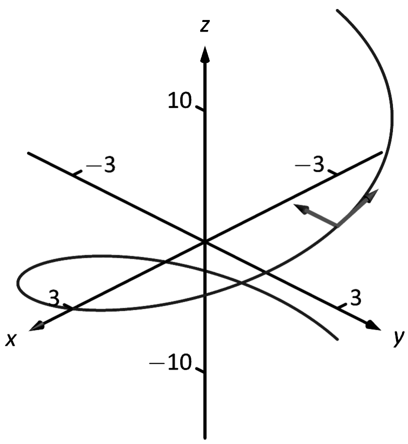


Figure 12.4.4: Plotting unit tangent and normal vectors in Example 12.4.3.

The previous example was once again "too nice." In general, the expression for $\vec{T}(t)$ contains fractions of square–roots, hence the expression of $\vec{T}\,'(t)$ is very messy. We demonstrate this in the next example.

Example 12.4.4 Computing the unit normal vector

Let $\vec{r}(t) = \big\langle t^2 - t, t^2 + t\big\rangle$ as in Example 12.4.2. Find $\vec{N}(t)$ and sketch $\vec{r}(t)$ with the unit tangent and normal vectors at $t = -1, 0$ and 1.

SOLUTION In Example 12.4.2, we found

$$\vec{T}(t) = \left\langle \frac{2t-1}{\sqrt{8t^2+2}}, \frac{2t+1}{\sqrt{8t^2+2}}\right\rangle.$$

Finding $\vec{T}\,'(t)$ requires two applications of the Quotient Rule:

$$\begin{aligned} T\,'(t) &= \left\langle \frac{\sqrt{8t^2+2}(2) - (2t-1)\left(\frac{1}{2}(8t^2+2)^{-1/2}(16t)\right)}{8t^2+2},\right. \\ &\qquad\qquad \left.\frac{\sqrt{8t^2+2}(2) - (2t+1)\left(\frac{1}{2}(8t^2+2)^{-1/2}(16t)\right)}{8t^2+2}\right\rangle \\ &= \left\langle \frac{4(2t+1)}{(8t^2+2)^{3/2}}, \frac{4(1-2t)}{(8t^2+2)^{3/2}}\right\rangle \end{aligned}$$

This is not a unit vector; to find $\vec{N}(t)$, we need to divide $\vec{T}\,'(t)$ by it's magnitude.

$$\begin{aligned} \parallel \vec{T}\,'(t) \parallel &= \sqrt{\frac{16(2t+1)^2}{(8t^2+2)^3} + \frac{16(1-2t)^2}{(8t^2+2)^3}} \\ &= \sqrt{\frac{16(8t^2+2)}{(8t^2+2)^3}} \\ &= \frac{4}{8t^2+2}. \end{aligned}$$

Finally,

$$\vec{N}(t) = \frac{1}{4/(8t^2+2)} \left\langle \frac{4(2t+1)}{(8t^2+2)^{3/2}}, \frac{4(1-2t)}{(8t^2+2)^{3/2}} \right\rangle$$
$$= \left\langle \frac{2t+1}{\sqrt{8t^2+2}}, -\frac{2t-1}{\sqrt{8t^2+2}} \right\rangle.$$

Using this formula for $\vec{N}(t)$, we compute the unit tangent and normal vectors for $t=-1, 0$ and 1 and sketch them in Figure 12.4.5.

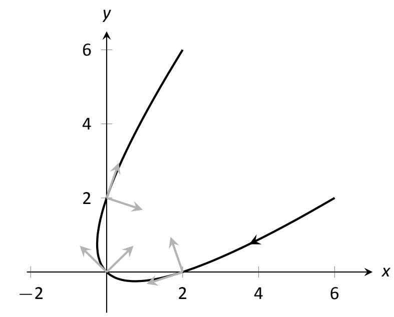


Figure 12.4.5: Plotting unit tangent and normal vectors in Example 12.4.4.

The final result for $\vec{N}(t)$ in Example 12.4.4 is suspiciously similar to $\vec{T}(t)$. There is a clear reason for this. If $\vec{u} = \langle u_1, u_2\rangle$ is a unit vector in $\mathbb{R}^2$, then the *only* unit vectors orthogonal to $\vec{u}$ are $\langle -u_2, u_1\rangle$ and $\langle u_2, -u_1\rangle$. Given $\vec{T}(t)$, we can quickly determine $\vec{N}(t)$ if we know which term to multiply by (-1).

Consider again Figure 12.4.5, where we have plotted some unit tangent and normal vectors. Note how $\vec{N}(t)$ always points "inside" the curve, or to the concave side of the curve. This is not a coincidence; this is true in general. Knowing the direction that $\vec{r}(t)$ "turns" allows us to quickly find $\vec{N}(t)$.

Theorem 12.4.1 Unit Normal Vectors in $\mathbb{R}^2$

Let $\vec{r}(t)$ be a vector–valued function in $\mathbb{R}^2$ where $\vec{T}\,'(t)$ is smooth on an open interval I. Let t_0 be in I and $\vec{T}(t_0) = \langle t_1, t_2\rangle$ Then $\vec{N}(t_0)$ is either

$$\vec{N}(t_0) = \langle -t_2, t_1\rangle \quad \text{or} \quad \vec{N}(t_0) = \langle t_2, -t_1\rangle\,,$$

whichever is the vector that points to the concave side of the graph of $\vec{r}$.

Application to Acceleration

Note: Keep in mind that both a_T and a_N are functions of t; that is, the scalar changes depending on t. It is convention to drop the "(t)" notation from $a_\text{T}(t)$ and simply write a_T.

Let $\vec{r}(t)$ be a position function. It is a fact (stated later in Theorem 12.4.2) that acceleration, $\vec{a}(t)$, lies in the plane defined by $\vec{T}$ and $\vec{N}$. That is, there are scalars a_T and a_N such that

$$\vec{a}(t) = a_\text{T}\vec{T}(t) + a_\text{N}\vec{N}(t).$$

The scalar a_T measures "how much" acceleration is in the direction of travel, that is, it measures the component of acceleration that affects the speed. The scalar a_N measures "how much" acceleration is perpendicular to the direction of travel, that is, it measures the component of acceleration that affects the direction of travel.

We can find a_T using the orthogonal projection of $\vec{a}(t)$ onto $\vec{T}(t)$ (review Definition 11.3.3 in Section 11.3 if needed). Recalling that since $\vec{T}(t)$ is a unit vector, $\vec{T}(t)\cdot\vec{T}(t) = 1$, so we have

$$\text{proj}_{\,\vec{T}(t)}\,\vec{a}(t) = \frac{\vec{a}(t)\cdot\vec{T}(t)}{\vec{T}(t)\cdot\vec{T}(t)}\vec{T}(t) = \underbrace{\left(\vec{a}(t)\cdot\vec{T}(t)\right)}_{a_\text{T}}\vec{T}(t).$$

Thus the amount of $\vec{a}(t)$ in the direction of $\vec{T}(t)$ is $a_\text{T} = \vec{a}(t)\cdot\vec{T}(t)$. The same logic gives $a_\text{N} = \vec{a}(t)\cdot\vec{N}(t)$.

While this is a fine way of computing a_T, there are simpler ways of finding a_N (as finding $\vec{N}$ itself can be complicated). The following theorem gives alternate formulas for a_T and a_N.

Theorem 12.4.2 Acceleration in the Plane Defined by $\vec{T}$ and $\vec{N}$

Let $\vec{r}(t)$ be a position function with acceleration $\vec{a}(t)$ and unit tangent and normal vectors $\vec{T}(t)$ and $\vec{N}(t)$. Then $\vec{a}(t)$ lies in the plane defined by $\vec{T}(t)$ and $\vec{N}(t)$; that is, there exists scalars a_T and a_N such that

$$\vec{a}(t) = a_T\vec{T}(t) + a_N\vec{N}(t).$$

Moreover,

$$a_T = \vec{a}(t)\cdot\vec{T}(t) = \frac{d}{dt}\Big(\parallel \vec{v}(t) \parallel\Big)$$
$$a_N = \vec{a}(t)\cdot\vec{N}(t) = \sqrt{\parallel \vec{a}(t) \parallel^2 - a_T^2} = \frac{\parallel \vec{a}(t)\times\vec{v}(t) \parallel}{\parallel \vec{v}(t) \parallel} = \parallel \vec{v}(t) \parallel \parallel \vec{T}\,'(t) \parallel$$

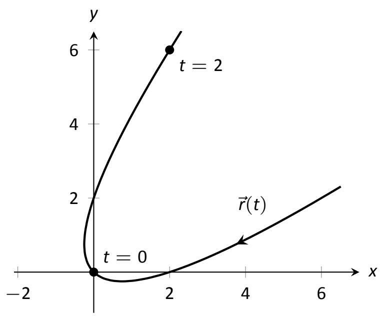


Figure 12.4.6: Graphing $\vec{r}(t)$ in Example 12.4.6.

Note the second formula for a_T: $\frac{d}{dt}\Big(\parallel \vec{v}(t) \parallel\Big)$. This measures the rate of change of speed, which again is the amount of acceleration in the direction of travel.

Example 12.4.5 Computing a_T and a_N

Let $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$ as in Examples 12.4.1 and 12.4.3. Find a_T and a_N.

Solution The previous examples give $\vec{a}(t) = \langle -3\cos t, -3\sin t, 0\rangle$ and

$$\vec{T}(t) = \left\langle -\frac{3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5}\right\rangle \quad\text{and}\quad \vec{N}(t) = \langle -\cos t, -\sin t, 0\rangle\,.$$

We can find a_T and a_N directly with dot products:

$$a_T = \vec{a}(t)\cdot\vec{T}(t) = \frac{9}{5}\cos t\sin t - \frac{9}{5}\cos t\sin t + 0 = 0.$$
$$a_N = \vec{a}(t)\cdot\vec{N}(t) = 3\cos^2 t + 3\sin^2 t + 0 = 3.$$

Thus $\vec{a}(t) = 0\vec{T}(t) + 3\vec{N}(t) = 3\vec{N}(t)$, which is clearly the case.

What is the practical interpretation of these numbers? $a_T = 0$ means the object is moving at a constant speed, and hence all acceleration comes in the form of direction change.

Example 12.4.6 Computing a_T and a_N

Let $\vec{r}(t) = \left\langle t^2 - t, t^2 + t\right\rangle$ as in Examples 12.4.2 and 12.4.4. Find a_T and a_N.

Solution The previous examples give $\vec{a}(t) = \langle 2, 2\rangle$ and

$$\vec{T}(t) = \left\langle \frac{2t-1}{\sqrt{8t^2+2}}, \frac{2t+1}{\sqrt{8t^2+2}}\right\rangle \quad\text{and}\quad \vec{N}(t) = \left\langle \frac{2t+1}{\sqrt{8t^2+2}}, -\frac{2t-1}{\sqrt{8t^2+2}}\right\rangle.$$

While we can compute a_N using $\vec{N}(t)$, we instead demonstrate using another formula from Theorem 12.4.2.

$$a_T = \vec{a}(t)\cdot\vec{T}(t) = \frac{4t-2}{\sqrt{8t^2+2}} + \frac{4t+2}{\sqrt{8t^2+2}} = \frac{8t}{\sqrt{8t^2+2}}.$$

$$a_N = \sqrt{\parallel \vec{a}(t) \parallel^2 - a_T^2} = \sqrt{8 - \left(\frac{8t}{\sqrt{8t^2+2}}\right)^2} = \frac{4}{\sqrt{8t^2+2}}.$$

When $t = 2$, $a_T = \dfrac{16}{\sqrt{34}} \approx 2.74$ and $a_N = \dfrac{4}{\sqrt{34}} \approx 0.69$. We interpret this to mean that at $t = 2$, the particle is accelerating mostly by increasing speed, not by changing direction. As the path near $t = 2$ is relatively straight, this should make intuitive sense. Figure 12.4.6 gives a graph of the path for reference.

Contrast this with $t = 0$, where $a_T = 0$ and $a_N = 4/\sqrt{2} \approx 2.82$. Here the particle's speed is not changing and all acceleration is in the form of direction change.

t	a_T	a_N
0	-16	27.7
1	0	32
2	16	27.7
3	24.2	20.9
4	27.7	16
5	29.4	12.7

Figure 12.4.8: A table of values of a_T and a_N in Example 12.4.7.

Example 12.4.7 Analyzing projectile motion

A ball is thrown from a height of 240 ft with an initial speed of 64 ft/s and an angle of elevation of 30°. Find the position function $\vec{r}(t)$ of the ball and analyze a_T and a_N.

Solution Using Key Idea 12.3.2 of Section 12.3 we form the position function of the ball:

$$\vec{r}(t) = \left\langle \big(64\cos 30^\circ\big)t, -16t^2 + \big(64\sin 30^\circ\big)t + 240 \right\rangle,$$

which we plot in Figure 12.4.7.

From this we find $\vec{v}(t) = \langle 64\cos 30^\circ, -32t + 64\sin 30^\circ\rangle$ and $\vec{a}(t) = \langle 0, -32\rangle$. Computing $\vec{T}(t)$ is not difficult, and with some simplification we find

$$\vec{T}(t) = \left\langle \frac{\sqrt{3}}{\sqrt{t^2-2t+4}}, \frac{1-t}{\sqrt{t^2-2t+4}} \right\rangle.$$

With $\vec{a}(t)$ as simple as it is, finding a_T is also simple:

$$a_T = \vec{a}(t)\cdot\vec{T}(t) = \frac{32t-32}{\sqrt{t^2-2t+4}}.$$

We choose to not find $\vec{N}(t)$ and find a_N through the formula $a_N = \sqrt{\|\,\vec{a}(t)\,\|^2 - a_T^2}$:

$$a_N = \sqrt{32^2 - \left(\frac{32t-32}{\sqrt{t^2-2t+4}}\right)^2} = \frac{32\sqrt{3}}{\sqrt{t^2-2t+4}}.$$

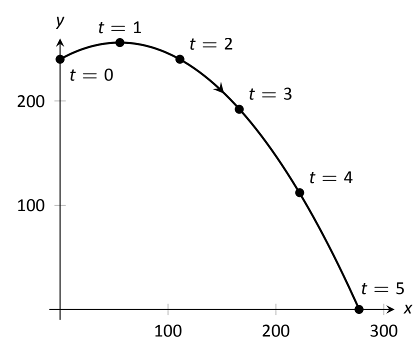


Figure 12.4.7: Plotting the position of a thrown ball, with 1s increments shown.

Figure 12.4.8 gives a table of values of a_T and a_N. When $t = 0$, we see the ball's speed is decreasing; when $t = 1$ the speed of the ball is unchanged. This corresponds to the fact that at $t = 1$ the ball reaches its highest point.

After $t = 1$ we see that a_N is decreasing in value. This is because as the ball falls, it's path becomes straighter and most of the acceleration is in the form of speeding up the ball, and not in changing its direction.

Our understanding of the unit tangent and normal vectors is aiding our understanding of motion. The work in Example 12.4.7 gave quantitative analysis of what we intuitively knew.

The next section provides two more important steps towards this analysis. We currently describe position only in terms of time. In everyday life, though, we often describe position in terms of distance ("The gas station is about 2 miles ahead, on the left."). The *arc length parameter* allows us to reference position in terms of distance travelled.

We also intuitively know that some paths are straighter than others – and some are curvier than others, but we lack a measurement of "curviness." The arc length parameter provides a way for us to compute *curvature*, a quantitative measurement of how curvy a curve is.

Exercises 12.4

Terms and Concepts

1. If $\vec{T}(t)$ is a unit tangent vector, what is $\parallel \vec{T}(t) \parallel$?

2. If $\vec{N}(t)$ is a unit normal vector, what is $\vec{N}(t) \cdot \vec{r}\,'(t)$?

3. The acceleration vector $\vec{a}(t)$ lies in the plane defined by what two vectors?

4. a_T measures how much the acceleration is affecting the ________ of an object.

Problems

In Exercises 5 – 8 , given $\vec{r}(t)$, find $\vec{T}(t)$ and evaluate it at the indicated value of t.

5. $\vec{r}(t) = \langle 2t^2, t^2 - t \rangle, \quad t = 1$

6. $\vec{r}(t) = \langle t, \cos t \rangle, \quad t = \pi/4$

7. $\vec{r}(t) = \langle \cos^3 t, \sin^3 t \rangle, \quad t = \pi/4$

8. $\vec{r}(t) = \langle \cos t, \sin t \rangle, \quad t = \pi$

In Exercises 9 – 12 , find the equation of the line tangent to the curve at the indicated t-value using the unit tangent vector. Note: these are the same problems as in Exercises 5 – 8.

9. $\vec{r}(t) = \langle 2t^2, t^2 - t \rangle, \quad t = 1$

10. $\vec{r}(t) = \langle t, \cos t \rangle, \quad t = \pi/4$

11. $\vec{r}(t) = \langle \cos^3 t, \sin^3 t \rangle, \quad t = \pi/4$

12. $\vec{r}(t) = \langle \cos t, \sin t \rangle, \quad t = \pi$

In Exercises 13 – 16 , find $\vec{N}(t)$ using Definition 12.4.2. Confirm the result using Theorem 12.4.1.

13. $\vec{r}(t) = \langle 3\cos t, 3\sin t \rangle$

14. $\vec{r}(t) = \langle t, t^2 \rangle$

15. $\vec{r}(t) = \langle \cos t, 2\sin t \rangle$

16. $\vec{r}(t) = \langle e^t, e^{-t} \rangle$

In Exercises 17 – 20 , a position function $\vec{r}(t)$ is given along with its unit tangent vector $\vec{T}(t)$ evaluated at $t = a$, for some value of a.

(a) Confirm that $\vec{T}(a)$ is as stated.

(b) Using a graph of $\vec{r}(t)$ and Theorem 12.4.1, find $\vec{N}(a)$.

17. $\vec{r}(t) = \langle 3\cos t, 5\sin t \rangle; \quad \vec{T}(\pi/4) = \left\langle -\frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle.$

18. $\vec{r}(t) = \left\langle t, \frac{1}{t^2+1} \right\rangle; \quad \vec{T}(1) = \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle.$

19. $\vec{r}(t) = (1 + 2\sin t)\,\langle \cos t, \sin t \rangle; \quad \vec{T}(0) = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle.$

20. $\vec{r}(t) = \langle \cos^3 t, \sin^3 t \rangle; \quad \vec{T}(\pi/4) = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle.$

In Exercises 21 – 24 , find $\vec{N}(t)$.

21. $\vec{r}(t) = \langle 4t, 2\sin t, 2\cos t \rangle$

22. $\vec{r}(t) = \langle 5\cos t, 3\sin t, 4\sin t \rangle$

23. $\vec{r}(t) = \langle a\cos t, a\sin t, bt \rangle; \quad a > 0$

24. $\vec{r}(t) = \langle \cos(at), \sin(at), t \rangle$

In Exercises 25 – 30 , find a_T and a_N given $\vec{r}(t)$. Sketch $\vec{r}(t)$ on the indicated interval, and comment on the relative sizes of a_T and a_N at the indicated t values.

25. $\vec{r}(t) = \langle t, t^2 \rangle$ on $[-1, 1]$; consider $t = 0$ and $t = 1$.

26. $\vec{r}(t) = \langle t, 1/t \rangle$ on $(0, 4]$; consider $t = 1$ and $t = 2$.

27. $\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle$ on $[0, 2\pi]$; consider $t = 0$ and $t = \pi/2$.

28. $\vec{r}(t) = \langle \cos(t^2), \sin(t^2) \rangle$ on $(0, 2\pi]$; consider $t = \sqrt{\pi/2}$ and $t = \sqrt{\pi}$.

29. $\vec{r}(t) = \langle a\cos t, a\sin t, bt \rangle$ on $[0, 2\pi]$, where $a, b > 0$; consider $t = 0$ and $t = \pi/2$.

30. $\vec{r}(t) = \langle 5\cos t, 4\sin t, 3\sin t \rangle$ on $[0, 2\pi]$; consider $t = 0$ and $t = \pi/2$.

12.5 The Arc Length Parameter and Curvature

In normal conversation we describe position in terms of both *time* and *distance*. For instance, imagine driving to visit a friend. If she calls and asks where you are, you might answer "I am 20 minutes from your house," or you might say "I am 10 miles from your house." Both answers provide your friend with a general idea of where you are.

Currently, our vector–valued functions have defined points with a parameter t, which we often take to represent time. Consider Figure 12.5.1(a), where $\vec{r}(t) = \langle t^2 - t, t^2 + t\rangle$ is graphed and the points corresponding to $t = 0,\ 1$ and 2 are shown. Note how the arc length between $t = 0$ and $t = 1$ is smaller than the arc length between $t = 1$ and $t = 2$; if the parameter t is time and $\vec{r}$ is position, we can say that the particle travelled faster on $[1, 2]$ than on $[0, 1]$.

Now consider Figure 12.5.1(b), where the same graph is parametrized by a different variable s. Points corresponding to $s = 0$ through $s = 6$ are plotted. The arc length of the graph between each adjacent pair of points is 1. We can view this parameter s as distance; that is, the arc length of the graph from $s = 0$ to $s = 3$ is 3, the arc length from $s = 2$ to $s = 6$ is 4, etc. If one wants to find the point 2.5 units from an initial location (i.e., $s = 0$), one would compute $\vec{r}(2.5)$. This parameter s is very useful, and is called the **arc length parameter**.

How do we find the arc length parameter?

Start with any parametrization of $\vec{r}$. We can compute the arc length of the graph of $\vec{r}$ on the interval $[0, t]$ with

$$\text{arc length } = \int_0^t \parallel \vec{r}\,'(u) \parallel du.$$

We can turn this into a function: as t varies, we find the arc length s from 0 to t. This function is

$$s(t) = \int_0^t \parallel \vec{r}\,'(u) \parallel du. \qquad (12.1)$$

This establishes a relationship between s and t. Knowing this relationship explicitly, we can rewrite $\vec{r}(t)$ as a function of s: $\vec{r}(s)$. We demonstrate this in an example.

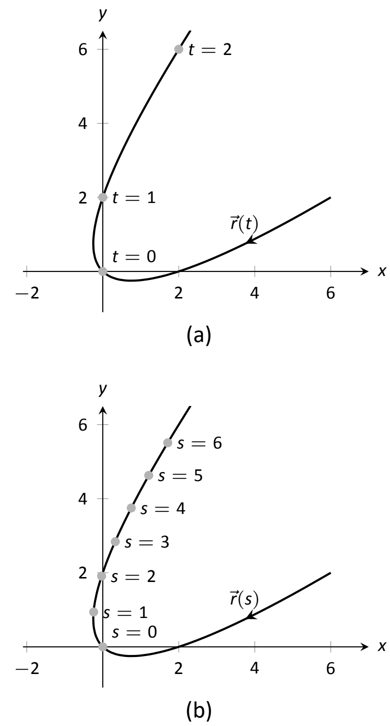


Figure 12.5.1: Introducing the arc length parameter.

Example 12.5.1 Finding the arc length parameter

Let $\vec{r}(t) = \langle 3t - 1, 4t + 2\rangle$. Parametrize $\vec{r}$ with the arc length parameter s.

Solution Using Equation (12.1), we write

$$s(t) = \int_0^t \parallel \vec{r}\,'(u) \parallel du.$$

We can integrate this, explicitly finding a relationship between s and t:

$$\begin{aligned} s(t) &= \int_0^t \parallel \vec{r}\,'(u) \parallel du \\ &= \int_0^t \sqrt{3^2 + 4^2}\, du \\ &= \int_0^t 5\, du \\ &= 5t. \end{aligned}$$

Since $s = 5t$, we can write $t = s/5$ and replace t in $\vec{r}(t)$ with $s/5$:

$$\vec{r}(s) = \langle 3(s/5) - 1, 4(s/5) + 2\rangle = \left\langle \frac{3}{5}s - 1, \frac{4}{5}s + 2\right\rangle.$$

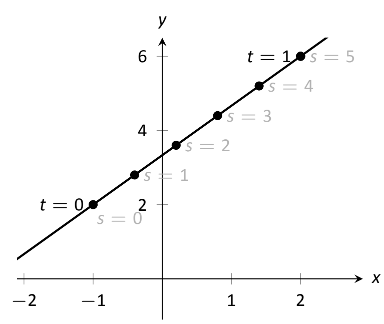


Figure 12.5.2: Graphing $\vec{r}$ in Example 12.5.1 with parameters t and s.

Clearly, as shown in Figure 12.5.2, the graph of $\vec{r}$ is a line, where $t = 0$ corresponds to the point $(-1, 2)$. What point on the line is 2 units away from this initial point? We find it with $\vec{r}(2) = \langle 1/5, 18/5 \rangle$.

Is the point $(1/5, 18/5)$ really 2 units away from $(-1, 2)$? We use the Distance Formula to check:

$$d = \sqrt{\left(\frac{1}{5} - (-1)\right)^2 + \left(\frac{18}{5} - 2\right)^2} = \sqrt{\frac{36}{25} + \frac{64}{25}} = \sqrt{4} = 2.$$

Yes, $\vec{r}(2)$ is indeed 2 units away, in the direction of travel, from the initial point.

Things worked out very nicely in Example 12.5.1; we were able to establish directly that $s = 5t$. Usually, the arc length parameter is much more difficult to describe in terms of t, a result of integrating a square–root. There are a number of things that we can learn about the arc length parameter from Equation (12.1), though, that are incredibly useful.

First, take the derivative of s with respect to t. The Fundamental Theorem of Calculus (see Theorem 5.4.1) states that

$$\frac{ds}{dt} = s'(t) = \| \vec{r}\,'(t) \|. \tag{12.2}$$

Letting t represent time and $\vec{r}(t)$ represent position, we see that the rate of change of s with respect to t is speed; that is, the rate of change of "distance travelled" is speed, which should match our intuition.

The Chain Rule states that

$$\begin{aligned} \frac{d\vec{r}}{dt} &= \frac{d\vec{r}}{ds} \cdot \frac{ds}{dt} \\ \vec{r}\,'(t) &= \vec{r}\,'(s) \cdot \| \vec{r}\,'(t) \|. \end{aligned}$$

Solving for $\vec{r}\,'(s)$, we have

$$\vec{r}\,'(s) = \frac{\vec{r}\,'(t)}{\| \vec{r}\,'(t) \|} = \vec{T}(t), \tag{12.3}$$

where $\vec{T}(t)$ is the unit tangent vector. Equation 12.3 is often misinterpreted, as one is tempted to think it states $\vec{r}\,'(t) = \vec{T}(t)$, but there is a big difference between $\vec{r}\,'(s)$ and $\vec{r}\,'(t)$. The key to take from it is that $\vec{r}\,'(s)$ is a unit vector. In fact, the following theorem states that this characterizes the arc length parameter.

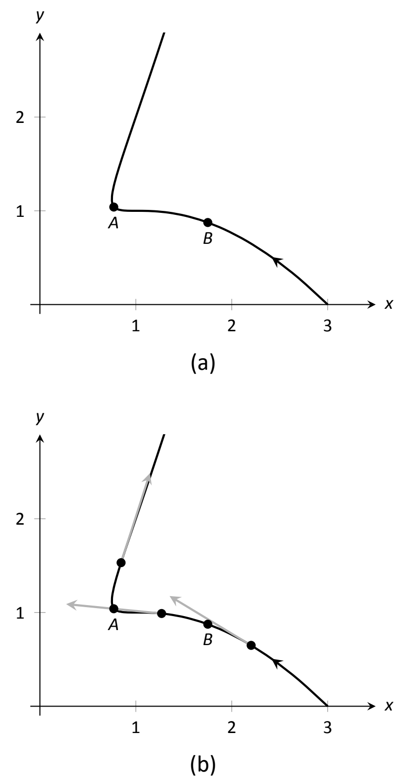


Figure 12.5.3: Establishing the concept of curvature.

Theorem 12.5.1 Arc Length Parameter

Let $\vec{r}(s)$ be a vector–valued function. The parameter s is the arc length parameter if, and only if, $\| \vec{r}\,'(s) \| = 1$.

Curvature

Consider points A and B on the curve graphed in Figure 12.5.3(a). One can readily argue that the curve curves more sharply at A than at B. It is useful to use a number to describe how sharply the curve bends; that number is the **curvature** of the curve.

We derive this number in the following way. Consider Figure 12.5.3(b), where unit tangent vectors are graphed around points A and B. Notice how the direction of the unit tangent vector changes quite a bit near A, whereas it does not change as much around B. This leads to an important concept: measuring the rate of change of the unit tangent vector with respect to arc length gives us a measurement of curvature.

Definition 12.5.1 Curvature

Let $\vec{r}(s)$ be a vector–valued function where s is the arc length parameter. The curvature κ of the graph of $\vec{r}(s)$ is

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\| = \| \vec{T}\,'(s) \| .$$

If $\vec{r}(s)$ is parametrized by the arc length parameter, then

$$\vec{T}(s) = \frac{\vec{r}\,'(s)}{\| \vec{r}\,'(s) \|} \quad \text{and} \quad \vec{N}(s) = \frac{\vec{T}\,'(s)}{\| \vec{T}\,'(s) \|}.$$

Having defined $\| \vec{T}\,'(s) \| = \kappa$, we can rewrite the second equation as

$$\vec{T}\,'(s) = \kappa\vec{N}(s). \tag{12.4}$$

We already knew that $\vec{T}\,'(s)$ is in the same direction as $\vec{N}(s)$; that is, we can think of $\vec{T}(s)$ as being "pulled" in the direction of $\vec{N}(s)$. How "hard" is it being pulled? By a factor of κ. When the curvature is large, $\vec{T}(s)$ is being "pulled hard" and the direction of $\vec{T}(s)$ changes rapidly. When κ is small, $T(s)$ is not being pulled hard and hence its direction is not changing rapidly.

We use Definition 12.5.1 to find the curvature of the line in Example 12.5.1.

Example 12.5.2 Finding the curvature of a line

Use Definition 12.5.1 to find the curvature of $\vec{r}(t) = \langle 3t - 1, 4t + 2 \rangle$.

Solution In Example 12.5.1, we found that the arc length parameter was defined by $s = 5t$, so $\vec{r}(s) = \langle 3s/5 - 1, 4s/5 + 2 \rangle$ parametrized $\vec{r}$ with the arc length parameter. To find κ, we need to find $\vec{T}\,'(s)$.

$$\begin{aligned} \vec{T}(s) &= \vec{r}\,'(s) \quad \text{(recall this is a unit vector)} \\ &= \langle 3/5, 4/5 \rangle . \end{aligned}$$

Therefore

$$\vec{T}\,'(s) = \langle 0, 0 \rangle$$

and

$$\kappa = \| \vec{T}\,'(s) \| = 0.$$

It probably comes as no surprise that the curvature of a line is 0. (How "curvy" is a line? It is not curvy at all.)

While the definition of curvature is a beautiful mathematical concept, it is nearly impossible to use most of the time; writing $\vec{r}$ in terms of the arc length parameter is generally very hard. Fortunately, there are other methods of calculating this value that are much easier. There is a tradeoff: the definition is "easy" to understand though hard to compute, whereas these other formulas are easy to compute though it may be hard to understand why they work.

Theorem 12.5.2 Formulas for Curvature

Let C be a smooth curve in the plane or in space.

1. If C is defined by $y = f(x)$, then
$$\kappa = \frac{|f''(x)|}{\Big(1 + \big(f'(x)\big)^2\Big)^{3/2}}.$$

2. If C is defined as a vector–valued function in the plane, $\vec{r}(t) = \langle x(t), y(t)\rangle$, then
$$\kappa = \frac{|x'y'' - x''y'|}{\big((x')^2 + (y')^2\big)^{3/2}}.$$

3. If C is defined in space by a vector–valued function $\vec{r}(t)$, then
$$\kappa = \frac{\| \vec{T}\,'(t) \|}{\| \vec{r}\,'(t) \|} = \frac{\| \vec{r}\,'(t) \times \vec{r}\,''(t) \|}{\| \vec{r}\,'(t) \|^3} = \frac{\vec{a}(t) \cdot \vec{N}(t)}{\| \vec{v}(t) \|^2}.$$

We practice using these formulas.

Example 12.5.3 Finding the curvature of a circle

Find the curvature of a circle with radius r, defined by $\vec{c}(t) = \langle r\cos t, r\sin t\rangle$.

Solution Before we start, we should expect the curvature of a circle to be constant, and not dependent on t. (Why?)

We compute κ using the second part of Theorem 12.5.2.

$$\begin{aligned}\kappa &= \frac{|(-r\sin t)(-r\sin t) - (-r\cos t)(r\cos t)|}{\big((-r\sin t)^2 + (r\cos t)^2\big)^{3/2}}\\ &= \frac{r^2(\sin^2 t + \cos^2 t)}{\big(r^2(\sin^2 t + \cos^2 t)\big)^{3/2}}\\ &= \frac{r^2}{r^3} = \frac{1}{r}.\end{aligned}$$

We have found that a circle with radius r has curvature $\kappa = 1/r$.

Example 12.5.3 gives a great result. Before this example, if we were told "The curve has a curvature of 5 at point A," we would have no idea what this really meant. Is 5 "big" – does is correspond to a really sharp turn, or a not-so-sharp turn? Now we can think of 5 in terms of a circle with radius 1/5. Knowing the units (inches vs. miles, for instance) allows us to determine how sharply the curve is curving.

Let a point P on a smooth curve C be given, and let κ be the curvature of the curve at P. A circle that:

- passes through P,
- lies on the concave side of C,

- has a common tangent line as C at P and
- has radius $r = 1/\kappa$ (hence has curvature κ)

is the **osculating circle**, or **circle of curvature**, to C at P, and r is the **radius of curvature**. Figure 12.5.4 shows the graph of the curve seen earlier in Figure 12.5.3 and its osculating circles at A and B. A sharp turn corresponds to a circle with a small radius; a gradual turn corresponds to a circle with a large radius. Being able to think of curvature in terms of the radius of a circle is very useful. (The word "osculating" comes from a Latin word related to kissing; an osculating circle "kisses" the graph at a particular point. Many beautiful ideas in mathematics have come from studying the osculating circles to a curve.)

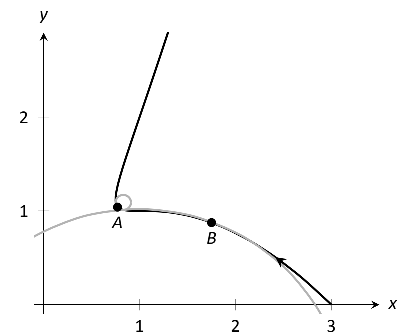


Figure 12.5.4: Illustrating the osculating circles for the curve seen in Figure 12.5.3.

Example 12.5.4 Finding curvature

Find the curvature of the parabola defined by $y = x^2$ at the vertex and at $x = 1$.

SOLUTION We use the first formula found in Theorem 12.5.2.

$$\begin{aligned}\kappa(x) &= \frac{|2|}{\left(1+(2x)^2\right)^{3/2}}\\ &= \frac{2}{\left(1+4x^2\right)^{3/2}}.\end{aligned}$$

At the vertex ($x = 0$), the curvature is $\kappa = 2$. At $x = 1$, the curvature is $\kappa = 2/(5)^{3/2} \approx 0.179$. So at $x = 0$, the curvature of $y = x^2$ is that of a circle of radius $1/2$; at $x = 1$, the curvature is that of a circle with radius $\approx 1/0.179 \approx 5.59$. This is illustrated in Figure 12.5.5. At $x = 3$, the curvature is 0.009; the graph is nearly straight as the curvature is very close to 0.

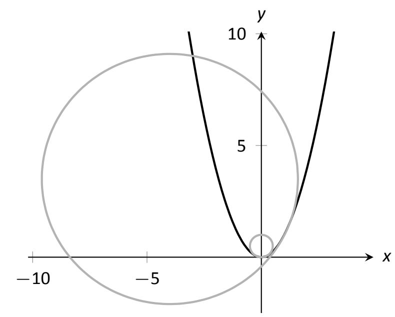


Figure 12.5.5: Examining the curvature of $y = x^2$.

Example 12.5.5 Finding curvature

Find where the curvature of $\vec{r}(t) = \left\langle t, t^2, 2t^3\right\rangle$ is maximized.

SOLUTION We use the third formula in Theorem 12.5.2 as $\vec{r}(t)$ is defined in space. We leave it to the reader to verify that

$$\vec{r}\,'(t) = \left\langle 1, 2t, 6t^2\right\rangle, \quad \vec{r}\,''(t) = \langle 0, 2, 12t\rangle, \quad \text{and} \quad \vec{r}\,'(t)\times\vec{r}\,''(t) = \left\langle 12t^2, -12t, 2\right\rangle.$$

Thus

$$\begin{aligned}\kappa(t) &= \frac{\|\,\vec{r}\,'(t)\times\vec{r}\,''(t)\,\|}{\|\,\vec{r}\,'(t)\,\|^3}\\ &= \frac{\|\left\langle 12t^2, -12t, 2\right\rangle\|}{\|\left\langle 1, 2t, 6t^2\right\rangle\|^3}\\ &= \frac{\sqrt{144t^4+144t^2+4}}{\left(\sqrt{1+4t^2+36t^4}\right)^3}\end{aligned}$$

While this is not a particularly "nice" formula, it does explicitly tell us what the curvature is at a given t value. To maximize $\kappa(t)$, we should solve $\kappa'(t) = 0$ for t. This is doable, but *very* time consuming. Instead, consider the graph of $\kappa(t)$ as given in Figure 12.5.6(a). We see that κ is maximized at two t values; using a numerical solver, we find these values are $t \approx \pm 0.189$. In part (b) of the figure we graph $\vec{r}(t)$ and indicate the points where curvature is maximized.

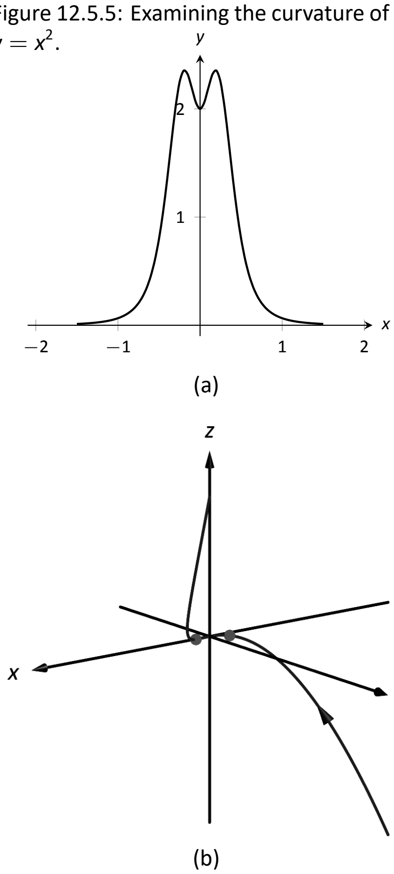


Figure 12.5.6: Understanding the curvature of a curve in space.

Curvature and Motion

Let $\vec{r}(t)$ be a position function of an object, with velocity $\vec{v}(t) = \vec{r}\,'(t)$ and acceleration $\vec{a}(t) = \vec{r}\,''(t)$. In Section 12.4 we established that acceleration is in the plane formed by $\vec{T}(t)$ and $\vec{N}(t)$, and that we can find scalars a_T and a_N such that

$$\vec{a}(t) = a_\text{T}\vec{T}(t) + a_\text{N}\vec{N}(t).$$

Theorem 12.4.2 gives formulas for a_T and a_N:

$$a_\text{T} = \frac{d}{dt}\Big(\|\,\vec{v}(t)\,\|\Big) \quad \text{and} \quad a_\text{N} = \frac{\|\,\vec{v}(t)\times\vec{a}(t)\,\|}{\|\,\vec{v}(t)\,\|}.$$

We understood that the amount of acceleration in the direction of $\vec{T}$ relates only to how the speed of the object is changing, and that the amount of acceleration in the direction of $\vec{N}$ relates to how the direction of travel of the object is changing. (That is, if the object travels at constant speed, $a_\text{T} = 0$; if the object travels in a constant direction, $a_\text{N} = 0$.)

In Equation (12.2) at the beginning of this section, we found $s\,'(t) = \|\,\vec{v}(t)\,\|$. We can combine this fact with the above formula for a_T to write

$$a_\text{T} = \frac{d}{dt}\Big(\|\,\vec{v}(t)\,\|\Big) = \frac{d}{dt}\big(s\,'(t)\big) = s\,''(t).$$

Since $s\,'(t)$ is speed, $s\,''(t)$ is the rate at which speed is changing with respect to time. We see once more that the component of acceleration in the direction of travel relates only to speed, not to a change in direction.

Now compare the formula for a_N above to the formula for curvature in Theorem 12.5.2:

$$a_\text{N} = \frac{\|\,\vec{v}(t)\times\vec{a}(t)\,\|}{\|\,\vec{v}(t)\,\|} \quad \text{and} \quad \kappa = \frac{\|\,\vec{r}\,'(t)\times\vec{r}\,''(t)\,\|}{\|\,\vec{r}\,'(t)\,\|^3} = \frac{\|\,\vec{v}(t)\times\vec{a}(t)\,\|}{\|\,\vec{v}(t)\,\|^3}.$$

Thus

$$\begin{aligned} a_\text{N} &= \kappa\|\,\vec{v}(t)\,\|^2 \qquad (12.5)\\ &= \kappa\Big(s\,'(t)\Big)^2 \end{aligned}$$

This last equation shows that the component of acceleration that changes the object's direction is dependent on two things: the curvature of the path and the speed of the object.

Imagine driving a car in a clockwise circle. You will naturally feel a force pushing you towards the door (more accurately, the door is pushing you as the car is turning and you want to travel in a straight line). If you keep the radius of the circle constant but speed up (i.e., increasing $s\,'(t)$), the door pushes harder against you (a_N has increased). If you keep your speed constant but tighten the turn (i.e., increase κ), once again the door will push harder against you.

Putting our new formulas for a_T and a_N together, we have

$$\vec{a}(t) = s\,''(t)\vec{T}(t) + \kappa\|\,\vec{v}(t)\,\|^2\vec{N}(t).$$

This is not a particularly practical way of finding a_T and a_N, but it reveals some great concepts about how acceleration interacts with speed and the shape of a curve.

Operating Speed (mph)	Minimum Radius (ft)
35	310
40	430
45	540

Figure 12.5.7: Operating speed and minimum radius in highway cloverleaf design.

Example 12.5.6 Curvature and road design

The minimum radius of the curve in a highway cloverleaf is determined by the operating speed, as given in the table in Figure 12.5.7. For each curve and speed, compute a_N.

Solution Using Equation (12.5), we can compute the acceleration normal to the curve in each case. We start by converting each speed from "miles per hour" to "feet per second" by multiplying by $5280/3600$.

$$\begin{aligned} \text{35mph, 310ft} &\Rightarrow \text{51.33ft/s}, \quad \kappa = 1/310 \\ a_N &= \kappa \parallel \vec{v}(t) \parallel^2 \\ &= \frac{1}{310}\big(51.33\big)^2 \\ &= 8.50\text{ft/s}^2. \end{aligned}$$

$$\begin{aligned} \text{40mph, 430ft} &\Rightarrow \text{58.67ft/s}, \quad \kappa = 1/430 \\ a_N &= \frac{1}{430}\big(58.67\big)^2 \\ &= 8.00\text{ft/s}^2. \end{aligned}$$

$$\begin{aligned} \text{45mph,540ft} &\Rightarrow \text{66ft/s}, \quad \kappa = 1/540 \\ a_N &= \frac{1}{540}\big(66\big)^2 \\ &= 8.07\text{ft/s}^2. \end{aligned}$$

Note that each acceleration is similar; this is by design. Considering the classic "Force $=$ mass $\times$ acceleration" formula, this acceleration must be kept small in order for the tires of a vehicle to keep a "grip" on the road. If one travels on a turn of radius 310 ft at a rate of 50 mph, the acceleration is double, at 17.35 ft/s^2. If the acceleration is too high, the frictional force created by the tires may not be enough to keep the car from sliding. Civil engineers routinely compute a "safe" design speed, then subtract 5-10 mph to create the posted speed limit for additional safety.

We end this chapter with a reflection on what we've covered. We started with vector–valued functions, which may have seemed at the time to be just another way of writing parametric equations. However, we have seen that the vector perspective has given us great insight into the behaviour of functions and the study of motion. Vector–valued position functions convey displacement, distance travelled, speed, velocity, acceleration and curvature information, each of which has great importance in science and engineering.

Exercises 12.5

Terms and Concepts

1. It is common to describe position in terms of both ________ and/or ________.

2. A measure of the "curviness" of a curve is ________.

3. Give two shapes with constant curvature.

4. Describe in your own words what an "osculating circle" is.

5. Complete the identity: $\vec{T}\,'(s) =$ ________$\vec{N}(s)$.

6. Given a position function $\vec{r}(t)$, how are a_{T} and a_{N} affected by the curvature?

Problems

In Exercises 7 – 10 , a position function $\vec{r}(t)$ is given, where $t = 0$ corresponds to the initial position. Find the arc length parameter s, and rewrite $\vec{r}(t)$ in terms of s; that is, find $\vec{r}(s)$.

7. $\vec{r}(t) = \langle 2t, t, -2t \rangle$

8. $\vec{r}(t) = \langle 7\cos t, 7\sin t \rangle$

9. $\vec{r}(t) = \langle 3\cos t, 3\sin t, 2t \rangle$

10. $\vec{r}(t) = \langle 5\cos t, 13\sin t, 12\cos t \rangle$

In Exercises 11 – 22 , a curve C is described along with 2 points on C.

(a) Using a sketch, determine at which of these points the curvature is greater.

(b) Find the curvature κ of C, and evaluate κ at each of the 2 given points.

11. C is defined by $y = x^3 - x$; points given at $x = 0$ and $x = 1/2$.

12. C is defined by $y = \dfrac{1}{x^2+1}$; points given at $x = 0$ and $x = 2$.

13. C is defined by $y = \cos x$; points given at $x = 0$ and $x = \pi/2$.

14. C is defined by $y = \sqrt{1-x^2}$ on $(-1, 1)$; points given at $x = 0$ and $x = 1/2$.

15. C is defined by $\vec{r}(t) = \langle \cos t, \sin(2t) \rangle$; points given at $t = 0$ and $t = \pi/4$.

16. C is defined by $\vec{r}(t) = \left\langle \cos^2 t, \sin t \cos t \right\rangle$; points given at $t = 0$ and $t = \pi/3$.

17. C is defined by $\vec{r}(t) = \left\langle t^2 - 1, t^3 - t \right\rangle$; points given at $t = 0$ and $t = 5$.

18. C is defined by $\vec{r}(t) = \langle \tan t, \sec t \rangle$; points given at $t = 0$ and $t = \pi/6$.

19. C is defined by $\vec{r}(t) = \langle 4t+2, 3t-1, 2t+5 \rangle$; points given at $t = 0$ and $t = 1$.

20. C is defined by $\vec{r}(t) = \left\langle t^3 - t, t^3 - 4, t^2 - 1 \right\rangle$; points given at $t = 0$ and $t = 1$.

21. C is defined by $\vec{r}(t) = \langle 3\cos t, 3\sin t, 2t \rangle$; points given at $t = 0$ and $t = \pi/2$.

22. C is defined by $\vec{r}(t) = \langle 5\cos t, 13\sin t, 12\cos t \rangle$; points given at $t = 0$ and $t = \pi/2$.

In Exercises 23 – 26 , find the value of x or t where curvature is maximized.

23. $y = \dfrac{1}{6}x^3$

24. $y = \sin x$

25. $\vec{r}(t) = \left\langle t^2 + 2t, 3t - t^2 \right\rangle$

26. $\vec{r}(t) = \langle t, 4/t, 3/t \rangle$

In Exercises 27 – 30 , find the radius of curvature at the indicated value.

27. $y = \tan x$, at $x = \pi/4$

28. $y = x^2 + x - 3$, at $x = \pi/4$

29. $\vec{r}(t) = \langle \cos t, \sin(3t) \rangle$, at $t = 0$

30. $\vec{r}(t) = \langle 5\cos(3t), t \rangle$, at $t = 0$

In Exercises 31 – 34 , find the equation of the osculating circle to the curve at the indicated t-value.

31. $\vec{r}(t) = \left\langle t, t^2 \right\rangle$, at $t = 0$

32. $\vec{r}(t) = \langle 3\cos t, \sin t \rangle$, at $t = 0$

33. $\vec{r}(t) = \langle 3\cos t, \sin t \rangle$, at $t = \pi/2$

34. $\vec{r}(t) = \left\langle t^2 - t, t^2 + t \right\rangle$, at $t = 0$

13: FUNCTIONS OF SEVERAL VARIABLES

13.1 Introduction to Multivariable Functions

Definition 13.1.1 Function of Two Variables

Let D be a subset of $\mathbb{R}^2$. A **function f of two variables** is a rule that assigns each pair (x, y) in D a value $z = f(x, y)$ in $\mathbb{R}$. D is the **domain** of f; the set of all outputs of f is the **range**.

Example 13.1.1 Understanding a function of two variables

Let $z = f(x, y) = x^2 - y$. Evaluate $f(1, 2)$, $f(2, 1)$, and $f(-2, 4)$; find the domain and range of f.

SOLUTION Using the definition $f(x, y) = x^2 - y$, we have:

$$\begin{aligned} f(1,2) &= 1^2 - 2 = -1 \\ f(2,1) &= 2^2 - 1 = 3 \\ f(-2,4) &= (-2)^2 - 4 = 0 \end{aligned}$$

The domain is not specified, so we take it to be all possible pairs in $\mathbb{R}^2$ for which f is defined. In this example, f is defined for *all* pairs (x, y), so the domain D of f is $\mathbb{R}^2$.

The output of f can be made as large or small as possible; any real number r can be the output. (In fact, given any real number r, $f(0, -r) = r$.) So the range R of f is $\mathbb{R}$.

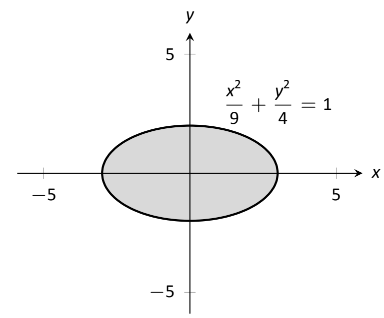


Figure 13.1.1: Illustrating the domain of $f(x, y)$ in Example 13.1.2.

Example 13.1.2 Understanding a function of two variables

Let $f(x, y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$. Find the domain and range of f.

SOLUTION The domain is all pairs (x, y) allowable as input in f. Because of the square–root, we need (x, y) such that $0 \le 1 - \frac{x^2}{9} - \frac{y^2}{4}$:

$$\begin{aligned} 0 &\le 1 - \frac{x^2}{9} - \frac{y^2}{4} \\ \frac{x^2}{9} + \frac{y^2}{4} &\le 1 \end{aligned}$$

The above equation describes an ellipse and its interior as shown in Figure 13.1.1. We can represent the domain D graphically with the figure; in set notation, we can write $D = \{(x, y) | \frac{x^2}{9} + \frac{y^2}{4} \le 1\}$.

The range is the set of all possible output values. The square–root ensures that all output is ≥ 0. Since the x and y terms are squared, then subtracted, inside the square–root, the largest output value comes at $x = 0$, $y = 0$: $f(0, 0) = 1$. Thus the range R is the interval $[0, 1]$.

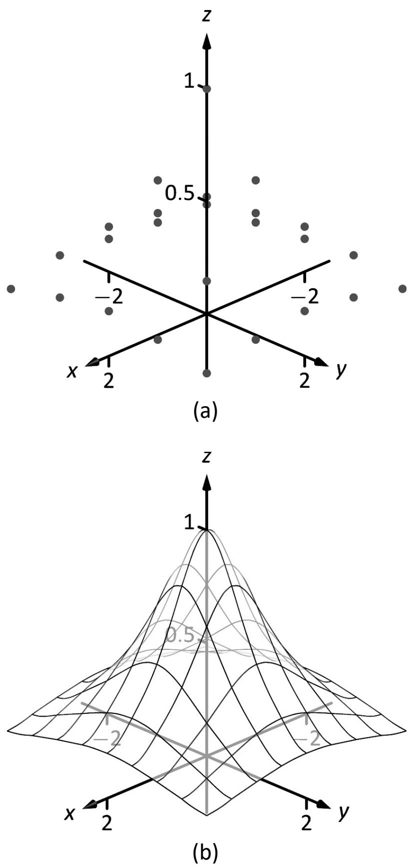


Figure 13.1.2: Graphing a function of two variables.

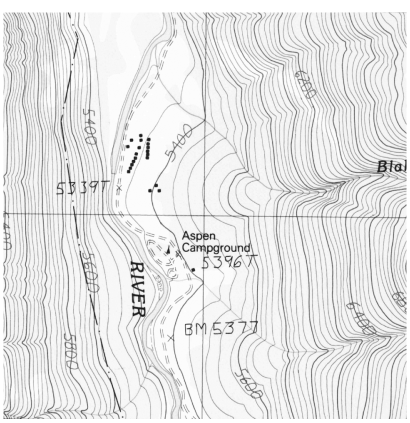


Figure 13.1.3: A topographical map displays elevation by drawing contour lines, along with the elevation is constant. Sample taken from the public domain USGS Digital Raster Graphics, `http://topmaps.usgs.gove/drg/`.

Graphing Functions of Two Variables

The **graph** of a function f of two variables is the set of all points $\big(x, y, f(x, y)\big)$ where (x, y) is in the domain of f. This creates a **surface** in space.

One can begin sketching a graph by plotting points, but this has limitations. Consider Figure 13.1.2(a) where 25 points have been plotted of

$$f(x, y) = \frac{1}{x^2 + y^2 + 1}.$$

More points have been plotted than one would reasonably want to do by hand, yet it is not clear at all what the graph of the function looks like. Technology allows us to plot lots of points, connect adjacent points with lines and add shading to create a graph like Figure 13.1.2b which does a far better job of illustrating the behaviour of f.

While technology is readily available to help us graph functions of two variables, there is still a paper–and–pencil approach that is useful to understand and master as it, combined with high–quality graphics, gives one great insight into the behaviour of a function. This technique is known as sketching **level curves**.

Level Curves

It may be surprising to find that the problem of representing a three dimensional surface on paper is familiar to most people (they just don't realize it). Topographical maps, like the one shown in Figure 13.1.3, represent the surface of Earth by indicating points with the same elevation with **contour lines**. The elevations marked are equally spaced; in this example, each thin line indicates an elevation change in 50 ft increments and each thick line indicates a change of 200 ft. When lines are drawn close together, elevation changes rapidly (as one does not have to travel far to rise 50 ft). When lines are far apart, such as near "Aspen Campground," elevation changes more gradually as one has to walk farther to rise 50 ft.

Given a function $z = f(x, y)$, we can draw a "topographical map" of f by drawing **level curves** (or, contour lines). A level curve at $z = c$ is a curve in the x-y plane such that for all points (x, y) on the curve, $f(x, y) = c$.

When drawing level curves, it is important that the c values are spaced equally apart as that gives the best insight to how quickly the "elevation" is changing. Examples will help one understand this concept.

Example 13.1.3 Drawing Level Curves

Let $f(x, y) = \sqrt{1 - \dfrac{x^2}{9} - \dfrac{y^2}{4}}$. Find the level curves of f for $c = 0, 0.2, 0.4, 0.6,$ 0.8 and 1.

Solution Consider first $c = 0$. The level curve for $c = 0$ is the set of all points (x, y) such that $0 = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$. Squaring both sides gives us

$$\frac{x^2}{9} + \frac{y^2}{4} = 1,$$

an ellipse centred at $(0, 0)$ with horizontal major axis of length 6 and minor axis of length 4. Thus for any point (x, y) on this curve, $f(x, y) = 0$.

Now consider the level curve for $c = 0.2$

$$\begin{aligned} 0.2 &= \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \\ 0.04 &= 1 - \frac{x^2}{9} - \frac{y^2}{4} \\ \frac{x^2}{9} + \frac{y^2}{4} &= 0.96 \\ \frac{x^2}{8.64} + \frac{y^2}{3.84} &= 1. \end{aligned}$$

This is also an ellipse, where $a = \sqrt{8.64} \approx 2.94$ and $b = \sqrt{3.84} \approx 1.96$.

In general, for $z = c$, the level curve is:

$$\begin{aligned} c &= \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \\ c^2 &= 1 - \frac{x^2}{9} - \frac{y^2}{4} \\ \frac{x^2}{9} + \frac{y^2}{4} &= 1 - c^2 \\ \frac{x^2}{9(1-c^2)} + \frac{y^2}{4(1-c^2)} &= 1, \end{aligned}$$

ellipses that are decreasing in size as c increases. A special case is when $c = 1$; there the ellipse is just the point $(0, 0)$.

The level curves are shown in Figure 13.1.4(a). Note how the level curves for $c = 0$ and $c = 0.2$ are very, very close together: this indicates that f is growing rapidly along those curves.

In Figure 13.1.4(b), the curves are drawn on a graph of f in space. Note how the elevations are evenly spaced. Near the level curves of $c = 0$ and $c = 0.2$ we can see that f indeed is growing quickly.

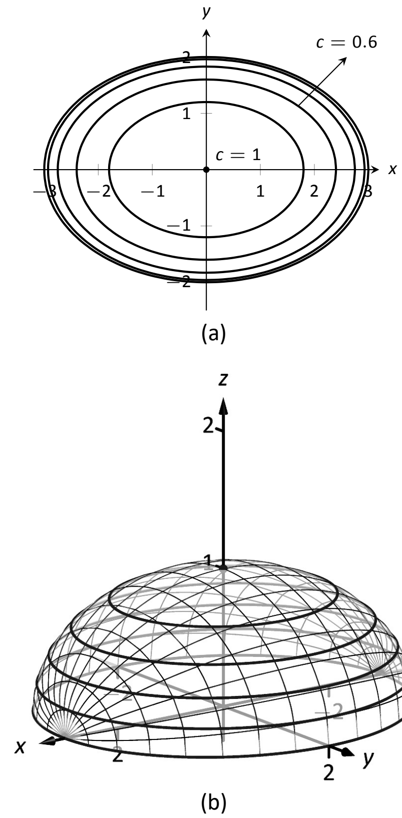


Figure 13.1.4: Graphing the level curves in Example 13.1.3.

Example 13.1.4 Analyzing Level Curves

Let $f(x, y) = \dfrac{x + y}{x^2 + y^2 + 1}$. Find the level curves for $z = c$.

Solution We begin by setting $f(x, y) = c$ for an arbitrary c and seeing if algebraic manipulation of the equation reveals anything significant.

$$\begin{aligned} \frac{x + y}{x^2 + y^2 + 1} &= c \\ x + y &= c(x^2 + y^2 + 1). \end{aligned}$$

We recognize this as a circle, though the center and radius are not yet clear. By completing the square, we can obtain:

$$\left(x - \frac{1}{2c}\right)^2 + \left(y - \frac{1}{2c}\right)^2 = \frac{1}{2c^2} - 1,$$

a circle centred at $\big(1/(2c), 1/(2c)\big)$ with radius $\sqrt{1/(2c^2) - 1}$, where $|c| < 1/\sqrt{2}$. The level curves for $c = \pm 0.2$, ± 0.4 and ± 0.6 are sketched in Figure 13.1.5(a). To help illustrate "elevation," we use thicker lines for c values near 0, and dashed lines indicate where $c < 0$.

There is one special level curve, when $c = 0$. The level curve in this situation is $x + y = 0$, the line $y = -x$.

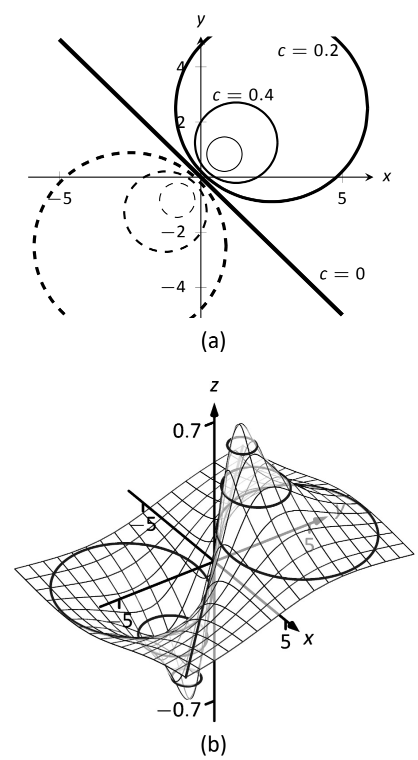


Figure 13.1.5: Graphing the level curves in Example 13.1.4.

In Figure 13.1.5(b) we see a graph of the surface. Note how the y-axis is pointing away from the viewer to more closely resemble the orientation of the level curves in (a).

Seeing the level curves helps us understand the graph. For instance, the graph does not make it clear that one can "walk" along the line $y = -x$ without elevation change, though the level curve does.

Functions of Three Variables

We extend our study of multivariable functions to functions of three variables. (One can make a function of as many variables as one likes; we limit our study to three variables.)

Definition 13.1.2 Function of Three Variables

Let D be a subset of $\mathbb{R}^3$. A **function f of three variables** is a rule that assigns each triple (x, y, z) in D a value $w = f(x, y, z)$ in $\mathbb{R}$. D is the **domain** of f; the set of all outputs of f is the **range**.

Note how this definition closely resembles that of Definition 13.1.1.

Example 13.1.5 Understanding a function of three variables

Let $f(x, y, z) = \dfrac{x^2 + z + 3\sin y}{x + 2y - z}$. Evaluate f at the point $(3, 0, 2)$ and find the domain and range of f.

Solution $f(3, 0, 2) = \dfrac{3^2 + 2 + 3\sin 0}{3 + 2(0) - 2} = 11.$

As the domain of f is not specified, we take it to be the set of all triples (x, y, z) for which $f(x, y, z)$ is defined. As we cannot divide by 0, we find the domain D is

$$D = \{(x, y, z) \mid x + 2y - z \neq 0\}.$$

We recognize that the set of all points in $\mathbb{R}^3$ that *are not* in D form a plane in space that passes through the origin (with normal vector $\langle 1, 2, -1\rangle$).

We determine the range R is $\mathbb{R}$; that is, all real numbers are possible outputs of f. There is no set way of establishing this. Rather, to get numbers near 0 we can let $y = 0$ and choose $z \approx -x^2$. To get numbers of arbitrarily large magnitude, we can let $z \approx x + 2y$.

Level Surfaces

It is very difficult to produce a meaningful graph of a function of three variables. A function of *one* variable is a *curve* drawn in *2* dimensions; a function of *two* variables is a *surface* drawn in *3* dimensions; a function of *three* variables is a *hypersurface* drawn in *4* dimensions.

There are a few techniques one can employ to try to "picture" a graph of three variables. One is an analogue of level curves: **level surfaces**. Given $w = f(x, y, z)$, the level surface at $w = c$ is the surface in space formed by all points (x, y, z) where $f(x, y, z) = c$.

Example 13.1.6 Finding level surfaces

If a point source S is radiating energy, the intensity I at a given point P in space is inversely proportional to the square of the distance between S and P. That is, when $S = (0, 0, 0)$, $I(x, y, z) = \dfrac{k}{x^2 + y^2 + z^2}$ for some constant k.

Let $k = 1$; find the level surfaces of I.

Solution We can (mostly) answer this question using "common sense." If energy (say, in the form of light) is emanating from the origin, its intensity will be the same at all points equidistant from the origin. That is, at any point on the surface of a sphere centred at the origin, the intensity should be the same. Therefore, the level surfaces are spheres.

We now find this mathematically. The level surface at $I = c$ is defined by

$$c = \frac{1}{x^2 + y^2 + z^2}.$$

A small amount of algebra reveals

$$x^2 + y^2 + z^2 = \frac{1}{c}.$$

Given an intensity c, the level surface $I = c$ is a sphere of radius $1/\sqrt{c}$, centred at the origin.

Figure 13.1.6 gives a table of the radii of the spheres for given c values. Normally one would use equally spaced c values, but these values have been chosen purposefully. At a distance of 0.25 from the point source, the intensity is 16; to move to a point of half that intensity, one just moves out 0.1 to 0.35 – not much at all. To again halve the intensity, one moves 0.15, a little more than before.

Note how each time the intensity if halved, the distance required to move away grows. We conclude that the closer one is to the source, the more rapidly the intensity changes.

c	r
16.	0.25
8.	0.35
4.	0.5
2.	0.71
1.	1.
0.5	1.41
0.25	2.
0.125	2.83
0.0625	4.

Figure 13.1.6: A table of c values and the corresponding radius r of the spheres of constant value in Example 13.1.6.

In the next section we apply the concepts of limits to functions of two or more variables.

Exercises 13.1

Terms and Concepts

1. Give two examples (other than those given in the text) of "real world" functions that require more than one input.

2. The graph of a function of two variables is a ________.

3. Most people are familiar with the concept of level curves in the context of ________ maps.

4. T/F: Along a level curve, the output of a function does not change.

5. The analogue of a level curve for functions of three variables is a level ________.

6. What does it mean when level curves are close together? Far apart?

Problems

In Exercises 7 – 14, give the domain and range of the multivariable function.

7. $f(x,y)=x^2+y^2+2$

8. $f(x,y)=x+2y$

9. $f(x,y)=x-2y$

10. $f(x,y)=\dfrac{1}{x+2y}$

11. $f(x,y)=\dfrac{1}{x^2+y^2+1}$

12. $f(x,y)=\sin x\cos y$

13. $f(x,y)=\sqrt{9-x^2-y^2}$

14. $f(x,y)=\dfrac{1}{\sqrt{x^2+y^2-9}}$

In Exercises 15 – 22, describe in words and sketch the level curves for the function and given c values.

15. $f(x,y)=3x-2y$; $c=-2,0,2$

16. $f(x,y)=x^2-y^2$; $c=-1,0,1$

17. $f(x,y)=x-y^2$; $c=-2,0,2$

18. $f(x,y)=\dfrac{1-x^2-y^2}{2y-2x}$; $c=-2,0,2$

19. $f(x,y)=\dfrac{2x-2y}{x^2+y^2+1}$; $c=-1,0,1$

20. $f(x,y)=\dfrac{y-x^3-1}{x}$; $c=-3,-1,0,1,3$

21. $f(x,y)=\sqrt{x^2+4y^2}$; $c=1,2,3,4$

22. $f(x,y)=x^2+4y^2$; $c=1,2,3,4$

In Exercises 23 – 26, give the domain and range of the functions of three variables.

23. $f(x,y,z)=\dfrac{x}{x+2y-4z}$

24. $f(x,y,z)=\dfrac{1}{1-x^2-y^2-z^2}$

25. $f(x,y,z)=\sqrt{z-x^2+y^2}$

26. $f(x,y,z)=z^2\sin x\cos y$

In Exercises 27 – 30, describe the level surfaces of the given functions of three variables.

27. $f(x,y,z)=x^2+y^2+z^2$

28. $f(x,y,z)=z-x^2+y^2$

29. $f(x,y,z)=\dfrac{x^2+y^2}{z}$

30. $f(x,y,z)=\dfrac{z}{x-y}$

31. Compare the level curves of Exercises 21 and 22. How are they similar, and how are they different? Each surface is a quadric surface; describe how the level curves are consistent with what we know about each surface.

13.2 Limits and Continuity of Multivariable Functions

We continue with the pattern we have established in this text: after defining a new kind of function, we apply calculus ideas to it. The previous section defined functions of two and three variables; this section investigates what it means for these functions to be "continuous."

We begin with a series of definitions. We are used to "open intervals" such as $(1, 3)$, which represents the set of all x such that $1 < x < 3$, and "closed intervals" such as $[1, 3]$, which represents the set of all x such that $1 \leq x \leq 3$. We need analogous definitions for open and closed sets in the x-y plane.

Definition 13.2.1 Open Disk, Boundary and Interior Points, Open and Closed Sets, Bounded Sets

An **open disk** B in $\mathbb{R}^2$ centred at (x_0, y_0) with radius r is the set of all points (x, y) such that $\sqrt{(x - x_0)^2 + (y - y_0)^2} < r$.

Let S be a set of points in $\mathbb{R}^2$. A point P in $\mathbb{R}^2$ is a **boundary point** of S if all open disks centred at P contain both points in S and points not in S.

A point P in S is an **interior point** of S if there is an open disk centred at P that contains only points in S.

A set S is **open** if every point in S is an interior point.

A set S is **closed** if it contains all of its boundary points.

A set S is **bounded** if there is an $M > 0$ such that the open disk, centred at the origin with radius M, contains S. A set that is not bounded is **unbounded**.

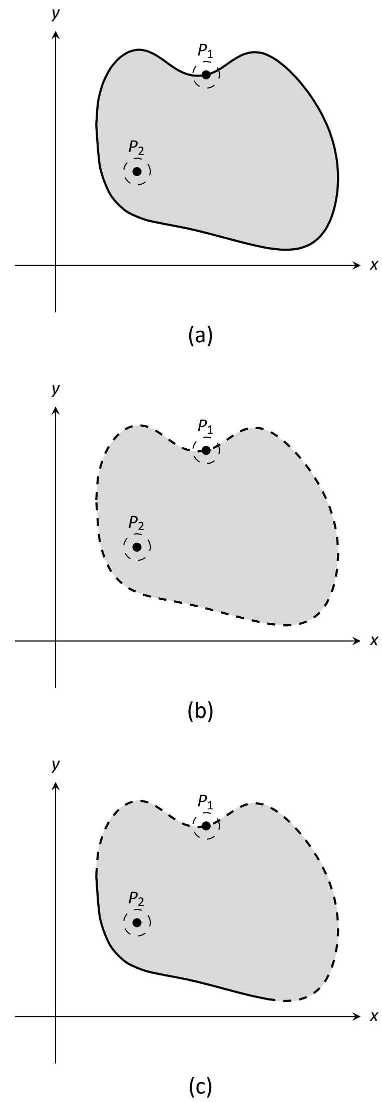


Figure 13.2.1: Illustrating open and closed sets in the x-y plane.

Figure 13.2.1 shows several sets in the x-y plane. In each set, point P_1 lies on the boundary of the set as all open disks centred there contain both points in, and not in, the set. In contrast, point P_2 is an interior point for there is an open disk centred there that lies entirely within the set.

The set depicted in Figure 13.2.1(a) is a closed set as it contains all of its boundary points. The set in (b) is open, for all of its points are interior points (or, equivalently, it does not contain any of its boundary points). The set in (c) is neither open nor closed as it contains some of its boundary points.

Example 13.2.1 Determining open/closed, bounded/unbounded

Determine if the domain of the function $f(x, y) = \sqrt{1 - x^2/9 - y^2/4}$ is open, closed, or neither, and if it is bounded.

Solution This domain of this function was found in Example 13.1.2 to be $D = \{(x, y) \mid \frac{x^2}{9} + \frac{y^2}{4} \leq 1\}$, the region *bounded* by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Since the region includes the boundary (indicated by the use of "$\leq$"), the set contains all of its boundary points and hence is closed. The region is bounded as a disk of radius 4, centred at the origin, contains D.

Example 13.2.2 Determining open/closed, bounded/unbounded

Determine if the domain of $f(x,y) = \frac{1}{x-y}$ is open, closed, or neither.

Solution As we cannot divide by 0, we find the domain to be $D = \{(x,y) \mid x - y \neq 0\}$. In other words, the domain is the set of all points (x,y) *not* on the line $y = x$.

The domain is sketched in Figure 13.2.2. Note how we can draw an open disk around any point in the domain that lies entirely inside the domain, and also note how the only boundary points of the domain are the points on the line $y = x$. We conclude the domain is an open set. The set is unbounded.

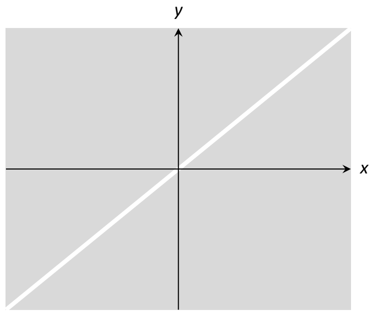


Figure 13.2.2: Sketching the domain of the function in Example 13.2.2.

Limits

Recall a pseudo–definition of the limit of a function of one variable: "$\lim_{x\to c} f(x) = L$" means that if x is "really close" to c, then $f(x)$ is "really close" to L. A similar pseudo–definition holds for functions of two variables. We'll say that

$$\text{“}\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L\text{”}$$

means "if the point (x,y) is really close to the point (x_0,y_0), then $f(x,y)$ is really close to L." The formal definition is given below.

Note: While our first limit definition was defined over an open interval, we now define limits over a set S in the plane (where S does not have to be open). As planar sets can be far more complicated than intervals, our definition adds the restriction ". . . where every open disk centred at P contains points in S other than P." In this text, all sets we'll consider will satisfy this condition and we won't bother to check; it is included in the definition for completeness.

Definition 13.2.2 Limit of a Function of Two Variables

Let S be a set containing $P = (x_0,y_0)$ where every open disk centred at P contains points in S other than P, let f be a function of two variables defined on S, except possibly at P, and let L be a real number. The **limit of** $f(x,y)$ **as** (x,y) **approaches** (x_0,y_0) **is** L, denoted

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L,$$

means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all (x,y) in S, where $(x,y) \neq (x_0,y_0)$, if (x,y) is in the open disk centred at (x_0,y_0) with radius δ, then $|f(x,y) - L| < \varepsilon$.

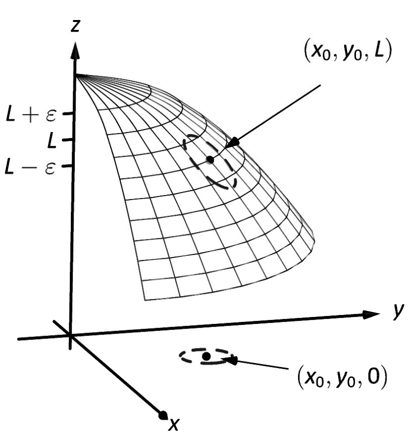


Figure 13.2.3: **Illustrating the definition of a limit.** The open disk in the x-y plane has radius δ. Let (x,y) be any point in this disk; $f(x,y)$ is within ε of L.

The concept behind Definition 13.2.2 is sketched in Figure 13.2.3. Given $\varepsilon > 0$, find $\delta > 0$ such that if (x,y) is any point in the open disk centred at (x_0,y_0) in the x-y plane with radius δ, then $f(x,y)$ should be within ε of L.

Computing limits using this definition is rather cumbersome. The following theorem allows us to evaluate limits much more easily.

Theorem 13.2.1 Basic Limit Properties of Functions of Two Variables

Let b, x_0, y_0, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = K.$$

The following limits hold.

1. Constants: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} b = b$
2. Identity $\displaystyle\lim_{(x,y)\to(x_0,y_0)} x = x_0; \quad \lim_{(x,y)\to(x_0,y_0)} y = y_0$
3. Sums/Differences: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \big(f(x,y) \pm g(x,y)\big) = L \pm K$
4. Scalar Multiples: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} b \cdot f(x,y) = bL$
5. Products: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) \cdot g(x,y) = LK$
6. Quotients: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)/g(x,y) = L/K$, $(K \neq 0)$
7. Powers: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)^n = L^n$

This theorem, combined with Theorems 1.3.2 and 1.3.3 of Section 1.3, allows us to evaluate many limits.

Example 13.2.3 Evaluating a limit

Evaluate the following limits:

1. $\displaystyle\lim_{(x,y)\to(1,\pi)} \left(\frac{y}{x} + \cos(xy)\right)$ 2. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}$

Solution

1. The aforementioned theorems allow us to simply evaluate $y/x + \cos(xy)$ when $x = 1$ and $y = \pi$. If an indeterminate form is returned, we must do more work to evaluate the limit; otherwise, the result is the limit. Therefore

$$\begin{aligned}\lim_{(x,y)\to(1,\pi)} \left(\frac{y}{x} + \cos(xy)\right) &= \frac{\pi}{1} + \cos\pi \\ &= \pi - 1.\end{aligned}$$

2. We attempt to evaluate the limit by substituting 0 in for x and y, but the result is the indeterminate form "0/0." To evaluate this limit, we must "do more work," but we have not yet learned what "kind" of work to do. Therefore we cannot yet evaluate this limit.

When dealing with functions of a single variable we also considered one–sided limits and stated

$$\lim_{x\to c} f(x) = L \quad \text{if, and only if,} \quad \lim_{x\to c^+} f(x) = L \quad \textbf{and} \quad \lim_{x\to c^-} f(x) = L.$$

That is, the limit is L if and only if $f(x)$ approaches L when x approaches c from **either** direction, the left or the right.

In the plane, there are infinitely many directions from which (x, y) might approach (x_0, y_0). In fact, we do not have to restrict ourselves to approaching (x_0, y_0) from a particular direction, but rather we can approach that point along a path that is not a straight line. It is possible to arrive at different limiting values by approaching (x_0, y_0) along different paths. If this happens, we say that $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist (this is analogous to the left and right hand limits of single variable functions not being equal).

Our theorems tell us that we can evaluate most limits quite simply, without worrying about paths. When indeterminate forms arise, the limit may or may not exist. If it does exist, it can be difficult to prove this as we need to show the same limiting value is obtained regardless of the path chosen. The case where the limit does not exist is often easier to deal with, for we can often pick two paths along which the limit is different.

Example 13.2.4 Showing limits do not exist

1. Show $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}$ does not exist by finding the limits along the lines $y = mx$.
2. Show $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$ does not exist by finding the limit along the path $y = -\sin x$.

Solution

1. Evaluating $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}$ along the lines $y = mx$ means replace all y's with mx and evaluating the resulting limit:

$$\begin{aligned}\lim_{(x,mx)\to(0,0)} \frac{3x(mx)}{x^2+(mx)^2} &= \lim_{x\to 0} \frac{3mx^2}{x^2(m^2+1)}\\ &= \lim_{x\to 0} \frac{3m}{m^2+1}\\ &= \frac{3m}{m^2+1}.\end{aligned}$$

   While the limit exists for each choice of m, we get a *different* limit for each choice of m. That is, along different lines we get differing limiting values, meaning *the* limit does not exist.

2. Let $f(x, y) = \frac{\sin(xy)}{x+y}$. We are to show that $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist by finding the limit along the path $y = -\sin x$. First, however, consider the limits found along the lines $y = mx$ as done above.

$$\begin{aligned}\lim_{(x,mx)\to(0,0)} \frac{\sin\big(x(mx)\big)}{x+mx} &= \lim_{x\to 0} \frac{\sin(mx^2)}{x(m+1)}\\ &= \lim_{x\to 0} \frac{\sin(mx^2)}{x} \cdot \frac{1}{m+1}.\end{aligned}$$

By applying L'Hospital's Rule, we can show this limit is 0 *except* when $m = -1$, that is, along the line $y = -x$. This line is not in the domain of f, so we have found the following fact: along every line $y = mx$ in the domain of f, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

Now consider the limit along the path $y = -\sin x$:

$$\lim_{(x,-\sin x)\to(0,0)} \frac{\sin\big(-x\sin x\big)}{x-\sin x} = \lim_{x\to 0} \frac{\sin\big(-x\sin x\big)}{x-\sin x}$$

Now apply L'Hospital's Rule twice:

$$\begin{aligned}
&= \lim_{x\to 0} \frac{\cos\big(-x\sin x\big)(-\sin x - x\cos x)}{1-\cos x} \qquad (\text{“} = 0/0\text{”})\\
&= \lim_{x\to 0} \frac{-\sin\big(-x\sin x\big)(-\sin x - x\cos x)^2 + \cos\big(-x\sin x\big)(-2\cos x + x\sin x)}{\sin x}\\
&= \text{“}-2/0\text{”} \Rightarrow \text{the limit does not exist.}
\end{aligned}$$

Step back and consider what we have just discovered. Along any line $y = mx$ in the domain of the $f(x,y)$, the limit is 0. However, along the path $y = -\sin x$, which lies in the domain of $f(x,y)$ for all $x \neq 0$, the limit does not exist. Since the limit is not the same along every path to $(0,0)$, we say $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x+y}$ does not exist.

Example 13.2.5 Finding a limit

Let $f(x,y) = \dfrac{5x^2y^2}{x^2+y^2}$. Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.

Solution It is relatively easy to show that along any line $y = mx$, the limit is 0. This is not enough to prove that the limit exists, as demonstrated in the previous example, but it tells us that if the limit does exist then it must be 0.

To prove the limit is 0, we apply Definition 13.2.2. Let $\varepsilon > 0$ be given. We want to find $\delta > 0$ such that if $\sqrt{(x-0)^2+(y-0)^2} < \delta$, then $|f(x,y) - 0| < \varepsilon$.

Set $\delta < \sqrt{\varepsilon/5}$. Note that $\left|\dfrac{5y^2}{x^2+y^2}\right| < 5$ for all $(x,y) \neq (0,0)$, and that if $\sqrt{x^2+y^2} < \delta$, then $x^2 < \delta^2$.

Let $\sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2} < \delta$. Consider $|f(x,y) - 0|$:

$$\begin{aligned}
|f(x,y) - 0| &= \left|\frac{5x^2y^2}{x^2+y^2} - 0\right|\\
&= \left|x^2 \cdot \frac{5y^2}{x^2+y^2}\right|\\
&< \delta^2 \cdot 5\\
&< \frac{\varepsilon}{5}\cdot 5\\
&= \varepsilon.
\end{aligned}$$

Thus if $\sqrt{(x-0)^2+(y-0)^2} < \delta$ then $|f(x,y) - 0| < \varepsilon$, which is what we wanted to show. Thus $\lim\limits_{(x,y)\to(0,0)} \dfrac{5x^2y^2}{x^2+y^2} = 0$.

Continuity

Definition 1.6.1 defines what it means for a function of one variable to be continuous. In brief, it meant that the graph of the function did not have breaks, holes, jumps, etc. We define continuity for functions of two variables in a similar way as we did for functions of one variable.

Definition 13.2.3 Continuous

Let a function $f(x,y)$ be defined on a set S containing the point (x_0,y_0).

1. f is **continuous** at (x_0,y_0) if $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.
2. f is **continuous on** S if f is continuous at all points in S. If f is continuous at all points in $\mathbb{R}^2$, we say that f is **continuous everywhere**.

Example 13.2.6 Continuity of a function of two variables

Let $f(x,y) = \begin{cases} \frac{\cos y \sin x}{x} & x \neq 0 \\ \cos y & x = 0 \end{cases}$. Is f continuous at $(0,0)$? Is f continuous everywhere?

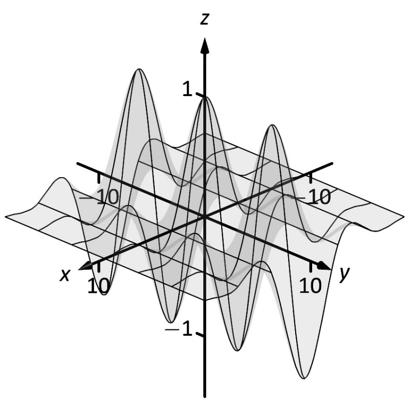


Figure 13.2.4: A graph of $f(x,y)$ in Example 13.2.6.

Solution To determine if f is continuous at $(0,0)$, we need to compare $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ to $f(0,0)$.

Applying the definition of f, we see that $f(0,0) = \cos 0 = 1$.

We now consider the limit $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$. Substituting 0 for x and y in $(\cos y \sin x)/x$ returns the indeterminate form "0/0", so we need to do more work to evaluate this limit.

Consider two related limits: $\displaystyle\lim_{(x,y)\to(0,0)} \cos y$ and $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin x}{x}$. The first limit does not contain x, and since $\cos y$ is continuous,

$$\lim_{(x,y)\to(0,0)} \cos y = \lim_{y\to 0} \cos y = \cos 0 = 1.$$

The second limit does not contain y. By Theorem 1.3.5 we can say

$$\lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Finally, Theorem 13.2.1 of this section states that we can combine these two limits as follows:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{\cos y \sin x}{x} &= \lim_{(x,y)\to(0,0)} (\cos y)\left(\frac{\sin x}{x}\right) \\
&= \left(\lim_{(x,y)\to(0,0)} \cos y\right)\left(\lim_{(x,y)\to(0,0)} \frac{\sin x}{x}\right) \\
&= (1)(1) \\
&= 1.
\end{aligned}$$

We have found that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\cos y \sin x}{x} = f(0,0)$, so f is continuous at $(0,0)$.

A similar analysis shows that f is continuous at all points in $\mathbb{R}^2$. As long as $x \neq 0$, we can evaluate the limit directly; when $x = 0$, a similar analysis shows that the limit is $\cos y$. Thus we can say that f is continuous everywhere. A graph of f is given in Figure 13.2.4. Notice how it has no breaks, jumps, etc.

The following theorem is very similar to Theorem 1.6.1, giving us ways to combine continuous functions to create other continuous functions.

Theorem 13.2.2 Properties of Continuous Functions

Let f and g be continuous on a set S, let c be a real number, and let n be a positive integer. The following functions are continuous on S.

1. Sums/Differences: $f \pm g$
2. Constant Multiples: $c \cdot f$
3. Products: $f \cdot g$
4. Quotients: f/g (as longs as $g \neq 0$ on S)
5. Powers: f^n
6. Roots: $\sqrt[n]{f}$ (if n is even then $f \geq 0$ on S; if n is odd, then true for all values of f on S.)
7. Compositions: Adjust the definitions of f and g to: Let f be continuous on S, where the range of f on S is J, and let g be a single variable function that is continuous on J. Then $g \circ f$, i.e., $g(f(x,y))$, is continuous on S.

Example 13.2.7 Establishing continuity of a function

Let $f(x,y) = \sin(x^2 \cos y)$. Show f is continuous everywhere.

SOLUTION We will apply both Theorems 1.6.1 and 13.2.2. Let $f_1(x,y) = x^2$. Since y is not actually used in the function, and polynomials are continuous (by Theorem 1.6.1), we conclude f_1 is continuous everywhere. A similar statement can be made about $f_2(x,y) = \cos y$. Part 3 of Theorem 13.2.2 states that $f_3 = f_1 \cdot f_2$ is continuous everywhere, and Part 7 of the theorem states the composition of sine with f_3 is continuous: that is, $\sin(f_3) = \sin(x^2 \cos y)$ is continuous everywhere.

Functions of Three Variables

The definitions and theorems given in this section can be extended in a natural way to definitions and theorems about functions of three (or more) variables. We cover the key concepts here; some terms from Definitions 13.2.1 and 13.2.3 are not redefined but their analogous meanings should be clear to the reader.

Definition 13.2.4 Open Balls, Limit, Continuous

1. An **open ball** in $\mathbb{R}^3$ centred at (x_0, y_0, z_0) with radius r is the set of all points (x, y, z) such that $\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} = r$.

2. Let D be an open set in $\mathbb{R}^3$ containing (x_0, y_0, z_0) where every open ball centred at (x_0, y_0, z_0) contains points of D other than (x_0, y_0, z_0), and let $f(x, y, z)$ be a function of three variables defined on D, except possibly at (x_0, y_0, z_0). The **limit** of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0) is L, denoted
$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = L,$$
means that given any $\varepsilon > 0$, there is a $\delta > 0$ such that for all (x, y, z) in D, $(x, y, z) \neq (x_0, y_0, z_0)$, if (x, y, z) is in the open ball centred at (x_0, y_0, z_0) with radius δ, then $|f(x, y, z) - L| < \varepsilon$.

3. Let $f(x, y, z)$ be defined on a set D containing (x_0, y_0, z_0). f is **continuous** at (x_0, y_0, z_0) if $\displaystyle\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = f(x_0, y_0, z_0)$; if f is continuous at all points in D, we say f is **continuous on** D.

These definitions can also be extended naturally to apply to functions of four or more variables. Theorem 13.2.2 also applies to function of three or more variables, allowing us to say that the function

$$f(x, y, z) = \frac{e^{x^2+y}\sqrt{y^2+z^2+3}}{\sin(xyz)+5}$$

is continuous everywhere.

When considering single variable functions, we studied limits, then continuity, then the derivative. In our current study of multivariable functions, we have studied limits and continuity. In the next section we study derivation, which takes on a slight twist as we are in a multivarible context.

Exercises 13.2

Terms and Concepts

1. Describe in your own words the difference between boundary and interior points of a set.

2. Use your own words to describe (informally) what $\lim_{(x,y)\to(1,2)} f(x,y) = 17$ means.

3. Give an example of a closed, bounded set.

4. Give an example of a closed, unbounded set.

5. Give an example of a open, bounded set.

6. Give an example of a open, unbounded set.

Problems

In Exercises 7 – 10, a set *S* is given.

(a) Give one boundary point and one interior point, when possible, of *S*.

(b) State whether *S* is open, closed, or neither.

(c) State whether *S* is bounded or unbounded.

7. $S = \left\{(x,y) \;\middle|\; \dfrac{(x-1)^2}{4} + \dfrac{(y-3)^2}{9} \leq 1\right\}$

8. $S = \left\{(x,y) \,|\, y \neq x^2\right\}$

9. $S = \left\{(x,y) \,|\, x^2 + y^2 = 1\right\}$

10. $S = \{(x,y) | y > \sin x\}$

In Exercises 11 – 14:

(a) Find the domain *D* of the given function.

(b) State whether *D* is an open or closed set.

(c) State whether *D* is bounded or unbounded.

11. $f(x,y) = \sqrt{9 - x^2 - y^2}$

12. $f(x,y) = \sqrt{y - x^2}$

13. $f(x,y) = \dfrac{1}{\sqrt{y - x^2}}$

14. $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

In Exercises 15 – 20, a limit is given. Evaluate the limit along the paths given, then state why these results show the given limit does not exist.

15. $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$
    (a) Along the path $y = 0$.
    (b) Along the path $x = 0$.

16. $\lim_{(x,y)\to(0,0)} \dfrac{x + y}{x - y}$
    (a) Along the path $y = mx$.

17. $\lim_{(x,y)\to(0,0)} \dfrac{xy - y^2}{y^2 + x}$
    (a) Along the path $y = mx$.
    (b) Along the path $x = 0$.

18. $\lim_{(x,y)\to(0,0)} \dfrac{\sin(x^2)}{y}$
    (a) Along the path $y = mx$.
    (b) Along the path $y = x^2$.

19. $\lim_{(x,y)\to(1,2)} \dfrac{x + y - 3}{x^2 - 1}$
    (a) Along the path $y = 2$.
    (b) Along the path $y = x + 1$.

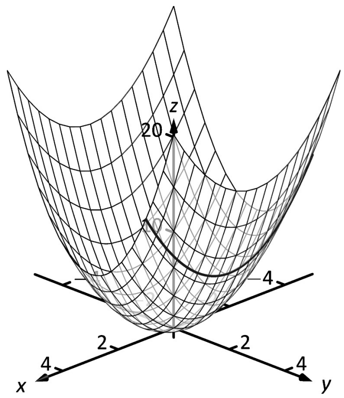
20. $\lim_{(x,y)\to(\pi,\pi/2)} \dfrac{\sin x}{\cos y}$
    (a) Along the path $x = \pi$.
    (b) Along the path $y = x - \pi/2$.

13.3 Partial Derivatives

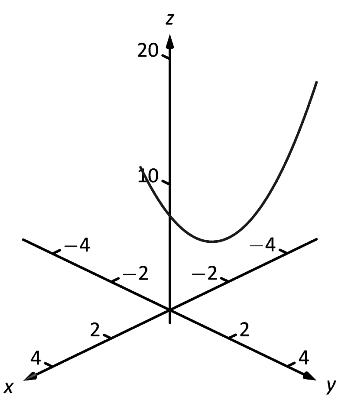
Let y be a function of x. We have studied in great detail the derivative of y with respect to x, that is, $\frac{dy}{dx}$, which measures the rate at which y changes with respect to x. Consider now $z = f(x, y)$. It makes sense to want to know how z changes with respect to x and/or y. This section begins our investigation into these rates of change.

Consider the function $z = f(x, y) = x^2 + 2y^2$, as graphed in Figure 13.3.1(a). By fixing $y = 2$, we focus our attention to all points on the surface where the y-value is 2, shown in both parts (a) and (b) of the figure. These points form a curve in space: $z = f(x, 2) = x^2 + 8$ which is a function of just one variable. We can take the derivative of z with respect to x along this curve and find equations of tangent lines, etc.

The key notion to extract from this example is: by treating y as constant (it does not vary) we can consider how z changes with respect to x. In a similar fashion, we can hold x constant and consider how z changes with respect to y. This is the underlying principle of **partial derivatives**. We state the formal, limit–based definition first, then show how to compute these partial derivatives without directly taking limits.



(a)



(b)

Figure 13.3.1: By fixing $y = 2$, the surface $f(x, y) = x^2 + 2y^2$ is a curve in space.

Definition 13.3.1 Partial Derivative

Let $z = f(x, y)$ be a continuous function on an open set S in $\mathbb{R}^2$.

1. The **partial derivative of f with respect to x** is:
$$f_x(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h}.$$
2. The **partial derivative of f with respect to y** is:
$$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

Alternate notations for $f_x(x, y)$ include:

$$\frac{\partial}{\partial x} f(x, y), \quad \frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x}, \quad \text{and } z_x,$$

with similar notations for $f_y(x, y)$. For ease of notation, $f_x(x, y)$ is often abbreviated f_x.

Example 13.3.1 Computing partial derivatives with the limit definition

Let $f(x, y) = x^2y + 2x + y^3$. Find $f_x(x, y)$ using the limit definition.

Solution Using Definition 13.3.1, we have:

$$\begin{aligned} f_x(x, y) &= \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h} \\ &= \lim_{h\to 0} \frac{(x + h)^2y + 2(x + h) + y^3 - (x^2y + 2x + y^3)}{h} \\ &= \lim_{h\to 0} \frac{x^2y + 2xhy + h^2y + 2x + 2h + y^3 - (x^2y + 2x + y^3)}{h} \\ &= \lim_{h\to 0} \frac{2xhy + h^2y + 2h}{h} \\ &= \lim_{h\to 0} 2xy + hy + 2 \\ &= 2xy + 2. \end{aligned}$$

We have found $f_x(x, y) = 2xy + 2$.

Example 13.3.1 found a partial derivative using the formal, limit–based definition. Using limits is not necessary, though, as we can rely on our previous knowledge of derivatives to compute partial derivatives easily. When computing $f_x(x, y)$, we hold y fixed – it does not vary. Therefore we can compute the derivative with respect to x by treating y as a constant or coefficient.

Just as $\frac{d}{dx}\big(5x^2\big) = 10x$, we compute $\frac{\partial}{\partial x}\big(x^2y\big) = 2xy$. Here we are treating y as a coefficient.

Just as $\frac{d}{dx}\big(5^3\big) = 0$, we compute $\frac{\partial}{\partial x}\big(y^3\big) = 0$. Here we are treating y as a constant. More examples will help make this clear.

Example 13.3.2 Finding partial derivatives

Find $f_x(x, y)$ and $f_y(x, y)$ in each of the following.

1. $f(x, y) = x^3y^2 + 5y^2 - x + 7$
2. $f(x, y) = \cos(xy^2) + \sin x$
3. $f(x, y) = e^{x^2y^3}\sqrt{x^2+1}$

Solution

1. We have $f(x, y) = x^3y^2 + 5y^2 - x + 7$.
   Begin with $f_x(x, y)$. Keep y fixed, treating it as a constant or coefficient, as appropriate:
   $$f_x(x, y) = 3x^2y^2 - 1.$$
   Note how the $5y^2$ and 7 terms go to zero.

   To compute $f_y(x, y)$, we hold x fixed:
   $$f_y(x, y) = 2x^3y + 10y.$$
   Note how the $-x$ and 7 terms go to zero.

2. We have $f(x, y) = \cos(xy^2) + \sin x$.
   Begin with $f_x(x, y)$. We need to apply the Chain Rule with the cosine term; y^2 is the coefficient of the x-term inside the cosine function.
   $$f_x(x, y) = -\sin(xy^2)(y^2) + \cos x = -y^2\sin(xy^2) + \cos x.$$
   To find $f_y(x, y)$, note that x is the coefficient of the y^2 term inside of the cosine term; also note that since x is fixed, $\sin x$ is also fixed, and we treat it as a constant.
   $$f_y(x, y) = -\sin(xy^2)(2xy) = -2xy\sin(xy^2).$$

3. We have $f(x, y) = e^{x^2y^3}\sqrt{x^2+1}$.
   Beginning with $f_x(x, y)$, note how we need to apply the Product Rule.
   $$\begin{aligned} f_x(x, y) &= e^{x^2y^3}(2xy^3)\sqrt{x^2+1} + e^{x^2y^3}\frac{1}{2}\big(x^2+1\big)^{-1/2}(2x) \\ &= 2xy^3e^{x^2y^3}\sqrt{x^2+1} + \frac{xe^{x^2y^3}}{\sqrt{x^2+1}}. \end{aligned}$$
   Note that when finding $f_y(x, y)$ we do not have to apply the Product Rule; since $\sqrt{x^2+1}$ does not contain y, we treat it as fixed and hence becomes a coefficient of the $e^{x^2y^3}$ term.
   $$f_y(x, y) = e^{x^2y^3}(3x^2y^2)\sqrt{x^2+1} = 3x^2y^2e^{x^2y^3}\sqrt{x^2+1}.$$

We have shown *how* to compute a partial derivative, but it may still not be clear what a partial derivative *means*. Given $z = f(x, y)$, $f_x(x, y)$ measures the rate at which z changes as only x varies: y is held constant.

Imagine standing in a rolling meadow, then beginning to walk due east. Depending on your location, you might walk up, sharply down, or perhaps not change elevation at all. This is similar to measuring z_x: you are moving only east (in the "x"-direction) and not north/south at all. Going back to your original location, imagine now walking due north (in the "y"-direction). Perhaps walking due north does not change your elevation at all. This is analogous to $z_y = 0$: z does not change with respect to y. We can see that z_x and z_y do not have to be the same, or even similar, as it is easy to imagine circumstances where walking east means you walk downhill, though walking north makes you walk uphill.

The following example helps us visualize this more.

Example 13.3.3 Evaluating partial derivatives
Let $z = f(x, y) = -x^2 - \frac{1}{2}y^2 + xy + 10$. Find $f_x(2, 1)$ and $f_y(2, 1)$ and interpret their meaning.

SOLUTION We begin by computing $f_x(x, y) = -2x + y$ and $f_y(x, y) = -y + x$. Thus

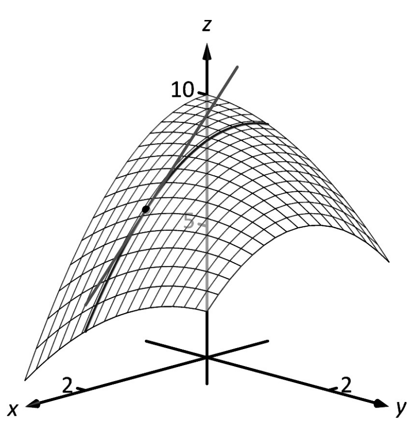
$$f_x(2, 1) = -3 \quad \text{and} \quad f_y(2, 1) = 1.$$

It is also useful to note that $f(2, 1) = 7.5$. What does each of these numbers mean?

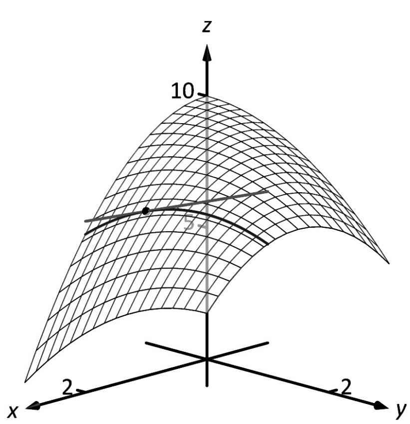
Consider $f_x(2, 1) = -3$, along with Figure 13.3.2(a). If one "stands" on the surface at the point $(2, 1, 7.5)$ and moves parallel to the x-axis (i.e., only the x-value changes, not the y-value), then the instantaneous rate of change is -3. Increasing the x-value will decrease the z-value; decreasing the x-value will increase the z-value.

Now consider $f_y(2, 1) = 1$, illustrated in Figure 13.3.2(b). Moving along the curve drawn on the surface, i.e., parallel to the y-axis and not changing the x-values, increases the z-value instantaneously at a rate of 1. Increasing the y-value by 1 would increase the z-value by approximately 1.

Since the magnitude of f_x is greater than the magnitude of f_y at $(2, 1)$, it is "steeper" in the x-direction than in the y-direction.



(a)



(b)

Figure 13.3.2: Illustrating the meaning of partial derivatives.

Second Partial Derivatives

Let $z = f(x,y)$. We have learned to find the partial derivatives $f_x(x,y)$ and $f_y(x,y)$, which are each functions of x and y. Therefore we can take partial derivatives of them, each with respect to x and y. We define these "second partials" along with the notation, give examples, then discuss their meaning.

Definition 13.3.2 Second Partial Derivative, Mixed Partial Derivative

Let $z = f(x,y)$ be continuous on an open set S.

1. The **second partial derivative of f with respect to x then x** is
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \left(f_x\right)_x = f_{xx}$$

2. The **second partial derivative of f with respect to x then y** is
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \left(f_x\right)_y = f_{xy}$$

Similar definitions hold for $\dfrac{\partial^2 f}{\partial y^2} = f_{yy}$ and $\dfrac{\partial^2 f}{\partial x \partial y} = f_{yx}$.

The second partial derivatives f_{xy} and f_{yx} are **mixed partial derivatives**.

Note: The terms in Definition 13.3.2 all depend on limits, so each definition comes with the caveat "where the limit exists."

The notation of second partial derivatives gives some insight into the notation of the second derivative of a function of a single variable. If $y = f(x)$, then $f''(x) = \dfrac{d^2y}{dx^2}$. The "$d^2y$" portion means "take the derivative of y twice," while "dx^2" means "with respect to x both times." When we only know of functions of a single variable, this latter phrase seems silly: there is only one variable to take the derivative with respect to. Now that we understand functions of multiple variables, we see the importance of specifying which variables we are referring to.

Example 13.3.4 Second partial derivatives

For each of the following, find all six first and second partial derivatives. That is, find
$$f_x, \quad f_y, \quad f_{xx}, \quad f_{yy}, \quad f_{xy} \quad \text{and} \quad f_{yx}\,.$$

1. $f(x,y) = x^3y^2 + 2xy^3 + \cos x$
2. $f(x,y) = \dfrac{x^3}{y^2}$
3. $f(x,y) = e^x \sin(x^2y)$

Solution In each, we give f_x and f_y immediately and then spend time deriving the second partial derivatives.

1. $f(x,y) = x^3y^2 + 2xy^3 + \cos x$

   $f_x(x,y) = 3x^2y^2 + 2y^3 - \sin x$

$f_y(x,y) = 2x^3y + 6xy^2$

$f_{xx}(x,y) = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\big(3x^2y^2 + 2y^3 - \sin x\big) = 6xy^2 - \cos x$

$f_{yy}(x,y) = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\big(2x^3y + 6xy^2\big) = 2x^3 + 12xy$

$f_{xy}(x,y) = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\big(3x^2y^2 + 2y^3 - \sin x\big) = 6x^2y + 6y^2$

$f_{yx}(x,y) = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\big(2x^3y + 6xy^2\big) = 6x^2y + 6y^2$

2. $f(x,y) = \dfrac{x^3}{y^2} = x^3y^{-2}$

$f_x(x,y) = \dfrac{3x^2}{y^2}$

$f_y(x,y) = -\dfrac{2x^3}{y^3}$

$f_{xx}(x,y) = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\Big(\frac{3x^2}{y^2}\Big) = \frac{6x}{y^2}$

$f_{yy}(x,y) = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\Big(-\frac{2x^3}{y^3}\Big) = \frac{6x^3}{y^4}$

$f_{xy}(x,y) = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\Big(\frac{3x^2}{y^2}\Big) = -\frac{6x^2}{y^3}$

$f_{yx}(x,y) = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\Big(-\frac{2x^3}{y^3}\Big) = -\frac{6x^2}{y^3}$

3. $f(x,y) = e^x\sin(x^2y)$

Because the following partial derivatives get rather long, we omit the extra notation and just give the results. In several cases, multiple applications of the Product and Chain Rules will be necessary, followed by some basic combination of like terms.

$f_x(x,y) = e^x\sin(x^2y) + 2xye^x\cos(x^2y)$

$f_y(x,y) = x^2e^x\cos(x^2y)$

$f_{xx}(x,y) = e^x\sin(x^2y) + 4xye^x\cos(x^2y) + 2ye^x\cos(x^2y) - 4x^2y^2e^x\sin(x^2y)$

$f_{yy}(x,y) = -x^4e^x\sin(x^2y)$

$f_{xy}(x,y) = x^2e^x\cos(x^2y) + 2xe^x\cos(x^2y) - 2x^3ye^x\sin(x^2y)$

$f_{yx}(x,y) = x^2e^x\cos(x^2y) + 2xe^x\cos(x^2y) - 2x^3ye^x\sin(x^2y)$

Notice how in each of the three functions in Example 13.3.4, $f_{xy} = f_{yx}$. Due to the complexity of the examples, this likely is not a coincidence. The following theorem states that it is not.

Theorem 13.3.1 Clairaut's Theorem

Let f be defined such that f_{xy} and f_{yx} are continuous on an open set S. Then for each point (x, y) in S, $f_{xy}(x, y) = f_{yx}(x, y)$.

Finding f_{xy} and f_{yx} independently and comparing the results provides a convenient way of checking our work.

Understanding Second Partial Derivatives

Now that we know *how* to find second partials, we investigate *what* they tell us.

Again we refer back to a function $y = f(x)$ of a single variable. The second derivative of f is "the derivative of the derivative," or "the rate of change of the rate of change." The second derivative measures how much the derivative is changing. If $f''(x) < 0$, then the derivative is getting smaller (so the graph of f is concave down); if $f''(x) > 0$, then the derivative is growing, making the graph of f concave up.

Now consider $z = f(x, y)$. Similar statements can be made about f_{xx} and f_{yy} as could be made about $f''(x)$ above. When taking derivatives with respect to x twice, we measure how much f_x changes with respect to x. If $f_{xx}(x, y) < 0$, it means that as x increases, f_x decreases, and the graph of f will be concave down *in the x-direction*. Using the analogy of standing in the rolling meadow used earlier in this section, f_{xx} measures whether one's path is concave up/down when walking due east.

Similarly, f_{yy} measures the concavity in the y-direction. If $f_{yy}(x, y) > 0$, then f_y is increasing with respect to y and the graph of f will be concave up in the y-direction. Appealing to the rolling meadow analogy again, f_{yy} measures whether one's path is concave up/down when walking due north.

We now consider the mixed partials f_{xy} and f_{yx}. The mixed partial f_{xy} measures how much f_x changes with respect to y. Once again using the rolling meadow analogy, f_x measures the slope if one walks due east. Looking east, begin walking *north* (side–stepping). Is the path towards the east getting steeper? If so, $f_{xy} > 0$. Is the path towards the east not changing in steepness? If so, then $f_{xy} = 0$. A similar thing can be said about f_{yx}: consider the steepness of paths heading north while side–stepping to the east.

The following example examines these ideas with concrete numbers and graphs.

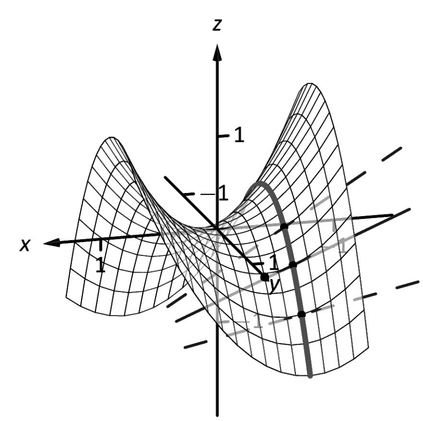
Example 13.3.5 Understanding second partial derivatives

Let $z = x^2 - y^2 + xy$. Evaluate the 6 first and second partial derivatives at $(-1/2, 1/2)$ and interpret what each of these numbers mean.

Solution We find that:

$f_x(x, y) = 2x + y$, $f_y(x, y) = -2y + x$, $f_{xx}(x, y) = 2$, $f_{yy}(x, y) = -2$ and $f_{xy}(x, y) = f_{yx}(x, y) = 1$. Thus at $(-1/2, 1/2)$ we have

$$f_x(-1/2, 1/2) = -1/2, \qquad f_y(-1/2, 1/2) = -3/2.$$



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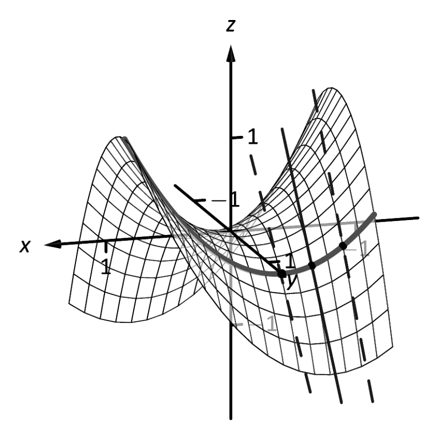


(b)

Figure 13.3.3: Understanding the second partial derivatives in Example 13.3.5.

The slope of the tangent line at $(-1/2, 1/2, -1/4)$ in the direction of x is $-1/2$: if one moves from that point parallel to the x-axis, the instantaneous rate of change will be $-1/2$. The slope of the tangent line at this point in the direction of y is $-3/2$: if one moves from this point parallel to the y-axis, the instantaneous rate of change will be $-3/2$. These tangents lines are graphed in Figure 13.3.3(a) and (b), respectively, where the tangent lines are drawn in a solid line.

Now consider only Figure 13.3.3(a). Three directed tangent lines are drawn (two are dashed), each in the direction of x; that is, each has a slope determined by f_x. Note how as y increases, the slope of these lines get closer to 0. Since the slopes are all negative, getting closer to 0 means the *slopes are increasing.* The slopes given by f_x are increasing as y increases, meaning f_{xy} must be positive.

Since $f_{xy} = f_{yx}$, we also expect f_y to increase as x increases. Consider Figure 13.3.3(b) where again three directed tangent lines are drawn, this time each in the direction of y with slopes determined by f_y. As x increases, the slopes become less steep (closer to 0). Since these are negative slopes, this means the slopes are increasing.

Thus far we have a visual understanding of f_x, f_y, and $f_{xy} = f_{yx}$. We now interpret f_{xx} and f_{yy}. In Figure 13.3.3(a), we see a curve drawn where x is held constant at $x = -1/2$: only y varies. This curve is clearly concave down, corresponding to the fact that $f_{yy} < 0$. In part (b) of the figure, we see a similar curve where y is constant and only x varies. This curve is concave up, corresponding to the fact that $f_{xx} > 0$.

Partial Derivatives and Functions of Three Variables

The concepts underlying partial derivatives can be easily extend to more than two variables. We give some definitions and examples in the case of three variables and trust the reader can extend these definitions to more variables if needed.

Definition 13.3.3 Partial Derivatives with Three Variables

Let $w = f(x, y, z)$ be a continuous function on an open set S in $\mathbb{R}^3$.
The **partial derivative of** f **with respect to** x is:

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}.$$

Similar definitions hold for $f_y(x, y, z)$ and $f_z(x, y, z)$.

By taking partial derivatives of partial derivatives, we can find second partial derivatives of f with respect to z then y, for instance, just as before.

Example 13.3.6 Partial derivatives of functions of three variables
For each of the following, find f_x, f_y, f_z, f_{xz}, f_{yz}, and f_{zz}.

1. $f(x, y, z) = x^2y^3z^4 + x^2y^2 + x^3z^3 + y^4z^4$
2. $f(x, y, z) = x\sin(yz)$

Solution

1. $f_x = 2xy^3z^4 + 2xy^2 + 3x^2z^3$; $\quad f_y = 3x^2y^2z^4 + 2x^2y + 4y^3z^4$;
   $f_z = 4x^2y^3z^3 + 3x^3z^2 + 4y^4z^3$; $\quad f_{xz} = 8xy^3z^3 + 9x^2z^2$;

$f_{yz} = 12x^2y^2z^3 + 16y^3z^3; \quad f_{zz} = 12x^2y^3z^2 + 6x^3z + 12y^4z^2$

2. $f_x = \sin(yz); \quad f_y = xz\cos(yz); \quad f_z = xy\cos(yz);$
   $f_{xz} = y\cos(yz); \quad f_{yz} = x\cos(yz) - xyz\sin(yz); \quad f_{zz} = -xy^2\sin(xy)$

Higher Order Partial Derivatives

We can continue taking partial derivatives of partial derivatives of partial derivatives of …; we do not have to stop with second partial derivatives. These higher order partial derivatives do not have a tidy graphical interpretation; nevertheless they are not hard to compute and worthy of some practice.

We do not formally define each higher order derivative, but rather give just a few examples of the notation.

$$f_{xyx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) \quad \text{and}$$
$$f_{xyz}(x,y,z) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right).$$

Example 13.3.7 Higher order partial derivatives

1. Let $f(x,y) = x^2y^2 + \sin(xy)$. Find f_{xxy} and f_{yxx}.
2. Let $f(x,y,z) = x^3e^{xy} + \cos(z)$. Find f_{xyz}.

Solution

1. To find f_{xxy}, we first find f_x, then f_{xx}, then f_{xxy}:

$$\begin{aligned} f_x &= 2xy^2 + y\cos(xy) \qquad f_{xx} = 2y^2 - y^2\sin(xy) \\ f_{xxy} &= 4y - 2y\sin(xy) - xy^2\cos(xy). \end{aligned}$$

   To find f_{yxx}, we first find f_y, then f_{yx}, then f_{yxx}:

$$\begin{aligned} f_y &= 2x^2y + x\cos(xy) \qquad f_{yx} = 4xy + \cos(xy) - xy\sin(xy) \\ f_{yxx} &= 4y - y\sin(xy) - \big(y\sin(xy) + xy^2\cos(xy)\big) \\ &= 4y - 2y\sin(xy) - xy^2\cos(xy). \end{aligned}$$

   Note how $f_{xxy} = f_{yxx}$.

2. To find f_{xyz}, we find f_x, then f_{xy}, then f_{xyz}:

$$\begin{aligned} f_x &= 3x^2e^{xy} + x^3ye^{xy} \qquad f_{xy} = 3x^3e^{xy} + x^3e^{xy} + x^4ye^{xy} = 4x^3e^{xy} + x^4ye^{xy} \\ f_{xyz} &= 0. \end{aligned}$$

In the previous example we saw that $f_{xxy} = f_{yxx}$; this is not a coincidence. While we do not state this as a formal theorem, as long as each partial derivative is continuous, it does not matter the order in which the partial derivatives are taken. For instance, $f_{xxy} = f_{xyx} = f_{yxx}$.

This can be useful at times. Had we known this, the second part of Example 13.3.7 would have been much simpler to compute. Instead of computing f_{xyz} in the x, y then z orders, we could have applied the z, then x then y order (as $f_{xyz} = f_{zxy}$). It is easy to see that $f_z = -\sin z$; then f_{zx} and f_{zxy} are clearly 0 as f_z does not contain an x or y.

A brief review of this section: partial derivatives measure the instantaneous rate of change of a multivariable function with respect to one variable. With $z = f(x, y)$, the partial derivatives f_x and f_y measure the instantaneous rate of change of z when moving parallel to the x- and y-axes, respectively. How do we measure the rate of change at a point when we do not move parallel to one of these axes? What if we move in the direction given by the vector $\langle 2, 1\rangle$? Can we measure that rate of change? The answer is, of course, yes, we can. This is

Exercises 13.3

Terms and Concepts

1. What is the difference between a constant and a coefficient?

2. Given a function $z = f(x, y)$, explain in your own words how to compute f_x.

3. In the mixed partial fraction f_{xy}, which is computed first, f_x or f_y?

4. In the mixed partial fraction $\dfrac{\partial^2 f}{\partial x \partial y}$, which is computed first, f_x or f_y?

Problems

In Exercises 5 – 8, evaluate $f_x(x, y)$ and $f_y(x, y)$ at the indicated point.

5. $f(x, y) = x^2y - x + 2y + 3$ at $(1, 2)$

6. $f(x, y) = x^3 - 3x + y^2 - 6y$ at $(-1, 3)$

7. $f(x, y) = \sin y \cos x$ at $(\pi/3, \pi/3)$

8. $f(x, y) = \ln(xy)$ at $(-2, -3)$

In Exercises 9 – 26, find f_x, f_y, f_{xx}, f_{yy}, f_{xy} and f_{yx}.

9. $f(x, y) = x^2y + 3x^2 + 4y - 5$

10. $f(x, y) = y^3 + 3xy^2 + 3x^2y + x^3$

11. $f(x, y) = \dfrac{x}{y}$

12. $f(x, y) = \dfrac{4}{xy}$

13. $f(x, y) = e^{x^2+y^2}$

14. $f(x, y) = e^{x+2y}$

15. $f(x, y) = \sin x \cos y$

16. $f(x, y) = (x + y)^3$

17. $f(x, y) = \cos(5xy^3)$

18. $f(x, y) = \sin(5x^2 + 2y^3)$

19. $f(x, y) = \sqrt{4xy^2 + 1}$

20. $f(x, y) = (2x + 5y)\sqrt{y}$

21. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

22. $f(x, y) = 5x - 17y$

23. $f(x, y) = 3x^2 + 1$

24. $f(x, y) = \ln(x^2 + y)$

25. $f(x, y) = \dfrac{\ln x}{4y}$

26. $f(x, y) = 5e^x \sin y + 9$

In Exercises 27 – 30, form a function $z = f(x, y)$ such that f_x and f_y match those given.

27. $f_x = \sin y + 1, \quad f_y = x \cos y$

28. $f_x = x + y, \quad f_y = x + y$

29. $f_x = 6xy - 4y^2, \quad f_y = 3x^2 - 8xy + 2$

30. $f_x = \dfrac{2x}{x^2 + y^2}, \quad f_y = \dfrac{2y}{x^2 + y^2}$

In Exercises 31 – 34, find f_x, f_y, f_z, f_{yz} and f_{zy}.

31. $f(x, y, z) = x^2e^{2y-3z}$

32. $f(x, y, z) = x^3y^2 + x^3z + y^2z$

33. $f(x, y, z) = \dfrac{3x}{7y^2z}$

34. $f(x, y, z) = \ln(xyz)$

13.4 Tangent Lines, Normal Lines, and Tangent Planes

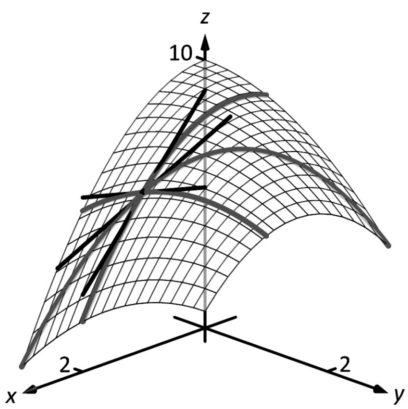


Figure 13.4.1: Showing various lines tangent to a surface.

Derivatives and tangent lines go hand–in–hand. Given $y = f(x)$, the line tangent to the graph of f at $x = x_0$ is the line through $\big(x_0, f(x_0)\big)$ with slope $f'(x_0)$; that is, the slope of the tangent line is the instantaneous rate of change of f at x_0.

When dealing with functions of two variables, the graph is no longer a curve but a surface. At a given point on the surface, it seems there are many lines that fit our intuition of being "tangent" to the surface.

In Figure 13.4.1 we see lines that are tangent to curves in space. Since each curve lies on a surface, it makes sense to say that the lines are also tangent to the surface. The next definition formally defines what it means to be "tangent to a surface."

Definition 13.4.1 Directional Tangent Line

Let $z = f(x, y)$ be differentiable on an open set S containing (x_0, y_0) and let $\vec{u} = \langle u_1, u_2\rangle$ be a unit vector.

1. The line ℓ_x through $\big(x_0, y_0, f(x_0, y_0)\big)$ parallel to $\langle 1, 0, f_x(x_0, y_0)\rangle$ is the **tangent line to f in the direction of x at** (x_0, y_0).
2. The line ℓ_y through $\big(x_0, y_0, f(x_0, y_0)\big)$ parallel to $\langle 0, 1, f_y(x_0, y_0)\rangle$ is the **tangent line to f in the direction of y at** (x_0, y_0).
3. The line $\ell_{\vec{u}}$ through $\big(x_0, y_0, f(x_0, y_0)\big)$ parallel to $\langle u_1, u_2, D_{\vec{u}}f(x_0, y_0)\rangle$ is the **tangent line to f in the direction of $\vec{u}$ at** (x_0, y_0).

It is instructive to consider each of three directions given in the definition in terms of "slope." The direction of ℓ_x is $\langle 1, 0, f_x(x_0, y_0)\rangle$; that is, the "run" is one unit in the x-direction and the "rise" is $f_x(x_0, y_0)$ units in the z-direction. Note how the slope is just the partial derivative with respect to x. A similar statement can be made for ℓ_y. The direction of $\ell_{\vec{u}}$ is $\langle u_1, u_2, D_{\vec{u}}f(x_0, y_0)\rangle$; the "run" is one unit in the $\vec{u}$ direction (where $\vec{u}$ is a unit vector) and the "rise" is the directional derivative of z in that direction.

Definition 13.4.1 leads to the following parametric equations of directional tangent lines:

$$\ell_x(t) = \begin{cases} x = x_0 + t \\ y = y_0 \\ z = z_0 + f_x(x_0, y_0)t \end{cases}, \quad \ell_y(t) = \begin{cases} x = x_0 \\ y = y_0 + t \\ z = z_0 + f_y(x_0, y_0)t \end{cases} \quad \text{and} \quad \ell_{\vec{u}}(t) = \begin{cases} x = x_0 + u_1t \\ y = y_0 + u_2t \\ z = z_0 + D_{\vec{u}}f(x_0, y_0)t \end{cases}.$$

Example 13.4.1 Finding directional tangent lines

Find the lines tangent to the surface $z = \sin x \cos y$ at $(\pi/2, \pi/2)$ in the x and y directions and also in the direction of $\vec{v} = \langle -1, 1\rangle$.

Solution The partial derivatives with respect to x and y are:

$$\begin{aligned} f_x(x, y) &= \cos x \cos y \quad \Rightarrow \quad f_x(\pi/2, \pi/2) = 0 \\ f_y(x, y) &= -\sin x \sin y \quad \Rightarrow \quad f_y(\pi/2, \pi/2) = -1. \end{aligned}$$

At $(\pi/2, \pi/2)$, the z-value is 0.

Thus the parametric equations of the line tangent to f at $(\pi/2, \pi/2)$ in the directions of x and y are:

$$\ell_x(t) = \begin{cases} x = \pi/2 + t \\ y = \pi/2 \\ z = 0 \end{cases} \quad \text{and} \quad \ell_y(t) = \begin{cases} x = \pi/2 \\ y = \pi/2 + t \\ z = -t \end{cases}.$$

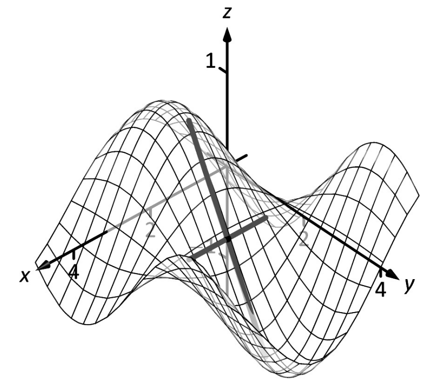
The two lines are shown with the surface in Figure 13.4.2(a). To find the equation of the tangent line in the direction of $\vec{v}$, we first find the unit vector in the direction of $\vec{v}$: $\vec{u} = \left\langle -1/\sqrt{2}, 1/\sqrt{2}\right\rangle$. The directional derivative at $(\pi/2, \pi, 2)$ in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(\pi/2, \pi, 2) = \langle 0, -1\rangle \cdot \left\langle -1/\sqrt{2}, 1/\sqrt{2}\right\rangle = -1/\sqrt{2}.$$

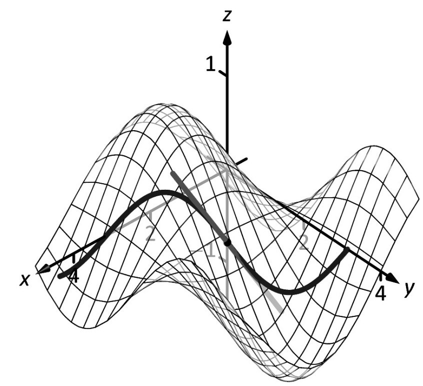
Thus the directional tangent line is

$$\ell_{\vec{u}}(t) = \begin{cases} x = \pi/2 - t/\sqrt{2} \\ y = \pi/2 + t/\sqrt{2} \\ z = -t/\sqrt{2} \end{cases}.$$

The curve through $(\pi/2, \pi/2, 0)$ in the direction of $\vec{v}$ is shown in Figure 13.4.2(b) along with $\ell_{\vec{u}}(t)$.



(a)



(b)

Figure 13.4.2: A surface and directional tangent lines in Example 13.4.1.

Example 13.4.2 Finding directional tangent lines

Let $f(x, y) = 4xy - x^4 - y^4$. Find the equations of *all* directional tangent lines to f at $(1, 1)$.

Solution First note that $f(1, 1) = 2$. We need to compute directional derivatives, so we need ∇f. We begin by computing partial derivatives.

$$f_x = 4y - 4x^3 \Rightarrow f_x(1, 1) = 0; \quad f_y = 4x - 4y^3 \Rightarrow f_y(1, 1) = 0.$$

Thus $\nabla f(1, 1) = \langle 0, 0\rangle$. Let $\vec{u} = \langle u_1, u_2\rangle$ be any unit vector. The directional derivative of f at $(1, 1)$ will be $D_{\vec{u}} f(1, 1) = \langle 0, 0\rangle \cdot \langle u_1, u_2\rangle = 0$. It does not matter what direction we choose; the directional derivative is always 0. Therefore

$$\ell_{\vec{u}}(t) = \begin{cases} x = 1 + u_1 t \\ y = 1 + u_2 t \\ z = 2 \end{cases}.$$

Figure 13.4.3 shows a graph of f and the point $(1, 1, 2)$. Note that this point comes at the top of a "hill," and therefore every tangent line through this point will have a "slope" of 0.

That is, consider any curve on the surface that goes through this point. Each curve will have a relative maximum at this point, hence its tangent line will have a slope of 0. The following section investigates the points on surfaces where all tangent lines have a slope of 0.

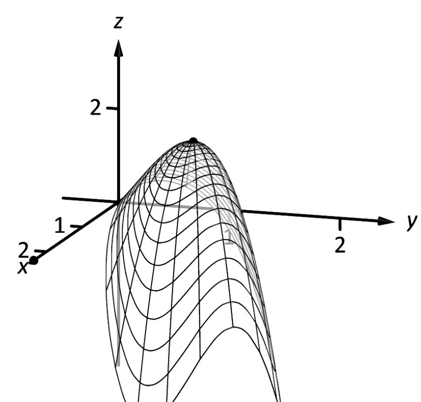


Figure 13.4.3: Graphing f in Example 13.4.2.

Normal Lines

When dealing with a function $y = f(x)$ of one variable, we stated that a line through $(c, f(c))$ was *tangent* to f if the line had a slope of $f'(c)$ and was *normal* (or, *perpendicular, orthogonal*) to f if it had a slope of $-1/f'(c)$. We extend the concept of normal, or orthogonal, to functions of two variables.

Let $z = f(x, y)$ be a differentiable function of two variables. By Definition 13.4.1, at (x_0, y_0), $\ell_x(t)$ is a line parallel to the vector $\vec{d}_x = \langle 1, 0, f_x(x_0, y_0)\rangle$ and $\ell_y(t)$ is a line parallel to $\vec{d}_y = \langle 0, 1, f_y(x_0, y_0)\rangle$. Since lines in these directions through $\big(x_0, y_0, f(x_0, y_0)\big)$ are *tangent* to the surface, a line through this point and orthogonal to these directions would be *orthogonal*, or *normal*, to the surface. We can use this direction to create a normal line.

The direction of the normal line is orthogonal to $\vec{d}_x$ and $\vec{d}_y$, hence the direction is parallel to $\vec{d}_n = \vec{d}_x \times \vec{d}_y$. It turns out this cross product has a very simple form:

$$\vec{d}_x \times \vec{d}_y = \langle 1, 0, f_x\rangle \times \langle 0, 1, f_y\rangle = \langle -f_x, -f_y, 1\rangle\,.$$

It is often more convenient to refer to the opposite of this direction, namely $\langle f_x, f_y, -1\rangle$. This leads to a definition.

Definition 13.4.2 Normal Line

Let $z = f(x, y)$ be differentiable on an open set S containing (x_0, y_0) where

$$a = f_x(x_0, y_0) \quad \text{and} \quad b = f_y(x_0, y_0)$$

are defined.

1. A nonzero vector parallel to $\vec{n} = \langle a, b, -1\rangle$ is **orthogonal to f at** $P = \big(x_0, y_0, f(x_0, y_0)\big)$.
2. The line ℓ_n through P with direction parallel to $\vec{n}$ is the **normal line to f at P**.

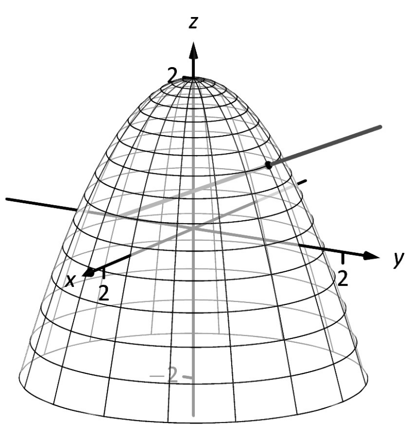


Figure 13.4.4: Graphing a surface with a normal line from Example 13.4.3.

Thus the parametric equations of the normal line to a surface f at $\big(x_0, y_0, f(x_0, y_0)\big)$ is:

$$\ell_n(t) = \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = f(x_0, y_0) - t \end{cases}.$$

Example 13.4.3 Finding a normal line

Find the equation of the normal line to $z = -x^2 - y^2 + 2$ at $(0, 1)$.

Solution We find $z_x(x, y) = -2x$ and $z_y(x, y) = -2y$; at $(0, 1)$, we have $z_x = 0$ and $z_y = -2$. We take the direction of the normal line, following Definition 13.4.2, to be $\vec{n} = \langle 0, -2, -1\rangle$. The line with this direction going through the point $(0, 1, 1)$ is

$$\ell_n(t) = \begin{cases} x = 0 \\ y = -2t + 1 \\ z = -t + 1 \end{cases} \quad \text{or} \quad \ell_n(t) = \langle 0, -2, -1\rangle\, t + \langle 0, 1, 1\rangle\,.$$

The surface $z = -x^2 - y^2 + 2$, along with the found normal line, is graphed in Figure 13.4.4.

The direction of the normal line has many uses, one of which is the definition of the **tangent plane** which we define shortly. Another use is in measuring distances from the surface to a point. Given a point Q in space, it is a general geometric concept to define the distance from Q to the surface as being the length of the shortest line segment $\overline{PQ}$ over all points P on the surface. This, in turn, implies that $\overrightarrow{PQ}$ will be orthogonal to the surface at P. Therefore we can measure the distance from Q to the surface f by finding a point P on the surface such that $\overrightarrow{PQ}$ is parallel to the normal line to f at P.

Example 13.4.4 Finding the distance from a point to a surface

Let $f(x,y) = 2 - x^2 - y^2$ and let $Q = (2,2,2)$. Find the distance from Q to the surface defined by f.

Solution This surface is used in Example 13.4.2, so we know that at (x,y), the direction of the normal line will be $\vec{d}_n = \langle -2x, -2y, -1\rangle$. A point P on the surface will have coordinates $(x, y, 2-x^2-y^2)$, so $\overrightarrow{PQ} = \left\langle 2-x, 2-y, x^2+y^2\right\rangle$. To find where $\overrightarrow{PQ}$ is parallel to $\vec{d}_n$, we need to find x, y and c such that $c\overrightarrow{PQ} = \vec{d}_n$.

$$\begin{aligned} c\overrightarrow{PQ} &= \vec{d}_n \\ c\left\langle 2-x, 2-y, x^2+y^2\right\rangle &= \langle -2x, -2y, -1\rangle . \end{aligned}$$

This implies

$$\begin{aligned} c(2-x) &= -2x \\ c(2-y) &= -2y \\ c(x^2+y^2) &= -1 \end{aligned}$$

In each equation, we can solve for c:

$$c = \frac{-2x}{2-x} = \frac{-2y}{2-y} = \frac{-1}{x^2+y^2}.$$

The first two fractions imply $x = y$, and so the last fraction can be rewritten as $c = -1/(2x^2)$. Then

$$\begin{aligned} \frac{-2x}{2-x} &= \frac{-1}{2x^2} \\ -2x(2x^2) &= -1(2-x) \\ 4x^3 &= 2-x \\ 4x^3 + x - 2 &= 0. \end{aligned}$$

This last equation is a cubic, which is not difficult to solve with a numeric solver. We find that $x = 0.689$, hence $P = (0.689, 0.689, 1.051)$. We find the distance from Q to the surface of f is

$$\parallel \overrightarrow{PQ} \parallel = \sqrt{(2-0.689)^2 + (2-0.689)^2 + (2-1.051)^2} = 2.083.$$

We can take the concept of measuring the distance from a point to a surface to find a point Q a particular distance from a surface at a given point P on the surface.

Example 13.4.5 Finding a point a set distance from a surface

Let $f(x,y) = x - y^2 + 3$. Let $P = \big(2, 1, f(2,1)\big) = (2,1,4)$. Find points Q in space that are 4 units from the surface of f at P. That is, find Q such that $\parallel \overrightarrow{PQ} \parallel = 4$ and $\overrightarrow{PQ}$ is orthogonal to f at P.

SOLUTION We begin by finding partial derivatives:

$$\begin{aligned} f_x(x,y) &= 1 & \Rightarrow \quad & f_x(2,1) = 1 \\ f_y(x,y) &= -2y & \Rightarrow \quad & f_y(2,1) = -2 \end{aligned}$$

The vector $\vec{n} = \langle 1, -2, -1\rangle$ is orthogonal to f at P. For reasons that will become more clear in a moment, we find the unit vector in the direction of $\vec{n}$:

$$\vec{u} = \frac{\vec{n}}{\parallel \vec{n} \parallel} = \Big\langle 1/\sqrt{6}, -2/\sqrt{6}, -1/\sqrt{6}\Big\rangle \approx \langle 0.408, -0.816, -0.408\rangle .$$

Thus a the normal line to f at P can be written as

$$\ell_n(t) = \langle 2, 1, 4\rangle + t\,\langle 0.408, -0.816, -0.408\rangle .$$

An advantage of this parametrization of the line is that letting $t = t_0$ gives a point on the line that is $|t_0|$ units from P. (This is because the direction of the line is given in terms of a unit vector.) There are thus two points in space 4 units from P:

$$\begin{aligned} Q_1 &= \ell_n(4) & \qquad Q_2 &= \ell_n(-4) \\ &\approx \langle 3.63, -2.27, 2.37\rangle & &\approx \langle 0.37, 4.27, 5.63\rangle \end{aligned}$$

The surface is graphed along with points P, Q_1, Q_2 and a portion of the normal line to f at P.

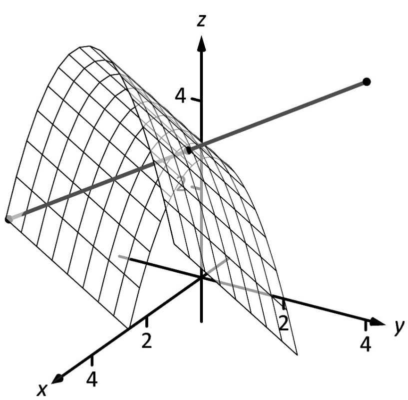


Figure 13.4.5: Graphing the surface in Example 13.4.5 along with points 4 units from the surface.

Tangent Planes

We can use the direction of the normal line to define a plane. With $a = f_x(x_0, y_0)$, $b = f_y(x_0, y_0)$ and $P = \big(x_0, y_0, f(x_0, y_0)\big)$, the vector $\vec{n} = \langle a, b, -1\rangle$ is orthogonal to f at P. The plane through P with normal vector $\vec{n}$ is therefore **tangent** to f at P.

Definition 13.4.3 Tangent Plane

Let $z = f(x,y)$ be differentiable on an open set S containing (x_0, y_0), where $a = f_x(x_0, y_0)$, $b = f_y(x_0, y_0)$, $\vec{n} = \langle a, b, -1\rangle$ and $P = \big(x_0, y_0, f(x_0, y_0)\big)$.

The plane through P with normal vector $\vec{n}$ is the **tangent plane to** f **at** P. The standard form of this plane is

$$a(x - x_0) + b(y - y_0) - \big(z - f(x_0, y_0)\big) = 0.$$

Example 13.4.6 Finding tangent planes

Find the equation of the tangent plane to $z = -x^2 - y^2 + 2$ at $(0, 1)$.

Solution Note that this is the same surface and point used in Example 13.4.3. There we found $\vec{n} = \langle 0, -2, -1\rangle$ and $P = (0, 1, 1)$. Therefore the equation of the tangent plane is

$$-2(y-1) - (z-1) = 0.$$

The surface $z = -x^2 - y^2 + 2$ and tangent plane are graphed in Figure 13.4.6.

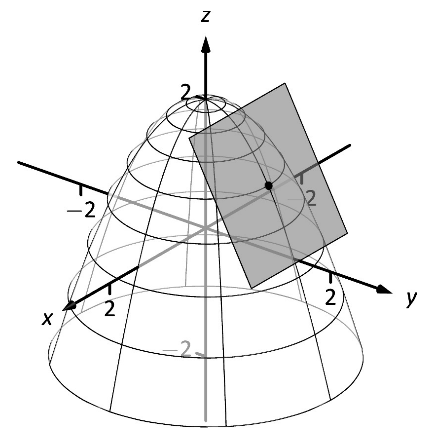


Figure 13.4.6: Graphing a surface with tangent plane from Example 13.4.6.

Example 13.4.7 Using the tangent plane to approximate function values

The point $(3, -1, 4)$ lies on the surface of an unknown differentiable function f where $f_x(3, -1) = 2$ and $f_y(3, -1) = -1/2$. Find the equation of the tangent plane to f at P, and use this to approximate the value of $f(2.9, -0.8)$.

Solution Knowing the partial derivatives at $(3, -1)$ allows us to form the normal vector to the tangent plane, $\vec{n} = \langle 2, -1/2, -1\rangle$. Thus the equation of the tangent line to f at P is:

$$2(x-3) - 1/2(y+1) - (z-4) = 0 \quad \Rightarrow \quad z = 2(x-3) - 1/2(y+1) + 4. \tag{13.1}$$

Just as tangent lines provide excellent approximations of curves near their point of intersection, tangent planes provide excellent approximations of surfaces near their point of intersection. So $f(2.9, -0.8) \approx z(2.9, -0.8) = 3.7$.

This is not a new method of approximation. Compare the right hand expression for z in Equation (13.1) to the total differential:

$$dz = f_x dx + f_y dy \quad \text{and} \quad z = \underbrace{\underbrace{2}_{f_x}\underbrace{(x-3)}_{dx} + \underbrace{-1/2}_{f_y}\underbrace{(y+1)}_{dy}}_{dz} + 4.$$

Thus the "new z-value" is the sum of the change in z (i.e., dz) and the old z-value (4). As mentioned when studying the total differential, it is not uncommon to know partial derivative information about a unknown function, and tangent planes are used to give accurate approximations of the function.

Tangent lines and planes to surfaces have many uses, including the study of instantaneous rates of changes and making approximations. Normal lines also have many uses. In this section we focused on using them to measure distances from a surface. Another interesting application is in computer graphics, where the effects of light on a surface are determined using normal vectors.

Exercises 13.4

Terms and Concepts

1. Explain how the vector $\vec{v} = \langle 1, 0, 3\rangle$ can be thought of as having a "slope" of 3.

2. Explain how the vector $\vec{v} = \langle 0.6, 0.8, -2\rangle$ can be thought of as having a "slope" of -2.

3. T/F: Let $z = f(x, y)$ be differentiable at P. If $\vec{n}$ is a normal vector to the tangent plane of f at P, then $\vec{n}$ is orthogonal to ℓ_x and ℓ_y at P.

4. Explain in your own words why we do not refer to *the* tangent line to a surface at a point, but rather to *directional* tangent line*s* to a surface at a point.

Problems

In Exercises 5 – 8, a function $z = f(x, y)$, a vector $\vec{v}$ and a point P are given. Give the parametric equations of the following directional tangent lines to f at P:

(a) $\ell_x(t)$

(b) $\ell_y(t)$

(c) $\ell_{\vec{u}}(t)$**, where $\vec{u}$ is the unit vector in the direction of $\vec{v}$.**

5. $f(x, y) = 2x^2y - 4xy^2$, $\vec{v} = \langle 1, 3\rangle$, $P = (2, 3)$.

6. $f(x, y) = 3\cos x \sin y$, $\vec{v} = \langle 1, 2\rangle$, $P = (\pi/3, \pi/6)$.

7. $f(x, y) = 3x - 5y$, $\vec{v} = \langle 1, 1\rangle$, $P = (4, 2)$.

8. $f(x, y) = x^2 - 2x - y^2 + 4y$, $\vec{v} = \langle 1, 1\rangle$, $P = (1, 2)$.

In Exercises 9 – 12, a function $z = f(x, y)$ and a point P are given. Find the equation of the normal line to f at P. Note: these are the same functions as in Exercises 5 – 8.

9. $f(x, y) = 2x^2y - 4xy^2$, $P = (2, 3)$.

10. $f(x, y) = 3\cos x \sin y$, $P = (\pi/3, \pi/6)$.

11. $f(x, y) = 3x - 5y$, $P = (4, 2)$.

12. $f(x, y) = x^2 - 2x - y^2 + 4y$, $P = (1, 2)$.

In Exercises 13 – 16, a function $z = f(x, y)$ and a point P are given. Find the two points that are 2 units from the surface f at P. Note: these are the same functions as in Exercises 5 – 8.

13. $f(x, y) = 2x^2y - 4xy^2$, $P = (2, 3)$.

14. $f(x, y) = 3\cos x \sin y$, $P = (\pi/3, \pi/6)$.

15. $f(x, y) = 3x - 5y$, $P = (4, 2)$.

16. $f(x, y) = x^2 - 2x - y^2 + 4y$, $P = (1, 2)$.

In Exercises 17 – 20, a function $z = f(x, y)$ and a point P are given. Find the equation of the tangent plane to f at P. Note: these are the same functions as in Exercises 5 – 8.

17. $f(x, y) = 2x^2y - 4xy^2$, $P = (2, 3)$.

18. $f(x, y) = 3\cos x \sin y$, $P = (\pi/3, \pi/6)$.

19. $f(x, y) = 3x - 5y$, $P = (4, 2)$.

20. $f(x, y) = x^2 - 2x - y^2 + 4y$, $P = (1, 2)$.

A: Solutions To Selected Problems

Chapter 10

Section 10.1

1. Answers will vary.
3. Answers will vary.
5. $2, \frac{8}{3}, \frac{8}{3}, \frac{32}{15}, \frac{64}{45}$
7. $-\frac{1}{3}, -2, -\frac{81}{5}, -\frac{512}{3}, -\frac{15625}{7}$
9. $a_n = 3n + 1$
11. $a_n = 10 \cdot 2^{n-1}$
13. $1/7$
15. 0
17. diverges
19. converges to 0
21. diverges
23. converges to e
25. converges to 0
27. converges to 2
29. bounded
31. bounded
33. neither bounded above or below
35. monotonically increasing
37. never monotonic
39. Let $\{a_n\}$ be given such that $\lim\limits_{n\to\infty} |a_n| = 0$. By the definition of the limit of a sequence, given any $\varepsilon > 0$, there is a m such that for all $n > m$, $\big| |a_n| - 0 \big| < \varepsilon$. Since $\big| |a_n| - 0 \big| = |a_n - 0|$, this directly implies that for all $n > m$, $|a_n - 0| < \varepsilon$, meaning that $\lim\limits_{n\to\infty} a_n = 0$.
41. A sketch of one proof method:

    Let any $\varepsilon > 0$ be given. Since $\{a_n\}$ and $\{b_n\}$ converge, there exists an $N > 0$ such that for all $n \geq N$, both a_n and b_n are within $\varepsilon/2$ of L; we can conclude that they are at most ε apart from each other. Since $a_n \leq c_n \leq b_n$, one can show that c_n is within ε of L, showing that $\{c_n\}$ also converges to L.

Section 10.2

1. Answers will vary.
3. One sequence is the sequence of terms $\{a_n\}$. The other is the sequence of n^{th} partial sums, $\{S_n\} = \{\sum_{i=1}^{n} a_i\}$.
5. F
7. (a) $-1, -\frac{1}{2}, -\frac{5}{6}, -\frac{7}{12}, -\frac{47}{60}$
   (b) Plot omitted
9. (a) $-1, 0, -1, 0, -1$
   (b) Plot omitted
11. (a) $1, \frac{3}{2}, \frac{5}{3}, \frac{41}{24}, \frac{103}{60}$
    (b) Plot omitted
13. (a) $-0.9, -0.09, -0.819, -0.1629, -0.75339$
    (b) Plot omitted
15. $\lim\limits_{n\to\infty} a_n = 3$; by Theorem 10.2.4 the series diverges.
17. $\lim\limits_{n\to\infty} a_n = \infty$; by Theorem 10.2.4 the series diverges.
19. $\lim\limits_{n\to\infty} a_n = 1/2$; by Theorem 10.2.4 the series diverges.
21. Converges; p-series with $p = 5$.
23. Diverges; geometric series with $r = 6/5$.
25. Diverges; fails n^{th} term test
27. F
29. Diverges; by Theorem 10.2.3 this is half the Harmonic Series, which diverges by growing without bound. "Half of growing without bound" is still growing without bound.
31. (a) $S_n = \frac{1-(1/4)^n}{3/4}$
    (b) Converges to $4/3$.
33. (a) $S_n = \left(\frac{n(n+1)}{2}\right)^2$
    (b) Diverges
35. (a) $S_n = 5\frac{1-1/2^n}{1/2}$
    (b) Converges to 10.
37. (a) $S_n = \frac{1-(-1/3)^n}{4/3}$
    (b) Converges to $3/4$.
39. (a) With partial fractions, $a_n = \frac{3}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$. Thus $S_n = \frac{3}{2}\left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$.
    (b) Converges to 9/4
41. (a) $S_n = \ln\left(1/(n+1)\right)$
    (b) Diverges (to $-\infty$).
43. (a) $a_n = \frac{1}{n(n+3)}$; using partial fractions, the resulting telescoping sum reduces to $S_n = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right)$
    (b) Converges to $11/18$.
45. (a) With partial fractions, $a_n = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$. Thus $S_n = \frac{1}{2}\left(3/2 - \frac{1}{n} - \frac{1}{n+1}\right)$.
    (b) Converges to $3/4$.
47. (a) The n^{th} partial sum of the odd series is $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$. The n^{th} partial sum of the even series is $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}$. Each term of the even series is less than the corresponding term of the odd series, giving us our result.
    (b) The n^{th} partial sum of the odd series is $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$. The n^{th} partial sum of 1 plus the even series is $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2(n-1)}$. Each term of the even series is now greater than or equal to the corresponding term of the odd series, with equality only on the first term. This gives us the result.
    (c) If the odd series converges, the work done in (a) shows the even series converges also. (The sequence of the n^{th} partial sum of the even series is bounded and monotonically increasing.) Likewise, (b) shows that if the even series converges, the odd series will, too. Thus if either series converges, the other does.

    Similarly, (a) and (b) can be used to show that if either series diverges, the other does, too.

(d) If both the even and odd series converge, then their sum would be a convergent series. This would imply that the Harmonic Series, their sum, is convergent. It is not. Hence each series diverges.

Section 10.3

1. continuous, positive and decreasing
3. The Integral Test (we do not have a continuous definition of $n!$ yet) and the Limit Comparison Test (same as above, hence we cannot take its derivative).
5. Converges
7. Diverges
9. Converges
11. Converges
13. Converges; compare to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$, as $1/(n^2+3n-5)\leq 1/n^2$ for all $n>1$.
15. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$, as $1/n\leq \ln n/n$ for all $n\geq 3$.
17. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$. Since $n=\sqrt{n^2}>\sqrt{n^2-1}$, $1/n\leq 1/\sqrt{n^2-1}$ for all $n\geq 2$.
19. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$:
$$\frac{1}{n}=\frac{n^2}{n^3}<\frac{n^2+n+1}{n^3}<\frac{n^2+n+1}{n^3-5},$$
for all $n\geq 1$.
21. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$. Note that
$$\frac{n}{n^2-1}=\frac{n^2}{n^2-1}\cdot\frac{1}{n}>\frac{1}{n},$$
as $\frac{n^2}{n^2-1}>1$, for all $n\geq 2$.
23. Converges; compare to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$.
25. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n}$.
27. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$.
29. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$. Just as $\displaystyle\lim_{n\to 0}\frac{\sin n}{n}=1$, $\displaystyle\lim_{n\to\infty}\frac{\sin(1/n)}{1/n}=1$.
31. Converges; compare to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$.
33. Converges; Integral Test
35. Diverges; the n^{th} Term Test and Direct Comparison Test can be used.
37. Converges; the Direct Comparison Test can be used with sequence $1/3^n$.
39. Diverges; the n^{th} Term Test can be used, along with the Integral Test.
41. (a) Converges; use Direct Comparison Test as $\frac{a_n}{n}<n$.
    (b) Converges; since original series converges, we know $\lim_{n\to\infty}a_n=0$. Thus for large n, $a_na_{n+1}<a_n$.
    (c) Converges; similar logic to part (b) so $(a_n)^2<a_n$.
    (d) May converge; certainly $na_n>a_n$ but that does not mean it does not converge.
    (e) Does not converge, using logic from (b) and n^{th} Term Test.

Section 10.4

1. algebraic, or polynomial.
3. Integral Test, Limit Comparison Test, and Root Test
5. Converges
7. Converges
9. The Ratio Test is inconclusive; the p-Series Test states it diverges.
11. Converges
13. Converges; note the summation can be rewritten as $\displaystyle\sum_{n=1}^{\infty}\frac{2^n n!}{3^n n!}$, to which the Ratio Test or Geometric Series Test can be applied.
15. Converges
17. Converges
19. Diverges
21. Diverges. The Root Test is inconclusive, but the n^{th}-Term Test shows divergence. (The terms of the sequence approach e^2, not 0, as $n\to\infty$.)
23. Converges
25. Diverges; Limit Comparison Test with $1/n$.
27. Converges; Ratio Test or Limit Comparison Test with $1/3^n$.
29. Diverges; n^{th}-Term Test or Limit Comparison Test with 1.
31. Diverges; Direct Comparison Test with $1/n$
33. Converges; Root Test

Section 10.5

1. The signs of the terms do not alternate; in the given series, some terms are negative and the others positive, but they do not necessarily alternate.
3. Many examples exist; one common example is $a_n=(-1)^n/n$.
5. (a) converges
   (b) converges (p-Series)
   (c) absolute
7. (a) diverges (limit of terms is not 0)
   (b) diverges
   (c) n/a; diverges
9. (a) converges
   (b) diverges (Limit Comparison Test with $1/n$)
   (c) conditional
11. (a) diverges (limit of terms is not 0)
    (b) diverges
    (c) n/a; diverges
13. (a) diverges (terms oscillate between ± 1)
    (b) diverges
    (c) n/a; diverges
15. (a) converges

(b) converges (Geometric Series with $r = 2/3$)

(c) absolute

17. (a) converges

(b) converges (Ratio Test)

(c) absolute

19. (a) converges

(b) diverges (p-Series Test with $p = 1/2$)

(c) conditional

21. $S_5 = -1.1906$; $S_6 = -0.6767$;
$$-1.1906 \leq \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)} \leq -0.6767$$

23. $S_6 = 0.3681$; $S_7 = 0.3679$;
$$0.3681 \leq \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \leq 0.3679$$

25. $n = 5$

27. Using the theorem, we find $n = 499$ guarantees the sum is within 0.001 of $\pi/4$. (Convergence is actually faster, as the sum is within ε of $\pi/24$ when $n \geq 249$.)

Section 10.6

1. 1

3. 5

5. $1 + 2x + 4x^2 + 8x^3 + 16x^4$

7. $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

9. (a) $R = \infty$

(b) $(-\infty, \infty)$

11. (a) $R = 1$

(b) $(2, 4]$

13. (a) $R = 2$

(b) $(-2, 2)$

15. (a) $R = 1/5$

(b) $(4/5, 6/5)$

17. (a) $R = 1$

(b) $(-1, 1)$

19. (a) $R = \infty$

(b) $(-\infty, \infty)$

21. (a) $R = 1$

(b) $[-1, 1]$

23. (a) $R = 0$

(b) $x = 0$

25. (a) $f'(x) = \sum_{n=1}^{\infty} n^2 x^{n-1}; \quad (-1, 1)$

(b) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{n}{n+1} x^{n+1}; \quad (-1, 1)$

27. (a) $f'(x) = \sum_{n=1}^{\infty} \frac{n}{2^n} x^{n-1}; \quad (-2, 2)$

(b) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{1}{(n+1)2^n} x^{n+1}; \quad [-2, 2)$

29. (a) $f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}$; $(-\infty, \infty)$

(b) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}; \quad (-\infty, \infty)$

31. $1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4$

33. $1 + x + x^2 + x^3 + x^4$

35. $0 + x + 0x^2 - \frac{1}{6}x^3 + 0x^4$

Section 10.7

1. The Maclaurin polynomial is a special case of Taylor polynomials. Taylor polynomials are centered at a specific x-value; when that x-value is 0, it is a Maclauring polynomial.

3. $p_2(x) = 6 + 3x - 4x^2$.

5. $p_3(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$

7. $p_5(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5$

9. $p_4(x) = \frac{2x^4}{3} + \frac{4x^3}{3} + 2x^2 + 2x + 1$

11. $p_4(x) = x^4 - x^3 + x^2 - x + 1$

13. $p_4(x) = 1 + \frac{1}{2}(-1+x) - \frac{1}{8}(-1+x)^2 + \frac{1}{16}(-1+x)^3 - \frac{5}{128}(-1+x)^4$

15. $p_6(x) = \frac{1}{\sqrt{2}} - \frac{-\frac{\pi}{4}+x}{\sqrt{2}} - \frac{(-\frac{\pi}{4}+x)^2}{2\sqrt{2}} + \frac{(-\frac{\pi}{4}+x)^3}{6\sqrt{2}} + \frac{(-\frac{\pi}{4}+x)^4}{24\sqrt{2}} - \frac{(-\frac{\pi}{4}+x)^5}{120\sqrt{2}} - \frac{(-\frac{\pi}{4}+x)^6}{720\sqrt{2}}$

17. $p_5(x) = \frac{1}{2} - \frac{x-2}{4} + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \frac{1}{32}(x-2)^4 - \frac{1}{64}(x-2)^5$

19. $p_3(x) = \frac{1}{2} + \frac{1+x}{2} + \frac{1}{4}(1+x)^2$

21. $p_3(x) = x - \frac{x^3}{6}$; $p_3(0.1) = 0.09983$. Error is bounded by $\pm\frac{1}{4!} \cdot 0.1^4 \approx \pm 0.000004167$.

23. $p_2(x) = 3 + \frac{1}{6}(-9+x) - \frac{1}{216}(-9+x)^2$; $p_2(10) = 3.16204$. The third derivative of $f(x) = \sqrt{x}$ is bounded on $(8, 11)$ by 0.003. Error is bounded by $\pm\frac{0.003}{3!} \cdot 1^3 = \pm 0.0005$.

25. The n^{th} derivative of $f(x) = e^x$ is bounded by 3 on intervals containing 0 and 1. Thus $|R_n(1)| \leq \frac{3}{(n+1)!} 1^{(n+1)}$. When $n = 7$, this is less than 0.0001.

27. The n^{th} derivative of $f(x) = \cos x$ is bounded by 1 on intervals containing 0 and $\pi/3$. Thus $|R_n(\pi/3)| \leq \frac{1}{(n+1)!}(\pi/3)^{(n+1)}$. When $n = 7$, this is less than 0.0001. Since the Maclaurin polynomial of $\cos x$ only uses even powers, we can actually just use $n = 6$.

29. The n^{th} term is $\frac{1}{n!}x^n$.

31. The n^{th} term is: when n even, 0; when n is odd, $\frac{(-1)^{(n-1)/2}}{n!}x^n$.

33. The n^{th} term is $(-1)^n x^n$.

35. $1 + x + \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3 + \dfrac{1}{24}x^4$

37. $1 + 2x - 2x^2 + 4x^3 - 10x^4$

Section 10.8

1. A Taylor polynomial is a **polynomial**, containing a finite number of terms. A Taylor series is a **series**, the summation of an infinite number of terms.

3. All derivatives of e^x are e^x which evaluate to 1 at $x = 0$.
The Taylor series starts $1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$;
the Taylor series is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

5. The n^{th} derivative of $1/(1-x)$ is $f^{(n)}(x) = (n)!/(1-x)^{n+1}$, which evaluates to $n!$ at $x=0$.
The Taylor series starts $1 + x + x^2 + x^3 + \cdots$;
the Taylor series is $\displaystyle\sum_{n=0}^{\infty} x^n$

7. The Taylor series starts
$0 - (x-\pi/2) + 0x^2 + \frac{1}{6}(x-\pi/2)^3 + 0x^4 - \frac{1}{120}(x-\pi/2)^5$;
the Taylor series is $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-\pi/2)^{2n+1}}{(2n+1)!}$

9. $f^{(n)}(x) = (-1)^n e^{-x}$; at $x=0$, $f^{(n)}(0) = -1$ when n is odd and $f^{(n)}(0) = 1$ when n is even.
The Taylor series starts $1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \cdots$;
the Taylor series is $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}$.

11. $f^{(n)}(x) = (-1)^{n+1}\frac{n!}{(x+1)^{n+1}}$; at $x=1$, $f^{(n)}(1) = (-1)^{n+1}\frac{n!}{2^{n+1}}$
The Taylor series starts
$\frac{1}{2} + \frac{1}{4}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 \cdots$;
the Taylor series is $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{2^{n+1}}$.

13. Given a value x, the magnitude of the error term $R_n(x)$ is bounded by
$$\left|R_n(x)\right| \le \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|x^{(n+1)}\right|,$$
where z is between 0 and x.
If $x>0$, then $z<x$ and $f^{(n+1)}(z) = e^z < e^x$. If $x<0$, then $x<z<0$ and $f^{(n+1)}(z) = e^z < 1$. So given a fixed x value, let $M = \max\{e^x, 1\}$; $f^{(n)}(z) < M$. This allows us to state
$$\left|R_n(x)\right| \le \frac{M}{(n+1)!}\left|x^{(n+1)}\right|.$$
For any x, $\displaystyle\lim_{n\to\infty}\frac{M}{(n+1)!}\left|x^{(n+1)}\right| = 0$. Thus by the Squeeze Theorem, we conclude that $\displaystyle\lim_{n\to\infty}R_n(x) = 0$ for all x, and hence
$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \quad \text{for all } x.$$

15. Given a value x, the magnitude of the error term $R_n(x)$ is bounded by
$$\left|R_n(x)\right| \le \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|(x-1)^{(n+1)}\right|,$$
where z is between 1 and x.
Note that $\left|f^{(n+1)}(x)\right| = \frac{n!}{x^{n+1}}$.
Per the statement of the problem, we only consider the case $1<x<2$.
If $1<x<2$, then $1<z<x$ and $f^{(n+1)}(z) = \frac{n!}{z^{n+1}} < n!$. Thus
$$\left|R_n(x)\right| \le \frac{n!}{(n+1)!}\left|(x-1)^{(n+1)}\right| = \frac{(x-1)^{n+1}}{n+1} < \frac{1}{n+1}.$$
Thus
$$\lim_{n\to\infty}\left|R_n(x)\right| < \lim_{n\to\infty}\frac{1}{n+1} = 0,$$
hence
$$\ln x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n} \text{ on } (1,2).$$

17. Given $\cos x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$,
$\cos(-x) = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(-x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = \cos x$, as all powers in the series are even.

19. Given $\sin x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$,
$\displaystyle\frac{d}{dx}\big(\sin x\big) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right) = \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = \cos x$. (The summation still starts at $n=0$ as there was no constant term in the expansion of $\sin x$).

21. $1 + \dfrac{x}{2} - \dfrac{x^2}{8} + \dfrac{x^3}{16} - \dfrac{5x^4}{128}$

23. $1 + \dfrac{x}{3} - \dfrac{x^2}{9} + \dfrac{5x^3}{81} - \dfrac{10x^4}{243}$

25. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{(2n)!}$.

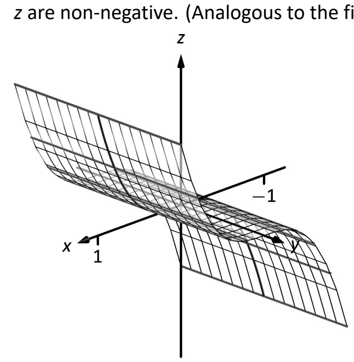
27. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(2x+3)^{2n+1}}{(2n+1)!}$.

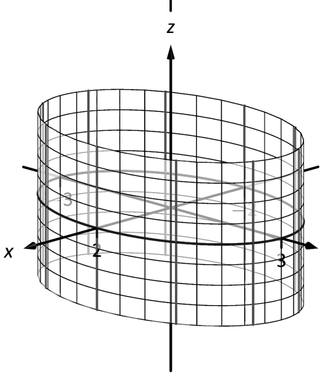
29. $x + x^2 + \dfrac{x^3}{3} - \dfrac{x^5}{30}$

31. $\displaystyle\int_0^{\sqrt{\pi}}\sin\left(x^2\right)dx \approx \int_0^{\sqrt{\pi}}\left(x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{5040}\right)dx = 0.8877$

Chapter 11

Section 11.1

1. right hand
3. curve (a parabola); surface (a cylinder)
5. a hyperboloid of two sheets
7. $\|\overline{AB}\| = \sqrt{6}$; $\|\overline{BC}\| = \sqrt{17}$; $\|\overline{AC}\| = \sqrt{11}$. Yes, it is a right triangle as $\|\overline{AB}\|^2 + \|\overline{AC}\|^2 = \|\overline{BC}\|^2$.
9. Center at $(4,-1,0)$; radius = 3
11. Interior of a sphere with radius 1 centered at the origin.
13. The first octant of space; all points (x,y,z) where each of x, y and z are non-negative. (Analogous to the first quadrant in the plane.)
15. 

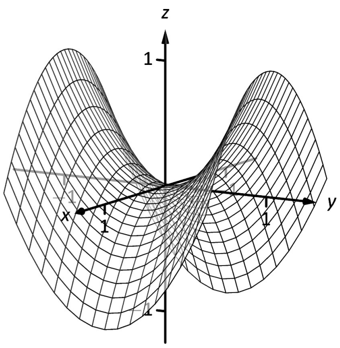
17. 

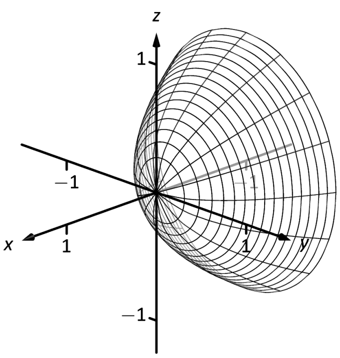
19. $x^2+z^2=\frac{1}{(1+y^2)^2}$

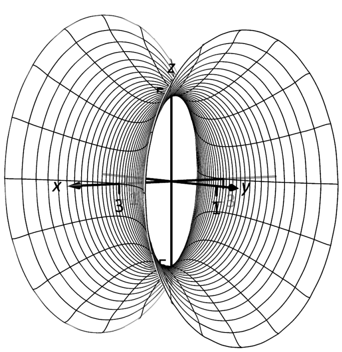
21. $z=(\sqrt{x^2+y^2})^2=x^2+y^2$

23. (a) $x=y^2+\dfrac{z^2}{9}$

25. (b) $x^2+\dfrac{y^2}{9}+\dfrac{z^2}{4}=1$

27. 

29. 

31. 

Section 11.2

1. Answers will vary.

3. A vector with magnitude 1.

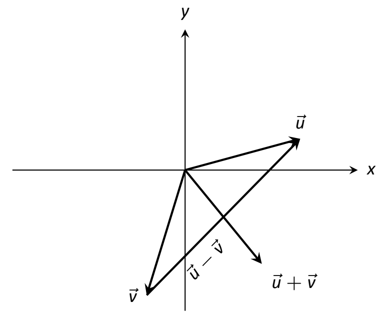
5. Their respective unit vectors are parallel; unit vectors $\vec{u}_1$ and $\vec{u}_2$ are parallel if $\vec{u}_1=\pm\vec{u}_2$.

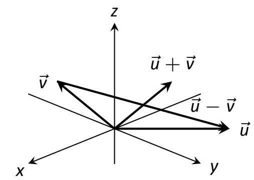
7. $\overrightarrow{PQ}=\langle 1,6\rangle=1\vec{i}+6\vec{j}$

9. $\overrightarrow{PQ}=\langle 6,-1,6\rangle=6\vec{i}-\vec{j}+6\vec{k}$

11. (a) $\vec{u}+\vec{v}=\langle 2,-1\rangle$; $\vec{u}-\vec{v}=\langle 0,-3\rangle$; $2\vec{u}-3\vec{v}=\langle -1,-7\rangle$.

    (c) $\vec{x}=\langle 1/2,2\rangle$.

13. 

15. 

17. $\|\vec{u}\|=\sqrt{5}$, $\|\vec{v}\|=\sqrt{13}$, $\|\vec{u}+\vec{v}\|=\sqrt{26}$, $\|\vec{u}-\vec{v}\|=\sqrt{10}$

19. $\|\vec{u}\|=\sqrt{5}$, $\|\vec{v}\|=3\sqrt{5}$, $\|\vec{u}+\vec{v}\|=2\sqrt{5}$, $\|\vec{u}-\vec{v}\|=4\sqrt{5}$

21. When $\vec{u}$ and $\vec{v}$ have the same direction. (Note: parallel is not enough.)

23. $\vec{u}=\langle 0.6,0.8\rangle$

25. $\vec{u}=\left\langle 1/\sqrt{3},-1/\sqrt{3},1/\sqrt{3}\right\rangle$

27. $\vec{u}=\langle \cos 120^\circ,\sin 120^\circ\rangle=\left\langle -1/2,\sqrt{3}/2\right\rangle$.

29. The force on each chain is $100/\sqrt{3}\approx 57.735$lb.

31. The force on the chain with angle θ is approx. 45.124lb; the force on the chain with angle φ is approx. 59.629lb.

33. $\theta=45^\circ$; the weight is lifted 0.29 ft (about 3.5in).

35. $\theta=45^\circ$; the weight is lifted 2.93 ft.

Section 11.3

1. Scalar

3. By considering the sign of the dot product of the two vectors. If the dot product is positive, the angle is acute; if the dot product is negative, the angle is obtuse.

5. -22

7. 3

9. not defined

11. Answers will vary.

13. $\theta=0.3218\approx 18.43^\circ$

15. $\theta=\pi/4=45^\circ$

17. Answers will vary; two possible answers are $\langle -7,4\rangle$ and $\langle 14,-8\rangle$.

19. Answers will vary; two possible answers are $\langle 1,0,-1\rangle$ and $\langle 4,5,-9\rangle$.

21. $\text{proj}_{\vec{v}}\,\vec{u}=\langle -1/2,3/2\rangle$.

23. $\text{proj}_{\vec{v}}\,\vec{u}=\langle -1/2,-1/2\rangle$.

25. $\text{proj}_{\vec{v}}\,\vec{u}=\langle 1,2,3\rangle$.

27. $\vec{u}=\langle -1/2,3/2\rangle+\langle 3/2,1/2\rangle$.

29. $\vec{u}=\langle -1/2,-1/2\rangle+\langle -5/2,5/2\rangle$.

31. $\vec{u}=\langle 1,2,3\rangle+\langle 0,3,-2\rangle$.

33. 1.96lb

35. 141.42ft–lb

37. 500ft–lb

39. 500ft–lb

Section 11.4

1. vector

3. “Perpendicular” is one answer.

5. Torque

7. $\vec{u} \times \vec{v} = \langle 12, -15, 3\rangle$

9. $\vec{u} \times \vec{v} = \langle -5, -31, 27\rangle$

11. $\vec{u} \times \vec{v} = \langle 0, -2, 0\rangle$

13. $\vec{u} \times \vec{v} = \langle 0, 0, ad - bc\rangle$

15. $\vec{i} \times \vec{k} = -\vec{j}$

17. Answers will vary.

19. 5

21. 0

23. $\sqrt{14}$

25. 3

27. $5\sqrt{2}/2$

29. 1

31. 7

33. 2

35. $\pm\frac{1}{\sqrt{6}}\langle 1, 1, -2\rangle$

37. $\langle 0, \pm 1, 0\rangle$

39. 87.5ft–lb

41. $200/3 \approx 66.67$ft–lb

43. With $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$, we have

$$\begin{aligned}\vec{u}\cdot(\vec{u}\times\vec{v}) &= \langle u_1, u_2, u_3\rangle \cdot (\langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1\rangle)\\ &= u_1(u_2v_3 - u_3v_2) - u2(u_1v_3 - u_3v_1) + u3(u_1v_2 - u_2v_1)\\ &= 0.\end{aligned}$$

Section 11.5

1. A point on the line and the direction of the line.

3. parallel, skew

5. vector: $\ell(t) = \langle 2, -4, 1\rangle + t\langle 9, 2, 5\rangle$
parametric: $x = 2 + 9t, y = -4 + 2t, z = 1 + 5t$
symmetric: $(x - 2)/9 = (y + 4)/2 = (z - 1)/5$

7. Answers can vary: vector: $\ell(t) = \langle 2, 1, 5\rangle + t\langle 5, -3, -1\rangle$
parametric: $x = 2 + 5t, y = 1 - 3t, z = 5 - t$
symmetric: $(x - 2)/5 = -(y - 1)/3 = -(z - 5)$

9. Answers can vary; here the direction is given by $\vec{d}_1 \times \vec{d}_2$: vector:
$\ell(t) = \langle 0, 1, 2\rangle + t\langle -10, 43, 9\rangle$
parametric: $x = -10t, y = 1 + 43t, z = 2 + 9t$
symmetric: $-x/10 = (y - 1)/43 = (z - 2)/9$

11. Answers can vary; here the direction is given by $\vec{d}_1 \times \vec{d}_2$: vector:
$\ell(t) = \langle 7, 2, -1\rangle + t\langle 1, -1, 2\rangle$
parametric: $x = 7 + t, y = 2 - t, z = -1 + 2t$
symmetric: $x - 7 = 2 - y = (z + 1)/2$

13. vector: $\ell(t) = \langle 1, 1\rangle + t\langle 2, 3\rangle$
parametric: $x = 1 + 2t, y = 1 + 3t$
symmetric: $(x - 1)/2 = (y - 1)/3$

15. parallel

17. intersecting; $\vec{\ell}_1(3) = \vec{\ell}_2(4) = \langle 9, -5, 13\rangle$

19. skew

21. same

23. $\sqrt{41}/3$

25. $5\sqrt{2}/2$

27. $3/\sqrt{2}$

29. Since both P and Q are on the line, $\overrightarrow{PQ}$ is parallel to $\vec{d}$. Thus $\overrightarrow{PQ} \times \vec{d} = \vec{0}$, giving a distance of 0.

31. (a) The distance formula cannot be used because since $\vec{d}_1$ and $\vec{d}_2$ are parallel, $\vec{c}$ is $\vec{0}$ and we cannot divide by $\|\vec{0}\|$.
(b) Since $\vec{d}_1$ and $\vec{d}_2$ are parallel, $\overrightarrow{P_1P_2}$ lies in the plane formed by the two lines. Thus $\overrightarrow{P_1P_2} \times \vec{d}_2$ is orthogonal to this plane, and $\vec{c} = (\overrightarrow{P_1P_2} \times \vec{d}_2) \times \vec{d}_2$ is parallel to the plane, but still orthogonal to both $\vec{d}_1$ and $\vec{d}_2$. We desire the length of the projection of $\overrightarrow{P_1P_2}$ onto $\vec{c}$, which is what the formula provides.
(c) Since the lines are parallel, one can measure the distance between the lines at any location on either line (just as to find the distance between straight railroad tracks, one can use a measuring tape anywhere along the track, not just at one specific place.) Let $P = P_1$ and $Q = P_2$ as given by the equations of the lines, and apply the formula for distance between a point and a line.

Section 11.6

1. A point in the plane and a normal vector (i.e., a direction orthogonal to the plane).

3. Answers will vary.

5. Answers will vary.

7. Standard form: $3(x - 2) - (y - 3) + 7(z - 4) = 0$
general form: $3x - y + 7z = 31$

9. Answers may vary;
Standard form: $8(x - 1) + 4(y - 2) - 4(z - 3) = 0$
general form: $8x + 4y - 4z = 4$

11. Answers may vary;
Standard form: $-7(x - 2) + 2(y - 1) + (z - 2) = 0$
general form: $-7x + 2y + z = -10$

13. Answers may vary;
Standard form: $2(x - 1) - (y - 1) = 0$
general form: $2x - y = 1$

15. Answers may vary;
Standard form: $2(x - 2) - (y + 6) - 4(z - 1) = 0$
general form: $2x - y - 4z = 6$

17. Answers may vary;
Standard form: $(x - 5) + (y - 7) + (z - 3) = 0$
general form: $x + y + z = 15$

19. Answers may vary;
Standard form: $3(x + 4) + 8(y - 7) - 10(z - 2) = 0$
general form: $3x + 8y - 10z = 24$

21. Answers may vary:
$$\ell = \begin{cases} x = 14t \\ y = -1 - 10t \\ z = 2 - 8t \end{cases}$$

23. $(-3, -7, -5)$

25. No point of intersection; the plane and line are parallel.

27. $\sqrt{5/7}$

29. $1/\sqrt{3}$

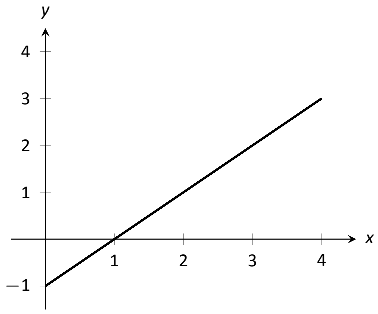
31. If P is any point in the plane, and Q is also in the plane, then $\overrightarrow{PQ}$ lies parallel to the plane and is orthogonal to $\vec{n}$, the normal vector. Thus $\vec{n} \cdot \overrightarrow{PQ} = 0$, giving the distance as 0.

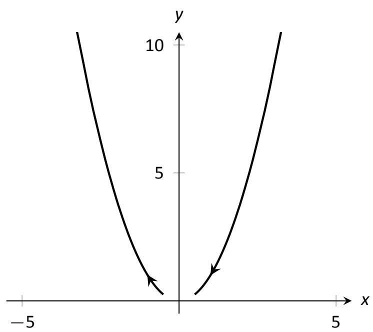
Chapter 12

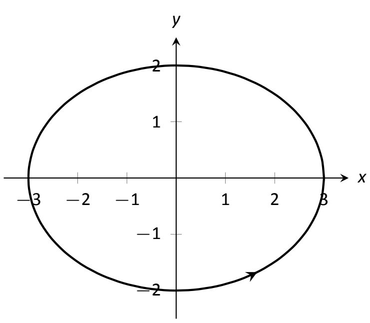
Section 12.1

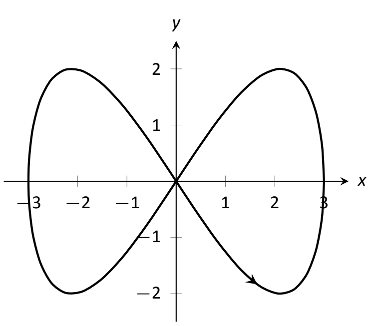
1. parametric equations

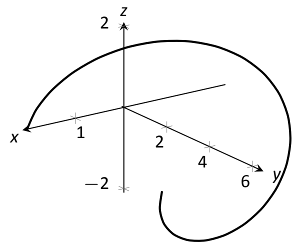
3. displacement

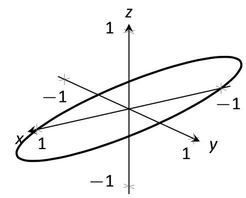
5. 

7. 

9. 

11. 

13. 

15. 

17. $\| \vec{r}(t) \| = \sqrt{t^2 + t^4} = |t|\sqrt{t^2 + 1}$.

19. $\| \vec{r}(t) \| = \sqrt{4\cos^2 t + 4\sin^2 t + t^2} = \sqrt{t^2 + 4}$.

21. Answers may vary, though most direct solution is $\vec{r}(t) = \langle 2\cos t + 1, 2\sin t + 2\rangle$.

23. Answers may vary, though most direct solution is $\vec{r}(t) = \langle 1.5\cos t, 5\sin t\rangle$.

25. Answers may vary, though most direct solutions are $\vec{r}(t) = \langle t, 5(t-2) + 3\rangle$ and $\vec{r}(t) = \langle t + 2, 5t + 3\rangle$.

27. Specific forms may vary, though most direct solutions are $\vec{r}(t) = \langle 1, 2, 3\rangle + t\,\langle 3, 3, 3\rangle$ and $\vec{r}(t) = \langle 3t + 1, 3t + 2, 3t + 3\rangle$.

29. Answers may vary, though most direct solution is $\vec{r}(t) = \langle 2\cos t, 2\sin t, 2t\rangle$.

31. $\langle 1, 0\rangle$

33. $\langle 0, 0, 1\rangle$

Section 12.2

1. component

3. It is difficult to identify the points on the graphs of $\vec{r}(t)$ and $\vec{r}\,'(t)$ that correspond to each other.

5. $\langle 11, 74, \sin 5\rangle$

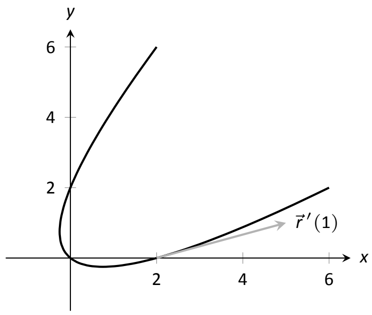
7. $\langle 1, e\rangle$

9. $(-\infty, 0)\bigcup(0, \infty)$

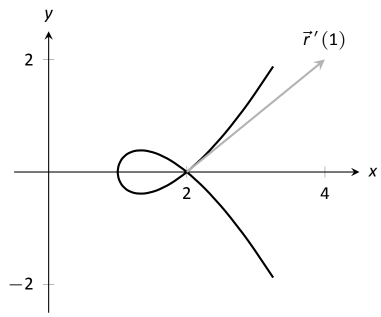
11. $\vec{r}\,'(t) = \langle -\sin t, e^t, 1/t\rangle$

13. $\vec{r}\,'(t) = (2t)\,\langle \sin t, 2t + 5\rangle + (t^2)\,\langle \cos t, 2\rangle = \left\langle 2t\sin t + t^2\cos t, 6t^2 + 10t\right\rangle$

15. $\vec{r}\,'(t) =$
$\langle 2t, 1, 0\rangle \times \langle \sin t, 2t + 5, 1\rangle + \left\langle t^2 + 1, t - 1, 1\right\rangle \times \langle \cos t, 2, 0\rangle =$
$\left\langle -1, \cos t - 2t, 6t^2 + 10t + 2 + \cos t - \sin t - t\cos t\right\rangle$

17. 

$\vec{r}\,'(t) = \langle 2t+1, 2t-1 \rangle$

19. 

$\vec{r}\,'(t) = \left\langle 2t, 3t^2-1 \right\rangle$

21. $\ell(t) = \langle 2, 0 \rangle + t\,\langle 3, 1 \rangle$

23. $\ell(t) = \langle -3, 0, \pi \rangle + t\,\langle 0, -3, 1 \rangle$

25. $t = 2n\pi$, where n is an integer; so $t = \ldots -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$

27. $\vec{r}(t)$ is not smooth at $t = 3\pi/4 + n\pi$, where n is an integer

29. Both derivatives return $\left\langle 5t^4, 4t^3-3t^2, 3t^2 \right\rangle$.

31. Both derivatives return $\left\langle 2t - e^t - 1, \cos t - 3t^2, (t^2+2t)e^t - (t-1)\cos t - \sin t \right\rangle$.

33. $\left\langle \frac{1}{4}t^4, \sin t, te^t - e^t \right\rangle + \vec{C}$

35. $\langle -2, 0 \rangle$

37. $\vec{r}(t) = \left\langle \frac{1}{2}t^2 + 2, -\cos t + 3 \right\rangle$

39. $\vec{r}(t) = \left\langle t^4/12 + t + 4,\ t^3/6 + 2t + 5,\ t^2/2 + 3t + 6 \right\rangle$

41. $2\sqrt{13}\pi$

43. $\frac{1}{54}\left((22)^{3/2} - 8\right)$

45. As $\vec{r}(t)$ has constant length, $\vec{r}(t) \cdot \vec{r}(t) = c^2$ for some constant c. Thus

$$\begin{aligned}
\vec{r}(t) \cdot \vec{r}(t) &= c^2 \\
\frac{d}{dt}\big(\vec{r}(t) \cdot \vec{r}(t)\big) &= \frac{d}{dt}\big(c^2\big) \\
\vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t) &= 0 \\
2\vec{r}(t) \cdot \vec{r}\,'(t) &= 0 \\
\vec{r}(t) \cdot \vec{r}\,'(t) &= 0.
\end{aligned}$$

Section 12.3

1. Velocity is a vector, indicating an objects direction of travel and its rate of distance change (i.e., its speed). Speed is a scalar.

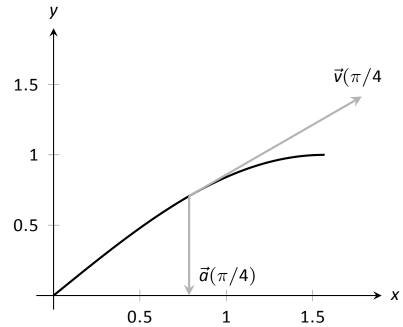
3. The average velocity is found by dividing the displacement by the time travelled – it is a vector. The average speed is found by dividing the distance travelled by the time travelled – it is a scalar.

5. One example is travelling at a constant speed s in a circle, ending at the starting position. Since the displacement is $\vec{0}$, the average velocity is $\vec{0}$, hence $\|\,\vec{0}\,\| = 0$. But travelling at constant speed s means the average speed is also $s > 0$.

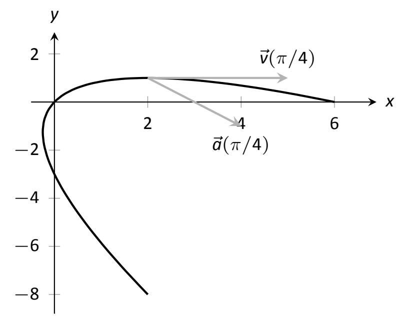
7. $\vec{v}(t) = \langle 2, 5, 0 \rangle$, $\vec{a}(t) = \langle 0, 0, 0 \rangle$

9. $\vec{v}(t) = \langle -\sin t, \cos t \rangle$, $\vec{a}(t) = \langle -\cos t, -\sin t \rangle$

11. $\vec{v}(t) = \langle 1, \cos t \rangle$, $\vec{a}(t) = \langle 0, -\sin t \rangle$



13. $\vec{v}(t) = \langle 2t+1, -2t+2 \rangle$, $\vec{a}(t) = \langle 2, -2 \rangle$



15. $\|\,\vec{v}(t)\,\| = \sqrt{4t^2+1}$.
Min at $t = 0$; Max at $t = \pm 1$.

17. $\|\,\vec{v}(t)\,\| = 5$.
Speed is constant, so there is no difference between min/max

19. $\|\,\vec{v}(t)\,\| = |\sec t|\sqrt{\tan^2 t + \sec^2 t}$.
min: $t = 0$; max: $t = \pi/4$

21. $\|\,\vec{v}(t)\,\| = 13$.
speed is constant, so there is no difference between min/max

23. $\|\,\vec{v}(t)\,\| = \sqrt{4t^2 + 1 + t^2/(1-t^2)}$.
min: $t = 0$; max: there is no max; speed approaches ∞ as $t \to \pm 1$

25. (a) $\vec{r}_1(1) = \langle 1, 1 \rangle$; $\vec{r}_2(1) = \langle 1, 1 \rangle$
(b) $\vec{v}_1(1) = \langle 1, 2 \rangle$; $\|\,\vec{v}_1(1)\,\| = \sqrt{5}$; $\vec{a}_1(1) = \langle 0, 2 \rangle$
$\vec{v}_2(1) = \langle 2, 4 \rangle$; $\|\,\vec{v}_2(1)\,\| = 2\sqrt{5}$; $\vec{a}_2(1) = \langle 2, 12 \rangle$

27. (a) $\vec{r}_1(2) = \langle 6, 4 \rangle$; $\vec{r}_2(2) = \langle 6, 4 \rangle$
(b) $\vec{v}_1(2) = \langle 3, 2 \rangle$; $\|\,\vec{v}_1(2)\,\| = \sqrt{13}$; $\vec{a}_1(2) = \langle 0, 0 \rangle$
$\vec{v}_2(2) = \langle 6, 4 \rangle$; $\|\,\vec{v}_2(2)\,\| = 2\sqrt{13}$; $\vec{a}_2(2) = \langle 0, 0 \rangle$

29. $\vec{v}(t) = \langle 2t+1, 3t+2 \rangle$, $\vec{r}(t) = \left\langle t^2+t+5, 3t^2/2 + 2t - 2 \right\rangle$

31. $\vec{v}(t) = \langle \sin t, \cos t \rangle$, $\vec{r}(t) = \langle 1 - \cos t, \sin t \rangle$

33. Displacement: $\langle 0, 0, 6\pi \rangle$; distance travelled: $2\sqrt{13}\pi \approx 22.65$ft; average velocity: $\langle 0, 0, 3 \rangle$; average speed: $\sqrt{13} \approx 3.61$ft/s

35. Displacement: $\langle 0, 0 \rangle$; distance travelled: $2\pi \approx 6.28$ft; average velocity: $\langle 0, 0 \rangle$; average speed: 1ft/s

37. At t-values of $\sin^{-1}(9/30)/(4\pi) + n/2 \approx 0.024 + n/2$ seconds, where n is an integer.

39. (a) Holding the crossbow at an angle of 0.013 radians, $\approx 0.745^\circ$ will hit the target 0.4s later. (Another solution exists, with an angle of 89°, landing 18.75s later, but this is impractical.)

(b) In the .4 seconds the arrow travels, a deer, travelling at 20mph or 29.33ft/s, can travel 11.7ft. So she needs to lead the deer by 11.7ft.

41. The position function is $\vec{r}(t) = \langle 220t, -16t^2 + 1000\rangle$. The y-component is 0 when $t = 7.9$; $\vec{r}(7.9) = \langle 1739.25, 0\rangle$, meaning the box will travel about 1740ft horizontally before it lands.

Section 12.4

1. 1
3. $\vec{T}(t)$ and $\vec{N}(t)$.
5. $\vec{T}(t) = \left\langle \frac{4t}{\sqrt{20t^2-4t+1}}, \frac{2t-1}{\sqrt{20t^2-4t+1}} \right\rangle$; $\vec{T}(1) = \langle 4/\sqrt{17}, 1/\sqrt{17}\rangle$
7. $\vec{T}(t) = \frac{\cos t \sin t}{\sqrt{\cos^2 t \sin^2 t}} \langle -\cos t, \sin t\rangle$. (Be careful; this cannot be simplified as just $\langle -\cos t, \sin t\rangle$ as $\sqrt{\cos^2 t \sin^2 t} \neq \cos t \sin t$, but rather $|\cos t \sin t|$.) $\vec{T}(\pi/4) = \langle -\sqrt{2}/2, \sqrt{2}/2\rangle$
9. $\ell(t) = \langle 2, 0\rangle + t\,\langle 4/\sqrt{17}, 1/\sqrt{17}\rangle$; in parametric form,
$\ell(t) = \begin{cases} x = 2 + 4t/\sqrt{17} \\ y = t/\sqrt{17} \end{cases}$
11. $\ell(t) = \langle \sqrt{2}/4, \sqrt{2}/4\rangle + t\,\langle -\sqrt{2}/2, \sqrt{2}/2\rangle$; in parametric form,
$\ell(t) = \begin{cases} x = \sqrt{2}/4 - \sqrt{2}t/2 \\ y = \sqrt{2}/4 + \sqrt{2}t/2 \end{cases}$
13. $\vec{T}(t) = \langle -\sin t, \cos t\rangle$; $\vec{N}(t) = \langle -\cos t, -\sin t\rangle$
15. $\vec{T}(t) = \left\langle -\frac{\sin t}{\sqrt{4\cos^2 t + \sin^2 t}}, \frac{2\cos t}{\sqrt{4\cos^2 t + \sin^2 t}} \right\rangle$;
$\vec{N}(t) = \left\langle -\frac{2\cos t}{\sqrt{4\cos^2 t + \sin^2 t}}, -\frac{\sin t}{\sqrt{4\cos^2 t + \sin^2 t}} \right\rangle$
17. (a) Be sure to show work
(b) $\vec{N}(\pi/4) = \langle -5/\sqrt{34}, -3/\sqrt{34}\rangle$
19. (a) Be sure to show work
(b) $\vec{N}(0) = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$
21. $\vec{T}(t) = \frac{1}{\sqrt{5}} \langle 2, \cos t, -\sin t\rangle$; $\vec{N}(t) = \langle 0, -\sin t, -\cos t\rangle$
23. $\vec{T}(t) = \frac{1}{\sqrt{a^2+b^2}} \langle -a\sin t, a\cos t, b\rangle$; $\vec{N}(t) = \langle -\cos t, -\sin t, 0\rangle$
25. $a_\text{T} = \frac{4t}{\sqrt{1+4t^2}}$ and $a_\text{N} = \sqrt{4 - \frac{16t^2}{1+4t^2}}$
At $t = 0$, $a_\text{T} = 0$ and $a_\text{N} = 2$;
At $t = 1$, $a_\text{T} = 4/\sqrt{5}$ and $a_\text{N} = 2/\sqrt{5}$.
At $t = 0$, all acceleration comes in the form of changing the direction of velocity and not the speed; at $t = 1$, more acceleration comes in changing the speed than in changing direction.
27. $a_\text{T} = 0$ and $a_\text{N} = 2$
At $t = 0$, $a_\text{T} = 0$ and $a_\text{N} = 2$;
At $t = \pi/2$, $a_\text{T} = 0$ and $a_\text{N} = 2$.
The object moves at constant speed, so all acceleration comes from changing direction, hence $a_\text{T} = 0$. $\vec{a}(t)$ is always parallel to $\vec{N}(t)$, but twice as long, hence $a_\text{N} = 2$.
29. $a_\text{T} = 0$ and $a_\text{N} = a$
At $t = 0$, $a_\text{T} = 0$ and $a_\text{N} = a$;
At $t = \pi/2$, $a_\text{T} = 0$ and $a_\text{N} = a$.
The object moves at constant speed, meaning that a_T is always 0. The object "rises" along the z-axis at a constant rate, so all acceleration comes in the form of changing direction circling the z-axis. The greater the radius of this circle the greater the acceleration, hence $a_\text{N} = a$.

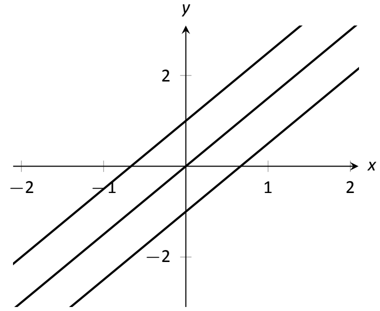
Section 12.5

1. time and/or distance
3. Answers may include lines, circles, helixes
5. κ
7. $s = 3t$, so $\vec{r}(s) = \langle 2s/3, s/3, -2s/3\rangle$
9. $s = \sqrt{13}t$, so $\vec{r}(s) = \langle 3\cos(s/\sqrt{13}), 3\sin(s/\sqrt{13}), 2s/\sqrt{13}\rangle$
11. $\kappa = \frac{|6x|}{\left(1+(3x^2-1)^2\right)^{3/2}}$;
$\kappa(0) = 0$, $\kappa(1/2) = \frac{192}{17\sqrt{17}} \approx 2.74$.
13. $\kappa = \frac{|\cos x|}{\left(1+\sin^2 x\right)^{3/2}}$;
$\kappa(0) = 1$, $\kappa(\pi/2) = 0$
15. $\kappa = \frac{|2\cos t\cos(2t)+4\sin t\sin(2t)|}{\left(4\cos^2(2t)+\sin^2 t\right)^{3/2}}$;
$\kappa(0) = 1/4$, $\kappa(\pi/4) = 8$
17. $\kappa = \frac{|6t^2+2|}{\left(4t^2+(3t^2-1)^2\right)^{3/2}}$;
$\kappa(0) = 2$, $\kappa(5) = \frac{19}{1394\sqrt{1394}} \approx 0.0004$
19. $\kappa = 0$;
$\kappa(0) = 0$, $\kappa(1) = 0$
21. $\kappa = \frac{3}{13}$;
$\kappa(0) = 3/13$, $\kappa(\pi/2) = 3/13$
23. maximized at $x = \pm\frac{\sqrt{2}}{\sqrt[4]{5}}$
25. maximized at $t = 1/4$
27. radius of curvature is $5\sqrt{5}/4$.
29. radius of curvature is 9.
31. $x^2 + (y - 1/2)^2 = 1/4$, or $\vec{c}(t) = \langle 1/2\cos t, 1/2\sin t + 1/2\rangle$
33. $x^2 + (y + 8)^2 = 81$, or $\vec{c}(t) = \langle 9\cos t, 9\sin t - 8\rangle$

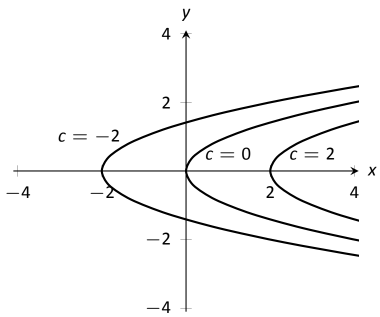
Chapter 13

Section 13.1

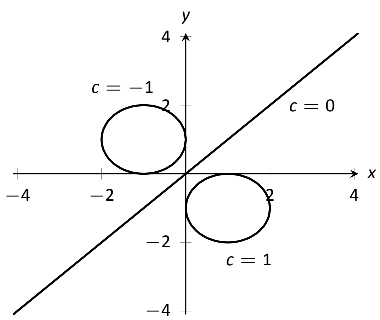
1. Answers will vary.
3. topographical
5. surface
7. domain: $\mathbb{R}^2$
range: $z \geq 2$
9. domain: $\mathbb{R}^2$
range: $\mathbb{R}$
11. domain: $\mathbb{R}^2$
range: $0 < z \leq 1$
13. domain: $\{(x, y) \mid x^2 + y^2 \leq 9\}$, i.e., the domain is the circle and interior of a circle centered at the origin with radius 3.
range: $0 \leq z \leq 3$
15. Level curves are lines $y = (3/2)x - c/2$.



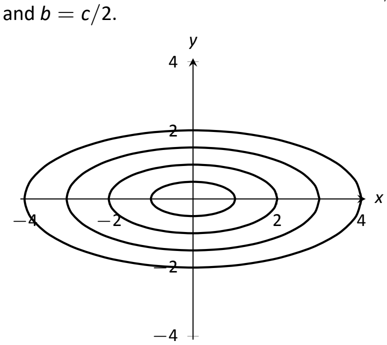
17. Level curves are parabolas $x = y^2 + c$.



19. When $c \neq 0$, the level curves are circles, centered at $(1/c, -1/c)$ with radius $\sqrt{2/c^2 - 1}$. When $c = 0$, the level curve is the line $y = x$.



21. Level curves are ellipses of the form $\frac{x^2}{c^2} + \frac{y^2}{c^2/4} = 1$, i.e., $a = c$ and $b = c/2$.



23. domain: $x + 2y - 4z \neq 0$; the set of points in $\mathbb{R}^3$ NOT in the domain form a plane through the origin.
range: $\mathbb{R}$

25. domain: $z \geq x^2 - y^2$; the set of points in $\mathbb{R}^3$ above (and including) the hyperbolic paraboloid $z = x^2 - y^2$.
range: $[0, \infty)$

27. The level surfaces are spheres, centered at the origin, with radius $\sqrt{c}$.

29. The level surfaces are paraboloids of the form $z = \frac{x^2}{c} + \frac{y^2}{c}$; the larger c, the "wider" the paraboloid.

31. The level curves for each surface are similar; for $z = \sqrt{x^2 + 4y^2}$ the level curves are ellipses of the form $\frac{x^2}{c^2} + \frac{y^2}{c^2/4} = 1$, i.e., $a = c$ and $b = c/2$; whereas for $z = x^2 + 4y^2$ the level curves are ellipses of the form $\frac{x^2}{c} + \frac{y^2}{c/4} = 1$, i.e., $a = \sqrt{c}$ and $b = \sqrt{c}/2$. The first set of ellipses are spaced evenly apart, meaning the function grows at a constant rate; the second set of ellipses are more closely spaced together as c grows, meaning the function grows faster and faster as c increases.
The function $z = \sqrt{x^2 + 4y^2}$ can be rewritten as $z^2 = x^2 + 4y^2$, an elliptic cone; the function $z = x^2 + 4y^2$ is a paraboloid, each matching the description above.

Section 13.2

1. Answers will vary.

3. Answers will vary.
One possible answer: $\{(x, y) | x^2 + y^2 \leq 1\}$

5. Answers will vary.
One possible answer: $\{(x, y) | x^2 + y^2 < 1\}$

7. (a) Answers will vary.
interior point: $(1, 3)$
boundary point: $(3, 3)$
(b) S is a closed set
(c) S is bounded

9. (a) Answers will vary.
interior point: none
boundary point: $(0, -1)$
(b) S is a closed set, consisting only of boundary points
(c) S is bounded

11. (a) $D = \{(x, y) \mid 9 - x^2 - y^2 \geq 0\}$.
(b) D is a closed set.
(c) D is bounded.

13. (a) $D = \{(x, y) \mid y > x^2\}$.
(b) D is an open set.
(c) D is unbounded.

15. (a) Along $y = 0$, the limit is 1.
(b) Along $x = 0$, the limit is -1.
Since the above limits are not equal, the limit does not exist.

17. (a) Along $y = mx$, the limit is $\dfrac{mx(1 - m)}{m^2x + 1} = 0$ for all m.
(b) Along $x = 0$, the limit is -1.
Since the above limits are not equal, the limit does not exist.

19. (a) Along $y = 2$, the limit is:
$$\begin{aligned}\lim_{(x,y)\to(1,2)} \frac{x + y - 3}{x^2 - 1} &= \lim_{x\to1} \frac{x - 1}{x^2 - 1}\\ &= \lim_{x\to1} \frac{1}{x + 1}\\ &= 1/2.\end{aligned}$$
(b) Along $y = x + 1$, the limit is:
$$\begin{aligned}\lim_{(x,y)\to(1,2)} \frac{x + y - 3}{x^2 - 1} &= \lim_{x\to1} \frac{2(x - 1)}{x^2 - 1}\\ &= \lim_{x\to1} \frac{2}{x + 1}\\ &= 1.\end{aligned}$$
Since the limits along the lines $y = 2$ and $y = x + 1$ differ, the overall limit does not exist.

Section 13.3

1. A constant is a number that is added or subtracted in an expression; a coefficient is a number that is being multiplied by a nonconstant function.

3. f_x

5. $f_x = 2xy - 1, f_y = x^2 + 2$
$f_x(1, 2) = 3, f_y(1, 2) = 3$

7. $f_x = -\sin x \sin y, f_y = \cos x \cos y$
$f_x(\pi/3, \pi/3) = -3/4, f_y(\pi/3, \pi/3) = 1/4$

9. $f_x = 2xy + 6x, f_y = x^2 + 4$
$f_{xx} = 2y + 6, f_{yy} = 0$
$f_{xy} = 2x, f_{yx} = 2x$

11. $f_x = 1/y, f_y = -x/y^2$
$f_{xx} = 0, f_{yy} = 2x/y^3$
$f_{xy} = -1/y^2, f_{yx} = -1/y^2$

13. $f_x = 2xe^{x^2+y^2}, f_y = 2ye^{x^2+y^2}$
$f_{xx} = 2e^{x^2+y^2} + 4x^2e^{x^2+y^2}, f_{yy} = 2e^{x^2+y^2} + 4y^2e^{x^2+y^2}$
$f_{xy} = 4xye^{x^2+y^2}, f_{yx} = 4xye^{x^2+y^2}$

15. $f_x = \cos x \cos y, f_y = -\sin x \sin y$
$f_{xx} = -\sin x \cos y, f_{yy} = -\sin x \cos y$
$f_{xy} = -\sin y \cos x, f_{yx} = -\sin y \cos x$

17. $f_x = -5y^3 \sin(5xy^3), f_y = -15xy^2 \sin(5xy^3)$
$f_{xx} = -25y^6 \cos(5xy^3),$
$f_{yy} = -225x^2y^4 \cos(5xy^3) - 30xy \sin(5xy^3)$
$f_{xy} = -75xy^5 \cos(5xy^3) - 15y^2 \sin(5xy^3),$
$f_{yx} = -75xy^5 \cos(5xy^3) - 15y^2 \sin(5xy^3)$

19. $f_x = \frac{2y^2}{\sqrt{4xy^2+1}}, f_y = \frac{4xy}{\sqrt{4xy^2+1}}$
$f_{xx} = -\frac{4y^4}{\sqrt{4xy^2+1}^3}, f_{yy} = -\frac{16x^2y^2}{\sqrt{4xy^2+1}^3} + \frac{4x}{\sqrt{4xy^2+1}}$
$f_{xy} = -\frac{8xy^3}{\sqrt{4xy^2+1}^3} + \frac{4y}{\sqrt{4xy^2+1}}, f_{yx} = -\frac{8xy^3}{\sqrt{4xy^2+1}^3} + \frac{4y}{\sqrt{4xy^2+1}}$

21. $f_x = -\frac{2x}{(x^2+y^2+1)^2}, f_y = -\frac{2y}{(x^2+y^2+1)^2}$
$f_{xx} = \frac{8x^2}{(x^2+y^2+1)^3} - \frac{2}{(x^2+y^2+1)^2}, f_{yy} = \frac{8y^2}{(x^2+y^2+1)^3} - \frac{2}{(x^2+y^2+1)^2}$
$f_{xy} = \frac{8xy}{(x^2+y^2+1)^3}, f_{yx} = \frac{8xy}{(x^2+y^2+1)^3}$

23. $f_x = 6x, f_y = 0$
$f_{xx} = 6, f_{yy} = 0$
$f_{xy} = 0, f_{yx} = 0$

25. $f_x = \frac{1}{4xy}, f_y = -\frac{\ln x}{4y^2}$
$f_{xx} = -\frac{1}{4x^2y}, f_{yy} = \frac{\ln x}{2y^3}$
$f_{xy} = -\frac{1}{4xy^2}, f_{yx} = -\frac{1}{4xy^2}$

27. $f(x, y) = x \sin y + x + C$, where C is any constant.

29. $f(x, y) = 3x^2y - 4xy^2 + 2y + C$, where C is any constant.

31. $f_x = 2xe^{2y-3z}, f_y = 2x^2e^{2y-3z}, f_z = -3x^2e^{2y-3z}$
$f_{yz} = -6x^2e^{2y-3z}, f_{zy} = -6x^2e^{2y-3z}$

33. $f_x = \frac{3}{7y^2z}, f_y = -\frac{6x}{7y^3z}, f_z = -\frac{3x}{7y^2z^2}$
$f_{yz} = \frac{6x}{7y^3z^2}, f_{zy} = \frac{6x}{7y^3z^2}$

Section 13.4

1. Answers will vary. The displacement of the vector is one unit in the x-direction and 3 units in the z-direction, with no change in y. Thus along a line parallel to $\vec{v}$, the change in z is 3 times the change in x – i.e., a "slope" of 3. Specifically, the line in the x-z plane parallel to z has a slope of 3.

3. T

5. (a) $\ell_x(t) = \begin{cases} x = 2 + t \\ y = 3 \\ z = -48 - 12t \end{cases}$

(b) $\ell_y(t) = \begin{cases} x = 2 \\ y = 3 + t \\ z = -48 - 40t \end{cases}$

(c) $\ell_{\vec{u}}(t) = \begin{cases} x = 2 + t/\sqrt{10} \\ y = 3 + 3t/\sqrt{10} \\ z = -48 - 66\sqrt{2/5}t \end{cases}$

7. (a) $\ell_x(t) = \begin{cases} x = 4 + t \\ y = 2 \\ z = 2 + 3t \end{cases}$

(b) $\ell_y(t) = \begin{cases} x = 4 \\ y = 2 + t \\ z = 2 - 5t \end{cases}$

(c) $\ell_{\vec{u}}(t) = \begin{cases} x = 4 + t/\sqrt{2} \\ y = 2 + t/\sqrt{2} \\ z = 2 - \sqrt{2}t \end{cases}$

9. $\ell_{\vec{n}}(t) = \begin{cases} x = 2 - 12t \\ y = 3 - 40t \\ z = -48 - t \end{cases}$

11. $\ell_{\vec{n}}(t) = \begin{cases} x = 4 + 3t \\ y = 2 - 5t \\ z = 2 - t \end{cases}$

13. $(1.425, 1.085, -48.078), (2.575, 4.915, -47.952)$

15. $(5.014, 0.31, 1.662)$ and $(2.986, 3.690, 2.338)$

17. $-12(x - 2) - 40(y - 3) - (z + 48) = 0$

19. $3(x - 4) - 5(y - 2) - (z - 2) = 0$ (Note that this tangent plane is the same as the original function, a plane.)

Index

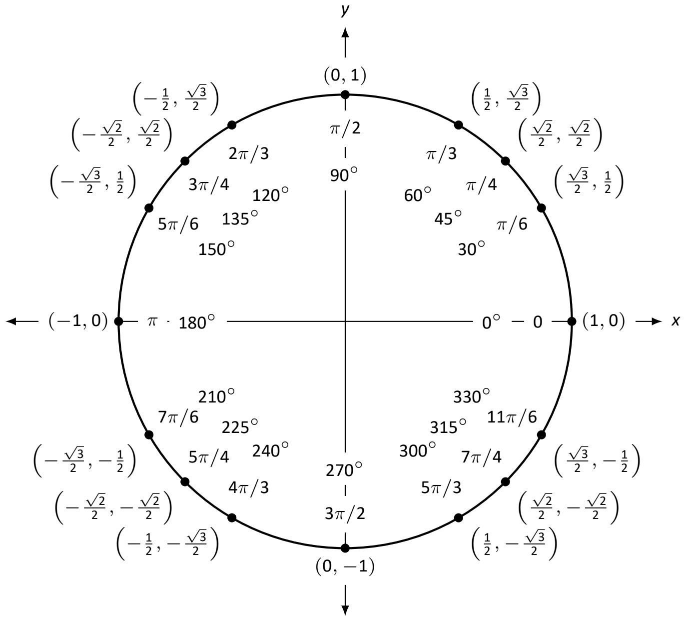
Differentiation Rules

1. $\frac{d}{dx}(cx) = c$
2. $\frac{d}{dx}(u \pm v) = u' \pm v'$
3. $\frac{d}{dx}(u \cdot v) = uv' + u'v$
4. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}(u(v)) = u'(v)v'$
6. $\frac{d}{dx}(c) = 0$
7. $\frac{d}{dx}(x) = 1$
8. $\frac{d}{dx}(x^n) = nx^{n-1}$
9. $\frac{d}{dx}(e^x) = e^x$
10. $\frac{d}{dx}(a^x) = \ln a \cdot a^x$
11. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
12. $\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$
13. $\frac{d}{dx}(\sin x) = \cos x$
14. $\frac{d}{dx}(\cos x) = -\sin x$
15. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
16. $\frac{d}{dx}(\sec x) = \sec x \tan x$
17. $\frac{d}{dx}(\tan x) = \sec^2 x$
18. $\frac{d}{dx}(\cot x) = -\csc^2 x$
19. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$
20. $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$
21. $\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{x\sqrt{x^2-1}}$
22. $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$
23. $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$
24. $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$
25. $\frac{d}{dx}(\cosh x) = \sinh x$
26. $\frac{d}{dx}(\sinh x) = \cosh x$
27. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
28. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
29. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
30. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
31. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$
32. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$
33. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}}$
34. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1+x^2}}$
35. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$
36. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2}$

Integration Rules

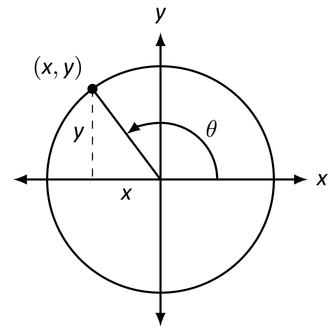
1. $\int c \cdot f(x)\, dx = c \int f(x)\, dx$
2. $\int f(x) \pm g(x)\, dx =$
   $\int f(x)\, dx \pm \int g(x)\, dx$
3. $\int 0\, dx = C$
4. $\int 1\, dx = x + C$
5. $\int x^n\, dx = \frac{1}{n+1} x^{n+1} + C,\ n \neq -1$
   $n \neq -1$
6. $\int e^x\, dx = e^x + C$
7. $\int a^x\, dx = \frac{1}{\ln a} \cdot a^x + C$
8. $\int \frac{1}{x}\, dx = \ln |x| + C$
9. $\int \cos x\, dx = \sin x + C$
10. $\int \sin x\, dx = -\cos x + C$
11. $\int \tan x\, dx = -\ln |\cos x| + C$
12. $\int \sec x\, dx = \ln |\sec x + \tan x| + C$
13. $\int \csc x\, dx = -\ln |\csc x + \cot x| + C$
14. $\int \cot x\, dx = \ln |\sin x| + C$
15. $\int \sec^2 x\, dx = \tan x + C$
16. $\int \csc^2 x\, dx = -\cot x + C$
17. $\int \sec x \tan x\, dx = \sec x + C$
18. $\int \csc x \cot x\, dx = -\csc x + C$
19. $\int \cos^2 x\, dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$
20. $\int \sin^2 x\, dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$
21. $\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
22. $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
23. $\int \frac{1}{x\sqrt{x^2-a^2}}\, dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$
24. $\int \cosh x\, dx = \sinh x + C$
25. $\int \sinh x\, dx = \cosh x + C$
26. $\int \tanh x\, dx = \ln(\cosh x) + C$
27. $\int \coth x\, dx = \ln |\sinh x| + C$
28. $\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \ln \left|x + \sqrt{x^2-a^2}\right| + C$
29. $\int \frac{1}{\sqrt{x^2+a^2}}\, dx = \ln \left|x + \sqrt{x^2+a^2}\right| + C$
30. $\int \frac{1}{a^2-x^2}\, dx = \frac{1}{2}\ln\left|\frac{a+x}{a-x}\right| + C$
31. $\int \frac{1}{x\sqrt{a^2-x^2}}\, dx = \frac{1}{a}\ln\left(\frac{x}{a+\sqrt{a^2-x^2}}\right) + C$
32. $\int \frac{1}{x\sqrt{x^2+a^2}}\, dx = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{x^2+a^2}}\right| + C$

The Unit Circle



Definitions of the Trigonometric Functions

Unit Circle Definition

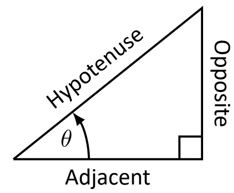


$\sin\theta = y \qquad \cos\theta = x$

$\csc\theta = \dfrac{1}{y} \qquad \sec\theta = \dfrac{1}{x}$

$\tan\theta = \dfrac{y}{x} \qquad \cot\theta = \dfrac{x}{y}$

Right Triangle Definition



$\sin\theta = \dfrac{\text{O}}{\text{H}} \qquad \csc\theta = \dfrac{\text{H}}{\text{O}}$

$\cos\theta = \dfrac{\text{A}}{\text{H}} \qquad \sec\theta = \dfrac{\text{H}}{\text{A}}$

$\tan\theta = \dfrac{\text{O}}{\text{A}} \qquad \cot\theta = \dfrac{\text{A}}{\text{O}}$

Common Trigonometric Identities

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

Cofunction Identities

$\sin\left(\dfrac{\pi}{2} - x\right) = \cos x \qquad \csc\left(\dfrac{\pi}{2} - x\right) = \sec x$

$\cos\left(\dfrac{\pi}{2} - x\right) = \sin x \qquad \sec\left(\dfrac{\pi}{2} - x\right) = \csc x$

$\tan\left(\dfrac{\pi}{2} - x\right) = \cot x \qquad \cot\left(\dfrac{\pi}{2} - x\right) = \tan x$

Double Angle Formulas

$$\sin 2x = 2\sin x\cos x$$

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x\\ &= 2\cos^2 x - 1\\ &= 1 - 2\sin^2 x\end{aligned}$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Sum to Product Formulas

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Power–Reducing Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Even/Odd Identities

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

$\csc(-x) = -\csc x$

$\sec(-x) = \sec x$

$\cot(-x) = -\cot x$

Product to Sum Formulas

$$\sin x\sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$

$$\cos x\cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$

$$\sin x\cos y = \frac{1}{2}\big(\sin(x+y) + \sin(x-y)\big)$$

Angle Sum/Difference Formulas

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$$

$$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$$

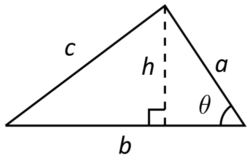
Areas and Volumes

Triangles

$h = a \sin \theta$

Area = $\frac{1}{2}bh$

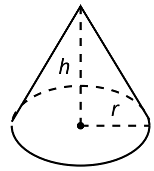
Law of Cosines:
$c^2 = a^2 + b^2 - 2ab \cos \theta$



Right Circular Cone

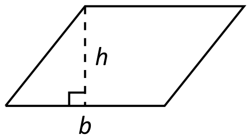
Volume = $\frac{1}{3}\pi r^2 h$

Surface Area =
$\pi r \sqrt{r^2 + h^2} + \pi r^2$



Parallelograms

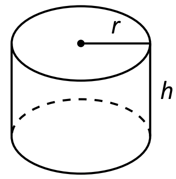
Area = bh



Right Circular Cylinder

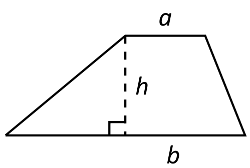
Volume = $\pi r^2 h$

Surface Area =
$2\pi rh + 2\pi r^2$



Trapezoids

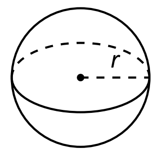
Area = $\frac{1}{2}(a + b)h$



Sphere

Volume = $\frac{4}{3}\pi r^3$

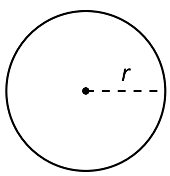
Surface Area =$4\pi r^2$



Circles

Area = πr^2

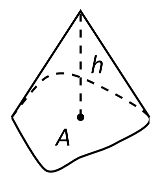
Circumference = $2\pi r$



General Cone

Area of Base = A

Volume = $\frac{1}{3}Ah$

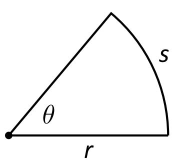


Sectors of Circles

θ in radians

Area = $\frac{1}{2}\theta r^2$

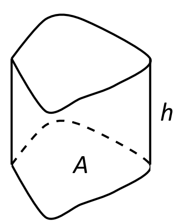
$s = r\theta$



General Right Cylinder

Area of Base = A

Volume = Ah



Algebra

Factors and Zeros of Polynomials
Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. If $p(a) = 0$, then a is a *zero* of the polynomial and a solution of the equation $p(x) = 0$. Furthermore, $(x - a)$ is a *factor* of the polynomial.

Fundamental Theorem of Algebra
An nth degree polynomial has n (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

Quadratic Formula
If $p(x) = ax^2 + bx + c$, and $0 \le b^2 - 4ac$, then the real zeros of p are $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

Special Factors
$x^2 - a^2 = (x-a)(x+a)$

$x^3 - a^3 = (x-a)(x^2 + ax + a^2)$

$x^3 + a^3 = (x+a)(x^2 - ax + a^2)$

$x^4 - a^4 = (x^2 - a^2)(x^2 + a^2)$

$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n$

$(x-y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \cdots \pm nxy^{n-1} \mp y^n$

Binomial Theorem
$(x+y)^2 = x^2 + 2xy + y^2$

$(x-y)^2 = x^2 - 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

Rational Zero Theorem
If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every *rational zero* of p is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Factoring by Grouping
$acx^3 + adx^2 + bcx + bd = ax^2(cx+d) + b(cx+d) = (ax^2+b)(cx+d)$

Arithmetic Operations
$$ab + ac = a(b+c) \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \qquad \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc} \qquad \frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc} \qquad \frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c} \qquad \frac{a-b}{c-d} = \frac{b-a}{d-c} \qquad \frac{ab+ac}{a} = b + c$$

Exponents and Radicals
$$a^0 = 1,\ \ a \ne 0 \qquad (ab)^x = a^x b^x \qquad a^x a^y = a^{x+y} \qquad \sqrt{a} = a^{1/2} \qquad \frac{a^x}{a^y} = a^{x-y} \qquad \sqrt[n]{a} = a^{1/n}$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \sqrt[n]{a^m} = a^{m/n} \qquad a^{-x} = \frac{1}{a^x} \qquad \sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b} \qquad (a^x)^y = a^{xy} \qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Additional Formulas

Summation Formulas:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^3}{12n^2}\left[\max\left|f''(x)\right|\right]$

Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \ldots + 2f(x_{n-1}) + 4f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^5}{180n^4}\left[\max\left|f^{(4)}(x)\right|\right]$

Arc Length:

$$L = \int_a^b \sqrt{1+f'(x)^2}\;\;dx$$

Surface of Revolution:

$$S = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\;\;dx$$

(where $f(x) \geq 0$)

$$S = 2\pi\int_a^b x\sqrt{1+f'(x)^2}\;\;dx$$

(where $a, b \geq 0$)

Work Done by a Variable Force:

$$W = \int_a^b F(x)\;\;dx$$

Force Exerted by a Fluid:

$$F = \int_a^b w\,d(y)\,\ell(y)\;\;dy$$

Taylor Series Expansion for $f(x)$:

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Maclaurin Series Expansion for $f(x)$, where $c = 0$:

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

Summary of Tests for Series:

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
*n*th-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} r^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum $= \dfrac{1}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+a})$	$\lim_{n\to\infty} b_n = L$		Sum $= \left(\sum_{n=1}^{a} b_n\right) - L$
p-Series	$\sum_{n=1}^{\infty} \dfrac{1}{(an+b)^p}$	$p > 1$	$p \leq 1$	
Integral Test	$\sum_{n=0}^{\infty} a_n$	$\int_1^{\infty} a(n)\, dn$ is convergent	$\int_1^{\infty} a(n)\, dn$ is divergent	$a_n = a(n)$ must be continuous
Direct Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$	$\sum_{n=0}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$	
Limit Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n\to\infty} a_n/b_n \geq 0$	$\sum_{n=0}^{\infty} b_n$ diverges and $\lim_{n\to\infty} a_n/b_n > 0$	Also diverges if $\lim_{n\to\infty} a_n/b_n = \infty$
Ratio Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} < 1$	$\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} a_{n+1}/a_n = \infty$
Root Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} \left(a_n\right)^{1/n} < 1$	$\lim_{n\to\infty} \left(a_n\right)^{1/n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} \left(a_n\right)^{1/n} = \infty$