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Geometry-Notes and Applications

Mathematics 3200-Fall 2020

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Trigonometric Identities

Euclidean Plane Geometry

for any triangle with sides *a, b, c*, angles *A, B, C*.

$\sin^2 ø + \cos^2 ø = 1$ for any value of ø

$c^2 = a^2 + b^2 - 2ab \cos C$ Cosine rule

$c^2 = a^2 + b^2$ If angle $C = 90°$ (Famous Pythagoras' Theorem)

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$ Sine rule

$\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$

$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$ Note change $\pm$ to $\mp$.

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

Area of triangle with sides a, b, c and angles A, B. C

$$= \tfrac{1}{2} ab \sin C = \tfrac{1}{2} bc \sin A = \tfrac{1}{2} ac \sin B$$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$ where $s = \tfrac{1}{2}(a + b + c)$

Spherical Geometry.

for triangle with sides *a, b, c*, angles *A, B, C*.

$\cos a = \cos b \cos c + \sin b \sin c \cos A$ *Cosine rule*

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$ *Sine rule*

$\cos A = -\cos B \cos C + \sin B \sin C \cos a$

$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A$

$\sin a \cos C = \cos c \sin b - \sin c \cos b \cos A$

$\cot a \sin b = \cos b \cos C + \cot A \sin C$

$$\sin\left(\frac{A}{2}\right) = \left(\frac{\sin(s-b)\sin(s-c)}{\sin(b)\sin(c)}\right)^{1/2}$$ where $s = \tfrac{1}{2}(a + b + c)$

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Chapter 1

Introduction to Geometry. Applications & Problems

From Brannan's "*Geometry*"

Geometry? For over two thousand years it was one of the criteria for recognition as an educated person to be acquainted with the subject of geometry. Euclidean geometry, of course. In the golden age of Greek civilisation around 400 BC, geometry was studied rigorously and put on a firm theoretical basis – for intellectual satisfaction, the intrinsic beauty of many geometrical results and the utility of the subject. It was written above the door of Plato's Academy "Let no-one ignorant of Geometry enter here!". As late as the 1950s translations of Euclid's Elements were being used as standard school geometry textbooks in many countries.

Geometry is the study of shape. It takes its name from the Greek for 'Earth measurement'. It is believed that geometry began with Egyptian surveyors of two or three millennia ago measuring the Earth, or at least the fertile expanse of it that was annually flooded by the Nile. It rapidly became more ambitious. Classical Greek geometry, called Euclidean geometry after Euclid, who organized an extensive collection of theorems into a definitive text, was regarded by all in the early modern world as the true geometry of space. Isaac Newton used it to formulate his Principia, the book that first set out the theory of gravity. Until the mid-19th Century, Euclidean geometry was regarded as one of the highest points of rational thought, as a foundation for practical mathematics as well as advanced science, and as a logical system splendidly adapted for the training of the mind.

Euclid's Elements.

From *Dan Pedoe*'s "*Geometry and Visual Arts*" (Dover 1983)

"Euclid's Elements is, without any doubt, the most influential mathematics book [13 books in fact] ever written. This was the first printed mathematical book of any importance. It had a margin of two and a half inches, and the figures relating to the Propositions were placed in the margin. This method has recently been revived with certain lush calculus books. The first, and most important, English translation, by Sir Henry Billingsley, appeared in London in 1570."

Euclid's Elements set the standard for axiomatic methods of 'theorem and proof'. Euclid's tools comprise a straightedge and a compass which collapses as soon as it is lifted from the page. Geometrical constructions using only these tools are referred to as Euclidean constructions. For over 2000 years geometers tried to trisect angles, square the circle (construct a square with the same area as a given circle) and to double a cube (construct a cube with volume double that of a given cube). Only in the 19th century was it shown that these constructions were impossible using only Euclid's tools.

Regular Pentagon.

It is alleged that to enter *Pythagoras*' school (~ 600 BC) would-be scholars had to know how to construct a regular pentagon using only a straightedge and compass. Could you enter this school? *Gauss* could have. In 1796 at age 19, *Gauss* looked at this problem further and produced a remarkable theorem related to the construction of regular polygons. *Gauss* was so pleased with this theorem he requested that a regular 17-sided polygon be chiseled on his gravestone. His request was not honoured. Perhaps there was no mason that could chisel a 17-sided regular polygon?

Chapter 1. *Introduction to Geometry & Applications*

Right angles.

Every house builder uses the converse of *Pythagoras*' theorem, sometimes several times daily. Builders of the pyramids 5,000 years ago and builders of today still use $3^2 + 4^2 = 5^2$ or $5^2 + 12^2 = 13^2$ or other so called *Pythagorean* triples. Can you come up with another integer triple (not a multiple of the above)? *Pythagoras*' theorem, Proposition #47 in *Euclid*'s Book I, is often referred to as the fundamental theorem of geometry. Euclid's Proposition #48 is the converse of Pythagoras' theorem.

Distance to the horizon.

Anyone who has ever sat in a window seat of an aeroplane, scaled a mountain or looked out from a high-rise building must have wondered how far is it to the horizon? There is a simple formula that is a direct consequence of Pythagoras' theorem to calculate that distance. What is this simple formula? It becomes a little more complicated if we take into account the refraction of the atmosphere.

Radius of the earth.

The Greek philosophers knew that Earth had to be round. Remarkably, Eratosthanes gave a very good estimate of Earth's radius around 230 BC. How did he do it? Strangely enough there were still many 'Flat-Earth' proponents in the 19^{th} and 20^{th} century.
In 1870, a Flat-Earth proponent *Johnathon Hampden* offered £500 (10 times an average worker's annual salary) to anyone who could prove that a body of water (river, lake, canal) could be curved. *Russell Wallace* gave a brilliant proof using Euclid's proposition III 36 (i.e. proposition #36 from book III). Alas *Hampden* would not pay up, even though the courts accepted *Wallace*'s proof and ordered *Hampden* to pay up. *Wallace* was also famous at the time for presenting with *Charles Darwin* their theory of evolution. Alas, *Wallace* seems to have been forgotten in recent years.

Builder's roofing problem.

When a builder needs to put a circular pipe (chimney?) through a sloping roof, he needs to cut an *ellipse* for the pipe to fit snugly. What is the relation between the ellipse, roof slope and pipe diameter? What simple instructions would we give to the builder to draw and cut the required ellipse?

Sundial construction.

Until 1900, sundials (heliochronometers) were used to regulate the time of some railroad stations of the French network. Readings were accurate to the nearest minute. Sundials are amazingly accurate if constructed for the latitude and set up correctly. How can we construct a sundial that works in Lethbridge? Will the same sundial work in Vancouver? Edmonton? New York? Sundial readings depend on latitude and longitude. Further the readings must be adjusted according to the value of the '*Equation of Time*' for that day of reading.

There is a large sundial at Sangudo, Alberta (117 km west of Edmonton, on hwy 43), the largest in Canada. The Sangudo sundial, large as it is, is dwarfed by a huge sundial in Jaipur India built in 1734 which is almost 30 metres high and still accurate to within seconds.
There is a relatively new type of sundial at CCHS, West Lethbridge. This is an analemmatic dial first designed in the 16^{th} century, quite recent as sundials go.

Shortest distance between cities.

To find the shortest distance between any two places on the globe we need Spherical Geometry. Simple Euclidean Geometry is okay only for calculating short distances. Euclidean Geometry is not accurate for ocean and airline navigators.
In 1569 Mercator used simple geometry to make the biggest advance in map making history. The Mercator Map is still used today by long distance navigators. How was the map made? What makes the Mercator map, loved by mariners, so special?

Sunrise times and positions.

Everyone has noticed that the Sun rises south east in winter and north east in summer. Only on two days a year does it rise from the east (and set in the west). Using Spherical Geometry we can calculate a simple formula to determine how far south and how far north of east the Sun will rise, for a given latitude on a given day. We can also get a simple formula for sunrise and sunset times.

Twilight.

Civil (evening) twilight is defined as that time from sunset to when the sun is 6° below the horizon. Generally, anything that can be seen and done between sunrise and sunset can be seen and done during civil twilight.
Nautical (evening) twilight is defined as that time from sunset to when the sun is 12° below the horizon. Generally, when the sun is 12° below the horizon the horizon can no longer be seen, hence a (nautical) sextant can no longer be used.
Astronomical (evening) twilight is defined as that time from sunset to when the sun is 18° below the horizon. Generally when the sun is 18° below the horizon, the Sun's light no longer interferes with astronomical observations.

Projectiles.

Simple geometry gives the maximum range of artillery shells when they are fired at 45° from the horizontal (assuming no air resistance). However, during WWI, it was found that 10° from the vertical gave maximum range. Also WWI pilots describe that they aimed slightly higher when firing their machine guns to their right and aimed slightly lower when firing at planes to their left? Why should this be? What is the *envelope of safety* pilots associated with artillery projectiles?

Kepler's Laws.

In 1609 *Johannes Kepler* postulated two fabulous empirical laws of astronomy, and a further law in 1619. Fifty years later Isaac Newton proved these laws followed from his theory of gravitational attraction. Most people believe that a stone when thrown follows a parabolic path, but according to Kepler's first law it would be an elliptical path. The following is Howard Eves talking about his Harvard geometry Professor: *J. L. Coolidge*:

"*Coolidge* engaged in all sorts of antics. I remember, he had a watch on a chain hanging from his vest. He would twirl the chain around and around his index finger, and then unwind it. One day the chain broke and the watch looped across the room and smashed on a windowsill. Without any hesitation he said, 'Gentlemen*, you have just witnessed a perfect parabola'.
Which wasn't really true. A lot of people think that projectiles travel on parabolic paths. It really is a little nose of an ellipse of which the centre of the earth is one focus, but so closely resembles a parabola that no one can tell the difference. I did not want to correct him, though." Coolidge was a top rate geometer; he of course would have known Kepler's law and the correct path. *Coolidge* possibly would have felt it a bit over the top to mention that the pathwas really an ellipse.

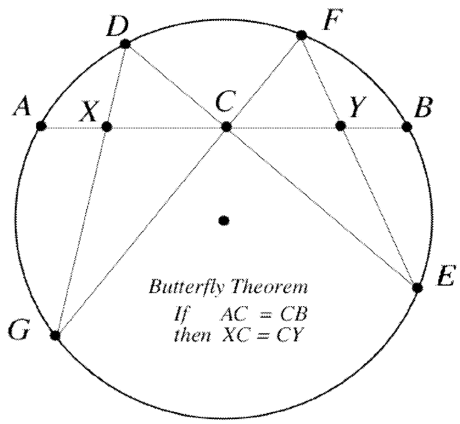
*This was university in the 1930s, there would have been no women in a math class.

Inversive Geometry.

A geometric inversion is a transformation of the plane which maps points inside a given circle to points outside the circle and conversely. Why would we map points in this way? It turns out that some difficult problems of geometry have simpler solutions when mapped inversely. The inverse problem is solved and the solution is then mapped back for the solution to the original problem. Two examples are: Steiner's Porism mentioned above and the famously difficult to prove Butterfly Theorem (worth a Google).

It is not unlike being given the problem of multiplying two Roman numerals: XIII and VI together, with the result to be in Roman numerals. Almost certainly, most people would map the numerals to their Arabic equivalents 13 and 6, multiply to get 78, then map back to Roman numerals to give the answer: XIII x VI = LXXVIII.

The Butterfly Theorem.

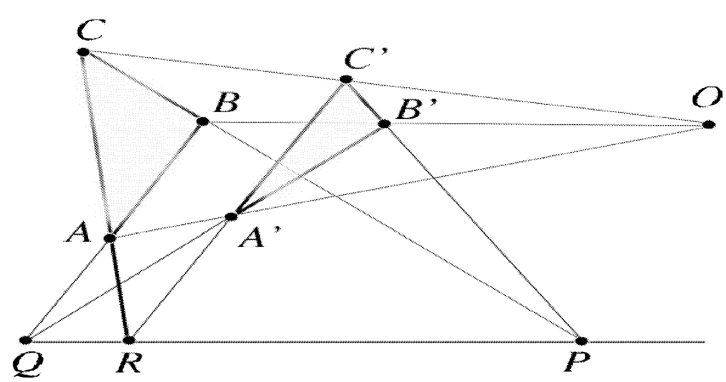


For any given circle and any given chord AB, take the centre point C.
Through C draw any two other (distinct) chords DCE and FCG say. Let the two chords DG and FE cut original chord AB at points X and Y.
Then XC = CY.

If the original chord AB is diameter, then the theorem is obvious.

Projective Geometry.

An early discovery (early 17th century) of projective geometry was the famous theorem of Desargues (1593-1662).



If in a plane two triangles ABC and $A^{/}B^{/}C^{/}$ are such that the straight line joining corresponding vertices are concurrent at some point O say, then the corresponding extended sides will intersect in three collinear points (Q, R, P in the above diagram).

The converse of this theorem is also true. Further, if the two triangles ABC and $A^{/}B^{/}C^{/}$ are in any two non-parallel planes, the theorem still holds.

Conics & Dandelin's Spheres

In 1830 *Germinal Dandelin* gave three magnificent proofs regarding conics (parabolas, ellipses, hyperbolas). The Greeks with rather difficult proofs knew these results. *Dandelin*'s proofs are so elegant, perhaps my favourite proofs in all geometry. Using these results we are able to deduce the path of the shadow of a gnomon tip for various locations on earth. We are able to solve the 'Builders' Roofing Problem' above.

Steiner's Porism.

I cannot see any applications of this remarkable result, but remarkable it is. I once described Steiner's porism to a language student, she was not impressed. I can only think that I must have failed to explain the porism properly? There is a wonderful proof for Steiner's porism in Inversive Geometry.

Euclidean Constructions.

A great game of the ancient Greeks was to construct various geometric figures and procedures using only a collapsible compass and straightedge, 'Greek constructions'. Three famous problems of the age were constructions to:

1. *Double a cube. That is given a cube of side 1, construct a cube of side* $\sqrt[3]{2}$.
2. *Trisect any given angle.*
3. *Construct a square with area the same as that of a given circle.*

Constructions were never found. After 2000 years of trying, in 1837, is was finally shown that the first two constructions were indeed impossible. Nearly fifty years later, 1882, it was shown that the third construction too was impossible.
All Greek constructions used a collapsible compass, that collapsed immediately it was removed from the page (sand?). The Greek compass would not transfer distances as we can with the modern compass. It is easily shown that the 'modern compass and straightedge' is equivalent to the 'Greek collapsible compass and straightedge'. This result is Proposition 2 of Euclid's Elements, Book I. Further, it can be shown, with a little difficulty, that the two compasses are equivalent with or without the straightedge. That is: a circle of fixed radius and centre can be drawn with the same radius centred on any other given point, using just the collapsible compass; no straightedge needed. However, what is remarkable is the so-called Mohr–Mascheroni theorem which says any Greek construction can be made using a compass alone. For example, try finding the mid point of the line segment AB using just a compass. It can be shown that this is impossible using just a straightedge. Further, it was shown that any Greek construction can be made using a so-called 'rusty compass' and straightedge.

Klein's Erlangen Programme.

In 1870 Felix Klein presented (at the University of Erlangen) a remarkable thesis connecting geometry with groups. Klein's great idea was to regard a geometry as a space with a group of transformations of that space. The properties of figures that are not altered by any transformation in the group are the geometrical properties of that geometry.
In the two-dimensional Euclidean geometry the space is the plane, and the group is the group of all length-preserving transformations of the plane. Klein showed that most geometries (affine, inversive, spherical and non-Euclidean) are all examples of projective geometry with extra conditions.

Non-Euclidean Geometry.

For over 2000 years geometers tried in vain to prove Euclid's fifth postulate. It was not until early in the 19th century that *Gauss*, Lobachevskii and Bolyai independently proved that the fifth postulate was indeed an axiom independent of the first four postulates of Euclid (Book I). This was a huge step for mathematics, because now we could define different 5th (parallel) postulates and get different geometries. These geometries were called non-Euclidean.

Farkas Bolyai to his son *Janos*, around 1820:

"*You must not attempt this approach to parallels. I know this way to its very end. I have traversed this bottomless night, which extinguished all light and joy of my life. I entreat you, leave the science of parallels alone ... I thought I would sacrifice myself for the sake of the truth. I was ready to become a martyr who would remove the flaw from geometry and return it purified to mankind. I accomplished monstrous, enormous labours; my creations are far better than those of others and yet I have not achieved complete satisfaction ... I turned back when I saw that no man can reach the bottom of the night. I turned back unconsoled, pitying myself and all mankind.*

I admit that I expect little from the deviation of your lines. It seems to me that I have been in these regions that I have travelled past all reefs of this infernal Dead Sea and have always come back with broken mast and torn sail. The ruin of my disposition and my fall date back to this time. I thoughtlessly risked my life and happiness..

Quoted from *M. J. Greenberg's* "***Euclidean and Non-Euclidean Geometries***".

Spherical Geometry is an excellent, very practical example of a non-Euclidean Geometry. Axiom 5 of Euclid's geometry does not hold for Spherical Geometry. Given a line l, and a point P not on l, there is no line through P that does not meet the given line l.

Spherical Geometry is essential for navigation on the planet (all GPS systems) and for mathematical astronomy determining sunrise/set times, twilight times, solar and lunar eclipses and much more.

Hyperbolic Geometry is another example of non-Euclidian Geometry. Euclid's Axiom 5 does not hold. Given a line l, and a point P not on l, there is an infinity of lines through P that do not meet the given line l. While this geometry is not so easy to imagine as Spherical Geometry, Poincare gave a rather nice example, defined at the end of Chapter 6. It is just possible that the geometry of our universe is Hyperbolic.

The Shadow Curve.

As the sun moves across the sky, the tip of a stake will cast a moving shadow. The path of this shadow, known as the *shadow curve,* varies due to the time of year and the location of the stake. Most texts suggest that the path is a hyperbola. This is because most people live between the arctic and antarctic circles latitudes 66.5° N and 66.5° S. It can be shown, using Spherical Geometry, that the path is one of the following: a straight line, a hyperbola, a parabola, or an ellipse. One sees that if a vertical stake were planted at the North or South Pole then the shadow path would be a circle (a special ellipse). Above the arctic circle in summer, when the sun does not set, the *shadow curve* will be an ellipse.
Dorrie, in his great book "***100 Great Problems of Elementary Mathematics***" gives a single, adorable equation that covers the the conic sections: hyperbola, parabola & ellipse.

$$y^2 = 2x \tan p - \left(1 - \frac{\cos^2\varphi}{\cos^2 p}\right)x^2 \qquad p \neq 90°$$

where ϕ is the latitude of the stake and $p = 90° - \delta$
where δ *is the* sun's declination for that particular day

The straight line *shadow curve* occurs at equinox (when $\delta = 0$, *i.e.* $p = 90°$).

Chapter 2

Euclidean Geometry

Euclid's Elements Euclid, fl c 300 BC, lived in Egypt in the time of Ptolemy I. He is said to have been younger than Plato and older than Archimedes.

From *Dan Pedoe*'s "*Geometry and Visual Arts*" (Dover 1983)

"Euclid's Elements is, without any doubt, the most influential mathematics book [13 books in fact] ever written. This was the first printed mathematical book of any importance. It had a margin of two and a half inches, and the figures relating to the Propositions were placed in the margin. This method has recently been revived with certain lush calculus books. The first, and most important, English translation, by Sir Henry Billingsley, appeared in London in 1570. It consists of 928 folio pages, not counting a long preface, and the figures for the Propositions in three-dimensional geometry are given twice, once in Euclid's version, and again with pieces of paper pasted at the edges so that the pieces can be turned up and made to show the real forms of the solid figures represented."

From *T. L. Heath*'s "Euclid's Elements" (1933).

"Euclid's Elements: what book in the world could be more suitable for inclusion in the Library than this, the greatest textbook of elementary mathematics that there has ever been or is likely to be, a book which, ever since it was written twenty-two centuries ago , has been read and appealed to as authoritative by mathematicians great and small, from Archimedes and Apollonius of Perga onwards? No textbook, presumably, can ever be without flaw (especially in a subject like geometry, where some first principles, postulates or axioms, have to be assumed without proof, and any number of alternative systems are possible), and flaws there are in Euclid; but it is safe to say that no alternative to the Elements has yet been produced which is open to fewer or less serious objections. The only general criticism of it which is deserving of consideration is that it is unsuitable as a textbook for very young boys and girls who are just beginning to learn the first things about geometry. This can be admitted without detracting in the least from the greatness of the permanent value of the book. The simple truth is that it was not written for schoolboys or schoolgirls, but for the grown man who would have the necessary knowledge and judgement to appreciate the highly contentious matters which have to be grappled with in any attempt to set out the essentials of Euclidean geometry as a strictly logical system, and, in particular the difficulty of making the best selection of unproved postulates or axioms to form the foundation of the subject. My advice would, therefore, be: if you must spoon-feed the very young, do so; but when they have shown a taste for the subject and attained the standard necessary for passing honours examinations, let them then be introduced to Euclid in his original form as an antidote to the more or less feeble echoes of him that are to be found in the ordinary school textbooks of 'geometry'. I should be surprised if such qualified readers, making the acquaintance of Euclid for the first time, did not find it fascinating, a book to be read in bed or on a holiday, a book as difficult as any detective story to put down when once begun.
I know of one actual case, that of an undergraduate at Cambridge suddenly presented with a copy of Euclid, where this happened. This is a true test of such a book. Nor does the reading of it require the "higher mathematics'. Any intelligent person with a fair recollection of school work in elementary geometry would find it (progressing as it does by gradual and nicely contrived steps) easy reading, and should feel a real thrill in following its development, always assuming that enjoyment of the book is not marred by any prospect of having to pass an examination in it!
This is why I applaud the addition of this great classic to Everyman's Library; for everybody ought to read it who can, that is all educated persons except the very few who are constitutionally incapable of mathematics."

From Philosopher Bertrand Russell 1883:
(See Fauvel & Gray's '*The History of Mathematics*' Page 140):

"At the age of eleven, I began Euclid, with my brother as my tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined that there was anything so delicious in the world. After I had learned the fifth proposition, my brother told me that it was generally considered difficult, but I had found no difficulty whatever. This was the first it had dawned upon me that I might have some intelligence. From that moment until Whitehead and I finished *Principia Mathematica*, when I was thirty-eight, mathematics was my chief interest, and my chief source of happiness. Like all happiness, however, it was not unalloyed. I had been told that Euclid proved things, and was much disappointed that he started with axioms. At first I refused to accept them unless my brother could offer me some reason for doing so, but he said: 'If you don't accept them we cannot go on', and as I wished to go on, I reluctantly admitted them *pro tem*. The doubt as to the premisses of mathematics which I felt at that moment remained with me, and determined the course of my subsequent work.
The beginnings of Algebra I found far more difficult, perhaps as a result of bad teaching. I was made to learn by heart: 'The square of the sum of two numbers is equal to the sum of their squares increased by twice their product'. I had not the vaguest idea what this meant, and when I could not remember the words, my tutor threw the book at my head, which did not stimulate my intellect in any way."

On-line:

Try **http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI5.html**
for a wonderful on-line introduction to Euclid's Elements, all thirteen books.

<u>Euclid begins in Book I with 23 definitions.</u>

1. A *point* is that which has no part.
2. A *line* is breadthless length. [Euclid's '*line*' can be any shape; it is only *straight* if he says it is straight as in the next definition # 3.]
3. The extremities of a line are points.
4. A *straight line* is a line which lies evenly with the points on itself.
5. A *surface* is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A *plane surface* is a surface which lies evenly with the straight lines of itself.
8. A *plane angle* is the inclination of one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called *rectilineal*.
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right*, and the straight line standing on the other is called a *perpendicular* to that on which it stands..
11. An *obtuse angle* is an angle greater than a right angle.
12. An *acute angle* is an angle less than a right angle.
13. A *term or boundary* is the extremity of any thing.
14. A *figure* is that which is enclosed by one or more boundaries.
15. A *circle* is a plane figure contained by one line such that all the straight lines falling upon it from one point lying within the figure are equal to one another.
16. And the point is called the *centre* of the circle.
17. The *diameter* of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.
18. A *semicircle* is the figure contained by the diameter and the part of the circumference cut off by the diameter.

19. *Rectilinear* figures are those which are contained by straight lines, *trilateral* figures (*triangles*) being those contained by three, *quadrilateral* those contained by four, and *multilateral* those contained by more than four straight lines.
20. Of trilateral figures, an *equilateral triangle* is that which has its three sides equal, an *isosceles triangle* that which has two of its sides equal and *scalene triangle* that which has its three sides unequal.
21. Further, of trilateral figures, a *right-angled triangle* is that which has a right angle, an *obtuse-angled triangle* that which has an obtuse angle, and an *acute-angled triangle* that which has its three angles acute.
22. Of quadrilateral figures, a *square* is that which is both equilateral and right-angled.
23. *Parallel* straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either directions. [Euclid's *straight lines* were *finite segments* and not infinite, as we think of lines today.]

Euclid's 5 Common Notions (obvious axioms) from Book I.

Common Notion 1	Things which equal the same thing also equal one another. [If $a = b$ & $c = b$, then $a = c$]
Common Notion 2	If equals are added to equals, then the wholes are equals. [If $a = b$ & $c = d$, then $a + c = b + d$]
Common Notion 3	If equals are subtracted from equals, then the remainders are equal. [If $a = b$ & $c = d$ then $a - c = b - d$]
Common Notion 4	Things which coincide with one another equal one another.
Common Notion 5	The whole is greater than the part. [$a + b > a$]

Euclid's 5 Axioms or postulates from Book I.

Axiom 1:	To draw a straight line from any point to any point.
Axiom 2:	To produce a finite straight line continuously in a straight line.
Axiom 3:	To describe a circle with any centre and any radius.
Axiom 4:	That all right angles are equal to one another.

Axiom 5: *That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

Axiom 5 was the cause of much interest for over 2000 years. Geometers tried to prove that *Axiom 5* was a natural consequence of the first four axioms. It seems obvious enough, but it turned out that Axiom 5 was indeed independent from the first four axioms. This led to non-Euclidean geometries, geometries with an axiom different to Axiom 5.

Geometry using only the first four axioms is referred to as *Neutral Geometry*.

Propositions: 1 – 48 (Euclid’s Elements Book I).

I.1 To construct an equilateral triangle on a given finite straight line.

I.2. To place a straight line [segment] equal to a given straight line [segment] with one end at a given point.

I.3. To cut off from the greater of two given unequal straight lines, a straight line equal to the lesser.

I.4. If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.
[We say the two triangles with these respective properties (corresponding sides, included angle and corresponding side or **SAS** for short, are congruent.]

I.5. In isosceles triangles the angles at the base are equal to one another, and if the equal sides are produced further, the angles under the base will be equal to one another. [This proposition is known as: *The Pons Asinorum*, or *Asses' Bridge*.]

I.6. If in a triangle two angles equal one another, then the sides which subtend the equal angles also equal one another.
[This is the converse of the previous proposition.]

I.7. Given two straight lines constructed from the ends of a straight line and meeting in a point, there cannot be constructed from the ends of the same straight line, and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each equal to that from the same end.

I.8. If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are constrained by the equal straight lines.
[We say the two triangles with these respective properties (three sides equal or **SSS** for short, are congruent.]

I.9. To bisect a given rectilinear angle.

I.10. To bisect a given finite straight line.

I.11. To draw a straight line at right angles to a given straight line from a given point on it.

I.12. To draw a straight line perpendicular to a given infinite straight line from a given point not on it.

I.13. If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.

I.14. If with any straight line [AB say] and at a point [B say] on it, two straight lines [through point B] not lying on the same side [of the straight line AB] make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.

I.15. If two straight lines cut one another, then they make the vertical angles equal to one another.

I.16. In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.

I.17. In any triangle the sum of any two angles is less than two right angles.

I.18. In any triangle the angle opposite the greater side is greater.

I.19. In any triangle the side opposite the greater angle is greater.

I.20. In any triangle the sum of any two sides is greater than the remaining one.

I.21. If from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.

I.22. To construct a triangle out of three straight lines which equal three given straight lines: thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one.

I.23. To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it.

I.24. If two triangles have two sides equal to two sides respectively, but have one of the angles contained by the equal straight lines greater than the other, then they also have the base greater than the base.

I.25. If two triangles have two sides equal to two sides respectively, but have the base greater than the base, then they also have one of the angles contained by the equal straight lines greater than the other.

I.26. If two triangles have two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the remaining sides equal the remaining sides and the remaining angle equals the remaining angle.
[We say the two triangles with these respective properties (two angles equal and one side equal or **AAS** for short, are congruent.]

I.27. If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.

I.28. If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.

I.29. A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles.

I.30. Straight lines parallel to the same straight line are also parallel to one another.

I.31. Through a given point to draw a straight line parallel to a given straight line.

I.32. In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.

I.33. The straight lines joining equal and parallel straight lines (at the extremities which are) in the same direction (respectively) are themselves also equal and parallel.

I.34. In parallelogrammic areas the opposite sides and angles are equal to one another, and the diameter bisects the areas.

I.35. Parallelograms which are on the same base and in the same parallels are equal to one another. [*i.e.* equal in area.]

I.36. Parallelograms which are on equal bases and in the same parallels are equal to one another.

I.37. Triangles which are on the same base and in the same parallels are equal to one another. [*i.e.* equal in area.]

I.38. Triangles which are on equal bases and in the same parallels are equal to one another.

I.39. `Equal triangles which are on the same base and on the same side are also in the same parallels.

I.40. Equal triangles which are on equal bases and on the same side are also in the same parallels.

I.41. If a parallelogram has the same base with a triangle and be in the same parallels, the parallelogram is double the triangle.

I.42. To construct, in a given rectilineal angle, a parallelogram equal to a given triangle.

I.43. In any parallelogram the complements of the parallelograms about the diameter are equal to one another.

I.44. To a given straight line to apply, in a given rectilineal angle, a parallelogram equal to a given triangle.

I.45. To construct, in a given rectilineal angle, a parallelogram equal to a given rectilineal figure.

I.46 On a given straight line to describe a square.

I.47 In right angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle. [*Pythagoras*' theorem.]

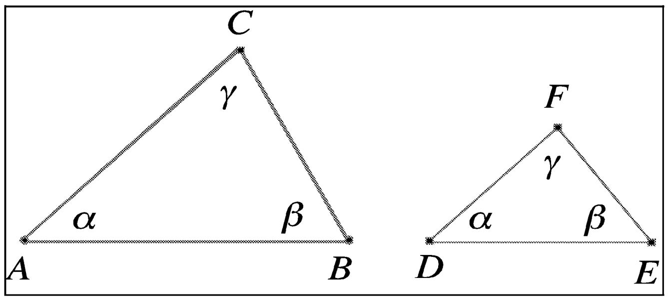
I.48. If in a triangle the square of one of the sides be equal to the square on the remaining two sides of the triangle, the angle contained by the remaining two sides of the triangle is right. [Converse of Pythagoras theorem, loved by all carpenters.]

Perhaps note that Axiom 5 (Parallel postulate) is not required for the first 28 propositions.

We will look at some simple constructions, the construction of the regular pentagon and *Gauss*' theorem on the construction of regular polygons. We will end with some compass only constructions.

Similar Triangles.

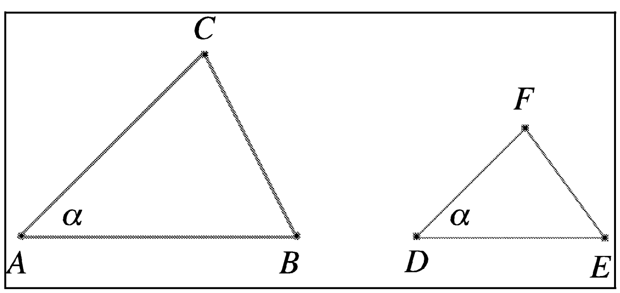
Triangles ABC and DEF are said to be ***similar*** if they have the same angles, or, equivalently, if the ratios of the 3 corresponding sides are equal. (Book VI #4)
We write Δ ABC ~ Δ DEF.



AB/AC = DE/DF
AB/BC = DE/EF
AC/CB = DF/FE

Of course, if two angles of two triangles are equal, then the third angle must also be equal. This follows from Proposition I. 32 above, which says that all triangles have their three angles adding to two right angles (or 180°).

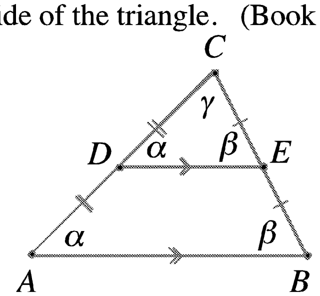
Two triangles are similar if two sides of one triangle are proportional, respectively, to two sides of another triangle and the angles included between the sides are equal. (Book VI #6)



If AC/AB = DF/DE
with angle CAB = angle FDE

then Δ CAB ~ Δ FDE

Proposition VI.2. A line through the mid points of any two sides of a triangle will be parallel to the third side of the triangle. (Book VI #2)



Proposition VI.3 A line through the mid point of any triangle and parallel to another side bisects the third side of the triangle. (Book VI #2)

Congruent Triangles.

Two triangles are ***congruent*** (identical) in the following **four cases**:

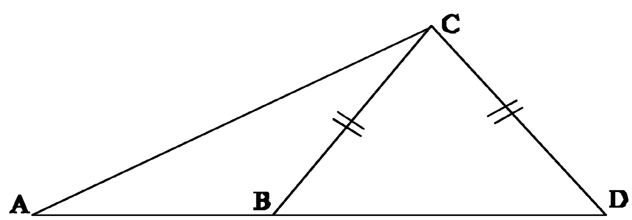
SSS When the three corresponding sides of the triangles are equal.
Proposition I.8 above (#8 of Euclid's Elements Book 1).

AAS When two corresponding angles and a corresponding side are equal.
Proposition I.26 above.
If two corresponding angles are equal then all three corresponding angles are equal, since by I.32 the sum of the three angles of any triangle is fixed, to that of two right-angles..

SAS When two corresponding sides and the *included* angle are equal.
Proposition I.4 above.

Note this does NOT include **ASS**, which is when two corresponding sides are equal, but an angle other than the included angle is equal, unless this common equal angle is a right angle.

As my high school teacher used to say, 'don't make an **ASS** of yourself with this mistake'. See diagram below:



We see Δ ABC and Δ ACD have corresponding equal sides: AC (common to both) and CB = CD and there is a common angle CAB. We have ASS. We do not have SAS. Clearly Δ ABC ≠ Δ ADC.

SS90° The right angle need not be the included angle, it can be anywhere.
This is equivalent to **SSS**, since if one angle is a right angle and two sides are equal, then all three sides must be equal. Propositions I.8 & I.47.

Euclidean Constructions.

Euclidean constructions are those figures that can be constructed using only a straightedge and a compass. Note the straightedge is not a ruler, it has no markings. Euclid's compass needs two given points to use. The compass requires a centre point and another point to establish the radius. The compass collapses once it is withdrawn from the page.

However, Euclid's Proposition #2 (Book 1) assures us that the modern compass which can be used to transfer fixed distances is equivalent to his collapsing compass when used with a straightedge. For all 'Euclidean constructions' we shall assume the use of a straightedge and a modern compass, since Euclid's collapsing compass and the modern compass are equivalent for Euclidean constructions. Of interest, it can be shown that a fixed (rusty) compass and a straightedge is all that is needed for all Euclidean constructions.

Try the following constructions:

1. *To construct an equilateral triangle with one side a given finite (straight) line*. I.1.
   In Euclid's text a 'line' was not necessarily straight, any curve 'without breadth' was a line. We shall use the current understanding of a 'line' as being always straight.
2. *To bisect a given angle*. (Proposition I.9)
3. *To bisect a given line segment*. (Proposition I.10)
4. *To construct a line through a given point P, which is perpendicular to a given line*.

Case a.	When P is on the given line.	(Proposition I.11)
Case b.	When P is not on the given line.	(Proposition I.12)

5. *To construct a line parallel to a given line through a point P not on the given line*. I.31.
6. *To construct an angle at a given point P on a given line, the angle the same as a given angle*. (Proposition I. 22)
7. *To construct a square with one side a given line segment*. (Proposition 1.46)
8. *To construct the tangent from a given point P to a given circle, P outside the circle*.
9. *To construct two pairs of tangents common to two given non-intersecting circles*.

10. *To construct a square with area twice that of a given square*.
11. *To trisect a right angle*.
12. *To trisect a given line segment*.
13. *To construct a regular hexagon*.
14. *To construct an equilateral triangle inscribed within a given circle*.
15. *To construct a regular pentagon*. See also *Gauss*' theorem below.

Constructions using the compass alone, no straightedge:

16. *To construct (find) the midpoint between two points A and B*.
17. *To find the centre of a given circle*.
18. *To inscribe a square inside a given circle*. (*Napoleon*'s problem.)
19. *To construct the corners of a square given two adjacent corners*.
20. *To construct the corners of a square given two diagonal corners*.

In fact, all points of any Euclidean constructions can be found with the compass alone. This is the *Mohr–Mascheroni* theorem mentioned in the Introduction.

Regular Polygons.

Euclid listed constructions for the regular polygons with 3 sides (triangle), 4 sides (square), 5 sides (pentagon), 6 sides (hexagon). Euclid showed that if you could construct a regular polygon of r sides and one of s sides, then you could construct one of rs sides, provided r and s were relatively prime (that is integers r and s having 1 as the largest common factor). The construction of a regular 15-sided polygon is listed in The Elements, Book IV, Proposition #15. You might wonder if it is possible to construct regular polygons of any number of sides? It is clear that having done 3 sides, 4 sides, 5 sides and 15 sides, it is a easy matter to construct regular polygons of sides $3\text{x}2^n$, $4\text{x}2^n$, $5\text{x}2^n$ and $15\text{x}2^n$ for any positive integer n by bisecting (repeatedly) the sides.

What about 7, 9, 11, 13, 17, 19, ... sides?
Gauss as a teenager proved the remarkable result that an m sided regular polygon can be constructed with Euclidean tools when m is prime and m is of the form:

$$m = 2^{2^n} + 1 \qquad \textit{any integer } n \geq 0.$$

Numbers of this form, denoted F(n), are known as *Fermat* numbers.

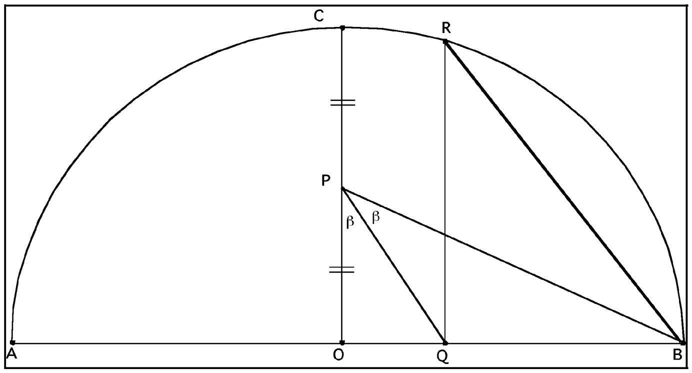
F(0) =	$2^{2^0} + 1$	= 3	prime	
F(1) =	$2^{2^1} + 1$	= 5	prime	
F(2) =	$2^{2^2} + 1$	= 17	prime	
F(3) =	$2^{2^3} + 1$	= 257	prime	
F(4) =	$2^{2^4} + 1$	= 65,537	prime	
F(5) =	$2^{2^5} + 1$	= 4,294,967,297	not prime	(641 is a factor)

Of interest, *Fermat* conjectured that all such numbers would be prime. It was *Euler* who noticed that F(5) was not prime. To date, no *Fermat* numbers, other than the first five are known to be prime. It is now thought that only the first five *Fermat* numbers are prime.
The constructions for the 17, 257 & 65,537-sided regular polygons are possible. They are not easy. Recall that *Gauss* requested a 17-sided regular polygon inscribed on his tombstone.

A mathematician (Professor *Hermes*) has described the construction of a regular 65,537-sided polygon; apparently no one has checked his thousands of pages of instructions. In 1837 *Pierre Wantzel* proved the converse of *Gauss*' remarkable result. *Wantzel* showed, using methods from Abstract Algebra, that regular polynomials could be constructed only if the number of sides was some multiple of 2^n with distinct *Fermat* primes.
Hence no need to waste time seeking the construction (using only *Euclid*'s straightedge and compass) of a 7, 9, 11, and 13-sided regular polygon, as we now know it is not possible.

<u>Theorem</u>. (*Gauss-Wantzel*)
An m-sided regular polygon ($m \geq 3$) can be constructed with Euclidean tools if and only if $m = 2^n p_1 p_2 p_3 \ldots p_i$ where the p_i are distinct *Fermat* primes and n is any integer, $n \geq 0$.

Regular Pentagon construction.



1. Construct radius OC perpendicular to a diameter AB of any circle.
2. Bisect radius OC at P.
3. Bisect angle BPO, bisector meeting OB at point Q.
4. Construct QR perpendicular to OB, meeting the circle at R.

BR is a side of a regular pentagon (with vertices on the circle of construction).

Proof (that the construction in the diagram above does indeed yield a regular pentagon)

We show that angle ORQ $= 18°$, hence angle ROQ $= 72°$ as required for a regular pentagon.

Let the circle have radius one unit, and let angle BPO $= 2\beta$

$\therefore$ $\tan 2\beta = OB/OP = 1/(1/2) = 2$ from right triangle Δ POB

also $\tan 2\beta = \dfrac{2\tan\beta}{1-\tan^2\beta}$ well known trigonometric identity.

$$\therefore \quad \frac{2\tan\beta}{1-\tan^2\beta} = 2 \qquad \text{or} \qquad \tan^2\beta - \tan\beta - 1 = 0$$

solving this quadratic in 'tan β' gives $\tan\beta = \dfrac{\sqrt{5}-1}{2}$

$\tan\beta = 2\,OQ$ (from right triangle Δ POQ) $\therefore$ $OQ = \dfrac{\sqrt{5}-1}{4}$

$\therefore$ $\sin ORQ = \dfrac{\sqrt{5}-1}{4}$ (from right triangle Δ ORQ.)

$\sin 18° = \dfrac{\sqrt{5}-1}{4}$ (*lemma below*)

i.e. $\sin ORQ = \sin 18°$

$\therefore$ Angle ORQ $= 18°$ $\Rightarrow$ ROQ $= 72°$. **QED**

The lemma used above comes from elementary trigonometry.
We prove this lemma on the next page.

Lemma. $\text{Sin } 18° = \dfrac{\sqrt{5}-1}{4}$

Proof. Let $\alpha = 18°$ and note that $5\alpha = 2\alpha + 3\alpha = 90°$.

$\sin 2\alpha = \cos 3\alpha$ (consider any right triangle with angles 2α, 3α & $90°$)

$\sin 2\alpha = \cos 2\alpha \cos \alpha - \sin 2\alpha \sin \alpha$

write $\cos 3\alpha = \cos(2\alpha + \alpha)$ & expand

$\sin 2\alpha\,(1 + \sin \alpha) = \cos 2\alpha \cos \alpha$

recall: $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

$2\sin \alpha \cos \alpha\,(1 + \sin \alpha) = (1 - 2\sin^2\alpha) \cos \alpha$ since $\cos 2\alpha = 1 - 2\sin^2\alpha$

$4 \sin^2 \alpha + 2 \sin \alpha - 1 = 0$ then solving this quadratic in '$\sin \alpha$'

$\sin \alpha = \dfrac{\sqrt{5}-1}{4} = \sin 18°$ since α was $18°$. **Q.E.D.**

Four basic, most often used trigonometric identities:

- $\sin^2 \alpha + \cos^2 \alpha = 1$ for <u>any</u> angle α.
- $\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \sin \beta \cos \alpha$ for <u>any</u> angles α & β.
- $\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$ for <u>any</u> angles α & β.
- $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \tan\beta}$ for <u>any</u> angles α & β.

Note the inversion of the *plus-minus* signs in the last two identities.

Impossible constructions, three famous problems.

1. To construct a cube with volume double that of a given cube.
2. To trisect any given angle.
3. To construct a square with the same area as that of a given circle.
   This problem was referred to as 'squaring the circle'.

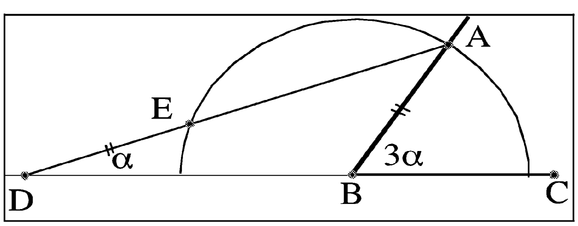
The first problem: also referred to as *'duplicating the cube'* or the *'Delian Problem'*. A reference to this problem occurs in a document from the famous Greek geometer Eratosthenes to King Ptolemy III around 240 BC: "To King Ptolemy, Eratosthanes sends greetings. It is said that one of the ancient tragic poets represented Minos as preparing a tomb for Glaucus and as declaring, when he learnt it was a 100 feet each way: "Small indeed is the tomb thou hast chosen for a royal burial. Let it be double [in volume]. And thou shalt not miss that fair form if thou quickly doublest each side of the tomb." But he was wrong. For when the sides are doubled, the surface area becomes four times as great and the volume eight times as great. It became a subject of inquiry among geometers in what manner one might double the given volume without changing the shape. And this problem was called the duplication of the cube, for a given cube they sought to double it." (You might note that it is a relatively easy problem to double a (two dimensional) square.)

It was finally shown by *Pierre Wantzel* in 1837 that this first problem is indeed impossible. Interestingly enough, *Wantzel* did not used methods of geometry, but techniques from the new subject of Abstract Algebra. See *A. Jones, S.A. Morris* and *K.R. Pearson's "Abstract Algebra and Famous Impossibilities"* (*Springer-Verlag* 1991).

The second problem: *Jones, Morris* and *Pearson* (above) suggest that this problem possibly arose from attempts to construct a 9-sided regular polygon which can be constructed if, and only if, the angle 60° can be trisected.
Angle 90° and various other angles can easily be trisected, but the general angle resisted trisection. Again, it was *Wantzel* in 1837 who showed that it is not possible to trisect an arbitrary angle, hence a regular 9-sided polygon is not possible to construct.

On trisecting an angle.

1. Archimedes' construction: To trisect given angle ABC

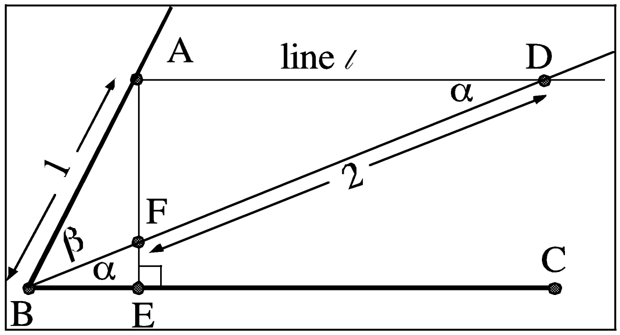


1. Draw any semi-circle centred on point B as shown.
2. Construct AD, D on BC extended, as shown, so that DE has the same length as the radius of the semi-circle.

Angle ADB is one third of the given angle ABC.

Proof.
Join EB.
Angle EBD = angle EDB = α
Angle AEB = 2α — proposition # 32 & isosceles Δ DEB
Angle AEB = Angle EAB — isosceles Δ EBA
Angle ABC = Angle D + Angle A = 3α. — proposition # 32 **Q.E.D.**
Why is this construction not classified as Euclidean?

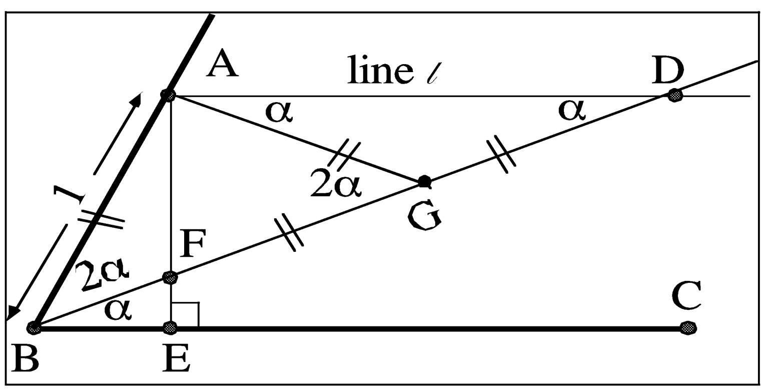
2. Hippocrates' construction: To trisect given angle ABC.



1. Construct any line l parallel to BC, through A say.
2. Drop AE perpendicular to BC as shown.
3. Construct BD, point D on line l, so that DF is twice the length of AB.
   Angle DBC is one-third the given angle ABC.

Proof.
Let angle ADB = α = angle DBC
Let angle ABD = β
Apply sine rule to triangle ABD
$\sin\beta / (2\cos\alpha) = \sin\alpha / 1$ (since AD = $2\cos\alpha$)
$\sin\beta = \sin 2\alpha$, therefore $\beta = 2\alpha$. **Q.E.D.**

An alternative proof.



In triangle AFD drop the median from A to side FD at G say. This creates two isosceles triangles AGD and AGB (since ΔADF is right).

It will be shown that the diameter of a circle subtends a right angle at the circumference and conversely. The angles are then as marked. **Q.E.D.**

Why is this construction not classified as Euclidean?

The third problem: also referred to as the *'quadrate of the circle*. This problem is linked to that of finding the area of a circle. A manuscript from 1650 BC proposes that the area of a circle is that of a square whose side is the diameter diminished by one ninth. That is, the area of a circle of radius r is given as $[(8/9)(2r)]^2$. This corresponds to a value of $\pi = (256)/(81) \approx 3.1604....$. *Archimedes* (287-212 BC) improved on this approximate value of π and showed that:

$3\frac{10}{71} < \pi < 3\frac{1}{7}$ or equivalently: $3.14084 \ldots < \pi < 3.14285\ldots$

we now know π to thousands (trillions) of decimal places: π = 3.14159 26535 89793
23846 83279 50288 41971 69399 37510 58209 74944 59230 78164 06286 20899 86280
34825 34211 70679 82148 08651 32823 06647 09384 46095 50582 23172 53594 08128
48111 74502 84102 70193 85211 05559 64462 29489 54930 38196 44288 10975 66593
34461 28475 64823 37867 83165 27120 19091 45648 56692 34603 48610 45432 66482
13393 60726 02491 41273 72458 70066 06315 58817 48815 20920 96282 92540 91715
36436 78925 90360 01133 05305 48820 46652 13841 46951 94151 16094 33057 27036
57595 91953 09218 61173 81932 61179 31051 18548 07446 23799 62749 56735 18857
52724 89122 79381 83011 94912 98336 **73362** 44065 66430 86021 39494 63952 24737
19070 21798 60943 70277 05392 17176 29317 67523 84674 81846 76694 05132 00056
81271 45263 56082 77857 71342 75778 96091 73637 17872 14684 40901 22495 34301
46549 58537 10507 92279 68925 89235 42019 95611 21290 21960 86403 44181 59813
62977 47713 09960 51870 72113 49999 99837 29780 49951 05973 17328 16096 31859
50244 59455 34690 83026 42522 30825 33446 85035 26193 11881 71010 00313 78387
52886 58753 32083 81420 61717 76691 47303 59825 34904 28755 46873 11595 62863
88235 37875 93751 95778 18577 80532 17122 68066 13001 92787 66111 95909 21642
0199...

The great 20th century mathematician *Alex Craig Aitken* could easily recite the above 1000 decimal places. Once he was asked to begin at the 501st decimal, Aitken instantly began: "7336244065664308....

F. P. Lindemann, using modern abstract algebra, showed in 1882 that it is impossible to construct (with Euclidean tools) a line of length $\sqrt{\pi}$, which in effect showed that the quadrature of a circle is impossible.

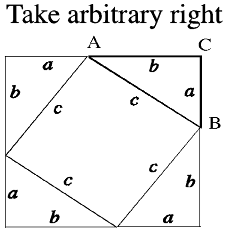
Pythagoras' Theorem. Proposition I. 47

If ABC is any right triangle with sides a, b, c, and side c is the hypotenuse, the side opposite the right angle at vertex C, then: $a^2 + b^2 = c^2$.

Proof.

There are hundreds of different proofs for this great theorem. See *E. S. Loomis* "***Pythagorean Proposition***" for a list of over 350 proofs of this fundamental theorem. We give one of the simplest proofs. Later, we give Euclid's proof.

Take arbitrary right triangle ABC and arrange three more copies as shown in the diagram.



Area of large square = Area of small square + 4 (Area of ΔABC)

$$(a + b)^2 = c^2 + 4\,(ab/2)$$

$$a^2 + b^2 = c^2$$ *Q.E.D.*

Historical note:

When I was at school, all geometry proofs ended with Q.E.D. for the Latin '*Quod erat demonstrandum*' for '*Which was to be proved*'. Many math books, Calculus texts especially, still mark the end of a proof, no longer with Q.E.D., but with some mark such as $\square$.
Many Geometry texts still use Q.E.D.
At times, the reader is unfamiliar with a complicated proof and it is helpful to be told that the proof has indeed ended. Sometimes the proof is so simple (like the very next proof) that it can be helpful to be told, especially at first reading, that this is all there is to the proof.

Converse of Pythagoras' Theorem Proposition #48 *Euclid's Elements* Book I

If $a^2 + b^2 = c^2$ for any triangle with sides a, b, c, then the angle between the two sides a and b is a right angle.

Proof:

Given Δ ABC with sides a, b, c and $a^2 + b^2 = c^2$
construct a right triangle A′B′C′ with two sides of length a and b, perpendicular to each other.

By Pythagoras' theorem the third side of Δ A′B′C′

Is $c = \sqrt{a^2 + b^2}$ so that Δ ABC = Δ A′B′C′ (**SSS**).
So that $\hat{C} = \hat{C}' = 90°$. ***Q.E.D.***

Builders, to frame structures at right angles to each other, use Pythagoras' converse theorem constantly. The most common triangle used is one having sides of 3K, 4K & 5K units of length, where K can be any non-negative number, almost always an integer for convenience.

Another example often used by builders is the 5, 12, 13 triangle.
Since $5^2 + 12^2 = 13^2$ there will be a right angle between the two shorter sides of any triangle with sides 5, 12, 13 metres (cms, feet or any unit of distance measurement).

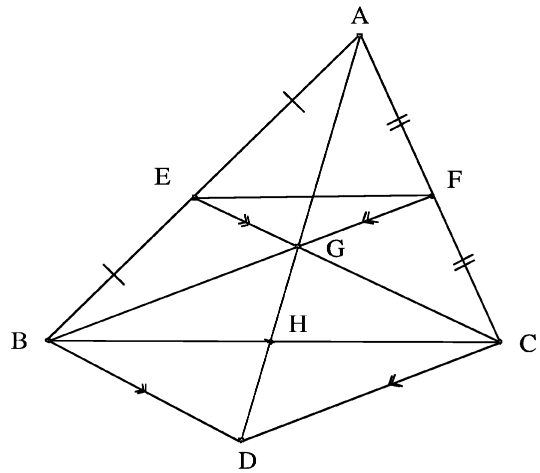
Definition.

Integer triples, $\{a,b,c\}$ with $a^2 + b^2 = c^2$, such as {3, 4, 5} and {5, 12, 13} are called *Pythagorean triples*.

It can be shown that there is an infinity of distinct Pythagorean triples, other than the obvious integer multiples of the above such as {6, 8, 10}, {30, 40, 50}, {50, 120, 130}.
{7, 24, 25}, {8, 15, 17}, {9, 40, 41}, {12, 35, 37}, {20, 21, 29}
are further examples of Pythagorean triples.

Definition.
A *median* of a triangle is the line segment from a vertex to the mid-point of the opposite side.

Theorem. *The medians of any triangle are concurrent.*



Proof:
Take arbitrary triangle ABC. Let BF & CE be any two medians meeting at G say.
We show that the third median passes through G, by extending AG meeting BC at H
and then showing BH = HC. i.e. we show GH is the 3rd median.

Extend AG further to point D such that AG = GD.
We see EG // BD since EG goes through the mid points of sides AB & AD of Δ BAD
FG // CD since FG goes through the mid points of sides AD & AC of Δ DAC.

Therefore BGCD is a parallelogram. The diagonals of any parallelogram bisect,
so that BH = HC. The third median does indeed go through G. **Q.E.D.**

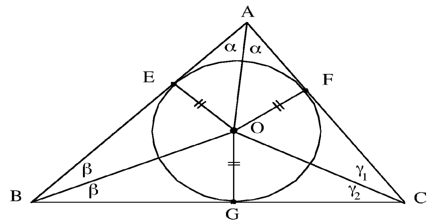
Corollary.
The point of concurrency of the medians of any triangle trisects each median.

Proof HG = HD since the diagonals of parallelogram BGCD bisect each other.
GA = GD by construction
HG = AG/2 = AH/3 so that G is the trisection of median AH.

Since there was nothing special about median AH, the proof would apply equally
well to any median, so that G trisects each of the three medians. **Q.E.D.**

Of interest. It can be shown that the point of concurrency of medians is the centre of gravity for a triangle lamina (of uniform density).

Theorem. *The angle bisectors of any triangle are concurrent.*



Proof. Take an arbitrary triangle A B C.
Bisect any two angles A & B say, with the bisectors meeting at O.
We show that O C bisects angle .

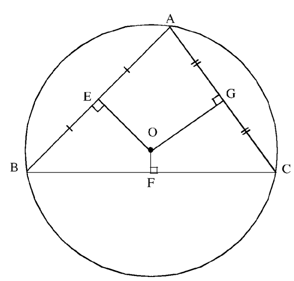
Drop perpendiculars from O to the three sides at points E, F, G as shown.

we see	Δ OBE = Δ OBG	**(SAA)**	∴	OE = OG
	Δ OAE = Δ OFF	**(SAA)**	∴	OE = OF
therefore	Δ OCF = Δ OCG	**(SS90°)**	since	OF = OG
therefore	OF = OG		**Q.E.D.**	

Corollary. The point of concurrency of the angle bisectors of any triangle is the centre of a circle (**the incircle**), which touches the three sides of the triangle.

Proof From the above proof, we see that segments OE = OF = OG and that each is perpendicular to the sides AB, BC, CA respectively. The radius of the incircle is OE and the three triangle sides are tangents to this circle. ***Q.E.D.***

Theorem. *The perpendicular side bisectors of any triangle are concurrent.*



Proof. Take arbitrary triangle ABC.
Take the perpendicular bisectors of any two sides, AB and AC say.
Let them be OE and OG as shown.
Drop a perpendicular from O to side BC at F. We show that OF bisects BC.

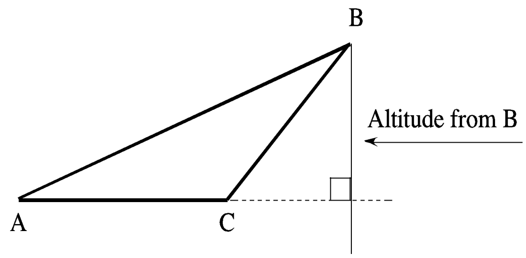
OA = OB	since Δ OEA = Δ OEB	**(SS90°)**	
OA = OC	since Δ OGA = Δ OGC	**(SS90°)**	
therefore	Δ OFB = Δ OFC	**(SS90°)**	
and so	FB = FC		***Q.E.D.***

Corollary:
The point of concurrency of the perpendicular bisectors of any triangle ABC is the centre the unique circle (circumcircle) through the three vertices A, B, C.

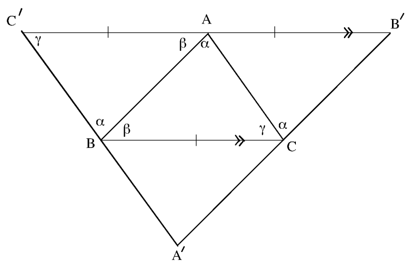
Proof:
From the above proof we see that OA = OB = OC. Therefore the circumcircle with centre O and radius OA goes through the points A, B, C. ***Q.E.D.***

Definition:

An *altitude* of a triangle is a line segment from a vertex perpendicular to the opposite side. An opposite triangle side may have to be extended outside of the triangle for perpendicular intersection.



<u>Theorem</u>. The altitudes of any triangle are concurrent.

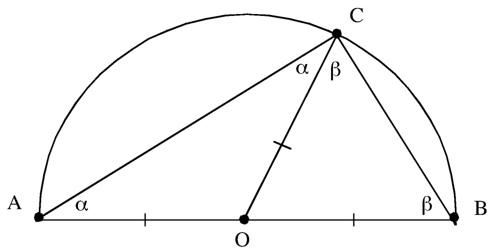


<u>Proof.</u>
Take arbitrary triangle ABC.
Construct triangle A'B'C' as shown with B'C' // CB, A'C' // CA and B'A' // AB.
If α, β, γ are the angles in Δ ABC we see angle ACB' = α since A'B' // BA.
Similarly, all the other angles follow as marked in the diagram.

Δ ABC = Δ ABC' = Δ CB'A therefore AC' = AB'

Since C'B' // BC (by construction) we see that the perpendicular bisector of side C'B' is an altitude of triangle ABC. Similarly, we see that the three perpendicular bisectors of the sides of triangle A'B'C' are the three altitudes of triangle ABC. The perpendicular bisectors of <u>any</u> triangle are concurrent, so it follows that the altitudes of ABC are concurrent. ***Q.E.D.***

Theorem. The diagonal of a circle subtends a right angle at the circumference.



Proof. Take any point C on an arbitrary circle with diameter AB and centre O.
We show angle ACB = $\alpha + \beta = 90°$.

Since	OA = OC	angle OAC = angle OCA = α
and	OC = OB	angle OCB = angle OBC = β

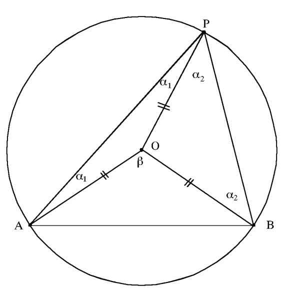
Summing the angles of Δ ABC: $\alpha + \alpha + \beta + \beta = 180°$
so that $\alpha + \beta = 90°$ ***Q.E.D.***

Corollary.
For any right triangle, the median to the hypotenuse is half the length of the hypotenuse.

The converse of this theorem is true and is often used by builders to check for right angles. This check requires no calculations, no square roots.

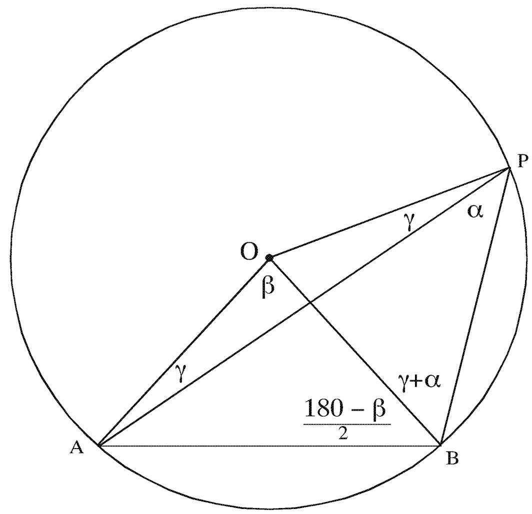
This result came in handy when we considered Hippocrates' trisection of an angle.

Theorem. Any chord of a circle subtends an angle at the centre twice that subtended at any point P on the circumference, P on the same side as the centre.



Proof.
Take arbitrary chord AB and arbitrary point P on the circumference,
the same side as the circle centre O. We show $\beta = 2\,(\alpha_1 + \alpha_2)$.
Note the two base angles of Δ OAB are $(180-\beta)/2$
Summing the angles of Δ APB: $180° = \alpha_1 + \alpha_1 + \alpha_2 + \alpha_2 + 2(180-\beta)/2$
$$\beta = 2(\alpha_1 + \alpha_2).$$ ***Q.E.D.***

If AP (or BP) is below AO the diagram (& proof) need to be modified slightly.



Again, we show $2\alpha = \beta$.
Mark in all equal angles of the two isosceles triangles.
Sum the 3 angles of Δ APB

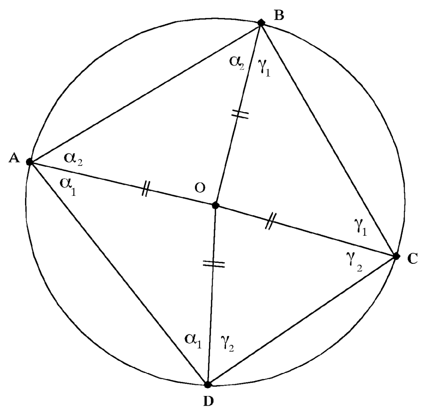
$$180° = \left[\frac{180-\beta}{2} - \gamma\right] + \alpha + \left[(\gamma + \alpha) + \frac{180-\beta}{2}\right]$$

i.e. $2\alpha = \beta$. ***Q.E.D.***

Corollary.
Any chord of a circle subtends at the circumference P, on the same side as the centre a fixed angle. [On the other side of the chord: It will be shown as a corollary to the next theorem, that a different constant angle is subtended by the chord at any point on the circumference, unless the chord is a diameter, in which case the angles subtended at the circumference on both sides of the chord are equal to 90°.]

Proof. The angle subtended at any point P on the circumference, on the same side as the centre, is always half the fixed angle subtended by the chord at the centre of the circle. This follows immediately from the above theorem. ***Q.E.D.***

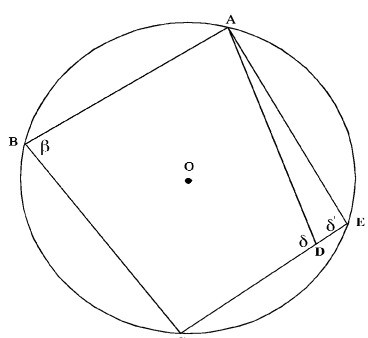
Theorem.
The opposite angles of a quadrilateral sum to 180° if and only if the vertices A, B, C, D lie on a circle. [We say ABCD is a *cyclic quadrilateral*.]



Proof. ($\Rightarrow$)
Suppose points A, B, C, D lie on a circle. We show
angle A + angle C = $(\alpha_1 + \alpha_2) + (\gamma_1 + \gamma_2) = 180°$
Mark in all equal angles of the four isosceles triangles and sum the interior angles of the given quadrilateral ABCD (which sum to 360°).

$$360° = 2\,\alpha_1 + 2\gamma_1 + 2\,\alpha_2 + 2\gamma_2$$
$$180° = (\alpha_1 + \alpha_2) + (\gamma_1 + \gamma_2)$$

Conversely ($\Leftarrow$)
Let ABCD be an arbitrary quadrilateral such that the opposite angles sum to 180°.
We show that the vertices A, B, C, D must lie on some circle.



Construct the unique circle through any three vertices, A, B, C say.
Let E be the point of intersection of this circle with CD (or CD extended).
Since A, B, C, E are on the circle, by the above result we must have:

$$\beta + \delta' = 180$$

We are given: $\beta + \delta = 180°$ $\therefore\ \delta = \delta'$ hence D = E
so that D lies on the circle through the three points A, B, C. **Q.E.D.**

Of interest:

1. We can construct a circle through any three non-collinear points.
   This is the *circumcircle* of the triangle formed by these three points.

2. We can construct a circle through the vertices of any rectangle since all angles are 90° (hence the opposite angles sum to 180°).

3. We can never construct a circle containing the vertices of a parallelogram unless it is a rectangle.

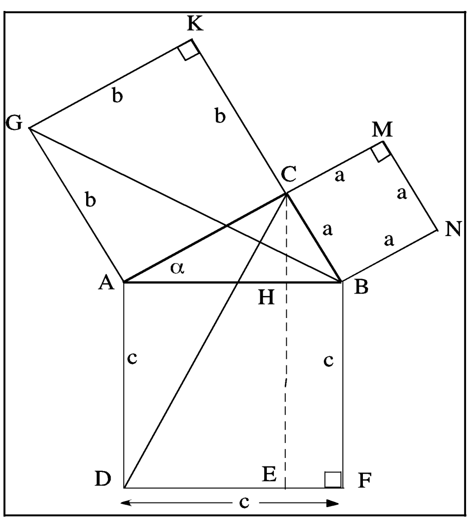
4. Given four random points A, B, C, and D on a plane, it is most unlikely that a circle can be constructed to contain each point. We need that extra condition of having the opposite angles of the quadrilateral ABCD sum to 180°. Perhaps note that if two opposite vertex angles of a quadrilateral sum to 180°, then the remaining two vertex angles also sum to180°.

***Euclid*'s Proof of *Pythagoras*' Theorem.**

Proposition #47 of Euclid's Elements Book I:

If ABC is any right triangle with sides *a, b, c* then $a^2 + b^2 = c^2$.
Where side *c* is the side opposite the right angle at vertex C.

Proof.



Take any right triangle ABC with sides *a, b, c* in the usual notation.
Construct squares adjoining the sides as shown
Let CHE be perpendicular to DF (and perp. to AB).
We show that the area of rectangle ADEH is b^2 .
Similarly we could show area of rectangle HEFB is a^2.

Δ AGB = Δ ACD (**SAS**) [AB=AC= c ,
angle DAC = angle GAB, AG = AC = *b*]

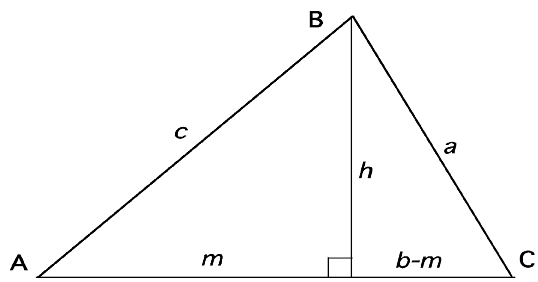
Area Δ GAB = Area Δ GAC = $b^2/2$ (both triangles have same base *b* & same height *b*)
Area Δ ADC = Area Δ ADH (both triangles have same base *c* and same height)

Therefore: 1/2 area of square AGKC = 1/2 area of rectangle ADEH = $b^2 / 2$
Similarly 1/2 area of square BCMN = 1/2 area of rectangle FEHB = $a^2 / 2$

Finally we see the area of square ABFD = $c^2 = a^2 + b^2$. ***Q.E.D.***

***Heron*'s Formula**.

For any triangle ABC with sides a, b, c where semi-perimeter $s = (a+b+c)/2$



Proof.

Area of triangle ABC $= b\,h/2 \quad = \dfrac{b}{2}\sqrt{c^2 - m^2}$

To find m, note $h^2 = c^2 - m^2 = a^2 - (b-m)^2$.

In terms of a,b,c $\quad m = \dfrac{c^2 + b^2 - a^2}{2b}$

Area of triangle ABC $= \dfrac{b}{2}\sqrt{c^2 - \dfrac{(c^2+b^2-a^2)^2}{(2b)^2}}$

This can be factored using (several times): $a^2 - b^2 = (a+b)(a-b)$

Area of triangle ABC
$$= \frac{1}{4}\sqrt{4b^2c^2 - (c^2+b^2-a^2)^2}$$
$$= \frac{1}{4}\sqrt{(2bc + c^2 + b^2 - a^2)(2bc - c^2 - b^2 + a^2)}$$
$$= \frac{1}{4}\sqrt{((c+b)^2 - a^2)(a^2 - (c-b)^2)}$$
$$= \frac{1}{4}\sqrt{(c+b+a)(c+b-a)(a+b-c)(a+c-b)}$$
$$= \sqrt{\frac{(c+b+a)}{2}\frac{(c+b-a)}{2}\frac{(a+b-c)}{2}\frac{(a+c-b)}{2}}$$
$$= \sqrt{s(s-a)(s-b)(s-c)}$$

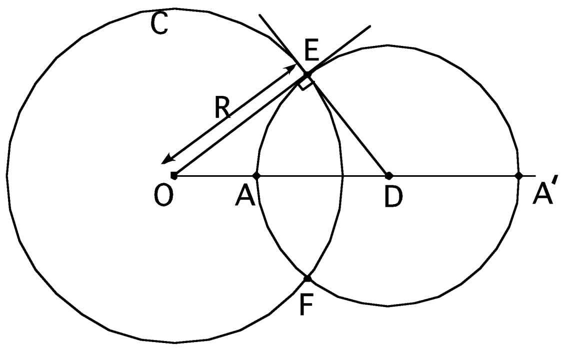
Q.E.D.

Perpendicular Circles construction.

We are given a circle C_1 and two arbitrary points A & B inside this circle, not on a diameter. Our aim is to construct a circle C_2 through points A & B such that $C_1 \perp C_2$ (at the two points of intersection). We begin with a warm-up theorem. This theorem is not needed, however it gives a guide to the proof of the main theorem.

Theorem 1. (Warm up)
Given a circle C with radius R, centre O and a point A inside. Extend OA to any point A′ outside C, let D be the mid point of AA′. The circle centre D and radius DA cuts the given circle at right angles iff $OA.OA' = R^2$.

[A & A′ are called inverse points with respect to C. We will come back to this in Inverse Geometry.]



Proof

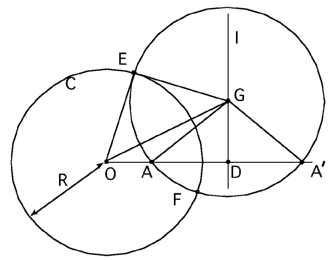
⇒ Suppose that the circles are perpendicular and meet at points E & F. OED is a right triangle, so that by Pythagoras' theorem:

$$R^2 = OD^2 - ED^2 = OD^2 - AD^2 = (OD - AD)(OD + AD) = OA.OA'$$

⇐ Suppose that $OA.OA' = R^2$. It follows that $R^2 = OD^2 - ED^2$ then using the converse of Pythagoras' theorem it follows that triangle OED is a right triangle, so that the two circles are indeed perpendicular.

Q.E.D.

Theorem 2. (The main event)
Given a circle C with radius R, centre O and a point A inside. Extend OA to any point A′ outside C. Let l be the line through D the mid point of AA′, with $l \perp AA'$. Then any circle with a centre G on l, and with radius GA is perpendicular to the given circle C *iff* $OA.OA' = R^2$



Proof

⇒ Suppose that the circles are perpendicular and meet at E and F as shown. We apply Pythagoras' theorem in succession to the right triangles OEG, ODG and ADG.

$$\begin{aligned} R^2 &= OG^2 - GE^2 && \text{Right triangle OEG} \\ &= OD^2 + GD^2 - GE^2 && \text{Right triangle ODG} \\ &= OD^2 + GA^2 - AD^2 - GE^2 && \text{Right triangle ADG} \\ &= OD^2 - AD^2 = (OD - AD)(OD + AD) = OA.OA' && \text{Since } GA = GE \end{aligned}$$

The converse follows immediately using the converse of *Pythagoras*' theorem.

⇐ If $R^2 = OA.OA'$, then working backwards it follows that $R^2 = OG^2 - GE^2$ so that angle OED is a right angle, and so the circles are perpendicular. Angle OFD is also a right angle by symmetry about OG. ***Q.E.D.***

The Construction.
We are given an arbitrary circle *C* radius R and centre O say, with two interior points A & B not on any diameter. We construct the unique circle through points A and B and perpendicular to the given circle *C*.

1. Extend either OA or OB, OA say, to point A′ such that $OA.OA' = R^2$
   (See the construction for inverse points below).
2. Construct line l the perpendicular bisector of segment AA′.
3. Construct line m the perpendicular bisector of segment AB.
4. The required circle is centred on the intersection of l & m
   point G say, having radius GA = GB = GA′.

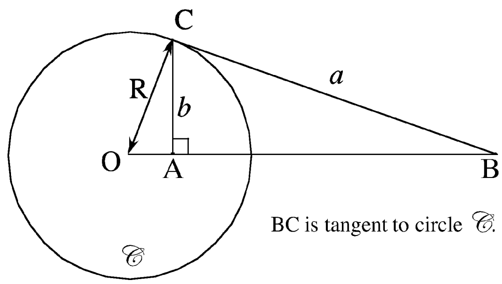
Stated more simply:

1. Extend either OA or OB, OA say, to point A′ such that $OA.OA' = R^2$.
2. The required circle is the unique circle through the points A, B, A′.

It is clear that the required circle is unique since there is only one circle possible through any given three (non collinear) points.

This construction will be used to construct the "*P-lines*" of *Poincare*'s model of hyperbolic geometry in the circle, end of Chapter 6.

Construction of Inverse Points:



It can be shown for the above diagram with the tangent from B to point C, that

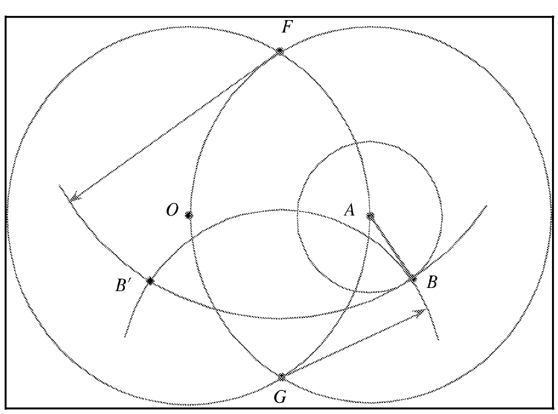
$$OA.OB = R^2. \qquad \text{i.e. A and B are inverse points.}$$

Hint: Use *Pythagoras*' theorem on the three right triangles to get an expression in R, OA and OB (= OA + AB).

Constructions with compass alone.
In the introduction we mentioned the Mohr–Mascheroni theorem which states that any Euclidean construction can be done with compass alone. We give four such constructions of interest.

Theorem.
The modern compass is equivalent to the Greek collapsible compass.
That is, any construction with the modern compass can be achieved with the Euclidean collapsing compass.

Proof. Essentially we prove Proposition 2 of Euclid's Elements Book I, but without the use of a straightedge. We show that for any given point O say, and any given circle of radius AB through the points A and B can be moved to have centre O, using only Euclid's collapsible compass.
We shall use Howard Eves' convenient notation: A(B) shall represent the circle centred on point A and passing through point B, any points AB. A(BC) shall represent the circle centred on point A with radius the length of BC, for any points A, B, C. [It follows A(B) = A(AB)]

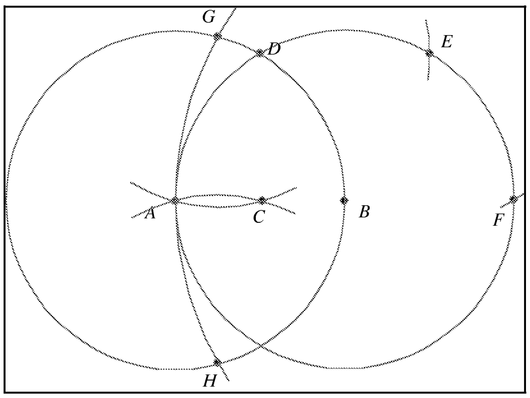


Let O be any point. Let A, B be any other two points.
We show that the circle A(AB) can be transferred from centre A to centre O.

Construct circles O(A) and A(O), meeting at F and G.
Construct circles F (B) and G (B), meeting at B′.

From symmetry we see OB′ = AB.
Q.E.D.

We give the construction, using only a compass to find the mid point between two given points A and B. Perhaps note, having proved the above theorem, we can use the modern compass free of any guilt. Euclid will not be looking down from above, glowering and shaking his head.



Let A and B be two arbitrary points.
Construct circles B(A) and A(B) meeting at D.

Construct circle D(A) meeting B(A) at E.

Construct circle E(D) meeting B(A) at F.

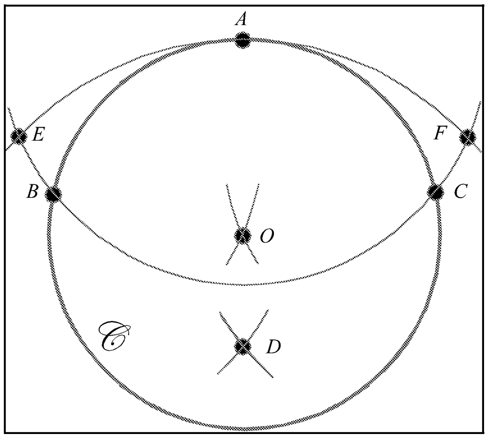
A, B, F will be collinear with AB = BF.

Construct circle F(A) meeting A(B) at G and H.
Construct circles G(A) and H(A) meeting at C.

We show AC = CB.
Isosceles Δ FAG ~ isosceles Δ CAG
AG/AF = AC/AG = AC/AB *i.e.*
AG/AF = ½ = AC/AB ***Q.E.D.***

Three cheers for Mascheroni.

We next find the centre of a given circle. No straightedge!



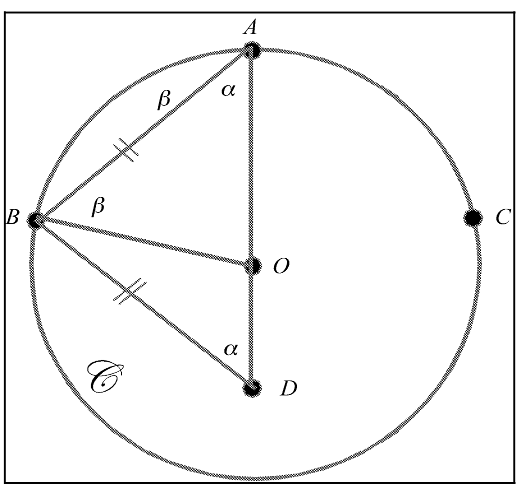
Let 𝒞 be any given circle.
We find the centre, point O.

Choose any point A on the given circle.
Construct any circle centred on A such that it cuts the given circle 𝒞 in two points B and C, circle A(B) say.
Construct circles B(A) and C(A) meeting at D say.
(It is possible that point D is outside circle 𝒞.)

Construct circle D(A) meeting A(B) at points E and F.
Construct circles E(A) and F(A) meeting at point O.
It can be shown that O is the centre of given circle 𝒞.

Proof.
Note that the two (isosceles) triangles: Δ DEA ~ Δ OEA hence have proportional sides.
DA = DE & EO = EA by construction, and angle EAO = angle EAD;
AE/AD = AO/AE hence AB/AD = AO/AB (since AE = AB by construction).

Next, note Δ ABO & Δ ABD are similar triangles, having one angle BAO in common and proportional sides. Hence Δ ABO is isosceles since Δ ABD was isosceles.



That is: AB/AD = AO/AB since Δ DEA ~ Δ OEA
Isosceles Δ DBA ~ Δ OBA since AB/AD = AO/AB
and angle BAO is the included (common) angle between the proportional sides.

So that Δ OBA must be isosceles too, *i.e.* OB = OA.

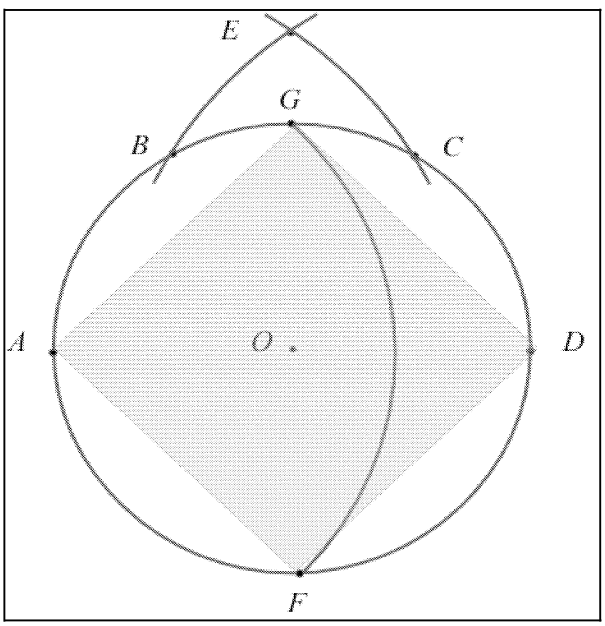
Finally, using symmetry it follows that OC = OA.

Point O is equidistant from 3 points on the circle, it follows that O must be the circle's centre. ***Q.E.D.***

Napoleon's Problem.

Napoleon Bonaparte (1769–1821) was a respected geometer. He was good friends with many of the great French mathematicians of the time (Gaspard Monge, Pierre Laplace, Pierre Mascheroni, Louis Lagrange, Victor Poncelet, ….).
Napoleon is considered to be one of the greatest military commanders of all time (along with my military hero *Alexander the Great* (356-323 BC) from Macedonia).
The Duke of Wellington, of Waterloo fame, is quoted as saying: "I would rather see 50,000 [French] troops arrive on the battlefield than Napoleon himself." Such was Napoleon's electrifying influence on his men.

'*Napoleon's Problem*' was to divide a circle into four equal parts using only the compass.



Equivalently, to inscribe a square in the circle.

Let us assume that the centre point O is given. If not, we could easily find it, using only a compass, by the above construction.

Choose any point A on the given circle with centre O.

Construct circles A(AO), B(AO) and C(AO) meeting the given circle at points B, C, and D respectively.
Construct circles A(C) and D(B) meeting at point E as shown.
Construct circle A(EO) meeting given circle at points F & G.

It can be shown that points A, G, D, F are the points of a square inscribed in the given circle.
Hint: Let the circle diameter AD be 2R. Show that AC = $\sqrt{3}$R, then show

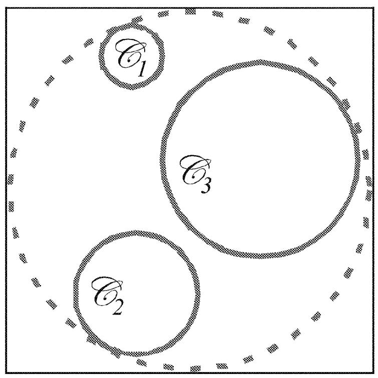
$$AD = GD = \sqrt{2}\ R.$$

Napoleon's Theorem.

Take any triangle, construct equilateral triangles on each of the three sides. The centroids of these three triangles are the three vertices of an equilateral triangle.
(The proof is not that easy.)

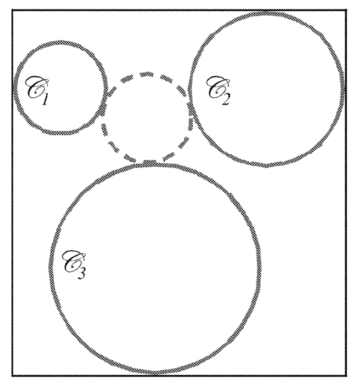
Appollonius' Problem.

Appollonius of Perga: (240 – 190 BC) Known as "The Great Geometer" whose treatise on Conics is one of the greatest scientific works from the ancient world.
Appollonius' circle problem was to construct (straightedge and compass only) a circle touching (tangent to) three given circles. No easy task, though there is an easy proof that it is possible (in most cases), using Inverse Geometry.



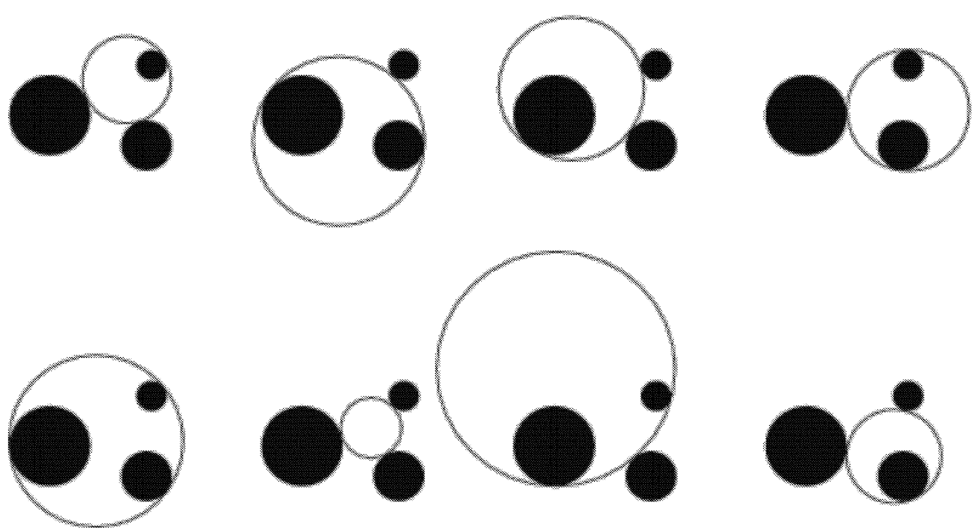
Circles $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$ are given (i.e. they are fixed on the page).

The problem is to construct (using Euclid's tools only) a circle (dotted) that just touches each of the given three circles $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$.



There are usually many possible cases for three given circles:

The following eight diagrams are the curtesy of Wolfram.

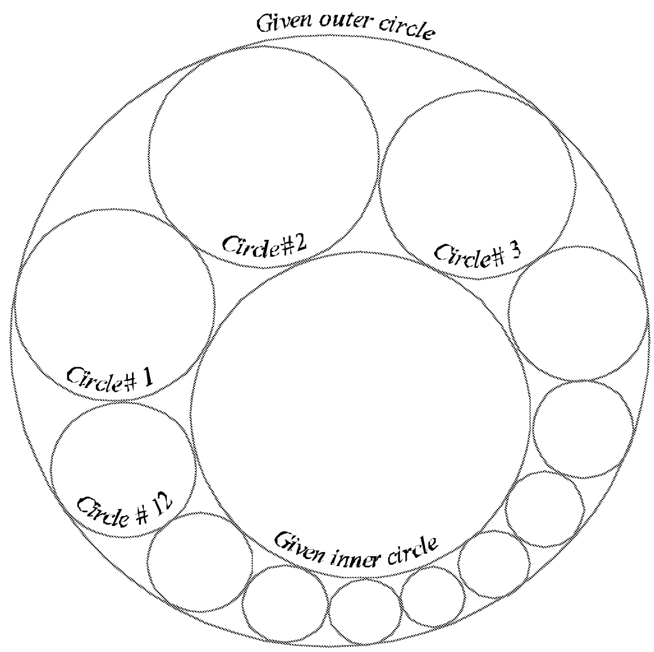


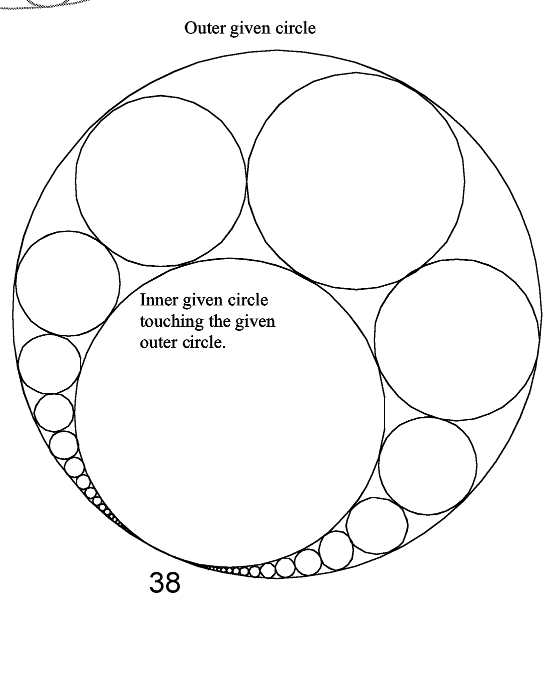
One tangent circle with the three given three circles inside.
One tangent circle with the three given three circles outside.
Three tangent circles with one of the given three circles inside.
Three tangent circles with two of the given three circles inside.

Steiner's Porism.

Take any two circles, one inside the other, either touching or not touching and not necessarily concentric. We can draw a circle touching both circles, circle # 1 below say. Then, using Appollonius' result we can draw a second circle touching the two given circles and circle #1, to get circle # 2, and so on. Will the last circle (#12 in the diagram below) just touch the first circle? Not always, but if it does, then we can begin with any circle, any size, for circle #1, so that there is either no way to make all circles mutually touching or there is an infinity of ways of making all circles mutually touching. I just love this result.

In the case of two given touching circles, we can begin anywhere with circle number one, then using *Appollonius*' result construct as many touching circles as we please heading towards the kissing point. There is no last circle of course, the touching circles just get smaller and smaller, an infinity of them on either side of the first circle constructed, circle #1 in the diagram (of any size, so long as it fits between the two given touching circles (one inside the other).

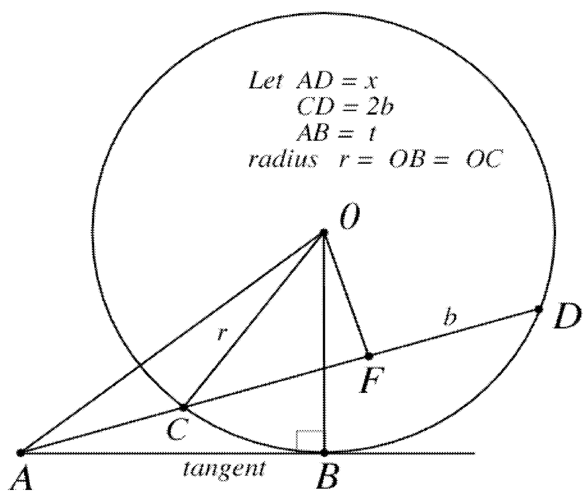




Finally, we give Wallace's 1870 answer to Hampton's £500 challenge to prove that water does in fact lie in a curved surface, using Euclid's III.36, #36 from book III. This example also shows how educated people were almost always familiar with Euclid's work. Even in the 20th century, 'Geometry' was usually simply referred to as 'Euclid'.

Euclid's Proposition III.36.

If a point is taken outside a circle and two straight lines fall from it on the circle, and if one of them cuts the circle and the other touches it, then the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference equals the square on the tangent.



Euclid Book III Postulate # 36
$AC.AD = AB^2$ *i.e.* $t^2 = (x-2b)\,x$

$AB^2 = AC \bullet AD = (x-2b)\,x$
Note: $(x-b)^2 - b^2 = (x-2b)\,x$

Wallace placed three poles of equal height along a canal. The section of canal was approximately 6 miles (10 km) long. The poles were placed so that the tops were all exactly the same height above the adjacent water level of the canal.

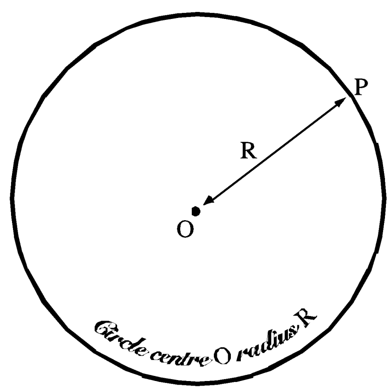
When one looked from the first pole to the final pole, it was fairly clear that the tops of the poles were not all in a straight line. The top of the furthest pole appeared to be slightly down from the line joining the tops of the first two poles.

The experiment was not as conclusive as one might expect. The refraction of light, especially when looking horizontally, complicates the experiment somewhat by bending the light rays to some extent.

Chapter 3

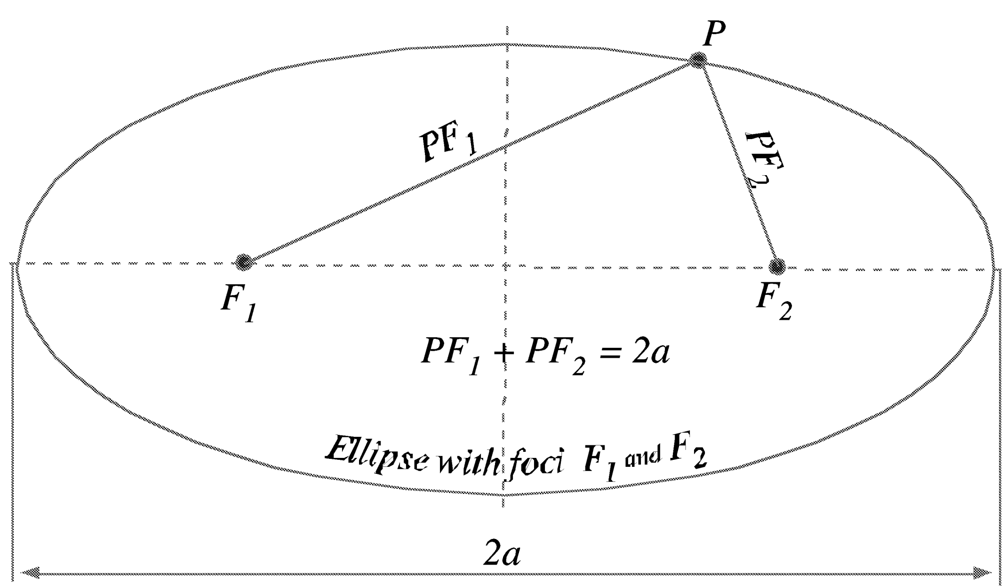
Conic Sections

Definition.
A circle is a set of points which are equidistant from a fixed point O.
Points in this set are said to lie on the circle *centre* O, with *radius* R.



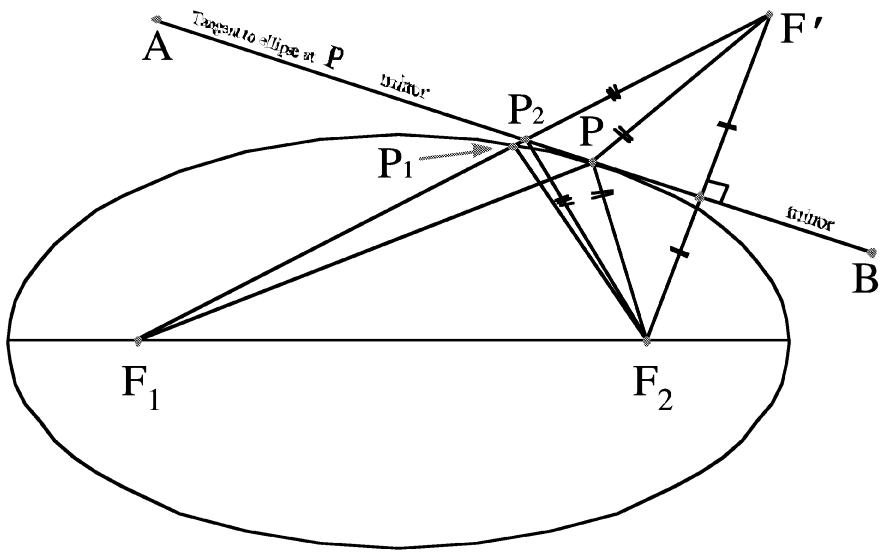
Definition.
An ellipse is the set of points P such that the distance $PF_1 + PF_2$ is a constant, 2a say, for any two points F_1 and F_2. F_1 and F_2 are called *focus points* for the ellipse.

When $F_1 = F_2$ this special case is of course the circle defined above.



Reflection property of the ellipse.

Theorem.
Take any point P on the ellipse with foci F_1 and F_2.
Let APB be the tangent at point P, to the ellipse,
then angle APF_1 equals angle BPF_2.



Proof.
Let F' be the reflection of F_2 in 'mirror' APB, so that $PF_2 = PF'$.
It will suffice to show that F_1PF' is a straight line.

Suppose that F_1PF' is not a straight line. Suppose that the straight line joining F_1 to F' meets the ellipse at P_1, $P_1 \neq P$ and meets the tangent line AB at P_2.
We get a contradiction by showing that the length F_1F' is greater than $F_1P + PF'$ which is impossible since the third side of any triangle ($\Delta F_1PF'$ in this case) cannot exceed the sum of the other two sides.

Using the property of the ellipse we have:

$$F_1P_1 + P_1F_2 = 2a = F_1P + PF_2 \qquad \text{(some constant } a\text{)}$$

$$F_1F' = F_1P_1 + (P_1P_2 + P_2F') > F_1P_1 + P_1F_2 = 2a$$

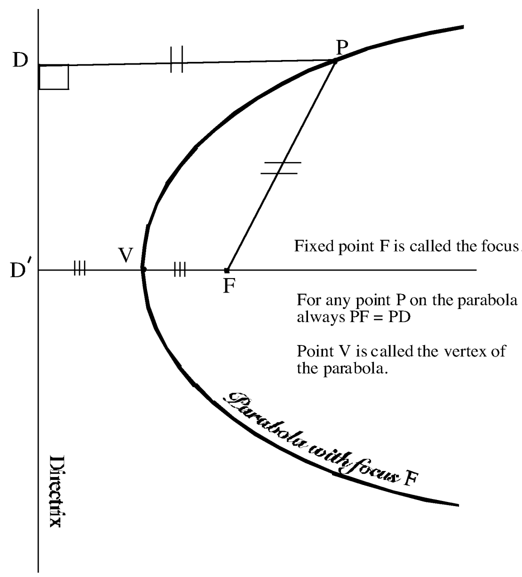
$$\text{note } (P_1P_2 + P_2F') = P_1P_2 + P_2F_2 > P_1F_2$$

$$F_1F' > 2a = F_1P + PF'$$

This is impossible, hence F_1PF' must be a straight line, we must have $P_1 = P$.
Q.E.D.

That is, any ray of light from F_1 reflects from the mirrored ellipse to F_2 and conversely.
There is an elliptical church in Sicily, with a confessional box placed at one focus!

***Definition*.**
A parabola is the set of points P such that the distance PF from P to a fixed point F equals the distance PD the distance from P to a fixed line l. The fixed point F is called the *focus* of the parabola and the fixed line l the *directrix* of the parabola.



The reflection property of the parabola is that any ray of light (say) parallel to the axis reflects from a parabolic mirror to its focus F.

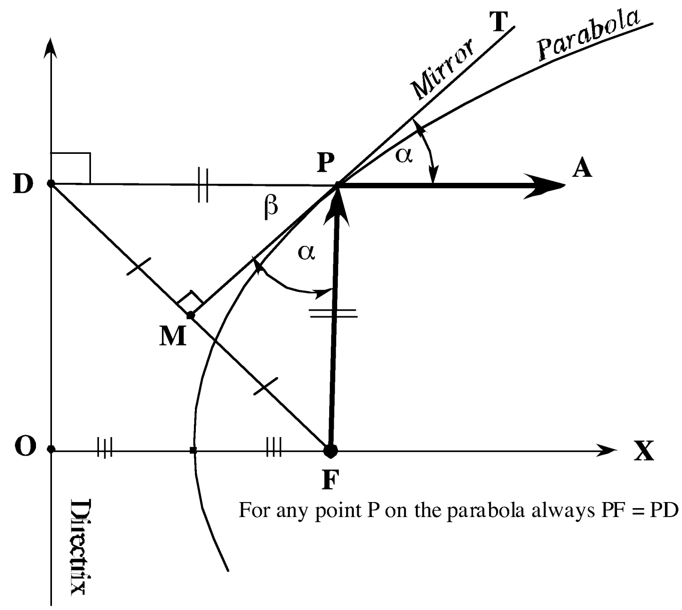
This property is used in all satellite dishes where the satellite signal is concentrated by the parabolic dish to a focus point then transmitted to the television receiver.

This property is also used in parabolic mirror telescopes with the same effect of gathering light from a distant object and concentrating it to a focal point for better vision.

The reverse is used in all auto headlights; when the bulb is placed at the focus of the parabolic headlight mirror, a fairly parallel beam emerges to light up the road ahead. Military searchlights have the same parabolic shape to focus intense carbon arc light sources high into the sky.

Proof of the reflection property of the Parabola.

Theorem.
Any ray of light from the focus F of a parabola reflects parallel to the axis of the parabola.



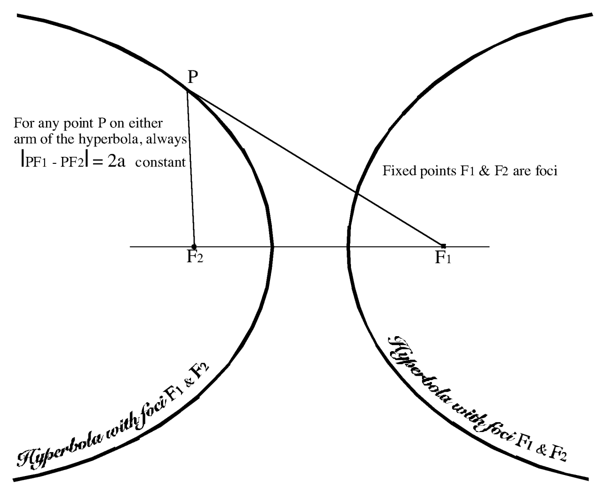
Proof.
Consider a ray of light from focus F reflecting from any point P on the parabola.
In the diagram above, with PD perpendicular to the directrix, we show $\alpha = \beta$,
i.e. we show that DPA is a straight line.

Construct the perpendicular bisector of DF, TM say, M the mid-point of DF.
TM contains all points equidistant from points D and F and so contains point P.
It is readily seen that TPM is the tangent to the parabola at point P.
[Assume that MP cuts the parabola at P′ as well as P, then we would quickly arrive at a contradiction.]

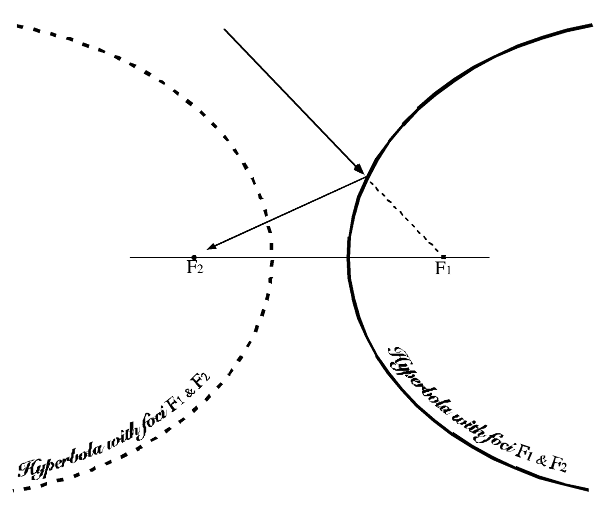
angle TPA = angle MPF = α (say)	(mirror reflection property)
angle DPM = angle FPM	(since Δ DPM = Δ FPM (SSS)).
angle DPM = angle TPA	DPA is a straight line // OFX.

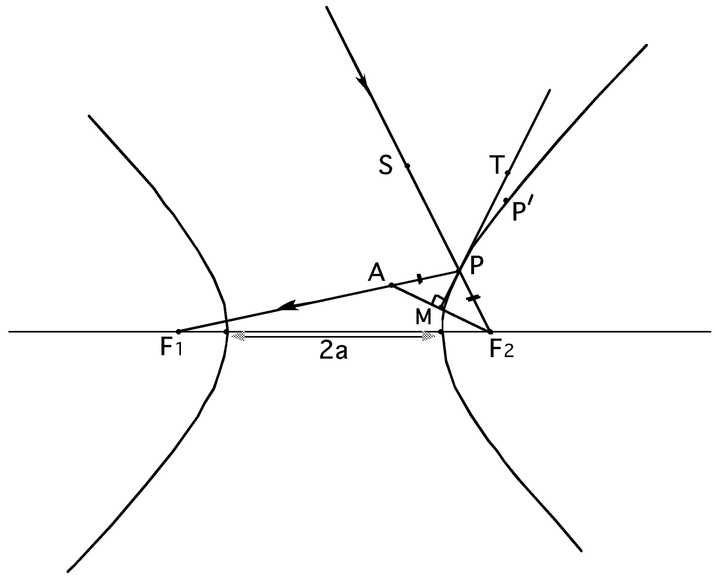
Q.E.D.

Definition.
A hyperbola is the set of points P such that the positive difference $|PF_1 - PF_2|$ between the distances PF_1 and PF_2 is a constant.
F_1 and F_2 are called *focus points* for the hyperbola.



The reflection property of the hyperbola: Any ray of light, coming towards a focal point F_1 say, is reflected from that arm of the hyperbola (with that focal point F_1) towards the other focus. Similarly, any ray of light coming from a focus would be reflected from the hyperbola mirror as though it came from the other focus. See diagram below. This property is used along with the parabolic reflection property in some telescopes.



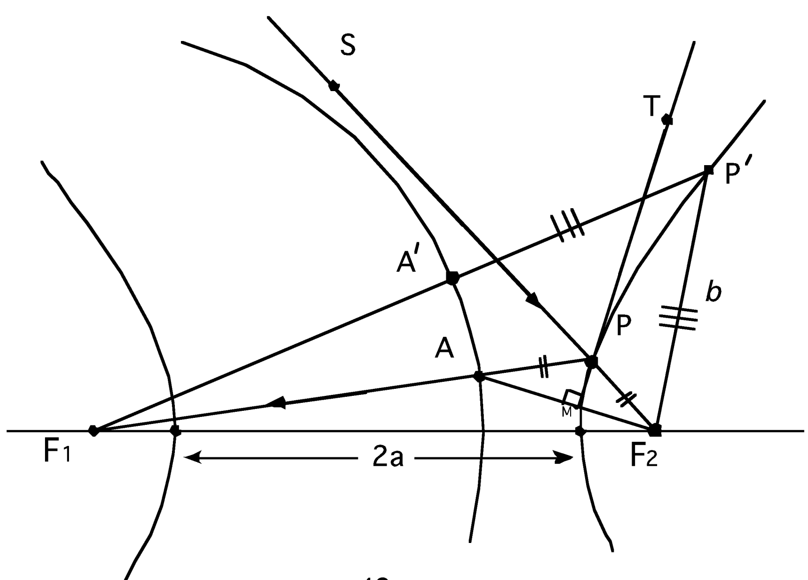


Proof of the reflection property for the hyperbola.

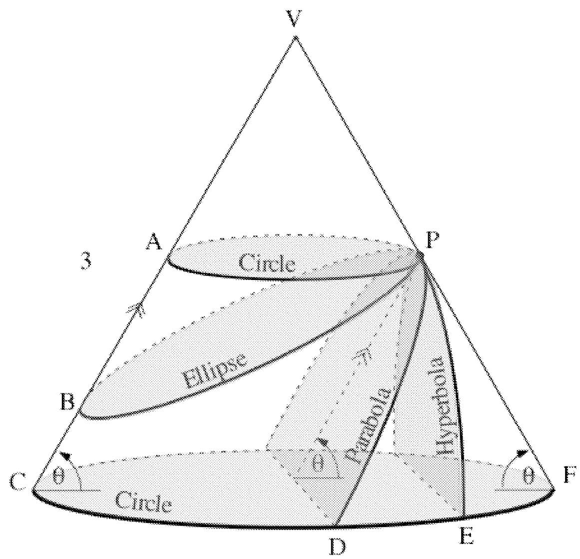
Let P be any point on the right branch of the hyperbola with foci F_1 and F_2 as shown above. Let SP be a ray of light with S, P and F_2 collinear. We show that SP is reflected from the tangent to the hyperbola at P to F_1.

Let PA = PF_2 on the line PF_1 and let M be the mid-point of AF_2.
If TM is the perpendicular bisector of AF_2 then P is on TM, since TM contains all points equidistant from A and from F_2. We show that TM is indeed the tangent at P, which will complete the proof.

Suppose that TM is not a tangent at P, then TM meets the hyperbola at P and at P′ say.
Let $P'F_2 = P'A = b$ say. $P'F_1 = 2a + b$ (property of hyperbola) which contradicts $P'A + AF_1 = b + 2a$, unless $P = P'$.



The conic sections.
The ellipse, parabola and hyperbola are called *conic sections* because they can be formed by the intersection of a plane with the surface of a right cylindrical cone.
See diagram below.



The circle.
The circle is a special ellipse (with zero eccentricity) and is obtained when a horizontal plane cuts the cone.

The Ellipse
Think of a horizontal plane going through point P. This plane could be rotated about an axis coming directly out of the page from point P. When the rotated plane is between PV and PD, the curve of intersection of the plane and cone is an ellipse. The horizontal plane through point A gives the special ellipse, the circle with diameter PA.

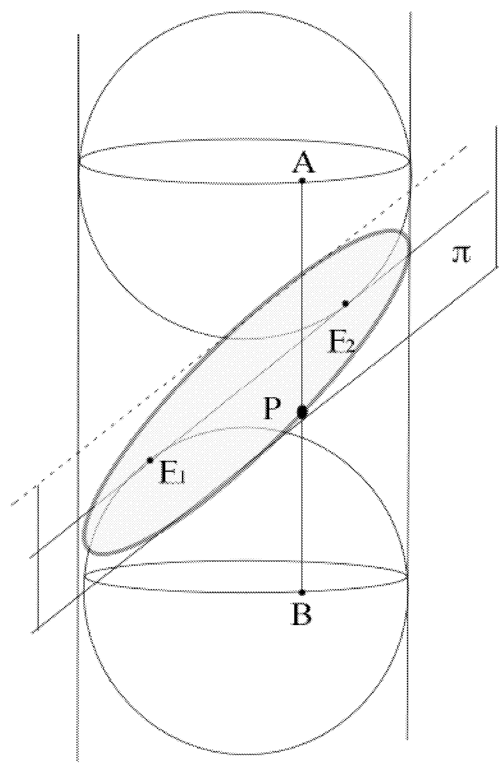
The Parabola
When, and only when, the rotated plane is parallel to VC is a parabola formed.

The Hyperbola.
When the plane through P is rotated further, anywhere between points D and F, one arm of a hyperbola is formed. Most texts show the line PE perpendicular to the horizontal, but any angle obtained between points D and F will yield a hyperbola. If we consider a double cone, vertex to vertex (both with the same vertex angle) then the plane cuts out the other side of the hyperbola on the upper cone.

We now give *Dandelin*'s elegant proofs of the above results (known to the Greeks but with much more difficult proofs).

We begin with a warm-up theorem.

Theorem.
If a plane cuts a (hollow) circular cylinder the points of intersection form an ellipse.
If the plane is perpendicular to the axis of the cylinder then the ellipse is special, a circle.



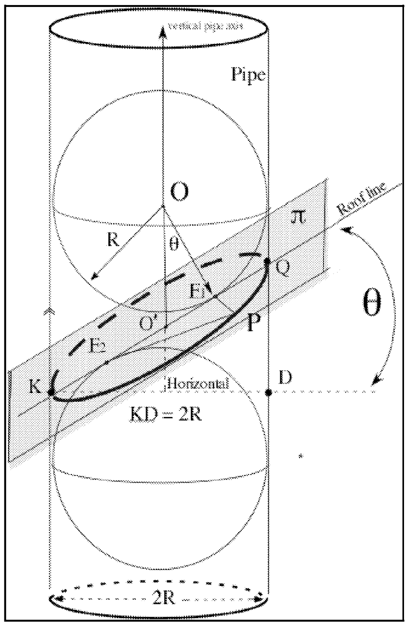
Let π be any plane cutting the given circular cylinder above.
Consider two *Dandelin* spheres with the same radius as the cylinder, inserted into the cylinder so that the plane π is tangent to each sphere as shown.

Let P be any point on the intersection of the plane and the cylinder.
We see $PF_2 = PA$ (since both are tangents to the upper sphere from point P)
We see $PF_1 = PB$ (since both are tangents to the lower sphere from point P)

So that $PF_1 + PF_2 = PA + PB = AB$ constant.
That is, the locus of points P must be an ellipse with focal points F_1 and F_2. **Q.E.D.**

The Plumber's Problem.

The plumber wants to put a vertical pipe of radius R into a sloping roof, sloping at $\theta°$ to the horizontal. How would he draw the appropriate elliptical shape to cut from the roof so that the pipe will fit snugly through?



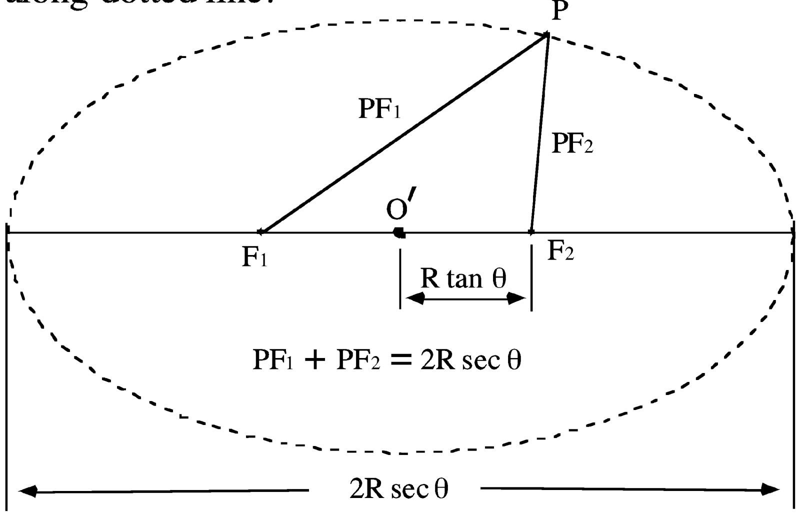
Let the plane π of the roof be sloping θ ° to the horizontal. We see that the ellipse going through points QPK is the required shape to cut. The ellipse will be centred at O′ on the axis of the pipe. We need to find the distance $O'F_1$ (which equals $O'F_2$) and we need to find the distance KQ.

We see:

$O'F_1 = R \tan\theta$ right triangle $OO'F_1$
$KQ = 2R \sec\theta$ right triangle KQD

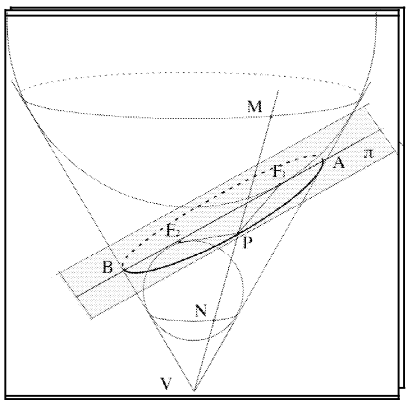
We instruct the plumber to put nails into points F_1 and F_2 a distance $R\tan(\theta)$ from the centre of where the pipe is to be. F_1 and F_2 are on the line of steepest descent through O′. Then tie the ends of a piece of string to each nail so that the length of string between the nails is $2R\sec(\theta)$. Hold a pencil or chalk taut to the string and run the chalk on the roof around the nails. The resulting chalk curve will be the required ellipse, though which the vertical pipe of radius R should fit snugly.

Cut along dotted line!



Theorem.
If a plane cuts a circular cone as shown in the diagram below, the resulting points of intersection are the points of an ellipse.

Proof:

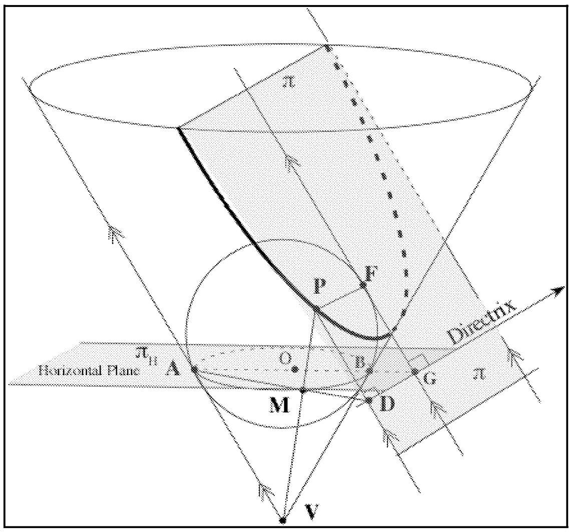


Let π be the plane cutting the given cone as shown. Insert two Dandelin spheres: the upper sphere touching the plane π at F_1 and touching the cone in an upper circle C_U say. The lower sphere touching the plane π at F_2 and the cone in a lower circle C_L say.
Let P be any point of intersection of the plane and cone.
Draw line PV which cuts the lower circle C_L at N and the upper circle C_U at M.

$PF_1 = PM$ since both are tangents to the upper sphere from point P.
$PF_2 = PN$ since both are tangents to the lower sphere from point P.
$PF_1 + PF_2 = PM + PN = MN$ a constant. **Q.E.D.**

***Theorem*.**
If a plane cuts a circular cone as shown in the diagram below, with the plane parallel to a cone generator, the resulting points of intersection are the points of a parabola.

Proof:



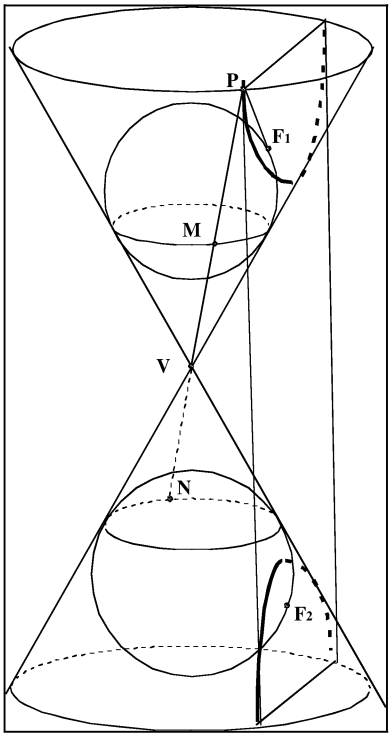
Let π be the plane cutting the given cone with vertex V as shown, so that FG is parallel to VA (i.e. the distance from any point on the line VA to the plane π is always the same). Inflate a Dandelin sphere in the lower part of the cone until it just touches the given plane π at the point F say, and the cone in a horizontal circle C_H. Let C_H have centre O and diagonal points A and B as shown.
We show that the line of intersection of plane π and the horizontal plane π_H (containing the circle C_H) is the directrix of the parabola of intersection of the cone and plane π.
Let P be any point on the intersection in question.
We show that PF equals PD the distance from P to the 'directrix'. Let PV intersect C_H at point M (M on the Dandelin sphere and on the cone and M in the horizontal plane). The line joining D to A must contain point M, since it meets PV at the point where PV goes through the horizontal plane π_H.

We see angle DPM = angle MVA (since AV // PD)
angle AMV = angle PMD (vertical angles)
so that triangles AMV and DMP are similar, isosceles triangles (since VA = VM).
Therefore PM = PD .
PM = PF (tangents from the same point P) so that PF = PD. **Q.E.D.**

Perhaps note:
The four points P, D, V, A lie on the one plane. That lines PD and AV are parallel (essential for this theorem) and that two lines PV and AM will intersect at a point M and that M is indeed on the Dandelin sphere where it touches the cone.

Theorem.
The intersection of a plane and a double cone, as shown in the diagram, is a hyperbola.

Proof:



Inflate two Dandelin spheres inside the cones from the vertex V until they just touch the plane π in question, at points F_1 and F_2 say. The upper sphere touches the upper cone in a horizontal circle C_U, the lower sphere touches the lower cone in a horizontal circle C_L.

Let P be any point on intersection of plane π and the cones.
We show, for the diagram above:
$PF_2 - PF_1 =$ MN a constant.

Let PV running on the surface of the two cones (a so called generator) intersect the upper circle C_U at point M say, and the lower circle C_L at point N say.

$PF_1 = PM$ (two tangents to same (upper) sphere from point P)
$PF_2 = PN$ (two tangents to same (lower) sphere from point P)

$PF_2 = PN = PM + MN = PF_1 + MN$
$PF_2 - PF_1 = MN.$ **Q.E.D.**

Perhaps note:
1. That the two Dandelin spheres will only be the same diameter if the plane π is parallel to the axis of the cones.

2. We could equally well choose arbitrary point P on the lower points of intersection with plane π and the (lower) cone, which would give: $PF_1 - PF_2 = MN$, so that always $| PF_2 - PF_1 | = MN =$ constant.

The study of conic sections suddenly gained major attention following Kepler's laws and Newton's Principia. Newton proved that all planets did indeed move according to Kepler's empirical laws, consequently planet positions could be computed years in advance. Newton showed not only planets but comets and asteroids follow conic paths with the Sun at one focus. If the asteroid is in the solar system permanently, it follows an elliptical path. If the asteroid is just visiting the solar system for a fly past the Sun, then the path is hyperbolic or possibly (though most unlikely) parabolic.

Chapter 4

Vector Geometry in three dimensions

We shall call the ordered triple $v = < x, y, z >$ with x, y, z real numbers, a 3 dimensional vector. '*Ordered triple'* just means that order is important, so that $< 1, 2, 3 > \neq < 3, 2, 1 >$ for example.
Vectors will be always be denoted by a bold letter: $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{v}$, etc. Physicists often use a v with an arrow on top. We shall let V **denote the set of all three dimensional vectors:**

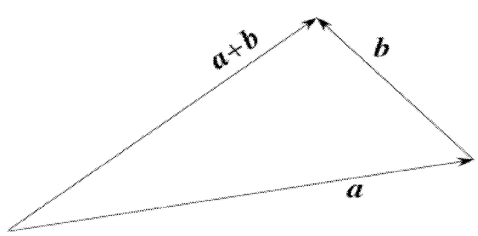
$$V = \{< x, y, z > : x, y, z \text{ any real numbers i.e. } x, y, z \in \mathcal{R}\}$$

$\mathcal{R}$ denotes the set of real numbers i.e. $\mathcal{R} = \{ x : -\infty < x < \infty \}$

The three vectors: $\boldsymbol{i} = < 1, 0, 0 >$, $\boldsymbol{j} = < 0, 1, 0 >$, $\boldsymbol{k} = < 0, 0, 1 >$ are said to form a basis for V, since every vector $< x, y, z >$ of V can be written as a linear combination of these three vectors, simply: $< x, y, z > = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$.
There is an infinity of other sets of three vectors that will form a basis for V.
Vectors ***i, j & k*** have some obvious benefits and later we see when we associate a direction with a vector that these three vectors are mutually perpendicular.
We can picture a vector $< x, y, z >$ as an arrow from point $(0, 0, 0)$ to point (x, y, z).
This vector (arrow) has length and direction. This arrow or vector can be moved to any convenient position in the plane so long as its length is unchanged and it remains parallel to the line from $(0, 0, 0)$ to (x, y, z).

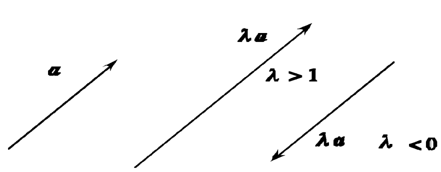
Definitions.

Addition of two vectors: $< x_1, y_1, z_1 > + < x_2, y_2, z_2 > = < x_1 + x_2, y_1 + y_2, z_1 + z_2 >$



Sometimes called 'triangle addition'.

Multiplication by a scalar: $\lambda < x, y, z > = < \lambda x, \lambda y, \lambda z >$ for any $\lambda \in \mathcal{R}$



Definition

Length of vector $\mathbf{a} = < x, y, z >$ is defined: $\sqrt{x^2 + y^2 + z^2}$ denoted by $|\mathbf{a}|$ which is the distance between points $(0, 0, 0)$ and (x, y, z), the length of the arrow.

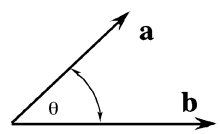
Multiplication of two vectors. There are two useful products of vectors:

Definition 1. **The '*Dot Product*' or so called '*Scalar Product*'.**

For two vectors $\mathbf{a} = < a_1, a_2, a_3 >$ and $\mathbf{b} = < b_1, b_2, b_3 >$

(a) define: $\mathbf{a} \bullet \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$ or

(b) define: $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\theta)$ where θ is the angle between $\mathbf{a}$ *and* $\mathbf{b}$.



Recall that we can slide vectors **a** and **b** around to begin at a common point.
It can be shown that these two definitions (a) and (b) are equivalent.
We see immediately that this product is commutative i.e. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

The Dot Product is particularly useful when we need to find the angle θ between the two vectors **a** and **b**. The following theorem is especially useful.

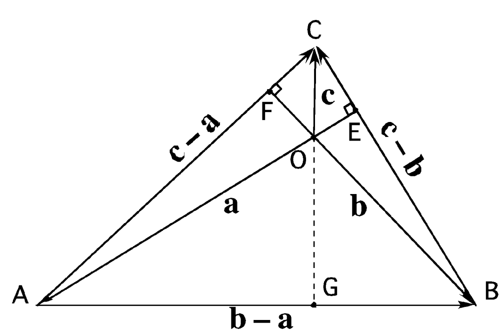
Theorem. $\mathbf{a} \perp \mathbf{b}$ if and only if the scalar product $\mathbf{a} \bullet \mathbf{b} = 0$.

The proof follows immediately from the definition.

We give vector proofs for four theorems proved earlier. These proofs are not improvements, and not necessarily easier, but they help get us acquainted with vectors and vector methods.

Theorem. *The altitudes of any triangle are concurrent.*

Proof: Take any triangle ABC as shown.

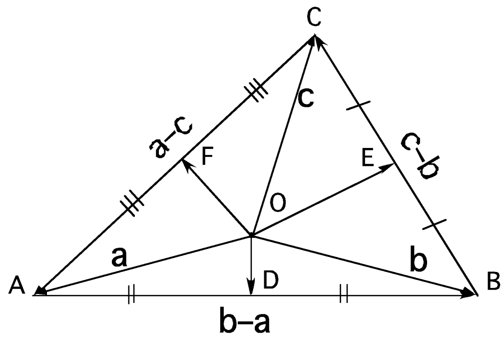


Let AE and BF be altitudes from vertices A and B, meeting at point O say.

Let $\boldsymbol{OA} = \mathbf{a}$, $\boldsymbol{OB} = \mathbf{b}$, $\boldsymbol{OC} = \boldsymbol{c}$,
then $\boldsymbol{AB} = \mathbf{b} - \mathbf{a}$,
$\boldsymbol{BC} = \mathbf{c} - \mathbf{b}$,
$\boldsymbol{AC} = \mathbf{c} - \mathbf{a}$.

Since $\mathbf{a} \perp (\mathbf{c} - \mathbf{b})$ $\mathbf{a} \cdot (\mathbf{c} - \mathbf{b}) = 0$
i.e. $\mathbf{a} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b}$
and $\mathbf{b} \perp (\mathbf{c} - \mathbf{a})$ $\mathbf{b} \cdot (\mathbf{c} - \mathbf{a}) = 0$ *i.e.* $\mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b}$
so that $\mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c}$ $(= \mathbf{a} \cdot \mathbf{b})$
hence $\mathbf{c} \cdot (\mathbf{b} - \mathbf{a}) = 0$, so that $\boldsymbol{OC}$ $(= \mathbf{c})$ is perpendicular to $\boldsymbol{AB}$ $(= \mathbf{b} - \mathbf{a})$.
COG is indeed the third altitude of triangle ABC. QED

Theorem. *The perpendicular bisectors of the sides of any triangle are concurrent.*



Proof.

Take any triangle ABC, let OD & OE be the perpendicular bisectors of sides AB & CB.

Let $\boldsymbol{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$, $\boldsymbol{OC} = \mathbf{c}$,

so that $\boldsymbol{AB} = \mathbf{b} - \mathbf{a}$, $\boldsymbol{BC} = \mathbf{c} - \mathbf{b}$ and $\boldsymbol{CA} = \mathbf{a} - \mathbf{c}$.

Let F be the mid-point of CA, we show that OF is perpendicular to CA.

$\boldsymbol{OD} = (\mathbf{a} + \mathbf{b})/2$ $\quad \boldsymbol{OF} = (\mathbf{a} + \mathbf{c})/2$ $\quad \boldsymbol{OE} = (\mathbf{c} + \mathbf{b})/2$.

$\boldsymbol{OD} \,.\, \boldsymbol{AB} = 0$ i.e. $(\mathbf{a} + \mathbf{b})/2 \,.\, (\mathbf{b} - \mathbf{a}) = (\mathbf{b}^2 - \mathbf{a}^2)/2 = 0$

$\therefore$ $|\mathbf{a}| = a = b = |\mathbf{b}|$

$\boldsymbol{OE} \,.\, \boldsymbol{BC} = 0$ i.e. $(\mathbf{c} + \mathbf{b})/2 \,.\, (\mathbf{c} - \mathbf{b}) = (\mathbf{c}^2 - \mathbf{b}^2)/2 = 0$

$\therefore$ $|\mathbf{c}| = c = b = |\mathbf{b}|$

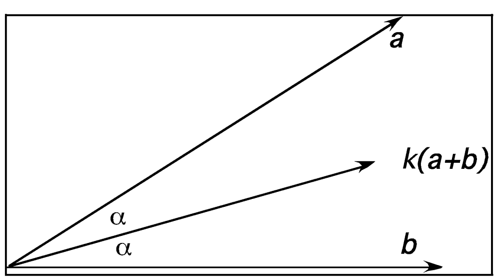
So that $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = a = b = c$.

$\boldsymbol{OF} \,.\, \boldsymbol{CA} = (\mathbf{a} + \mathbf{c})/2 \,.\, (\mathbf{a} - \mathbf{c}) = (\mathbf{a}^2 - \mathbf{c}^2)/2 = (a^2 - c^2)/2 = 0$

hence $\boldsymbol{OF}$ is perpendicular to $\boldsymbol{CA}$. **Q.E.D.**

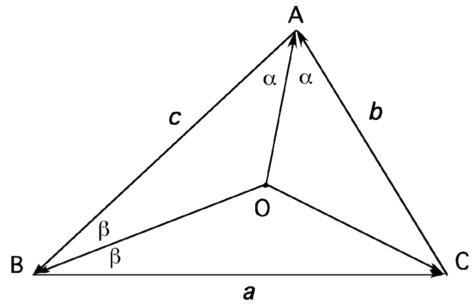
Corollary. *There exists a circle centre O through points A, B, C.* (Since $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$.)

Note: If $\boldsymbol{c}$ is a vector that bisects the angle between two vectors **a** and **b**, then $\mathbf{c} = k\,(\mathbf{a} + \mathbf{b})$ for any non-zero constant (scalar) k.



We use this result to prove the next theorem, that the vertex angle bisectors for any triangle are concurrent.

Theorem.
The vertex angle bisectors of any triangle are concurrent.



Proof.

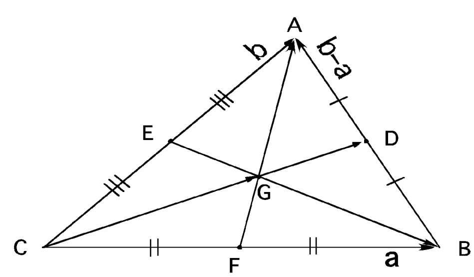
Let ABC be any triangle. Let OA bisect vertex A, let OB bisect vertex B.
Let ***BC* = a**, ***CA* = b**, ***AB* = c** so that **a + b + c = 0**.

OB $= k$ (**c** – **a**) some constant *k* since OB bisects angle ABC
OA $= h$ (**b** – **c**) some constant *h* since OA bisects angle CAB
we show that ***OC*** $= l$(**a** – **b**) *for some constant l.*

OC= ***OA*** + ***AC*** = h(**b** – **c**) – **b** = h(**b** + **a** + **b**) – **b** = h **a** + $(2h–1)$**b** (*using* **a+b+c = 0**)
also
OC = ***OB*** + ***BC*** = k (**c** – **a**) + **a** = k (–**a**–**b**–**a**) + **a** = $(1–2k)$ **a** – k **b** (*using* **a+b+c = 0**)

We must have: $h = (1–2k)$ *and* $(2h–1) = –k$ *so that* $k = h = 1/3$
OC = $(1/3)$(**a** – **b**) *so that OC bisects angle BCA.* ***Q.E.D.***

Theorem.
The medians of any triangle are concurrent.



Proof.
Take any triangle, ABC, let F, D, E be the midpoints of the sides, as shown.
Let ***CA* = a**, ***CB* = b**, then ***BA* = b – a**. Let median BE meet median AE at G.
We show ***CG*** = l ***CD*** some constant l. Note ***CD*** = **a** + ½(**b** – **a**) = (**a** + **b**) /2.

CG = ***CE*** + k ***EB*** = **b**/2 + k (**a**–**b**/2)= k **a** + ½$(1–k)$**b** some constant *k*.
also
CG = ***CF*** + h ***FA*** = **a** /2 + h(**b**–**a**/2) = ½$(1–h)$**a** + h**b** some constant *h*.

Since the coefficients of **a** and **b** must be identical, we have
½(1–h) = k & ½((1–k) = h i.e. h = k = 1/3, so that ***CG*** = $(1/3)$(**a+b**) =*(2/3)****CD*** **Q.E.D.**

Corollary. *Centroid G trisects each median.*
We see |***CG***| = (2/3) |***CD***|, so that G trisects median CD.
Similarly we could show that G trisects medians AF and BE.

Definition 2. '***Vector Product***' or '***Cross Product***' for two 3 D vectors **a** & **b**

a x b = | **a** | | **b** | $sin(\theta)$ ***c*** where θ is the angle between **a** *and* **b**
and **c** is the unit vector which is perpendicular to both **a** *&* **b**
and in a direction that a screw would progress if turned through
an angle $\theta < 180°$ from **a** *to* **b** into the plane containing vectors **a** *and* **b**.

To get the direction of axb physicists often use the ***Right-Hand-Rule***:
Point the index finger (of the right hand) along vector **a,** the middle finger along vector **b,** then the thumb, at right angles to both fingers points in the direction of **a x b.**

We see immediately that this product is not commutative i.e. **a x b** $\neq$ **b x a**

An alternative (equivalent) definition of the vector product of **a** and **b** is:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \boldsymbol{i}(a_2b_3 - b_2a_3) - \boldsymbol{j}(a_1b_3 - b_1a_3) + \boldsymbol{k}(a_1b_2 - b_1a_2)$$

The vector product is particularly useful for finding a vector that is perpendicular to two given vectors **a** and **b**. The vector product can also be used to find the angle θ between two vectors. However, it is usually easier to find angle θ between two vectors using the dot product.

Perhaps note that there are two unit vectors perpendicular to both **a** *&* **b**:
c and $-c$. Note |**a x b**| = | **a** | | **b** | | $sin(\theta)$ | *since **c** has length* | **c** | = *1*

Examples:

1. $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$ $\quad \boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$ $\quad \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$ *all easily seen using the definition.*

---

2. *Using the definition find the vector product of* $\boldsymbol{a} = <1, 2, 3>$ *and* $\boldsymbol{b} = <2, 0, 1>$
$\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\, sin(\theta)\, \boldsymbol{c}$ where $|\boldsymbol{c}| = 1$ *and* $\boldsymbol{c} \perp \boldsymbol{a}$ *and* $\boldsymbol{c} \perp \boldsymbol{b}$.

| ***a*** | and | ***b*** | are easy to find, we need to find unit vector **c** and the angle θ.
The dot product will give the angle θ :
$\mathbf{a} . \mathbf{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\, cos(\theta)$ $\qquad cos(\theta) = 5/\sqrt{70}$
therefore $sin(\theta) = (\sqrt{45})/(\sqrt{70}) = (\sqrt{45})/(|\boldsymbol{a}|\,|\boldsymbol{b}|)$

So that: $\boldsymbol{a} \times \boldsymbol{b} = \sqrt{(45)}\ \boldsymbol{c}$

To find vector ***c****:* *let* $\boldsymbol{c} = < c_1, c_2, c_3 >$
$\boldsymbol{c} \perp \boldsymbol{a}$ *gives* $\qquad c_1 + 2c_2 + 3c_3 = 0$
$\boldsymbol{c} \perp \boldsymbol{b}$ *gives* $\qquad 2c_1 + 0c_2 + 1c_3 = 0$
(2 equations 3 unknowns, we expect (and get) an infinity of solutions)
solving: $\boldsymbol{c} = t < 1/2, 5/4, -1>$ *for any constant* $t \neq 0$.
Since $|\boldsymbol{c}| = 1$ *we see* $t = \pm 4/(\sqrt{45})$ *i.e.* $\boldsymbol{c} = \pm 1/\sqrt{45} < 2, 5, -4 >$
Finally, using the right hand (screw) rule we see that the direction of ***c*** *is such that*

$\boldsymbol{c} = +1/\sqrt{45} < 2, 5, -4 >$ *so that* $\boldsymbol{a} \times \boldsymbol{b} = < 2, 5, -4 >$.

Alternatively, using the theorem above to find the vector product we have (much easier):

$$\boldsymbol{a}\text{x}\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 3 \\ 2 & 0 & 1 \end{vmatrix} = (2\text{x}1 - 3\text{x}0)\boldsymbol{i} - (1\text{x}1 - 3\text{x}2)\boldsymbol{j} + (1\text{x}0 - 2\text{x}2)\boldsymbol{k}$$

$$= 2\boldsymbol{i} + 5\boldsymbol{j} - 4\boldsymbol{k} \qquad = <2, 5, -4>.$$

---

3. $<1, 2, 3> \mathbf{x} <1, 1, 0> = <-3, 3, -1> \quad = -<3, -3, 1>$

This can be shown using the definition as in the above example # 2, or much easier (and more reliable) is to use the theorem above and expand the 3x3 determinant.

Maybe note: $<1, 2, 3> \perp <-3, 3, -1>$ *as it must, since* $<1, 2, 3> \bullet <-3, 3, -1> = 0$

and $<1, 1, 0> \perp <-3, 3, -1>$ *as it must, since* $<1, 1, 0> \bullet <-3, 3, -1> = 0$

---

4. $\mathbf{a} \text{ x } \mathbf{a} = \mathbf{0}$ *for any vector* **a** since the $sin(0) = 0$.

$\mathbf{a} \text{ x } \mathbf{b} = \mathbf{0}$ *for any parallel vectors* **a** *and* **b** since the $sin(0°) = sin(180°) = 0$

---

5. $|\,\mathbf{a} \text{ x } \mathbf{b}\,| = |\,\mathbf{a}\,|\,|\,\mathbf{b}\,|$ *iff vectors* ***a*** *and* ***b*** *are perpendicular.*

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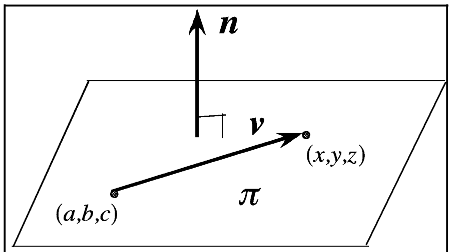
6. $\mathbf{a} \text{ x } \mathbf{b} \,.\, \mathbf{c}$ is called a *triple scalar product*.

Brackets are not needed here as the vector product must be done before the dot product. The final result is always a scalar.
It is not difficult to follow the proof that the absolute value of the triple scalar product is the volume of a parallelepiped with sides (edges) **a**, **b** and **c**.

Equation of any plane with a *normal* $\boldsymbol{n} = < n_1, n_2, n_3 >$ *is:* $n_1x + n_2y + n_3z = d$ where d is some real number. If the plane contains the origin then, $d = 0$.

Let π be the plane containing point (a, b, c) and with normal $\boldsymbol{n}$. Then for any point (x, y, z) in π, the vector $< x - a,\ y - b,\ z - c >$ is always perpendicular to $\boldsymbol{n}$.



No matter where point (x,y,z) is on the plane always $\boldsymbol{n}.\boldsymbol{v} = 0$.

That is $\boldsymbol{n}. < x\text{-}a, y\text{-}b, z\text{-}c > = 0$ or

$$n_1x + n_2y + n_3z = d$$

where $d = n_1a + n_2b + n_3c$ is some constant.

Example 1.
Find the equation of the plane through $(1, 2, 3)$ with normal $\boldsymbol{n} = < 2, 1, 4 >$.

The equation of the plane must be: $2x + 1y + 4z = d$ for some constant d.
Since the point $(1, 2, 3)$ satisfies this equation we have $d = 16$, so that the required equation of the plane is: $2x + 1y + 4z = 16$.
More formally we might write: $\{(x, y, z): 2x + y + 4z = 16\}$
the set of all points (x, y, z) that satisfy that equation.

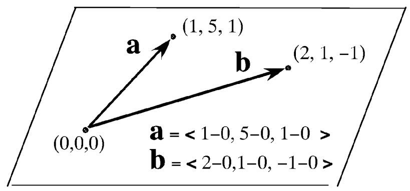
Example 2.
$3x + 4y - z = 2$ is the equation of a plane with a normal $\boldsymbol{n} = < 3, 4, -2 >$.
We say "*a* normal" rather than "*the* normal" since $< -3, -4, 2 >$ is also a normal to this plane, in fact $\lambda < 3, 4, -2 >$ is a normal to the plane for any $\lambda \neq 0$. Every plane has an infinity of normals, though only two *unit* normals exist for any given plane.

$3x + 4y - z = 6$ is the equation of a plane parallel to $3x + 4y - z = 2$. Since both planes have a parallel normal. $-3x - 4y + z = 7$ is another parallel plane since the equation could equally well be written: $3x + 4y - z = -7$ etc.

Examples 3.
$x = 5$ is the equation of the plane with a normal $< 1, 0, 0 >$
$x = z$ is the equation of the plane with a normal $< 1, 0, -1 >$
$y = 2z$ is the equation of the plane with a normal $< 0, 1, -2 >$ etc.

$y = 0$ or $0x + 1y + 0z = 0$ is the equation of the x-z plane containing the *x-axis* & *z-axis*.
$y = 1$ or $0x + 1y + 0z = 1$ is the equation of the plane parallel to plane $= 0$, one unit above.

Example 4
Find the equation of the plane containing points: $(1, 5, 1)$, $(0, 0, 0)$ and $(2, 1, -1)$.



From the diagram, we see that $< 1, 5, 1 >$ and $< 2, 1, -1 >$ are vectors in the plane so that $< 1, 5, 1 > \mathbf{x} < 2, 1, -1 > \; = \; < -6, \; 3, -9 >$ is a normal to the required plane. We see that $< 2, -1, 3 >$ is also a normal, hence the required plane is $2x - y + 3z = d$. To find d, we substitute any of the three points given in the plane. Since we can substitute any point, the obvious choice is $(0, 0, 0)$ which gives $d = 0$.

Example 5.
Show that the following 4 points are coplanar: $(2, 0, 0)$, $(1, 1, 0)$, $(2, -1, \; 1)$ & $(35, -35, 2)$.

We take any three points and show (as in example 4 above) that the equation of the plane containing these three points is $x + y + z = 2$. We then substitute the coordinates of the fourth point into this equation, to see that this point is also in the plane. Hence all four points are indeed coplanar.

To make life easier, we would normally choose the three points (above) to find the plane, those points having the smallest coordinate values. We especially love points with zero coordinates.

Example 6.
Write down 3 points (any three) lying in the plane: $x + y - 2z = 3$.
We see $(3, 0, 0)$, $(0, 3, 0)$ and $(3, 2, 1)$ are three easy points.

Example 7.
Write down 2 points (any two) lying in the two planes: $x + y - 2z = 3$ & $x + y = 1$.

We must solve the *two* simultaneous equations: $x + y - 2z = 3$ & $x + y = 1$
in three variables (so we expect a possible infinity of solutions)

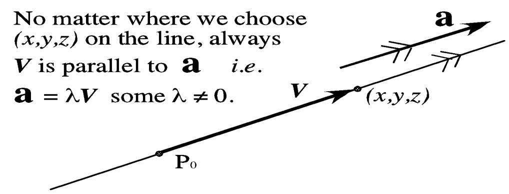
Solving, we easily see $(1, 0, -1)$ and $(0, 1, -1)$ are indeed two of an infinity of possible solutions.

Example 8.
Find a point lying in the three planes:
$x + y + z = 3, \qquad x + y = 2 \qquad$ *and* $\; 2x + y + 2z = 5$.

We solve the *three* simultaneous equations in three variables.
We expect at most one solution. Solving we get $x = 1, \; y = 1, \; z = 1$,
hence the only point lying on the three planes is the point $(1, 1, 1)$.

Equation(s) of a line parallel to a vector **a** $= < a_1, a_2, a_3 >$ and containing the point $P_0 = (x_0, y_0, z_0)$ is $\dfrac{x-x_0}{a_1} = \dfrac{y-y_0}{a_2} = \dfrac{z-z_0}{a_3}$ this is the 'standard' form.

To see this consider the following diagram:



We see that if (x, y, z) is any point on the line, then the vector $< x - x_0, y - y_0, z - z_0 >$ is always parallel to vector **a**. That is $< x- x_0, y- y_0, z- z_0 > = \lambda \mathbf{a}$ some constant λ.

$x - x_0 = \lambda a_1$

$y - y_0 = \lambda a_2$

$z - z_0 = \lambda a_3$ eliminating λ gives: $\dfrac{x-x_0}{a_1} = \dfrac{y-y_0}{a_2} = \dfrac{z-z_0}{a_3}$

Note that the literature refers to the 'equation of a line', but always two equations (of a plane) are necessary to define a line. The standard form given above is really the intersection of two planes:

Plane: $\dfrac{x-x_0}{a_1} = \dfrac{y-y_0}{a_2}$ *i.e.* $a_2 x - a_1 y = d_1$. *With* $d_1 = a_2 x_0 - a_1 y_0$

and

Plane: $\dfrac{y-y_0}{a_2} = \dfrac{z-z_0}{a_3}$ *i.e.* $a_3 y - a_2 z = d_2$ *With* $d_2 = a_3 y_0 - a_2 z_0$

Another form of the 'equation of a line' could be the two equations of a plane.

Any point (x, y, z) satisfying these two equations would lie on the line of intersection of these two planes. However, the standard form is usually preferred since we can immediately read off the direction of the line and one point on the line.

Example.

$x = y/2 = z$ or $\frac{x-0}{1} = \frac{y-0}{2} = \frac{z-0}{1}$ is the standard 'equation' of a line.

These represent planes $2x = y$ & $x = z$ or planes $2x = y$ & $y = 2z$ or .....

We see that the line is parallel to the vector $< 1, 2, 1 >$ and contains point $(0, 0, 0)$.
We easily see that the following points for example, lie on this line:
$(2, 4, 2)$, $(20, 40, 20)$, $(e, 2e, e)$, $(0.4, 0.8, 0.4)$ etc.

Example $x - 1 = 2y = z + 3$ or $\frac{x-1}{1} = \frac{y-0}{1/2} = \frac{z-(-3)}{1}$

is the standard equation of a line parallel to $< 1, 1/2, 1 >$,
or parallel to $< 2, 1, 2 >$ a vector without fractions.
We immediately see this line contains the point $(1, 0, -3)$.

Perhaps note if the line is parallel to $\boldsymbol{a} = < a_1, a_2, a_3 >$ and one or two of the components are zero, then the equation of the line cannot be put into the standard form.
We use an alternative form (the intersection of two planes).

Example

Suppose $a_3 = 0$, the equation of the line parallel to $\boldsymbol{a} = < a_1, a_2, a_3 >$
through $P_0 = (x_0, y_0, z_0)$

is $\frac{x - x_0}{a_1} = \frac{y - y_0}{a_2}$ and $z = z_0$ the intersection of two planes.

Example

Suppose $a_2 = 0$ & $a_3 = 0$, *the 'equation' of the line // to* $\boldsymbol{a} = < a_1, a_2, a_3 >$ and
through $P_0 = (x_0, y_0, z_0)$ *is* $y = y_0$ *and* $z = z_0$
again, the intersection of planes $y = y_0$ & $z = z_0$.

Example

" $y = 2$ *and* $z = 7$ " is the 'equation' of the line parallel to vector $< a, 0, 0 >$
any $a \neq 0$, and contains points $(x, 2, 7)$ for any x in $\mathcal{R}$.
This is the line of intersection of the two planes $y = 2$ & $z = 7$.

Example

The equation of the *y–axis* is: $z = 0$ *and* $x = 0$
since these two planes intersect along the *y–axis*. $< 0, \pm 1, 0 > = \pm \boldsymbol{j}$ are the two unit vectors parallel to this line. The line, the *y-axis* contains only points of the form $(0, y, 0)$ for values of y in $\mathcal{R}$.

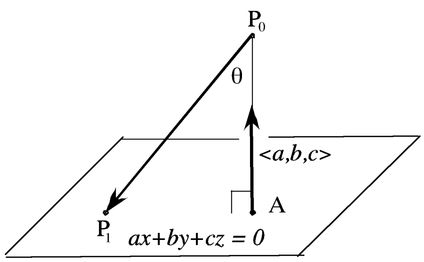
Example

The equation of the <u>*z*–axis</u> is: $x = 0$ *and* $y = 0$.
since these two planes intersect along the *z-axis*. $< 0, 0, \pm 1 > = \pm \boldsymbol{k}$ are the two unit vectors // to this line. This line, the *z-axis*, contains only points of the form $(0, 0, z)$ for values of z in $\mathcal{R}$.

Distances: points to lines and points to planes.

1. **Distance from point** $P_0 = (x_0, y_0, z_0)$ to plane $ax + by + cz = d$.

Take any point P_1 in the plane, $P_1 = (x_1, y_1, z_1)$ say.



From the diagram, we see the required distance D is

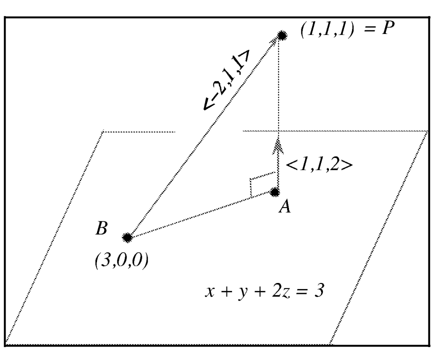
D = distance from P_0 to point A. (Of course if we knew A it would be very easy.)

$= |\mathbf{P_0P_1}| \cos\theta$ (Using right triangle P_0AP_1)

$$= \left| \mathbf{P_0P_1} \cdot \frac{<a,b,c>}{|<a,b,c>|} \right|$$

where $\frac{<a,b,c>}{|<a,b,c>|}$ *is a unit vector normal to the plane.*

Example.
Find the distance from point P = (1,1,1) to the plane $x + y + 2z = 3$.



Let A be the closest point in the given plane to the point P = (1,1,1).

Choose any point B on the plane, B = (3, 0, 0) say. Then the vector **BP** = **<** –2, 1, 1 **>**

Distance PA = | **<** –2, 1, 1 **>** . **<** 1, 1, 2 **>** /√6 |

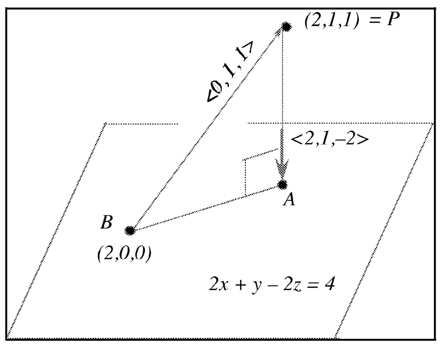
= 1/√6.

Example.
Find the distance from point P = (1, 0, –2) to the plane $z = 0$.
We see immediately that the distance is | –2 | = 2.

Example.
Find the distance from point P = (1, 0, –2) to the plane $y = 0$.
We see immediately that point P is on the plane $y = 0$. The distance is zero.

Example.
Find the point on the plane $2x + y - 2z = 4$ closest to the point P = (2, 1, 1).

Solution:



Let point A, as shown in the diagram, be the closest point in the given plane, to point P = (2, 1, 1). Choose any point B on the plane, B = (2, 0, 0) say, so that vector **BP** = < 0, 1, 1 >

Distance PA = | < 0, 1, 1 > . < 2, 1, –2 > / √9 |
= 1/3

< 2, 1, –2 > /3 is the unit vector in the direction PA or AP; we have drawn it as PA, though it could be AP.

So that vector **PA** is either is one of two points:
< 2, 1, 1 > + 1/3 of unit vector < 2, 1, –2 >/3
or < 2, 1, 1 > – 1/3 of unit vector < 2, 1, –2 > /3
hence point A is either: (2+2/9, 1+1/9, 1–2/9) or (2–2/9, 1–1/9, 1+2/9)

We see that the first point lies on the given plane, so that this point (20/9, 10/9, 7/9) is the required point.

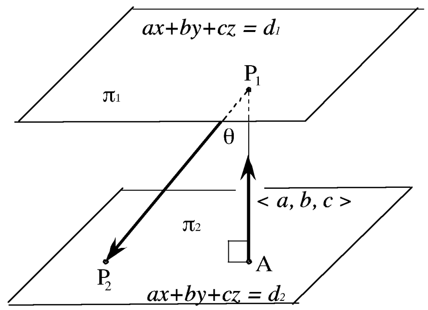
Notes on the above two diagrams.
When drawing just the plane $2x + y - 2z = 4$ (immediately above), it is customary to draw a normal < 2, 1, –2 > going upwards from the plane. However, once another point is drawn, such as (2,1,1) it is possible to see whether the normal < 2, 1, –2 > is directed 'up' or directed 'down'.

In the course of finding point A above, we saw that if we drew the point (2, 1, 1) above the plane, then the vector < 2, 1, –2 > is properly directed downwards. So, I went back and changed the direction of < 2, 1, –2 > in the diagram, with the arrowhead pointing down.

When we drew just the plane $x + y + 2z = 3$ (previous page), we drew the normal < 1, 1, 2 > going upwards from the plane, as is customary. However, once another point is drawn, such as (1, 1, 1), it is possible to see whether the normal < 1, 1, 2 > is directed 'up' or directed 'down'. Since it does not matter for the solution to this example, we usually do not bother, and leave the normal < 1, 1, 2 > pointing up. Strictly speaking, if we do not bother to check the correct direction, we should write in the diagram that the normal vector shown is either < 1, 1, 2 > or < –1, –1, –2 >. If we were to check it out, it turns out (by luck) that the diagram is correct in this case.

2 Distance between two (parallel) planes $\pi_1 : \; ax + by + cz = d_1$

& $\pi_2: \ ax + by + cz = d_2$



Take any point in π_1, $P_1 = (x_1, y_1, z_1)$ say,
and any point in π_2, $P_2 = (x_2, y_2, z_2)$ say.

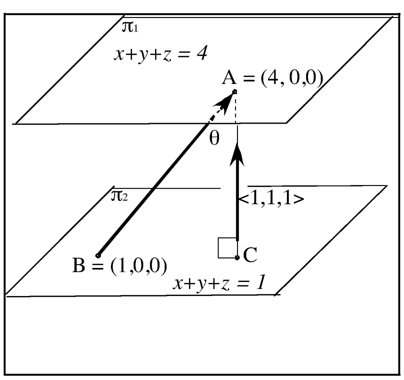
From the diagram, we see the required distance

$$D = |\mathbf{P_1P_2}| \cos\theta = \left| \mathbf{P_1P_2} \bullet \frac{< a, b, c >}{|< a, b, c >|} \right|$$

Example.
Find the distance between the x–y plane and the plane: $z = -5$.

We see immediately that the distance is 5.

Example.
Find the distance between planes: π_1: $x + y + z = 1$ and π_2: $x + y + z = 4$.



Take any point A = (4, 0, 0) say, in π_1.
Take any point B = (1, 0, 0) say, in $\pi_)$

We see the required distance AC (C as in diagram)
Distance AC

$$= | < 3, 0, 0 > . < 1, 1, 1 > /\sqrt{3} |$$
$$= \sqrt{3}.$$

Example:
The distance between planes π_1: $x + y + 2z = 2$ & π_2: $x + y + z = 1$ is zero.
The two planes are not parallel, they have different normals and so meet (in a line).

Example.

Find the point on the plane π: $x + y - z = 2$ which is closest to point $P = (5, 5, 0)$.

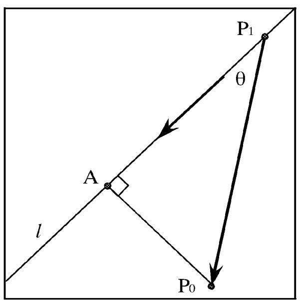
Solution. First we find the distance from point P to the given plane π.

We see the distance = $| <3, 5, 0> . <1, 1, -1> / \sqrt{3} | = 8 / \sqrt{3}$.
So that the nearest point in the plane π to point $(5, 5, 0)$ is either:
$(5+8/3, 5+8/3, 0-8/3)$ or $(5-8/3, 5-8/3, 0+8/3)$ since
$< 1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3} >$ and $< -1/\sqrt{3}, -1/\sqrt{3}, +1/\sqrt{3} >$
are the two unit normals for π.

We easily see, by substitution, that the second point: $(7/3, 7/3, 8/3)$ is the required point on π, which is closest to the point $(5, 5, 0)$.

---

3. Distance from a point $P_0 = (x_0, y_0, z_0)$ to a line l :



$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Take any point on l, $P_1 = (x_1, y_1, z_1)$ say.

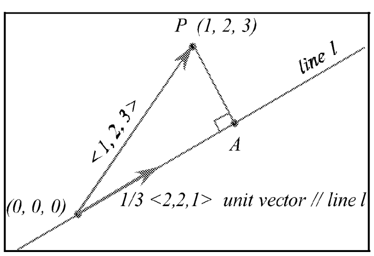
From the diagram, we see the required distance

$$D = |\mathbf{P_0 P_1}| \sin\theta = \left| \mathbf{P_0P_1} \times \frac{<a,b,c>}{|<a,b,c>|} \right|$$

Example:

Find the distance from point $P = (1, 2, 3)$ to the line l : $x = y = 2z$.

Solution:



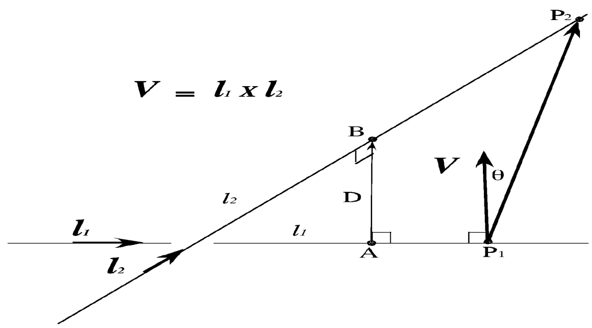
Choose any point, $(0, 0, 0)$ say, lying on line l
(l is parallel to $< 1, 1, ½ >$ // *unit vector* *(1/3)* $< 2, 2, 1 >$).

Required distance PA $= | < 1, 2, 3 > \times < 2/3, 2/3, 1/3 > |$
$= (1/3) | < -4, 5, -2 > |$
$= \sqrt{5}$

4. Distance between two lines

$l_1 : \dfrac{x-x_1}{a_1} = \dfrac{y-y_1}{b_1} = \dfrac{z-z_1}{c_1}$ *and* $l_2 : \dfrac{x-x_2}{a_2} = \dfrac{y-y_2}{b_2} = \dfrac{z-z_2}{c_2}$

Take any point on l_1, $P_1 = (x_1, y_1, z_1)$ *say*, and any point on l_2, $P_2 = (x_2, y_2, z_2)$ say.



From the diagram, we see the required distance D

$D = |\mathbf{P_1P_2}| \cos\theta = \dfrac{\boldsymbol{l_1} \times \boldsymbol{l_2}}{|\boldsymbol{l_1} \times \boldsymbol{l_2}|} \,|\, \mathbf{P_1P_2} \cdot \;|$

where $\quad \boldsymbol{l_1} = \langle a_1, b_1, c_1 \rangle \quad$ parallel to line l_1

& $\quad \boldsymbol{l_2} = \langle a_2, b_2, c_2 \rangle \quad$ parallel to line l_2

To see this, I imagine finding the distance between two parallel planes π_1 and π_2 containing lines l_1 and l_2 respectively.

$\dfrac{\boldsymbol{l_1} \times \boldsymbol{l_2}}{|\boldsymbol{l_1} \times \boldsymbol{l_2}|}$ is a unit normal to the two parallel planes π_1 and π_2

Example.

Find the distance between the two lines: $x = y = 2z$ and $x-1 = y = z + 2$.

Solution:

Choose arbitrary point $P_1 = (0, 0, 0)$ on the line: $x = y = 2z$ which is parallel to $\langle 2, 2, 1 \rangle$.

Choose arbitrary point $P_2 = (1, 0, -2)$ on line: $x-1 = y = z+2$ which is parallel to $\langle 1, 1, 1 \rangle$.

Distance = $| \langle 1-0, 0-0, -2-0 \rangle \cdot \langle 2, 2, 1 \rangle \times \langle 1, 1, 1 \rangle | / \sqrt{2} = 1/\sqrt{2}$.

Note $| \langle 2,2,1 \rangle \times \langle 1,1,1 \rangle | = \sqrt{2}$.

Example.

Show that the lines $x-1 = 2y-2 = z/4$ and $x-2 = y-1 = z-7$ intersect and find the point of intersection.

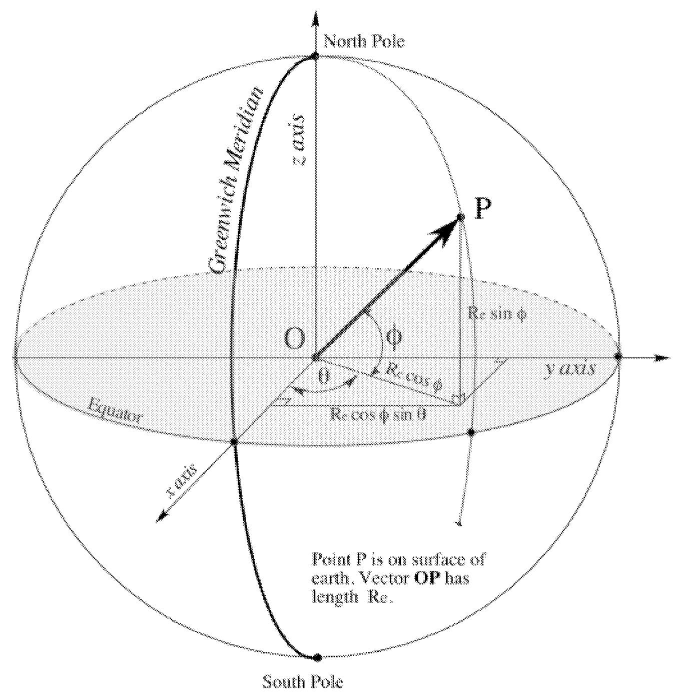
Solution:

Similar to the above example, we show that the distance between the lines is zero. We then solve the four sets of equations to find the (common) point of intersection to be (3, 2, 8).

To find the distance between two locations on earth, using vectors.

Here we set up the *x-y-z coordinate axes* with the origin at the centre of the earth, the *z-axis* running through the North and South poles and the *x-axis* meeting the equator at the Greenwich meridian.

We find the vector $\boldsymbol{v}$ associated with any city with longitude θ and latitude φ, then use vector methods to solve the above problem. Most would agree that the spherical geometry methods are easier to manipulate and interpret, but this method of vectors is a nice application (revision) of vector methods, using the dot product to find angles and the cross product to find perpendicular vectors.
Longitude θ will be measured anticlockwise from the *x–axis,* so that western longitudes will be $-\theta$. Latitude φ will be measured upwards from the *x–y plane*, so that southern latitudes will be $-\varphi$.



We see that the *x-y-z* coordinates for location (θ, φ) on the surface of the earth would be: $R_e\,(\cos(\varphi)\cos(\theta), \cos(\varphi)\sin(\theta), \sin(\varphi))$. R_e = earth's radius.

The vector $\boldsymbol{v}$ from the centre of the earth O to location (θ, φ) on the surface is

$$\boldsymbol{v} = R_e < \cos(\varphi)\cos(\theta), \cos(\varphi)\sin(\theta), \sin(\varphi) >$$

Note that $\boldsymbol{< \cos(\theta), \sin(\theta), 0 >}$ is a unit vector in the *x-y plane* going through longitude θ.
$\boldsymbol{< \sin(\theta), -\cos(\theta), 0 >}$ is a perpendicular unit vector to the plane containing longitude θ,

(since $\boldsymbol{< \cos(\theta),\ \sin(\theta),\ 0 > . < \sin(\theta), -\cos(\theta),\ 0 >} = 0$).
This will be used in part **(c)** below.

Example.

For a flight from Lethbridge to Tokyo along the connecting great circle find:

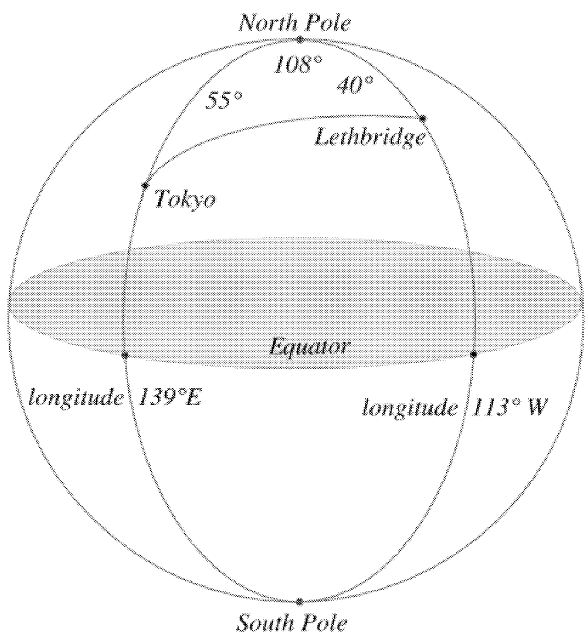
(a) The (shortest) distance between the two cities *Lethbridge* & *Tokyo*.

(b) The initial bearing setting off from *Lethbridge*.

(c) The final bearing upon landing in *Tokyo*.

(d) The most northern latitude reached on this shortest route.

Given co-ordinates: *Lethbridge* (113°W, 50°N) & *Tokyo* (139°E, 35°N).



To make the calculations slightly easier, we set the zero longitude at either city, *Lethbridge* say, so that *Lethbridge* coordinates become **(0°, 50°N)**, and the coordinates for *Tokyo* become **(108°W, 35°N).**

Let **L** be the vector from the centre of the earth O to *Lethbridge*.
Let **T** be the vector from the earth's centre O to *Tokyo*.
Let R_e denote the radius of the earth, approximately 6400 kms.

Then $$\mathbf{L} = R_e < \cos(50)\cos(0),\ \cos(50)\sin(0),\ \sin(50) >$$

$$= R_e < \cos(50),\ 0,\ \sin(50) >$$

and $$\mathbf{T} = R_e < \cos(35)\cos(-108), \cos(35)\sin(-108), \sin(35) >$$

(a) To find the distance between the two cities, we find angle α between the vectors **L** & **T**. We use the dot product.

$$\mathbf{L} \bullet \mathbf{T} = |\mathbf{L}|\,|\mathbf{T}|\cos(\alpha).$$

$$\alpha = cos^{-1}\left[cos(50)\,cos(35)\,cos(-108) + 0 + sin(50)\,sin(35)\right] \approx 73.94°$$

So that the great circle (shortest) distance is $\approx$ **8,259 kms.**

(b) To find the bearing with which the aeroplane sets off from *Lethbridge*, we find the angle γ between the plane containing **L** & **T** and the plane containing the zero meridian (remember that we have adjusted our axes, so that Lethbridge is on the zero meridian).

$$\cos\gamma = \frac{\mathbf{TxL} \bullet \langle 0,-1,0 \rangle}{|\mathbf{TxL}|}$$

$$|\mathbf{Tx\,L}| = R^2 \sin\alpha \qquad \text{(see below)}$$

$$\gamma = \cos^{-1}\left\{\frac{sin(35)\,cos(50) - cos(35)\,cos(-108)\,sin(50)}{sin(73.938)}\right\}$$

$$= 54.165° \approx \underline{\mathbf{54°09'54''}} \qquad \textbf{West of North.}$$

(c) To find the bearing that the aeroplane has on landing in *Tokyo,* we find the angle β between the plane containing **L** & **T** and the plane containing the 108°W meridian, remembering that we have adjusted our axes, so that *Tokyo* is not on 139°E, but on 108°West (of *Lethbridge*).

$$\cos\beta = \frac{\mathbf{TxL} \bullet \langle \sin(-108), -\cos(-108), 0 \rangle}{|\mathbf{TxL}|} \qquad |\mathbf{T\,x\,L}| = R^2 \sin\alpha$$

$$\beta = \cos^{-1} A \qquad \text{where angle A is given by:}$$

$$A = \left\{\frac{cos(35)\,sin(-108)\,sin(50)\,sin(-108) + [cos(35)\,cos(-108)\,sin(50) - sin(35)\,cos(50)]\,cos(-108)}{sin(73.938)}\right\}$$

$$\approx 39.5061322875....°$$

$$\approx \underline{39°30'22'' \quad \text{East of North}}.$$

Note:
A normal to the plane containing meridian $\theta°$ is $< sin(\theta), - cos(\theta), 0 >$.
To see this, think of a unit normal to the plane containing the Greenwich meridian, this could be $\boldsymbol{j}$ or $-\boldsymbol{j}$. Let us take $-\boldsymbol{j}$ (as it is a little easier to draw the diagram), then if this plane is rotated anticlockwise $\theta°$ around the *z-axis* (the line joining North and South Poles), the unit normal to this plane will change from $-\boldsymbol{j}$ to $< sin(\theta), - cos(\theta), 0 >$.

Had we taken the unit normal to be $\boldsymbol{j}$, then on rotation anticlockwise ø° the unit normal would have become $< -sin(\theta), cos(\theta), 0 >$.

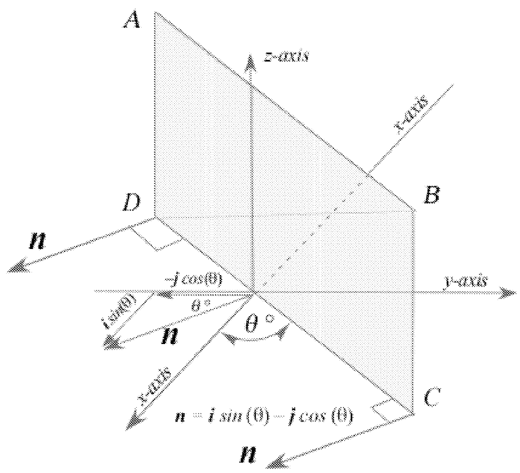


figure 4.23

Plane ABCD contains the *z-axis*, has normal $\boldsymbol{n}$.

If $\theta = 0°$ then plane ABCD contains the *z-axis* and the *x-axis*, and $\boldsymbol{n} = -\boldsymbol{j}$

We see for any angle θ : $\boldsymbol{n} = \boldsymbol{i}\, sin(\theta) - \boldsymbol{j}\, cos(\theta) = < sin(\theta), - cos(\theta), 0 >$

(d) To find the northernmost latitude, we find the angle δ between the plane containing **L** & **T** and the horizontal plane containing the equator. That is the angle between the normals to these two planes, the angle between the two vectors: $\mathbf{T} \times \mathbf{L}$ and $\boldsymbol{k}$.

$$\mathbf{T} \times \mathbf{L} \cdot \boldsymbol{k} = |\mathbf{T} \times \mathbf{L}|\,|\boldsymbol{k}| \cos(\delta)$$

$$|\mathbf{T} \times \mathbf{L}| = R^2 sin\, \alpha \qquad \text{with} \quad \alpha = 73.94°$$

$$\delta = cos^{-1}\left[\frac{-\cos(50)\cos(0)\cos(35)\sin(-100)}{sin(73.94°)}\right]$$

$$\approx 58.5930998215...° \qquad \underline{\approx 58°35'35'' \text{ North.}}$$

Note: $|\mathbf{T} \times \mathbf{L}| = R^2 sin(\alpha)$
since $\mathbf{T} \times \mathbf{L} = |\mathbf{T}||\mathbf{L}|\, sin(\alpha)\, \mathbf{c}$ with $|\mathbf{T}| = |\mathbf{L}| = R$ and $|\boldsymbol{c}| = 1$.

Note:
The above method gives the latitude of the point nearest the North Pole on the great circle connecting the two cities. It could happen that this point is on the larger of the two great circle arcs connecting the cities. In which case the 'upper' city would be the highest latitude on that particular journey.

For example, if the two cities were on the same meridian, then $\delta = 90°$. The point of highest latitude on the great circle connecting the two cities would be the North Pole, which would not be on the shortest route between the two cities. Note that a great circle through the North Pole comprises two meridians.

It might seem surprising that on a flight from Vancouver at 50°N to Tokyo at 35°N, the plane goes close to Alaska on 60° North. The Aleutian Islands can be seen on such a flight.

Of interest, the great circle Los Angeles to Sydney Australia contains two convenient refueling islands, Hawaii and Fiji. However these refuelling stops are no longer necessary with today's long-range jets, which fly non-stop Los Angeles to Sydney.

Find the direct distance from Los Angeles to Sydney and compare with the sum of the three distances: L.A. to Hawaii, Hawaii to Fiji, Fiji to Sydney.

Chapter 6.
Non-Euclidean Geometry
Spherical Geometry

Euclids Axiom 5: The parallel postulate:
That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Axiom 5 was the cause of much interest for over 2000 years. Geometers tried to prove that *Axiom 5* was a natural consequence of the first four axioms. It seems obvious enough, but it turned out impossible to prove and led to non-Euclidean geometries.

Playfair's Axiom:
At most one line can be drawn through any point not on a given line parallel to the given line in a plane. This axiom is equivalent to Euclid's Axiom 5.

1. The sum of the angles in every triangle is 180°
2. There exists a triangle whose angles add up to 180°.
3. The sum of the angles is the same for every triangle.
4. There exists a pair of similar, but not congruent, triangles.
5. Every triangle can be circumscribed.
6. If three angles of a quadrilateral are right angles, then the fourth angle is also a right angle.
7. There exists a quadrilateral in which all angles are right angles.
8. There exists a pair of straight lines that are at constant distance from each other.
9. Two lines that are parallel to the same line are also parallel to each other.
10. In a right angled triangle, the square of the hypotenuse equals the sum of the squares of the other two sides. (Pythagoras' theorem.)
11. There is no upper limit to the area of a triangle. (Wallis Axiom.)
12. The summit angles of the Saccheri quadrilateral are 90°.
13. If a line intersects one of two parallel lines, both of which are coplanar with the original line, then it also intersects the other. (Proclus' axiom).

Most people are aware of these properties, if not the fact that they are all equivalent to Euclid's famous Axiom 5. At the beginning of the 19th century *Gauss, Bolyai* and *Lobachesky* began to look at alternative geometries to Euclid; Geometries where Axiom 5 did not hold:
Geometries where no line could be drawn parallel to the given line l, through a given point P not on l.
Or Geometries where an infinity of lines could be drawn parallel to a given line l, through a given point not on l.
These new geometries were labelled Non-Euclidean, elliptical and hyperbolic respectively.

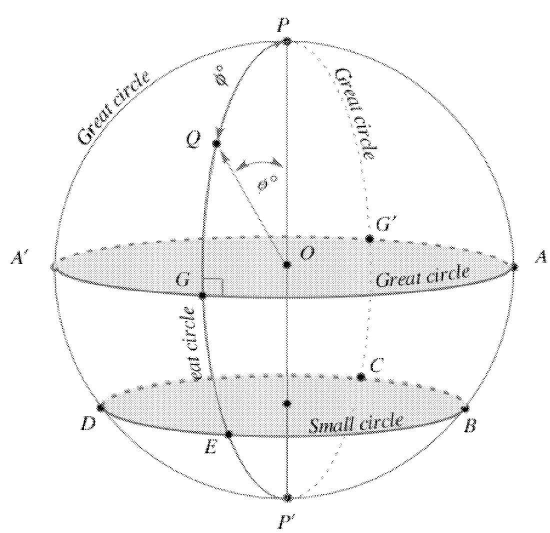
It is a little perplexing that it took so long to come up with these Non-Euclidean Geometries when the properties of Spherical Geometry were so well known and much used. Spherical Geometry is an example of the Non-Euclidean elliptical geometry. In Spherical Geometry (described in some detail below) no line can be drawn through parallel to the given line through a given point P not on the given line. I n Spherical Geometry all triangles have more than 180°, different triangles have different angle sums, and Pythagoras' theorem does not hold, ... We will see that none of the above listed 14 equivalent statements hold.

One clearly sees that Playfair's axiom does not hold in Spherical Geometry. However, I find it more difficult to imagine the hyperbolic geometry where you can draw more more than one line through any point P not on a given line and parallel to the given line. The great mathematician Poincare gives a very nice example which helps us to imagine hyperbolic geometry. We will consider this example at the end of this chapter. Let us first look at the Non-Euclidean Spherical Geometry.

Spherical Geometry.

A *sphere* is defined as a surface, all points on which are at the same distance from a certain fixed point. This point is the *centre*, and the constant distance is the *radius*.
A *great circle* of a sphere is the circle in which the sphere is cut by any plane passing through the centre of the sphere. A *small circle* is the circle in which the sphere is cut by any plane not passing through the centre.

The *axis* of a great or small circle is the diameter of the sphere perpendicular to the plane of the circle. The *poles* of the circle are the extremities of this diameter. Secondaries to a circle (great or small) are great circles passing through its poles.



If the radius of the sphere is R and the angle POQ subtended by the arc PQ at the centre O is denoted by ϕ, the length of the arc PQ is Rϕ, where ϕ is expressed in circular measure, or radians.
It is convenient and conventional to take the radius of the sphere as unit length; the length of the arc PQ is then equal to ϕ. Thus the angular distance between two points on a sphere is measured by the arc of the great circle joining them or by the angle which this arc subtends at the centre of the sphere.
If the angle POQ is 60°, we can say that the length of the great circle PQ is $\pi/3$ radians, or that it is 60°.
Note the custom to denote points diametrically opposite with a prime:
A and A′, P and P′, G and G′ are diametrically opposite pairs of points.

A great circle on a sphere is analogous to a straight line in a plane. A straight line joining two points in a plane is the shortest distance between those two points; so also the shortest distance between two points on a sphere is the arc of the great circle passing through those two points.

Small circles on a sphere are analogous in their general properties to circles in a plane. Secondaries to a great circle are analogous to perpendiculars to a straight line. The distance of a point from any great circle is the length of the arc of a secondary drawn from the point to the circle.
The angle between two great circles is the angle between their planes.
The angle between two great circles is equal to any of the following:

1. The angle between the tangents to them at their points of intersection.
2. The arc, which they intercept on a great circle to which they are both secondaries.
3. The angular distance between their poles.

Spherical Triangles.
A *spherical triangle* is a portion of a spherical surface bounded by three arcs of great circles. Perhaps note that P'DE in the above diagram, is not a spherical triangle since side DE is not an arc of a great circle.
A spherical triangle, like a plane triangle, has six parts, its three sides and its three angles. The sides are generally measured by the angles they subtend at the centre of the sphere, so that the six parts are all expressed as angles. No part is supposed to exceed 180°. No angle of a spherical triangle can exceed 180°.
A spherical triangle has the property, in common with plane triangles, that the sum of any two sides is greater than the third side. But whereas the sum of the three angles of a plane triangle is equal to 180°, the sum of the angles of a spherical triangle is always greater than 180°. The amount by which the sum of the three angles exceeds 180° is termed the *spherical excess*.
A plane triangle may have one angle a right angle. A spherical triangle, on the other hand, may have one, two or three angles that are right angles.

Terrestrial Longitude.
The *longitude* of a place on Earth is the angle between the terrestrial meridian through that place, and a certain meridian fixed on the Earth, and called the Prime Meridian or the Greenwich Meridian. By international agreement, the prime meridian from which the longitudes of all places on the Earth are measured is defined as the meridian passing through the Airy transit instrument at the Royal Observatory Greenwich.
As all places on a given meridian have the same longitude, the terrestrial meridians are often called meridians of longitude.
Longitudes are measured eastwards and westwards from the prime meridian, from 0° to 180°.

Terrestrial Latitude.
The *latitude* of a place on the Earth is its angular distance from the equator, measured along the meridian. All points on a small circle parallel to the equator, have the same latitude. For this reason, parallels to the equator are usually termed *parallels of latitude*. Latitudes are measured both northwards and southwards, from 0° to 90°. The complement (with respect to 90°) of the latitude is called the *colatitude*. The colatitude is the angular radius of the parallel of latitude.

Solving Spherical Triangles.

The formulae for solving spherical triangles are in every case different from the analogous formulae in plane trigonometry. A further difference is that a spherical triangle is completely determined if its three angles are given.

Two spherical triangles will, in general, be equal if they have the following parts equal:

1. Three sides
2. Two sides & included angle.
3. Two sides & one opposite angle
4. Three angles
5. Two angles & adjacent side.
6. Two angles & one opposite side.

Cases 3 and 6 may be ambiguous.

Cosine rule for spherical trigonometry.

This is the most important and most used rule in spherical trig. namely:

For any triangle with vertex angles A, B, C and opposite sides *a, b, c*,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \qquad \text{..... 1}$$

We give a proof on the following page.

Sine rule for spherical trigonometry.

For any triangle with vertex angles A, B, C and opposite sides *a, b, c*:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \qquad \text{.... 2}$$

We give a proof later.

Of interest, we list a few more trigonometric identities from Spherical Trigonometry

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a \qquad \text{..... 3}$$

$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A \qquad \text{..... 4}$$

$$\sin a \cos C = \cos c \sin b - \sin c \cos b \cos A \qquad \text{..... 5}$$

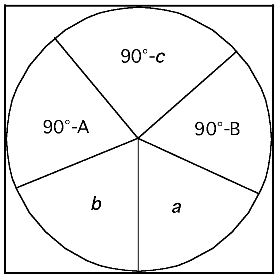
$$\cot a \sin b = \cos b \cos C + \cot A \sin C \qquad \text{..... 6}$$

$$\sin(\frac{A}{2}) = \left(\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}\right)^{1/2} \qquad \text{..... 7} \quad \text{etc.}$$

where $s = (a + b + c)/2$.

<u>For any right-Angled Triangles</u> ABC with C = 90°, and with parts

a, b, 90 – A, 90 – c, 90 – B arranged around the circle below:



It can be shown that

sin (Any) = *tan* (Adjacent 1) *tan* (Adjacent 2)

e.g. *sin* (a) = *tan* (b) *tan* (90 – B)

and

sin (Any) = *cos* (Opposite 1) *cos* (Opposite 2)

e.g. *sin* (90 – A) = *cos* (a) *cos* (90 – B)

In many cases the simplest way to solve a general triangle is to draw a great circle through one of the angles at right angles to the opposite side, and then apply the formulae applicable to right-angled triangles.

<u>**Proof** of the Cosine Rule in Spherical Geometry</u>

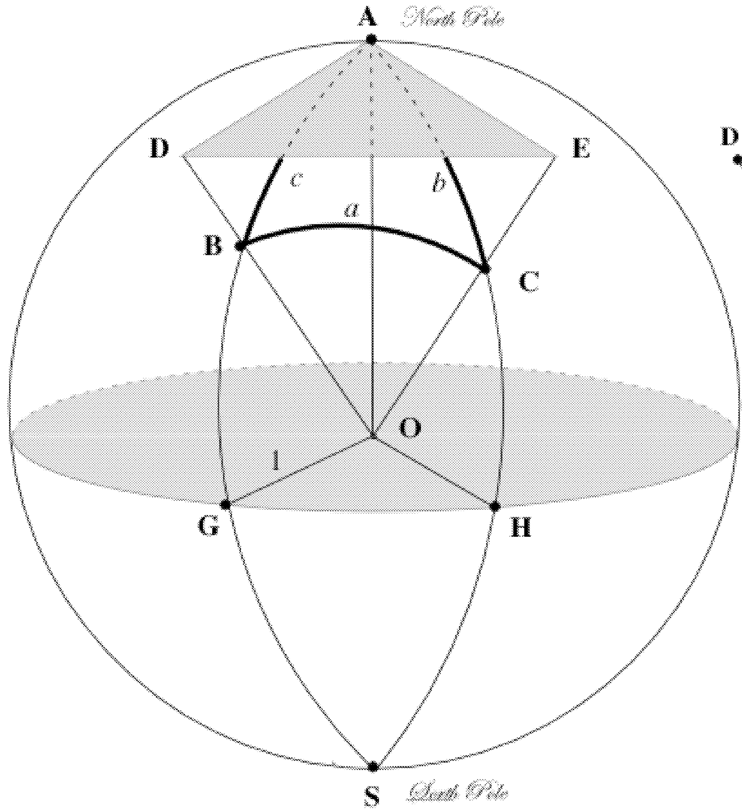
For arbitrary spherical triangle ABC with sides a, b, c

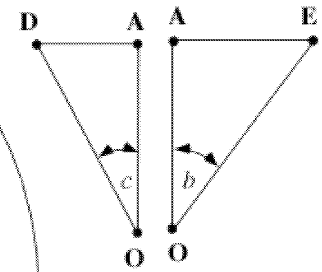
$$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(\mathrm{A})$$

To help interpret diagram below, rotate the sphere (unit radius) with arbitrary triangle ABC so that point A is the North Pole. Then think of triangle sides AB and AC arcs of the longitudes through the North and South Poles.

Note that the 4 triangles: ADE, ODE, ADO and AEO are in 4 distinct planes.

Triangle ADE is in a tangent plane to the sphere. This plane touches the sphere at the north pole A. This plane is parallel to the plane of the equator and tangent to sides AB and AC, so that Euclidean triangles ADO and AEO are right-angled triangles.





Proof:

We see $DE^2 = OD^2 + OE^2 - 2OD.OE\cos(a)$ *cosine rule on* $\Delta\, ODE$

$DE^2 = AD^2 + AE^2 - 2AD.AE\cos(A)$ *cosine rule on* $\Delta\, ADE$

so that $OD^2 + OE^2 - 2OD.OE\cos(a) = DE^2 = AD^2 + AE^2 - 2AD.AE\cos(A)$

or

$$2OD.OE\cos(a) = OD^2 + OE^2 - AD^2 - AE^2 + 2AD.AE\cos(A)$$

$$2OD.OE\cos(a) = (OD^2 - AD^2) + (OE^2 - AE^2) + 2AD.AE\cos(A)$$

$$2OD.OE\cos(a) = OA^2 + OA^2 + 2AD.AE\cos(A)$$

$$OD.OE\cos(a) = OA^2 + AD.AE\cos(A)$$

$$\cos(a) = \frac{OA.OA}{OD.OE} + \frac{AD.AE}{OD.OE}\cos(A)$$

$$\cos(a) = \cos(c)\cos(b) + \sin(c)\sin(b)\cos(A)$$

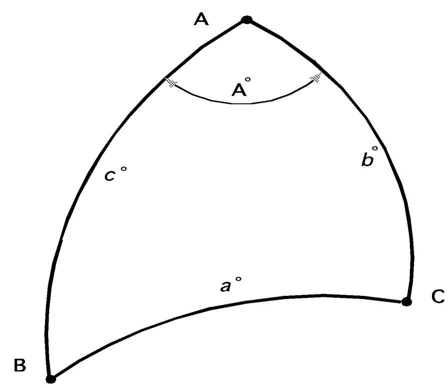
Q.E.D.

Note: $\cos(c) = OA/OD$
$\cos(b) = OA/OE$
$\sin(c) = AD/OD$
$\sin(b) = AE/OE$ *using right triangles* ADO & AEO.

Proof of the Sine Rule for Spherical Geometry

For arbitrary spherical triangle ABC with sides a, b, c $\dfrac{\sin(a)}{\sin(A)} = \dfrac{\sin(b)}{\sin(B)} = \dfrac{\sin(c)}{\sin(C)}$

Proof: *(unusual)*



For arbitrary spherical triangle ABC we have (the cosine rule)

$$\cos(a) = \cos(c)\cos(b) + \sin(c)\sin(b)\cos(A)$$

or $\sin(c)\sin(b)\cos(A) = \cos(c)\cos(b) - \cos(a)$ *squaring both sides:*

$$\sin^2(c)\sin^2(b)\cos^2(A) = \cos^2(c)\cos^2(b) - 2\cos(a)\cos(b)\cos(c) + \cos^2(a)$$

$$\sin^2(c)\sin^2(b)(1 - \sin^2(A)) = \cos^2(c)\cos^2(b) - 2\cos(a)\cos(b\cos(c) + \cos^2(a)$$

$$\sin^2(c)\sin^2(b)\sin^2(A) = -\cos^2(c)\cos^2(b) + 2\cos(a)\cos(b)\cos(c) - \cos^2(a) + \sin^2(c)\sin^2(b)$$

$$\sin^2(c)\sin^2(b)\sin^2(A) = -\cos^2(c)\cos^2(b) + 2\cos(a)\cos(b)\cos(c) - \cos^2(a) + (1 - \cos^2(c))(1 - \cos^2(b))$$

$$\sin^2(c)\sin^2(b)\sin^2(A) = 1 - \cos^2(a) - \cos^2(b) - \cos^2(c) + 2\cos(a)\cos(b)\cos(c)$$

The RHS is symmetric in variables a, b, c so that rotating $a \to b$, $b \to c$, *and* $c \to a$ *etc. we get a total of three different expressions on the LHS, but the right hand side stays unchanged (equal) so that we have:*

$$\sin^2(c)\sin^2(b)\sin^2(A) = \sin^2(a)\sin^2(c)\sin^2(B) = \sin^2(b)\sin^2(a)\sin^2(C)$$

dividing through by $\sin^2(a)\sin^2(b)\sin^2(c)$

$$\frac{\sin^2(a)}{\sin^2(A)} = \frac{\sin^2(b)}{\sin^2(B)} = \frac{\sin^2(c)}{\sin^2(C)} \quad or \quad \frac{\sin(a)}{\sin(A)} = \frac{\sin(b)}{\sin(B)} = \frac{\sin(c)}{\sin(C)} \qquad Q.E.D.$$

Chapter 6 *Non-Euclidean Geometry – Spherical Geometry*

Of interest:

Note that there are no parallel lines in Spherical geometry.
Playfair's axiom is a non-starter here.

Note that we can construct a triangle with more than 180°. It fact, it can be shown that all triangles in this geometry have more than 180° and less than 540°, that is $180° < A + B + C < 540°$.

A Nautical Mile (6, 082 feet) is one minute of arc along a great circle of the earth's surface (taking the earth's radius as 3,960 (land) miles; a land mile is 5,280 feet).

A Knot is a speed of one nautical mile per hour.
Perhaps note that knots per hour is not a speed, but a measure of acceleration of one nautical mile/hour/hour.

How long would it take to cross the Atlantic (3000 nautical miles) at a rate of 1 knot per hour? (Answer: Just under 80 hours).

A Metre was originally defined to be one ten-millionth of the quadrant of the of the great circle on Earth's surface running from the north pole to the equator through Paris.

A slight error was made in the measurement of this quadrant (1792–1799), so that the official metre at the time, a platinum bar in Paris, was just short of one forty millionth of a great circle going through the north and south poles.

Previous to this definition, a metre was defined to be the length of a pendulum with a period of precisely 2 seconds. This definition was abandoned when it was realised that the acceleration of gravity was not constant over the surface of the earth. Consequently, a pendulum with period 2 seconds would have lengths that depended upon the location of the pendulum.

A metre is now defined as the distance light travels in a certain fraction of a second. Equivalently, light travels 299, 792, 458 metres in one second.

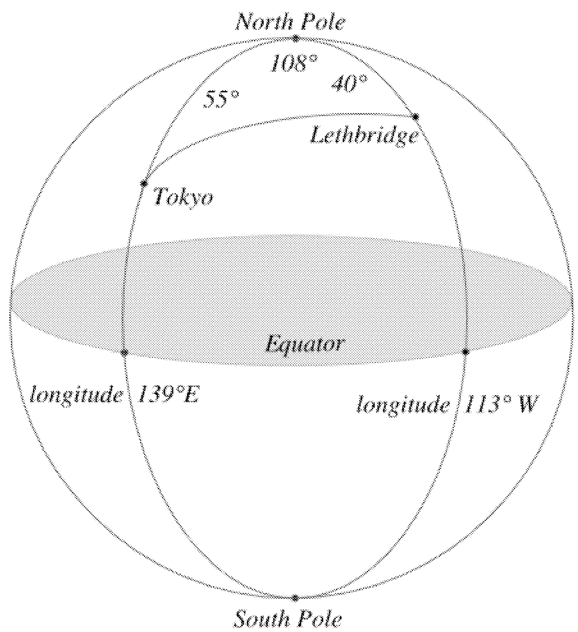
Chapter 6 Non-Euclidean Geometry – Spherical Geometry

Global distances. Using spherical geometry.

<u>Example</u>.
We give the same example from the previous chapter 5 where the problem was solved using vectors. Let us hope that we get the identical answers.

Consider the two cities as in Chapter 4:

***Lethbridge* (113°W, 50°N)** and ***Tokyo* (139°E, 35°N).**



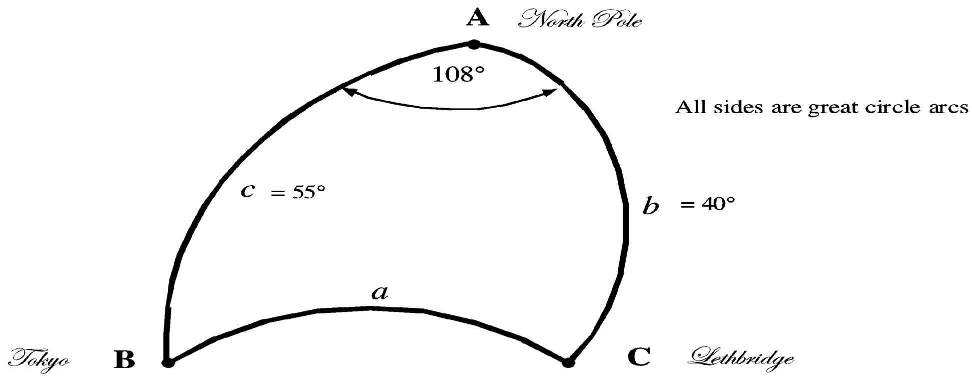
Find

(a) The shortest (great circle) distance *Lethbridge to Tokyo*.

(b) The initial bearing setting off from *Lethbridge*.

(c) The final bearing upon landing in *Tokyo*.

(d) The most northern latitude reached on this shortest route.



(a) Consider the spherical triangle ABC as shown:
Using the spherical cosine rule:
$cos\ (a) = cos\ (b)\ cos\ (c)\ +\ sin\ (b)\ sin\ (c)\ cos\ (A)$ $\quad b = 40°,\ c = 55°,\ A = 108°.$

$a\ =\ cos^{-1}[cos\ (40)\ cos\ (55) + sin\ (40)\ sin\ (55)\ cos\ (108)]\ \approx\ 73.93814240...°$
So that the distance from *Lethbridge* to *Tokyo* is $\approx$ **8,259 kms**.

(b) We want to find angle C.
We could use the cosine rule or the sine rule on Δ ABC.

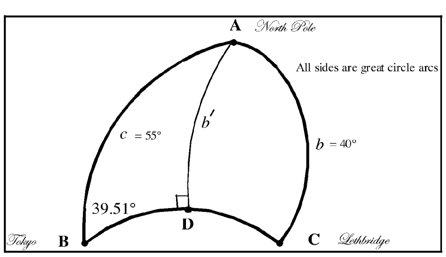
Using the sine rule: $C = sin^{-1}(sin\ (c)\ sin\ (A)\ /\ sin\ (a)$
$= sin^{-1}[sin\ (55)\ sin\ (108)\ /\ sin\ (73.938)]\ \approx$ **54.1650659459...°**
So that the aeroplane heads off from *Lethbridge* **54°09'54.2" West of North**.

(c) We want to find angle B. Similar to finding angle C, we use the sine rule on Δ ABC.

$$\frac{\sin(A)}{\sin(a)} = \frac{\sin(B)}{\sin(b)}$$

$B = \sin^{-1}[sin\ (b)\ sin\ (A)\ /\ sin\ (a)]$ $\quad = sin^{-1}[sin\ (40)\ sin\ (108)\ /sin\ (73.938)]$
$\approx$ **39.5061322875...°**
So that the aeroplane would be heading **39°30'22" East of North** landing in *Tokyo*.

(d) To find the most northern point of the journey, drop a line (great circle) from A, perpendicular to side BC at point D.



Using the sine rule on Δ ABD

$$\frac{\sin(b')}{\sin(B)} = \frac{\sin(c)}{\sin(90)}$$

(We could have used Δ ACD)

$b'\ = sin^{-1}\ (sin\ (c)\ sin\ (B))$
$=$ **31.406900178°**

So that the most northern latitude reached is **58° 35' 35" N** (of the equator)

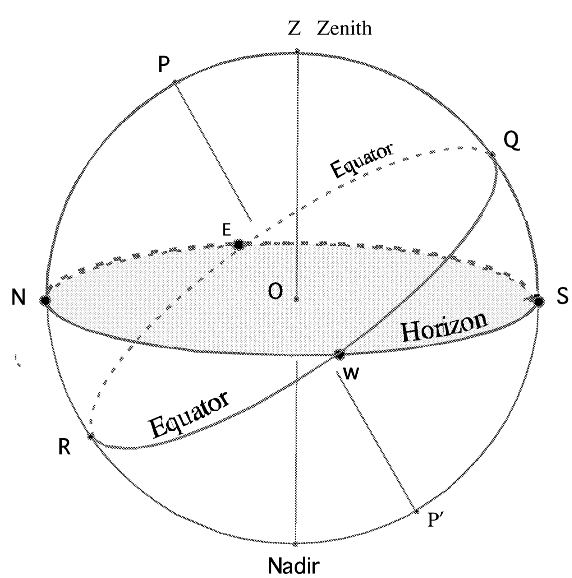
The Celestial Sphere.

On observing the stars it is not difficult to imagine that they are bright points dotted about on the inside of a hollow spherical dome, whose centre is at the eye of the observer. Such a sphere is called the *Celestial Sphere*. The sphere may be taken as a dome upon which the stars *appear* to lie.
If through the observer, a line be drawn in the direction in which gravity acts, that is, the direction indicated by a plumb line, it will meet the celestial sphere in two points. One of these is vertically above the observer, and is called the *Zenith*; the other is vertically below the observer, and is called the *Nadir*. For an observer at the earth's north pole, the zenith Z and Celestical Pole P coincide. For an observer on earth's equator, the Celestial Pole P lies on the celestial horizon. The celestial equator is at right angles to the horizon.
The plane through the observer perpendicular to the direction to the zenith will cut the celestial sphere in a great circle. This great circle is called the *Celestial Horizon*. Its poles are the zenith and the nadir.

If we observe the sky at different intervals during the night, we find that the stars always maintain the same configurations relative to one another, but that their actual situations in the sky, relative to the horizon, are continually changing. Some stars will set in the west, others will rise in the east. One star remains almost fixed. This star is called Polaris, or the Pole Star. All the other stars north and south of the equator describe on the celestial sphere small circles having a common pole P very near the Pole Star, and the revolutions are performed in the same period of time (about 23 hours 56 minutes of our ordinary time).

The common motion of the stars may most easily be conceived by imagining them to be attached to the surface of a sphere which is made to revolve uniformly about the diameter PP' (see diagram below). The extremities of this diameter P and P' are called the *Celestial Poles*. That pole P, which is above the horizon in northern latitudes is called the *North Pole*, the other, diametrically opposite, P', is called the *South Pole*.

The great circle EQWR, having these two points for its poles, is called the *Celestial Equator*. It is, therefore, the circle which would be traced out by the diurnal path of a star distant 90° from either pole. The *Meridian* is the great circle PZP'N passing through the zenith and the nadir and the celestial poles. It cuts both the horizon and the equator at right angles, since it passes through their poles.



Cardinal Points.
The East and West Points are the points of the intersection of the equator and the horizon. The North and South Points are the intersections of the meridian with the horizon.

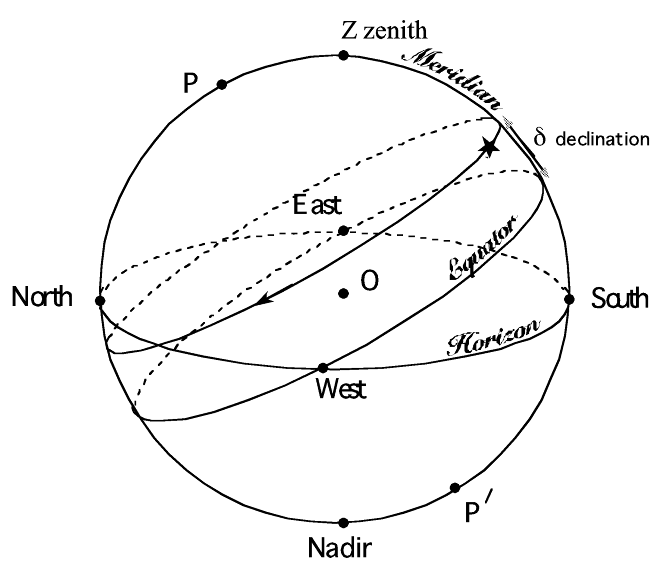
Verticals.
Secondaries to the horizon, that is, great circles through the zenith and nadir, are called *Vertical Circles*, or, briefly, *Verticals*. Thus, the meridian is a vertical. The *Prime Vertical* is the vertical circle passing though the east and west points.

The *North Polar Distance* of a star is its angular distance from the celestial pole.

The *Declination* is the angular distance from the equator, measured along a secondary, and is, therefore, the complement of the north polar distance. The declination may be considered positive or negative, according to the star's position to the north or south of the equator; it is also customary to specify this by the letter N. or S., as the case may be, and this is called the name of the declination. South declinations are always to be regarded as negative.

The great circle through the pole and the star is called the star's *Declination Circle*.

The *Hour Angle* of the star is the angle which the star's declination circle makes with the meridian. The hour angle is generally measured along the equator from the meridian towards the west, and is reckoned from 0° to 360°. The hour angle of a star is generally measured by the number of hours, minutes, and seconds of sidereal time (see below). The hour angle of a star, when expressed in time, is the interval of sidereal time that has elapsed since the star was on the meridian.
The *declination* and the *hour angle* may be taken as the two coordinates of a star.
This is referred to as the *Equatorial System of Coordinates*.



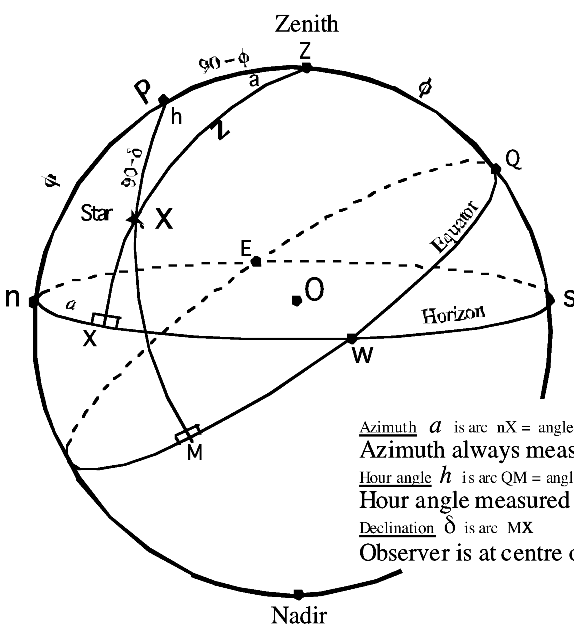
<u>Sidereal Day and Sidereal Time</u>.
The rotation of the Earth causes the stars to transit in succession across any given meridian. The interval between two successive passages of a fixed star over the meridian of a place is called a sidereal day. The sidereal day is the true period of the Earth's rotation. Like the civil day, it is divided into 24 hours, and these are divided into 60 minutes and 60 seconds each. The sidereal day is about 4 minutes shorter than the civil day.
It should be noted that at any instant the sidereal time is different for two different meridians. Thus the sidereal time at any instant is related to the observer's meridian and for this reason it is often called *local sidereal time*.

<u>Transits</u>.
The passage of a star across the meridian is called its *Transit*.

<u>Azimuth and Altitude. Horizontal System of Coordinates.</u>
The *Azimuth* of a star is the arc of the horizon intercepted between the north point and the vertical of the star, or the angle, which the star's vertical makes with the meridian. The azimuth is measured from 0° to 180° eastwards or westwards.

The *Altitude* of a star is its angular distance from the horizon, measured along a vertical.
The *Zenith Distance* is its angular distance from the zenith, or the complement of the altitude.

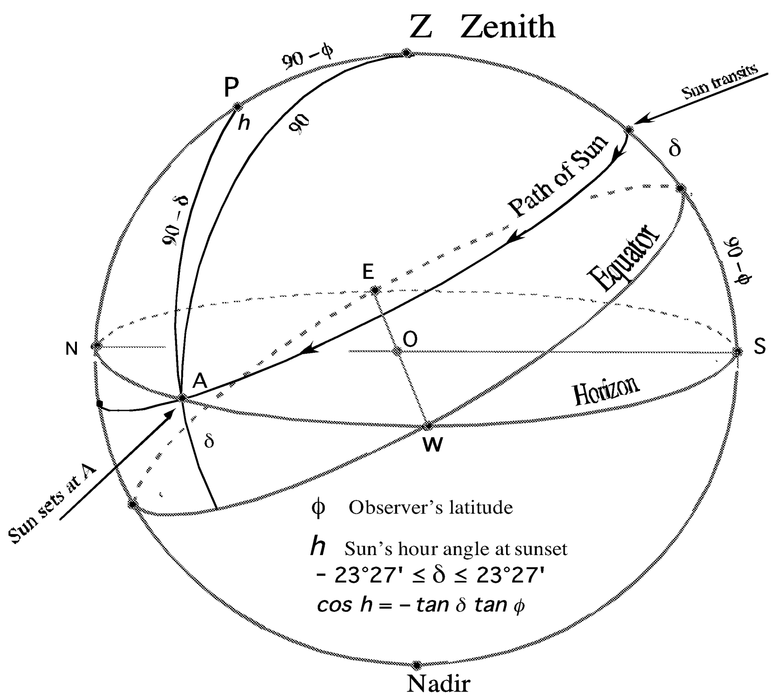
Star west
$0° \le h \le 180°$
$0° \le a \le 180$ W



Azimuth a is arc nX = angle PZX
Azimuth always measured from n, west or east
Hour angle h is arc QM = angle ZPX
Hour angle measured from meridian westwards.
Declination δ is arc MX
Observer is at centre of Celestial Sphere O.

Hour-angle and Azimuth of the Sun's Rising and Setting.

When the latitude of the observer, ϕ, and the declination of the star, δ, is known, the hour-angles and azimuths at rising and setting are easily found.



a The distance NA along the **horizon** is *azimuth*, measured east or west from N ranging 0°–180°

h The *hour angle* APZ measured westwards along the equator from the meridian, ranging 0°– 360°

ϕ Latitude of observer measured north or south from equator, + 90° to – 90°

δ Sun's declination ranging from + 23° 27′ to – 23°27′.

A Point of Sun's setting, as it goes below the horizon.

Using the cosine rule of spherical trigonometry on triangle PNA we can find:

1. The hour angle of the Sun's setting with $cos(h) = -tan(\delta)\, tan(\phi)$
2. The azimuth of the Sun's setting with $sin(\delta) = cos(a)\, cos(\phi)$.

(Perhaps recall: $sin(90 - \alpha) = cos(\alpha)$ and $cos(90 - \alpha) = sin(\alpha)$.)

Example:

(a) Find the time (summer time) of the Sun's transit over *Brooks* (AB) on June 21.

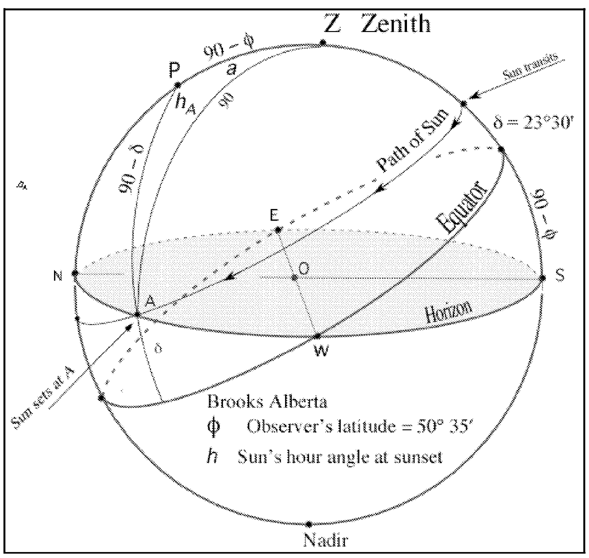
(b) Find the *time* (summer time) and *direction* of sunrise in *Brooks* on June 21.

(c) Approximately how long (in hours) is maximum nautical twilight in *Brooks*?
Nautical twilight is defined: sunset until the Sun is 12° below the horizon.
The horizon is generally not visible after nautical twilight. Sailors cannot use their sextants after nautical twilight as the horizon is no longer visible. Civil twilight ends when the sun is 6° below the horizon and astronomical twilight ends when the sun is 18° below the horizon (and the sun no longer interferes with astronomical observations).

Brooks:* $50° 35'$*N,* $111° 54'$*W. Sun's declination (June 21) is approx. *+23° 30'.*

Solution:

(a) The sun will transit at 1:00 PM summer time + equation of time June 21 + Brooks' longitude adjustment. That is: 1:00 PM + 2 minutes + (6+54/60)x4 minutes. Approximately: **1:30 PM. Summer time.** (We had to look up the Equation of Time values.)



(b) We will compute the time of sunset and azimuth of sunset first, because the diagram is a little easier to draw and read.

The key triangle to look at is ZPA. This triangle gives us the azimuth a of sunset and the hour angle h_A of sunset.

We apply the cosine rule in two cases.

(b) 1. To find h_A $\cos(90) = \cos(90-\phi)\cos(90-\delta) + \sin(90-\phi)\sin(90-\delta)\cos(h_A)$

$$0 = \sin(\phi)\sin(\delta) + \cos(\phi)\cos(\delta)\cos(h_A)$$

$$\cos(h_A) = -\tan(\phi)\tan(\delta) = -\tan(50°35')\tan(23°30') = -0.5290$$

$$h_A = 121.94°$$

$$\approx 8.129 \text{ hours} \approx 8 \text{ hours } 8 \text{ min.}$$ *(Recall 15° = 1 hour)*

So that sunrise would be 8 hours 8 mins. before transit, approx: ***5:22 AM.***
(The sun would set at 8 hrs 8 mins after transit, approx. 9:38 PM.)

(b) 2. *To find* a: $\cos(90-\delta) = \cos(90-\phi)\cos(90) + \sin(90-\phi)\,\text{sinv}(90)\cos(a)$

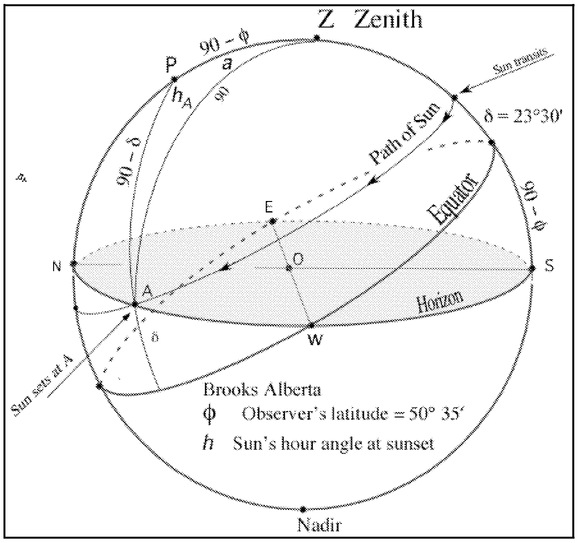
$$\sin(\delta) = \cos(\phi)\cos(a)$$

$$\sin(23°30') = \cos(50°35')\cos(a)$$

$$\cos(a) \approx 0.627995 \qquad a \approx 51.0976° \approx 51°5'51''$$

So that the sun would rise in Brooks, June 21, ***51°5′51″ East of North.***
(The sun would set approx. 51°5′51″ West of North.)

(c) To find the amount of nautical twilight in Brooks on June 21, consider the following diagram. The diagram is not to scale, as the 12° marking has been enlarged to make the point T more visible relative to the horizon.



The key triangle here is PZT; all sides are sections of great circles. P is the pole for the great circle 'Equator' and hence PT goes on to meet the equator at right angles.

Similarly Z is the pole for the great circle 'Horizon' so that ZT meets the horizon at right angles.

From the diagram, we see that nautical twilight ends when the Sun passes through point T 12° below the horizon. Sunset was when the Sun passed through point A. We have computed the hour angle h_A when the Sun was on the horizon at this point A.
We need to calculate the hour angle h_T of the Sun as it passes through point T.

We apply the cosine rule to the triangle PZT.

$$\cos(102°) = \cos(90-\phi)\cos(90-\delta) + \sin(90-\phi)\sin(90-\delta)\cos(h_T)$$

$$\cos(h_T) = \frac{\cos(102) - \cos(90-\phi)\cos(90-\delta)}{\sin(90-\phi)\sin(90-\delta)} = \frac{\cos(102) - \sin(\phi)\sin(\delta)}{\cos(\phi)\cos(\delta)}$$

$$= \frac{\cos(102) - \sin(50°35')\sin(23°30')}{\cos(50°35')\cos(23°30')} \qquad \textit{for Brooks on June 21.}$$

$$\approx -0.88609\ldots$$

$$h_T \approx \cos^{-1}(-0.88609\ldots) = 152.3863\ldots°$$

$$\approx 10.15908\ldots \text{ hours} \qquad (\text{recall } 15° = 1 \text{ hour}).$$

$$\approx 10 \text{ hours } 9 \text{ mins } 32 \text{ seconds.}$$

$$h_A = 8 \text{ hours } 8 \text{ mins} \qquad (\text{calculated above, for sunset (b)})$$

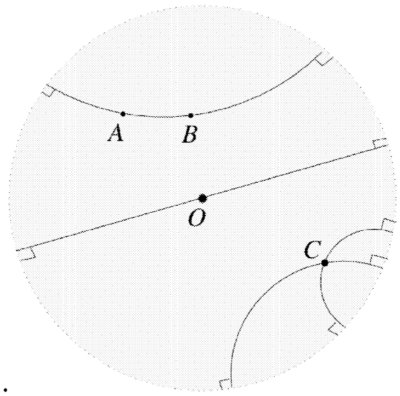
Amount of nautical twilight
for Brooks at Summer Solstice $\approx h_T - h_A$ $\approx$ ***2 hours 2 minutes.*** ***QED***

Poincare's Model of Hyperbolic (Non-Euclidean) Geometry.

Our *space* is the set of all points in the *interior* of a circle: $D = \{(x, y): x^2 + y^2 < 1\}$

The *lines* or so called *P-lines* in this space are straight lines within D which are perpendicular to the boundary of D, or any part of a circle (a Euclidean circle) within D, that is perpendicular to the boundary of D. We saw how to construct such circles towards the end of Chapter 2, section: Euclidean Constructions of Perpendicular Circles.

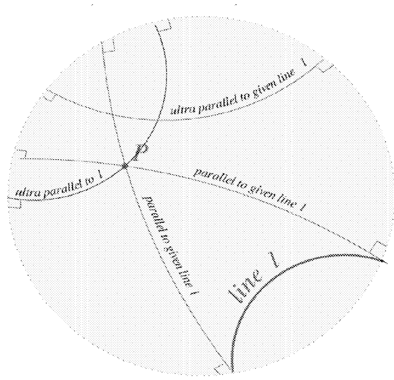
We immediately see that all diameters through point O of the circle enclosing D are *P-lines*. It can be shown that there are an infinity of lines through any one point C say, in D and that there is only one line through any two distinct points (A and B shown) in D



***Parallel lines** are lines that* do not meet. We divide parallel lines into two categories:

i. We say two *P-lines* are (simply) *parallel* if they do not meet in D, but would meet at a point on the boundary of D if extended. Recall that boundary points are not in our space.

ii. We say two *P-lines* are *ultra parallel* if they do not meet in D nor on the boundary of D if extended.



For any given line l and any given point P (not on l) there are an infinity of parallel lines that go through P and do not meet line l.

*It can be shown that **all** triangles in this geometry have their three angles sum to less than 180°. Recall that in Euclidean geometry all triangle have angle sums to exactly 180° and in Elliptical geometry, all triangle have angle sums to more than 180°.*

Appendix I Induction

Axiom of Induction:

Let $P(n)$ be any statement defined for positive integers n.
If i. $P(n_0)$ is true
and ii. $P(n) \Rightarrow P(n+1)$
then the statement $P(n)$ is true for all $n \geq n_0$.

To prove $P(n) \Longrightarrow P(n+1)$, we usually write down the statement $P(n+1)$ *for arbitrary n* and ***rearrange*** it, often with elementary algebra, so that we can see that $P(n+1)$ is clearly true **if** $P(n)$ were a true statement. Or equivalently, we use $P(n)$ to rearrange the statement $P(n+1)$ so that it is obviously true, such as a statement that "$2 < 3$" or maybe "$a^2 \geq 0$" etc. two obviously true statements.

Exercises.

1. Show that $\sum_{1}^{n} r = n(n+1)/2$ is true for all n.

2. Show $n^7/7 + n^3/3 + 11n/(21)$ is an integer all n.

3. Show that $P(n) \Rightarrow P(n+1)$ *for* the statement $P(n)$: " $n^2 + n + 1$ is even ".

4. Use induction to prove $$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}^n = \begin{bmatrix} \cos n\phi & -\sin n\phi \\ \sin n\phi & \cos n\phi \end{bmatrix}$$

5. Show that $\sum_{1}^{n} r^3 = [\sum_{1}^{n} r]^2$ is true for all $n \geq 1$.

6. The famous binomial theorem: $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$ is true all $n \geq 1$.

Of interest:

1. *Euler*'s formula for primes: $E_1(n) = n^2 + n + 41$
Gives 40 different prime numbers for $n = 0, 1, 2, \dots 39$.
Obviously $E_1(41)$ cannot be prime.

2. $E_2(n) = n^2 - 79n + 1601$ is prime for $n = 0, 1, 2, \dots, 79$
Not all 80 primes are distinct. Obviously $E_2(1601)$ is not prime.

3. *Fermat* numbers: $F_n = 2^{2^n} + 1 \qquad n = 0, 1, 2, 3, \ldots$

Pierre *Fermat* conjectured in 17th century that such numbers were always prime. L. *Euler* noticed that F_5 had a factor of 641.

To this day, no *Fermat* number other than the first five has been found to be prime. These numbers are the same *Fermat* numbers in *Gauss*' theorem on regular polygon construction.

4. Consider the statement $P(n)$: $n \le 10^{100}$

This statement is true for $n = 1, 2, 3, \ldots, 10^{100}$.

Clearly $P(n)$ is false for $n = 10^{100} + 1$.

5. Two examples showing the necessity of induction.

Example 5.1

Place *n* distinct points on a circle. Join all *n* points together with lines.
How many segments of area are obtained?

Solution: Draw diagrams for the following cases. We see:

For	$n = 1$	we get $S(1)$ the number of segments = 1 = 2^0
	$n = 2$	we get $S(2)$ the number of segments = 2 = 2^1
	$n = 3$	we get $S(3)$ the number of segments = 4 = 2^2
	$n = 4$	we get $S(4)$ the number of segments = 8 = 2^3
	$n = 5$	we get $S(5)$ the number of segments = 16 = 2^4

Surely $S(n) = 2^{n-1}$ *all* $n \ge 1$.

It turns out that this is true only for $n \le 5$

and that $S(n) = \dfrac{n^4 - 6n^3 + 23n^2 - 18n + 24}{24}$ *is true for all* $n \ge 1$.

*See Martin Gardner's "**Mathematical Circus**" Pages 177, 180,181.*

Example 5.2
We demonstrate a remarkable statement $P(n)$ which is true for
$n = 1, 2, 3, \ldots, 26{,}860$ then suddenly fails for the case $n = 26{,}861$.

Let $\pi_1(n)$ denote the number of primes of the form $4k + 1$ which are not greater than n.
Let $\pi_3(n)$ denote the number of primes of the form $4k + 3$ which are not greater than n.

$\pi_1(1) = 0$		$\pi_3(1) = 0$	
$\pi_1(2) = 0$		$\pi_3(2) = 0$	
$\pi_1(3) = 0$		$\pi_3(3) = 1$	*3 (when k = 0)*
$\pi_1(4) = 0$		$\pi_3(4) = 1$	
$\pi_1(5) = 1$	*5 (when k = 1)*	$\pi_3(5) = 1$	
$\pi_1(6) = 1$		$\pi_3(6) = 1$	
$\pi_1(7) = 1$		$\pi_3(7) = 2$	*3, 7 (when k = 0, 2)*
......		....	
$\pi_1(10) = 1$		$\pi_3(10) = 2$	
$\pi_1(11) = 1$		$\pi_3(11) = 3$	*3, 7, 11 (when k = 0,2,3)*
$\pi_1(17) = 3$	*5, 13, 17 (when k = 1, 3, 4)*	$\pi_3(17) = 3$	*3, 7, 11 (when k = 0,2,3)*
......		....	
$\pi_1(100) = 11$		$\pi_3(100) = 13$	
5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97		3, 7, 11, 19, 23, 31, 43, 47, 59, 67, 71, 79, 83	
......		....	
......		....	

Consider the statement $P(n)$: $\pi_1(n) \le \pi_3(n)$

We see that $P(n)$ is true for $n = 1, 2, 3, \ldots, 7$ and for $n = 10, 11, 17$ & 100. With much patience, the inequality can be shown to hold for $n = 1, 2, \ldots, 26{,}860$. Surely the statement is always true?

In 1957 J. Leech showed that
$\pi_1(26{,}861) = 1473$ and $\pi_3(26{,}861) = 1472$,
i.e. $\pi_1(n) > \pi_3(n)$ for $n = 26{,}861$. $P(n)$ is false for $n = 26{,}861$.

Induction is certainly necessary if we want to sleep soundly at night and not have nightmares about possible exceptions to various equations and inequalities.

Induction examples.

Example $\sum_1^n r = n(n+1)/2$ is true for all $n \geq 1$.

Proof by induction. Let $P(n)$ be the statement: $\sum_1^n r = n(n+1)/2$

We easily see that $P(1)$ is true.
We next show that $P(n) \Longrightarrow P(n+1)$ (for arbitrary $n \geq 1$.)
$P(n+1)$ is the statement: $\sum_1^{n+1} r = (n+1)(n+2)/2$

Our aim is to rearrange this statement $P(n+1)$ somehow so that we can see:

If $P(n)$ *were* true, then the statement $P(n+1)$ would be obviously true.
Consider LHS of the statement $P(n+1)$:

$$\sum_1^{n+1} r = \sum_1^{n} r + (n+1) \qquad \textit{summation definition}$$

$$= n(n+1)/2 + (n+1) \qquad \textbf{\textit{if P(n) were} true}$$

$$= (n+1)(n+2)/2$$

$$= \text{RHS of statement } P(n+1)$$

Therefore $P(n) \Longrightarrow P(n+1)$, *and recall* $P(1)$ *is true*.
By the axiom of induction $P(n)$ *is true all* $n \geq 1$. ***Q.E.D.***

Example: $\sum_1^n r^2 = n(n+1)(2n+1)/6$ is true for all $n \geq 1$.

Proof by induction. Let $P(n)$ be the statement: $\sum_1^n r^2 = n(n+1)(2n+1)/6$

We easily see that $P(1)$ is true.
We next show that $P(n) \Longrightarrow P(n+1)$ (for arbitrary $n \geq 1$.)
$P(n+1)$ is the statement: $\sum_1^{n+1} r^2 = (n+1)(n+2)(2n+3)/6$

Consider LHS of $P(n+1)$: $\sum_1^{n+1} r^2 = \sum_1^{n} r^2 + (n+1)^2$ *summation definition*

$$= n(n+1)(2n+1)/6 + (n+1)^2 \qquad \textbf{\textit{if P(n) were} true}$$

$$= (n+1)(n+2)(2n+3)/6 \qquad \textit{basic algebra}$$

$$= \text{RHS of statement } P(n+1)$$

Therefore $\mathbf{P(n)} \Longrightarrow P(n+1)$. *By the axiom of induction* $P(n)$ *is true all* $n \geq 1$. ***Q.E.D.***

Example: $\sum_{1}^{n} r^3 = (n+1)^2/4$ is true for all $n \geq 1$.

Proof by induction. Let $P(n)$ be the statement: $\sum_{1}^{n} r^3 = n^2(n+1)^2/4$

We easily see that $P(1)$ is true.
We next show that $P(n) \Longrightarrow P(n+1)$ (For arbitrary $n \geq 1$.)

$P(n+1)$ is the statement: $\sum_{1}^{n+1} r^3 = (n+1)^2(n+2)^2/4$

Consider LHS of $P(n+1)$:

$$\sum_{1}^{n+1} r^3 = \sum_{1}^{n} r^3 + (n+1)^3 \qquad \textit{summation definition}$$

$$= n^2(n+1)^2/4 + (n+1)^3 \qquad \textit{if } \boldsymbol{P(n)} \textit{ were } \textbf{true}$$

$$= (n+1)^2(n+2)^2/4 \qquad \textit{basic algebra}$$

$$= \text{RHS of statement } P(n+1)$$

Therefore $P(n) \Longrightarrow P(n+1)$. *By axiom of induction P (n) is indeed true n ≥ 1.* ***Q.E.D.***

Example If $P(n)$ is the statement: " n^2+n+1 is even " then $P(n) \Longrightarrow P(n+1)$.

Proof Statement $P(n+1)$ is " $(n+1)^2+(n+1)+1$ is even "
or "$(n^2+n+1)+(2n+2)$ is even " simple rearrangement.

Clearly, **if** $\boldsymbol{P(n)}$ **were true**, then $P(n+1)$ would also be true, i.e. $P(n) \Longrightarrow P(n+1)$
Q.E.D.

Perhaps note that $P(n)$ is never a true statement.
But, **if** $\boldsymbol{P(n)}$ **were true**, then $P(n+1)$ would also be true for any positive integer n.

A most common error is saying:

'*P(n) is even*' or '*P(n) is divisible by 5*' or '*n (n+1) / 2 is true*' etc.
$P(n)$ is a statement (equation, inequality, etc.). The statement can be *true* or *false*;
a *statement* is never *even nor odd*, we do not divide statements by numbers etc.

Whereas a *number* can be *even* or *odd*, or divisible by 5 etc.
We never say a number is *true* or *false*.
What would it mean to say $(n+1)(n+4)$ is true, or that $(n+5)$ is false?

Example. $n^7/7 + n^3/3 + 11n/(21)$ is an integer for all $n \geq 1$.

Proof Let $P(n)$ be the statement: " $n^7/7 + n^3/3 + 11n/(21)$ is an integer "
We immediately see: $P(1)$: "$(1^7/7 + 1^3/3 + 11/(21))$ is an integer "
is a true statement.
We show that $P(n) \Longrightarrow P(n+1)$, *for arbitrary* $n \geq 1$.

$P(n+1)$ is the statement: " $(n+1)^7/7 + (n+1)^3/3 + 11(n+1)/(21)$ is an integer "

In this form, it is certainly not obviously true, even if we assumed $P(n)$ to be true.
We try and ***rearrange*** the statement $P(n+1)$ somehow, so that we can see that $P(n+1)$ is indeed obviously true if $P(n)$ were true.

Consider the number: $(n+1)^7/7 + (n+1)^3/3 + 11(n+1)/(21)$ this number can be rearranged to: $n^7/7 + n^3/3 + 11n/(21) + m$ where m is some integer.

In this form, it is clear that **if** $P(n)$ **were true**, then $P(n+1)$ would also be a true statement.
$P(1)$ is true and $P(n) \Longrightarrow P(n+1)$ so that by (mathematical) induction $P(n)$ is true all $n \geq 1$.
Q.E.D.

The difficulty above is in rearranging $(n+1)^7/7 + (n+1)^3/3 + 11(n+1)/(21)$ *to*
$(n^7/7 + n^3/3 + 11n/(21)) + m$ (m integer)
The binomial theorem is most useful here with the fact that the binomial coefficients ${}^7C_r$ are easily seen to be divisible by 7 for $1 \leq r \leq 6$.

Example: $11^n - 4^n$ is divisible by 7 for all $n = 1, 2, 3, 4, ...$

Proof: (By induction) Let $P(n)$ be the statement: " $11^n - 4^n$ is divisible by 7 "

We see: $P(1)$: "$11 - 4$ is divisible by 7 " is a true statement.
Consider $P(n+1)$: " $11^{n+1} - 4^{n+1}$ is divisible by 7 "

We try and rearrange the statement $P(n+1)$ somehow, so that we can see that $P(n+1)$ is indeed clearly true if $P(n)$ were true.
That is, we try to rearrange the number $11^{n+1} - 4^{n+1}$ in $P(n+1)$ somehow, so that it is obvious that **if** $P(n)$ **were true**, then $P(n+1)$ would also be true.

$$11^{n+1} - 4^{n+1} = 11.11^n - 4.4^n$$
$$= 11.\,11^n - 11.4^n + 7.4^n$$
$$= 11(11^n - 4^n) + 7.4^n \qquad \text{(using basic algebra)}$$

$P(n+1)$ can be written: "$11(11^n - 4^n) + 7.4^n$ is divisible by 7 ".

In this form it is now obvious that **if** $P(n)$ **were true**, then $P(n+1)$ would also be true.

$P(1)$ is true and $P(n) \Longrightarrow P(n+1)$
so that by (mathematical) induction $P(n)$ is true all $n \geq 1$. **Q.E.D.**

Appendices. I Induction. II Kepler's Laws. III Sundials.

Example: $7^n - 2^n$ is divisible by 5 for all $n = 1, 2, 3, 4, \ldots$

Proof: (By induction) Let *P(n)* be the statement: " $7^n - 2^n$ is divisible by 5 "

We see: $P(1)$: " $7 - 2$ is divisible by 5 " is a true statement.

We show that $P(n) \Longrightarrow P(n+1)$

Consider *P(n+1):* " $7^{n+1} - 2^{n+1}$ is divisible by 5 "
we see $7^{n+1} - 2^{n+1} = 7(7^n) - 2(2^n) = 7(7^n - 2^n) + 5(2^n)$

So that P (n+1) is the statement: " $7(7^n - 2^n) + 5(2^n)$ is divisible by 5 "

With *P(n+1)* in this form, it is clear that **if *P(n)* were true**, then *P(n+1)* is also true.

P(1) is true and $P(n) \Longrightarrow P(n+1)$ so that by induction *P (n)* is true all $n \geq 1$. **Q.E.D.**

Example. Show that $P(n) \Longrightarrow P(n+1)$ for the statement
$P(n)$: " $n^2 + 5n + 2$ is odd "

Consider *P(n+1)*: $(n+1)^2 + 5(n+1) + 2$ is odd
or $(n^2 + 5n + 2) + 2(n+3)$ is odd (using basic algebra)

With *P(n+1)* in this form it is obvious that **if *P* (*n*) were true**, then *P (n+1)* would also be true, i.e. $P(n) \Longrightarrow P(n+1)$.

Of course P (n) is never true, but :

If* P(n) were true, then we know that P(n+1) would also be true.*

Induction proof of the Binomial Theorem (not easy, but a nice example to go over).

We show the statement $P(n)$: " $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$ " is true all $n \geq 1$.

We easily see that $P(1)$ is true.
We next show that $P(n) \Rightarrow P(n+1)$ (For arbitrary $n \geq 1$.)

$P(n+1)$ is the statement: $(a+b)^{n+1} = \sum_{r=0}^{n+1} \binom{n+1}{r} a^{n+1-r} b^r$

Our aim is to rearrange this statement $P(n+1)$ somehow so that we can see:
If $P(n)$ *were* true, then the statement $P(n+1)$ would be (clearly) true.

Consider $(a+b)^{n+1}$ the LHS of the statement $P(n+1)$:

$$(a+b)^{n+1} = (a+b)(a+b)^n$$

Now *if* $P(n)$ *were* true, then we could write the LHS of $P(n+1)$ as follows:

$$\begin{aligned}(a+b)^{n+1} &= (a+b) \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \\ &= a \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r + b \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \\ &= \sum_{r=0}^{n} \binom{n}{r} a^{n+1-r} b^r + \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^{r+1} \quad \ldots\ldots \quad \textit{equation } 1\end{aligned}$$

Our aim is to somehow combine these two sums to look identical to the one sum on the RHS of $P(n+1)$. We note that the term $a^{n+1-r} b^r$ in the left sum of *equation 1* above looks closer to our desired form than does the term $a^{n-r} b^{r+1}$ in the right sum of *equation 1*. This suggests that we fiddle with the right sum, change the dummy variable r hoping that we can combine these two sums into one sum.

Set $r+1 = k$ in the sum $\sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^{r+1}$ to get $\sum_{k=1}^{k=n+1} \binom{n}{k-1} a^{n+1-k} b^k$

since $k = 1$ when $r = 0$, & $k = n+1$ when $r = n$, & $k = r-1$.

So that *equation 1* becomes: $(a+b)^{n+1} = \sum_{r=0}^{n} \binom{n}{r} a^{n+1-r} b^r + \sum_{k=1}^{k=n+1} \binom{n}{k-1} a^{n+1-k} b^k$

$(a+b)^{n+1} = \ldots\ldots \quad \ldots\ldots$ (*3 more lines*)

$(a+b)^{n+1} = \sum_{r=0}^{n+1} \binom{n+1}{r} a^{n+1-r} b^r$ *QED*

We have shown that *if* $P(n)$ *were* true, then $P(n+1)$ would be true,
i.e. we've shown $P(n) \Rightarrow P(n+1)$.

This implication (which was done for arbitrary n), together with statement $P(1)$ being true, guarantees (by the principle of induction) that $P(n)$ is indeed true for all integers $n \geq 1$. The usual difficulty in a proof by induction is rearranging $P(n+1)$; this is especially true in the above example.

Appendices. I Induction. II Kepler's Laws. III Sundials.

Some fun examples from Polya's Mathematics and Plausible Reasoning and Colin Clark's Elementary Mathematical Analysis.

1. $\sum_1^n \frac{1}{4r^2-1} = n/(2n+1)$ *Show true for all n = 1,2,3, ...*

2. $\prod_{r=2}^n (1-\frac{1}{r^2}) = (n+1)/2n$ *Show true for all n = 2, 3, 4....*

3. $\prod_{r=2}^n (1-\frac{4}{(2r-1)^2}) = (2n+2)/(2n-1)$ *Show true for all n = 2, 3, ...*

4. $\sum_1^n (2r-1) = n^2$ *Show true for all n = 1, 2, 3, ...*

5. $\sum_{r=0}^n a^r = \frac{1-a^{n+1}}{1-a}$ $a \neq 1$ of course. *Show true for all n = 1, 2, 3, ...*

6. What is the formula for #7 above when $a = 1$?

7. $(1+\varepsilon)^n \geq 1+n\varepsilon$ for any $\varepsilon > 0$. *Show true for all n = 1, ,2, 3, ...*

8. $2^n > n^2$ What values of n is this statement true?
Show true for all integers $n \geq 1$.

9. *If* $0 < a_i < 1$ *for all* $i = 1, 2, 3, \dots, n$ *with* $n \geq 2$ *then*

$(1-a_1)(1-a_2)\dots(1-a_n) > 1-a_1-a_2-a_3-\dots-a_n$.
Show true all n = 2, 3, …

i .e. $\prod_{i=1}^n (1-a_i) > 1-\sum_{i=1}^n a_i$ *is true for all n = 2, 3, 4, ...*

Appendix II Kepler's Laws.

Kepler's Laws 1 & 2 were proposed in year 1609 (the year the telescope was invented).
Law #3 was proposed in year 1619.

1. **The [six] planets describe ellipses about the Sun as focus.**

2. **The radius vector drawn from a planet to the Sun describes equal areas in equal times.**

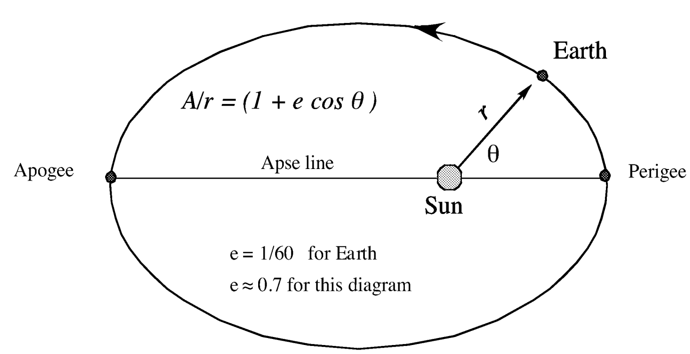
3. **The squares of the periodic times of the different planets are proportional to the cubes of the respective mean distances from the Sun.**

Kepler based these results on naked eye observations, his own and those of Tycho Brahe. These were incredibly accurate observations considering that the telescope was not invented until 1609. Isaac Newton's theory of gravity confirmed, 60 years later, that indeed the planets should follow these rules very closely.

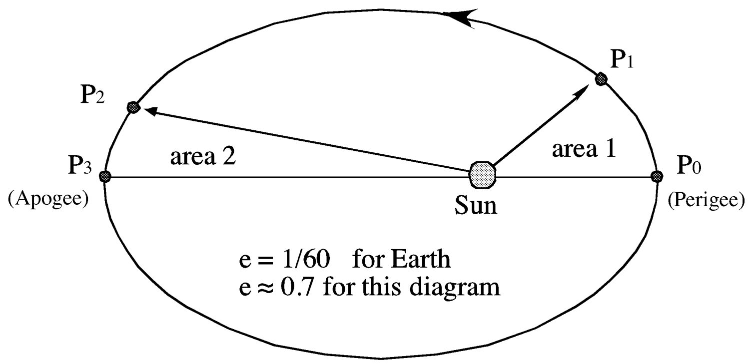
Kepler noticed that the Sun's diameter as observed from earth varied ever so slightly. The change was around 0.4 seconds of arc per day, as seen from earth. If the earth was orbiting in a circle around the sun, then the observed diameter of the Sun should remain constant. From earth, the maximum angular diameter of the Sun is $32'\ 36''$ (at Perigee), the minimum is $31'22''$ (at Apogee).

Kepler observed that the Sun's diameter varied inversely with the quantity $(1+\ e\ cos\ \theta)$ where θ was the angle between Earth's radius vector and its apse line, e is a constant (eccentricity of the ellipse). Using the maximum and minimum angular diameter of the sun, Kepler showed that $e \approx 1/60$ for Earth's orbit.
Since the polar equation of an ellipse is $A/r = (1+ e\ cos\ \theta)$ some constant A, it follows that the Earth does indeed describe an ellipse around the sun.

For the ellipse in the diagram below, e ≈ 0.7. A diagram with an ellipse with e = 1/60 would be indistinguishable from a circle, both being within the thickness of the boundary lines drawn by the finest laser printer.



Newton's theory of gravity showed that Kepler's second law was essentially the fact that angular momentum for any planet is constant.



If area 1 equals area 2, then the time taken for a planet to travel from P_0 to P_1 is the same as taken to travel from P_2 to P_3. The planet travels fastest when nearest the Sun and slowest when furthest from the sun.

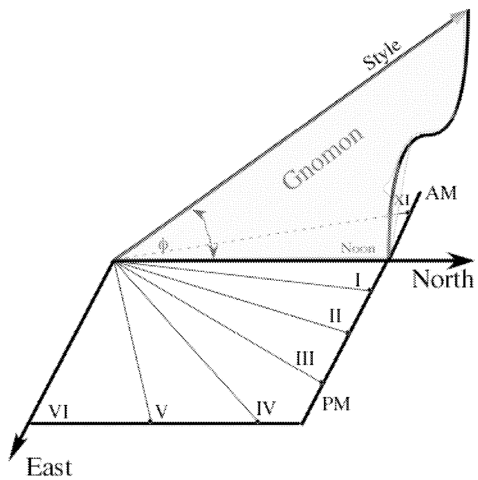
Corrections to Kepler's Third Law. (From Horace Lamb's "*Dynamics*")

When prolonged observations are made, it is found that Kepler's laws do not give a perfect description of the planetary motions; and the theory of universal gravitation supplies a reason why they should be departed from. The laws in question would, on this theory, hold rigorously for a system of planets which were themselves destitute of attractive power; but the accelerations usually produced by the planets in one another, and in the Sun, though comparatively slight, are sufficient to produce modifications of the orbits. As some of the effects are cumulative, the changes may in time become considerable.
Comparing the mass of two planets m, m′ revolving round the Sun, whose mass is S (say). With D, D′ average distances to the Sun and T, T′ periods of revolution about Sun for the respective planets, we have the amended form of Kepler's third law:

$$\left(\frac{D}{D'}\right)^2 = \frac{(S+m)}{(S+m')}\left(\frac{T}{T'}\right)^3$$

Appendix III. Sundials

(a) The Horizontal Sundial.



There are five adjustments needed to set up and read an accurate sundial. Once set up, the sundial will be accurate for at least 10, 000 years.

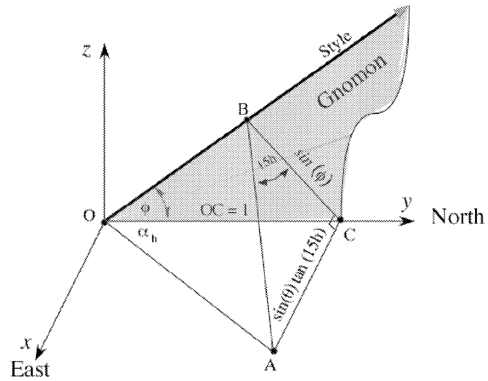
1. The style must have the same inclination as the latitude of the location of the sundial. Latitude at Lethbridge is 50° North. The high end of the style is North, the low end South.

2. The plane of the gnomon must be vertical and must align along the local meridian. That is, the style must point true north, not magnetic north which could be quite different. Magnetic north is 15° East of True North in Lethbridge, 2013. This difference varies slightly year to year.

3. The hour lines on the horizontal base of the sundial must be calculated for the appropriate latitude of the sundial. The sundial will then work for any location on that latitude (northern or southern hemisphere). At latitude ϕ the hour angles are seen to be

$$\alpha_h = tan^{-1}[tan\ (15h)\ sin\ (\phi)] \qquad \text{for hours} \quad h = 1, 2, 3, \ldots..$$

see diagram below.



$\tan(\alpha_h) = \sin(\theta) \tan(15h)$

$\alpha_h = \tan^{-1}[\sin(\theta) \tan(15h)]$

Angle OBC = 90°
Angle OCA = 90°
Angle BCA = 90°

At noon (sundial noon) the sun is directly above the meridian of the sundial location. The shadow of the style will fall along the North–South line directly below the style.

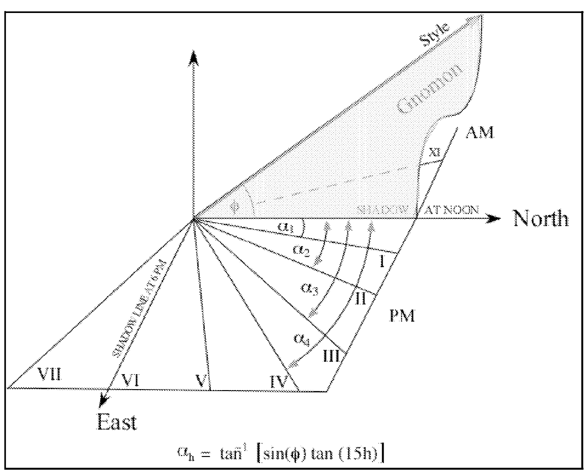
When the Sun is h hours past noon, the shadow of the style will fall along OA. To construct the sundial, we need to find the hour angles α_h for various values of h.

Let OC = 1 unit, then BC = $\sin\phi$ (right triangle OBC)

AC = $\sin(\phi)\tan(15h)$ (right triangle ACB)

$\tan\alpha_h = \sin(\phi)\tan(15h)$ (right triangle OCA)

$\alpha_h = \tan^{-1}(\sin(\phi)\tan(15h))$



Recall that the sun moves 15° per hour, hence the argument *15h* in the above functions.

For the half hour markings, we can use $h = 0.5, 1.5,$ etc.
and for the quarter hour markings we use $h = 0.25, 0.75, 1.25,$ etc.
(however, it is usually accurate enough to simply bisect the hour angles, repeatedly).

Always the 6 PM hour line is due East.

Always the 6 AM hour line is due West.

The 7, 8, 9, … PM lines are extensions of the 7, 8, 9, … AM lines.

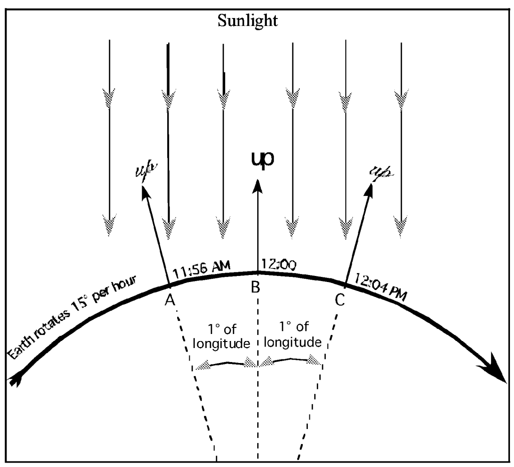
4. Longitude adjustment.

As well as the latitude, we need to know the longitude of the sundial location.

To every sundial reading we must add four minutes for each degree west of a meridian that is a multiple of 15°. Meridian 105° (Moose Jaw) is for Lethbridge a convenient reference meridian. Since Lethbridge is 113°W, we add 4 minutes for each of the eight degrees west of Moose Jaw, to the time showing on the sundial. So that for Lethbridge, we will always add 32 minutes to the time showing on the sundial. This will then give the same time (after the equation of time adjustment) as given by all clocks in the 7th time zone.

If we wanted Pacific Time, we would subtract 4 minutes for each of the the 7 degrees east of the 120° meridian, of the 8th time zone. This would give us Pacific Time, one hour earlier than Mountain Standard Time of zone 7.

Of interest: for sundials in Lethbridge, or anywhere along the 113° W meridian, it is usually convenient to add 32 minutes to each value in the equation of time. This adjusted equation of time table with 32 minutes added to each entry is good for all sundials along longitude 113°W (but not for other longitudes).



If a sundial is reading 12 noon at point B, we see that a sundial gnomon is casting a shadow on the PM side at point C. The sundial will read 4 minutes past noon if C is 1° different from B's longitude. Recall that the gnomon of the sundial is always *up* in the vertical plane.

Similarly, if a sundial is reading 12 noon at point B (above), then at point A, the sun has not yet reached transit. The shadow of the gnomon at A will be on the AM side, by an amount of 4 minutes for every 1° difference A's longitude is from B's longitude.

5. Equation of Time adjustment.

We must add (or subtract) the amount of time indicated by the Equation of Time tables for the day of reading the sundial. This adjustment varies day to day. This adjustment is the same for all longitudes and all latitudes around the globe. The one set of adjustments is good for all sundial owners.

Although the Earth rotates about its axis at a very constant rate, it moves around the Sun at various speeds. It moves fastest when closest to the Sun (Kepler's second law), that is, near the end of December. This varying speed along with the tilt of the earth's axis combines to make sundial noon differ from clock noon in a range from
–16 to +14 minutes (to the nearest minute). The daily differences are pretty well constant from year to year for all locations on Earth. These minor differences are available in all almanacs. See below.

Summer Time.

If summer time is in effect, we must add a further hour to the time shown by the sundial. Perhaps note that Saskatchewan has no summer time adjustments. Summer time in Canada is in effect from the second week in March through through first week of November.

Setting up.

The first two requirements 1 and 2 above, ensure that the style is aligned parallel to the line (axis) joining the North pole to the South pole that the earth spins around once every 24 hours, 15° every hour, 1° every 4 minutes. In Australia the gnomon is set pointing true South, with the high end South, since the Australian sun shines from the North (in most locations of Australia).

Finding True North.

1. If you know the deviation from the magnetic north you can find True North with a magnetic compass. For Lethbridge, True North is approximately 17° west of Magnetic North. The amount of magnetic deviation varies around the globe; some locations have no deviation (if they are on a so called Agonic line).

2. Another method to find True North is to find the time of the Sun's transit, i.e. when it is highest in the sky, directly over the local meridian. This is the mid point in time between sunrise and sunset. Sunrise-sunset times are usually given in the local newspapers. At the sun's transit the shadow of any vertical pole points true north.
For Lethbridge December 2/2005: Sunrise was 8:09 AM, Sunset was 4:33 PM so that the Sun's transit (over the meridian 113°) was 12:21 PM on Dec. 2. In fact, this holds true for any year.

3. Another method, perhaps the easiest: Add the location longitude adjustment and the appropriate equation of time adjustment to 12:00. At this adjusted time, on your watch, the shadow of any vertical pole points true north, i.e. the shadow falls along the meridian of the pole's location.

For example: In Lethbridge the longitude adjustment of time is +32 minutes. The equation of time adjustment on December 2 for example, is –11 minutes. So that at 12:21 PM MST, December 2, the shadow of any vertical pole in Lethbridge falls along the meridian 113° W. (through Lethbridge) thus pointing due North. Your sundial should be set so that it reads 12:00 noon when your watch reads 12:21 PM on December 2.

Example 1.

The ***Jim Coutts*** horizontal sundial near Nanton, Alberta, at 50.35° N, 113.75°W has its style set at 50.35° pointing North. The hour lines I to VI are at: 12°, 24°, 38°, 53°, 71° & 90°. The daily longitude adjustment is (113.75 – 105) x 4 = + 35 minutes for all readings. Always 35 minutes must be added to every sundial reading.

Example 2.

The ***Bill Hird*** horizontal sundial near Nerang, Qld. Australia, at 28° S, 153. 25°E has its style set at 28° pointing South. The hour lines I to VI are at: 7°, 15°, 25°, 39°, 60° & 90°. The daily longitude adjustment is (150 – 153.25)x 4 = – 13 minutes for all readings. Always 13 minutes must be subtracted from every sundial reading.

Example 3.

When a Lethbridge (50°N, 113°W) sundial reads 1:46 PM on December 11, then MST = 1:46 – 7 minutes + 32 minutes = 2:11 PM. (Mountain Standard Time.)
Since the equation of time for December 11th is –7 minutes.

Example 4.

When a Lethbridge sundial reads 12:00 noon (the sun is at transit) on February 5th, then MST = 12:00 + 14 minutes + 32 minutes = 12:46 PM
Since the equation of time for February 5th is +14 minutes.

Example 5.

When a Lethbridge sundial in Medicine Hat (50°N, 110.5° W) reads 8:21 AM on April 12th, then MST = 8:21 + 1 minute + 22 minutes = 8:44 AM

Note 1. Since Medicine Hat and Lethbridge share the same latitude 50°N, a Lethbridge sundial works equally well in Medicine Hat.
Note 2. Since Medicine Hat is on longitude 110°30'W, it is 5.5° west of longitude 105° W, the longitude adjustment for any sundial reading in Medicine Hat is always + 22 minutes.

Example 6.

When a Lethbridge sundial in Vancouver (49°N, 123° W) reads 11:02 AM on August 1st, then PST = 11:02 + 6 minutes + 12 minutes + 1 hour = 12:20 PM (Pacific Summer Time)
Note 1. Vancouver's latitude is close enough to that of Lethbridge so that the sundial works well enough.
Note 2. Vancouver is 3°W of longitude 120° W, so that 12 minutes must be added to all sundial readings.
Note 3. The date is August, summer time (+1 hour) will be in effect in British Columbia.

Example 7.

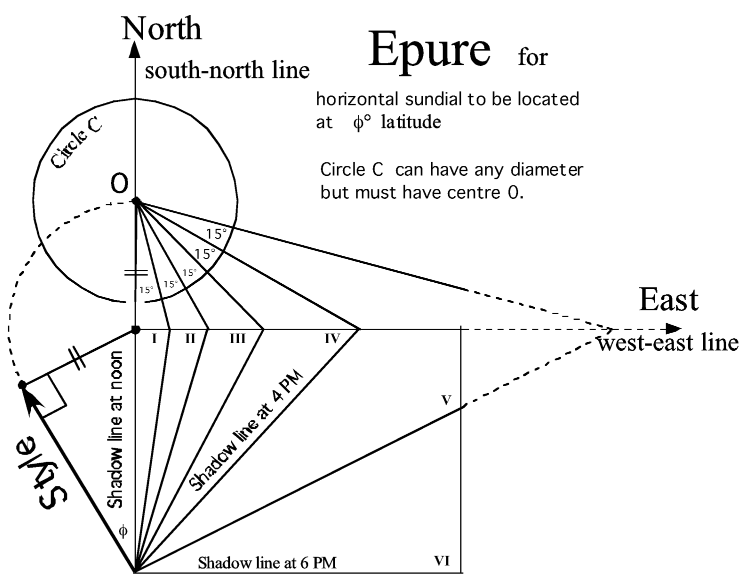
When a Los Angeles sundial in L.A. (33°N, 118° W) reads 4:22 PM on April 15th, then Pacific Time = 4:22 + 0 minutes – 8 minutes = 4:14 PM

Note 1. A Lethbridge sundial is no use in L.A. The style is too steep and the hour angles all wrong. We must use an L.A. sundial with a 33° style and corresponding hour angles.
Note 2. Los Angeles is 2° East of 120° W. L.A. uses the 8th time zone (Pacific Time) so that 8 minutes must be *subtracted* from every sundial reading taken in L.A.
If Los Angeles chose to use Mountain Standard Time, then we would note that L.A. is 13° West of longitude 105° W and add 52 minutes to every sundial reading to get MST.

<u>Constructing the hour lines for the horizontal sundial using an *epure*.</u>

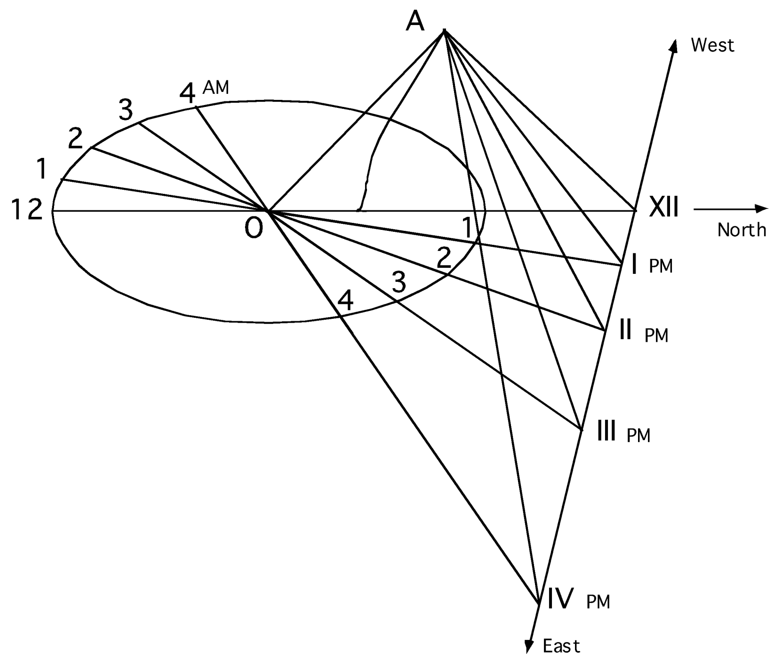
The following diagram is an epure, a device for constructing the hour lines on the base of the horizontal sundial. This diagram is on the cover of *Rohr*'s great book on sundials.
To understand the diagram, imagine a circular disc with centre O at the tip of the style, the normal to this disc (going through O) lying along the style. The hour markings on the circumference of this disk would be uniformly distributed every 15°. This set up is the same as in an armillary sphere sundial.
Now imagine the hour lines from point O extended to meet the front, west-east line of the sundial plate. Finally, imagine the circular disk along with these extended hour lines rotated about this west-east line until the disk is horizontal and in the same plane as the sundial base. In the picture below, the gnomon has been laid horizontal, 90° about the south-north line for esthetic reasons only; this is not necessary.

We see, as with the armillary sundial, that circle C can have any diameter; however, for practical purposes, the larger the diameter the better the accuracy. These epures were used by sundial makers as they required no trigonometric tables to construct their dials.
Perhaps note, any carpenter can easily make 15° angles.



The following diagram on the next page may help to understand the epure. Imagine an armillary circle attached to the end of the gnomon, such that the circle touches the horizontal plane. Recall that the hourly markings on the armillary circle are evenly spaced, every 15°. Next, imagine that the armillary circle is rotated about an East-West horizontal line through the point where the armillary circle touches the base, rotated (away from the gnomon) until it lies flat with on the horizontal plane. So that the centre of the armillary circle pt. sssA of the figure below goes to point O in the diagram of the epure above.

Recall that the radius of this circle is not relevant, only the position of its centre on the horizontal plane.



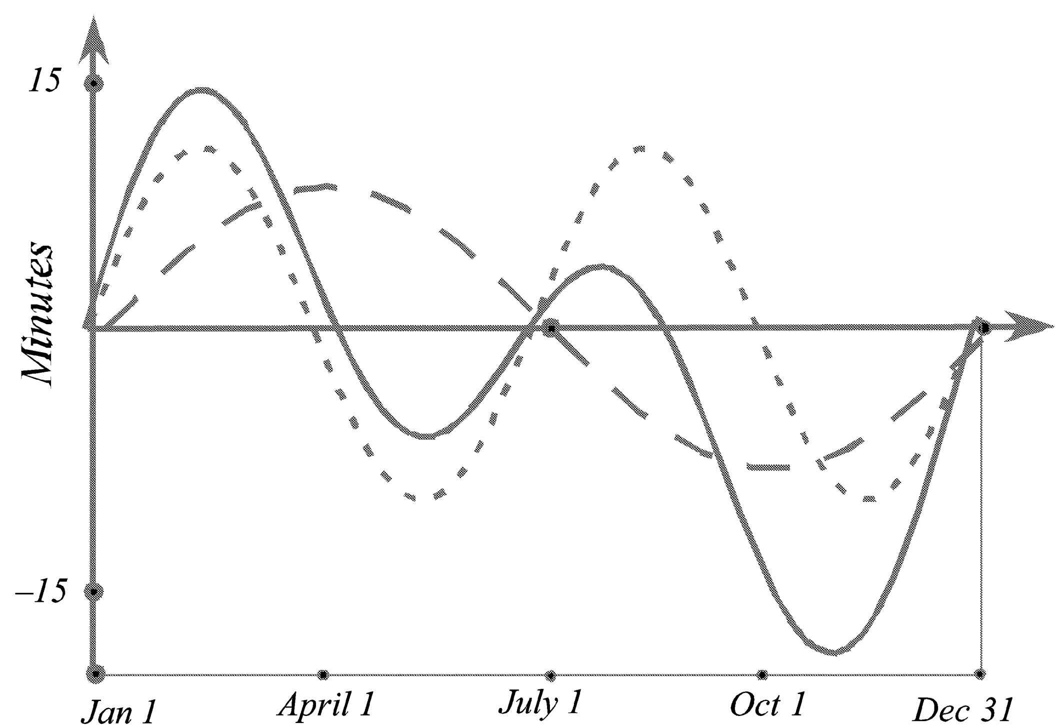
In the above diagram, the style OA is perpendicular to the line A to XII.
The style OA is normal to the plane containing A and the East-West line.

The angles XIIAI, IAII, IIAIII, IIIAIV etc. are each 15°.

Recall as the sun rotates around the style OA, it moves 15° each hour for a total of 360° in 24 hours (of *apparent time*).

***Equation of Time*.** From Whitaker's Almanack.

"Time shown by a sundial is called Apparent Solar Time. It differs from Mean Solar Time (clock time if on a longitude being a multiple of 15°) by an amount known as the Equation of Time, which is the total effect of two causes, which make the length of the apparent solar day non-uniform. One cause of variation is that the orbit of the Earth is not a circle, but an ellipse, having the Sun at one focus. As a consequence, the angular speed of the Earth in its orbit is not constant; it is greatest at the beginning of January when the Earth is nearest the Sun. The other cause is due to the obliquity of the ecliptic, as the plane of the equator (which is at right angles to the axis of the Earth) does not coincide with the ecliptic (the plane defined by the apparent annual motion of the Sun around the celestial sphere) but is inclined at an angle of 23° 27′. As a result the apparent solar day is shorter than average at the equinoxes and longer in the solstices. From the combined effects of the components due to obliquity and eccentricity, the Equation of Time reaches its maximum values in February (–14 minutes) and early November (+16 minutes). It has a zero value on four days of the year, and it is only on those dates (approx. April 15, June 14, Sept. 1 and Dec. 25) that a sundial shows Mean Solar Time."



The two dashed curves are representations of the components that sum to make the solid curve:
The smaller dashed curve represents the changes due to the 23° tilt of Earth to the plane of motion.
The larger dashed curve represents the changes due to the varying speed of Earth about the Sun.

The solid curve above is a graphical representation of the Equation of Time.

(b) The Analemmatic Sundial at CCHS West Lethbridge.

This rather new type of sundial is from the 18th century. The time is read by the shadow of a vertical post (or person) standing on the North-South axis of an ellipse mathematically configured for our latitude of 49.75° North. A person stands at the very centre of the ellipse on March 22 and Sept. 22. In the 'summer' months March 20 onwards to Sept. 24, one stands (always on the meridian line) a little north of centre, and a little south of centre in the 'winter' months September 24 to March 20. These different positions are related to the height of the sun (its declination) as it passes overhead. In mid summer, the overhead sun is some 47° higher than it is in mid winter.

To read the solar time, i.e. sundial time, stand on the North-South axis of the ellipse, positioned according to the date. The hour markers here have been set so as to take into account the sundial's longitude (113° West). The noon marker is 32 minutes to the left of the North-South axis (the meridian) of the ellipse. To get watch time, read the sundial time from your shadow, then add the appropriate '*Equation of Time*' value for the date of reading. Add one hour if Daylight Savings is in effect. The *Equation of Time* values can range from plus 14 minutes (in February) to minus16 minutes (in late October and early November).

Alberta Daylight Savings: Begins the **2nd week of March** and ends the **1st week of Nov**.

Example 1: Suppose the date is **February 1**.

a. Stand on the north-south axis, 50 centimetres ***south*** of centre of the ellipse.

Since $2.5\ tan\ (-17°)\ cos\ (49.67°) = 0.495$ metres = approx. 50 cms see general dial below
since $2a = 5$ metres for the width (main axis) of the CCHS dial
Declination of the sun on February 1 is $-17°$
Latitude of CCHS dial is $49°\ 40'$ North

b. Read from the dial the time that the middle of your shadow falls upon: 10:30 AM say.

For MST: Add the equation of time adjustment for February 1: + 14 minutes.
Mountain **S**tandard **T**ime: 10:44 AM

Example 2: Suppose the date is **May 21**.

a. Stand on the north-south axis, 59 centimetres ***north*** of the centre of the ellipse.

Since $2.5\ tan\ (20°)\ cos(49.67°) = 0.589$ metres = approx. + 59 cms
Declination of the sun on May 21 is $+20°$

b. Read from the dial the time that the centre of your shadow falls upon: 3:00 PM say.

For MST: Add the equation of time adjustment for May 20: 3 minutes.
Add 1 hour for summer time.
Mountain **S**tandard **T**ime: 4:03 PM

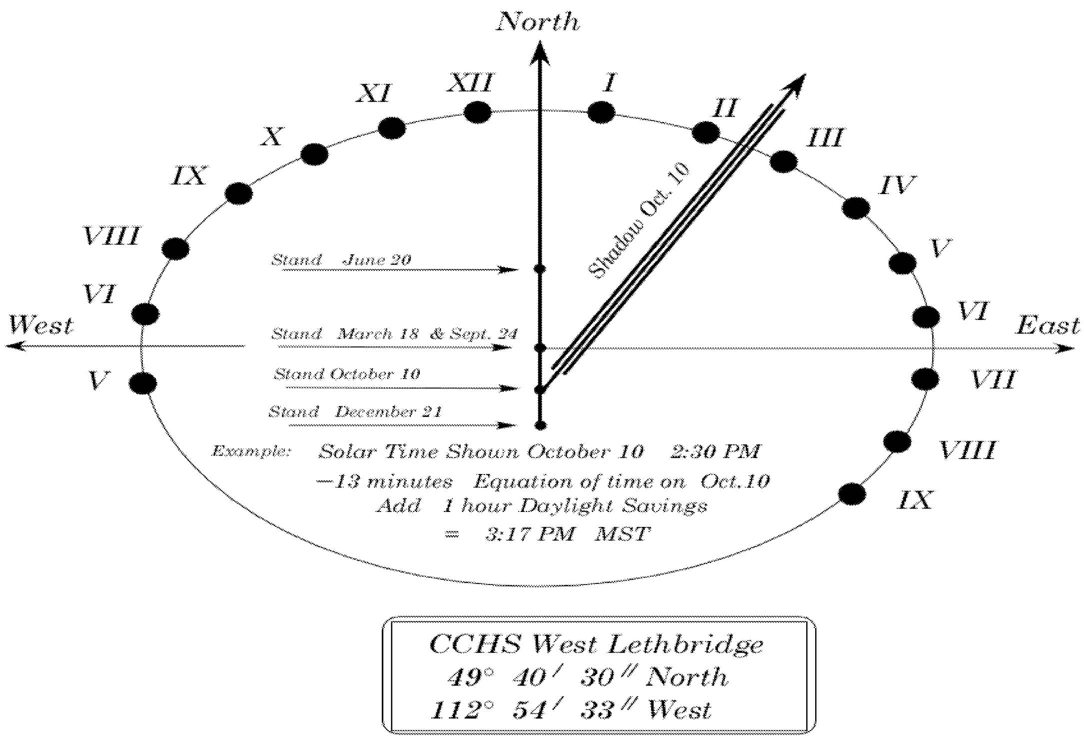
Example 3 Suppose the date is **October 10**. *See diagram below.*

a. Stand on the north-south axis, 18 centimetres **south** of the centre of the ellipse.

Since $2.5\ tan(-6.5°)\ cos(49.67°) = -0.184$ metres = approx. – 18 cms
Declination of the sun on October 10 is $-6.5°$

b. Read from the dial the time that the centre of your shadow falls upon: 2:30 PM say.

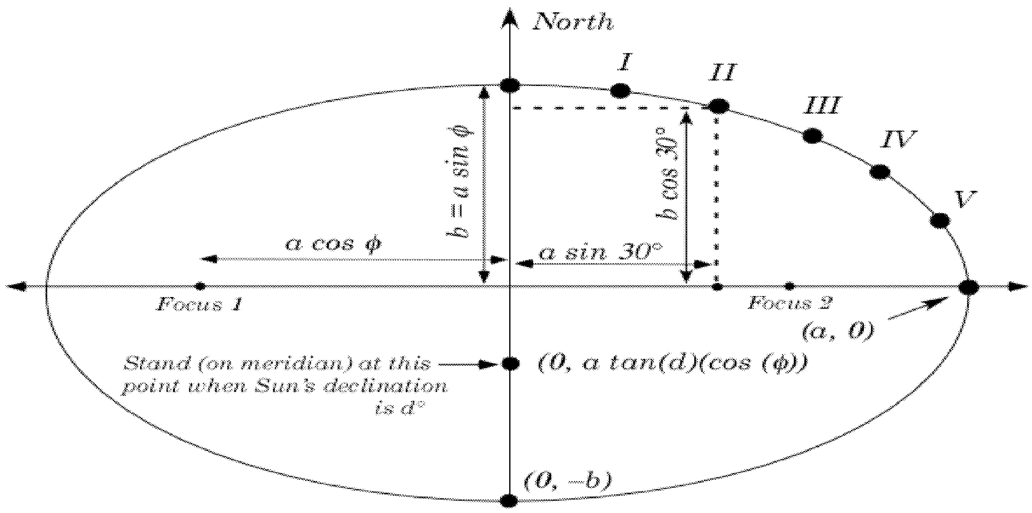
For MST: 'Add' the equation of time adjustment for October 10: –13 minutes.
Add 1 hour for summer time.
Mountain **S**tandard **T**ime: 3:17 PM



Width (major, East-West, axis of the ellipse) of CCHS West dial is	*5 metres.*
Height (minor, North-South, axis of the ellipse) of CCHS West dial is	3.8 metres

Analemmatic *Sundial Construction at latitude* $\phi°$

Draw ellipse with minor axis on the meridian:
Semi major axis (East-West) size a *metres, say.*
and semi-minor axis (North-South) size $b = a \sin \phi$
foci are $a \cos \phi$ *either side of ellipse centre.*



d° angle of Sun's declination
Hour points have coordinates $(a \sin \theta, b \cos \theta)$
where $\theta = 15h$ $(h = \text{hour } 1, 2, 3, \ldots)$
only x-coordinate is needed if ellipse is drawn.

The two focal points are used only to easily draw the ellipse. The semi major axis length ***a*** *metres can be any length, but is usually chosen to be the length of an average person's shadow around 6 PM.*
Note the standing point on the meridian through the centre of the ellipse varies with the Sun's daily declination. In summer, when the declination d° is positive, one stands $\boldsymbol{a \tan(d) \cos(\phi)}$ *metres above the major axis. In winter, the declination d° is negative, so that* $\boldsymbol{a \tan(d) \cos(\phi)}$ *is negative, hence one stands below the centre of the ellipse in winter.*
Note that one always stands along the minor axis which is aligned along the meridian $112°55'$ *West. Only the distance along the meridan, above or below the centre of the ellipse varies. This standing point depends on the Sun's declination for the day of reading.*

*Recall 2**a** metres is the East-West width of the ellipse and that* $\phi°$ *is the dial's latitude.*
For the CCHS West dial, $2a = 5$ *metres.*

Perhaps again note that the CCHS dial has each hourly marker set 32 minutes (in time) to the left (anticlockwise on the ellipse) of hourly markers of the general dial described above. This CCHS setting incorporates the longitude adjustment for the location of the CCHS dial, which is 8° west of longitude 105° (= 7x15°). So that when the shadow falls on the meridian, the time reads 12:32 PM, hence only the *equation of time* and *summer hour* (if in effect) need be added to the sundial time. The longitude adjustment for this dial has been incorporated into the hourly markers. The equation of time adjustments vary throughout the year and cannot be incorporated into the fixed dial.

A Sumo wrestler would have difficulty reading this dial; for greater accuracy, he might hold a thin vertical rod to cast the shadow.

Equation of Time

Day	Jan	Feb	Mar	April	May	June	July	Aug	Sept	Oct	Nov	Dec
1	3	14	13	4	-3	-2	4	6	0	-10	-16	-11
2	4	14	12	4	-3	-2	4	6	0	-10	-16	-10
3	4	14	12	4	-3	-2	4	6	0	-11	-16	-10
4	5	14	12	3	-3	-2	4	6	-1	-11	-16	-10
5	6	14	12	3	-3	-2	4	6	-1	-11	-16	-10
6	6	14	11	3	-3	-2	5	6	-1	-12	-16	-9
7	6	14	11	2	-3	-1	5	6	-2	-12	-16	-9
8	6	14	11	2	-4	-1	5	5	-2	-12	-16	-8
9	7	14	11	2	-4	-1	5	5	-2	-13	-16	-8
10	7	14	10	1	-4	-1	5	5	-3	-13	-16	-8
11	8	14	10	1	-4	-1	5	5	-3	-13	-16	-7
12	8	14	10	1	-4	0	6	5	-3	-13	-16	-7
13	8	14	10	0	-4	0	6	5	-4	-13	-16	-6
14	9	14	9	0	-4	0	6	5	-4	-14	-16	-6
15	9	14	9	0	-4	0	6	4	-5	-14	-16	-5
16	10	14	9	0	-4	0	6	4	-5	-14	-15	-5
17	10	14	9	0	-4	1	6	4	-5	-14	-15	-4
18	10	14	8	-1	-4	1	6	4	-6	-15	-15	-4
19	11	14	8	-1	-4	1	6	4	-6	-15	-15	-3
20	11	14	8	-1	-4	1	6	3	-6	-15	-15	-3
21	11	14	7	-1	-3	2	6	3	-7	-15	-14	-2
22	11	14	7	-1	-3	2	6	3	-7	-15	-14	-2
23	12	13	7	-2	-3	2	6	3	-7	-16	-13	-1
24	12	13	7	-2	-3	2	6	2	-8	-16	-13	-1
25	12	13	6	-2	-3	2	6	2	-8	-16	-13	0
26	12	13	6	-2	-3	3	6	2	-8	-16	13	0
27	13	13	6	-2	-3	3	6	1	-9	-16	-13	1
28	13	13	5	-2	-3	3	6	1	-9	-16	-12	1
29	13		5	-3	-3	3	6	1	-9	-16	-12	2
30	13		5	-3	-3	3	6	1	-10	-16	-11	2
31	13		4		-3		6	1		-16		3

The time (in minutes) must be added to the sundial time to give clock time.
A longitude adjustment must be made if not on a longitude of a multiple of 15°.

Examples: If Sundial location is on longitude 0°, 15° , 30° or some multiple of 15°

Sundial reads Noon on January 10, then clock time is 12:07 PM
Sundial reads 2:30 PM on October 20, then clock time is 2:15 PM

The sundial of course does not know about summer time addition of 1 hour..

Christmas Day, Boxing Day, mid June, mid April, early September:
on these days the sundial and wall clock read the same time, no adjustment needed.

Equation of Time in minutes and seconds.

Values change slightly (seconds) year to year.

DAY ↓	JAN	FEB	MAR	APR	MAY	JUNE	JULY	AUG	SEPT	OCT	NOV	DEC
1	+ 3:24	+ 13:36	+ 12:30	+ 4:05	— 2:48	— 2:18	+ 3:36	+ 6:18	+ 0:12	— 10:06	— 16:18	— 11:12
2	+ 3:54	+ 13:42	+ 12:18	+ 3:48	— 3:00	— 2:12	+ 3:48	+ 6:12	— 0:06	— 10:24	— 16:23	— 10:48
3	+ 4:18	+ 13:48	+ 12:06	+ 3:30	— 3:06	— 2:00	+ 4:00	+ 6:12	— 0:30	— 10:48	— 16:23	— 10:24
4	+ 4:48	+ 13:54	+ 11:54	+ 3:12	— 3:12	— 1:54	+ 4:12	+ 6:06	— 0:42	— 11:06	— 16:23	— 10:00
5	+ 5:12	+ 14:00	+ 11:42	+ 2:54	— 3:18	+ 1:42	+ 4:24	+ 6:00	— 1:06	— 11:24	— 16:23	— 9:36
6	— 5:42	+ 14:06	+ 11:30	+ 2:36	— 3:24	— 1:30	+ 4:36	+ 5:54	— 1:30	— 11:42	— 16:18	— 9:12
7	+ 6:06	+ 14:12	+ 11:12	+ 2:18	— 3:24	— 1:18	+ 4:42	+ 5:48	— 1:48	— 12:00	— 16:18	— 8:48
8	+ 6:30	+ 14:12	+ 11:00	+ 2:06	— 3:30	— 1:12	+ 4:54	+ 5:42	— 2:06	— 12:18	— 16:18	— 8:18
9	+ 6:54	+ 14:18	+ 10:42	+ 1:48	— 3:36	— 1:00	+ 5:00	+ 5:30	— 2:30	— 12:36	— 16:12	— 7:54
10	+ 7:18	+ 14:18	+ 10:30	+ 1:30	— 3:36	— 0:48	+ 5:12	+ 5:24	— 2:48	— 12:48	— 16:06	— 7:30
11	+ 7:48	+ 14:18	+ 10:12	+ 1:12	— 3:42	— 0:36	+ 5:18	+ 5:12	— 3:12	— 13:06	— 16:00	— 7:00
12	+ 8:11	+ 14:18	+ 10:00	+ 0:05	— 3:42	— 0:24	+ 5:24	+ 5:06	— 3:30	— 13:24	— 15:54	— 6:30
13	+ 8:30	+ 14:12	+ 9:42	+ 0:42	— 3:42	— 0:12	+ 5:36	+ 4:54	— 3:54	— 13:36	— 15:48	— 6:06
14	+ 8:54	+ 14:12	+ 9:24	+ 0:24	— 3:42	0	+ 5:42	+ 4:42	— 4:12	— 13:48	— 15:36	— 5:36
15	+ 9:18	+ 14:06	+ 9:06	+ 0:12	— 3:42	+ 0:12	+ 5:48	+ 4:30	— 4:36	— 14:06	— 15:30	— 5:06
16	+ 9:36	+ 14:00	+ 8:54	— 0:06	— 3:42	+ 0:24	+ 5:54	+ 4:18	— 5:00	— 14:18	— 15:18	— 4:36
17	+ 9:54	+ 13:54	+ 8:36	— 0:12	— 3:42	+ 0:42	+ 6:00	+ 4:05	— 5:18	— 14:30	— 15:06	— 4:05
18	+ 10:18	+ 13:48	+ 8:18	— 0:30	— 3:42	+ 0:54	+ 6:06	+ 3:54	— 5:30	— 14:42	— 14:54	— 3:36
19	+ 10:36	+ 13:42	+ 8:00	— 0:42	— 3:36	+ 1:06	+ 6:12	+ 3:42	— 6:00	— 14:54	— 14:42	— 3:12
20	+ 10:54	+ 13:36	+ 7:42	— 0:54	— 3:36	+ 1:18	+ 6:12	+ 3:30	— 6:24	— 15:06	— 14:30	— 2:42
21	+ 11:12	+ 13:30	+ 7:24	— 1:12	— 3:30	+ 1:30	+ 6:18	+ 3:12	— 6:42	— 15:12	— 14:18	— 2:12
22	+ 11:30	+ 13:24	+ 7:06	— 1:24	— 3:30	+ 1:42	+ 6:18	+ 3:00	— 7:06	— 15:24	— 14:00	— 1:42
23	+ 11:48	+ 13:30	+ 6:48	— 1:36	— 3:24	+ 2:00	+ 6:24	+ 2:48	— 7:24	— 15:36	— 13:42	— 1:12
24	+ 12:00	+ 13:24	+ 6:30	— 1:48	— 3:18	+ 2:12	+ 6:24	+ 2:30	— 7:48	— 15:42	— 13:24	— 0:42
25	+ 12:18	+ 13:12	+ 6:12	— 1:54	— 3:12	+ 2:24	+ 6:24	+ 2:12	— 8:06	— 15:48	— 13:06	— 0:12
26	+ 12:30	+ 13:06	+ 5:54	— 2:06	— 3:06	+ 2:36	+ 6:24	+ 1:54	— 8:24	— 15:54	— 12:54	+ 0:18
27	+ 12:42	+ 12:54	+ 5:36	— 2:18	— 3:00	+ 2:48	+ 6:24	+ 1:42	— 8:48	— 16:00	— 12:30	+ 0:48
28	+ 12:54	+ 12:42	+ 5:18	— 2:24	— 2:54	+ 3:00	+ 6:24	+ 1:24	— 9:06	— 16:06	— 12:12	+ 1:18
29	+ 13:06		+ 5:00	— 2:36	— 2:48	+ 3:12	+ 6:24	+ 1:06	— 9:30	— 16:12	— 11:54	+ 1:48
30	+ 13:18		+ 4:42	— 2:42	— 2:36	+ 3:24	+ 6:24	+ 0:48	— 9:48	— 16:18	— 11:30	+ 2:18
31	+ 13:24		+ 4:24		— 2:30		+ 6:18	+ 0:30		— 16:18		+ 2:48

Equation of Time (nearest minute) with 31 minutes added to each value to account for Lethbridge longitude

Lethbridge Longitude is 112° 48' W . 7° 48' west of longitude 105°W (Leth. latitude: 49° 42' N.)

DAY	JAN	FEB	MAR	APR	MAY	JUNE	JULY	AUG	SEPT	OCT	NOV	DEC
1	33	44	43	34	27	28	34	36	30	20	14	19
2	34	44	42	34	27	28	34	36	30	20	14	19
3	34	44	42	34	27	28	34	36	30	19	14	20
4	35	44	42	33	27	28	34	36	29	19	14	20
5	35	44	42	33	27	28	34	36	29	19	14	20
6	36	44	42	33	27	29	35	36	29	18	14	21
7	36	44	41	32	27	29	35	36	28	18	14	21
8	37	44	41	32	27	29	35	36	28	18	14	22
9	37	44	41	32	26	29	35	36	28	17	14	22
10	37	44	41	32	26	29	35	35	27	17	14	23
11	38	44	40	31	26	29	35	35	27	17	14	23
12	38	44	40	30	26	30	35	35	27	17	14	24
13	39	44	40	31	26	30	36	35	26	16	14	24
14	39	44	39	30	26	30	36	35	26	16	14	24
15	39	44	39	30	26	30	36	35	25	16	15	25
16	40	44	39	30	26	30	36	34	25	16	15	25
17	40	44	39	30	26	31	36	34	25	16	15	26
18	40	44	38	30	26	31	36	34	25	15	15	26
19	41	44	38	29	26	31	36	34	24	15	15	27
20	41	44	38	29	26	31	36	34	24	15	16	27
21	41	44	37	29	27	32	36	33	23	15	16	28
22	42	43	37	29	27	32	36	33	23	15	16	28
23	42	44	37	28	27	32	36	33	23	14	16	29
24	42	43	37	28	27	32	36	33	22	14	17	29
25	42	43	36	28	27	32	36	32	22	14	17	30
26	43	43	36	28	27	33	36	32	22	14	17	30
27	43	43	36	28	27	33	36	32	21	14	18	31
28	43	43	35	28	27	33	36	31	21	14	18	31
29	43	43	35	27	27	33	36	31	21	14	18	32
30	43		35	27	27	33	36	31	20	14	19	32
31	43		34		28		36	31		14		33

Examples:.

Lethbridge sundial reads **Noon** Nov 15, then Mountain Standard Time (MST) is **12:15 PM**

Lethbridge sundial reads **2:15 PM** June 20 then MST is 2:45 + 1 hr summer time:**3:45 PM**

	Declination of the Sun											
Day	**Jan**	**Feb**	**Mar**	**April**	**May**	**June**	**July**	**Aug**	**Sept**	**Oct**	**Nov**	**Dec**
1	-23	-17	-8	5	15	22	23	18	8	-3	-14	-22
2	-23	-17	-7	5	15	22	23	18	8	-4	-14	-22
3	-23	-17	-7	5	16	22	23	17	8	-4	-15	-22
4	-23	-17	-6	6	16	22	23	17	7	-4	-15	-22
5	-23	-16	-6	6	16	23	23	17	7	-5	-16	-22
6	-23	-16	-6	6	16	23	23	17	6	-5	-16	-22
7	-23	-16	-5	7	17	23	23	16	6	-5	-16	-23
8	-22	-15	-5	7	17	23	22	16	6	-6	-16	-23
9	-22	-15	-5	8	17	23	22	16	5	-6	-17	-23
10	-22	-15	-4	8	18	23	22	16	5	-7	-17	-23
11	-22	-14	-4	8	18	23	22	15	5	-7	-17	-23
12	-22	-14	-4	9	18	23	22	15	4	-7	-18	-23
13	-22	-14	-3	9	18	23	22	15	4	-7	-18	-23
14	-21	-13	-3	9	19	23	22	14	3	-8	-18	-23
15	-21	-13	-2	10	19	23	22	14	3	-8	-18	-23
16	-21	-13	-2	10	19	23	21	14	3	-9	-19	-23
17	-21	-12	-1	10	19	23	21	13	2	-9	-19	-23
18	-21	-12	-1	11	20	23	21	13	2	-10	-19	-23
19	-21	-12	-1	11	20	23	21	13	2	-10	-19	-23
20	-20	-11	0	12	20	23	21	12	1	-10	-20	-23
21	-20	-11	0	12	20	23°27'	20	12	1	-11	-20	-23°27'
22	-20	-11	1	12	20	23	20	12	0	-11	-20	-23
23	-20	-10	1	13	21	23	20	11	0	-11	-20	-23
24	-20	-10	1	13	21	23	20	11	0	-12	-21	-23
25	-19	-9	2	13	21	23	20	11	-1	-12	-21	-23
26	-19	-9	2	14	21	23	19	10	-1	-12	-21	-23
27	-19	-9	3	14	21	23	19	10	-2	-13	-21	-23
28	-19	-8	3	14	21	23	19	10	-2	-13	-21	-23
29	-18	-8	3	14	22	23	19	9	-2	-13	-21	-23
30	-18		4	15	22	23	19	9	-3	-14	-22	-23
31	-18		4		22		18	9		-14		-23

The above figures, to the nearest degree, are from Whitaker's Almanack.

Coordinates

Adelaide	34° 56′ S	138° 56′ E
Alice Springs	23.698° S,	133.88° E
Austin, Texas	30° 20′ N	47° 46′ E
Auckland NZ	35° 45′ S	174° 45′ E
Battle (U.K.)	50° 55′ N	03° 30′ W
Berlin	52° 32’ N	13° 25′ E
Boston	42.360° N	71.059° W
Brooks (Alta):	50° 35’ N	111° 54′ W
Calgary	51° 05′ N	114° 05′ W
Calcutta	22.573°N	88.3639°E
Darwin	12.463° S	130.846° E
Delhi:	28° 40′ N	77° 14′ E
Empress (AB)	50.953°N,	110.009° W
Glasgow	55° 52′ N	4° 15′ W
Halifax:	44° 38′ N	63° 35′ W
Hong Kong	22° 30′ N	114° 10′ E
Kabol:	34° 30′ N	69° 10′ E
Lethbridge	**49° 43′ N**	**112° 48′ W**
Lisbon	38° 44′ N	9° 08′ W
Los Angeles	34.052° N	118.244° W
Mecca:	21° 26′ N	39° 49′ E
Medicine Hat	50° 03′ N	110° 47′ W
Moscow	55.756° N	37.617° E
Mt Everest	27.988° N	86.925° E
Nairobi	1° 17′ S	36° 50′ E
New York	40° 40′ N	73° 50′ W
Paris	48° 52′ N	2° 20′ E
Reykjavik	64° 09′ N	21° 58′ W
Rio de Janeiro	22° 53′ S	43° 17′ W
Saint Johns	47° 34′ N	52° 41′ W
Salem Oregon	44°56'34"N	123°2' 6"W
Shanghai	34° 27′ N	121° 22′ E
Singapore	1° 22′ N	103° 55′ E
South Pole	90° 00′ 00″ S	
Sydney (Aust.)	33° 55′ S	151° 10′ E
N. Magnetic pole	86.39° N	169.80°W
Tallai (Qld)	28.0678° S,	153.3300° E
Tokyo	35° 27′ N	139° 22′ E
Vancouver	49° 13′ N	123° 06′ W
Wagga Wagga	**35° 07′ S**	**147° 24′ E**
Walla Walla	46°3'52.5"N	18°20' 35' W

Alexandria	31° 13′ N	29° 55′ E
Aswan	24° 05′ N	32° 56′ E
Athens	37° 59′ N	23° 42′ E
Bagdad	33° 20′ N	44° 26′ E
Beijing	39° 55′ N	116° 25′ E
Bern	46.948° N	7.447° E
Brisbane	27.470° S,	153.025° E
Burnley (U.K.)	53° 47′ N	2° 15′ W
Canberra ACT	35° 18′ S	149° 08′ E
Cardston AB	49.195°N	113.302° W
Dunedin	45° 52′ S	170° 30′ E
Edmonton	53° 34′ N	113° 25′ W
Fernie (B.C.)	49.504°N	115.063° W
Greenwich	51° 28′ 36.7″ N	0° 0′ 1.8″ W
Hobart	42.8821° S	147.3272° E
Juneau	58.3019° N	134.4197° W
Kingaroy (Qld)	26.5309° S	151.8400° E
Lands End:	50° 03′ N	5° 45′ W
London	51° 32′ N	0° 06′ W
Mexico City	19° 25′ N	99° 10′ W
Moose Jaw	50.3916° N	105.5349° W
Mt Rushmore	43.879°N	103.459° W
Moscow	55° 45′ N	37° 42′ E
North Pole	90° 00′ 00″ N	
Perth WA	31.9505° S	115.8605° E
Rimbey AB	52°38′27″ N	114°14′ 8.7″W
Rome	41° 54′ N	12° 29′ E
San Francisco	37° 48′ N	122° 24′ W
Salzberg	47° 54′ N	13° 03′ W
Shiraz Iran	29° 36′ N	52° 33′ E
Skagway US	59° 23′ N	135° 20′ W
Stewart B.C	55° 56′ N	130° 01′ W
Sundre AB	51°47′50″ N	114°38′26″ W
S. Magnetic pole	64.11°S	135.76° E
Tehran	35° 40′ N	51° 26′ E
Tuktoyaktuk	68° 24′ N	133° 01′ W
Whitehorse	60° 41′ N	135° 08′ W